Lab 2 – Prelab Answer Key

$$L = T - V$$
$$= \left(\tfrac{1}{2} m_p L_r^2 + \tfrac{1}{8} m_p L_p^2 \sin^2(\alpha) + \tfrac{1}{2} J_r\right)\dot{\theta}^2 + \left(\tfrac{1}{2} J_p + \tfrac{1}{8} m_p L_p^2\right)\dot{\alpha}^2$$
$$- \tfrac{1}{2} m_p L_p L_r \cos(\alpha)\dot{\theta}\dot{\alpha} - \tfrac{1}{2} m_p L_p g \cos(\alpha)$$

$$\frac{\partial L}{\partial \theta} = 0$$
$$\frac{\partial L}{\partial \alpha} = \tfrac{1}{8} m_p L_p^2 (2\sin(\alpha)\cos(\alpha))\dot{\theta}^2 + \tfrac{1}{2} m_p L_p L_r \sin(\alpha)\dot{\theta}\dot{\alpha} + \tfrac{1}{2} m_p L_p g \sin(\alpha)$$
$$= \tfrac{1}{8} m_p L_p^2 \sin(2\alpha)\dot{\theta}^2 + \tfrac{1}{2} m_p L_p L_r \sin(\alpha)\dot{\theta}\dot{\alpha} + \tfrac{1}{2} m_p L_p g \sin(\alpha)$$
$$\frac{\partial L}{\partial \dot{\theta}} = \left(\tfrac{1}{2} m_p L_r^2 + \tfrac{1}{8} m_p L_p^2 \sin^2(\alpha) + \tfrac{1}{2} J_r\right)(2\dot{\theta}) - \tfrac{1}{2} m_p L_p L_r \cos(\alpha)\dot{\alpha}$$
$$= \left(m_p L_r^2 + \tfrac{1}{4} m_p L_p^2 \sin^2(\alpha) + J_r\right)\dot{\theta} - \tfrac{1}{2} m_p L_p L_r \cos(\alpha)\dot{\alpha}$$
$$\frac{\partial L}{\partial \dot{\alpha}} = \left(\tfrac{1}{2} J_p + \tfrac{1}{8} m_p L_p^2\right)(2\dot{\alpha}) - \tfrac{1}{2} m_p L_p L_r \cos(\alpha)\dot{\theta}$$
$$= \left(J_p + \tfrac{1}{4} m_p L_p^2\right)\dot{\alpha} - \tfrac{1}{2} m_p L_p L_r \cos(\alpha)\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \left(\tfrac{1}{4} m_p L_p^2 (2\sin(\alpha)\cos(\alpha)\dot{\alpha})\right)\dot{\theta} + \left(m_p L_r^2 + \tfrac{1}{4} m_p L_p^2 \sin^2(\alpha) + J_r\right)\ddot{\theta}$$
$$- \tfrac{1}{2} m_p L_p L_r (-\sin(\alpha)\dot{\alpha})\dot{\alpha} - \tfrac{1}{2} m_p L_p L_r \cos(\alpha)\ddot{\alpha}$$
$$= \tfrac{1}{4} m_p L_p^2 \sin(2\alpha)\dot{\theta}\dot{\alpha} + \left(m_p L_r^2 + \tfrac{1}{4} m_p L_p^2 \sin^2(\alpha) + J_r\right)\ddot{\theta}$$
$$+ \tfrac{1}{2} m_p L_p L_r \sin(\alpha)\dot{\alpha}^2 - \tfrac{1}{2} m_p L_p L_r \cos(\alpha)\ddot{\alpha}$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\alpha}}\right) = \left(J_p + \tfrac{1}{4} m_p L_p^2\right)\ddot{\alpha} - \tfrac{1}{2} m_p L_p L_r (-\sin(\alpha)\dot{\alpha})\dot{\theta} - \tfrac{1}{2} m_p L_p L_r \cos(\alpha)\ddot{\theta}$$
$$= \left(J_p + \tfrac{1}{4} m_p L_p^2\right)\ddot{\alpha} + \tfrac{1}{2} m_p L_p L_r \sin(\alpha)\dot{\theta}\dot{\alpha} - \tfrac{1}{2} m_p L_p L_r \cos(\alpha)\ddot{\theta}$$

1. $Q_\theta = \tau - B_r\dot{\theta}$
$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta}$$
$$= \tfrac{1}{4} m_p L_p^2 \sin(2\alpha)\dot{\theta}\dot{\alpha} + \left(m_p L_r^2 + \tfrac{1}{4} m_p L_p^2 \sin^2(\alpha) + J_r\right)\ddot{\theta}$$
$$+ \tfrac{1}{2} m_p L_p L_r \sin(\alpha)\dot{\alpha}^2 - \tfrac{1}{2} m_p L_p L_r \cos(\alpha)\ddot{\alpha}$$

2. $Q_\alpha = -B_p\dot{\alpha}$
$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\alpha}}\right) - \frac{\partial L}{\partial \alpha}$$
$$= \left(J_p + \tfrac{1}{4} m_p L_p^2\right)\ddot{\alpha} + \tfrac{1}{2} m_p L_p L_r \sin(\alpha)\dot{\theta}\dot{\alpha} - \tfrac{1}{2} m_p L_p L_r \cos(\alpha)\ddot{\theta}$$
$$- \left(\tfrac{1}{8} m_p L_p^2 \sin(2\alpha)\dot{\theta}^2 + \tfrac{1}{2} m_p L_p L_r \sin(\alpha)\dot{\theta}\dot{\alpha} + \tfrac{1}{2} m_p L_p g \sin(\alpha)\right)$$
$$= \left(J_p + \tfrac{1}{4} m_p L_p^2\right)\ddot{\alpha} - \tfrac{1}{2} m_p L_p L_r \cos(\alpha)\ddot{\theta} - \tfrac{1}{8} m_p L_p^2 \sin(2\alpha)\dot{\theta}^2 - \tfrac{1}{2} m_p L_p g \sin(\alpha)$$

3. Controllable: can steer system from any initial condition to any final condition
Observable: can deduce system state from given output.