1) [Quiz 4 Q2] Let $\{u_1, \ldots, u_p\}$ be a subset of a vector space V, and let $\mathcal{B}$ be a basis for V of size n. Prove that the vectors $u_1, \ldots u_p$ are linearly independent if and only if the coordinate vectors $[u_1]_\mathcal{B}, \ldots [u_p]_\mathcal{B}$ are linearly independent in $\mathbb{R}^n$.

$T(\underline{x}) = [\underline{x}]_\beta$

Contrapositive

If $u_1, \ldots, u_p$ are dependent, then $\exists \overbrace{c_1, \ldots, c_p}^{\text{Not all } 0} \in \mathbb{R}$ s.t.

$$c_1 u_1 + \cdots + c_p u_p = 0$$

but then $T(c_1 u_1 + \cdots + c_p u_p) = T(0) = 0$

$$= c_1 T(u_1) + \cdots + c_p T(u_p)$$

$$= c_1 [u_1]_\mathcal{B} + \cdots + c_p [u_p]_\mathcal{B} = 0$$

$\Rightarrow [u_1]_\mathcal{B}, \ldots, [u_p]_\mathcal{B}$ dependent

Reverse direction works because coordinate map T is an isomorphism.

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2) [3.1] Compute the determinant of the matrix A via cofactor expansion.

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\det(A) = 1 \cdot \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= 1(5 \cdot 9 - 8 \cdot 6) - 2(4 \cdot 9 - 6 \cdot 7) + 3(4 \cdot 8 - 5 \cdot 7)$$

$$= (45 - 48) - 2(36 - 42) + 3(32 - 35)$$

$$= -3 - 2(-6) + 3(-3)$$

$$= -3 + 12 - 9$$

$$= 0$$

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 4 & 5 & 6 \\ 1 & 7 & 8 & 9 \end{bmatrix}$$

$$\det(B) = 1 \cdot \det(A) - 1 \begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}$$

$$= 0 - \begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}$$

repeated row $\Rightarrow$ not invertible $\Rightarrow \det = 0$

$\det(B) = 0$

3) [3.1] For $n \geq 2$, show that an $n \times n$ matrix with two identical rows has determinant 0.

Several ways to see this:

1) As above, repeated rows $\Rightarrow$ non-invertible $\Leftrightarrow$ det 0

2) By induction $\quad \underline{n=2}$ $A = \begin{bmatrix} a & b \\ a & b \end{bmatrix} \to \det(A) = ab - ab = 0$

$\underline{n=3}$ $A = \begin{bmatrix} a & b & c \\ a & b & c \\ d & e & f \end{bmatrix} \to \det(A) = d \begin{vmatrix} b & c \\ b & c \end{vmatrix} - e \begin{vmatrix} a & c \\ a & c \end{vmatrix} + f \begin{vmatrix} a & b \\ a & b \end{vmatrix}$

$= 0$

Assume true for $(n-1)\times(n-1)$ matrix

then, for the n case. If we choose a row which is not repeated then, the $(n-1)\times(n-1)$ minor still has repeated rows

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \\ \vdots & & & \\ c_1 & c_2 & \cdots & c_n \end{bmatrix}$$

[Midterm Questions?]

$c_1 (\square) + \cdots (0)$

(det 0)

3) We'll see that swapping rows of matrix multiplies det by -1

$\det\left(\begin{bmatrix} 1 & 2 \\ 0 & 7 \end{bmatrix}\right) = 7 \qquad \det\left(\begin{bmatrix} 0 & 7 \\ 1 & 2 \end{bmatrix}\right) = -7$

if $A = \begin{bmatrix} — a — \\ — a — \\ — b — \\ — c — \\ \vdots \end{bmatrix}$ then $\det(A) = -\det\underbrace{\begin{pmatrix} — a — \\ — a — \\ — b — \\ — c — \end{pmatrix}}_{= A}$

$\det(A) = -\det(A)$

$\Rightarrow \det(A) = 0$