1) [4.3 Q11] Find a basis for the set of vectors in $\mathbb{R}^3$ in the plane $x + 2y + z = 0$

$x + 2y + z = 0$

$\Rightarrow x = -2y - z$

two free variables

$y = s \in \mathbb{R}$

$z = t \in \mathbb{R}$

$x = -2s - t$
$y = s$
$z = t$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Spanning set $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

are they linearly ind.? yes! $\longrightarrow$ $\boxed{\text{Basis: } \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}}$

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2) [4.3 Q22,33] (i) (T/F) A linearly independent set in a subspace H is a basis for H. (ii) Suppose $\mathbb{R}^4 = \text{Span}\{v_1, \ldots, v_4\}$. Explain why $\{v_1, \ldots, v_4\}$ is a basis for $\mathbb{R}^4$.

(i) FALSE

Counterexample: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ is linearly independent $\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \neq k \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right)$

but is not a basis for $H = \text{Span}\{e_1, e_2, e_3\}$

, as $e_3 \notin \text{span}\{e_1, e_2\}$

(ii) $\mathbb{R}^4$ is 4-dimensional

if $\mathbb{R}^4 = \text{Span}\{v_1, \ldots, v_4\}$, so $\dim(\text{Span}\{v_1, \ldots, v_4\}) = 4$

$\Rightarrow v_1, v_2, v_3, v_4$ lin ind.

$\text{Span}\{v_1, \ldots, v_4\} = \mathbb{R}^4$ by def, so $\{v_1, v_2, v_3, v_4\}$ is a basis for $\mathbb{R}^4$

$\{v_1, \ldots, v_n\}$ lin ind. iff $\dim(\text{Span}\{v_1, \ldots, v_n\}) = n$

3) [4.4 Q3] Find the vector x determined by the given coordinate vector $[x]_{\mathcal{B}}$ and the given basis $\mathbb{B}$.

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ -7 \\ 0 \end{bmatrix} \right\}, [x]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$$

$$[X]_B = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \Rightarrow X = 3\begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix} + 0\begin{bmatrix} 5 \\ 2 \\ -2 \end{bmatrix} + (-1)\begin{bmatrix} 4 \\ -7 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ -12 \\ 9 \end{bmatrix} + \begin{bmatrix} -4 \\ 7 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -5 \\ 9 \end{bmatrix}$$

(alt.) $$X = \begin{bmatrix} 1 & 5 & 4 \\ -4 & 2 & -7 \\ 3 & -2 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \\ 9 \end{bmatrix}$$

↑ change of basis matrix

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4) [4.4 Q5] Find the coordinate vector $[x]_{\mathcal{B}}$ of x relative to the given basis $\mathcal{B} = \{b_1, \ldots, b_n\}$.

$$b_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, b_2 = \begin{bmatrix} 2 \\ -5 \end{bmatrix}, x = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$x = \begin{bmatrix} -2 \\ 1 \end{bmatrix} = a\begin{bmatrix} 1 \\ -3 \end{bmatrix} + b\begin{bmatrix} 2 \\ -5 \end{bmatrix} \qquad [X]_B = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\left.\begin{aligned} -2 &= a + 2b \\ 1 &= -3a - 5b \end{aligned}\right\} \quad \left[\begin{array}{cc|c} 1 & 2 & -2 \\ -3 & -5 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & -2 \\ 0 & 1 & -5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 8 \\ 0 & 1 & -5 \end{array}\right]$$

$$\Rightarrow [X]_B = \begin{bmatrix} 8 \\ -5 \end{bmatrix}$$

(alt.) $$[X]_B = \begin{bmatrix} 1 & 2 \\ -3 & -5 \end{bmatrix}^{-1}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 & -2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ -5 \end{bmatrix}$$