4062
Parameter Estimation (Supervised and Unsupervised)

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September 21, 2022

Supervised Maximum Likelihood Estimation(MLE)

- First scenario: (toss a 'coin' Z)^D
- 2nd scenario: (toss Z; (then A or B)^{10})^D

Parameter Estimation

Outline

Common-sense and relative frequency

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if there were (30 a, 70 b) in $\boldsymbol{d}$, 'common-sense' says $P(Z = a) = 30/100$

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ie. you 'define' or 'estimate' the probability by the *relative frequency*

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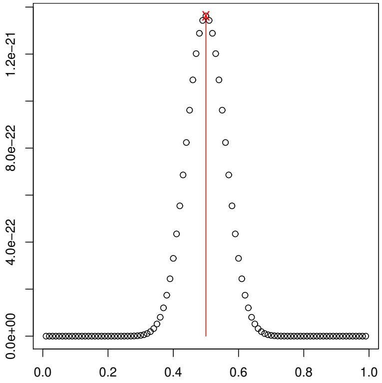
different settings of θ_a and θ_b will give different values for $p(\boldsymbol{d})$

following slides investigate this empirically

$p(\boldsymbol{d})$ for 50 a, 50 b



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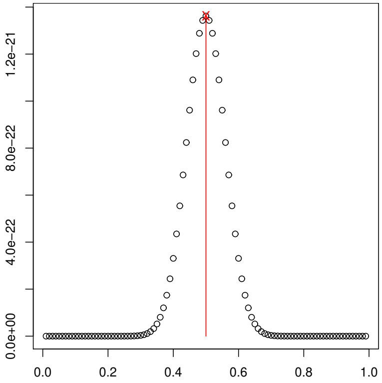

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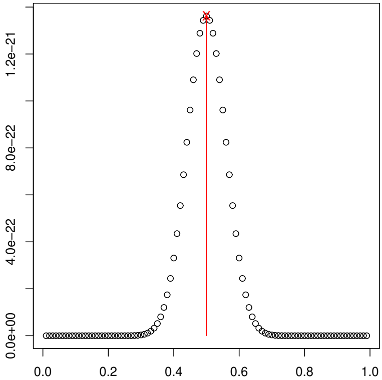

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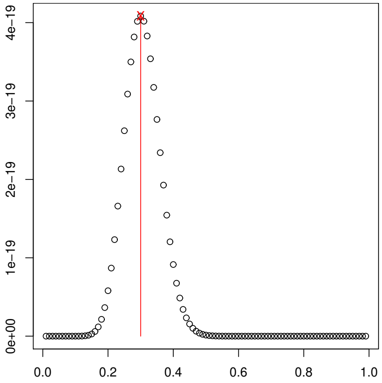

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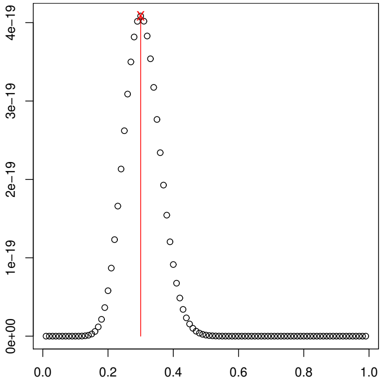

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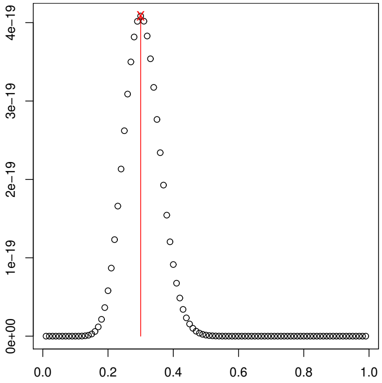

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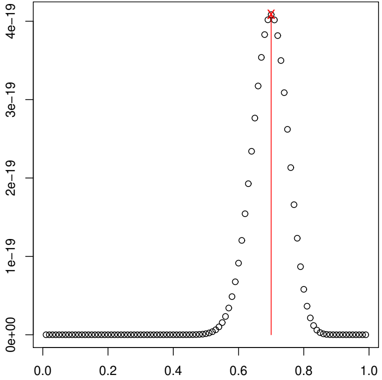

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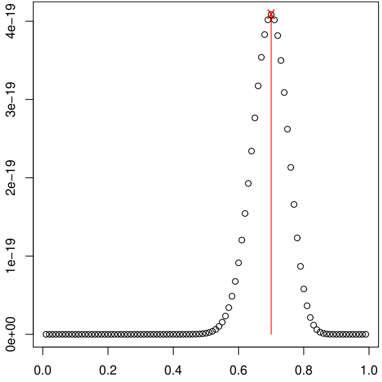

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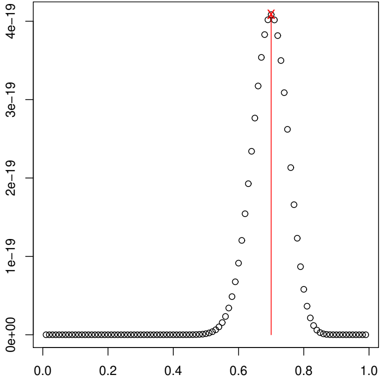

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- technically expressed as: the relative frequency is a *maximum likelihood estimator* of the parameters

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and because $\theta_b = 1 - \theta_a$ can really write this in terms of just parameter θ_a

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Yes $\Rightarrow$ take the log of this – the **log-likelihood** and use calculus to maximize *that* w.r.t. θ_a – this turns out to be (relatively) easy

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now want to consider slightly more complex scenario

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$$est(\theta_{h|b}) = \frac{\sum_{d:Z=B} \#(d,h)}{\sum_{d:Z=B} 10} = \frac{6}{30} = \frac{1}{5} = 0.2 \tag{4}$$

[$\#(d,h)$ is 'count' of h in row d, $\#(d,t)$ is 'count' of t in row d]

to make the comparision with the hidden variable version which will come up later, its worth noting that we can formulate all the restricted sums $\sum_{d:Z=A}(\Phi(d))$ with *unrestricted sums* if we put a so-called Kronecker-delta indicator function inside the sum $\sum_{d}(\delta(d,A)\Phi(d))$ where $\delta(d,A)=1$ if datum d had $Z=A$, and is 0 otherwise.

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$$est(\theta_a) = \frac{\sum_d \delta(d,A)}{D} \tag{5}$$

$$est(\theta_{h|a}) = \frac{\sum_d \delta(d,A)\#(d,h)}{\sum_d \delta(d,A)10} \tag{6}$$

$$est(\theta_{h|b}) = \frac{\sum_d \delta(d,B)\#(d,h)}{\sum_d \delta(d,B)10} \tag{7}$$

it turns out that in this scenario also, the 'common-sense', relative-frequency answers are also *maximum likelihood estimators* ie. values which maximise the probability of the data, and again it is (relatively) easy to show this by taking logs and using calculus.

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$$p(\mathbf{d}) = \prod_{d:Z=a} [\theta_a \theta_{h|a}^{\#(d,h)} \theta_{t|a}^{\#(d,t)}] \prod_{d:Z=b} [\theta_b \theta_{h|b}^{\#(d,h)} \theta_{t|b}^{\#(d,t)}]$$

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$$\sum_{d:Z=a} [log\theta_a + \#(d,h) log\theta_{h|a} + \#(d,t) log\theta_{t|a}] +$$

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call this $L(\theta_a, \theta_{h|a}, \theta_{h|b})$

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and this means that when you take the derivatives of $L(\theta_a, \theta_{h|a}, \theta_{h|b})$ wrt. θ_a, $\theta_{h|a}$ and $\theta_{h|b}$ in each case you can just look at one of the above terms.

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and this means that when you take the derivatives of $L(\theta_a, \theta_{h|a}, \theta_{h|b})$ wrt. θ_a, $\theta_{h|a}$ and $\theta_{h|b}$ in each case you can just look at one of the above terms. They are all really of the same form being $N(log(p)) + M(log(1-p))$, the same form as seen in the first simple scenario, and it has maximum value at $p = \frac{N}{N+M}$

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finally the denominators of these turn into D, $\sum_{d:Z=a} 10$ and $\sum_{d:Z=b} 10$ respectively and so are exactly the 'common sense' formulae we started with in (2), (3), (4)