The Moment of Inertia of a Cardboard Equilateral Triangle Around its Centroid

IB Physics IA

Introduction

The moment of inertia of an object around an axis is a measure of how much said object resists acceleration (Kognity, 2022, Moment of inertia). The IB curriculum does not require calculations for the moments of inertia for arbitrary rigid bodies. Instead, on exams, we are given the moments of inertia for various primitives. Notably, the moment of inertia for what is arguably one of the simplest primitives, an equilateral triangle around its centroid, is not commonly given. The aim of this paper is to experimentally determine how the moment of inertia of an equilateral triangle, rotating as seen in Figure 1, changes with the triangle's side length.

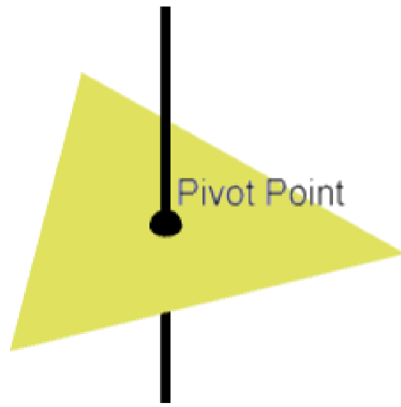


Figure 1. The Pivot Point and Axis of Rotation which will be Referenced in the Rest of this Paper. This figure was generated with Geogebra, and can be found at https://www.geogebra.org/calculator/ubpudqjs.

Research Question

What is the relationship between the side length of a corrugated cardboard equilateral triangular plane, rotating around its centroid on an axis normal to the plane, and its moment of inertia?

Background

The moment of inertia is dependent on the mass distribution within the given shape, around the axis of rotation. This leads to the moment of inertia also being dependent on the shape, and density of a uniformly dense rigid body (such as the one studied in this paper).

Any problem involving the prediction of the torque applied to, or energy of, a body in rotational dynamics involves the moment of inertia of the rotating body. Since most objects can be approximated with simple primitives, knowing the moment of inertia of said simple primitives is key to solving rotational dynamics problems.

In this paper, the moment of inertia of our equilateral triangles will be found by applying a constant torque to an object, and indirectly measuring the time it takes to achieve a certain angular displacement. Thus, we must define inertia in terms of a time interval, an angular displacement, and torque.

Force on a point mass (F) is defined as $F = ma$, where m is mass, and a is acceleration. Similarly, the torque on a rigid body (τ) is defined as $\tau = I\alpha$, where I is the moment of inertia, and α is angular acceleration (Kognity, 2022, Newton's second law revisited). We can rearrange this equation to

$$I = \frac{\tau}{\alpha}$$

For a rotating body, with no initial angular velocity or displacement, and a constant angular acceleration,

$$\theta = \frac{1}{2}\alpha t^2$$

where θ is the final angular displacement of the object, α is angular acceleration, and t time elapsed (Kognity, 2022, Uniform angular acceleration). We can rearrange this to

$$\alpha = 2\theta t^{-2}$$

Thus, we can calculate the moment of inertia in terms of the final angular displacement, time elapsed, and torque on the body,

$$I = \frac{\tau t^2}{2\theta}$$

Theoretical Value

The theoretical relationship between a given equilateral triangle's side length (L), mass (m), density (ρ), and moment of inertia rotating around the centroid, on an an axis normal to the triangle (I), is given as

$$I = \frac{1}{12}mL^2 \text{ or } I = \frac{\sqrt{3}}{48}\rho L^4$$

where ρ is the density of the triangle, and m is the mass of the triangle. The calculations to find this theoretical value are beyond the scope of IB Physics, but can be found in Appendix A.

Hypothesis

If corrugated cardboard equilateral triangles are spun around their centroid with constant torque, it will be found that the inertia vs. side length relationship will follow $I = \frac{\sqrt{3}}{48}\rho L^4$, where I is the moment of inertia, ρ is the density of the cardboard, and L is the side length of the equilateral triangle.

Methodology

To do this experiment, an apparatus is required that applies constant torque to a rotating body (See Figure 2 and Figure 3). This will be done by using a falling mass to pull a wound up string around a wheel. This experiment is heavily inspired by the apparatus found in Siedlecki's Lab, referred to as "Beck's Inertia Thing" (Siedlecki, 2016).

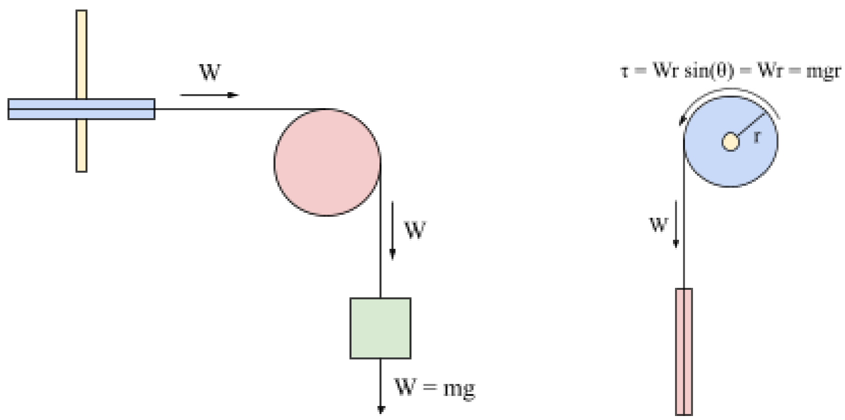


Figure 2. Diagrams of the apparatus used to apply constant acceleration to a rotating body. On the left, a side view, on the right, a bird's eye view. The green mass pulls the pulley down, the red wheel keeps the string tangential to the rotating body, and the blue wheel has string wound up around it that rotates the body. The yellow axel is where the triangle will eventually be attached to.

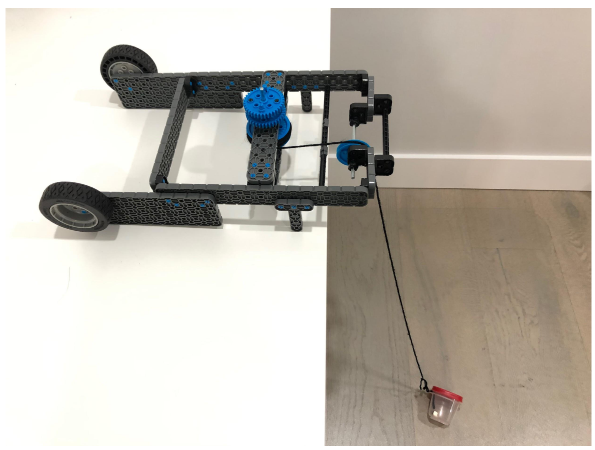


Figure 3. Actual apparatus, built with vex IQ

To calculate the torque on a the blue wheel seen in Figure 2,

$$\tau = Fr\,sin(\theta)$$

where F is force, r is radius of the wheel, and θ is the angle between the line of action, and line tangential to the point on the body the force is applied (Kognity, 2022, Torque). Since the guiding wheel keeps the string tangential to the rotating body, $\theta = 0$. Since the force acting on the wheel is the same as the weight of the falling mass, $F = W = mg$. Thus,

$$\tau = Wr = mgr$$

Substituting in mgr into $I = \frac{\tau t^2}{2\theta}$,

$$I = \frac{mgrt^2}{2\theta}$$

The angular displacement (θ) is constant for any trial, regardless of side length, and is dependent on the distance from the apparatus to the floor. Specifically,

$$\theta = \frac{s}{r}$$

where s is the distance from the apparatus to the floor, and r is the radius of the wheel. Substituting into $I = \frac{mgrt^2}{2\theta}$,

$$I = \frac{mgr^2t^2}{2s}$$

Using this equation, we can find the moment of inertia of the wheel in blue in Figure 2, as well as any object attached to it. Since inertia is additive, we will take a calibration trial, measuring the moment of internal inertia of the apparatus, and subtract it from the inertia from our experimental trials.

Variables

Controlled Variables:

1) Mass pulling the pulley down (m)
   - The mass will be unchanged throughout the experiment.
2) Radius of wheel rotating the triangle (r)
   - The wheel will remain unchanged throughout the experiment.
3) Distance the mass pulling the pulley travels (s)
   - The apparatus used in this experiment will be used from a constant height.
4) The density of the triangles (ρ)
   - The material of the triangles remains constant throughout the experiment.

Independent Variable: Side length of the triangle (L). The lengths tested for this experiment will be 30, 35, 40, 45, and 50 cm.

Dependent Variable: Time it takes for the mass pulling the pulley to hit the floor from release (t). This will be measured by taking a video, and recording the time of the drop and the time the mass hits the floor, relative to the start of the video. From there, taking the difference will yield the time elapsed.

Materials

- Constant Torque Apparatus (See Figure 2 and Figure 3)
- Corrugated Cardboard (Sourced from Home Depot), to make triangles of side lengths 30, 35, 40, 45, and 50 cm
- Tape Measure
- Protractor
- Ruler
- A computer, running video editing software, to measure time intervals in the video precisely.
- Blender (Software typically used for 3d modeling, with video editing capabilities, not the household appliance)
- Centigram balance

Procedure

1) Use a ruler and protractor to cut out 5 equilateral triangles, of side lengths 30, 35, 40, 45, and 50 cm. The uncertainty for the side lengths of these triangles will be $\pm\ 1\ cm$, due to errors in the cutting process. This is a source of random error.
2) Record mass at end of string, using a centigram balance.
3) Record radius of wheel rotating triangle, using a ruler.
4) Record the distance from the floor to the apparatus, using a tape measure.
5) Run the machine with no triangle and record a video. Record the time when the mass is dropped, and when it hits the ground, using Blender (software). (Do this 5 times)
6) Run the machine with each triangle, taking a video for each.
7) Record the time when the mass is dropped, and when it hits the ground. (5 times for each triangle).

Safety and Ethical Considerations

There are three main safety concerns:

1) Carpet Burn - The process of winding up the string can cause carpet burn, as you need to maintain tension to make sure it does not tangle. Due to this, touching the string will be avoided.
2) Toe Injury - Make sure to wear closed toe shoes. This is required in our school lab.
3) Tripping - The string of the pulley is a tripping hazard. Because of this, I will execute my experiment in a low traffic area of the lab.

Ethically, the disposal of materials must be considered. All materials are either reusable (vex robotics pieces) or recyclable (corrugated cardboard triangles).

Data and Observations

Qualitative Data

As the side length of the triangles increases, the mass drops significantly slower. With that being said, every trial, there was some amount of shaking, which may lead to an increase in friction, and is a source of error.

Table 1 - Controlled Variables

Mass of Weight at End of String ($\pm\ 0.01\ g$)	Radius of Wheel Rotating the Free Body ($\pm\ 0.1\ cm$)	Distance to Ground from Apparatus ($\pm\ 5\ cm$)
31.22	2.3	95

Table 2 - Masses of Each Triangle

Triangle Side Length ($\pm\ 1\ cm$)	Mass ($\pm\ 0.01\ g$)
30	21.12
35	27.95
40	34.64
45	46.19
50	58.50

Table 3 - Times of Start and End of Drop vs. Triangle Side Length

	Side Length of Triangle ($\pm\ 1\ cm$)	30	35	40	45	50	Calibration Trials
Trial 1	Start Time ($\pm\ 0.04\ s$)	1.77	3.17	1.63	2.17	1.90	1.93
	End Time ($\pm\ 0.04\ s$)	3.53	5.33	4.17	5.53	6.27	2.50
Trial 2	Start Time ($\pm\ 0.04\ s$)	1.70	2.10	1.70	1.83	1.03	1.37
	End Time ($\pm\ 0.04\ s$)	3.43	4.40	4.17	5.13	5.40	1.93
Trial 3	Start Time ($\pm\ 0.04\ s$)	1.30	1.70	1.37	2.73	1.80	1.03
	End Time ($\pm\ 0.04\ s$)	3.03	4.03	3.93	6.00	6.03	1.60
Trial 4	Start Time ($\pm\ 0.04\ s$)	0.73	1.57	2.63	2.63	1.27	0.80
	End Time ($\pm\ 0.04\ s$)	2.50	3.87	5.20	7.03	5.40	1.37
Trial 5	Start Time ($\pm\ 0.04\ s$)	1.43	1.4	1.17	3.70	3.23	2.10
	End Time ($\pm\ 0.04\ s$)	3.13	3.63	4.23	7.13	7.33	2.67

Data Analysis

Finding the Theoretical Relationship

The theoretical relationship this experiment compares to relies on the density of the triangles. The following is a sample calculation to find this density for the 30 cm triangle.

$$\rho = \frac{m}{A}$$

$$A = \frac{\sqrt{3}}{4}(30 \pm 1\ cm)^2 = 390 \pm 30\ cm^2$$

$$m = 21.12g \pm 0.01$$

$$\rho = \frac{21.12 \pm 0.01 g}{390 \pm 30\ cm^2} = 0.054 \pm 0.004\ g\ cm^{-2}$$

Table 4 - Area Density of Corrugated Cardboard

Triangle Side Length ($\pm\ 1\ cm$)	Density ($g\ cm^{-2}$)
30	0.054 ± 0.004
35	0.053 ± 0.004
40	0.050 ± 0.003
45	0.053 ± 0.003
50	0.054 ± 0.003
Average	0.053 ± 0.006

Recall that our theoretical relationship is $I = \frac{\sqrt{3}}{48}\rho L^4$. Since the average density, $\rho = 0.053 \pm 0.006\ g\ cm^{-2}$, our theoretical relationship is also $I = \frac{\sqrt{3}}{48}(0.053 \pm 0.006\ g\ cm^{-2})L^4$, or

$$I = (0.0019 \pm 0.0002 g\ cm^{-2})L^4$$

Finding the Moment of Inertia for Each Triangle

First, we must find the time interval for each drop. The following is a sample calculation for side length 30 cm, trial 1.

$$3.53s \pm 0.04s - 1.77 \pm 0.04s = 1.76 \pm 0.08s$$

Table 5 - Time Intervals for Each Drop

	Side Length of Triangle ($\pm\ 1\ cm$)	30	35	40	45	50	Calibration Trials
Trial 1	Time Interval ($\pm\ 0.08\ s$)	1.76	2.16	2.54	3.36	4.37	0.57
Trial 2	Time Interval ($\pm\ 0.08\ s$)	1.73	2.3	2.47	3.30	4.37	0.56
Trial 3	Time Interval ($\pm\ 0.08\ s$)	1.73	2.33	2.56	3.27	4.23	0.57
Trial 4	Time Interval ($\pm\ 0.08\ s$)	1.77	2.3	2.57	4.40	4.13	0.57
Trial 5	Time Interval ($\pm\ 0.08\ s$)	1.70	2.23	3.06	3.43	4.10	0.57

Next, we can average out the time intervals, and moments of inertia for each side length. The following is a sample calculation for the 30 cm side length, where t is the average time.

$$t = \frac{1.76+1.73+1.73+1.77+1.70}{5} \pm \frac{(1.77+0.08)-(1.70-0.08)}{2} cm = 1.7 \pm 0.2 \, cm$$

We can use $I = \frac{mgr^2t^2}{2s}$ to calculate inertia for the 30 cm side length,

$$I = \frac{mgr^2t^2}{2s}$$

$$= \frac{(31.22\pm0.01\, g)(981\, cm\, s^{-2})(2.3\pm0.1\, cm)^2(1.7\pm0.2\, s)^2}{2(95\pm5\, cm)}$$

$$= \frac{(31.22\, g)(981\, cm\, s^{-2})(2.3\, cm)^2(1.7\, s)^2}{2(95\, cm)} \pm \frac{I_{max}-I_{min}}{2}$$

$$= 2600\, g\, cm^2 \pm \frac{\frac{(31.22+0.01\, g)(981\, cm\, s^{-2})(2.3+0.1\, cm)^2(1.7+0.2\, s)^2}{2(95+5\, cm)} - \frac{(31.22-0.01\, g)(981\, cm\, s^{-2})(2.3-0.1\, cm)^2(1.7-0.2\, s)^2}{2(95-5\, cm)}}{2}$$

$$= 2600 \pm 900\, g\, cm^2$$

We can now use the same process to calculate the calibration moment of inertia (see table 6), and subtract it from the experimental moment of inertia, to correct for the internal moment of inertia of the constant torque apparatus.

$$I = (2600 \pm 900\, g\, cm^2) - (300 \pm 200\, g\, cm^2) = 2000 \pm 1000\, g\, cm^2$$

Table 6 - Time Intervals and Moments of Inertia for Each Side Length

Side Length of Triangle ($\pm$ 1 cm)	30	35	40	45	50	Calibration Trials
Time Interval (s)	1.7 ± 0.2	2.3 ± 0.2	2.6 ± 0.4	3.6 ± 0.7	4.3 ± 0.3	0.57 ± 0.01
Moment of Inertia ($g\, cm^2$)	2600 ± 900	4000 ± 2000	6000 ± 4000	10000 ± 9000	15000 ± 5000	300 ± 200
Moment of Inertia Corrected for Calibration ($g\, cm^2$)	2000 ± 1000	4000 ± 2000	6000 ± 4000	10000 ± 9000	15000 ± 5000	0

Because our theoretical relationship is a quartic power curve, $I = (0.0019 \pm 0.0002 g\, cm^{-2})L^4$, we expect a power curve to fit the experimental data. This will later be shown to be true using R^2, after the graph is generated.

While excel will generate the best fit line for us, maximum and minimum power lines must be done via a log-log plot. The following is a sample calculation for transforming the 30 cm side length datapoint onto a log-log plot, using e as a base. Note that for this step, as well as for any steps involving log-log plots, units will be omitted, as they become nonsensical through the natural log function.

Let $x =$ The Horizontal Coordinate of the Datapoint Transformed Onto the log-log Plot

Let $y =$ The Vertical Coordinate of the Datapoint Transformed Onto the log-log Plot

$$x = ln(30) = 3.40,\ y = ln(2000) = 7.60$$

Let $U_{h+} =$ Uncertainty of the Datapoint in the Positive Horizontal Direction

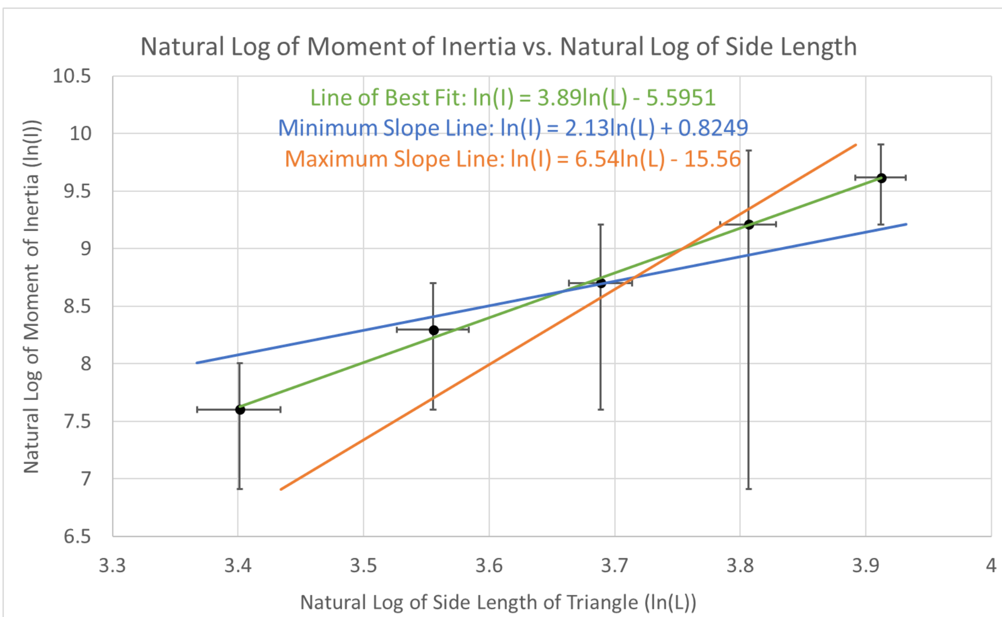
Let $U_{h-} =$ Uncertainty of the Datapoint in the Negative Horizontal Direction

Let $U_{v+} =$ Uncertainty of the Datapoint in the Positive Vertical Direction

Let $U_{v-} =$ Uncertainty of the Datapoint in the Negative Vertical Direction

$$U_{h+} = ln(30 + 1) - x = 0.0328,\ U_{h-} = x - ln(30 - 1) = 0.0339$$

$$U_{v+} = ln(2000 + 1000) - y = 0.405,\ U_{v-} = y - ln(2000 - 1000) = 0.693$$

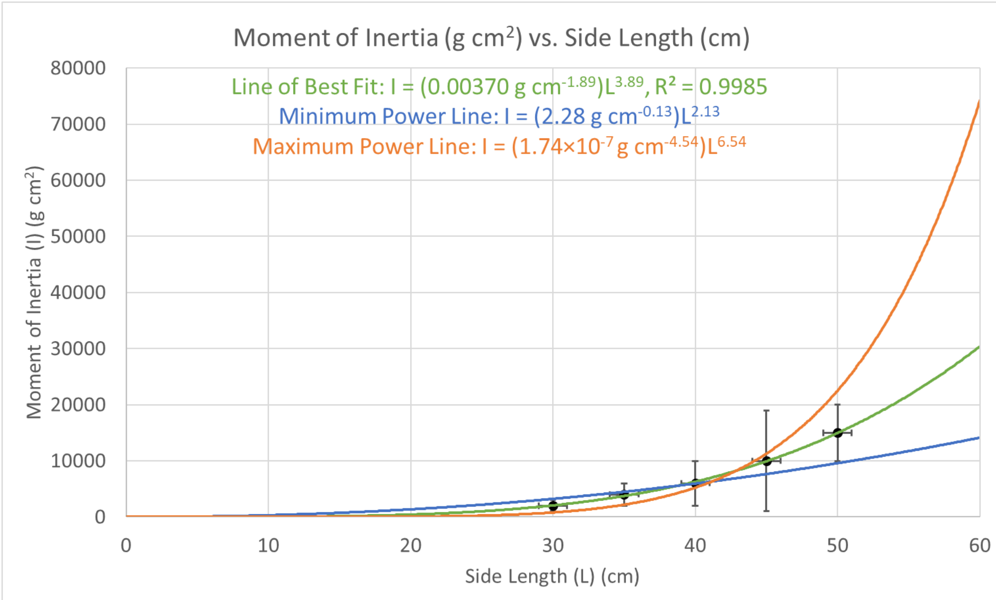


From here, given a line in the form $ln(y) = m\,ln(x) + b$, we can transform it back to a linear scale using $y = e^b x^m$. Using this we can find the best fit, maximum, and minimum exponent lines.

Best Fit Line: $ln(I) = 3.89\,ln(L) - 5.5951 \rightarrow I = 0.00370\,L^{3.89}$

Maximum Power Line: $ln(I) = 2.13\,ln(L) - 5.5951 \rightarrow I = 2.28\,L^{2.13}$

Minimum Power Line: $ln(I) = 6.54\,ln(L) - 5.5951 \rightarrow I = 1.74 \times 10^{-7}\,L^{6.54}$



It was found the line of best fit, following a power curve is, and using the Least Mean Squared (LMS) algorithm on excel, is $I = (0.00370^{+2.28}_{-0.00370}\, g\, (cm^{-1.89^{+1.76}_{-2.65}}))L^{3.89^{+2.65}_{-1.76}}$. The uncertainties were found by taking the difference in exponent and coefficient between the minimum power line and best fit line, and the maximum power line and best fit line. The data shows significant correlation, as $R^2 = 0.9985$, however there are significantly high uncertainties.

Note that the subscript/superscript uncertainty notation is used to denote asymmetrical uncertainties. Also note that, according to what we learned in class, uncertainties should be rounded up to 1 significant figure, and measurements should be rounded to match the uncertainty. Following this principle, the best fit line would be $I = (0^{+3}_{-1}\, g\, (cm^{-2^{+2}_{-3}}))L^{4^{+3}_{-2}}$, implying that inertia is always 0. With that being said, in this context, since the min and max lines are exact values, this principle no longer applies. Thus, instead of rounding uncertainties to 1 significant figure, they were rounded to 3, and values were also rounded to 3 significant figures, not matching the uncertainties, to better represent the data from the graph.

The exponent on the centimeter unit is calculated, such that the moment of inertia is in a unit of $mass \cdot distance^2$.

Conclusion

Experimentally, it was found that the relationship between the moment of inertia of an equilateral corrugated cardboard triangle rotating around its centroid, and the side length of triangle, is

$$I = (0.00370^{+2.28}_{-0.00370}\, g\, (cm^{-1.89^{+1.76}_{-2.65}}))L^{3.89^{+2.65}_{-1.76}}$$

Comparing this to the theoretical relationship, $(I = (0.0019 \pm 0.0002 g\ cm^{-2})L^{4}$, the percent error (E) can be found using the equation

$$E = (100\%)\frac{|(0.0019\ g\ cm^{-2})L^{4} - (0.00370\ g\ (cm^{-1.89}))L^{3.89}|}{(0.0019\ g\ cm^{-2})L^{4}}$$

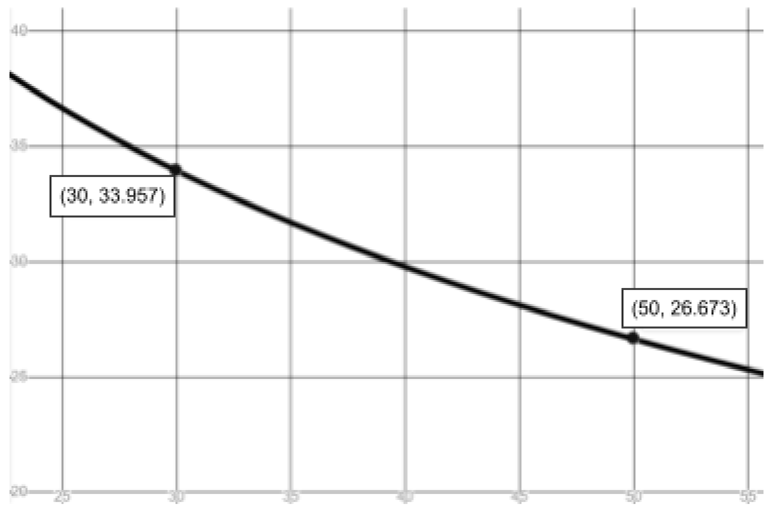


Figure 4. E in % plotted on the y-axis, and side length in cm on the x-axis. This diagram was generated in Desmos.

The worst case percent error for the experimented range (30-50 cm) is 34%, and remains under 34% until a side length of 18,700 cm.

The theoretical relationship falls within the possible range of measured relationships, thus supporting the hypothesis. With that being said, the results of this experiment are inconclusive, due to the significant uncertainties present in the measured relationship.

Sources of Error

1) Friction - The apparatus used to apply constant torque used in this experiment is a major source of friction, specifically from the rotation of the axels to which the wheels are attached. To mitigate this, O-rings were attached to every axel. This systematic error will lead to consistently higher measured moments of inertias than the theoretical value. This systematic error grows larger as the side length (and mass) of the triangles grows larger.
2) Triangle Construction - Constructing the corrugated cardboard triangles was done using a protractor and ruler. Small errors in reading the protractor were multiplied when finding the edges. This source of random error is accounted for, by adding an uncertainty of $\pm\ 1\ cm$ to the side lengths of the triangles.
3) Measuring the Angular Displacement of the Triangle - The theory discussed in this paper relies on inelastic string turning a wheel. In reality, the string used for this experiment is not completely inelastic. This source of random error is accounted for by adding a $\pm\ 5\ cm$ uncertainty to the distance between the apparatus and the floor.

4) Video Analysis - The camera used to record each datapoint had a framerate of 30 fps, thus each frame has a period of $\frac{1}{30}\ s$, or, rounded up to 1 significant figure, $0.04\ s$. This source of random error is accounted for by adding a $\pm\ 0.04\ s$ uncertainty to every time measurement, accounting for two frames.

Future Work and Improvements

This lab addresses the moment of inertia of equilateral triangular corrugated cardboard planes. Since only one density is tested, this experiment is not sufficient to back the theoretical relationship, $I = \frac{\sqrt{3}}{48}\rho L^4$. One major improvement that could be made is to add material density as an independent variable by changing the material of the equilateral triangle. Such an experiment would be sufficient to back the theoretical relationship.

The apparatus used in this lab has significant systematic errors, due to both internal friction within the apparatus, and air friction from the rotating body. Because of this, two improvements can be made. The first, which is out of scope for this lab, is to run the apparatus in a vacuum chamber. This would nearly eliminate air friction as an error. The second improvement would be to use bearings with grease in the apparatus, to minimize mechanical friction.

References

Kognity. (2022). *Option: Engineering physics*. IB Physics HL 2022-2023. Retrieved October 9, 2022, from https://kognity.com/

Siedlecki, R. (2016). *Experiment 9: Moments of Inertia*. General Physics Lab. Retrieved October 9, 2022, from https://www.phy.olemiss.edu/lab/genlab/labmanual/2021Manual/223/Experiments/MoIExp.pdf

Strang, G., & Herman, E. (2016, March 30). *5.6 calculating centers of mass and moments of inertia - calculus volume 3*. OpenStax. Retrieved November 24, 2022, from https://openstax.org/books/calculus-volume-3/pages/5-6-calculating-centers-of-mass-and-moments-of-inertia

Appendix A: Calculations for the theoretical Moment of Inertia

The moment of inertia (I) is defined such that

$$I = \sum_i m_i r_i^2$$

where m_i is the mass of a given particle i, and r_i is the distance between the axis of rotation and the given particle (Kognity, 2022, Moment of inertia) (Strang, G., & Herman, E., 2016, 5.6). This can be written in the form of the integral

$$I = \frac{m}{A}\int\int (x^2 + y^2)\, dA$$

where m is the net mass of the triangle, A is the net area of the shape.

The shape of half an equilateral triangle, oriented to point upwards, positioned such that $(0, 0)$ is the centroid of the triangle, split along the y axis, can be defined by the inequality,

$$-\frac{\sqrt{3}}{6}L < y < \sqrt{3}x + \frac{\sqrt{3}}{L}, \quad -\frac{L}{2} < x < 0 \tag{1}$$

where L is the side length of the triangle.

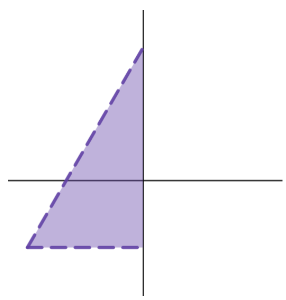


Figure 5. A diagram of inequality 1, generated with Desmos.

Since inertia is additive (Kognity, 2022, Moment of inertia), the moment of inertia of an equilateral triangle is defined by the finite double integral

$$I = 2\frac{m}{A}\int_{-\frac{L}{2}}^{0}\int_{-\frac{\sqrt{3}}{6}L}^{\sqrt{3}x+\frac{\sqrt{3}}{3}L} (x^2 + y^2)\, dy\, dx$$

Or, since the area of an equilateral triangle (A) is $\frac{\sqrt{3}}{4}L^2$,

$$I = \frac{8m}{\sqrt{3}L^2}\int_{-\frac{L}{2}}^{0}\int_{-\frac{\sqrt{3}}{6}L}^{\sqrt{3}x+\frac{\sqrt{3}}{3}L} (x^2 + y^2)\, dy\, dx$$

$$= \frac{8m}{\sqrt{3}L^2}\int_{-\frac{L}{2}}^{0} (x^2 y + \frac{y^3}{3})\Big|_{y=-\frac{\sqrt{3}}{6}L}^{y=\sqrt{3}x+\frac{\sqrt{3}}{3}L}\, dx$$

$$= \frac{8m}{\sqrt{3}L^2}\int_{-\frac{L}{2}}^{0}(x^2(\sqrt{3}x + \frac{\sqrt{3}}{3}L) + \frac{(\sqrt{3}x+\frac{\sqrt{3}}{3}L)^3}{3} - x^2(-\frac{\sqrt{3}}{6}L) - \frac{(-\frac{\sqrt{3}}{6}L)^3}{3})\,dx$$

$$= \frac{8m}{\sqrt{3}L^2}\int_{-\frac{L}{2}}^{0}(2\sqrt{3}x^3 + \frac{5\sqrt{3}}{6}Lx^2 + \frac{\sqrt{3}}{9}L^2x + \frac{\sqrt{3}}{24}L^3)\,dx$$

$$= \frac{8m}{L^2}\int_{-\frac{L}{2}}^{0}(2x^3 + \frac{5}{6}Lx^2 + \frac{1}{9}L^2x + \frac{1}{24}L^3)\,dx$$

$$= \frac{8m}{L^2}(\frac{1}{2}x^4 + \frac{5}{18}Lx^3 + \frac{1}{18}L^2x + \frac{1}{24}L^3x)\Big|_{x=-\frac{L}{2}}^{x=0}$$

$$= \frac{4m}{L^2}(x^4 + \frac{5}{9}Lx^3 + \frac{1}{9}L^2x + \frac{1}{12}L^3x)\Big|_{x=-\frac{L}{2}}^{x=0}$$

$$=-\frac{4m}{L^2}((-\frac{L}{2})^4 + \frac{5}{9}L(-\frac{L}{2})^3 + \frac{1}{9}L^2(-\frac{L}{2}) + \frac{1}{12}L^3(-\frac{L}{2}))$$

$$=-\frac{4m}{L^2}(\frac{1}{16}L^4 - \frac{5}{72}L^4 + \frac{1}{36}L^4 - \frac{1}{24}L^4)$$

$$= mL^2(-\frac{1}{4} + \frac{5}{18} - \frac{1}{9} + \frac{1}{6})$$

$$= \frac{1}{12}mL^2$$

$$= \frac{\sqrt{3}}{48}\rho L^4$$

where ρ is the density of the triangle, and m is the mass of the triangle.