5168 Final Project

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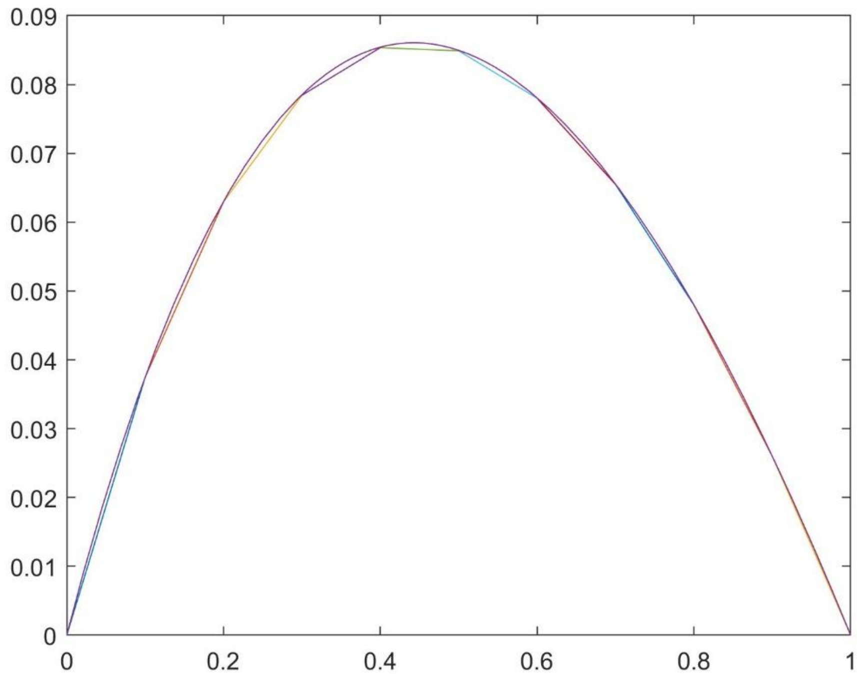
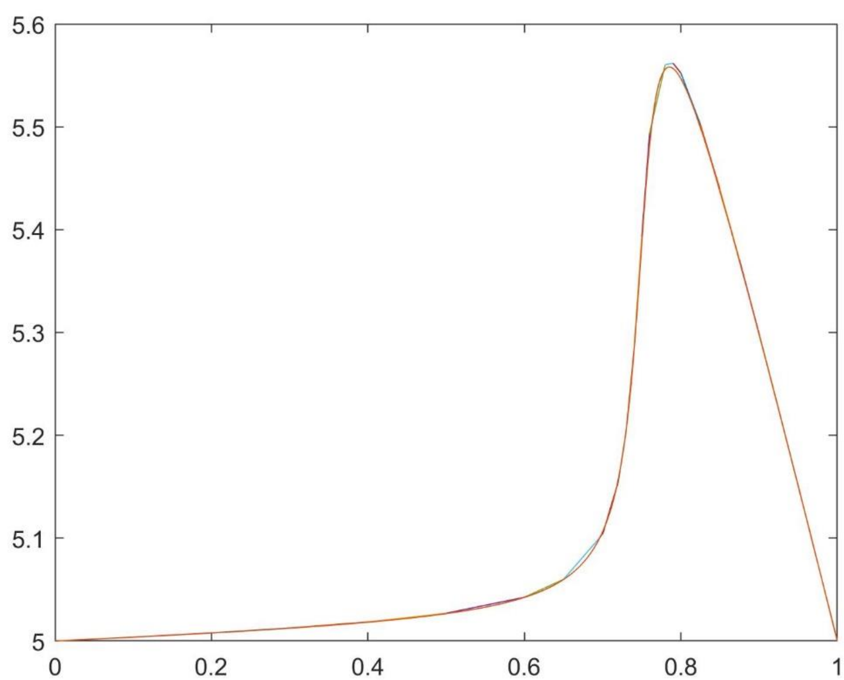
1.1

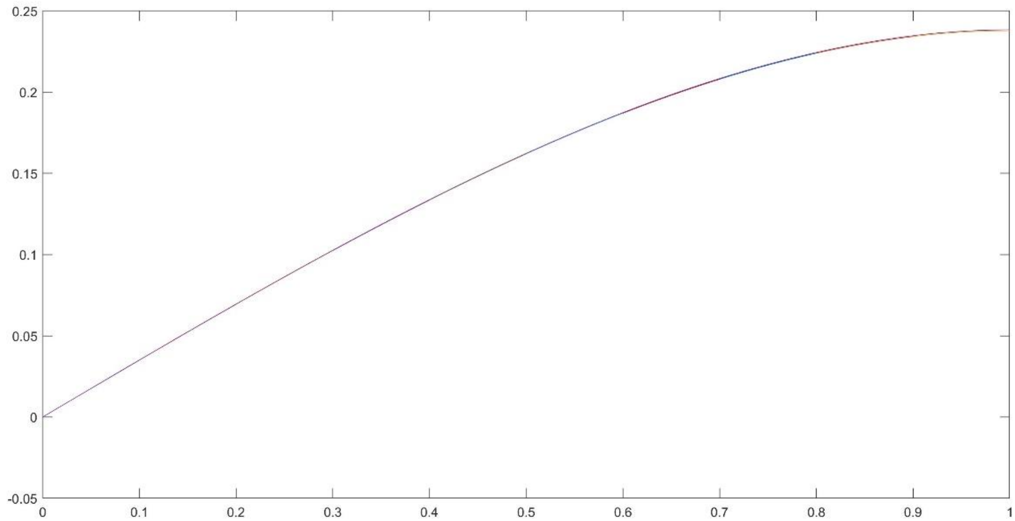
Homogenous Dirichlet:



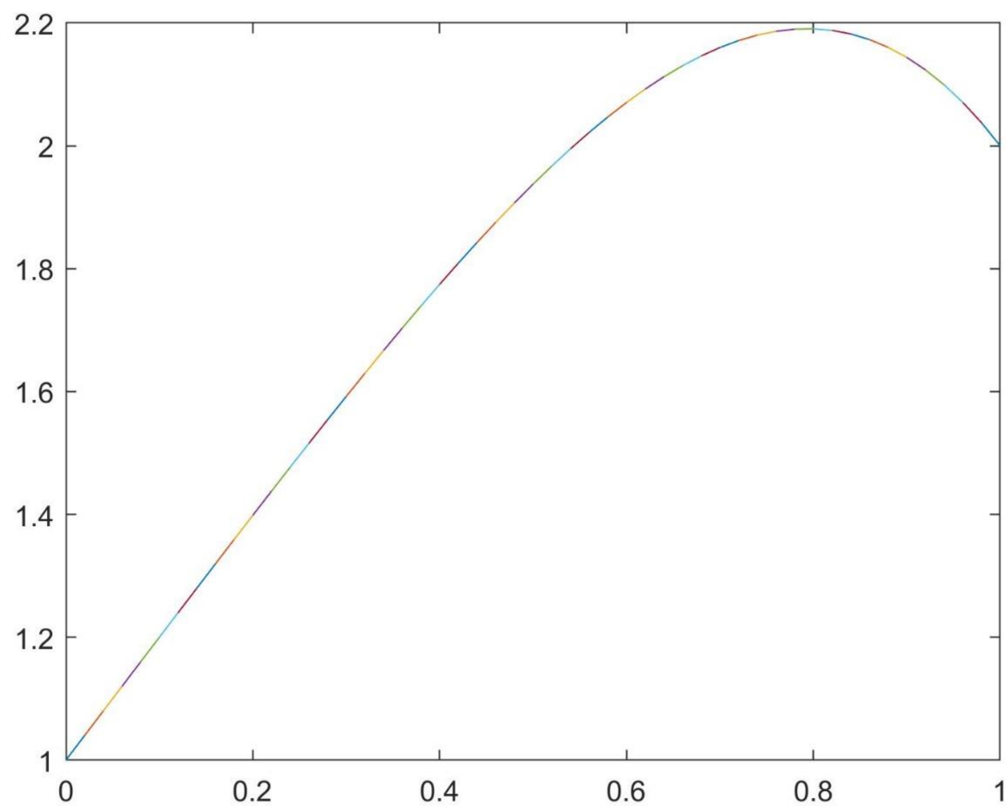
Inhomogenous Dirichlet:



Homogenous Mixed:



Inhomogenous Mixed (I plotted this one by itself because you can’t tell the difference between the FEM and the exact solution on the same plot):



1.2

a.)

1.2

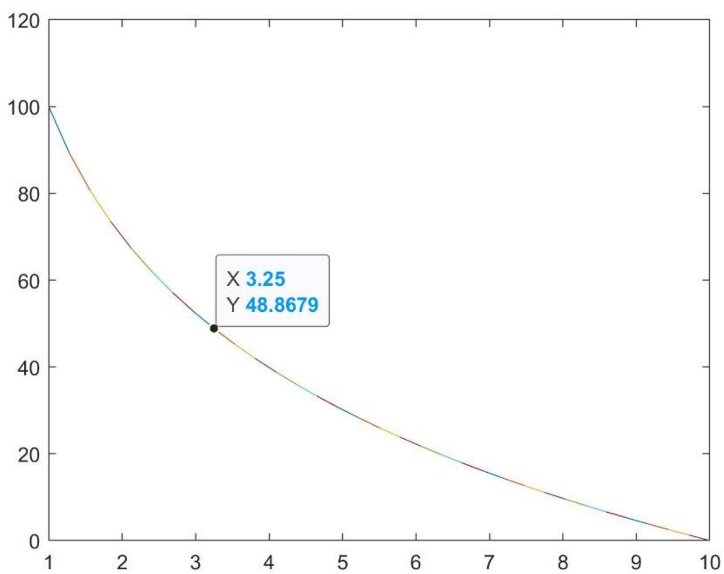
a.) strong form: $\frac{d}{dr}\left(-kr\frac{du}{dr}\right) = 0$, Ω: $1<r<10$, $u \in C^0(\Omega)$

$K = 1.7$, BC: $u(1) = 100$, $u(10) = 0$

Weak form: $\int_1^{10}\left(-kr\frac{du}{dr}\frac{dv}{dr}\right)dr = 0 \quad \forall v \in H_0^1$ where

$H_0^1 = \{g : g(1) = 100, g(10) = 0$ and $\int_1^{10} E\left(\frac{dg}{dr}\right)^2 dr < \infty\}$

b.)



Mesh 4 (left) gives an absolute error of |u(3.25)-48.8679|=0.0562 < 0.1

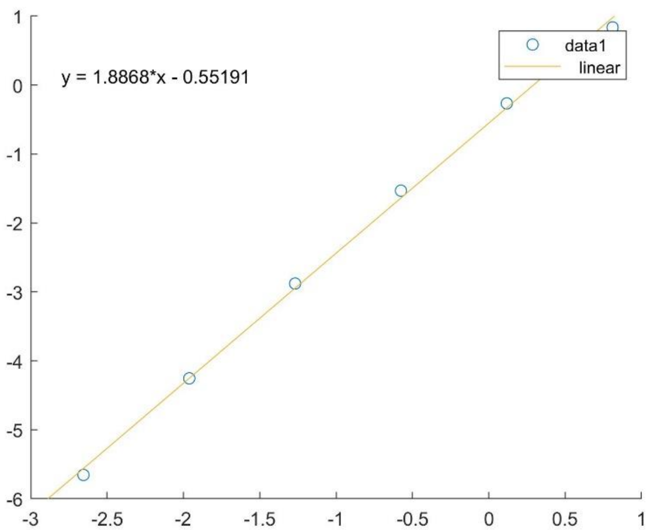
Mesh 3 gave an absolute error of |u(3.25)-49.0278|=0.216 > 0.1

So, Mesh 4 is the first mesh that gives the desired accuracy with 32 elements in the mesh.

c.)

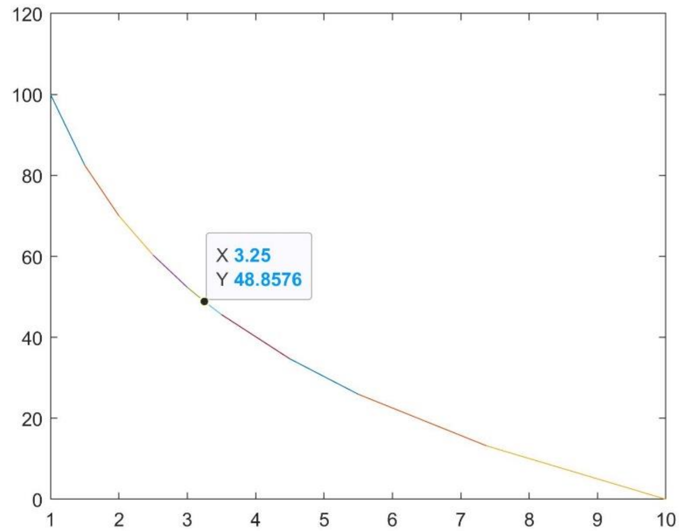
Absolute/Log error of meshes:

	Mesh 1	Mesh 2	Mesh 3	Mesh 4	Mesh 5	Mesh 6
Absolute	2.3102	0.7667	0.2161	0.0562	0.0142	0.0035
Log	0.8373	-0.2657	-1.5320	-2.8788	-4.2545	-5.6550



Average rate of convergence = 1.8868

d.)



Unstructured Mesh

The unstructured mesh gives an absolute error of |u(3.25)-48.8576|=0.0459

This is less than the absolute error given by Mesh 4 (0.0562). This is interesting, given that the unstructured mesh uses only 10 elements while Mesh 4 uses 32 elements. This is interesting, as it shows the benefits of using unstructured meshes over simply increasing the number of elements (and the finite-element method in general).

The accuracy at r=3.25 actually increases with the use of the unstructured mesh, because although there less elements overall, there is a higher "concentration" of elements near the point of interest.

e.)

Using the formulas given, we calculate the exact heat flux to be **73.8301**.

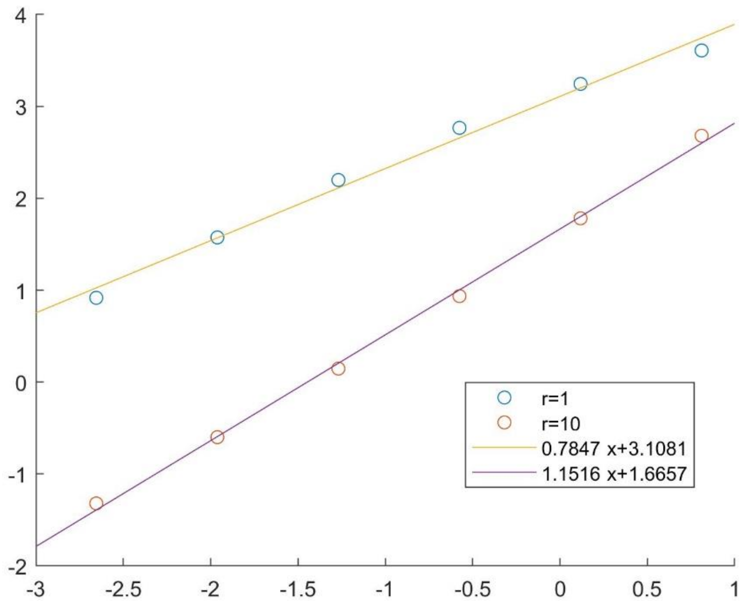
To calculate the heat flux using the FEM solution, this code was used:

```
flux_1 = -1.7*1*(100-u0(2))/(1-MESH.Points(2))
flux_10 = -1.7*10*(0-u0(nNodes-1))/(MESH.Points(nNodes)-
MESH.Points(nNodes-1))
```

u_h'(r) was calculated by determining the slope of the first/last piecewise element.

The first mesh that gave the desired accuracy (under 10% error) was Mesh 5, which had a relative error of (73.8301-69.0043)/73.8301 = **0.0654** at r = 1
and a relative error of (73.8301-74.3791)/73.8301 = **-0.0074** at r = 10.

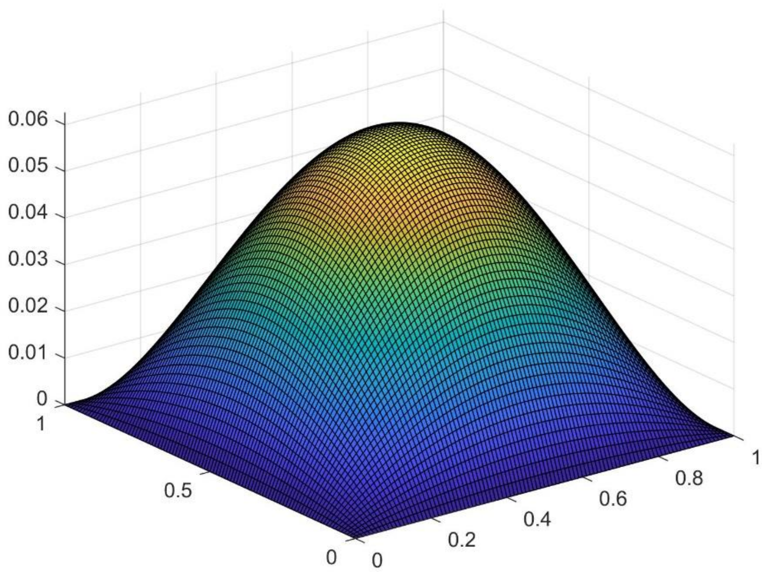
g.)



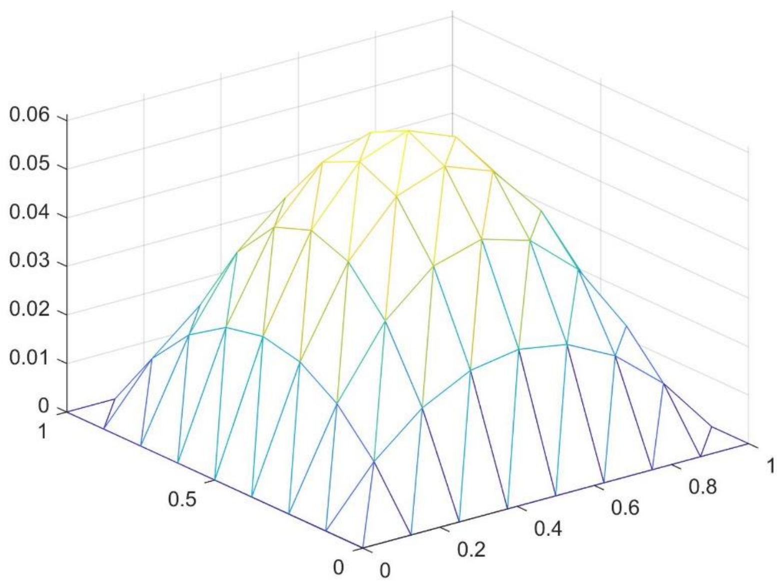
The rates of convergence are **0.7847** and **1.1516** for r = 1 and r = 10, respectively. These are both significantly smaller than the rate of convergence obtained in part c.

2.1

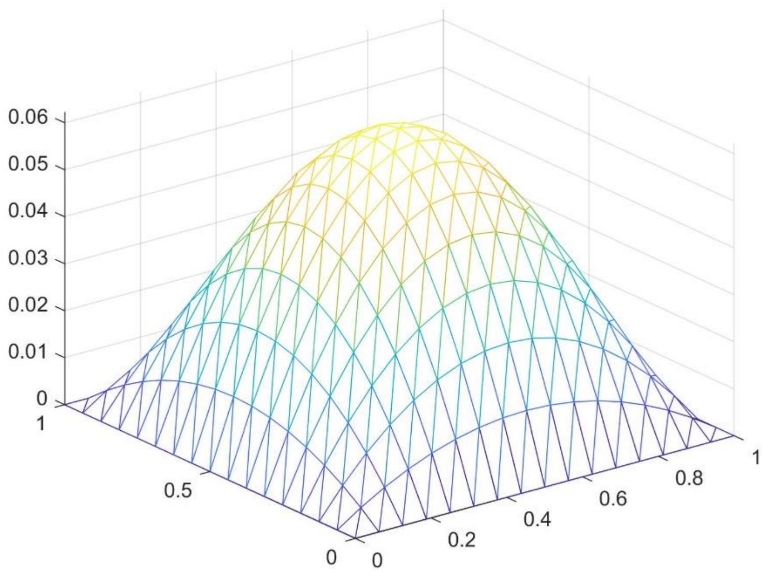
a.)



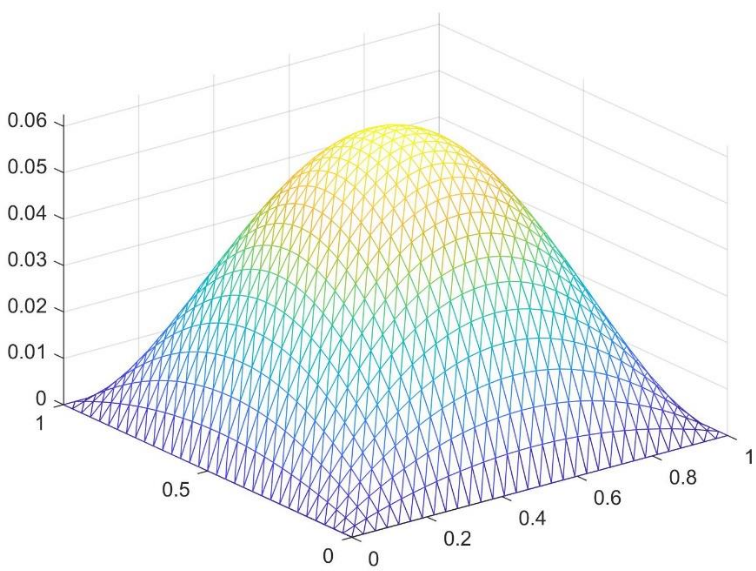
Exact solution



Mesh 1


0.06
0.05
0.04
0.03
0.02
0.01
0
1
0.5
0
0
0.2
0.4
0.6
0.8
1

Mesh 2


0.06
0.05
0.04
0.03
0.02
0.01
0
1
0.5
0
0
0.2
0.4
0.6
0.8
1

Mesh 3

b.) and c.)

L2 error norm	Mesh 1	Mesh 2	Mesh 3
Error	0.00169	0.00043	0.00011

0.00169 /0.00043 = 3.920 factor error decrease from Mesh 1 to Mesh 2

Rate of convergence = [log(0.00043)-log(0.00169)]/[log(0.002)-log(0.0078)] = 1.006

0.00043/0.00011 = 3.980 factor error decrease from Mesh 2 to Mesh 3

Rate of convergence = [log(0.00011)-log(0.00043)]/[log(4.8828e-4)-log(0.002)] = 0.967

L_inf error norm	Mesh 1	Mesh 2	Mesh 3
Error	0.00273	0.00069	0.00017

0.00273/0.00069 = 3.962 factor error decrease from Mesh 1 to Mesh 2

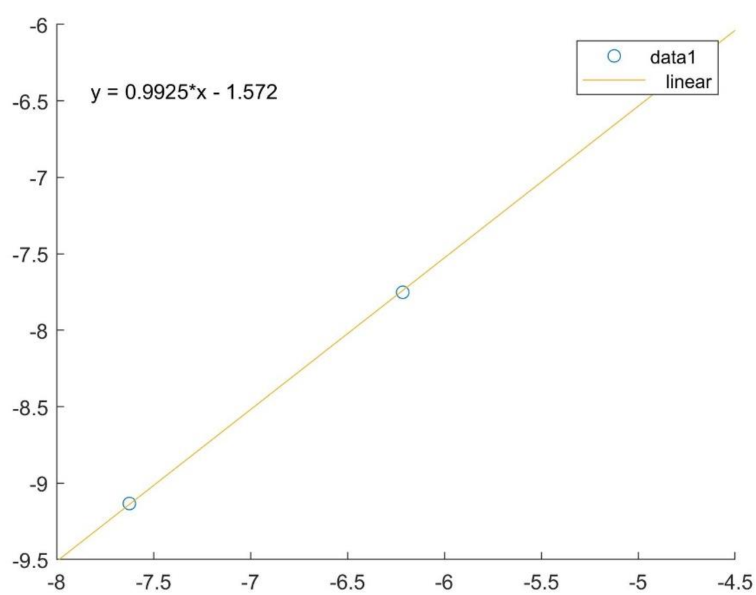
Rate of convergence = [log(0.00069)-log(0.00273)]/[log(0.002)-log(0.0078)] = 1.011

0.00069/0.00017 = 4.059 factor error decrease from Mesh 2 to Mesh 3

Rate of convergence = [log(0.00017)-log(0.00069)]/[log(4.8828e-4)-log(0.002)] = 0.994

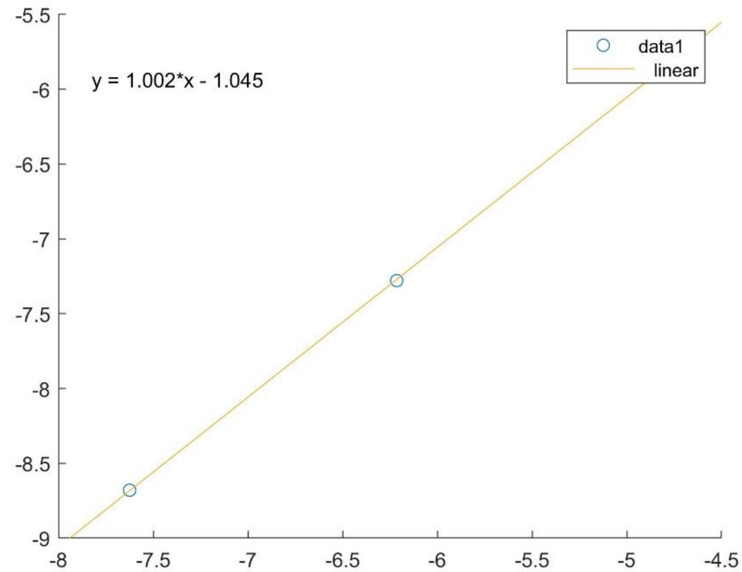
Since we are halving the element size with each refinement, we expect the error to theoretically decrease by a factor of Ch^2/(C(h/2)^2) = 4. This very closely matches with our actual error decrease.

d.)



log(L2 error) vs log(h)

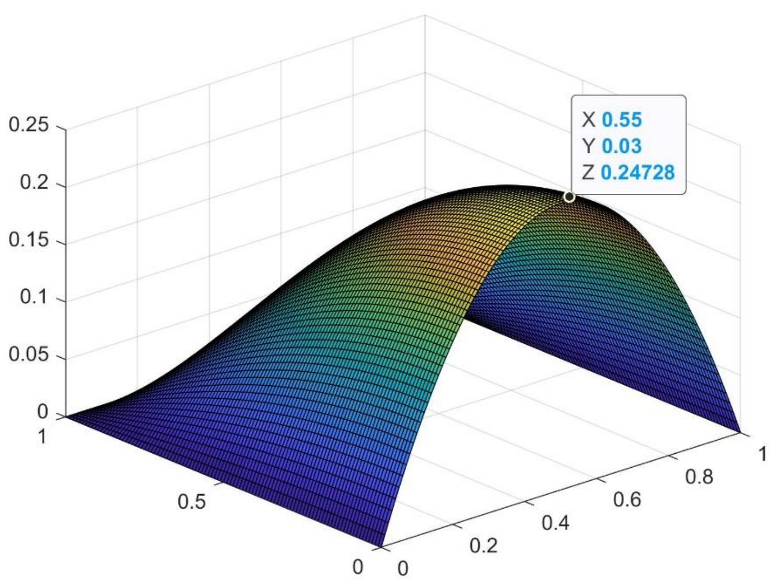
Average rate of convergence = 0.9925



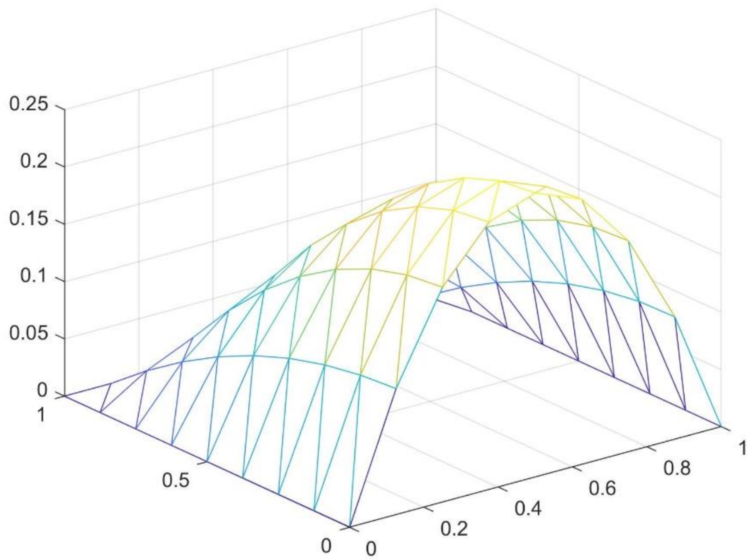
log(L_inf error) vs log(h)

Average rate of convergence = 1.002

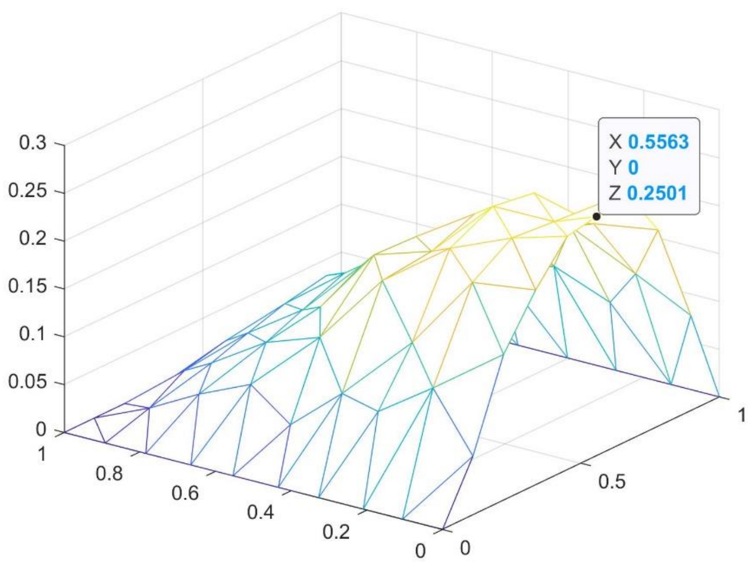
e.)



Exact solution



Structured mesh



Unstructured mesh

(see 'Appendices')

f.)

Structured Mesh:

L2 error norm = 0.00364

L_inf error norm = 0.00682

Unstructured Mesh:

L2 error norm = 0.00850

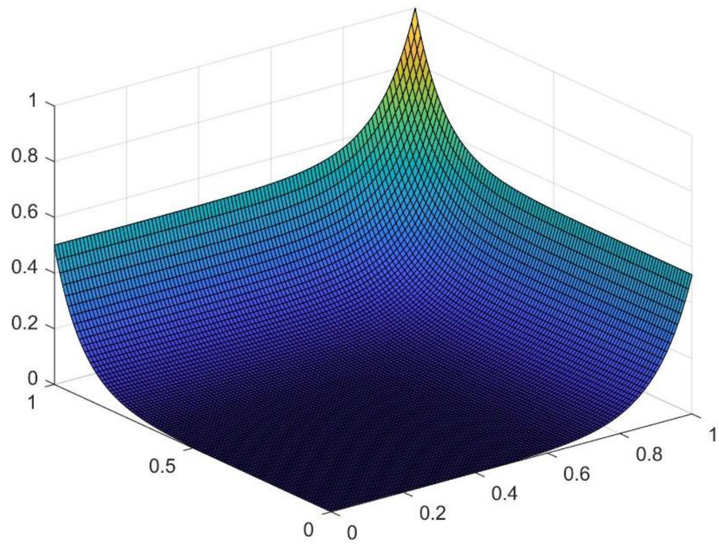
L_inf error norm = 0.02502

Although I found that the error norms for the structured mesh were lower than those of the unstructured mesh, this is most likely incorrect due to some unknown error in my code (see ‘Appendices’). The error most likely is related to my error calculations or my element matrix calculations (or both).

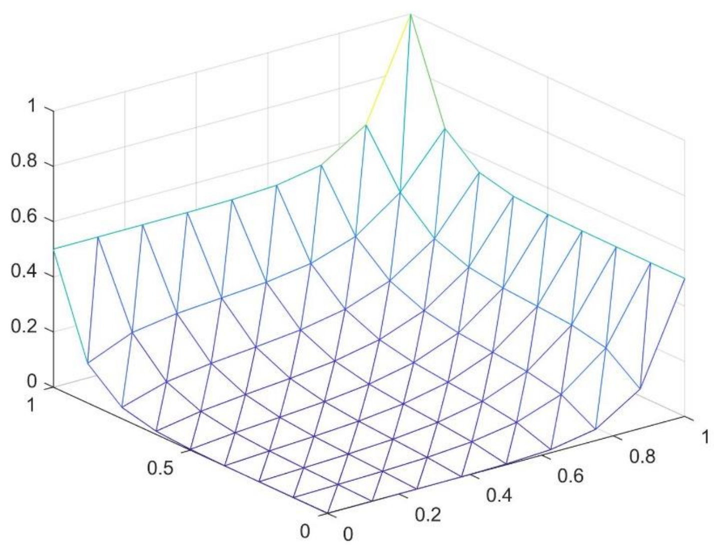
However, you would expect that the unstructured mesh would give better results. This is because for this problem, using equilateral triangle elements is symmetric with the solution and should produce more natural results compared to the structured mesh with diagonal triangle elements.

2.2

a.)

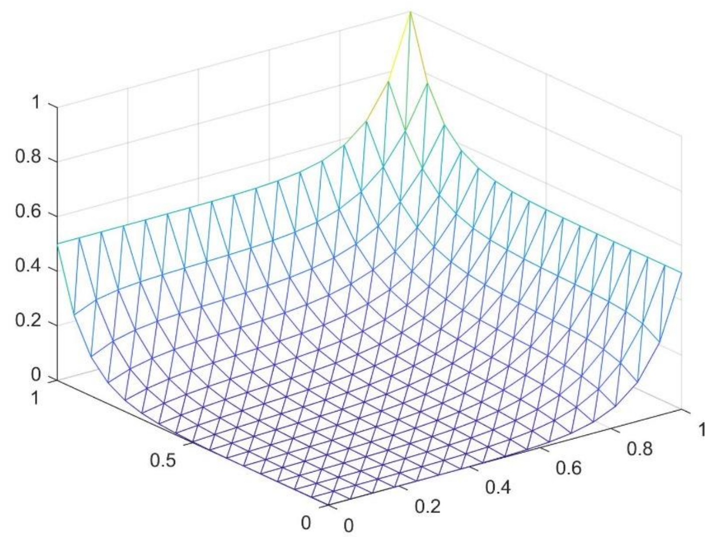


Exact solution



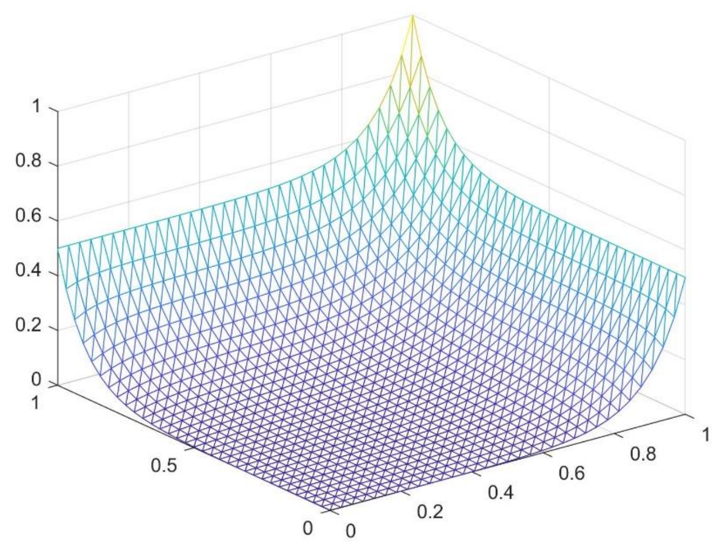
Mesh 1

Computational time: about 4 seconds



Mesh 2

Computational time: about 8 seconds



Mesh 3

Computational time: about 14 seconds

b.)

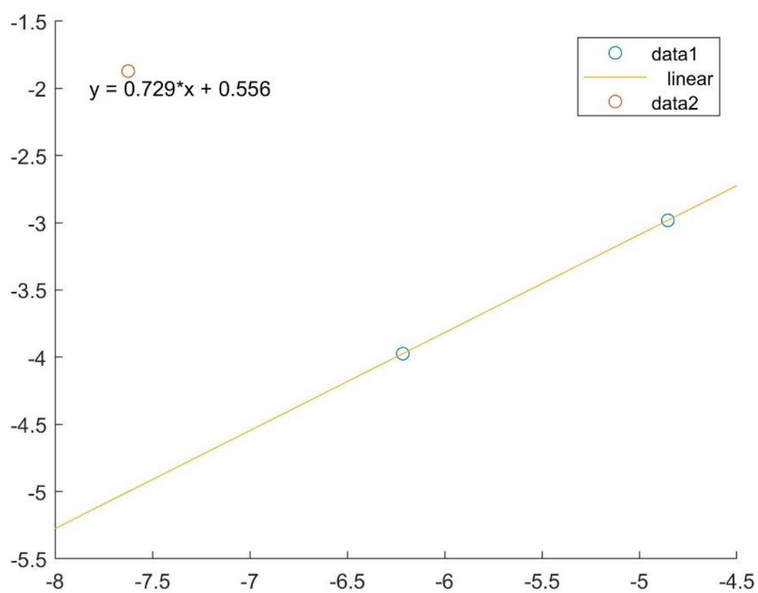
Although the problem solution does have a sharp edge, I do not observe any oscillations of any sort in any of the solutions. Since this problem has no advective term, you would not expect to see any oscillations.

c.)

	Mesh 1	Mesh 2	Mesh 3
L2 error norm	0.051	0.019	0.050
L_inf error norm	0.137	0.051	0.154

Note: These are most likely wrong due to some error in my code (See 'Appendices'). I am not sure why there is an increase in error norms from Mesh 2 to Mesh 3.

d.)

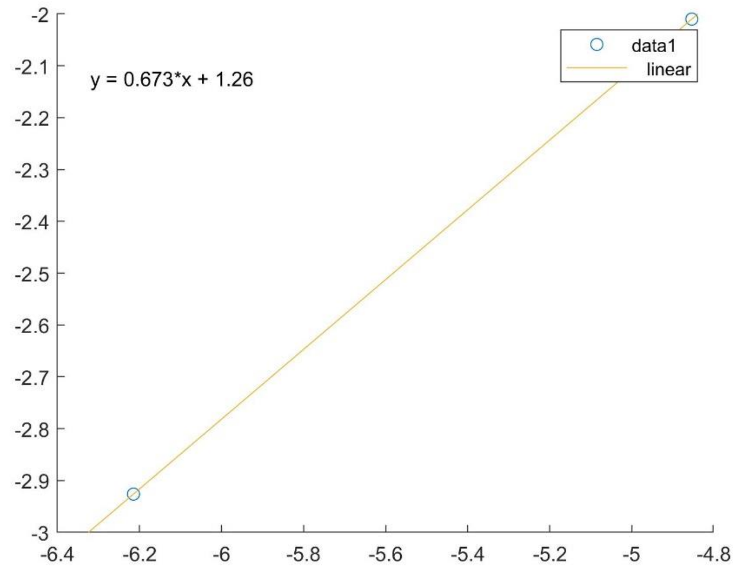


Log(L2) vs log(h)

For the reasons described in part c, I decided to omit my Mesh 3 error data from the plot.

This gives a rate of convergence of **0.729**.

Compared to the ROC of 2.1(d), we see that it is lower by about ~0.3.

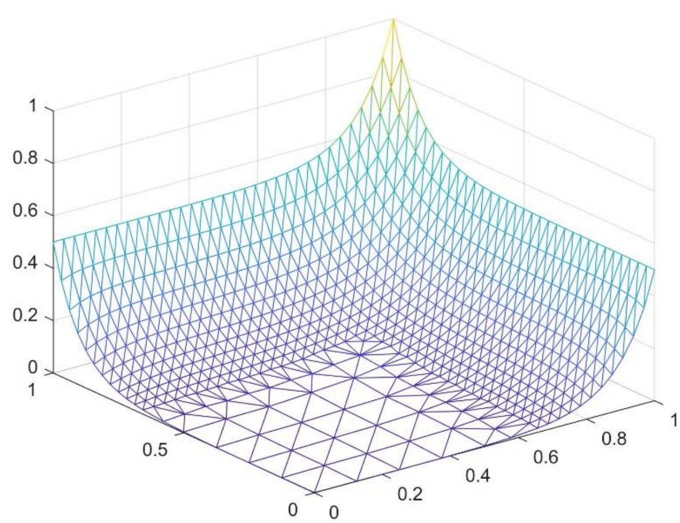


Log(L_inf) vs log(h)

Again omitting the Mesh 3 error data, we calculate an approximate rate of convergence of **0.673**.

Compared to the ROC of 2.1(d), it is lower by around ~0.33.

e.)



Unstructured mesh

L_2 error norm = 0.050

L_inf error norm = 0.154

Although the unstructured mesh took about half (~5.4 seconds) as long as Mesh 3 (~8.4 seconds), they had relatively similar L2 and L_inf errors (although the errors for Mesh 3 were very slightly lower). From my observations, this is because in the exact solution, there is a region that is nearly flat without much variation. The unstructured mesh takes advantage of this, setting fewer elements around that region because the results will differ very slightly for those elements. On the other hand, Mesh 3 simply uses a very fine uniform mesh across the entire domain, including over the flat region. Overall, the use of significantly more elements in Mesh 3 leads to a computational time that is not worth the extremely small decrease in error relative to the unstructured mesh solution. Once again, this problem emphasizes the benefits of using an unstructured mesh over a structured mesh in certain cases.

Appendices:

I am not too sure if some of my calculations are correct, so I figured I would attach the code I wrote for any possible partial credit. Sorry for the bad formatting (got messed up when I copied it over!)

```
%error
%u_function=(@(x,y) (x*y*(1-x)*(1-y))); %2.1 Dirichlet exact
u_function=(@(x,y) (x^2-x)*(y^2-1)); %2.1 Mixed exact
%u_function=(@(x,y) (cosh(10*x)+cosh(10*y))/(2*cosh(10))); %2.2 exact

L_2=0; L_inf=0;
q=quadtriangle(2,'Domain',[0 0; 1 0; 0 1],'Type','nonproduct');
polyvals = zeros(3,3);

for k=1:3
    for i=1:3

polyvals(k,i)=bipolyval(psi(i).fun,[q.Points(k,1),q.Points(k,2)]);
    end
end

for i=1:nElems
    currElement = MESH.ConnectivityList(i,:);
    one = currElement(1); two = currElement(2); three =
currElement(3); %local node numbers

    x=[MESH.Points(one,1) MESH.Points(two,1) MESH.Points(three,1)];
    y=[MESH.Points(one,2) MESH.Points(two,2) MESH.Points(three,2)];

    u_coeffs=[u(one) u(two) u(three)];

    element_area = polyarea([x(1) x(2) x(3)], [y(1) y(2) y(3)]);

    for j=1:3

        u_h=dot(u_coeffs,polyvals(j,:));
        x_j=dot(x,polyvals(j,:)); y_j=dot(y,polyvals(j,:));
        u_xy=u_function(x_j,y_j);

        L_2=L_2+q.Weights(j)*2*element_area*(u_xy-u_h)^2;

        L_inf=max(L_inf,abs(u_xy-u_h));
        %disp(L_2);
```

```
    end
end

L_2=sqrt(L_2);
```

```
function [K,F] = element2D(psi,KofXY,BofXY,FofXY,MESH)

fe = zeros(length(psi),1);
degree = length(psi)-1;

q = quadtriangle(3);

nElems = length(MESH.ConnectivityList);
nNodes = length(MESH.Points);
ke = zeros(length(psi));
K=zeros(nNodes);
F=zeros(nNodes,1);

for n = 1:nElems

    currElement = MESH.ConnectivityList(n,:);
    one = currElement(1); two = currElement(2); three =
currElement(3); %local node numbers
    one_xy = MESH.Points(one,:); two_xy = MESH.Points(two,:); three_xy
= MESH.Points(three,:); %global coords of local nodes
    element_area = polyarea([one_xy(1) two_xy(1) three_xy(1)],
[one_xy(2) two_xy(2) three_xy(2)]);
    y_31 = three_xy(2)-one_xy(2); y_21 = two_xy(2)-one_xy(2);
    x_31 = three_xy(1)-one_xy(1); x_21 = two_xy(1)-one_xy(1);
    %disp(element_area)
    for i = 1:length(psi)

        for j = 1:i


ke(i,j)=(1/(4*element_area))*KofXY(n)*((y_31*psi(i).xider(2,2)-
y_21*psi(i).etader(2,2))*(y_31*psi(j).xider(2,2)-
y_21*psi(j).etader(2,2))...
                +(x_31*psi(i).xider(2,2)-
x_21*psi(i).etader(2,2))*(x_31*psi(j).xider(2,2)-
x_21*psi(j).etader(2,2)))...

+2*element_area*BofXY(n)*dot(q.Weights,bipolyval(psi(i).fun,q.Points).
*bipolyval(psi(j).fun,q.Points));

            ke(j,i) = ke(i,j);

            K(currElement(i),currElement(j)) =
K(currElement(i),currElement(j))+ke(i,j);
```

```
K(currElement(j),currElement(i))=K(currElement(i),currElement(j));
        end


fe(i)=0.5*element_area*FofXY(n)*dot(q.Weights,bipolyval(psi(i).fun,q.P
oints));
        F(currElement(i)) = F(currElement(i))+fe(i);
    end
end
```

```
function [K,F] = enforce2DBCs(K,F,KofXY,boundaryValues,boundaryNodes)
numDirichlet = length(boundaryNodes('Dirichlet'));
nNodes = length(K);
dirichletValues = boundaryValues('Dirichlet');
dirichletNodes = boundaryNodes('Dirichlet');

for currDirichlet = 1:numDirichlet
    for j =1:nNodes %columns
        F(j)=F(j)-
K(j,dirichletNodes(currDirichlet))*dirichletValues(currDirichlet);
        K(j,dirichletNodes(currDirichlet))=0;
    end
end

for currDirichlet = 1:numDirichlet
    K(dirichletNodes(currDirichlet),:)=zeros(1,nNodes);

    K(dirichletNodes(currDirichlet),dirichletNodes(currDirichlet))=1;
    F(dirichletNodes(currDirichlet))=dirichletValues(currDirichlet);
    %disp(K(currDirichlet,:));

end

end
```

```
%main
psi = polyLagrange2D(p);
[K,F]=element2D(psi,KofXY,BofXY,FofXY,MESH);
[K,F]=enforce2DBCs(K,F,KofXY,boundaryValues,boundaryNodes);
u=K\F;
trimesh(MESH.ConnectivityList,MESH.Points(:,1),MESH.Points(:,2),u);
%FEM solution



figure(2);
%hold on;
```

```
x=0:0.01:1;
y=0:0.01:1;
[x,y]=meshgrid(x,y);

%2.1 Dirichlet exact solution
%u_exact=x.*y.*(1-x).*(1-y);
%2.1 Mixed exact solution
%u_exact=(x.^2-x).*(y.^2-1);
%2.2 exact solution (Inhomogenous Dirichlet)
u_exact=(cosh(10*x)+cosh(10*y))/(2*cosh(10));

surf(x,y,u_exact);
```