1) [Quiz 6 Q2] Suppose that A is an $n \times n$ matrix having n linearly independent eigenvectors. Show that A^T also has n linearly independent eigenvectors.

Diagonalization Thm: A diagonalizable $\Longleftrightarrow$ A has n L.I eigenvectors

If A has n L.I. evects, then A is diagonalizable i.e. $\exists$ D diagonal, & P inv. s.t. $A = PDP^{-1}$

then $A^T = (PDP^{-1})^T = (P^{-1})^T D^T P^T$

D diagonal, so $D^T = D$, & $(P^{-1})^T = (P^T)^{-1}$

so $A^T = (P^T)^{-1} D P^T \Rightarrow A^T$ diagonalizable

$\Rightarrow A^T$ has n L.I evecs $\square$

2) [Hand-in Proof?]

1)

$$A = \begin{array}{cc} & 9 \\ 14 & \begin{bmatrix} & & \end{bmatrix} \end{array}$$

Nullity(A) = 2

$\Rightarrow$ rank(A) = 7

if $B \sim A$, then rank(B) = 7

$\Rightarrow \#\text{rows}_B \geq 7$

Take $$U = \begin{array}{cc} & 9 \\ 7 & \begin{bmatrix} u_1 \\ \vdots \\ u_7 \end{bmatrix} \end{array}$$

where $u_1, \dots, u_7$ are L.I. rows of A

2) Use coordinate map

$T: \mathbb{P}_2 \to \mathbb{R}^3$

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ are L.I.

$v_1 \quad v_2 \quad v_3$

Show this

and $\{v_1, v_2, v_3\} \subseteq \text{span}\{p, q\}$

by Thm 4.10, $\{v_1, v_2, v_3\}$ are L.D.

$\Rightarrow\Leftarrow$

3) [5.5] Find the eigenvalues and eigenvectors of the following matrix:

$$\begin{bmatrix} 4 & -3 & 0 \\ 3 & 4 & 0 \\ 1 & 2 & 2 \end{bmatrix}$$

$$\begin{vmatrix} 4-\lambda & -3 & 0 \\ 3 & 4-\lambda & 0 \\ 1 & 2 & 2-\lambda \end{vmatrix} = (2-\lambda)\left[(4-\lambda)^2+9\right]$$

$$=(2-\lambda)\left[\lambda^2-8\lambda+25\right]$$

$$=(2-\lambda)(\lambda-(4-3i))(\lambda-(4+3i))$$

$$\Rightarrow \lambda = 2,\ 4-3i,\ 4+3i$$

$\lambda=2$

$$\begin{bmatrix} 2 & -3 & 0 \\ 3 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$$

$\lambda=4-3i$

$$\begin{bmatrix} 3i & -3 & 0 \\ 3 & 3i & 0 \\ 1 & 2 & -2+3i \end{bmatrix} \sim \begin{bmatrix} 3i & -3 & 0 \\ 3i & -3 & 0 \\ 1 & 2 & -2+3i \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -2+3i \\ 0 & -3-6i & -4-6i \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{-4-6i}{-3-6i}\cdot\frac{-3+6i}{-3+6i} = \frac{(-4-6i)(-3+6i)}{45}$$

$$\begin{bmatrix} 1 & 2 & -2+3i \\ 0 & 1 & \frac{-4-6i}{-3-6i} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{1}{45}(48+6i)$$

$$\frac{1}{15}(16+2i)$$

4) [6.1 Q14] Find the distance between $u = (0, -5, 2)^T$ and $v = (-4, -1, 4)^T$.

$$\begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix} - \begin{bmatrix} -4 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ -2 \end{bmatrix}$$

$$\left\| \begin{bmatrix} 4 \\ -4 \\ -2 \end{bmatrix} \right\| = \sqrt{16+16+4} = \sqrt{36} = 6$$