**1)** [Quiz Review] Let $T\colon V \to W$ be a linear transformation, let $Z$ be a subspace of $W$, and let

$$U = \{x \in V\colon T(x) \in Z\}.$$

Show that $U$ is a subspace of $V$.

Need to show: 1) $\mathcal{U}$ non-empty subset of $V$
2) Closed under $+$
3) Closed under scalar mult.

1) $\mathcal{U}$ is defined as subset of $V$, and $\underline{0} \in V$ and $T(\underline{0}) = \underline{0} \in Z$ as, $V$ and $Z$ are subspaces

$\therefore\ \underline{0} \in \mathcal{U} \subseteq V$ ✓

2) Let $\underline{x}, \underline{y} \in \mathcal{U}$

$\underline{x}+\underline{y} \in \mathcal{U}$, because $T(\underline{x}+\underline{y}) = \underbrace{T(\underline{x})}_{\in Z} + \underbrace{T(\underline{y})}_{\in Z} \in Z$ ✓

3) Let $\underline{x} \in \mathcal{U}$, $t \in \mathbb{R}$

then, $t\underline{x} \in \mathcal{U}$, because $T(t\underline{x}) = t\underbrace{T(\underline{x})}_{\in Z} \in Z$ ✓

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**2)** [4.6 Q16] In $\mathbb{P}_2$, find the change-of-coordinates matrix from the basis $\mathcal{B} = \{1-3t^2, 2+t-5t^2, l+2t\}$ to the standard basis. Then write $t^2$ as a linear combination of the polynomials in $\mathcal{B}$.

$$\mathcal{B} = \{1-3t^2, 2+t-5t^2, 1+2t\} = \left\{ \begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}2\\1\\-5\end{bmatrix}, \begin{bmatrix}1\\2\\0\end{bmatrix} \right\}$$

$$P_{\mathcal{E}\leftarrow\mathcal{B}} = \begin{bmatrix}1 & 2 & 1\\0 & 1 & 2\\-3 & -5 & 0\end{bmatrix}$$

So,

$$\left[\begin{array}{ccc|c}1 & 2 & 1 & 0\\0 & 1 & 2 & 0\\-3 & -5 & 0 & 1\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 2 & 1 & 0\\0 & 1 & 2 & 0\\0 & 1 & 3 & 1\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 0 & -3 & 0\\0 & 1 & 2 & 0\\0 & 0 & 1 & 1\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 0 & 0 & 3\\0 & 1 & 0 & -2\\0 & 0 & 1 & 1\end{array}\right] \Rightarrow \begin{bmatrix}3\\-2\\1\end{bmatrix}_{\mathcal{B}}$$

$3(1-3t^2) - 2(2+t-5t^2) + 1+2t$

$3 - 9t^2 - 4 - 2t + 10t^2 + 1 + 2t$

$(3-4+1) + (-2t+2t) + (10t^2-9t^2)$

$= t^2$

**3)** [4.5 Q13] Determine the dimensions of Null($A$), Col($A$), and Row($A$) of

$$A = \begin{bmatrix} 1 & 0 & 9 & 5 \\ 0 & 0 & 1 & -4 \end{bmatrix}$$

"rank"

$\shortparallel$

$\dim(\text{Col}(A)) = \#$ pivots of $A$ : 2

$\shortparallel$

$\dim(\text{Row}(A)) = 2$ $\begin{cases} \text{Col}(A) \subseteq \mathbb{R}^2 \\ \text{Row}(A) \subseteq \mathbb{R}^4 \end{cases}$ Both 2-dim though

$\dim(\text{Null}(A)) = \#\text{cols} - \dim(\text{Col}(A)) = 4-2 = \underline{2}$

$rk(A) + \text{Null}(A) = \#\text{cols}$

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**4)** [4.5 Q42] Show that the space $\mathbb{P}$ of all polynomial functions defined on the real line is an infinite-dimensional vector space.

$\mathbb{P} = \{$ all polynomial functions $\}$

<u>Before</u> $\mathbb{P}_2 = \{a + bx + cx^2 : a, b, c \in \mathbb{R}\}$

now no max exp.

<u>Is it a vector space?</u>

1) closed under addition

$(1+x) + (3x^2 + \sqrt{2}x + x^{20})$
$= 1 + (1+\sqrt{2})x + 3x^2 + x^{20} \in \mathbb{P}$ ✓

2) closed under scalar mult.

yeah ✓

<u>Infinite dimensional?</u>

Well, suppose $\mathbb{P}$ is dimension $N$ for some $N \in \mathbb{N}$

take, $1, x, x^2, \ldots, x^N, x^{N+1} \in \mathbb{P}$

These are lin. ind.

as. $\nexists$ non-zero

$c_0 + c_1x + c_2x^2 + \cdots + c_{N+1}x^{N+1} = \underline{0}$

$\Rightarrow \exists$ $N+1$ L.I vectors

$\Rightarrow$ $\mathbb{P}$ is at least $N+1$ dim

$\therefore$ $\mathbb{P}$ infinite-dim