1) [1.3 Q13] Determine if b is a linear combination of the vectors formed from the columns of the matrix A.

$$A = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 3 & 5 \\ -2 & 8 & -4 \end{bmatrix}, \; b = \begin{bmatrix} 3 \\ -7 \\ -3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & -4 & 2 & 3 \\ 0 & 3 & 5 & -7 \\ -2 & 8 & -4 & -3 \end{array}\right] \overset{R3 \mapsto R3+2R1}{\sim} \left[\begin{array}{ccc|c} 1 & -4 & 2 & 3 \\ 0 & 3 & 5 & -7 \\ 0 & 0 & \boxed{0} & \boxed{3} \end{array}\right] \overset{R2 \mapsto \frac{1}{3}R2}{\sim} \left[\begin{array}{ccc|c} 1 & -4 & 2 & 3 \\ 0 & 1 & 5/3 & -7/3 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 2+20/3 & 3-28/3 \\ 0 & 1 & 5/3 & -7/3 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Inconsistent (Theorem 2)

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2) [1.5 Q21] Describe and compare the solution sets of $x_1 + 9x_2 - 4x_3 = 0$ and $x_1 + 9x_2 - 4x_3 = -2$.

$x_1 + 9x_2 - 4x_3 = 0$

$x_1 = -9x_2 + 4x_3$

free variables

$x_1 + 9x_2 - 4x_3 = -2$

$x_1 = -2 - 9x_2 + 4x_3$

free variables

Vector form

$$\underline{x} = \cancel{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}} + t\begin{bmatrix} -9 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$$

$$\underline{x} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -9 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$$

Same plane of solutions – just shifted

3) [1.4 Q44] Suppose A is a 3×3 matrix and b is a vector in $\mathbb{R}^3$ with the property that $Ax = b$ has a unique solution. Explain why the columns of A must span $\mathbb{R}^3$?

$Ax=b$ has unique sol.

theorem 2

$$\iff \left[A \mid b \right] \sim \left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & \cdots \\ 0 & \boxed{1} & 0 & \\ 0 & 0 & \boxed{1} & \end{array} \right]$$

Theorem 4 A has pivot in every row $\Rightarrow$ cols of A span $\mathbb{R}^3$

$\square$

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4) [1.5 Q38] Suppose $Ax = b$ has a solution. Explain why the solution is unique precisely when $Ax = 0$ has only the trivial solution. .

Let x_0 be a solution to $Ax=b$, i.e. $Ax_0 = b$

if $\exists\, x_1 \neq x_0$ s.t. $Ax_1 = b$, then $Ax_1 - Ax_0 = b - b = 0$

Theorem 5 $\Rightarrow A\underbrace{(x_1 - x_0)}_{\text{non-zero as } x_1 \neq x_0} = 0$

$\Rightarrow Ax=0$ has non-triv. sol.

if $Ax=0$ has non-trivial sol. $y \neq 0$

then $Ay=0$, so $Ax_0 + Ay = b + \underline{0} = b$

$A\underbrace{(x_0+y)}_{\neq x_0 \text{, as } y \neq 0}$ $\Rightarrow A$ does not have unique sol.