Toomre Model of Galaxies

L. Ryder, 97988752

The Toomre model simplifies complex galaxies into massive core particles being orbited by massless stars. Newton's theory of gravitation shows that between two particles, the gravitational force $F_g = m\mathbf{a} = \frac{Gm_1m_2}{r^2}\hat{\mathbf{r}}$ where G is a gravitational constant, m_1, m_2 are masses, r is the distance between the masses, and $\hat{\mathbf{r}}$ is the radial unit vector. Generalizing to a system of particles, $m_i\mathbf{a}_i = \sum_{j=1,j\neq i}^{N} \frac{m_im_j}{r_{ij}^2}\hat{\mathbf{r}_{ij}}$ for the i-th particle. As we know, $\mathbf{a}_i$ can be expressed as the second derivative of the position, $\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2}$. If we substitute this, and simplify the unit vector, the acceleration due to gravitation is given as $\frac{d^2\mathbf{r}}{dt^2} = \sum_{j=1}^{N} \frac{m_j}{r_{ij}^3}\mathbf{r}_{ij}$.

We are looking for an approximate solution to this differential equation. Among the easiest way to find approximations computationally is a method called finite difference approximation. Instead of dealing with a continuum of data, such as with a pure function, instead calculate differences between points on a discrete grid with steps of some order. Since we are calculating a finite difference approximation for a second-order derivative, so we use a second-order approximation and the type we use is centred. A centred FDA is the average of forward and backwards approximations given by:

$$\frac{d^2\mathbf{r}(t)}{dt^2} \approx \frac{\mathbf{r}^{n+1} - 2\mathbf{r}^n + \mathbf{r}^{n-1}}{\Delta t^2}$$

If we substitute the acceleration found before, then for $n+1 = 3, 4, ..., n_t$, we get that for the i-th particle:

$$\sum_{j=1}^{N} \frac{m_j}{r_{ij}^3}\mathbf{r}_{ij} \approx \frac{\mathbf{r}_i^{n+1} - 2\mathbf{r}_i^n + \mathbf{r}_i^{n-1}}{\Delta t^2}$$

We can rearrange this formula for the next term:

$$\mathbf{r}_i^{n+1} = \Delta t^2 \sum_{j=1}^{N} \frac{m_j}{r_{ij}^3} + 2\mathbf{r}_i^n - \mathbf{r}_i^{n-1}$$

Typically, the biggest struggle in implementing physical models in code is dealing with optimization. Simulations large enough to show interesting collisions have at least 1000 particles. Thus, when something is not implemented correctly in the code, the script will take an inordinate amount of time to run. And with no output on that scale it can be tough to see where the issue is occurring. This is where the magic of the model comes in. Because all of the stars are massless, they only interact directly with the massive cores and not with each other. Since there are substantially more stars than cores in our case, that simplification allows us to code the sum term in such a way that we can skip a vast majority of the calculations using simple control structures.

The code starts off by initializing important conditions for the code, such as the number of stars, number of cores, and the level of the calculation. Then, an array of the positions and the masses of each particle must be initialized. Then as required by the FDA, the first two positions of each particle must be given. Then the script utilizes a for loop to step through all of the time steps, calculating the FDA of all of the particles at once, and calling a separate acceleration function that accounts for the different types of particles. The positions of each particle are then plotted for each of these steps.

To test the accuracy of the model to our physical universe, a convergence test is completed for the case of the massive cores orbiting each other - a simple 2-body problem. This is done by calculating the difference between values for multiple "levels" of finite grids. As the level increases, the number of time steps increases by a factor of 2, so we step through the higher level arrays as necessary to compare the data at the same time across levels. The figure shows the data outputted and scaled as required to show that the behaviour is the same at any level of FDA. Given that the data is calculated with a second-order approximation, we expect to see error on the order of Δt^2. As expected we see this quadratic convergence as the differences are coincident.

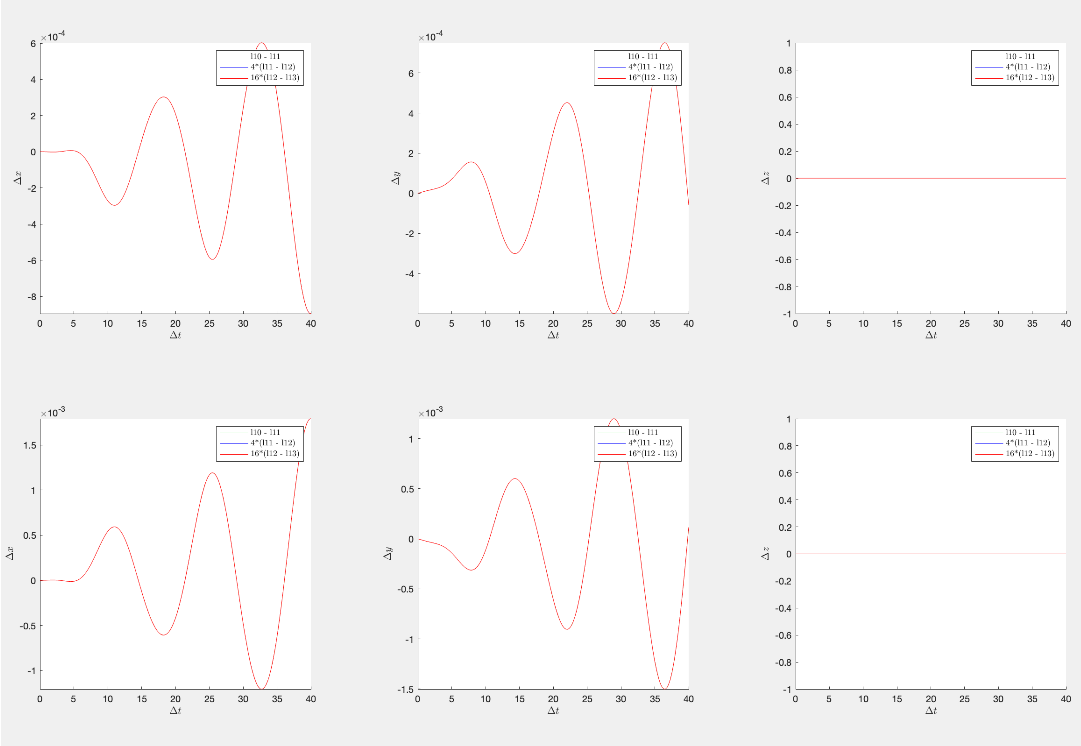

FIG. 1.