Problem Chosen	2020 MCM/ICM Summary Sheet	Team Control Number
A		2018224

In recent decades, due to climate change and ocean temperature increases, certain Scottish fish species have begun to migrate northwards to seek more suitable living environments. This is potentially a significant problem for Scotland, as they support a large fishing industry dependent on the plentiful supply of Scottish herring and mackerel. We have devised a model to predict the affect of aforementioned temperature changes and how the fish species might react spatially over time.

First, we devise a partial differential equation based on Fickian diffusion to model how fish might normally behave if not for temperature change: the fish tend towards a uniform distribution, which can be expected. Then, less trivially, we introduce a temperature gradient by letting our boundary conditions vary with respect to ocean temperature $Q(x, y, t)$, and thus, also time. These boundary conditions are set in a way such that fish will migrate towards a temperature that is closest to their preferred temperature Q_p. To solve these notoriously difficult and complex partial differential equations, we use a numerical method rather than an analytical one, allowing us to generalize to a wide variety of conditions.

We then use our model to solve a realistic problem based on the preferred temperatures of Scottish herring and mackerel. Based on the specific conditions used for that problem, we were able to conclude a significant northwards migration of Scottish fish species within the next decade or so, prompting us to suggest new changes in fishing operations for the future. We also analyze best-case, worst-case, and most-likely cases to better understand the severity of this problem.

Finally, we recognize that no mathematical model is always perfect, and the same applies for ours. That being said, we also provide a set of possible limitations and directions for future work in order to improve upon our current approach.

We sincerely hope you enjoy reading our paper.

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1 Introduction

Due to significant changes in global ocean temperatures, many various aquatic species, specifically Scottish herring and mackerel, have re-located to new habitats. As a result, small fisheries have struggled to maintain typical operations. In order to help predict the migration patterns of these fish based on temperature changes, we devised a mathematical model using ideas from Fick's diffusion equation [1, 3, 4].

1.1 General Assumptions

In order to justify our model, we have made a few broad assumptions on fish movement and migration:

- With a uniform heat distribution across the entire domain, 'normal' fish migration will be modelled completely by Fick's equation i.e. normal diffusion [1].
- All fish will instantly react to any ocean temperature change across the entire domain.
- In the case of no temperature change, fish will die and reproduce at exactly the same rate at all times i.e. no population change.

2 1-dimensional Model

2.1 Parameters, Variables, and Functions

**Throughout this paper's calculations and charts, we used 'tkm' as a unit of distance, meaning 'tens-of-kilometers'*

x in (tkm): variable distance in the x-direction

t in $(years)$: time variable

$\rho(x,t)$ in $(1/tkm)$: describes the concentration of fish at some position x and time t

k in $(tkm^2/years)$: fish diffusivity constant, where a larger k results in faster diffusion of fish

$Q(x,t)$ in (Celsius): the temperature of the ocean at some position x and time t

Q_p in (Celsius): the preferred ocean temperature of a given species

2.2 A simple 1-dimensional Solution and Example

Consider a 1-dimensional simplification of the problem such that $\forall x \forall t Q(x,t) = Q_p$. This would represent the typical movement of fish based upon their population distribution alone, assuming that the total number of fish does not change over time. We can model this directly using Fick's diffusion equation [1, 3, 4]:

$$\frac{\partial \rho(x,t)}{\partial t} = k\frac{\partial^2 \rho(x,t)}{\partial x^2}$$

where all fish are contained within a domain $\Omega : [0, L]$ and are initially normally distributed.

$$\rho(x,0) = \frac{m}{\sigma\sqrt{(2\pi)}}e^{-\frac{(x-\frac{L}{2})^2}{2\sigma^2}}$$

with Neumann boundary conditions

$$\frac{\partial \rho(0,t)}{\partial x} = \frac{\partial \rho(L,t)}{\partial x} = 0$$

Separating the PDE and solving for coefficients,

$$\rho(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{\pi n x}{2L}\right)$$

we acquire a general solution:

$$\rho(x,t) = A_0 + \sum_{n=1}^{\infty} A_n e^{\frac{-k\pi^2 n^2}{4L^2}} \cos\left(\frac{\pi n x}{2L}\right)$$

Solving this numerically using finite-difference methods [2], and using parameter values

$$k = 0.01, \sigma = 0.2, m = 2, L = 2$$

we can see that the steady-state approaches a uniform distribution (the fish disperse evenly away from $x = \frac{L}{2}$). Additionally, this model correctly exhibits conservation of mass:

$$\int_0^L \rho(x,0)dx = \lim_{t\to\infty} \int_0^L \rho(x,t)dx = 2\text{erf}\left(\frac{5}{\sqrt{2}}\right) = 2$$

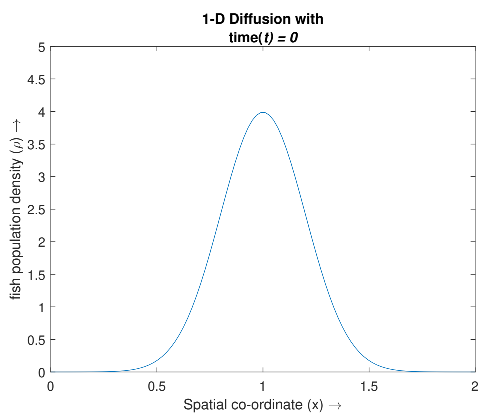

Figure 1: t=0

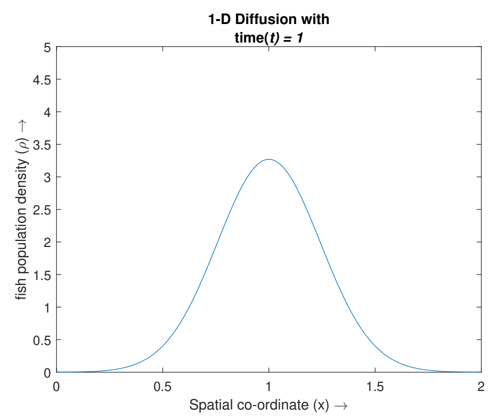

Figure 2: t=1

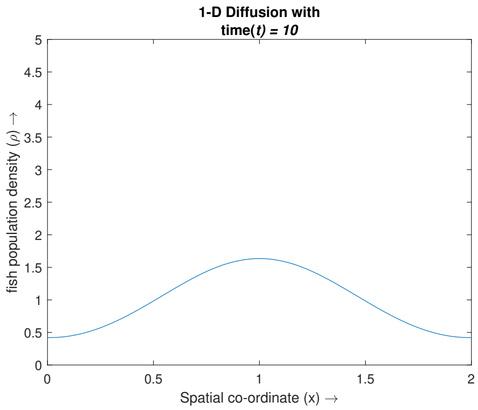

Figure 3: t=10

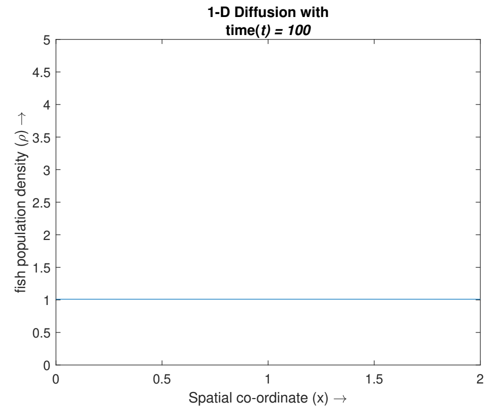

Figure 4: t=100

2.3 1-D solution with heat/time-dependent boundary conditions

Now we consider how a change in heat might affect the migration of fish in a 1-dimensional domain. Let $Q(x,t)$ describe the ocean temperature (in Celsius) at some position x and time t. Also let Q_p be a constant value that describes the preferred ocean temperature of a specific fish species. Then, we propose new boundary conditions:

$$\frac{\partial \rho}{\partial t}(0,t) = \frac{\partial \rho}{\partial t}(L,t) = \alpha(\mid Q(0,t) - Q_p \mid - \mid Q(L,t) - Q_p \mid)$$

It will be easier to see the reasoning behind these conditions with an example, as follows.

2.4 1-D example with heat/time-dependent boundary conditions

Maintaining similar parameters as before,

$$k = 0.01, \sigma = 0.2, m = 2, L = 2$$

and initial conditions,

$$\rho(x,0) = \frac{m}{\sigma\sqrt{(2\pi)}} e^{-\frac{(x-\frac{L}{2})^2}{2\sigma^2}}$$

but new boundary conditions,

$$\frac{\partial \rho}{\partial x}(0,t) = \frac{\partial \rho}{\partial x}(L,t) = \alpha(\mid Q(0,t) - Q_p \mid - \mid Q(L,t) - Q_p \mid), Q_p = 30, \alpha = 0.2$$

where

$$Q(L,t) = t, Q(0,t) = 2t$$

(The temperature at L is increasing linearly at twice the rate of the temperature at 0)

we now have a more complex but more accurate model. Solving numerically once again, we obtain these figures:

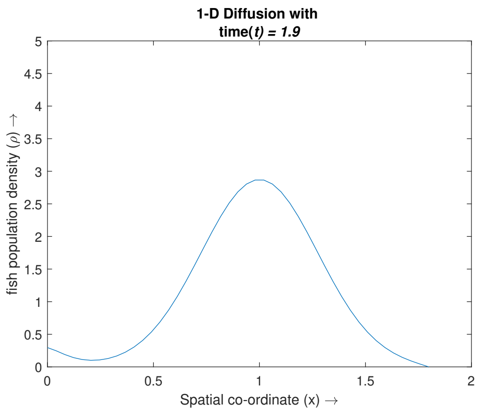

Figure 5: t=0

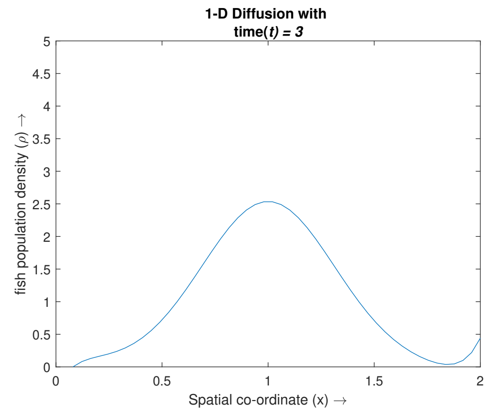

Figure 6: t=1

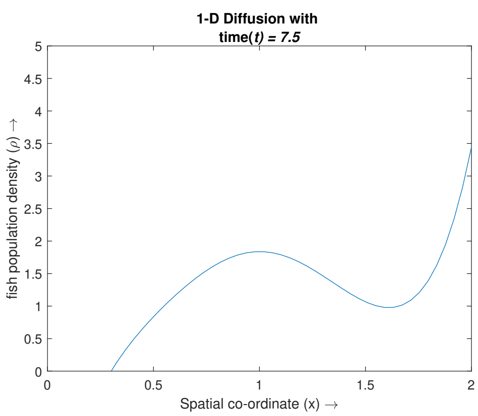

Figure 7: t=10

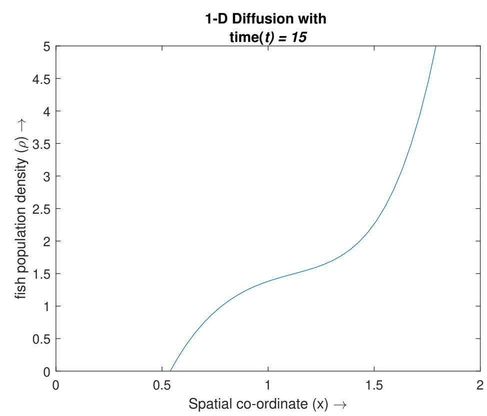

Figure 8: t=100

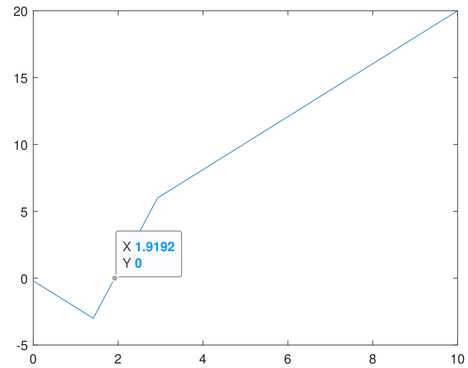

Figure 9: Boundary Conditions vs. Time(t)

As seen in Figure 9, $Q(L,t)$ is closer to Q_p up until $t = 1.9$ and therefore, fish will continue to diffuse towards $x = L$ until then, as seen in Figure 5. However, after this time, $Q(0,t)$ will always be closer to Q_p and so fish will then begin to diffuse towards $x = 0$ from then on, as seen in Figures 6-8.

It may be important to note that whenever $\rho(x,t) < 0$, as seen in Figure 8, we interpret that to represent a population density of zero; the fish have either died out or relocated from that region of the domain. Analogously, the population density on the opposite side of the domain is shown to have increased; the fish population has grown as a result of being within a region of preferred temperature.

With this methodology, we can now begin to model this problem within a 2-dimensional space.

3 2-dimensional Model

3.1 Parameters, Variables, and Functions

x in (tkm): variable distance in the x-direction

y in (tkm): variable distance in the y-direction

t in $(years)$: time variable

$\rho(x, y, t)$ in $(1/tkm^2)$: describes the concentration of fish at some position *(x,y)* and time *t*

k in $(tkm^2/years)$: fish diffusivity constant, where a larger k results in faster diffusion of fish

$Q(x, y, t)$ in (Celsius): the temperature of the ocean at some position *(x,y)* and time *t*

Q_p in (Celsius): the preferred ocean temperature of a given species

3.2 A Simple 2-dimensional Solution and Example

Let us now consider a 2-dimensional version of the problem such that $\forall x \forall y \forall t Q(x, y, t) = Q_p$. This will represent the movement of fish based upon the distribution of the population only, assuming that the total number of fish does not change over time. We model this directly by using Fick's law of diffusion $[3, 4]$:

$$\frac{\partial \rho}{\partial t}(x, y, t) = k\nabla^2 \rho(x, y, t)$$

where all fish are contained within a domain $\Omega : [0, L]$ and are normally distributed.

$$\rho(x, y, 0) = \frac{m^2}{\sigma_1 \sigma_2 (2\pi)} e^{-\frac{(x-\frac{L}{2})^2}{2\sigma_1^2} - \frac{(y-\frac{L}{2})^2}{2\sigma_2^2}}$$

with Neumann boundary conditions

$$\frac{\partial \rho(0, y, t)}{\partial x} = \frac{\partial \rho(x, 0, t)}{\partial y} = \frac{\partial \rho(L, y, t)}{\partial x} = \frac{\partial \rho(x, L, t)}{\partial y} = 0$$

Similar to the 1-dimensional case, we separate the PDE $[1, 3, 4]$ and apply our new boundary conditions to acquire a general solution in the form of

$$\rho(x, y, t) = A_0 + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_n e^{-k((\frac{n\pi}{2L})^2 + (\frac{m\pi}{2L})^2)} \cosh \frac{n\pi x}{2L} \cos \frac{m\pi y}{2L}$$

Once again, we solve this problem numerically with finite difference methods with the parameters

$$k = 0.1, \sigma_x = \sigma_y = 0.2, L = 2$$

to obtain the following plots:

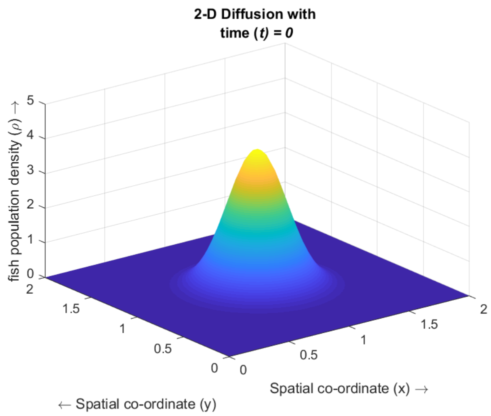

Figure 10: t=0

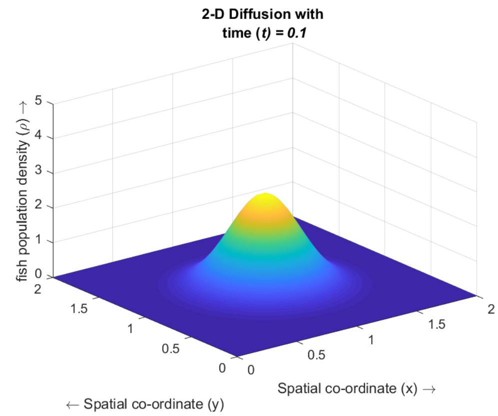

Figure 11: t=0.1

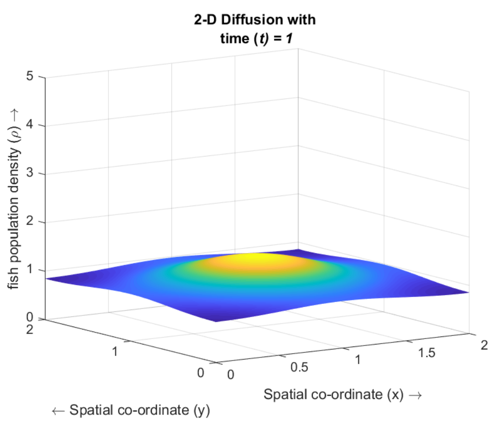

Figure 12: t=1

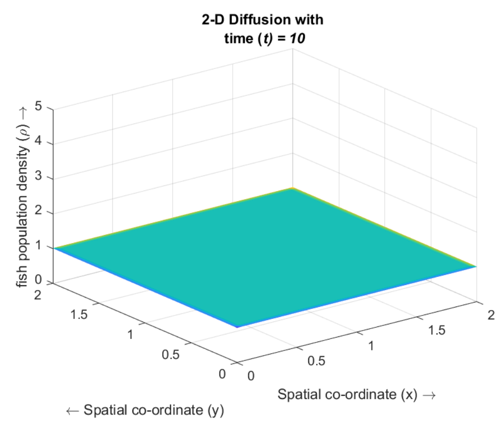

Figure 13: t=10

3.3 2-D solution with heat/time-dependent boundary conditions

We will continue with the parameters

$$k = 0.1, \sigma_x = \sigma_y = 0.2, L = 2, m = 1$$

Next, as before, we will illustrate our new 2-dimensional model using 2 pairs of heat/time-dependent boundary conditions (for horizontal and vertical boundaries),

$$\frac{\partial \rho(0, y, t)}{\partial x} = \frac{\partial \rho(L, y, t)}{\partial x} = 0.001[\mid 2t - 20 \mid - \mid t - 20 \mid]$$

$$\frac{\partial \rho(x, 0, t)}{\partial y} = \frac{\partial \rho(x, L, t)}{\partial y} = 0.001[\mid t - 20 \mid - \mid 2t - 20 \mid]$$

and a normally-distributed initial condition,

$$\rho(x,y,0) = \frac{m^2}{\sigma_1\sigma_2(2\pi)} e^{-\frac{(x-\frac{L}{2})^2}{2\sigma_1^2} - \frac{(y-\frac{L}{2})^2}{2\sigma_2^2}}$$

The figures below show our model:

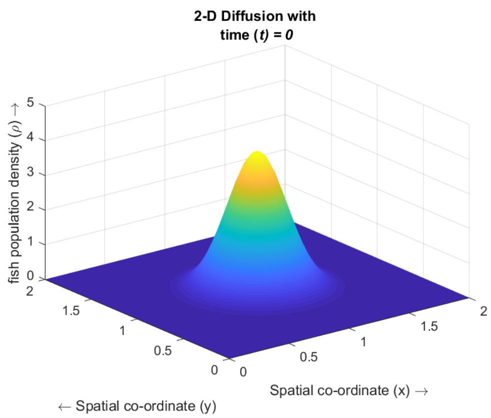

Figure 14: t=0

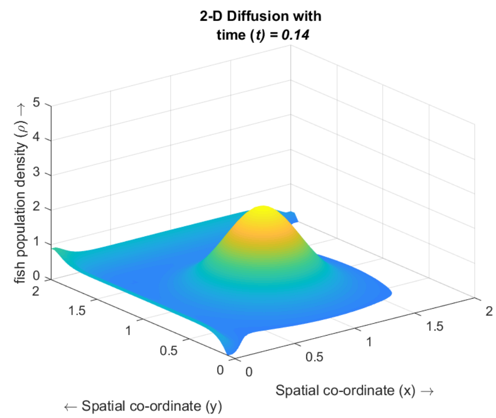

Figure 15: t=0.14

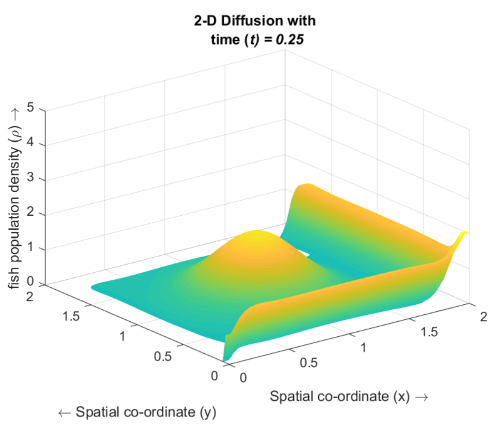

Figure 16: t=0.25

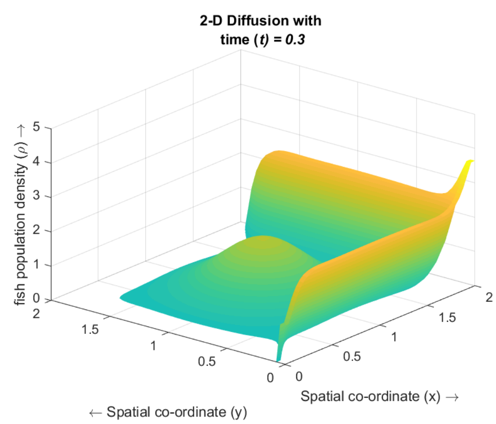

Figure 17: t=0.30

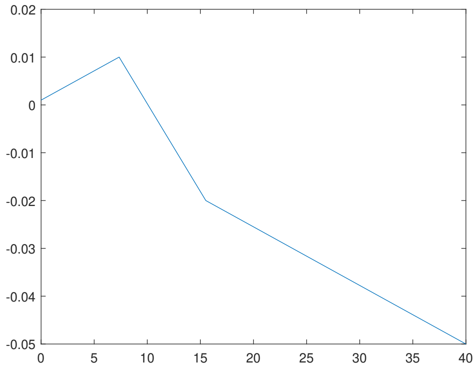

Figure 18: North-South Boundary Conditions vs. Time(t)

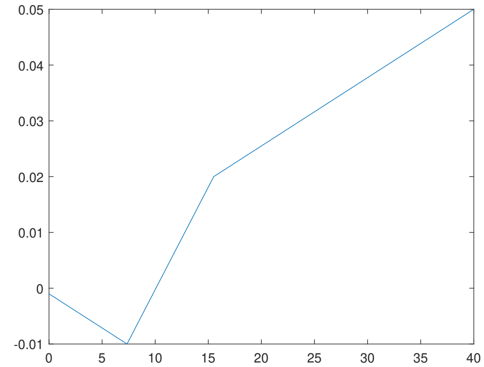

Figure 19: East-West vs. t

We can interpret this solution as essentially two 1-dimensional solutions combined into one. Each pair of boundary conditions describes the movement over time along their respective x,y co-ordinates as seen in Section 2.4. As seen in Figure 15, although the fish initially begin to move towards the 'north-west' corner, due to the change of boundary conditions at $t = 10$ (Figures 18, 19), the fish then change directions to move towards the 'south-east' corner, as expected. Meanwhile, the initial 'bump' continues to diffuse outwards as well, as modelled by Fick's 2-dimensional equation [3, 4].

4 Applicability

4.1 Another Example

In the previous sections, we showed several examples in order to justify our model and its results. However, we would also like to showcase our model's applicability with regards to other features such as various initial conditions, parameter values, time, continuity, and non-linear time-dependent boundary conditions, to name a few.

In this example, we use initial condition

$$\rho(x, y, 0) = \begin{cases} 4 & \text{if } (0 \le x \le 0.t) \wedge (0.75 \le y \le 1.25) \\ 0 & \text{otherwise} \end{cases}$$

and boundary conditions

$$0.05\sin(t/4)$$

for both horizontal and vertical boundaries. This might be practical in nature to represent periodic temperature variation such as seasonal climate change, for example (Figures 20-25). Furthermore, our model, being numerical, is able to handle discontinuous initial conditions as well, which might represent fish being released from a bounded captivity.

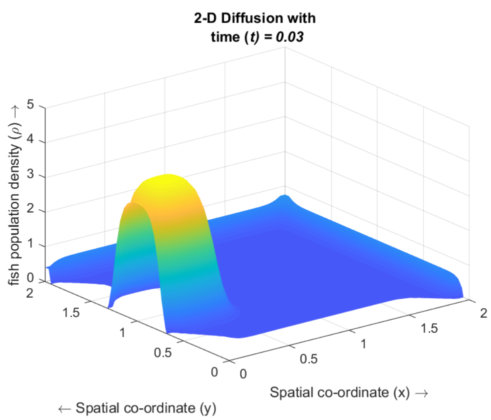

Figure 20: t=0.03

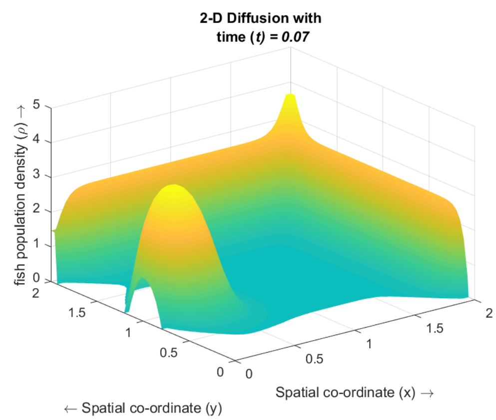

Figure 21: t=0.07

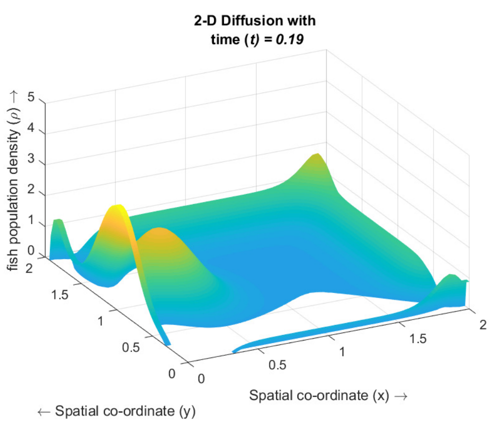

Figure 22: t=0.19

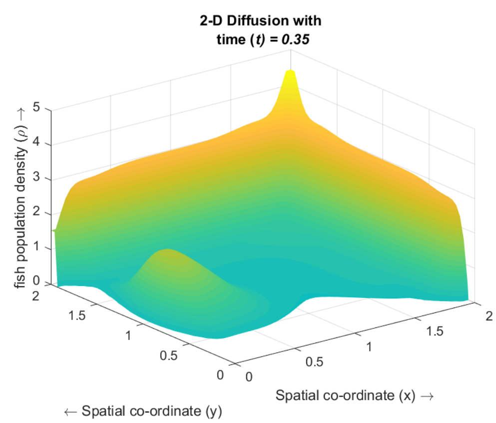

Figure 23: t=0.35

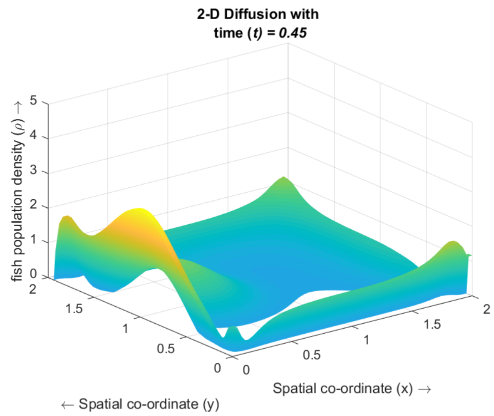

Figure 24: t=0.45

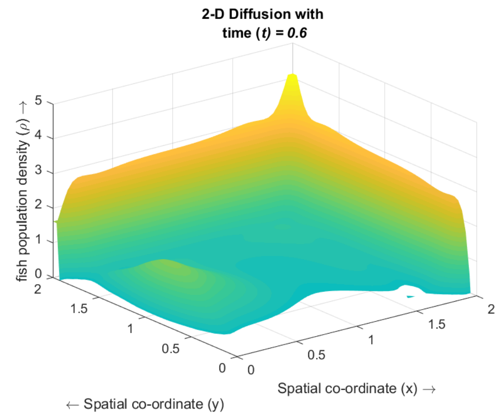

Figure 25: t=0.6

4.2 Robustness

Overall, we see that the model is robust under essentially all conditions regarding parameter variations of k and L, which allow us to consider different species of fish (different k values) as well as different areas (different L values). Additionally, the model functions with different heat/time-dependent boundary conditions, which apply to different possible changes in ocean temperatures. Finally, the initial conditions, specifically σ_1 and σ_2 while using a initial normal distribution, can also be changed in order to account for populations of fish with different starting distributions.

4.3 Limitations

4.3.1 Domain Shape

Our model inherently uses rectangular blocks for spatial domains. In reality, however, these domains may not be rectangular but rather more exotic shapes such as circles or ellipses (see Section 4.4.1).

4.3.2 Well-defined parameters

Although our parameters are well defined in terms of both physical meaning as well as units, it may be difficult in practice to obtain values and measurements for these parameters.

4.4 Future directions

4.4.1 Arbitrary Domains using a finite-element mesh

Since our model takes a numerical approach to find a solution, it will not be very difficult/impossible (contrary to many analytical approaches) to implement our model onto any arbitrary domain – not just squares/rectangles. We can do this by using finite-element methods since finite-difference methods, as we have used, are not usually used to solve across irregular geometries [5].

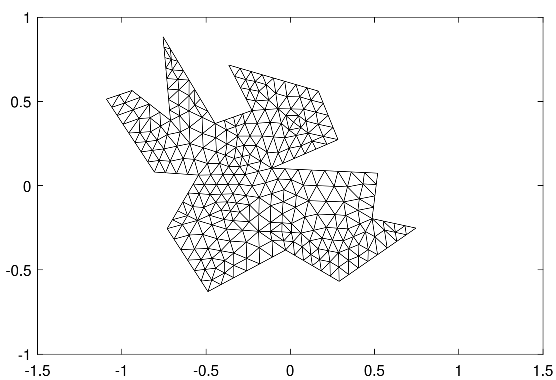

Figure 26: Arbitrary Finite Element Domain

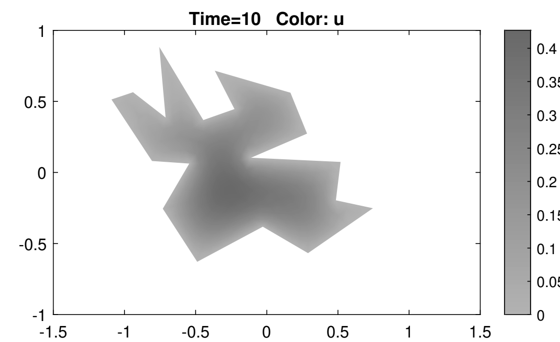

Figure 27: Solution

4.4.2 Forcing/Population growth

Our model can be revised to take into account different factors which affect the population growth of the fish species in question. In order to do this, we can introduce various forcing terms into our differential equation to model specific situations.

4.5 Application towards migration of Scottish herring and mackerel

Based on our research, we can attempt to model the migration of Scottish herring and mackerel across a 2tkm by 2tkm patch of sea, where we will let the positive x-axis represent the coastline and $x = 0$ be the southern boundary and $x = L$ be the northern boundary. We know that these fish species prefer a temperature $Q_p =$ 10 degrees Celsius [6], and ocean temperatures in the south have been steadily increasing, pushing the fish northwards. For this model, we will assume that for now, the fish are initially concentrated along the coastline, allowing for small fisheries to easily gather fish. Also, we will assume that the northern boundary has an ocean temperature of Q_p.

Therefore, we will let

$$\rho(x, y, 0) = \frac{1}{\sigma_1\sigma_2(2\pi)} e^{-\frac{(x-\frac{L}{2})^2}{2\sigma_1^2} - \frac{(y-\frac{L}{2})^2}{2\sigma_2^2}}, \sigma_1 = \sigma_2 = 0.4$$

$$\frac{\partial\rho}{\partial x}(0, y, t) = \frac{\partial\rho}{\partial x}(L, y, t) = 0$$

$$\frac{\partial\rho}{\partial y}(x, 0, t) = \frac{\partial\rho}{\partial y}(x, L, t) = \alpha(\mid Q(x, 0, t) - Q_p \mid - \mid Q(x, L, t) - Q_p \mid)$$

$$Q_p = 10, \alpha = 0.00013, Q(x, 0, t) = \sqrt{(t)}, Q(x, L, t) = 10$$

Resulting in this solution:

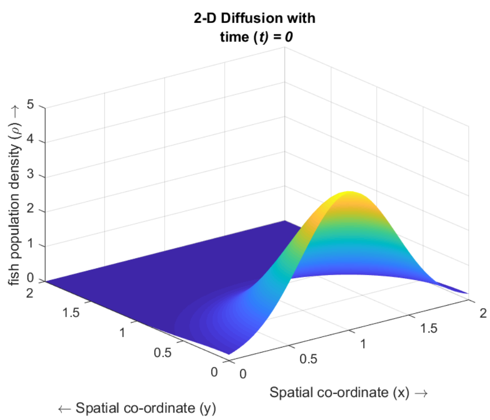

Figure 28: t=0

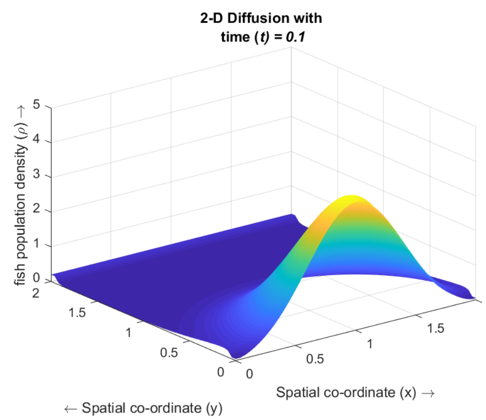

Figure 29: t=0.1

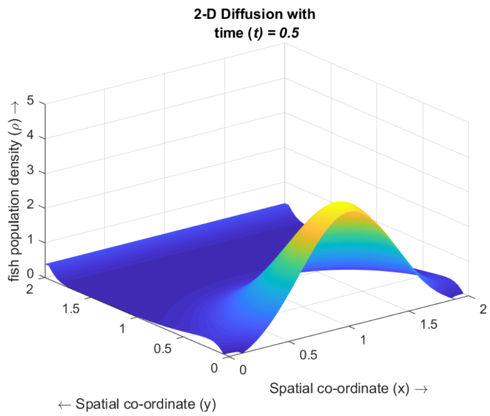

Figure 30: t=0.5

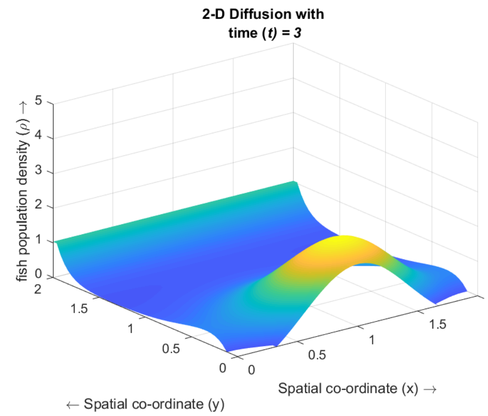

Figure 31: t=3

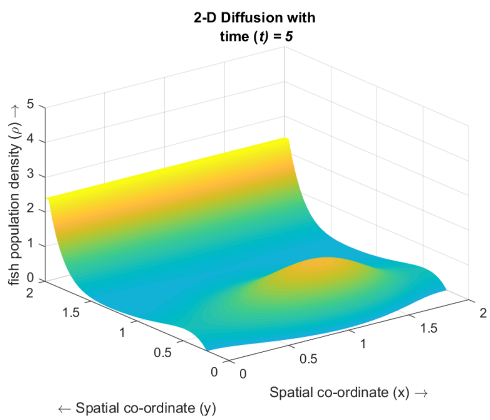

Figure 32: t=5

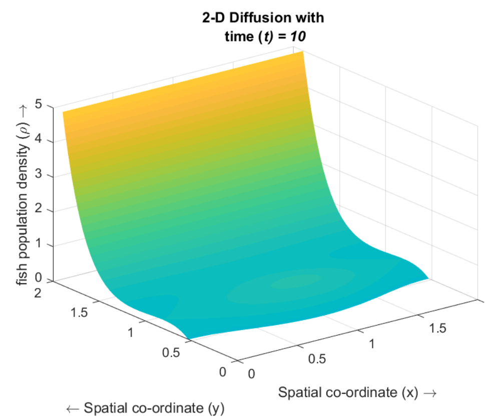

Figure 33: t=10

According to this solution, the fish will almost complete migrate away from the southern coastline in within 10 years time. However, these predictions are simply based on the prior assumptions and perhaps different conditions are needed to produce more realistic results.

5 Analysis

5.1 Best Case, Worst Case, and Most Likely

By our model, there are two main factors that affect the rate of fish migration: the fish diffusitivity constant (k) and the heat-dependent boundary conditions. More specifically, a greater k value will lead to a higher rate of migration (as a result of diffusion alone, not heat), and a greater difference between $Q(x, y, t)$ and Q_p at some boundary will also lead to a higher rate of migration (as a result of heat change alone, not diffusion). Both factors are somewhat arbitrary, and require much more data and real-world analysis in order to determine accurate measures for both. However, we can provide an estimated range of each factor for each case scenario.

5.1.1 k parameter

To test k, we remove variable boundary conditions and let the fish diffuse from a normal distribution as performed in Section 3.3. We then assess how long it takes for ρ to reach an approximately uniform distribution.

Here are the results from testing our model:

$$k = 1, t = \text{ 0.4 years}$$

$$k = 0.1, t = \text{ 3 years}$$

$$k = 0.01, t = \text{ 30} - \text{50 years}$$

With this, it is clear that the best-case scenario is to have a small k value so that fish migration is slower. On the contrary, the worst-case scenario would be when k is relatively large, as the fish would diffuse quicker. However, these are both extreme cases, as seen by the time elapsed, so we believe that on average, k will most likely be of the order of magnitude of 10^{-1}, as seen in the second calculation.

5.1.2 Boundary Conditions

It is especially hard to analyze different case scenarios with comparison to boundary conditions without the use of actual ocean temperature data. Still, we may still be able to provide some insight, based on our model, of what changing ocean temperatures may mean. As illustrated in Section 2.4, it is possible for fish to initially migrate one direction, only to change directions later on. However, we always see in the arbitrarily defined long-term scenario, the fish eventually move towards the boundary which has a greater rate of heat change with respect to time. On the other hand, since this 'long-term' case is somewhat arbitrary and dependent on data, it is also possible for the migration direction change to occur hundreds to thousands of years in the future, which is outside the scope of what the problem sees as important. From here, we suggest a few case-scenarios:

- Best-case: Q' at the boundary closest to the fisheries is significantly lower than Q' at the opposite boundary, causing fish to quickly migrate towards fishery locations (unlikely, but still possible).
- Worst-case: Q' at the closer boundary is significantly higher than Q' at the further boundary, causing fish to quickly migrate away from fishery locations.
- Most-likely: a combination of the best-case and worst-case scenarios; the fish do not migrate away at significant rates in the short-term, but if ocean temperature continues to increase at high rates, the fish eventually begin to migrate further away in the long-term.

5.2 Changes to Fishing Operations

Overall, our model predicts that the migration of fish will not have an effect on small fishing companies in the very near future, since the most likely scenario points to relatively low migration rates. However, if ocean temperatures keep rising at current rates, our model predicts that the fish migration will start to become appreciable after some time. After this point, the fishing companies will have to make changes to their current fishing operations. In order to combat this problem, we propose a two step solution:

- Firstly, the fishing companies should invest resources into determining accurate values for our models proposed parameters such as k and L. In addition to this, experiments should also be conducted to measure ocean temperatures over time. By taking this data and creating a fit, we can then precisely determine our boundary conditions, specifically $Q(x, y, t)$, in order to get valid models. Once all of the parameters and boundary conditions have been measured, our model can be solved numerically in order to get accurate results for the migration of herring and mackerel. This will then help guide the fishing companies to a proper course of action.
- After receiving information on the migration of the fish, we recommend the fishing companies to relocate their assets to a new suitable port. This port should be in a location that is closer to long term position of the mackerel and herring populations.

5.3 Foreign Territorial Waters

In the event that the fish populations migrate into the territorial waters of another country, our model presents two possible outcomes. One possibility is that the fish populations will diffuse over time, at least to some extent, to an area back in international waters. In this case, our recommendation for the fishing companies is to simply wait it out as the fish return back to viable locations. In the event that the fish populations remain in the territorial waters of another country indefinitely, we propose that the fishing companies and/or the government of the United Kingdom negotiate fishing rights with the foreign country in question. Since the fish will not naturally leave in this scenario, we view this as the only feasible solution.

5.4 Conclusion

Lastly, we would like to say that we view the problem of rising ocean temperatures as crucial to not only the well being of the fish populations and the fishing companies, but to the all of organisms on Earth. We believe that rather than working on models to deal with effects of these rising temperatures, more work should be done on models which help to curb climate change and its effects. It is these models that hold the key to our continued survival as not only a species but rather a planet.

6 Appendix

6.1 *Hook Line and Sinker* Article

The most important aspect of any successful fishing trip is knowing when and where the fish are going to bite. This is to say that any fisherman that lacks the knowledge of where the fish are will fail to produce a successful yield. In the case of Scotland and the Northern Atlantic Ocean, this information is constantly changing due to the fluctuation of global ocean temperatures. It is important to note that the location at which the highest concentration of Scottish herring and mackerel is not likely to stay the same. This is because of the tendency of these species to migrate toward habitats with more hospitable temperatures to increase the likelihood for reproduction in the future. However, as temperatures continue to increase in southern waters, the schools of fish are on a path away from the coast of Scotland, leaving many small Scottish fisheries at risk of economic failure. Ideally, the waters along the coast would be more hospitable than the adjacent water. This would allow for a large population of fish to reside close to the fishing companies that need to catch them; however, this is unlikely in the near future due to oceans becoming warmer. Unfortunately, we do not know how serious this problem is. As a result, we must turn to a solution that allows us to predict the future habitat locations of fish far from the coast depending on the current temperature change rates. Recognizing that population density and temperature are intrinsically linked and time-dependent, our purpose was to create a model that could be used to identify the location at which the highest concentration of fish can be found based on the temperatures of adjacent bodies of water. As time passes, we hope to recognize an area where the population of fish is mostly consolidated. Based on our model, small fishing companies will be able make informed decisions as to whether or not fishing should cease to occur based on the distance from their center of operations and their respective financial situations. We hope you both understand the importance of this problem and also the applicability of our mathematical model towards your situation.

6.2 References

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