Lab 3 – Prelab Answer Key

(Refer to lab 2 key for EOMs; find f by moving terms to one side)

$$\frac{\partial f_\theta}{\partial \theta} = 0$$
$$\frac{\partial f_\theta}{\partial \alpha} = \frac{1}{2} m_p L_p^2 \cos(2\alpha)\dot{\theta}\dot{\alpha} + \frac{1}{4} m_p L_p^2 \sin(2\alpha)\ddot{\theta} + \frac{1}{2} m_p L_p L_r \cos(\alpha)\dot{\alpha}^2 + \frac{1}{2} m_p L_p L_r \sin(\alpha)\ddot{\alpha}$$
$$\frac{\partial f_\theta}{\partial \dot{\theta}} = \frac{1}{4} m_p L_p^2 \sin(2\alpha)\dot{\alpha} + B_r$$
$$\frac{\partial f_\theta}{\partial \dot{\alpha}} = \frac{1}{4} m_p L_p^2 \sin(2\alpha)\dot{\theta} + m_p L_p L_r \sin(\alpha)\dot{\alpha}$$
$$\frac{\partial f_\theta}{\partial \ddot{\theta}} = m_p L_r^2 + \frac{1}{4} m_p L_p^2 \sin^2(\alpha) + J_r$$
$$\frac{\partial f_\theta}{\partial \ddot{\alpha}} = -\frac{1}{2} m_p L_p L_r \cos(\alpha)$$
$$\frac{\partial f_\theta}{\partial \tau} = -1$$

$$\frac{\partial f_\alpha}{\partial \theta} = 0$$
$$\frac{\partial f_\alpha}{\partial \alpha} = \frac{1}{2} m_p L_p L_r \sin(\alpha)\ddot{\theta} - \frac{1}{4} m_p L_p^2 \cos(2\alpha)\dot{\theta}^2 - \frac{1}{2} m_p L_p g \cos(\alpha)$$
$$\frac{\partial f_\alpha}{\partial \dot{\theta}} = -\frac{1}{4} m_p L_p^2 \sin(2\alpha)\dot{\theta}$$
$$\frac{\partial f_\alpha}{\partial \dot{\alpha}} = B_p$$
$$\frac{\partial f_\alpha}{\partial \ddot{\theta}} = -\frac{1}{2} m_p L_p L_r \cos(\alpha)$$
$$\frac{\partial f_\alpha}{\partial \ddot{\alpha}} = J_p + \frac{1}{4} m_p L_p^2$$
$$\frac{\partial f_\alpha}{\partial \tau} = 0$$

1. $f_\theta(0,0,0,0,0,0,0) = 0$

$\frac{\partial f_\theta}{\partial \theta}(0,0,0,0,0,0,0) = 0$

$\frac{\partial f_\theta}{\partial \dot{\theta}}(0,0,0,0,0,0,0,0) = B_r$

$\frac{\partial f_\theta}{\partial \ddot{\theta}}(0,0,0,0,0,0,0) = m_p L_r^2 + J_r$

$\frac{\partial f_\theta}{\partial \tau}(0,0,0,0,0,0,0) = -1$

$\frac{\partial f_\theta}{\partial \alpha}(0,0,0,0,0,0,0,0) = 0$

$\frac{\partial f_\theta}{\partial \dot{\alpha}}(0,0,0,0,0,0,0) = 0$

$\frac{\partial f_\theta}{\partial \ddot{\alpha}}(0,0,0,0,0,0,0) = -\frac{1}{2} m_p L_p L_r$

$$\Rightarrow f_{lin}(\theta, \alpha, \dot{\theta}, \dot{\alpha}, \ddot{\theta}, \ddot{\alpha}, \tau) = B_r \dot{\theta} + (m_p L_r^2 + J_r)\ddot{\theta} - \frac{1}{2} m_p L_p L_r \ddot{\alpha} - \tau$$

2. $f_\alpha(0,0,0,0,0,0,0) = 0$

$\frac{\partial f_\alpha}{\partial \theta}(0,0,0,0,0,0,0,0) = 0$

$\frac{\partial f_\alpha}{\partial \dot{\theta}}(0,0,0,0,0,0,0,0) = 0$

$\frac{\partial f_\alpha}{\partial \ddot{\theta}}(0,0,0,0,0,0,0,0) = -\frac{1}{2} m_p L_p L_r$

$\frac{\partial f_\alpha}{\partial \tau}(0,0,0,0,0,0,0,0) = 0$

$\frac{\partial f_\alpha}{\partial \alpha}(0,0,0,0,0,0,0,0) = -\frac{1}{2} m_p L_p g$

$\frac{\partial f_\alpha}{\partial \dot{\alpha}}(0,0,0,0,0,0,0,0) = B_p$

$\frac{\partial f_\alpha}{\partial \ddot{\alpha}}(0,0,0,0,0,0,0,0) = J_p + \frac{1}{4} m_p L_p^2$

$$\Rightarrow f_{lin}(\theta, \alpha, \dot{\theta}, \dot{\alpha}, \ddot{\theta}, \ddot{\alpha}, \tau) = -\frac{1}{2} m_p L_p g \alpha + B_p \dot{\alpha} - \frac{1}{2} m_p L_p L_r \ddot{\theta} + \left(J_p + \frac{1}{4} m_p L_p^2\right)\ddot{\alpha}$$

3. $$\underbrace{\begin{bmatrix} m_p L_r^2 + J_r & -\frac{1}{2} m_p L_p L_r \\ -\frac{1}{2} m_p L_p L_r & J_p + \frac{1}{4} m_p L_p^2 \end{bmatrix}}_{Q} \begin{bmatrix} \ddot{\theta} \\ \ddot{\alpha} \end{bmatrix} + \underbrace{\begin{bmatrix} B_r & 0 \\ 0 & B_p \end{bmatrix}}_{R} \begin{bmatrix} \dot{\theta} \\ \dot{\alpha} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{2} m_p L_p g \end{bmatrix}}_{S} \begin{bmatrix} \theta \\ \alpha \end{bmatrix} + \underbrace{\begin{bmatrix} -1 \\ 0 \end{bmatrix}}_{T} \tau = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \dot{\theta} \\ \dot{\alpha} \\ \ddot{\theta} \\ \ddot{\alpha} \end{bmatrix} = \begin{bmatrix} 0_{2\times 2} & I_{2\times 2} \\ -Q^{-1}S & -Q^{-1}R \end{bmatrix} \begin{bmatrix} \theta \\ \alpha \\ \dot{\theta} \\ \dot{\alpha} \end{bmatrix} + \begin{bmatrix} 0_{2\times 1} \\ -Q^{-1}T \end{bmatrix} \tau$$ (not worth expanding out)

4. $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Poles at (approximately) -48.6383, -5.8699, 0, and 7.0543. Since not all poles in $\mathbb{C}^-$, open-loop system is unstable.
↳ this equilibrium is the upright position/inverted pendulum, so it makes sense that this is unstable

5. Lab 2's equilibrium had poles at (approximately) -45.2297, $-1.1121 \pm 6.5797i$, and 0, yielding marginal stability.
↳ assuming no friction (not included in model), sufficiently small perturbations may continue indefinitely.

6. No specific answer? Looking for research/thoughts.