Supplementary Materials

Matrix Algebra

- A matrix is a rectangular array of numbers. More precisely, an *m* x *n* matrix has *m* rows and *n* columns. The positive integer *m* is called the *row dimension*, and *n* is called the *column dimension*.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Off-diagonal elements

Diagonal elements

- where a_{ij} represents the element in the *i*th row and the *j*th column. For example, a_{25} stands for the number in the second row and the fifth column of **A**.
- A **square matrix** has the same number of rows and columns. The dimension of a square matrix is its number of rows and columns.
  - Diagonal elements
  - Off-diagonal elements

Matrix Algebra

- A specific example of a 2 x 3 matrix is

$$A = \begin{bmatrix} 2 & -1 & 7 \\ -4 & 5 & 0 \end{bmatrix}$$

where $a_{13} = 7$. The shorthand $\mathbf{A} = [a_{ij}]$ is often used to define matrix operations.

- A 1 x m matrix is called a **row vector** (of dimension m) and can be written as $\mathbf{x} \equiv (x_1, x_2, \ldots, x_m)$.
- An n x 1 matrix is called a **column vector** and can be written as

$$y \equiv \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Matrix Algebra

- Two matrices **A** and **B**, each having dimension *m* x *n*, can be added element by element: $\mathbf{A} + \mathbf{B} = [a_{ij} + b_{ij}]$. More precisely,

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \vdots & & & \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mn} + b_{mn} \end{bmatrix}.$$

- Given any real number g (often called a scalar), **scalar multiplication** is defined as $\gamma\mathbf{A} = [\gamma a_{ij}]$, or

$$\gamma\mathbf{A} = \begin{bmatrix} \gamma a_{11} & \gamma a_{12} & \dots & \gamma a_{1n} \\ \gamma a_{21} & \gamma a_{22} & \dots & \gamma a_{2n} \\ \vdots & & & \\ \gamma a_{m1} & \gamma a_{m2} & \dots & \gamma a_{mn} \end{bmatrix}.$$

Matrix Algebra

- To multiply matrix **A** by matrix **B** to form the product **AB**, the *column* dimension of **A** must equal the *row* dimension of **B**. Therefore, let **A** be an m x n matrix and let **B** be an n x p matrix. Then, **matrix multiplication** is defined as

$$\mathbf{AB} = \left[\sum_{k=1}^{n} a_{ik}b_{kj}\right]. = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}.$$

- In other words, the (i, j)th element of the new matrix **AB** is obtained by multiplying each element in the ith row of **A** by the corresponding element in the jth column of **B** and adding these n products together. A schematic may help make this process more transparent:

$$i^{\text{th}}\text{ row} \rightarrow \overset{\mathbf{A}}{\begin{bmatrix} \\ a_{i1}\,a_{i2}\,a_{i3}\ldots a_{in} \\ \\ \end{bmatrix}} \underset{\substack{\uparrow \\ j^{\text{th}}\text{ column}}}{\overset{\mathbf{B}}{\begin{bmatrix} b_{1j} \\ b_{2j} \\ b_{3j} \\ \vdots \\ b_{nj} \end{bmatrix}}} = \underset{\substack{\uparrow \\ (i,j)^{\text{th}}\text{ element}}}{\overset{\mathbf{AB}}{\begin{bmatrix} \\ \sum_{k=1}^{n} a_{ik}b_{kj} \\ \\ \end{bmatrix}}},$$

Matrix Algebra

Properties of Matrix Multiplication

- $(a + b)\mathrm{A} = a\mathrm{A} + b\mathrm{A}$;
- $a(\mathrm{A} + \mathrm{B}) = a\mathrm{A} + a\mathrm{B}$;
- $(ab)\mathrm{A} = a(b\mathrm{A})$;
- $a(\mathrm{AB}) = (a\mathrm{A})\mathrm{B}$;
- $\mathrm{A} + \mathrm{B} = \mathrm{B} + \mathrm{A}$;
- $(\mathrm{A} + \mathrm{B}) + \mathrm{C} = \mathrm{A} + (\mathrm{B} + \mathrm{C})$;
- $(\mathrm{AB})\mathrm{C} = \mathrm{A}(\mathrm{BC})$;
- $\mathrm{A}(\mathrm{B} + \mathrm{C}) = \mathrm{AB} + \mathrm{AC}$;
- $(\mathrm{A} + \mathrm{B})\mathrm{C} = \mathrm{AC} + \mathrm{BC}$;
- $\mathrm{AB} \neq \mathrm{BA}$, even when both products are defined.

Matrix Algebra

- Let **A** = $[a_{ij}]$ be an m x n matrix. The **transpose** of **A,** denoted **A'** (called **A** *prime*), is the n x m matrix obtained by interchanging the rows and columns of **A**. We can write this as **A' =** $[a_{ji}]$.

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 7 \\ -4 & 5 & 0 \end{bmatrix}, \quad \mathbf{A}' = \begin{bmatrix} 2 & -4 \\ -1 & 5 \\ 7 & 0 \end{bmatrix}.$$

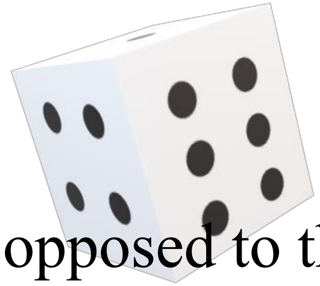
Matrix Algebra Examples

- $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1a + 2c & 1b + 2d \\ 3a + 4c & 3b + 4d \end{bmatrix}$
- $2 \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 * 1 & 2 * 2 \\ 2 * 3 & 2 * 4 \end{bmatrix}$

- AB ≠ BA
- A= $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and B= $\begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix}$
- AB= $\begin{bmatrix} 8 & 4 \\ 18 & 10 \end{bmatrix}$
- BA= $\begin{bmatrix} 8 & 12 \\ 6 & 10 \end{bmatrix}$

- I= $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- $A \times I = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = I \times A$

Random Variable



- We are interested mainly in some function of the outcome as opposed to the actual outcome itself.
- For instance, in tossing dice, we are often interested in the sum of the two dice and are not really concerned about the separate values of each die. That is, we may be interested in knowing that the sum is 7 and may not be concerned over whether the actual outcome was (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), or (6, 1). Also, in flipping a coin, we may be interested in the total number of heads that occur and not care at all about the actual head–tail sequence that results.
- These quantities of interest, or, more formally, these real-valued functions defined on the sample space, are known as ***random variables***.
- Because the value of a random variable is determined by the outcome of the experiment, we may assign probabilities to the possible values of the random variable.

A random variable is a variable that assumes numerical values associated with the random outcomes of an experiment, where one numerical value is assigned to each sample point.

Two Types of Random Variables

- Random variable that can assume a countable number of values are called Discrete.

  Examples of **Discrete** random variables:

  1. Number of voters in a sample of 500 who favor candidate A: x = 0, 1, 2, 3, ………, 500.
  2. The number of customers waiting to be served in a restaurant at a particular time: x = 0, 1, 2,…
  3. Number of accidents per month in a community: x = 0, 1, 2,…..

- Random variable that can assume values corresponding to any of the points contained in an interval are called **Continuous**.

  Examples of Continuous random variables:

  1. The length of time it takes a student to complete a 1-hour exam $0 \leq x \leq 60$
  2. Height of students in a class

Probability Distributions for Discrete Random Variables

- A complete description of a discrete random variable requires that we specify all the values the random variable can assume and the probability associated with each value.
- Probability mass function is a table or formula that specifies the probability associated with each possible value that the random variable can assume. Probability that a random variable X takes on a value x is denoted by $P(X=x) = p(x)$. The probability mass function for X is given by:

$$f(x) = P(X=x) = p(x)$$

- Two requirements must be satisfied by all the probability distributions for discrete random variables:

  1. $p(x) \geq 0$ for all values of x.
  2. $\Sigma\, p(x) = 1$, where the summation of $p(x)$ is over all possible values of x.

Probability Distributions for Continuous Random Variables

- The probability distribution function for a continuous random variable, x, can be represented by a smooth curve – a function of x, denoted $f(x)$. The probability that x falls between two values a and b, i.e., $P(a \leq x \leq b)$, is the area under the curve between a and b.

we don't have probability at a point of time, the area under curve is the probability

- Probability that x assumes a value in the interval $a \leq x \leq b$ is $P(a \leq x \leq b)$

$$P(a \leq x \leq b) = \int_b^a f(x)dx,$$

assuming that the integral exists. As with the requirements for a discrete probability distribution, we require that:

1. $f(x) \geq 0$ for all values of x.
2. $\int f(x)dx = 1$

Expected Values

- The expected value or expectation of a random variable is its mean or weighted average that is weighted according to the probability distribution.
- The expected value of a distribution can be thought of as a measure of center, as we think of averages as being middle values. By weighting the values of the random variable according to the probability distribution, we hope to obtain a number that summarizes a typical or expected value to an observation of the random variable.
- The expected value or mean of a random variable X denoted by $E(X)$ is

- $E(x) = \begin{cases} \sum_{x \in S} xf(x) = \sum_{x \in S} xP(X = x) & if\ X\ is\ discrete \\ \int_{-\infty}^{\infty} xf(x)dx & if\ X\ is\ continuous \end{cases}$

Example

- Random variable X takes 3 values: S = {1,2,3}

  P{X = 1} = 0.4

  P{X = 2} = 0.3

  P{X = 3} = 0.3

  $$E(X) = \sum_{x \in S} xP(X = x) = 0.4 \times 1 + 0.3 \times 2 + 0.3 \times 3 = 1.9$$

- Roll a standard die. The expectation of number that comes up:

  $$E(X) = \frac{1}{6} \times 1 + \frac{1}{6} \times 2 + \frac{1}{6} \times 3 + \frac{1}{6} \times 4 + \frac{1}{6} \times 5 + \frac{1}{6} \times 6 = 3.5$$

Properties of Expectation

- The process of taking expectations is a linear operation. Assuming X and Y are two random variables, a and b are constants

    1. $E(aX + b) = aE(X) + b$
    2. $E(X + Y) = E(X) + E(Y)$

Variance

- The variance of a random variable gives a measure of the degree of spread of a distribution around its mean. Variance of random variable X is given by:

$$Var(X) = E[X - E(X)]^2$$

- The positive square root of $Var(X)$ is the **standard deviation** of X.

$$\begin{aligned} Var(X) &= E[X - E(X)]^2 \\ &= E[X^2 + (E(X))^2 - 2XE(X)] \\ &= E(X^2) + (E(X))^2 - 2E(X)E(X)] \\ &= E(X^2) - (E(X))^2 \end{aligned}$$

Example

- Random variable X takes 3 values: S = {1,2,3}

  P{X = 1} = 0.4

  P{X = 2} = 0.3

  P{X = 3} = 0.3

$$\mathrm{Var(X)} = \sum_{x \in S} [x - E(X)]^2 P(X = x) = 0.4 \times [1 - 1.9]^2 + 0.3 \times [2 - 1.9]^2 + 0.3 \times [3 - 1.9]^2 = 0.69$$

- Roll a standard die. The expectation of number that comes up:

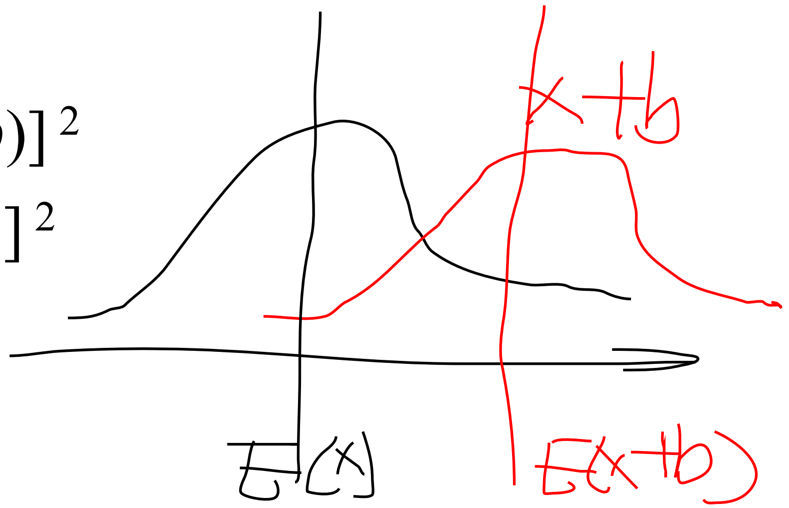
$$Var(X) = \frac{1}{6} \times [1 - 3.5]^2 + \frac{1}{6} \times [2 - 3.5]^2 + \frac{1}{6} \times [3 - 3.5]^2 + \frac{1}{6} \times [4 - 3.5]^2 + \frac{1}{6} \times [5 - 3.5]^2 + \frac{1}{6} \times [6 - 3.5]^2 = 2.92$$

Properties of Variance

- Assuming X *and* Y are random variables, a *and* b are constants:

1. $Var(aX + b) = a^2 Var(X)$

$$Var(aX + b) = E[aX + b - E(aX + b)]^2$$
$$= E[aX + b - aE(X) - b]^2$$
$$= E[aX - aE(X)]^2$$
$$= a^2 E[X - E(X)]^2$$
$$= a^2 Var(X)$$



2. $Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y)$

Note: For proof, use $Var(X) = E[X - E(X)]^2$ and rearrange terms.

Covariance and Correlation

linear

- Covariance and Correlation are numerical measures of the strength of relationship between two random variables.
- The Covariance between two random variables X and Y is given by:

$$Cov(X, Y) = E[(X - E(X))(Y - E(Y))]$$

- The Correlation between X and Y is given by:

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

Where σ_X and σ_Y are standard deviation of X and Y respectively.

- The value of ρ_{XY} is called the correlation coefficient. Correlation coefficient lies in the range [-1,1].
- $-1 \leq \rho_{XY} \leq 1$. If X and Y are independent then $Cov(X, Y) = 0$.

Properties of Covariance

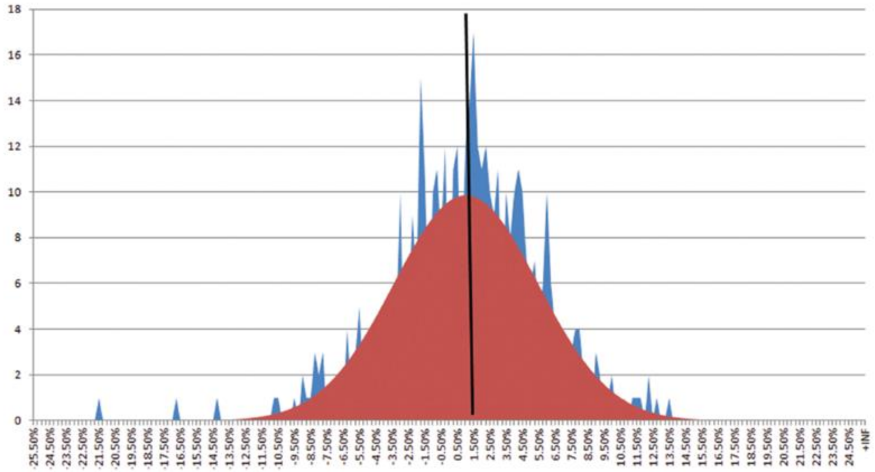
- Assuming X, Y, P and Q are random variables, a, b, c and d are constants:

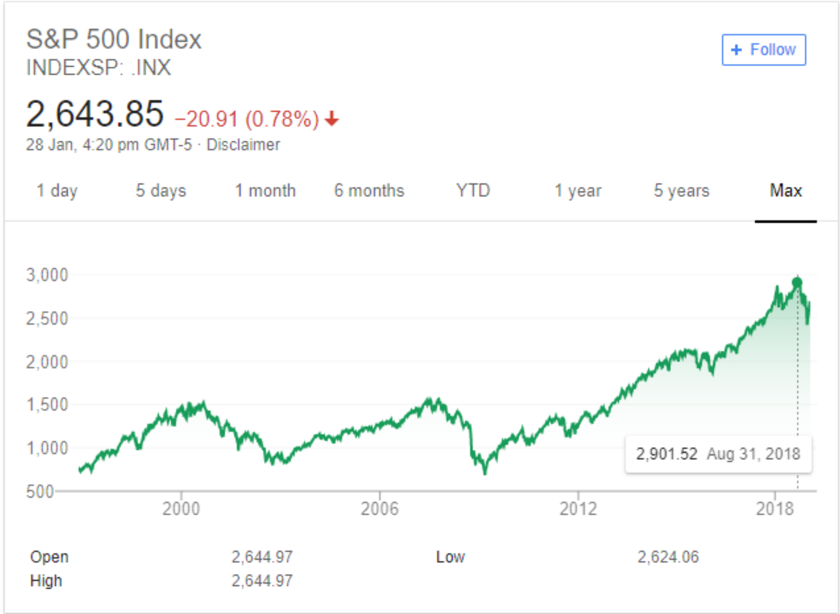
1. $Cov(X, Y) = E(XY) - E(X)E(Y)$
2. $Cov(X, X) = Var(X)$
3. $Cov(aX + b, cY + d) = \overset{ac}{\cancel{ab}}Cov(X, Y)$
4. $Cov(X + Y, P + Q) = Cov(X, P) + Cov(X, Q) + Cov(Y, P) + Cov(Y, Q)$

Note: For proof, apply $Cov(X, Y) = E[(X - E(X))(Y - E(Y))]$ and rearrange terms

Return and Risk

- Returns are the gains or losses from a security in a particular period and are usually quoted as a percentage.
- Risk is the chance that an investment's actual return will be different than expected. Risk means you have the possibility of losing some, or even all, of your original investment. Low levels of uncertainty (low risk) are associated with low potential returns. High levels of uncertainty (high risk) are associated with high potential returns.





Return and Risk

- Can you describe how S&P 500 index moves?
  - Upward? Anything else?
- Summarize the asset price movement in terms of return and risk
- Convert price into returns and make a histogram.

Rate of Return

- The return on an asset can be defined in two ways:
  - Percentage return: $R_t = \frac{P_t - P_{t-1}}{P_{t-1}}$
  - Log returns: $R_t = \ln(P_t) - \ln(P_{t-1})$

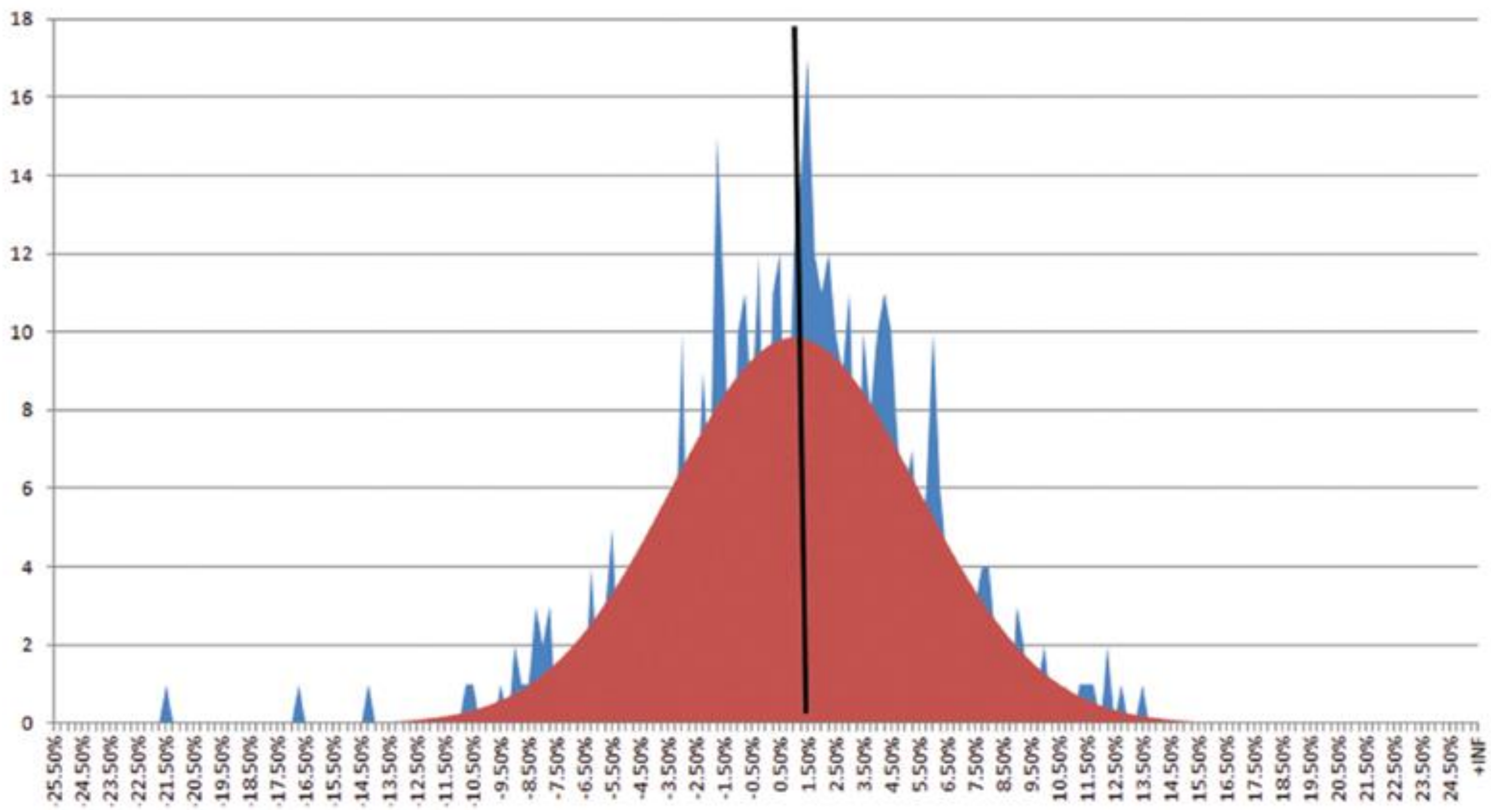
Total Return

- Assume that a share is bought at time t-1, a dividend is paid, then sold at time t.

- The total return on the share can be expressed as:

$$R_t = \frac{(P_t - P_{t-1}) + \text{dividend}}{P_{t-1}}$$

- More extreme negative returns
- Spike around the mean
- More obs on the left side of distribution



Monthly Return Distribution of S&P 500 index

Returns – Expectation and Variance

- Return on an investment a random variable.
  - Assume 3 possible scenarios:

		Return	
State of Market	Probability	Stock A	Stock B
Boom	0.25	28%	15%
Neutral	0.5	9%	10%
Recession	0.25	-25%	3%

$E(r^A) = ?$ $\qquad E(r^B) = ?$

$Var(r^A) = ?$ $\qquad Var(r^B) = ?$

$Cov(r^A, r^B) = ?$

Historical Time Series Returns

- We observe time series of realized returns and use these to estimate expected returns and variances.
- Treat each observation as an equally likely scenario.

$$E(r) = \frac{1}{n}\sum_{t=1}^{n} r_t = \bar{r}$$

$$Var(r) = \frac{1}{n-1}\sum_{t=1}^{n}(r_t - \bar{r})^2$$

$$Cov(r^A, r^B) = \frac{1}{n-1}\sum_{t=1}^{n}(r_t^A - \overline{r^A})(r_t^B - \overline{r^B})$$

- Use (n-1) in the denominator instead of n, since we do not know the “true” expected return so we use up a degree of freedom in calculating the expected return

Historical Time Series Returns - Example

	Return	
Year	Stock A	Stock B
2014	10%	8%
2015	8%	3%
2016	-2%	7%
2017	3%	9%
2018	7%	5%

$E(r^A) = \frac{1}{n}\sum_{t=1}^{n} r_t = \overline{r} = \frac{1}{5}(0.1 + 0.08 - 0.02 + 0.03 + 0.7) = 0.052$

$E(r^B) = \frac{1}{n}\sum_{t=1}^{n} r_t = \overline{r} = \frac{1}{5}(0.08 + 0.03 + 0.07 + 0.09 + 0.05) = 0.064$

$Var(r^A) = \frac{1}{n-1}\sum_{t=1}^{n}(r_t - \overline{r})^2 = \frac{1}{5-1}[\,(0.1 - 0.052)^2 + (0.08 - 0.052)^2 + (-0.02 - 0.052)^2 + (0.03 - 0.052)^2 + (0.07 - 0.052)^2] = 0.00227$

$Var(r^B) = \frac{1}{n-1}\sum_{t=1}^{n}(r_t - \overline{r})^2 = \frac{1}{5-1}[\,(0.08 - 0.064)^2 + (0.03 - 0.064)^2 + (0.07 - 0.064)^2 + (0.09 - 0.064)^2 + (0.05 - 0.064)^2] = 0.00058$

$Cov(r^A, r^B)$

$$= \frac{1}{5-1}\sum_{t=1}^{n}\left(r_t^A - \overline{r^A}\right)\left(r_t^B - \overline{r^B}\right) = \frac{1}{5-1}[(0.1 - 0.052)(0.08 - 0.064) + (0.08 - 0.052)(0.03 - 0.064) + (-0.02 - 0.052)(0.07 - 0.064) + (0.03 - 0.052)(0.09 - 0.064) + (0.07 - 0.052)(0.05 - 0.064) = -0.00036$$

Time Value of Money

- A dollar today is worth more than a dollar in the future.
  - If you invest the dollar today, you would get more than a dollar in the future.
- Imagine that you have $1000 today and invest it at a yearly interest rate of 5%.
  - This investment gives you $1050 after one-year.
  - $1000 today is worth $1050 in one year's time. In other words, the present value of $1050 is $1000.

Time Value of Money

- The relation between future and present values can be described as:

$$FV = PV \times (1 + r)$$

where:

FV	=	Future Value
PV	=	Present Value
r	=	Discount rate

- The future value t periods ahead can be estimated as:

$$FV = PV \times (1 + r)^t$$

- Rearranging the term, the expression for present value is given by:

$$PV = \frac{FV}{(1 + r)^t}$$

Annualized Return

- Effective Annual Rate (EAR)
  - Percentage increase in funds over 1-year horizon
  - Actual rate an investment grows
  - Accounts for compounding

$$EAR = (1 + \frac{Q}{m})^m - 1$$

Where

Q is the quoted interest rate

m is the number of compounding periods per year

- Annual Percentage Rate (APR)
  - Interest rate per period × number of periods per year
  - Ignores Compounding

Asset Classes

- Fixed Income
  - Money Markets
  - Capital Markets
- Equity
  - Common Stocks
  - Preferred Stocks
- Derivatives
  - Options – Put and Call
  - Futures/Forwards

Asset Classes

- Fixed Income
  - Money Markets
    - Short term, Liquid, Low risk, Often have large denominations
    - Examples: Treasury Bills, Commercial papers, Certificate of deposit
  - Capital Markets
    - Longer Term Borrowing
    - Examples: Treasury notes and bonds, Corporate bonds, Municipal Bonds
- Equity
- Derivatives

Asset Classes

- Fixed Income
- Equity
  - Common Stocks
    - Residual Claim
    - Limited Liability
  - Preferred Stocks
    - Priority over Common Stocks, rank below bonds (in terms of claim over firm assets)
    - Fixed Dividends
- Derivatives

Asset Classes

- Fixed Income
- Equity
- Derivatives
  - Options – Put and Call
    - Call option gives the holder the right to purchase the underlying asset at a specified prices
    - Put option gives the holder the right to sell the underlying asset at a specified prices
  - Futures/Forwards

Stock and Bond Market Indexes

- Uses
  - Track average returns
  - Compare performance of managers
  - Base of derivatives
- Bond Market Index
  - Merrill Lynch Global Bond Index
- US stock market index
  - S&P 500
- AUS stock market index
  - S&P/ASX 200

Investment Companies

- Functions
  - Record keeping and administration
  - Diversification and divisibility
  - Professional management
  - Lower transaction costs

- Definitions
  - Investment company: Financial intermediaries
  - Net asset value (NAV): Assets minus liabilities per share

Types of Investment Companies

- Unit Investment Trusts
  - Money pooled from many investors is invested in portfolio fixed for life of fund
- Managed Investment Companies
  - A professional investment firm that manages a portfolio for an annual fee
  - Load: Sales commission charged on mutual fund
  - Open-end or Closed-end fund
- Other Investment Organizations
  - Hedge funds
  - Real Estate Investment Trusts (REITs)