1) [3.1/3.2] Compute the determinant of the matrix A via cofactor expansion, then using properties of the determinant.

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Cofactor exp.

$$\det(A) = 1 \cdot \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= 1\cdot(5\cdot 9 - 8\cdot 6) - 2(4\cdot 9 - 6\cdot 7) + 3(8\cdot 4 - 5\cdot 7)$$

$$= (45-48) - 2(36-42) + 3(32-35)$$

$$= -3 - 2(-6) + 3(-3)$$

$$= -3 + 12 - 9$$

$$= 0$$

Theorem 3.3

$$\det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 7 & 8 & 9 \end{vmatrix} \quad R2 \mapsto R2 - R1$$

$$= \begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{vmatrix} \quad R3 \mapsto R3 - 2R1$$

$$= \begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 0 & 0 & 0 \end{vmatrix} \quad R3 \mapsto R3 - R2$$

not invertible

Theorem 3.4

$\Rightarrow \det(A) = 0$

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2) [3.1] For $n \geq 2$, show that an $n \times n$ matrix with two identical rows has determinant 0.

A few ways to do this. 1) Induction

but 2) Repeated Rows $\Rightarrow$ not-invertible $\Rightarrow \det = 0$

3) $A = \begin{bmatrix} - & r_1 & - \\ - & r_2 & - \\ & \vdots & \\ - & r_n & - \end{bmatrix}$

Say $r_1 = r_2$ (wlog)

then

$$\underbrace{\begin{bmatrix} - & r_1 & - \\ - & r_2 & - \\ & \vdots & \\ - & r_n & - \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} - & r_2 & - \\ - & r_1 & - \\ & \vdots & \\ - & r_n & - \end{bmatrix}}_{B}$$

but by Theorem 3.2.

$\det(A) = -1 \cdot \det(B)$

but $B = A$

So, $\det(A) = -\det(A)$

$\Rightarrow \det(A) = 0$

3) [5.1 Q33]Let λ be an eigenvalue of an invertible matrix A. Show that λ^{-1} is an eigenvalue of A^{-1}.

Let $\underline{x}$ be eigen vector such that

$$A\underline{x} = \lambda \underline{x}$$

then, $A^{-1}A\underline{x} = A^{-1}\lambda \underline{x}$

$$\underline{x} = \lambda A^{-1}\underline{x}$$

$$\lambda^{-1}\underline{x} = A^{-1}\underline{x}$$

A is invertible so eigenvalues are all non-zero (IMT)

$\Rightarrow \lambda^{-1}$ is an eigenvalue of A^{-1}

$\square$

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4) [5.2 Q11] Find the eigenvalues/vectors of the following matrix:

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 5 & 3 & 2 \\ -2 & 0 & 2 \end{bmatrix}$$

$\det(A - \lambda I) = 0$

$$= \begin{vmatrix} 4-\lambda & 0 & 0 \\ 5 & 3-\lambda & 2 \\ -2 & 0 & 2-\lambda \end{vmatrix}$$

$$= (4-\lambda)\begin{vmatrix} 3-\lambda & 2 \\ 0 & 2-\lambda \end{vmatrix} + 0 + 0$$

$$= (4-\lambda)[(3-\lambda)(2-\lambda) - 0]$$

$$= (4-\lambda)(3-\lambda)(2-\lambda)$$

$\Rightarrow \lambda = 2, 3, 4$ eigenvalues

all algebraic mult. 1

eigenvectors

$\mathrm{Null}(A - \lambda I)$ for each λ

$\lambda = 2$

$$\begin{bmatrix} 2 & 0 & 0 \\ 5 & 1 & 2 \\ -2 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \mathrm{Null}(A - 2I) = \mathrm{span}\left\{ \begin{bmatrix} 0 \\ -2c \\ c \end{bmatrix} \middle| c \in \mathbb{R} \right\}$

$v = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$ g.m. = 1

$\lambda = 3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 5 & 0 & 2 \\ -2 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\boxed{v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}$

$\lambda = 4$

$$\begin{bmatrix} 0 & 0 & 0 \\ 5 & -1 & 2 \\ -2 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$\boxed{v = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}}$