1) [5.2 Q18] Find h for the matrix A below such that the eigenspace corresponding to $\lambda = 5$ is 2-dimensional.

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & h & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$(A - 5I)$

$$= \begin{bmatrix} 0 & -2 & 6 & -1 \\ 0 & -2 & h & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

h where $\dim(\text{Null}(A-5I)) = 2$

$$\sim \begin{bmatrix} 0 & -2 & 6 & -1 \\ 0 & 0 & h-6 & 1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

h as 2 pivots exactly when $h - 6 = 0$

$\Rightarrow \underline{h = 6}$

$$\sim \begin{bmatrix} 0 & 1 & -3 & 1/2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & h-6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

---

2) [5.3 Q37] Construct a 3×3 matrix that is invertible but not diagonalizable.

$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ $\quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ $\quad (1-\lambda)(1-\lambda)$ $\quad \lambda = 1$

- Invert. iff $\det \neq 0$
- A diagonalizable $\Leftrightarrow$ $A = PDP^{-1}$, where $D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

take $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$ $\quad \det(A - \lambda I) = (1-\lambda)(\lambda^2 + 1)$

$\lambda = 1, \underline{\pm i}$

$\left(\text{or } \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \right)$

not diagonalizable

, but, $\underline{\det(A) = 1}$ $\Rightarrow$ $\underline{\text{invert.}}$

Q16 $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \lambda = 1, 2$

3) [5.3 ~~Q14~~] Diagonalize the following matrix B, where the eigenvalues of B are ~~$\lambda = 3, 4$~~.

$$B = \begin{bmatrix} 0 & -4 & -6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{bmatrix} \qquad \cancel{B = \begin{bmatrix} 4 & 0 & 2 \\ 2 & 3 & 4 \\ 0 & 0 & 3 \end{bmatrix}} \qquad \begin{bmatrix} 0 & -4 & -6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{bmatrix}$$

$\lambda = 2$

$$\begin{bmatrix} -2 & -4 & -6 \\ -1 & -2 & -3 \\ 1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$B = \begin{bmatrix} -2 & -3 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -2 & -3 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}^{-1}$$

$\lambda = 2, 1$

$\lambda = 1$

$$\begin{bmatrix} -1 & -4 & -6 \\ -1 & -1 & -3 \\ 1 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 0 & -3 & -3 \\ 1 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \right\}$$

---

4) [5.4 Q11] Define $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x) = Cx$. Find a basis $\mathcal{B}$ for $\mathbb{R}^2$ with the property that $[T]_{\mathcal{B}}$ is diagonal, where:

$$C = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$$

$$\det(C - \lambda I) = \begin{vmatrix} 4-\lambda & -2 \\ -1 & 3-\lambda \end{vmatrix} = (4-\lambda)(3-\lambda) - 2 = \lambda^2 - 7\lambda + 10 = (\lambda - 5)(\lambda - 2)$$

$\lambda = 2, 5$

$\lambda = 2$

$$\begin{bmatrix} 4-2 & -2 \\ -1 & 3-2 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix} \to \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

$\lambda = 5$

$$\begin{bmatrix} 4-5 & -2 \\ -1 & 3-5 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ -1 & -2 \end{bmatrix} \to \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$$

$$\therefore\ C = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}^{-1},$$

$$\therefore \text{ take } \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$$

$$\Rightarrow \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}^{-1} C \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$$