Lab 1 – Prelab Answer Key

$$L = T - V$$
$$= \tfrac{1}{2}J_r\dot{\theta}^2 + \tfrac{1}{2}J_b(\dot{\theta}+\dot{\alpha})^2 - \tfrac{1}{2}K_b\alpha^2$$

$$\frac{\partial L}{\partial \theta} = 0$$
$$\frac{\partial L}{\partial \alpha} = -\tfrac{1}{2}K_b(2\alpha) = -K_b\alpha$$
$$\frac{\partial L}{\partial \dot{\theta}} = \tfrac{1}{2}J_r(2\dot{\theta}) + \tfrac{1}{2}J_b(2(\dot{\theta}+\dot{\alpha})) = J_r\dot{\theta} + J_b(\dot{\theta}+\dot{\alpha}) = (J_r+J_b)\dot{\theta} + J_b\dot{\alpha}$$
$$\frac{\partial L}{\partial \dot{\alpha}} = \tfrac{1}{2}J_b(2(\dot{\theta}+\dot{\alpha})) = J_b(\dot{\theta}+\dot{\alpha})$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = (J_r+J_b)\ddot{\theta} + J_b\ddot{\alpha}$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\alpha}}\right) = J_b(\ddot{\theta}+\ddot{\alpha})$$

1. $Q_\theta = \tau - B_r\dot{\theta}$
$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta}$$
$$= (J_r+J_b)\ddot{\theta} + J_b\ddot{\alpha} - 0 \implies \tau - B_r\dot{\theta} = (J_r+J_b)\ddot{\theta} + J_b\ddot{\alpha} \qquad (\theta)$$

2. $Q_\alpha = -B_b\dot{\alpha}$
$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\alpha}}\right) - \frac{\partial L}{\partial \alpha}$$
$$= J_b(\ddot{\theta}+\ddot{\alpha}) - (-K_b\alpha) \implies -B_b\dot{\alpha} = J_b(\ddot{\theta}+\ddot{\alpha}) + K_b\alpha \qquad (\alpha)$$

3. $(\theta)-(\alpha)$: $\tau - B_r\dot{\theta} + B_b\dot{\alpha} = J_r\ddot{\theta} - K_b\alpha$

$$\iff \ddot{\theta} = \frac{K_b}{J_r}\alpha - \frac{B_r}{J_r}\dot{\theta} + \frac{B_b}{J_r}\dot{\alpha} + \frac{1}{J_r}\tau$$

(θ): $\tau - B_r\dot{\theta} = (J_r+J_b)\ddot{\theta} + J_b\ddot{\alpha}$

$$\iff \ddot{\alpha} = -\frac{B_r}{J_b}\dot{\theta} - \frac{J_r+J_b}{J_b}\ddot{\theta} + \frac{1}{J_b}\tau$$
$$= -\frac{B_r}{J_b}\dot{\theta} - \frac{J_r+J_b}{J_b}\left(\frac{K_b}{J_r}\alpha - \frac{B_r}{J_r}\dot{\theta} + \frac{B_b}{J_r}\dot{\alpha} + \frac{1}{J_r}\tau\right) + \frac{1}{J_b}\tau$$
$$= -K_b\left(\frac{1}{J_r}+\frac{1}{J_b}\right)\alpha + \frac{B_r}{J_r}\dot{\theta} - B_b\left(\frac{1}{J_r}+\frac{1}{J_b}\right)\dot{\alpha} - \frac{1}{J_r}\tau$$

$$\begin{bmatrix}\dot{\theta}\\ \dot{\alpha}\\ \ddot{\theta}\\ \ddot{\alpha}\end{bmatrix} = \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & \frac{K_b}{J_r} & -\frac{B_r}{J_r} & \frac{B_b}{J_r}\\ 0 & -K_b\left(\frac{1}{J_r}+\frac{1}{J_b}\right) & \frac{B_r}{J_r} & -B_b\left(\frac{1}{J_r}+\frac{1}{J_b}\right)\end{bmatrix}\begin{bmatrix}\theta\\ \alpha\\ \dot{\theta}\\ \dot{\alpha}\end{bmatrix} + \begin{bmatrix}0\\ 0\\ \frac{1}{J_r}\\ -\frac{1}{J_r}\end{bmatrix}\tau \qquad (*)$$