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Linear Regression, Logistic Regression & Naïve Bayes

04 – ML MODELS

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Agenda Today

- Linear Regression
  - Gradient Descent
- Logistic Regression
- Naïve Bayes

Recap: Bias – Variance tradeoff

> What is the difference between Bias and Variance?
> What is the difference between under- and over-fitting?
> How can we overcome these issues?

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Recap: Canonical Types // output size // un/supervised?

How would you tackle the following tasks? What *canonical problem type* are they? What is their output space *size*?

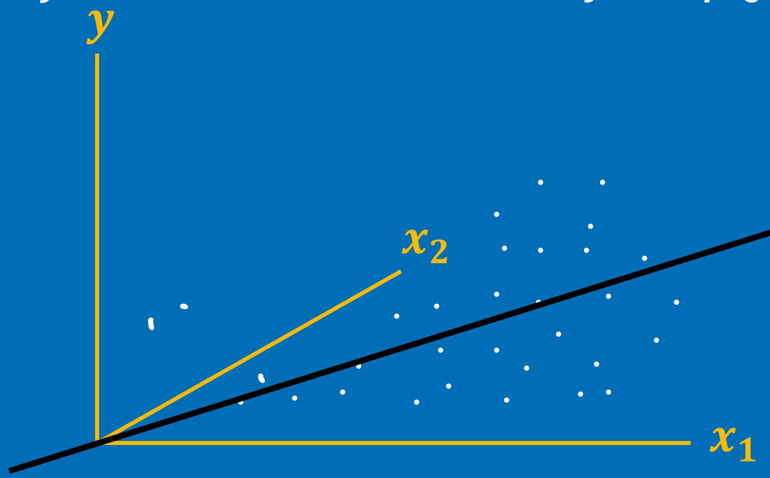
> Assign one of 36 POS tags to each word of a given sentence.
> Decide if an email is a spam.
> Determine if a comment is hateful, toxic, threat, insult, or a combination.
> Automatically extract and assign a set of 10 topics from 1000 documents.
> Given a comment about a product, predict the score the user gave.

Linear Regression

> A **Supervised Learning** method

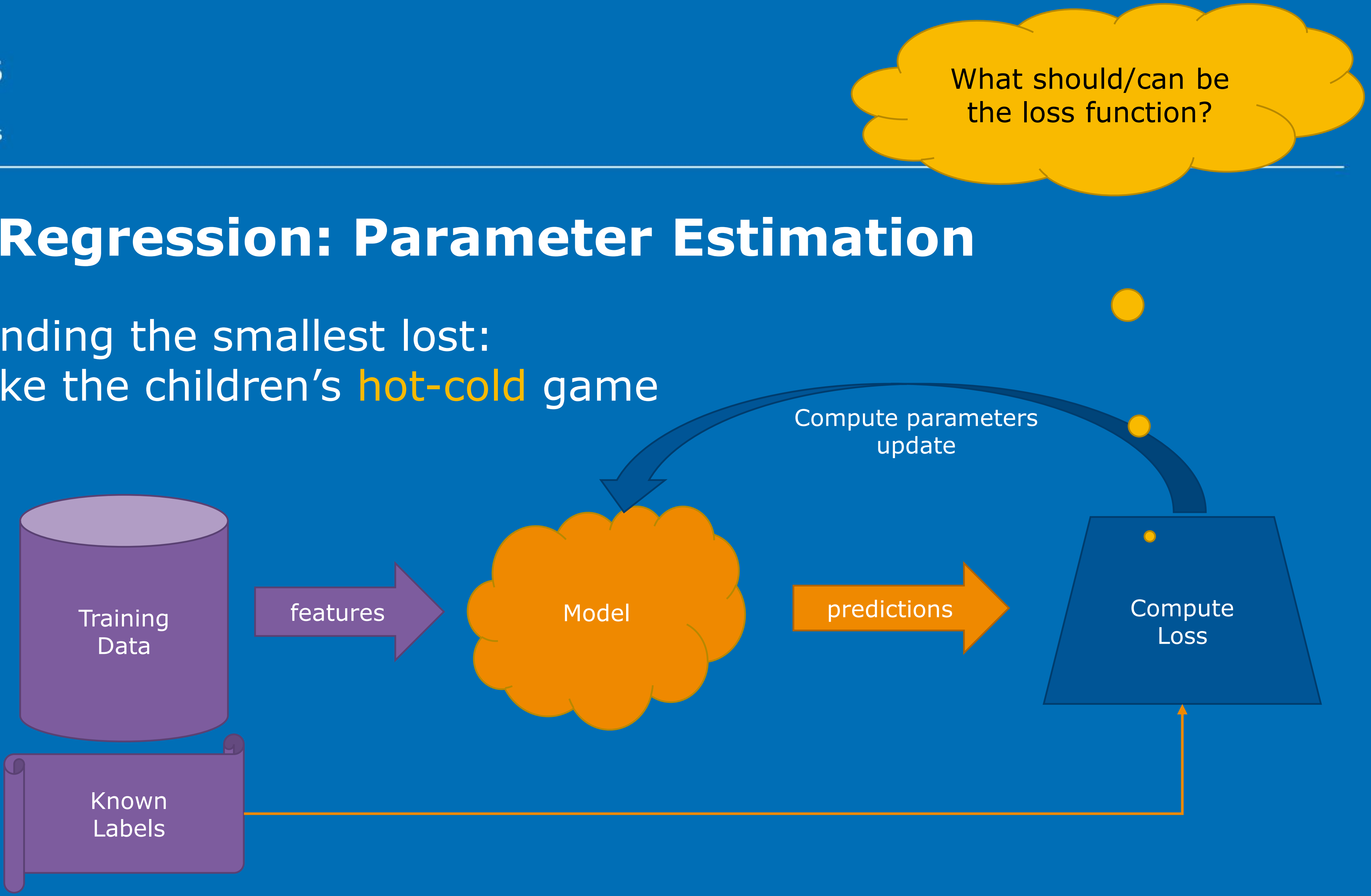
> Predicting a value $\hat{y} \in \mathbb{R}$ from m features $x \in \mathbb{R}^m$ by learning a linear function: $f(x) = y$

> f is of the form: $\hat{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 \ldots = \beta_0 + x^T \beta$



Linear Regression: Parameter Estimation

Finding the smallest lost:
Like the children's hot-cold game



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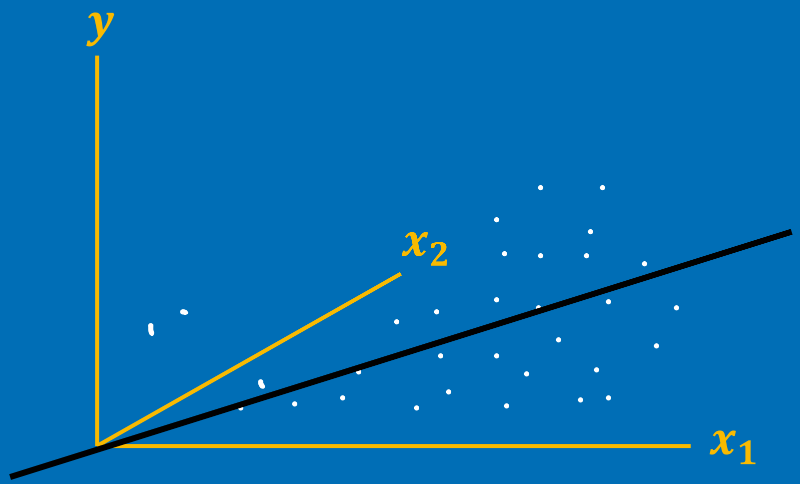
Linear Regression: Parameter Estimation

We can estimate the best parameters $\langle \beta_0, \beta_1 \rangle$ by minimizing a *loss* function.

A possible *loss*: the average of squared errors (MSE) between the true values and the predictions with these parameters:



$$\hat{\theta} = \arg \min_{\theta = \langle \beta_0, \beta_1 \rangle} \frac{1}{2n} \sum_{i=1}^{n} \left(y_i - \left(\beta_0 + x_i^T \beta \right) \right)^2$$

Linear Regression: Parameter Estimation

Simplified case with 2 parameters:

$$\hat{y} = \beta_0 + \beta_1 x_1$$

How to find the best parameters $\langle \beta_0, \beta_1 \rangle$?

1. Initialize with random values
2. Calculate the Loss: e.g., $\sum_{i=1}^{n} \left(y_{true} - y_{pred}\right)^2$
3. Calculate the necessary parameters change.
4. Repeat, until the average loss is very small, or the parameter change is minimal.

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Linear Regression: Parameter Estimation

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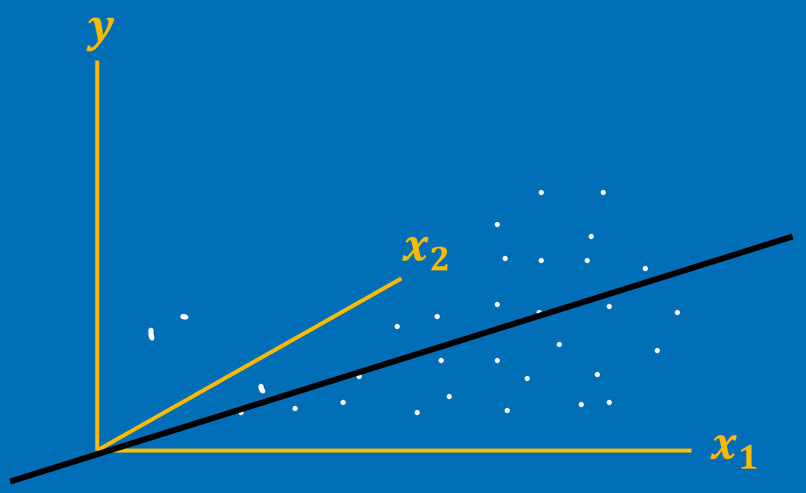
Linear Regression: Least Mean Squares (LMS)

Parameter Estimation is done through calculating the cost(=loss) function *partial derivatives* with respect to each β and finding its global minima with a method called:
Gradient Descent



$$\hat{\theta} = \arg \min_{\theta=\langle\beta_0,\beta_1\rangle} \frac{1}{2n} \sum_{i=1}^{n} \left(y_i - \left(\beta_0 + x_i^T \beta\right)\right)^2$$

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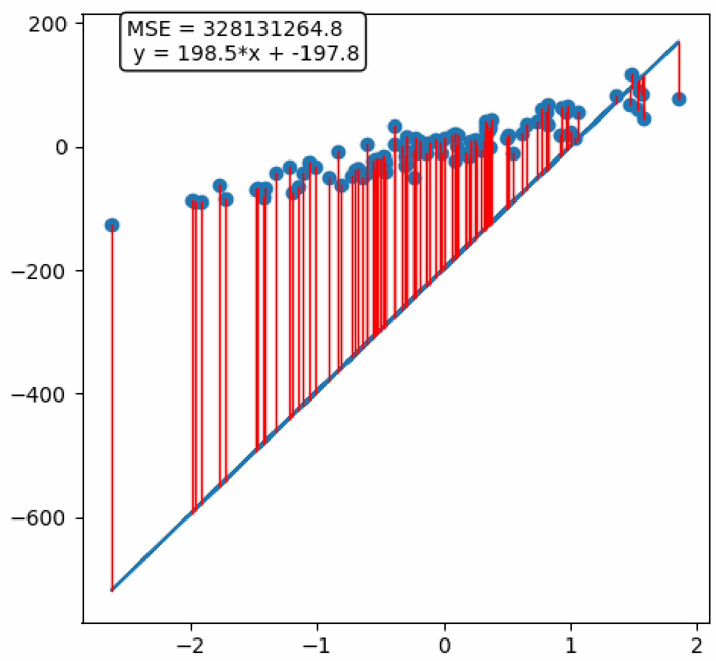
Optimizing Linear Regression with Gradient Descent

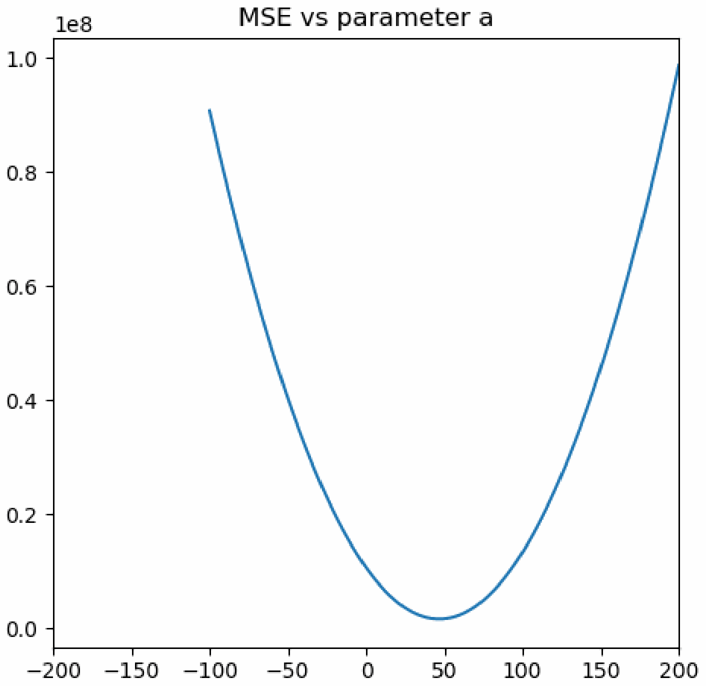
- $\hat{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2$ …
- Calculating partial derivatives **(gradient)**
- Finding the slope – which direction minimizes **(descent)** the gradients?
  And by how much?
  (Multiplied by some fraction - Learning Rate)

Gradient Descent

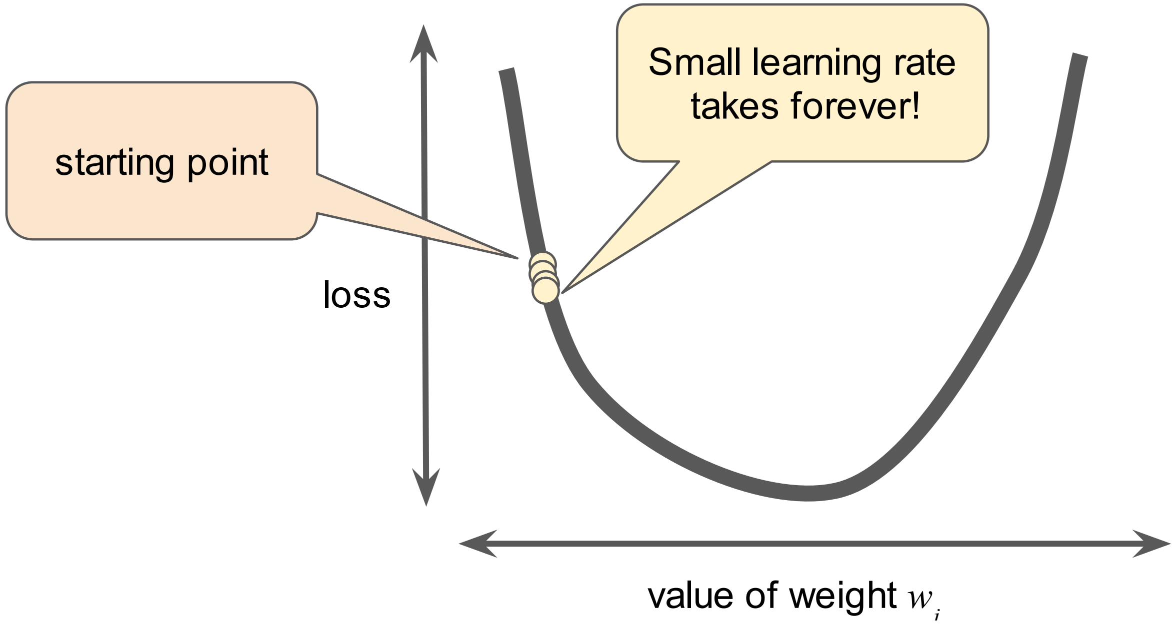
> Minimizing the partial derivatives (like in high school) – to find the global minimum

> Often is multiplied by some **learning-rate**
if too big – jumps too far
if too small – takes too long





Learning Rate – too small



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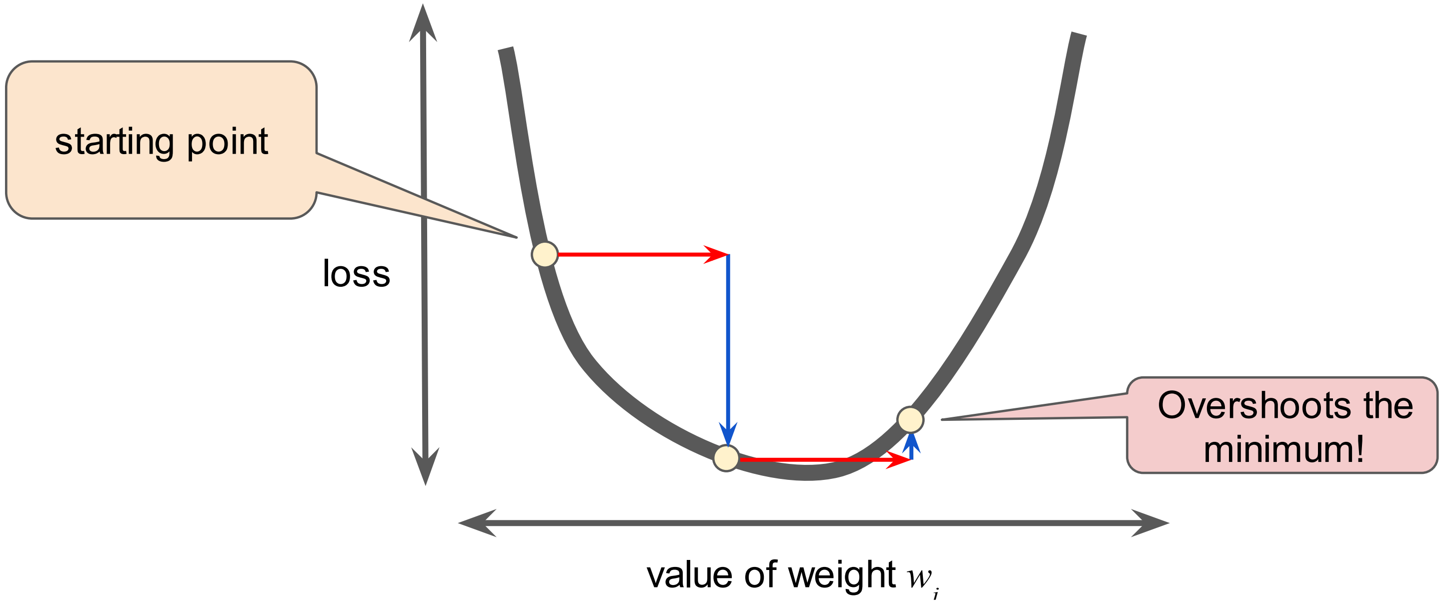
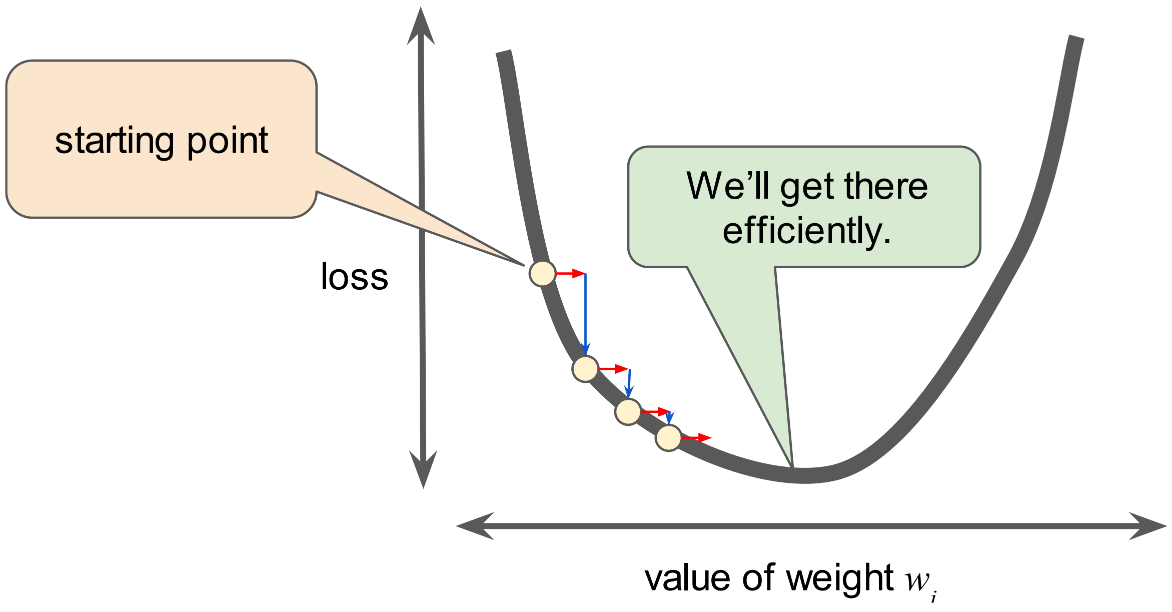
Learning Rate – too large



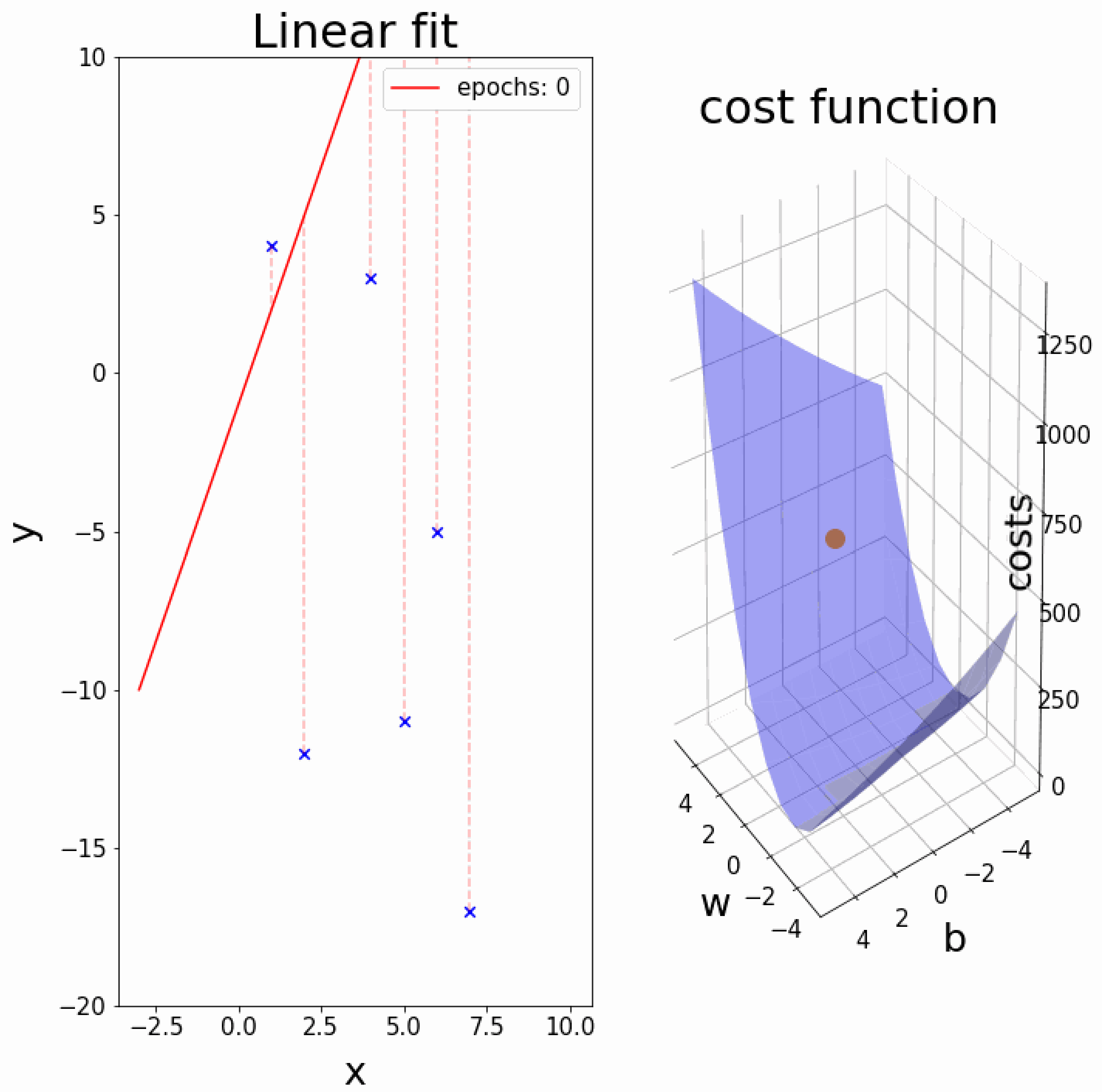
Learning Rate – just right



Two-dimensional

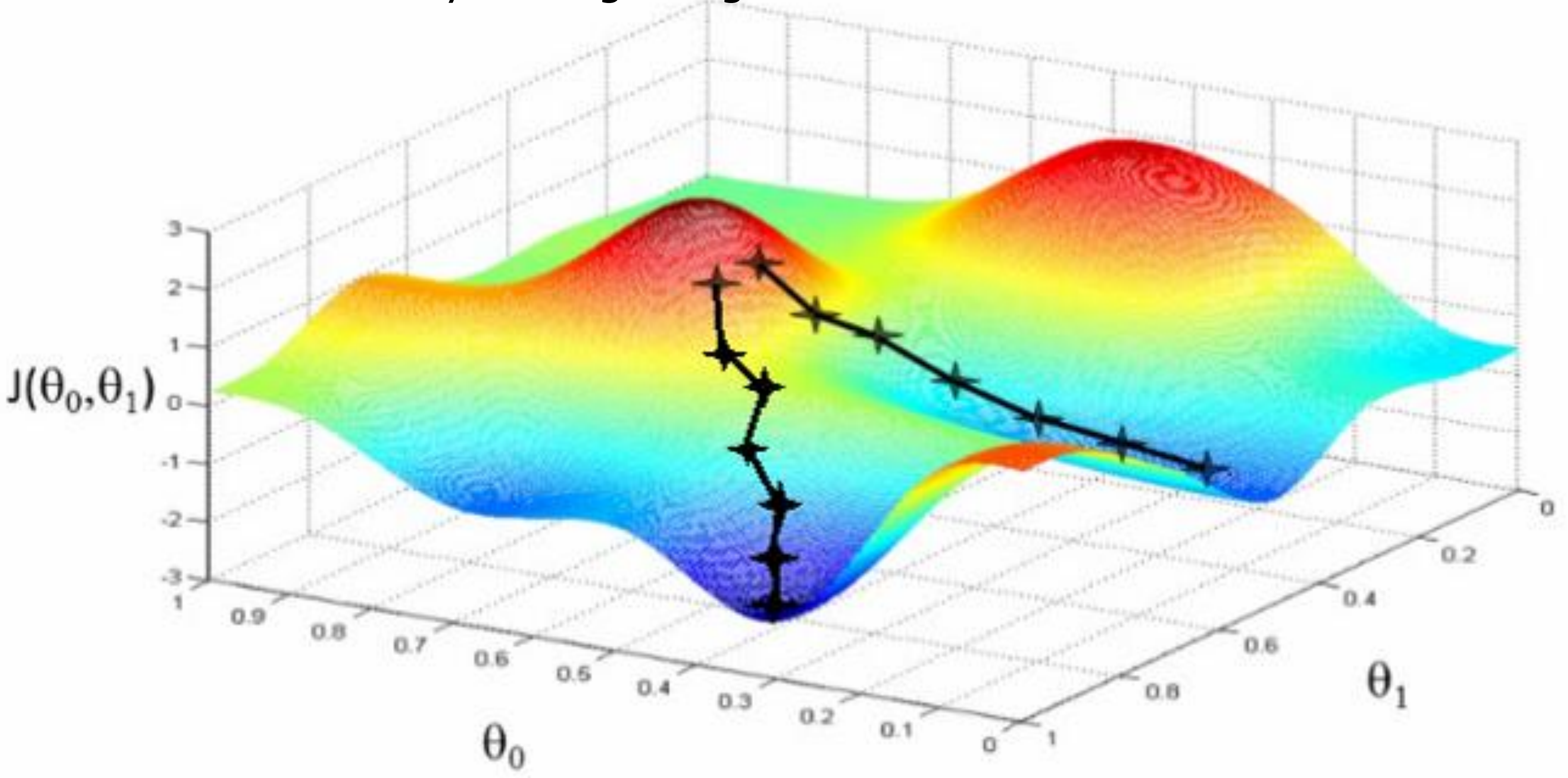
> $\hat{y} = \beta_0 + \beta_1 x$

> Minimizing both - β_0 & β_1

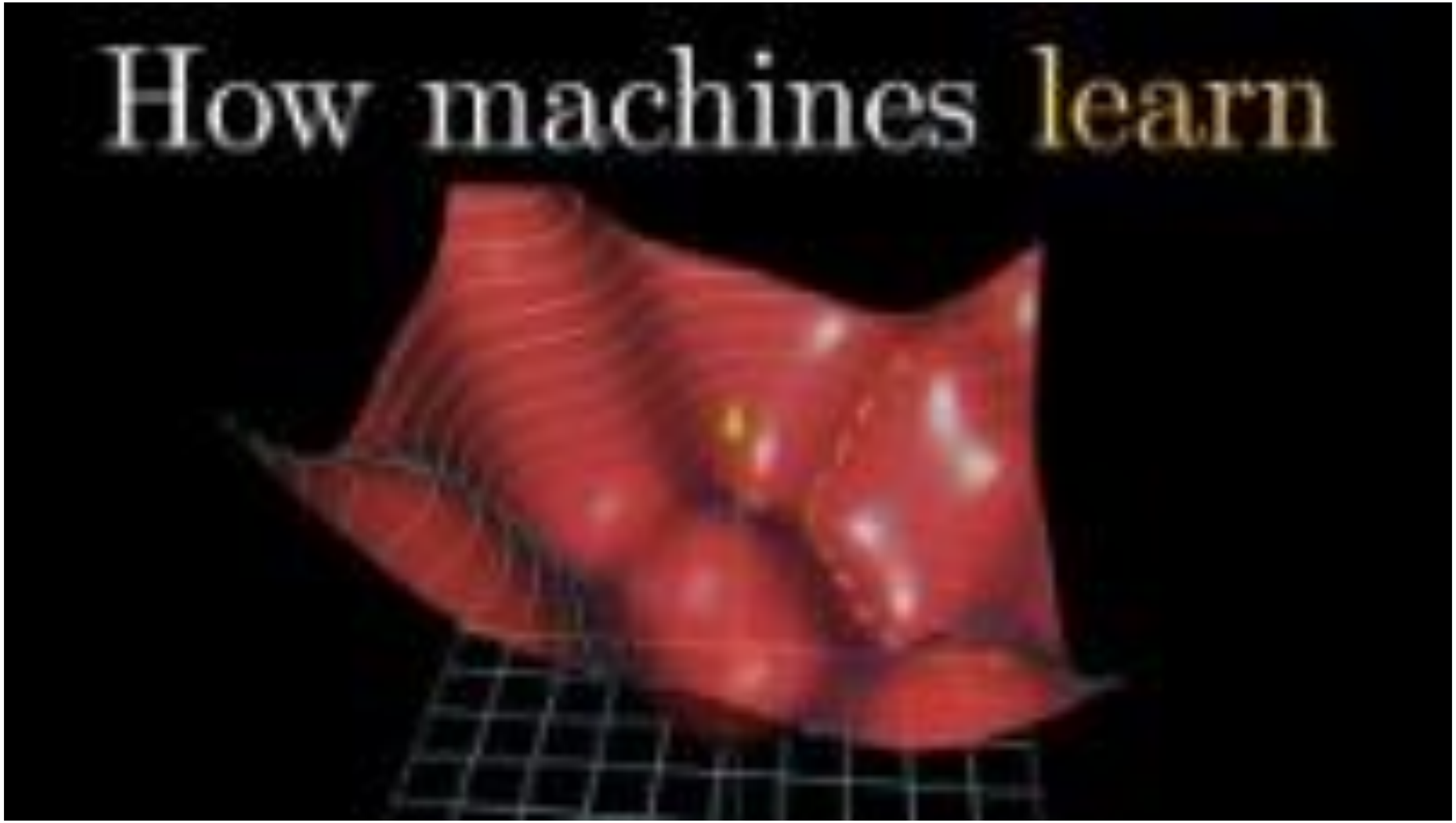


Gradient Descent

Minimizes the cost function by finding the global minima of the derivatives



Gradient Descent: Recommended Video



Back to Linear Regression

- Calculates the best linear fit of the given features
- Assumes the features are **independent (!)**
- Requires numerical features
  - Which features would you craft for the following problem: Given an essay, predict the student’s score (0-100)

Linear Regression - Regularization

> ML training can cause overfitting.
  > Q: What would overfitting look like for **Linear** regression?

> How can we prevent it?

Regularization!

$$\hat{\theta} = \arg \min_{\theta=\langle\beta_0,\beta_1\rangle} \frac{1}{2n} \sum_{i=1}^{n} \left(y_i - \left(\beta_0 + x_i^T \beta\right)\right)^2$$

Linear Regression - Regularization

> ML training can cause overfitting.
> > What would overfitting look like for **Linear** regression?

> How can we prevent it?

L_2 Regularization: Euclidean Distance

$$\hat{\theta} = \arg \min_{\theta=\langle \beta_0, \beta_1 \rangle} \frac{1}{2n} \sum_{i=1}^{n} \left(y_i - \left(\beta_0 + x_i^T \beta \right) \right)^2 + \frac{\lambda}{2} \sqrt{\sum_{j=1}^{m} \beta_j^2}$$

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Linear Regression - Regularization

> ML training can cause overfitting.
>
> > What would overfitting look like for **Linear** regression?

> How can we prevent it?

L_1 Regularization: Manhattan (Taxicab) Distance

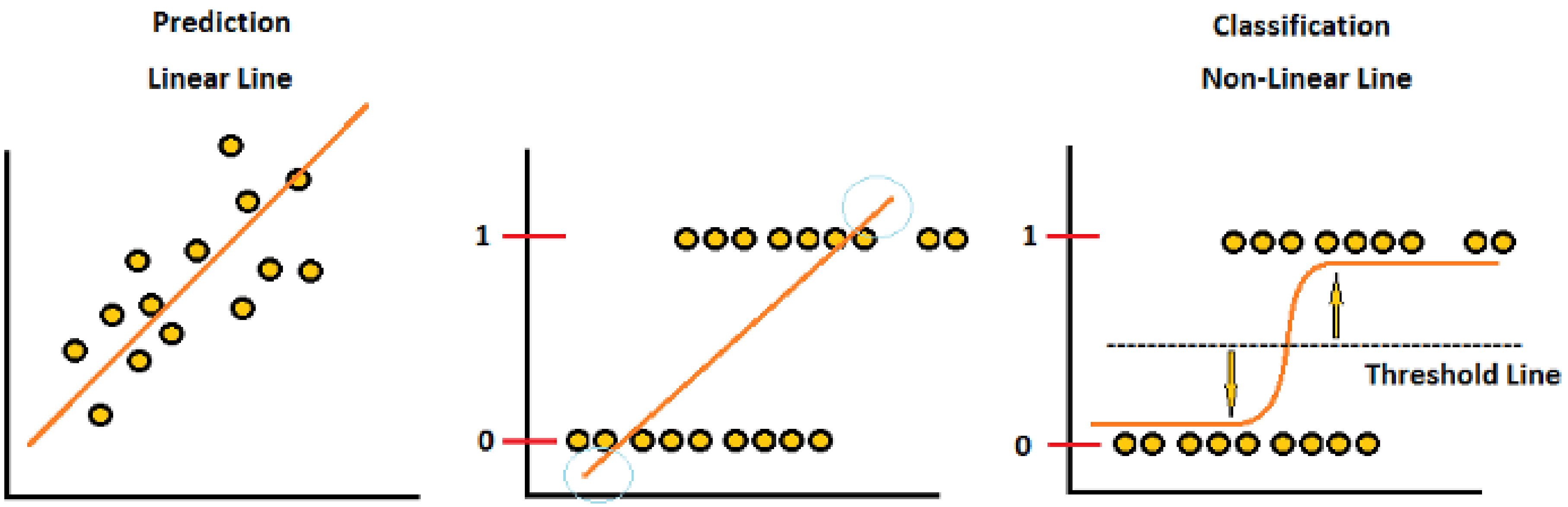
$$\hat{\theta} = \arg\min_{\theta=\langle\beta_0,\beta_1\rangle} \frac{1}{2n}\sum_{i=1}^{n}\left(y_i - \left(\beta_0 + x_i^T\beta\right)\right)^2 + \lambda\sum_{j=1}^{m}\left|\beta_j\right|$$

From Linear to Logistic

Classification using
Logistic Regression

Binary Logistic Regression

> Linear Regression: Predict an essay’s **score** (e.g., readability)
> Logistic Regression: Predict a **probability** (Readable or not)



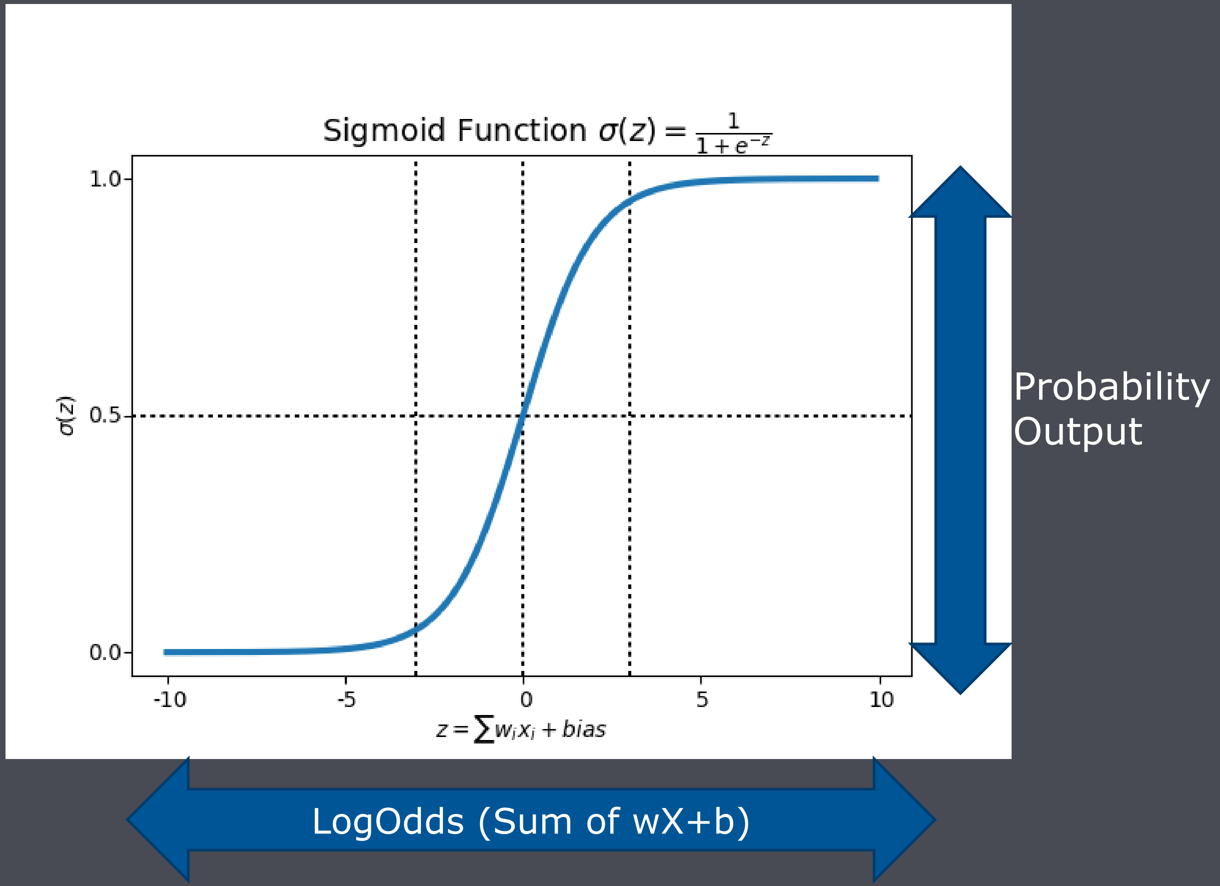
Linear Logistic Regression

> Gives a **probability** estimation

> By wrapping the linear function in a non-linear function.
E.g., *sigmoid function*:

$$y' = \frac{1}{1 + e^{-(w^T x + b)}}$$

> Squishing the linear output through sigmoid



Linear Logistic Regression – Sigmoid Function

- Probability vs Odds.
- Probability: 0…1

- Odds: $\frac{P}{1-P}$
- Logistic Function:

$$\ln\left(\frac{P}{1-P}\right) = \beta_0 + \beta_1 x$$

$$\frac{P}{1-P} = e^{\beta_0+\beta_1 x}$$

$$P = \frac{1}{1+e^{-(\beta_0+\beta_1 x)}}$$

Y	*1*	*0*
Pr(Y=1)	*P*	*1- P*

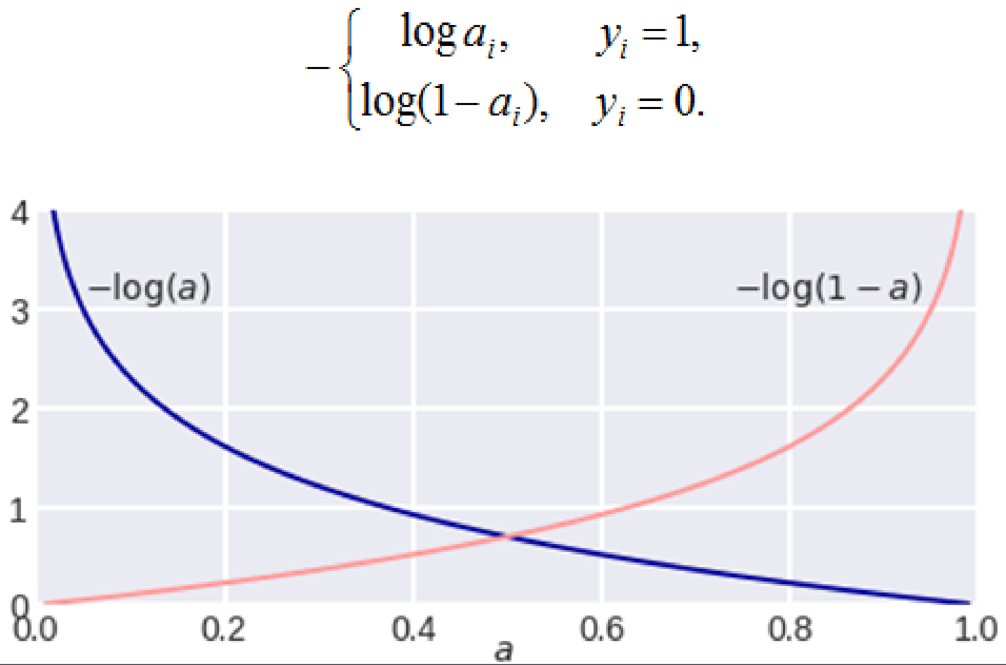
**P= Success, 1-P = Failure*

Linear Logistic Regression - *Loss* Function

- In Linear Regression we used **Mean Squared Error (MSE).**
- In Logistic Regression we need a different *loss*:

$$Log\ Loss = \sum_{(x,y)\in D} -y\log(\hat{y}) - (1-y)\log(1-\hat{y})$$

- As we reach closer to the side of the bars, the probability gets high quickly.



$(x, y) \in D$ – data and its labels
y – the true label (between 0 and 1)
y’ – the predicted label (between 0 and 1)

Regularization

> Regularization is very important in logistic regression modeling.

> Without it, the loss would go towards 0 in high dimensions (due to the asymptotic nature of logistic regression)

> Strategies to dampen model complexity:

1. L_2 regularization.
2. Early stopping

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Example: Sentiment Classification of movie reviews

> Problem definition: Estimate if a review is positive or not.

> Features:

1. count(positive lexicon words ∈ doc)
2. count(negative lexicon words ∈ doc)
3. { 1 – “no” found; 0 – Otherwise }
4. count(1st and 2nd pronouns ∈ doc)
5. { 1 – “!” found; 0 – Otherwise }
6. log(word count of doc)

Example: Sentiment Classification of movie reviews

It's hokey . There are virtually no surprises , and the writing is second-rate . So why was it so enjoyable ? For one thing , the cast is great . Another nice touch is the music . I was overcome with the urge to get off the couch and start dancing . It sucked me in , and it'll do the same to you .

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Example: Sentiment Classification of movie reviews

It's hokey . There are virtually no surprises , and the writing is second-rate . So why was it so enjoyable ? For one thing , the cast is great . Another nice touch is the music . I was overcome with the urge to get off the couch and start dancing . It sucked me in , and it'll do the same to you .

1.	count(positive lexicon words ∈ doc)	3
2.	count(negative lexicon words ∈ doc)	2
3.	{ 1 – "no" found; 0 – Otherwise }	1
4.	count(1st and 2nd pronouns ∈ doc)	3
5.	{ 1 – "!" found; 0 – Otherwise }	0
6.	log(word count of doc)	ln(66)=4.19

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Example: Sentiment Classification of movie reviews

Assuming we have a trained model with the following weights:

$$\beta_0 = 0.1, \qquad \beta = \begin{matrix} 2.5 \\ -5.0 \\ -1.2 \\ 0.5 \\ 2.0 \\ 0.7 \end{matrix}$$

1. 3
2. 2
3. 1
4. 3
5. 0
6. ln(66)=4.19

$$\sigma = \frac{1}{1 + e^{-(\beta_0 + \beta x)}}$$

$$P(y = 1|x) = \sigma(\beta \cdot x + \beta_0) = \sigma(.833) = 0.70$$

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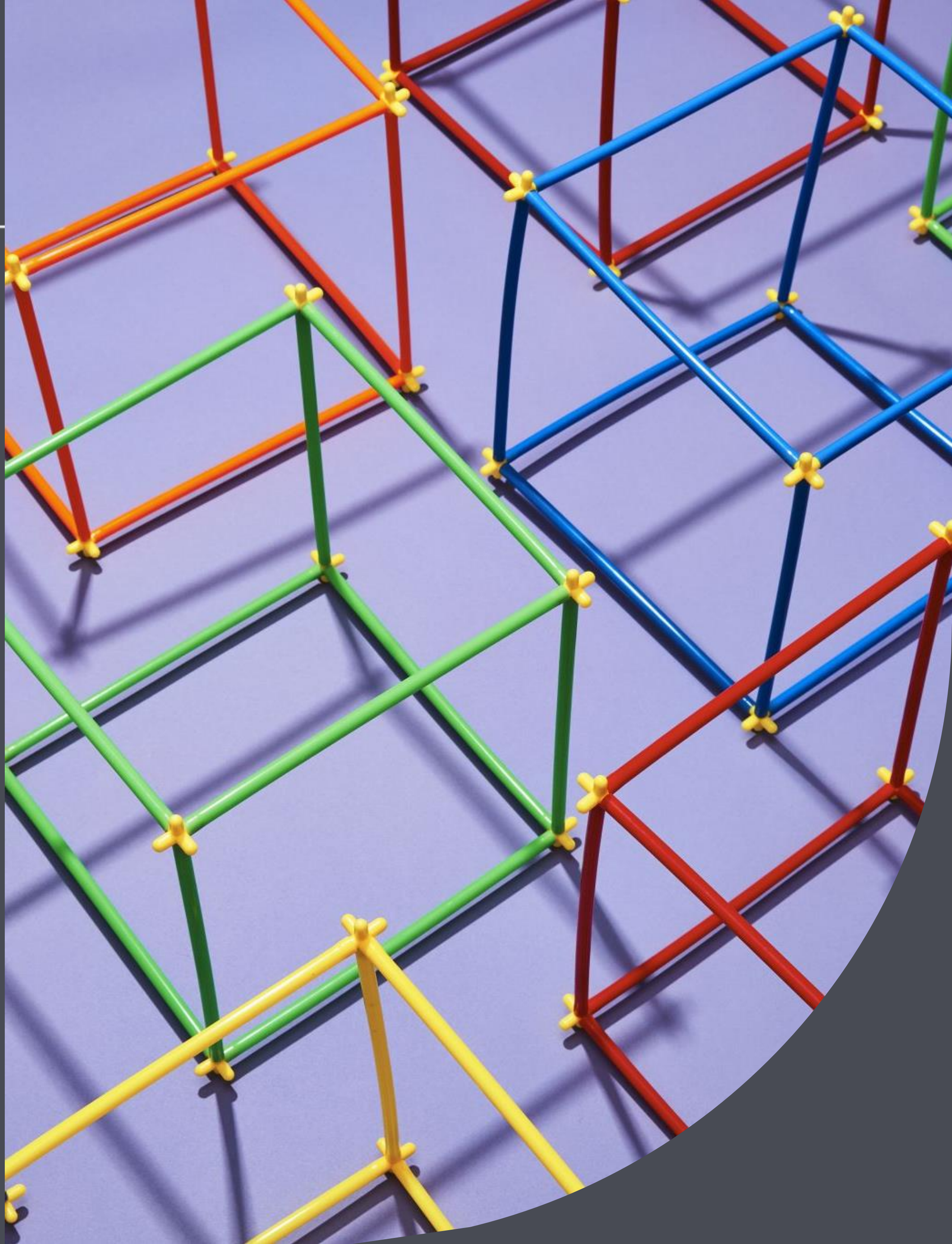
Demo

> [A Neural Network Playground (tensorflow.org)]



What about multiclass?

> For *k* classes, we need *k* models
> y will be a 1-hot vector: [0 0 0 1 0 0 0 0 0]
> Multinomial Logistic Regression

Softmax instead of sigmoid:

$$softmax\ (z_i) = \frac{\exp(z_i)}{\sum_{j=1}^{k} \exp(z_j)} = \frac{\exp(w_k \cdot x + b_k)}{\sum_{j=1}^{k} \exp(w_j \cdot x + b_j)}$$

What about multiclass?

Cross entropy loss:

$$-y \cdot log(p) + (1-y)\log(1-p)$$

Which generalizes to:

$$-\sum_{c=1}^{K} y_c \log(p_c)$$

Softmax instead of sigmoid:

$$softmax\ (z_i) = \frac{\exp(z_i)}{\sum_{j=1}^{k} \exp(z_j)} = \frac{\exp(w_k \cdot x + b_k)}{\sum_{j=1}^{k} \exp(w_j \cdot x + b_j)}$$

Understand the math

> Demo

Linear Logistic Regression – Final Notes

- Logistic Regression:
  Linear regression wrapped with a non-linear function:
  - Sigmoid for binary classification
  - Softmax for multiclass classification

- Very fast to train
- Efficient in RAM consumption
- The basis of Neural Networks

Recommended Read & Watch for Deeper Dive

- Speech and Language Processing. Daniel Jurafsky & James H. Martin. Chapter 5 – Logistic Regression.pdf
- Logistic Regression -- Why sigmoid function? (sebastianraschka.com)
- (17) Softmax Function Explained In Depth with 3D Visuals - YouTube
- https://ml-cheatsheet.readthedocs.io/en/latest/logistic_regression.html
- Loss Functions — ML Glossary documentation (ml-cheatsheet.readthedocs.io)

Naïve Bayes

Classification with pure Bayesian statistics

Recap: Maximum Likelihood Estimation (MLE)

> Recall your MLE with Christian

Naïve Bayes – an Introduction

> Similar ideas to MLE

> A probabilistic **classifier**

  > (a **generative** one)

> Makes a naïve assumption about the features

> Assumes feature independence

> Uses the **Bayes Rule**:

$$P(x|y) = \frac{P(y|x)P(x)}{P(y)}$$

Bayesian vs Frequentists

Bayesians:	Frequentists:
• Conditional Probability • Belief-based • A parameter is a random variable • Handle uncertainty with opinions • Goal: Update opinions	• Based on data • Parameters are static • Probability = How many times am I right? • Uncertainty requires data collection (null hypothesis) • Goal: decide which action to take

Bayesian vs Frequentists

Classification using Naïve Bayes

> Given a document, classify its author:

$$p(Mary\ Shelly \mid \text{"}The\ fallen\ angel\ becomes\ a\ malignant\ devil\text{"})$$

$$\hat{c} = \arg\max P(c|d) = \arg\max \frac{P(d|c)P(c)}{P(d)}$$

Since our document is the same for all classes, it can be simplified to:

$$\hat{c} = \arg\max P(c|d) = \arg\max P(d|c)P(c)$$

Classification using Naïve Bayes

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Likelihood Prior

Classification using Naïve Bayes

> Given a document, classify its author:

$$p(Mary\ Shelly \mid \text{"The fallen angel becomes a malignant devil"})$$

$$\hat{c} = \arg\max P(c|d) = \arg\max P(sentence\ \uparrow)P(Mary\ Shelly)$$

Since our document is the same for all classes, it can be simplified to:

$$\hat{c} = \arg\max P(c|d) = \arg\max \underbrace{P(d|c)}_{\text{Likelihood}}\underbrace{P(c)}_{\text{Prior}}$$

Classification using Naïve Bayes

How are these probabilities obtained?

> Given a document, classify its author:

$$p(Mary\ Shelly \mid \text{"}The\ fallen\ angel\ becomes\ a\ malignant\ devil\text{"})$$

$$P(sentence \uparrow)P(Mary\ Shelly)$$
$$P(sentence \uparrow)P(Edgar\ Ellen\ Poe)$$
$$P(sentence \uparrow)P(HP\ Lovecraft)$$

Since our document is the same for all classes, it can be simplified to:

$$\hat{c} = \arg\max P(c|d) = \arg\max P(d|c)P(c)$$

Likelihood Prior

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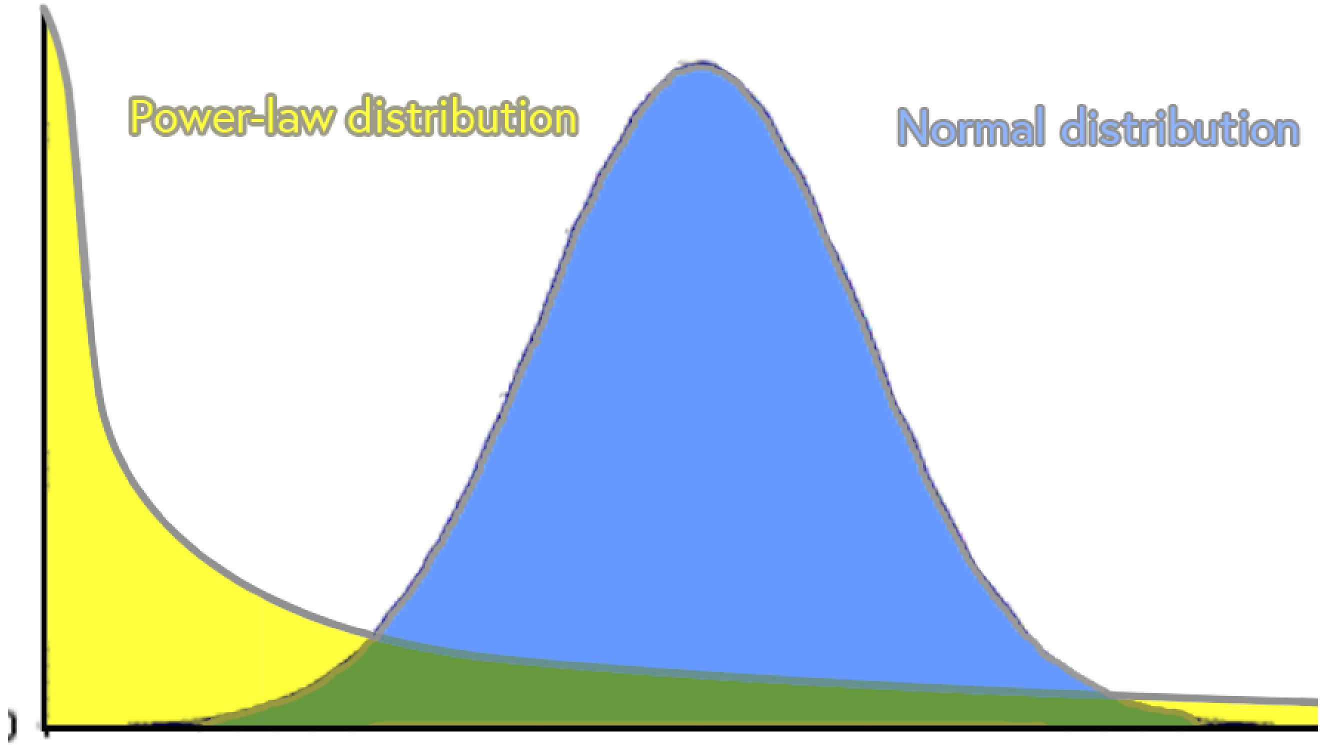
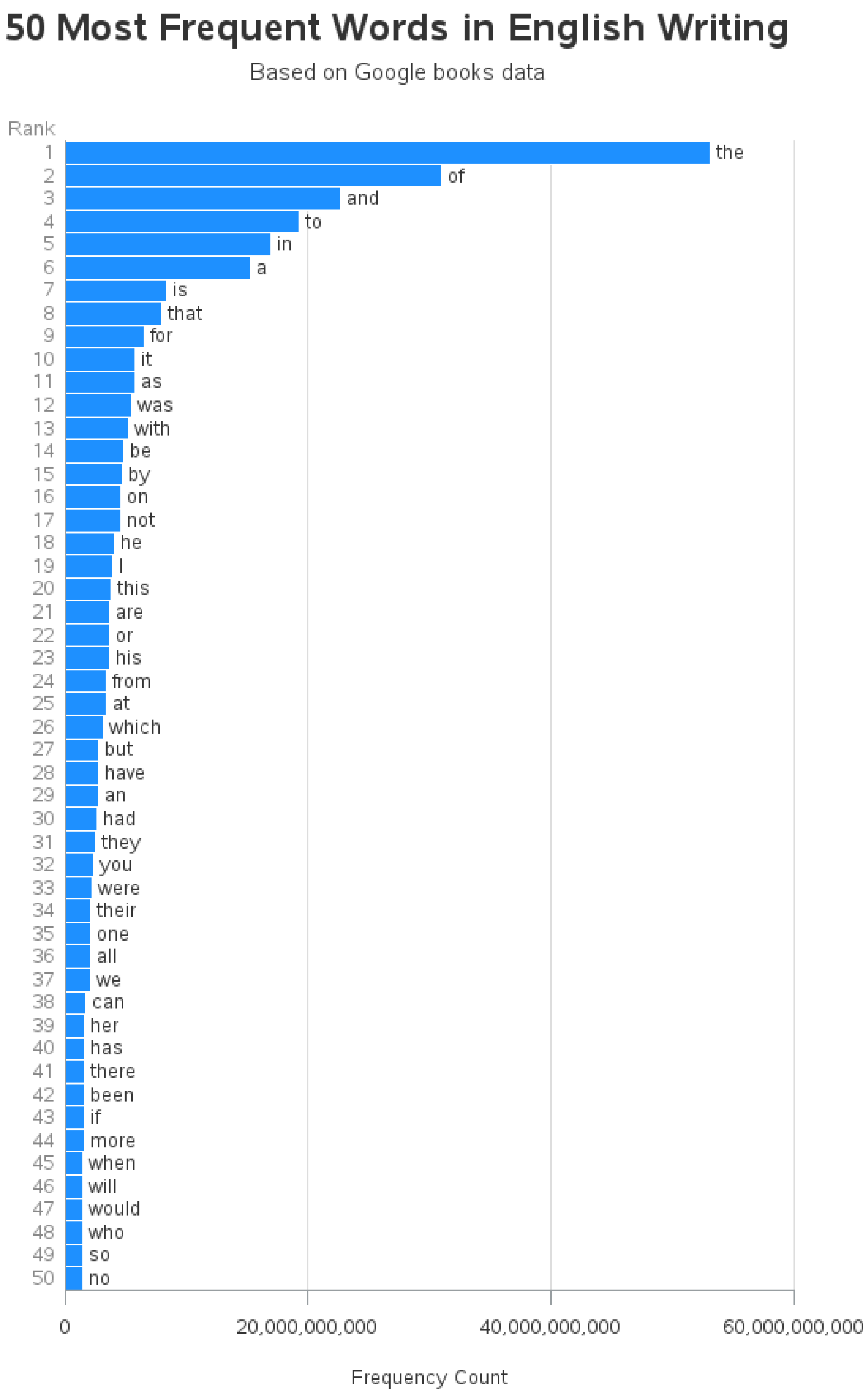
Language data properties



- Zipf Law
- Word frequencies follow a "power-law distribution":
  - **Long tail**
  - Most events rarely occur



Zipf Law

frequency of a word is inversely proportional to its rank in the frequency table

$$n(r) \propto \frac{1}{r^z} \qquad z \approx 1$$

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Zipf Law

In a 43M words text, there are:

- 316,710 unique words (types)
- 144,999 words occur only once
- 42,525 words occur 2 times
- 21,618 words occur 3 times
- 13,306 words occur 4 times
- 9,488 words occur 5 times
- 26,024 words appear >50 times

Zipf Law

No matter how large the training corpus is:

> It's likely to contain previously unseen word forms
> There will be many previously unseen word-*pairs*
> There will be even more previously unseen word-*triplets*
> There will be even more previously unseen *sentences*

Representing text as Features

"to be or not to be"

Bag of Words (BoW) - word counts:

{ "be": 2, "to": 2, "not": 1, "or": 1, "something": 0, "else": 0 }

(0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, … ,1, 0, 1, 0, …. ,0)

('a', 'the', 'there', 'to', …, 'be', 'if', 'an' … , 'or', 'as', 'not'…, 'zebra')

Representing text as Features

"to be or not to be"

Bag of Words (BoW) - word counts:

{ "be": 2, "to": 2, "not": 1, "or": 1, "something": 0, "else": 0 }

Reweighting:

TF/IDF or Pointwise Mutual Information – PMI

{ "be": 1.2, "to": 0.1, "not": 0.9, "or": 0.3 }

TF / IDF + PMI

{ "be": 2, "to": 2, "not": 1, "or": 1 }

$$w_{i,j} = tf_{i,j} \times \log\left(\frac{N}{df_i}\right)$$

$tf_{i,j}$ = number of occurrences of i in j
df_i = number of documents containing i
N = total number of documents

$$PMI(a,b) = \log\left(\frac{P(a,b)}{P(\mathrm{a})\mathrm{P}(\mathrm{b})}\right)$$

Classification using Naïve Bayes

> Given a document, classify its author:

$$p(Mary\ Shelly \mid \text{"}The\ fallen\ angel\ becomes\ a\ malignant\ devil\text{"})$$

$$P(sentence\ \ \uparrow)P(Mary\ Shelly)$$

(Naively) Assuming feature independence:

$$P(sentence) =$$
$$P(\text{The}) \times P(\text{“}fallen\text{”}) \times$$
$$P(\text{“}angel\text{”}) \times P(\text{“}becomes\text{”}) \times P(\text{“}a\text{”}) \times$$
$$P(\text{“}malignant\text{”}) \times P(\text{“}devil\text{”})$$

Classification using Naïve Bayes

> Given a document, classify its author:

$$p(Mary\ Shelly\,|\,\text{"}The\ fallen\ angel\ becomes\ a\ malignant\ devil\text{"})$$

$$P(sentence\ \ \uparrow)P(Mary\ Shelly)$$

(Naively) Assuming feature independence:

$$P(sentence|Mary\ Shelly) =$$
$$P(\text{“}The\text{”}|Mary\ Shelly) \times P(\text{“}fallen\text{”}|Mary\ Shelly) \times$$
$$P(\text{“}angel\text{”}|Mary\ Shelly) \times P(\text{“}becomes\text{”}|Mary\ Shelly) \times P(\text{“}a\text{”}|Mary\ Shelly) \times$$
$$P(\text{“}malignant\text{”}|Mary\ Shelly) \times P(\text{“}devil\text{”}|Mary\ Shelly) = \prod_{w \in sentence} P(w|Mary\ Shelly)$$

What happens if a word doesn't appear in the text at all?

Classification using Naïve Bayes

> Given a document, classify its author:

$$p(Mary\ Shelly \mid \text{"}The\ fallen\ angel\ becomes\ a\ malignant\ devil\text{"})$$

$$P(sentence\ \uparrow)P(Mary\ Shelly)$$

(Naively) Assuming feature independence:

Obtaining an ***a-priori*** by counting word occurrences per class.

$$P(\text{“}devil\text{”} \mid Mary\ Shelly) = \frac{5}{1000}$$

"devil" occurrences

Total words in Mary Shelly documents

Classification using Naïve Bayes - Unseen Words

> Given a document, classify its author:

$$p(Mary\ Shelly\,|\,\text{"}The\ fallen\ angel\ becomes\ a\ malignant\ devil\text{"})$$

$$P(sentence\ \uparrow)P(Mary\ Shelly)$$

(Naively) Assuming feature independence:

$$\frac{0}{1000} = 0$$

$$P(sentence|Mary\ Shelly) =$$
$$P(\text{“}The\text{”}|Mary\ Shelly) \times P(\text{“}fallen\text{”}|Mary\ Shelly) \times$$
$$P(\text{“}angel\text{”}|Mary\ Shelly) \times P(\text{“}becomes\text{”}|Mary\ Shelly) \times P(\text{“}a\text{”}|Mary\ Shelly) \times$$
$$P(\text{“}malignant\text{”}|Mary\ Shelly) \times P(\text{“}devil\text{”}|Mary\ Shelly) \Rightarrow 0$$

Classification using Naïve Bayes - OoV Solution

> Given a document, classify its author:

$$p(Mary\ Shelly \mid \text{"}The\ fallen\ angel\ becomes\ a\ malignant\ devil\text{"})$$

$$P(sentence\ \uparrow)P(Mary\ Shelly)$$

(Naively) Assuming feature independence:

Obtaining an ***a-priori*** with *Laplace smoothing*:
Counting word occurrences + 1 per class,
divided by all class words + sentence-words.

$$P(\text{“}devil\text{”} \mid Mary\ Shelly) = \frac{5+1}{1000+7}$$

“devil” occurrences + 1

Total words in Mary Shelly documents + sentence’s tokens

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Classification using Naïve Bayes – Log Probability

> Working with log probability:

$$\hat{c} = \arg\max P(c|d) = \arg\max P(d|c)P(c)$$

For all classes and all words:

$$argmax_{c \in Classes} P(c) \prod_{w \in sentence} P(w_i|c)$$

To ease calculation and speed, we can take logs from both sides:

$$argmax_{c \in Classes} \log P(c) + \sum_{w \in sentence} \log P(w_i|c)$$

Math Recap: Log rules

$$\log_b(MN) = \log_b(M) + \log_b(N)$$

$$\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N)$$

$$\log_b(M^p) = p\log_b(M)$$

Log rules: Justifying the logarithm properties (article) | Khan Academy

Naïve Bayes - Generative

> The bayes formula:

$$\hat{c} = \arg\max P(c|d) = \arg\max P(d|c)P(c)$$

Likelihood Prior

If the prior is known (category/class is given - $P(c) = 1$), we can generate words based on the Likelihood: $P(d|c)$

Reminder:
Discriminative → $P(\boldsymbol{c}lass|\boldsymbol{d}ocument)$ or

Generative → $P(\boldsymbol{d}ocument|\boldsymbol{c}lass)$

Naïve Bayes - Generative

On Discriminative vs. Generative classifiers: A comparison of logistic regression and naive Bayes

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nips01-discriminativegenerative.pdf (stanford.edu)

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Naïve Bayes – Final Notes

> A linear & probabilistic classifier

> Implementation exists in NLTK & Scikit-Learn (SKL)

> On SKL one can choose the distribution. The best performance is normally the *Multinomial* Naive Bayes
[A Comparison of Event Models for Naive Bayes Text Classification: binomial.dvi (washington.edu)]

A Comparison of Event Models for Naive Bayes Text Classification

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Naïve Bayes – Final Notes

- Can act as a *discriminative* or *generative* model
- Still widely used in Linguistics for classification

- Good summary/deeper dive: [The Optimality of Naive Bayes]() – sections: Abstract & Naive Bayes and Augmented Naive Bayes (you can safely ignore the augmented Naïve Bayes)

The Optimality of Naive Bayes

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Dive deeper

- Bayes theorem, the geometry of changing beliefs – YouTube
- 1.9. Naive Bayes — scikit-learn 1.1.3 documentation
- 6. Learning to Classify Text (nltk.org)
- Dan Jourafsky’s book – Chapter 4: https://web.stanford.edu/~jurafsky/slp3/4.pdf