Displacment with angle change

Our detector is at $z = L$, and our phase mask is at $z = 0$.

Each ray has a directional vector of the light $\hat{k}$. The point $\vec{P}$ of intersection of the light ray and the detecor is at $\vec{P}_z = \left(\vec{0} + l\hat{k}\right)_z = L$. Therefore:

$$l = \frac{L}{k_z}$$

And therefore:

$$\vec{P} = \frac{L}{k_z}\hat{k}$$

We notice that the angle at each point of each ray could be found using:

$$\frac{k_z}{k_x} = \tan\theta_x$$

$$\frac{k_z}{k_y} = \tan\theta_y$$

Now, we will find the displacement as a function of the angle difference:

$$P'_x - P_x = L\left(\frac{k'_x}{k'_z} - \frac{k_x}{k_z}\right) = L\left(\frac{1}{\tan\theta'_x} - \frac{1}{\tan\theta_x}\right)$$

Therefore:

$$\Delta P_x = L\left(\frac{1}{\tan\left(\theta_x + \Delta\theta_x\right)} - \frac{1}{\tan\left(\theta_x\right)}\right)$$

$$\Delta P_y = L\left(\frac{1}{\tan\left(\theta_y + \Delta\theta_y\right)} - \frac{1}{\tan\left(\theta_y\right)}\right)$$

For small changes we can approximate:

$$\Delta P_x = -\frac{L}{\cos^2\left(\theta_x\right)} \cdot \Delta\theta_x$$

$$\Delta P_y = -\frac{L}{\cos^2\left(\theta_y\right)} \cdot \Delta\theta_x$$

No mask Reconstruction

Using the equations above, we can reconstruct with no mask at all. Each point in the LF that we reconstruct, we scan $\Delta\vec{\theta}$ and gather light scanning over relevant $\Delta\vec{P}$. This is independent of the mask.

angle change with phase

Paraxial approximation

In the paraxial approximation, we will get the direction of a ray is:

$$E \propto e^{jk\cdot r}$$

Multiplying by phase$p = e^{j\phi(r)}$ we get:

$$E \propto e^{jk\cdot r}p = e^{jk\phi(r)}e^{jk\cdot r} = e^{j(k\cdot r+\phi(r))}$$

Assuming that the phase is slowly changing in r (high derivatives are negligable), or if we are looking at a small section(high powers of r are negligable) (my guess is that both of these are right. We assume the the phase changes slowly, and we will look for each ray at a certain small region) , we can approximate using taylor:

$$\phi(r) \approx \phi(0) + \vec{\nabla}\phi \cdot r$$

Therefore:

$$E \propto e^{j\left(k\cdot r+\phi(0)+\vec{\nabla}\phi\cdot r\right)} = e^{j\left(\vec{k}+\vec{\nabla}\phi\right)\cdot r}e^{j\phi(0)}$$

We might be able to ignore $\phi(0)$ as it is a global phase. However, it isnt really, as it only locally a global phase. I think these would be appropriate if we don't care about the phase (do we? speckle is dependent on the phase). Therefore

$$k_{new} = \vec{k} + \vec{\nabla}\phi$$

Now:

$$\theta_x = \arctan\left(\frac{k_z}{k_x}\right)$$

$$\theta_y = \arctan\left(\frac{k_z}{k_y}\right)$$

$$k'_z = \sqrt{1 - {k'_x}^2 - {k'_y}^2} = \sqrt{1 - \left(k_x + \frac{d}{dx}\phi\right)^2 - \left(k_y + \frac{d}{dy}\phi\right)^2}$$

$$= \sqrt{1 - k_x^2 - 2k_x\frac{d}{dx}\phi - \frac{d}{dx}^2\phi - k_y^2 - 2k_y\frac{d}{dy}\phi - \frac{d}{dy}^2\phi}$$

$$= \sqrt{1 - k_x^2 - k_y^2 - 2\vec{k} \cdot \vec{\nabla}\phi - \nabla^2\phi}$$

$$\approx 1 - 2k'_x - 2k'_y = 1 - 2k_x - 2k_y - 2\vec{k} \cdot \vec{\nabla}\phi = k_z - 2\vec{k} \cdot \vec{\nabla}\phi$$

Therefore:

$$\Delta P_x = L \left(\frac{k_x + \frac{d}{dx}\phi}{k_z - 2\vec{k} \cdot \vec{\nabla}\phi} - \frac{k_x}{k_z} \right)$$

$$\Delta P_y = L \left(\frac{k_y + \frac{d}{dy}\phi}{k_z - 2\vec{k} \cdot \vec{\nabla}\phi} - \frac{k_y}{k_z} \right)$$

And overall:

$$\Delta \vec{P} = L \left(\frac{k_x + \vec{\nabla}\phi}{k_z - 2\vec{k} \cdot \vec{\nabla}\phi} - \frac{k_x}{k_z} \right)$$

General expression

Assuming we have a ray coming at angle θ_1 to a prism with a slope at an angle both in x and y, so that there is a refraction in both x and y.

In order to make sense of everything we will write snell's law like so:

$$\vec{k} - \left(\vec{k} \cdot \hat{n}\right) \hat{n} = \vec{k'} - \left(\vec{k'} \cdot \hat{n}\right) \hat{n}$$

Where $\hat{n}$ is the normal vector. This is true as:

$$\vec{k} - \left(\vec{k} \cdot \hat{n}\right) \hat{n} = \vec{k} - (k_\perp)\, \hat{n} = \vec{k} - \vec{k_\perp} = \vec{k}_\parallel$$

Now, $\hat{n_1} = \hat{z}$ and $\hat{n_2}$ is angled with θ_x, θ_y. Therefore, we have 2 instances of snells law(ray in and ray out):

The first (I)

$$k_x = k'_x$$

$$k_y = k'_y$$

And the second (II):

$$\vec{k'} - \left(\vec{k'} \cdot \hat{n}\right) \hat{n} = \left(k'_x, k'_y, k'_z\right) - \left(k'_x n_x + k'_y n_y + k'_z n_z\right) (n_x, n_y, n_z)$$

$$= \left(k_x' - k_x' n_x^2, k_y' - k_y' n_y^2, k_z' - k_z' n_z^2\right)$$

$$\left(k_x'\left(1 - n_x^2\right), k_y'\left(1 - n_y^2\right), k_z'\left(1 - n_z^2\right)\right)$$

We can find n_x,n_y, n_z using the angles θ_x, θ_y:

$$n_x = n_z \tan\left(\theta_x\right)$$

$$n_y = n_z \tan\left(\theta_y\right)$$

We will find n_z using normalization:

$$|n|^2 = 1 \Rightarrow n_z^2\left(1 + \tan^2\left(\theta_x\right) + \tan^2\left(\theta_y\right)\right) = 1$$

$$\Rightarrow n_z^2 = \frac{1}{1 + \tan^2\left(\theta_x\right) + \tan^2\left(\theta_y\right)}$$

Therefore:

$$\vec{k'} - \left(\vec{k'} \cdot \hat{n}\right)\hat{n} =$$

$$\left(k_x'\left(1 - \tfrac{\tan^2(\theta_x)}{1+\tan^2(\theta_x)+\tan^2(\theta_y)}\right), k_y'\left(1 - \tfrac{\tan^2(\theta_y)}{1+\tan^2(\theta_x)+\tan^2(\theta_y)}\right), k_z'\left(1 - \tfrac{1}{1+\tan^2(\theta_x)+\tan^2(\theta_y)}\right)\right)$$

$$= \frac{\left(k_x'\left(1 + \tan^2\left(\theta_y\right)\right), k_y'\left(1 + \tan^2\left(\theta_x\right)\right), k_z'\left(\tan^2\left(\theta_x\right) + \tan^2\left(\theta_y\right)\right)\right)}{1 + \tan^2\left(\theta_x\right) + \tan^2\left(\theta_y\right)}$$

We can see that for $\theta_x = \theta_y = 0$ we get $\left(k_x', k_y', 0\right)$ as expected.
We will complete the second snell's law instance:

$$\frac{\left(k_x'\left(1 + \tan^2\left(\theta_y\right)\right), k_y'\left(1 + \tan^2\left(\theta_x\right)\right), k_z'\left(\tan^2\left(\theta_x\right) + \tan^2\left(\theta_y\right)\right)\right)}{1 + \tan^2\left(\theta_x\right) + \tan^2\left(\theta_y\right)}$$

$$= \frac{\left(k_x''\left(1 + \tan^2\left(\theta_y\right)\right), k_y''\left(1 + \tan^2\left(\theta_x\right)\right), k_z''\left(\tan^2\left(\theta_x\right) + \tan^2\left(\theta_y\right)\right)\right)}{1 + \tan^2\left(\theta_x\right) + \tan^2\left(\theta_y\right)}$$

Test

$$\left(k_x', k_y', k_z'\right) - \left(k_x' n_x + k_y' n_y + k_z' n_z\right)\left(n_x, n_y, n_z\right)$$

$$= \left(k_x' - \left(k_x' n_x + k_y' n_y + k_z' n_z\right) n_x, k_y' - \left(k_x' n_x + k_y' n_y + k_z' n_z\right) n_y, k_z' - \left(k_x' n_x + k_y' n_y + k_z' n_z\right) n_z\right)$$

Therefore the second instance of snell's law (II):

$$k'_x - \left(k'_x n_x + k'_y n_y + k'_z n_z\right) n_x = k''_x - \left(k''_x n_x + k''_y n_y + k''_z n_z\right) n_x$$

$$k'_y - \left(k'_x n_x + k'_y n_y + k'_z n_z\right) n_y = k''_y - \left(k''_x n_x + k''_y n_y + k''_z n_z\right) n_y$$

$$k'_z - \left(k'_x n_x + k'_y n_y + k'_z n_z\right) n_z = k''_z - \left(k''_x n_x + k''_y n_y + k''_z n_z\right) n_z$$

$$\left(k'_x n_x + k'_y n_y + k'_z n_z\right) n_x$$

Therefore:

$$(\beta - \alpha)\, n_x = \frac{k''_x - k'_x}{}$$

Path to solution, transform to a different coordinate system, so that we only have 2 vectors. Try solving for this, in 2d. Or think in terms of angles as in snell's law in 3d.

Transforming to k-space:

We have seen:

$$\theta_i = \arctan\left(\frac{k_z}{k_i}\right)$$

$$\frac{k_z}{k_i} = \tan\theta_i \Rightarrow k_i = \frac{k_z}{\tan\theta_i}$$

Using normalization:

$$1 = |k|^2 = k_z^2\left(1 + \frac{1}{\tan^2\theta_x} + \frac{1}{\tan^2\theta_y}\right)$$

Therefore:

$$k_z = \frac{1}{\sqrt{1 + \frac{1}{\tan^2\theta_x} + \frac{1}{\tan^2\theta_y}}}$$

Therefore:

$$k_x = \frac{k_z}{\tan\theta_x} = \frac{1}{\sqrt{1 + \tan^2\theta_x + \left(\frac{\tan\theta_x}{\tan\theta_y}\right)^2}}$$

LGFT

The DFT way

We want to find $x\left[k_x,k_y\right]$. First we analyze the case for DFT ($\sigma=\infty$), in 1-D.

$$x\left[k_x\right]=\sum_{n=0}^{N-1}x\left[n\right]e^{-j\frac{2\pi}{N}k_x n}$$

We know that:

$$x^{DFT}\left[k_x\right]\propto x^{DTFT}\left[k_x\cdot L\right]=x^{DTFT}\left[\frac{2\pi}{\lambda}\sin\theta_x\cdot L\right]=x^{DTFT}\left[2\pi\sin\theta_x\cdot\frac{L}{\lambda}\right]$$

Where L is the sampling distance.
We want to uphold nyquist:

$$\left|k_x\right|<\frac{\pi}{L}\Rightarrow\left|\frac{2\pi}{\lambda}\sin\theta_x\right|<\frac{\pi}{L}\Rightarrow\left|\sin\theta_x\right|<\frac{\lambda}{2L}$$

Of course $\left|\sin\theta_x\right|<1$ therefore:

$$1<\frac{\lambda}{2L}\Rightarrow L<\frac{\lambda}{2}$$

Which is pretty understandable.
We want our values of $\sin\theta_x$ to be between $\left[-\frac{1}{2},\frac{1}{2}\right]$ with 7 values. In order to get this, we can run the algorithm for $[-1,1]$ 14 values of k_x, without calculating the values we don't care about. Thus, we need $N=14$, or we could use zero padding with $N=7$

After this, we can just apply the gaussian. Notice that the gaussian is not dependent on the frequency.

Of course what we are doing can be described as a DFT with a wavlet:

$$W_{LGFT}=W_{DFT}e^{-\frac{x^2+y^2}{2\sigma^2}}=W_{DFT}\cdot g\left(x,y\right)$$

Where we defined $g=e^{-\frac{x^2+y^2}{2\sigma^2}}$.
The DFT of course is just applying a convolution, therefore, we can write:

$$x^{LGFT}\left[k_x\right]=x\otimes\left(h\cdot g\right)$$

Applying a fourier on the RHS:

$$DFT\left\{x\otimes\left(h\cdot g\right)\right\}=DFT\left\{x\right\}\cdot DFT\left\{\left(h\cdot g\right)\right\}$$

$$=DFT\left\{x\right\}\cdot DFT\left\{h\right\}\otimes DFT\left\{g\right\}$$

By definition DFT of the DFT kernel is just 1. We will define $G = DFT\{g\}$. Therefore:

$$DFT^{-1}\{LGFT\{x\}\} = DFT^{-1}\{DFT\{x\} \cdot DFT\{g\}\}$$

And therefore:

$$LGFT\{x\} = DFT\{x\} \cdot DFT\{g\}$$

We of course need to use the cyclic convolution in all of this, and therefore need to use the relevant padding.

The Non-DFT way

Intuitively, we would find the inner product between the mask and $u(k)$.

We need 49 u's for the 49 angles. let us look in 1-D.

$$u_k(x) = e^{jkx}e^{-\frac{x^2}{2\sigma^2}}$$

We need to replace $x \to n$. $x = Ln$ Therefore:

$$u_k[n] = e^{jkLn}e^{-\frac{(Ln)^2}{2\sigma^2}} = e^{jkLn}e^{-\frac{n^2}{2\left(\frac{\sigma}{L}\right)^2}} = e^{j\frac{2\pi}{\lambda}\sin\theta Ln}e^{-\frac{n^2}{2\left(\frac{\sigma}{L}\right)^2}}$$

Therefore:

$$x[n] \cdot u_k[n] = \sum_n x[n]\,e^{-jkLn}e^{-\frac{n^2}{2\left(\frac{\sigma}{L}\right)^2}} = \sum_n x[n]\,e^{j\frac{2\pi}{\lambda}\sin\theta Ln}e^{-\frac{n^2}{2\left(\frac{\sigma}{L}\right)^2}}$$

Notice that in the regular case:

$$x^d[k] = \sum_n x[n]\,e^{-j\frac{2\pi}{N}kn}$$

If we want $k_{dft} = k \Rightarrow \frac{2\pi}{N}k_{dft} = \frac{2\pi}{\lambda}\sin\theta L \Rightarrow k_{dft} = \frac{N}{\lambda}\sin\theta L = \left(\frac{LN}{\lambda}\right)\sin\theta$

Now, as said, we want to sample for $\sin\theta = -\frac{1}{2} + \frac{1}{6}n$ for $0 \le n \le 6$.

Therefore:

$$k_{dft} = \left(\frac{LN}{\lambda}\right)\left(-\frac{1}{2} + \frac{1}{6}n\right)$$

k_{dft} is of course whole. Therefore:

$$\left(\frac{LN}{\lambda}\right)\left(-\frac{1}{2} + \frac{1}{6}n\right) \in \mathbb{Z}$$

$$\Rightarrow -\frac{1}{2}\frac{LN}{\lambda} \in \mathbb{Z}$$

$$\Rightarrow \frac{1}{6}\frac{LN}{\lambda}n \in \mathbb{Z}$$

The second one is more severe. It tells us:

$$\frac{1}{6}\frac{LN}{\lambda} = m \Rightarrow N = \frac{6m\lambda}{L}$$

Taking $m = 1$:

$$N = \frac{6\lambda}{L}$$

If $\lambda = p \cdot L$ we get:

$$N = 6 \cdot \left(\frac{\lambda}{L}\right) = 6 \cdot p$$

We already showed that $L \leq \frac{\lambda}{2}$ in order to not get aliasing. Therefore:

$$p = \frac{\lambda}{L} \geq 2$$

$$N \geq 12$$

Therefore, if we use DFT with $N = 6 \cdot \left(\frac{\lambda}{L}\right)$ and choose $k = \frac{2\pi}{\lambda}\sin\theta L$ we should approximately get the result of $x[n] \cdot u_k[n]$

This is not true, as we didn't take the gaussian into account

The Naive way

This would not work, as we didn't take the gaussian into account

We apply a DFT. Thus we get the frequencies:

$$x^{DFT}[k] = x^{DTFT}\left[\frac{2\pi}{N}k\right] = x^{F}\left[\frac{2\pi}{N}k \cdot L\right]$$

We want to find the frequency with respect to θ, as in:

$$k_x = \frac{2\pi}{\lambda}\sin\theta$$

Therefore:

$$\frac{2\pi}{N}k \cdot L = \frac{2\pi}{\lambda}\sin\theta \Rightarrow k = \frac{L}{\lambda N}\sin\theta$$

Therefore we will choose $k = \left\lfloor \frac{L}{\lambda}N\sin\theta \right\rfloor$. We have seen that $L \leq \frac{\lambda}{2}$ Therefore $\frac{L}{\lambda} \leq 2$. Our frequency resolution is:

$$\Delta k = \frac{L}{\lambda} N \Delta \sin\theta$$

Where $\Delta \sin\theta = \frac{1}{6}$. Therefore:

$$\Delta k = \frac{L}{\lambda}\frac{N}{6}$$

We enforce $\Delta k \geq 1$:

$$\frac{L}{\lambda}\frac{N}{6} \geq 1 \Rightarrow N \geq 6 \cdot \frac{\lambda}{L}$$

$$\sin\theta = \frac{\lambda}{L}\frac{k}{N}$$

$$\Rightarrow \frac{\lambda}{L}\frac{1}{N} = \frac{1}{6} \Rightarrow N\frac{\lambda}{L}$$

Classic case:

When applying a DFT we are sampling in the following frequencies:

$$k = \frac{1}{L}\frac{2\pi}{N} \cdot m$$

Therefore:

$$\frac{2\pi}{\lambda}\sin\theta = \frac{1}{L}\frac{2\pi}{N} \cdot m \Rightarrow \sin\theta = \frac{\lambda}{L} \cdot \frac{m}{N}$$

we remember the range of values for m:

$$0 \leq m \leq N-1 \Rightarrow -\left\lfloor \frac{N}{2} \right\rfloor \leq m \leq N-1-\left\lfloor \frac{N}{2} \right\rfloor$$

Therefore:

$$-\frac{\lambda}{L}\frac{\left\lfloor \frac{N}{2} \right\rfloor}{N} \leq \sin\theta \leq \frac{\lambda}{L}\frac{\left(N-1-\left\lfloor \frac{N}{2} \right\rfloor\right)}{N}$$

Where $\Delta \sin\theta = \frac{\lambda}{L}\frac{1}{N}$. We want $\Delta \sin\theta = \frac{1}{n \cdot 6}$. Therefore

$$-\frac{1}{n \cdot 6}\left\lfloor \frac{N}{2} \right\rfloor \leq \sin\theta \leq \frac{1}{n \cdot 6}\left(N-1-\left\lfloor \frac{N}{2} \right\rfloor\right)$$

We want $-\frac{1}{n \cdot 6}\left\lfloor \frac{N}{2} \right\rfloor \leq -\frac{1}{2} \Rightarrow n \cdot 3 \leq \left\lfloor \frac{N}{2} \right\rfloor$.
We got 2 conditions:

$$\frac{\lambda}{L}\frac{1}{N} = \frac{1}{n\cdot 6} \Rightarrow N = 6n\cdot\frac{\lambda}{L}$$

$$n\cdot 3 \leq \left\lfloor \frac{N}{2} \right\rfloor \Rightarrow n\cdot 6 < N$$

For $n = 1$:

$$N = 6\frac{\lambda}{L}$$

$$N > 6$$

Mathematical developement:

I have $x[n] = e^{jk'Ln}$. I have $u_k[n] = e^{jkLn}$

Therefore:

$$x[n]\cdot u_k[n] = \sum_{n=0}^{N-1} e^{jk'Ln}e^{-jkLn}$$

$$= \sum_{n=0}^{N-1} e^{j\left(k'-k\right)Ln} = \sum_{n=0}^{N-1}\left(e^{j\left(k'-k\right)L}\right)^n$$

$$= \frac{e^{j\left(k'-k\right)LN}-1}{e^{j(k'-k)L}-1} = \frac{e^{\frac{j\left(k'-k\right)LN}{2}}}{e^{\frac{j\left(k'-k\right)L}{2}}}\frac{e^{\frac{j\left(k'-k\right)LN}{2}}-e^{-\frac{j\left(k'-k\right)LN}{2}}}{e^{\frac{j\left(k'-k\right)L}{2}}-e^{-\frac{j\left(k'-k\right)L}{2}}}$$

$$= e^{\frac{j\left(k'-k\right)L(N-1)}{2}}\frac{\sin\left(\frac{\left(k'-k\right)LN}{2}\right)}{\sin\left(\frac{(k'-k)L}{2}\right)}$$

This is the Dirichlet kernel. Looking at:

$$|x[n]\cdot u_k[n]| = \frac{\sin\left(\frac{\left(k'-k\right)LN}{2}\right)}{\sin\left(\frac{(k'-k)L}{2}\right)}$$

$$k = \frac{2\pi}{\lambda}\cdot\sin\theta$$

$$arr = 0, ..., N-1$$

$$0, \frac{2\pi}{N}, 2\frac{2\pi}{N}, 3\frac{2\pi}{N}, 4\frac{2\pi}{N}, 5\frac{2\pi}{N}$$

$$-3\frac{2\pi}{N}, -2\frac{2\pi}{N}, -\frac{2\pi}{N}, 0, \frac{2\pi}{N}, 2\frac{2\pi}{N}$$

$$-3\frac{2\pi}{N} = -\frac{\pi}{2}$$

$$0, \frac{2\pi}{N}, 2\frac{2\pi}{N}, 3\frac{2\pi}{N}, 4\frac{2\pi}{N}, 5\frac{2\pi}{N}$$

$$-3\frac{2\pi}{N}, -2\frac{2\pi}{N}, -\frac{2\pi}{N}, 0, \frac{2\pi}{N}, 2\frac{2\pi}{N}$$