UNSW Business School/ Banking & Finance

FINS2624 Lecture 2

Term Structure of Interest Rates II and Duration

Lecture Outline

- Arbitrage
  - Exploiting mispriced bonds
- Future Interest Rates
  - Short rates and forward rates
- Term Structure Hypotheses
  - Expectations Hypothesis
  - Liquidity Preference Hypothesis
- Interest rate risk
- Duration
  - Definition and calculation
  - Determinants
  - Duration as a measure of interest rate risk
  - Convexity
  - Immunisation

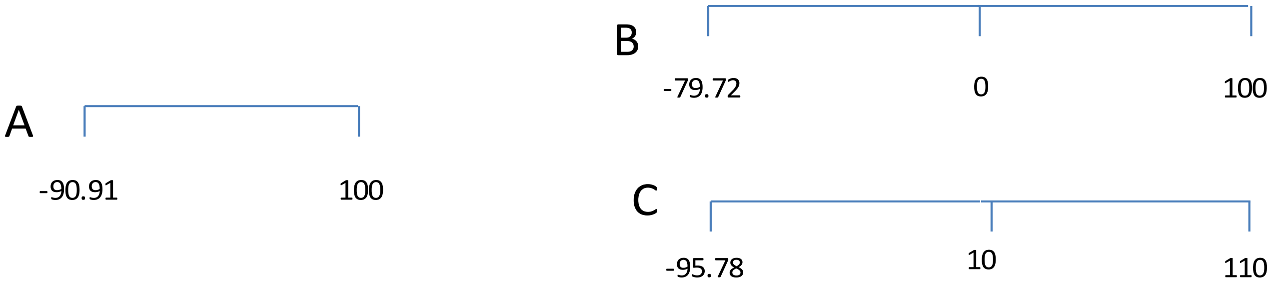
UNSW | AGSM
Business School

Arbitrage and the Term Structure

- To arbitrage on a potentially mispriced asset, we need to replicate its future cash flows with a portfolio of other traded assets (ie construct a replicating portfolio or synthetic asset)
- If the market value of the asset differs from the market value of the replicating portfolio, then an arbitrage opportunity arises
  - **Buy the cheaper** asset(s) and **sell the more expensive** one(s)
- With regard to the term structure of interest rates: if the prices of different bonds imply inconsistent spot rates, an arbitrage opportunity arises
  - The arbitrager can replicate a bond's future cash flows with a replicating portfolio of other bonds
  - The arbitrage trades will remove any mispricing in equilibrium, and reimpose a consistent term structure of interest rates

Arbitrage Example

- Suppose the following bonds trade in the market



- We can infer the term structure implied by the prices of A and B:

$$P_A = 90.91 = \frac{100}{(1+y_1)} \Leftrightarrow y_1 = \frac{100}{90.91} - 1 \approx 10\%$$

$$P_B = 79.72 = \frac{100}{(1+y_2)^2} \Leftrightarrow y_2 = \sqrt{\frac{100}{79.72}} - 1 \approx 12\%$$

- But, using these spot rates to derive P_C, implies its observed price is too low:

$$P_C^{\text{Implied}} = \frac{10}{(1+y_1)} + \frac{110}{(1+y_2)^2} = \frac{10}{1.1} + \frac{110}{1.12^2} \approx 96.78 > 95.78 = P_C^{\text{Observed}}$$

Example: Set up the Arbitrage Trades

- P_C is too low relative to the arbitrage-free price calculated from the term structure implied by P_A and P_B
  - There is an arbitrage opportunity
  - Bond C is "too cheap"
- Strategy: Construct a synthetic version of Bond C from bonds A and B
- Buy the underpriced real bond and sell the overpriced synthetic bond
  - One way of understanding selling a bond (that we don't have) is to view it as the opposite of buying the bond
  - **Buying** a bond means we are **lending** money to the issuer (ie lending the FV of the bond)
  - **Selling** a bond that we don't have is essentially the same as issuing the bond – we are **borrowing** from the bondholder

Example: Construct the Synthetic Bond

- A synthetic bond replicates the cash flows of Bond C:

$$CF_1^S = X_A CF_1^A + X_B CF_1^B = CF_1^C$$
$$CF_2^S = X_A CF_2^A + X_B CF_2^B = CF_2^C$$

$$\begin{cases} X_A \cdot 100 + X_B \cdot 0 = 10 \\ X_A \cdot 0 + X_B \cdot 100 = 110 \end{cases} \Rightarrow \begin{cases} X_A = \dfrac{10}{100} = 0.1 \\ X_B = \dfrac{110}{100} = 1.1 \end{cases}$$

- Market value of this synthetic bond:

$$P_c^S = X_A P_A + X_B P_B = 0.1 * 90.91 + 1.1 * 79.72 \approx 96.78 = P_c^{Implied}$$

Example: Exploit the Mispricing

- **Arbitrage trades:** Sell the synthetic bond (ie 0.1 Bond A + 1.1 Bond B) and buy the real bond C
- Cash flows of the arbitrage trade are as follows:

$$CF_0 = (X_A P_A + X_B P_B) - P_C = (0.1 \cdot 90.91 + 1.1 \cdot 79.72) - 95.78 \approx 96.78 - 95.78 = 1$$

$$CF_1 = -X_A CF_1^A + CF_1^C = -0.1 \cdot 100 + 10 = 0$$

$$CF_2 = -X_B CF_2^B + CF_2^C = -1.1 \cdot 100 + 110 = 0$$

- Arbitrageurs receive a positive cash flow at time 0 (of $1), but zero cash flows in the future. They will make a $1 arbitrage profit

Achieving Equilibrium

- Arbitrageurs can scale these trades up, and will continue to do so as long as prices do not converge
  - As these trades have no net cash outlay, they can theoretically be scaled up to infinity (ignoring transaction costs)
  - Therefore a small number of traders executing large scale arbitrage can theoretically restore a whole market to equilibrium
- Supply and demand from arbitrageurs will push bond prices to their no-arbitrage values
- Depending on the nature of the mispricing, the prices of bonds A, B and C may all change, so that in equilibrium:

$$P_C = P_C^S$$

That is, Bond C trades at the price implied by bonds A and B

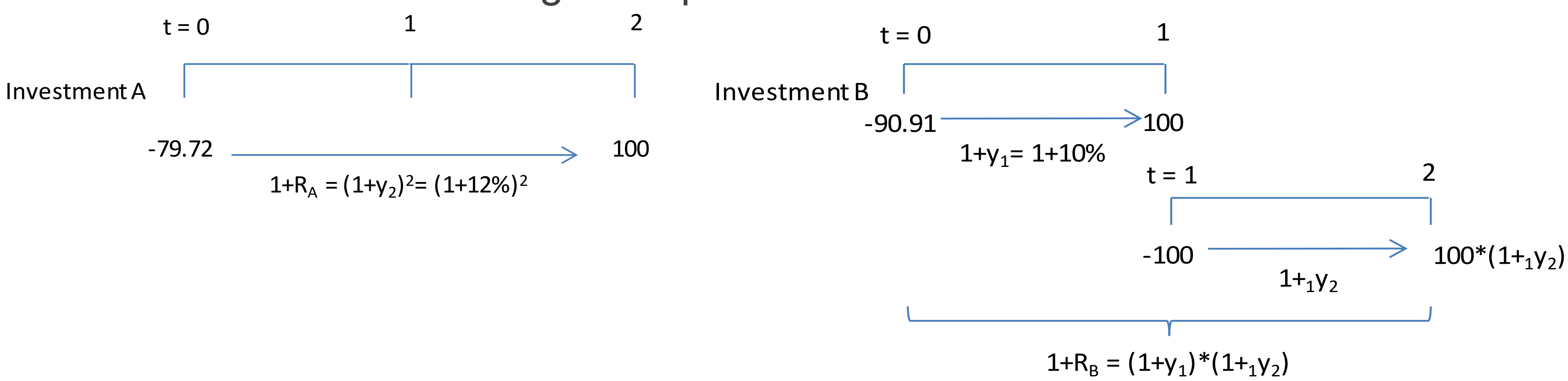
Future Interest Rates

- When given a series of spot rates, we can use them to derive interest rates predicted in the future
  - For example y_1 and y_2 are spot rates for 1-year and 2-year investments respectively
  - Therefore, from y_1 and y_2 we can infer the interest rate for an investment starting from the end of year 1 and ending at the end of year 2
- For further detail on pricing a bond, bootstrapping, and deriving future interest rates off a pure yield curve see file "L2 Yield Curve"

BKM 15.2

Future Interest Rates

- We can replicate the cash flows of long-term bonds by re-investing the cash flows from short-term bonds
- Suppose we want to invest for 2 years. We have two strategies:
  - Buy and hold a 2-year zero (A) **OR**
  - Buy a 1-year bond and roll it over into a new 1-year bond at maturity (B)
  - Under equilibrium (no-arbitrage) both strategies give the same return
- Consider the following example:



BKM 15.2

Future Interest Rates Under Certainty

- Please note the notation:
  - ${}_0y_t$ (or simply y_t) is the spot rate at time 0 for a *t*-period bond (maturing at t)
  - ${}_1y_t$ is the spot rate at time 1 for a (*t-1*)-period bond
- If everything is certain (ie ${}_1y_2$ is known for certain at time 0), the two investment returns should be equal:

$$1+R_A = (1 + y_2)^2 = 1+R_B = (1 + y_1)^*(1 + {}_1y_2)$$

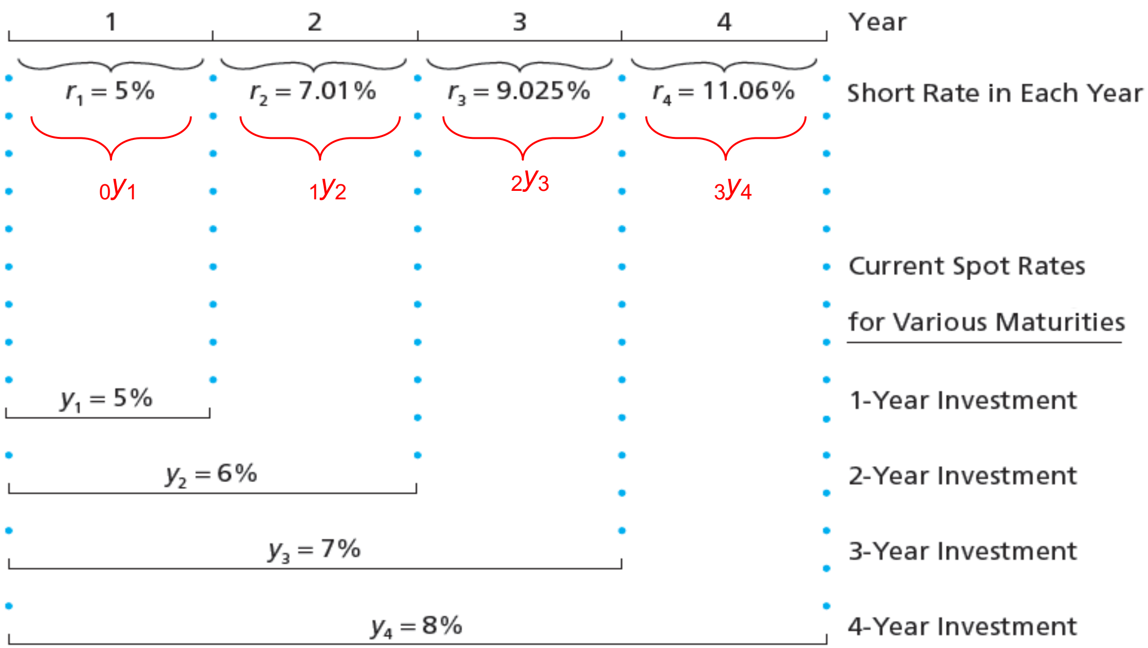
Rearranging: $${}_1y_2 = (1 + y_2)^2 / (1 + y_1) - 1$$

- However, under uncertainty we will not know what ${}_1y_2$ will be until time 1
  - Therefore it is more likely the actual returns from investments A and B will not be equal (≠)
- Note that under certainty BKM refers to ${}_1y_2$ as the **short rate**, denoted as r_2

BKM 15.2

Spot Rates vs Short Rates

- Please note that under certainty, BKM refers to ${}_0y_1$ as the short rate r_1, similarly ${}_1y_2$ would be referred to as the short rate r_2 etc
  - As we saw on the previous slide, the spot rate is the geometric average of its component short rates



- For example: Investing in a 3 year zero at 7% YTM provides the same proceeds (under certainty) as rolling over 1 year zeros at ${}_0y_1$ = 5%, ${}_1y_2$ = 7.01% and ${}_2y_3$ = 9.025%

BKM 15.2

Forward Rates

- Under **uncertainty**, we can still infer the **expected** future interest rates
  - These expected interest rates are known as **Forward Rates**
  - Forward rates are agreed upon interest rates commencing at some point in the future for a defined future time period
- Forward rates are interest rates agreed today but take place in the future
  - The forward rate (determined today) for an investment that starts at time *s* and ends at time *t* is denoted ${}_sf_t$
  - For example, a bank could offer you a loan commencing at time 1 for 1 year at the forward interest rate of ${}_1f_2$
- Using a similar methodology to the one we used to derive the future 1 year spot rate in 1 year's time ${}_1y_2$ (analogous to the short rate r_2 from BKM) we can see the 1 year forward rate in one year is given by:

$${}_1f_2 = (1 + y_2)^2 / (1 + y_1) - 1$$

BKM 15.3

Forward Rates

- How about the 1 year forward rate commencing in year 2, ${}_2f_3$?

$$ {}_2f_3 = (1 + y_3)^3 / (1 + y_2)^2 - 1 $$

  - Hint: derived by replicating cash flows of a 3-year zero by using a 2-year zero and reinvesting at the forward rate between year 2 and 3

- How about the forward rate for any future period t, ${}_{t-1}f_t$?

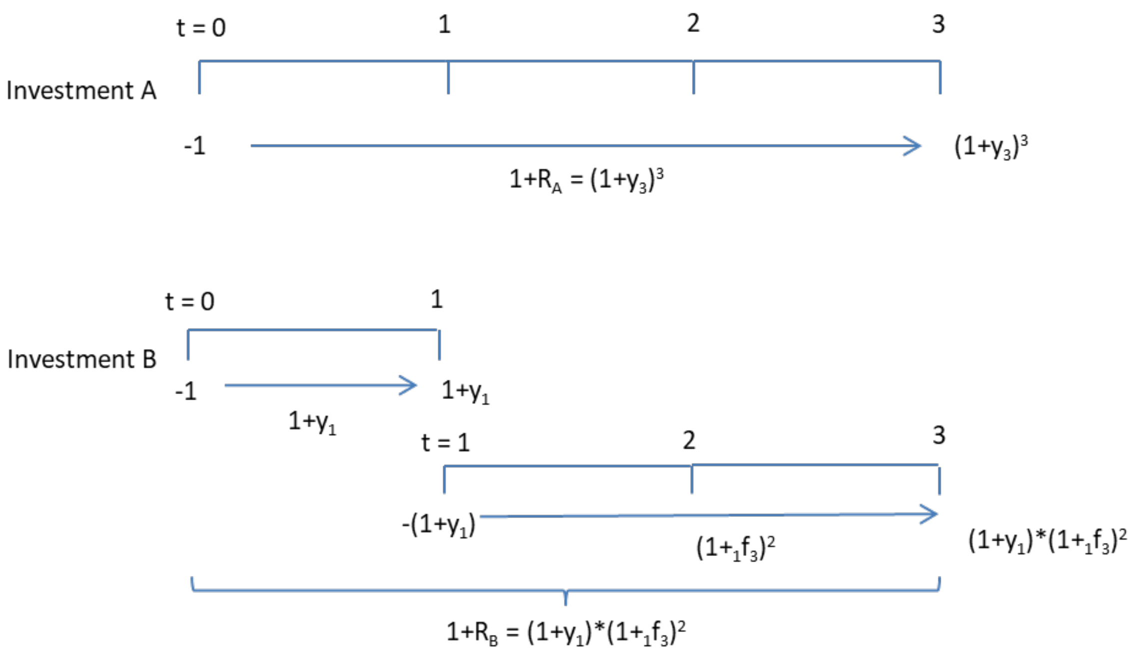
$$ {}_{t-1}f_t = \frac{(1+y_t)^t}{(1+y_{t-1})^{t-1}} - 1 $$

  - Hint: derived by replicating cash flows of a *t*-year zero using a (*t* -1)-year zero and reinvesting at the forward rate between year *t* -1 and *t*

BKM 15.3

Multi-Period Forward Rates

- How about the 2 year forward rate commencing in year 1, ${}_1f_3$?
  - Again, let's try to replicate the cash flows of a 3-year zero with a 1-year zero reinvested at the forward rate between year 1 and 3



BKM 15.3

Multi-Period Forward Rates

- The cash flows from both investments can be fixed today, and so under the no-arbitrage condition the returns from both strategies should be equal:

$$1 + R_A = (1 + y_3)^3 = 1 + R_B = (1 + y_1)^*(1 + {}_1f_3)^2$$

Rearranging: $${}_1f_3 = [(1 + y_3)^3 / (1 + y_1)]^{1/2} - 1$$

- Now let's generalise: to derive the forward rate for an investment that commences at any future time *s* and ends at any time *t* (and surely *t* > *s*), ${}_sf_t$ we can use the following generalised formula:

$${}_sf_t = \left[\frac{(1+y_t)^t}{(1+y_s)^s}\right]^{1/(t-s)} - 1$$

BKM 15.3

UNSW | AGSM Business School

Expectations Hypothesis

- We showed previously that under certainty, we can exactly replicate cash flows of long-term bonds by reinvesting cash flows from short-term bonds:

$$(1 + y_2)^2 = (1 + y_1)^*(1 + {}_1y_2)$$

  - However, in reality ${}_1y_2$ is not known with certainty at t = 0
  - The final realized cash flow of the 2-year zero (the LHS) is known at time 0. But the final realized cash flow of the 1 year rollover strategy (the RHS) is not known with certainty at time 0

- However the market has an **expectation** that the (certain) cash flow from the 2-year zero will equal the expected cash flow from the 1-year rollover. That is:

$$(1 + y_2)^2 = (1 + y_1)^*(1 + E({}_1y_2))$$

- This is known as the **Expectations Hypothesis**

* Note BKM would describe $E({}_1y_2)$ as the expected short rate, denoted $E(r_2)$

BKM 15.4

Expectations Hypothesis

- Since forward rates are inferred from the term structure, a common way to denote the expectations hypothesis EH is:

$$ {}_{s}f_{t} = E({}_{s}y_{t}) $$

We know from the earlier generalised forward rate equation that:

$$ (1+y_t)^t = [(1+y_s)^s * (1+{}_{s}f_t)^{(t-s)}] $$

Therefore under EH: $(1+y_t)^t = [(1+y_s)^s * (1+E({}_{s}y_t))^{(t-s)}]$

- However, the yield curve is typically upward sloping
  - If EH were to hold, this means the market expects actual interest rates in the future to be higher than today almost all the time
  - Perhaps there is some other explanation for the typically upward sloping yield curve

BKM 15.4

Liquidity Preference Hypothesis

- Suppose we have an investment horizon of **1-year**, we can either:
  - Strategy 1: buy a **1-year zero**
  - Strategy 2: buy a **2-year zero, and sell** it in 1 year's time (at t=1)

  Our Holding Period Returns (HPRs) from the two strategies would be:
  - Strategy 1: HPR is certain to be y_1
  - Strategy 2: HPR is uncertain because ${}_1y_2$, and therefore the price we can sell the bond for (P_1) in one year (at t=1), is uncertain today (at t=0)

- Now suppose we have an investment horizon of **2-years**, we can either:
  - Strategy 1: buy a **1-year zero, and reinvest** in a 1-year zero in 1 year's time
  - Strategy 2: buy a **2-year zero**

  Our HPRs from the two strategies would be:
  - Strategy 1: HPR is uncertain because ${}_1y_2$, and therefore the price of a 1-year zero in 1 years time (ie the bond we reinvest in) is uncertain today
  - Strategy 2: HPR is certain to be $(1 + y_2)^2 - 1$

BKM 15.4

Liquidity Preference Hypothesis

- In the first example, for investors with shorter term horizons, Strategy 2 (the buy and sell strategy) was riskier. We call this **Liquidity Risk**
- In the second example, for investors with longer term horizons, Strategy 1 (the reinvestment strategy) was riskier. We call this **Reinvestment Risk**
- The **Liquidity preference hypothesis LPH** refers to the case where most investors in the market have shorter horizons and therefore prefer short-term investments to avoid the liquidity risk of holding long-term bonds
  - Investors with shorter term horizons will therefore require a risk premium – called the **Liquidity Premium** - to hold a longer-term bond. Thus:

$$ {}_s f_t > E({}_s y_t) $$

OR

$$ {}_s f_t = E({}_s y_t) + L $$

(note $E({}_s y_t)$ is denoted as $E(r_n)$ – the expected "short rate" in BKM)

Where L = Liquidity Premium

BKM 15.4

UNSW | AGSM Business School

Summary of Theories

EH

- ${}_sf_t = E({}_sy_t)$
- Forward rates are market expectations of future spot rates
- Investors with long-term horizons offset liquidity premium requirements of investors with short-term horizons
- Does not explain typically upwards sloping yield curve

LPH

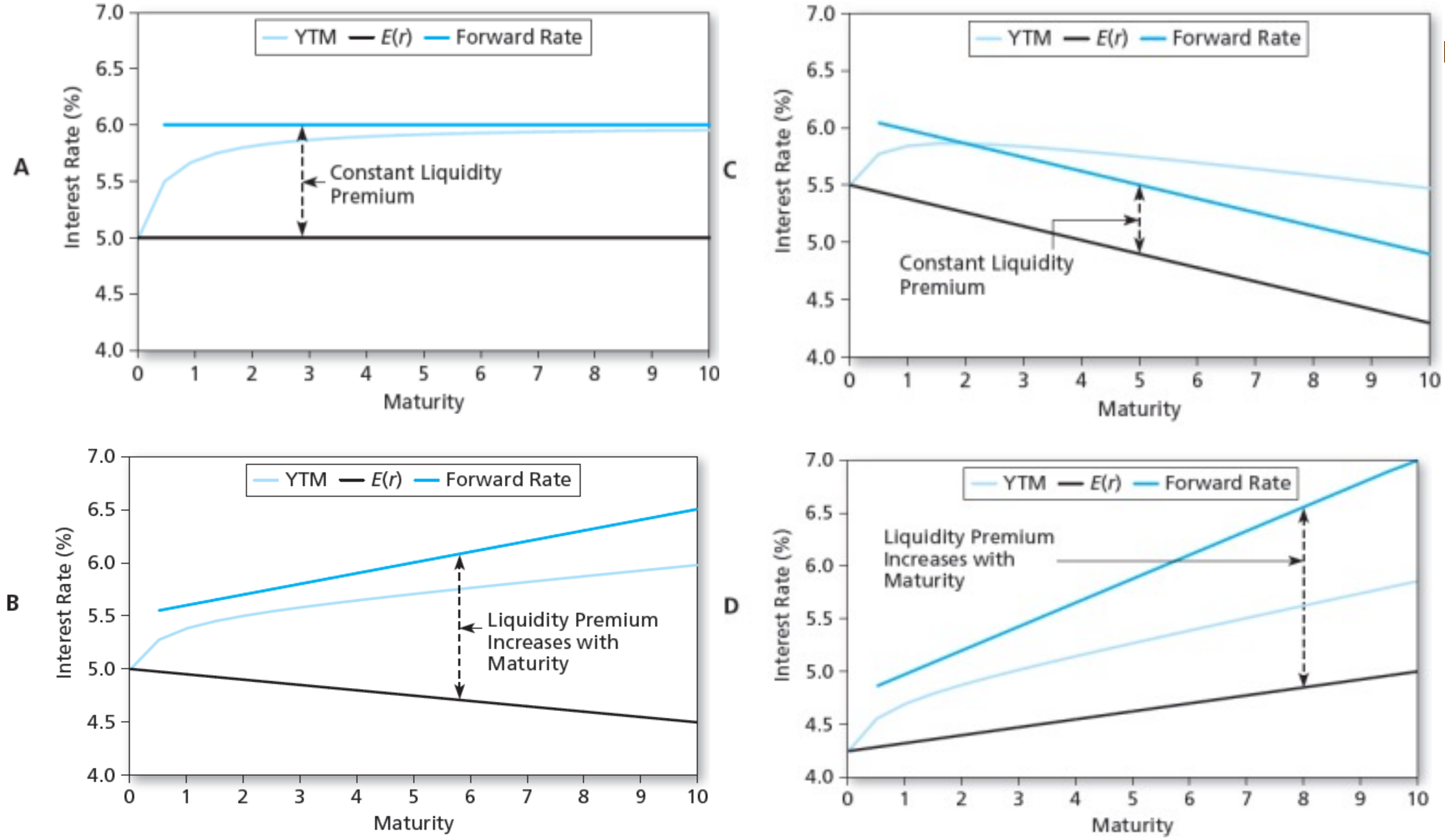
- ${}_sf_t = E({}_sy_t) + L \quad (L>0)$
- Forward rates are market expectations of future spot rates plus a liquidity premium
- Investors with short-term horizons dominate the market requiring a liquidity premium to avert liquidity risk
- Explains the typically upward sloping yield curve

Reinvestment Risk

- ${}_sf_t = E({}_sy_t) + L \quad (L<0)$
- Forward rates are market expectations of future spot rates less a discount for short-term investments
- Investors with long-term horizons dominate requiring a discount for short-term investments to avert reinvestment risk
- Explains a downward sloping yield curve

BKM 15.5

Liquidity Preference and the Yield Curve



- If there is a liquidity premium the yield curve is biased upward, particularly if the premium increases with maturity

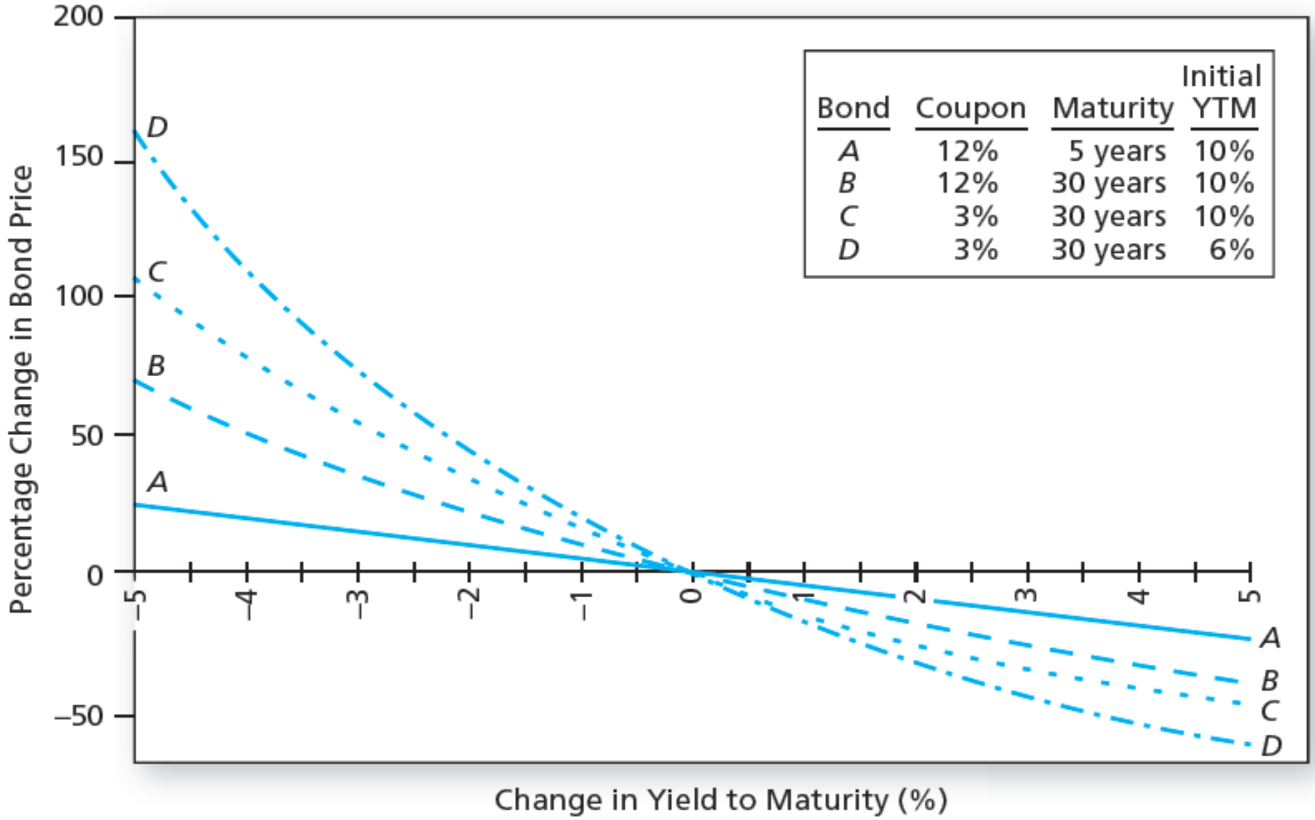
BKM 15.4

Interest Rate Risk

- Interest Rate Risk refers to how bond prices change in response to interest rate movements. This **interest rate sensitivity** has a number of aspects:
  1. Bond prices and yields are inversely related **P ↑ YTM ↓**
  2. Bonds are more (upwardly) price sensitive to interest rate falls than (downwardly) price sensitive to interest rate increases (**convexity**)
  3. Long-term bonds are more price sensitive than short-term bonds to a change in interest rates ie **ΔP ↑ T ↑**
  4. As maturity increases, price sensitivity increases at a decreasing rate
  5. Low coupon bonds are more price sensitive than high coupon bonds to a change in interest rates ie **ΔP ↑ C ↓**
  6. Bonds with lower YTM are more price sensitive than bonds with higher YTM to a change in interest rates ie **ΔP ↑ YTM ↓**

BKM 16.1

Bond Price Interest Rate Sensitivity



Bond	Coupon	Maturity	Initial YTM
A	12%	5 years	10%
B	12%	30 years	10%
C	3%	30 years	10%
D	3%	30 years	6%

1. Downward sloping left to right
2. Convex
3. B more sensitive than A
4. BKM Table 16.1
5. C more sensitive than B
6. D more sensitive than C

BKM 16.1

Duration

- The "effective" maturity of a bond is known as its **Duration**
- Duration is a measure of the weighted average time that cash flows (CF) on a bond are received:
  - In other words, it measures "effective" maturity by taking into account when the payments on a bond are actually made
  - As all cash flow on a zero-coupon bond is back-ended, it is always the case that duration = maturity for zero coupon bonds
  - For coupon bonds, it is always the case that duration < maturity
- As we will see, duration is a key concept for at least 3 reasons:
  1. A measure of the effective maturity (or "payback") of a bond
  2. A measure of the interest rate sensitivity of a portfolio
  3. An essential tool in immunizing portfolios from interest rate risk

BKM 16.1

UNSW AGSM Business School

Duration Definition

- Duration is a measure of interest rate risk - it is the sensitivity or change in the market value of a bond to a change in interest rates given by:

$$D = -\frac{\partial P / P}{\partial y / (1 + y)}$$

- Elasticity of bond value to (gross) yield: Given a 1% relative change in the yield (change in yield (∂y) divided by the level of yield ($1+y$)), what is the relative change in the bond value ($\partial P/P$) ie % the return on the bond?
- If you are not comfortable with differential calculus, simply substitute in change (Δ) (ie simply view ∂ as Δ – the change). That is, duration is a measure of the change in bond price to a change in yield
- Also referred to as *Macaulay duration* (in contrast with *modified duration* to be discussed later)

BKM 16.1

Duration Calculation

- Duration calculation:

$$D = \sum_{t=1}^{T} t \times w_t$$

Where w_t is equal to:

$$w_t = \frac{CF_t / (1+y)^t}{\text{Price}}$$

- Steps for calculating duration (see file "L2 Duration and Convexity") are:
  1. Calculate CF's on the bond
  2. Discount each CF by the YTM to derive PV of each CF (the sum of the CF PV's is the Bond Price)
  3. Divide each period's CF PV by Bond Price to derive weights
  4. Multiply each weight by the time period (ie 1, 2, 3…n)
  5. Sum the total

BKM 16.1

Duration Determinants

① *Rule 1*

- The duration of a **zero-coupon** bond equals its time to maturity

② *Rule 2*

- All else being equal, a bond's duration is *higher* when the **coupon rate** is *lower* **D ↑ C ↓**

③ *Rule 3*

- All else being equal, a bond's duration generally *increases* with its **time to maturity D ↑ T ↑** (note that it *increases* at a *decreasing* rate)

④ *Rule 4*

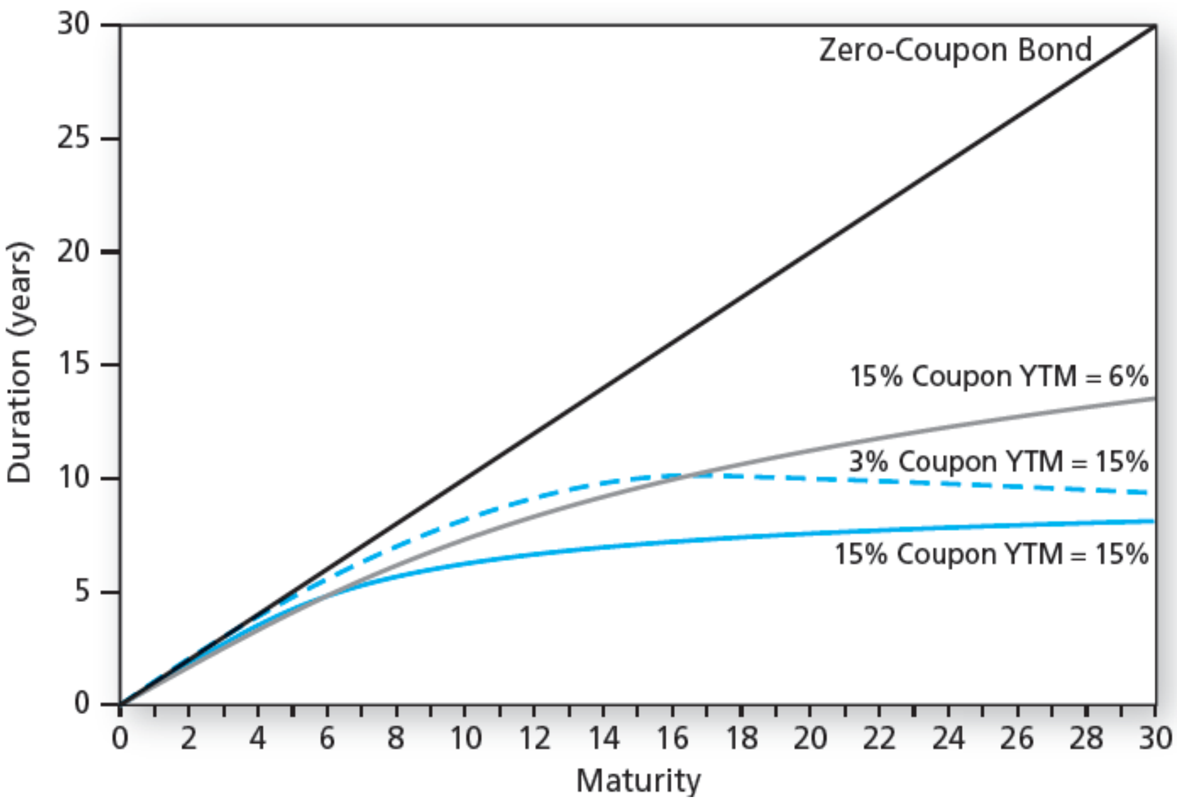
- All else being equal, the duration of a coupon bond is *higher* when the bond's **yield to maturity** is *lower* **D ↑ YTM ↓**

⑤ *Rule 5*

- The duration of a **perpetuity** is equal to: $(1 + y) / y$

BKM 16.1

Duration Determinants



① Zero coupon duration equals maturity (45 degree and linear)

② Low coupon bond ( ) has higher Duration than high coupon bond ( )

③ All curves slope upwards left to right

④ Low YTM bond ( ) has higher Duration than high YTM bond ( )

BKM 16.1

UNSW | AGSM Business School

Duration Determinants: Coupon & YTM

- What is the conceptual logic for the inverse relationship between duration and coupon rate?
  - The higher the coupon, the higher the weights put on those cash flows received before maturity relative to the principal repayment at maturity
  - This "front-ends" the return and so shortens the payback on the bond (shortening its effective maturity)

- What is the conceptual logic for the inverse relationship between duration and YTM?
  - If YTM decreases, the PV of all CFs increases, but the value of CFs in later periods increases relatively more (stretching its effective maturity)
  - If YTM increases, the PV of all CFs decreases, but the value of CFs in later periods decreases relatively more (shortening its effective maturity)

BKM 16.1

Duration as a Measure of IR Risk

- Duration can be used to determine the bond price sensitivity to changes in the YTM
- **Duration-Price Relationship**
  - Price change is proportional to duration (not to maturity):

$$\frac{\Delta P}{P} = -D\left[\frac{\Delta(1+y)}{1+y}\right]$$

  - If we define **Modified Duration D*** as: **D* = D / (1 + *y*)**

    We can simply write the above equation as follows:

$$\frac{\Delta P}{P} = -D^{*}\,\Delta y$$

BKM 16.1

Duration-Price Relationship Example

- Two bonds have duration of 4.2814 years
  - One is a 5-year, 8% coupon bond with YTM=10%
  - Second is a zero-coupon bond with maturity 4.2814 years YTM=10%
- Therefore both bonds have the same modified duration
  - Modified D^* = 4.2814/(1 + 0.1) = 3.892
  - Suppose the interest rate increases by 0.1%. Bond prices fall by:

$$\frac{\Delta P}{P} \approx -D^* \Delta y$$

$$\approx -3.892 \times 0.1\%$$

$$\approx -0.3892\%$$

- Bonds with equal *D* have the same interest rate sensitivity
- Buy why the $\approx$? Why is this relationship just an approximation?

BKM 16.1

Convexity

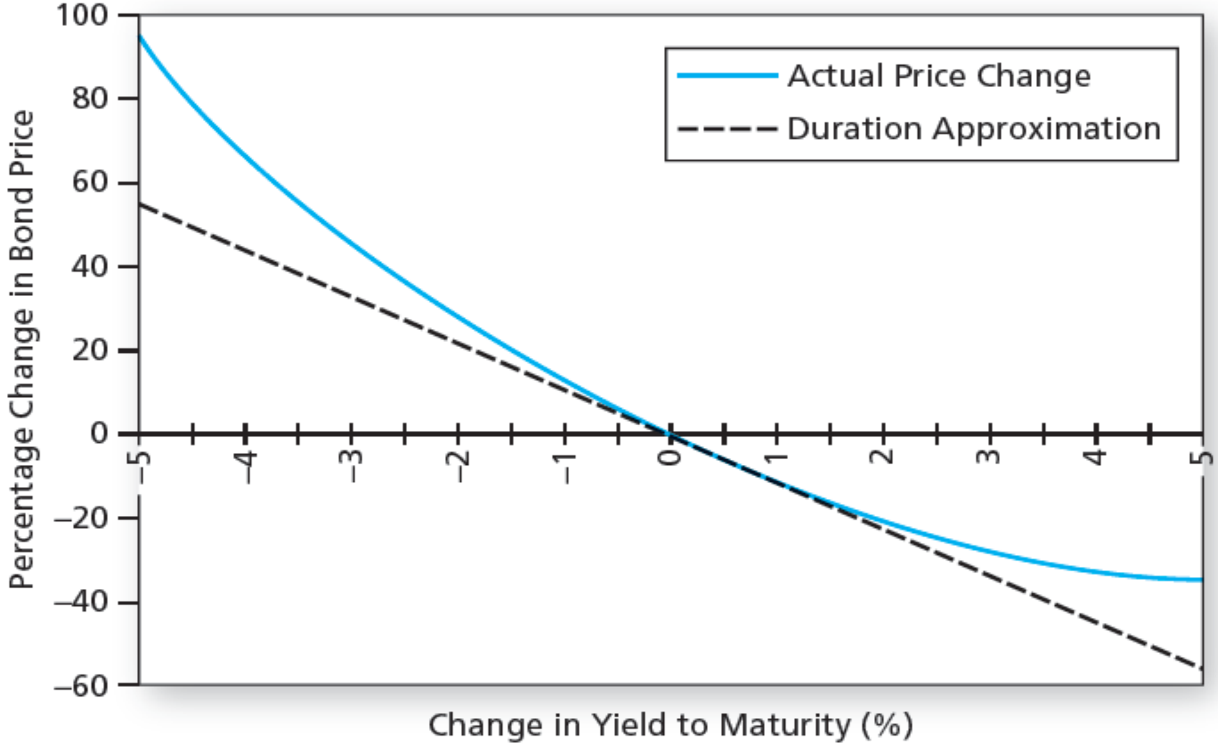
- The relationship between bond prices and yields is not linear
- However the duration-price relationship below implies it is linear:

$$\frac{\Delta P}{P} = -D^* \Delta y$$

- Duration (at a specific yield point) is essentially the slope of the price-yield curve only at that point
- Therefore the duration-price relationship (duration rule) above is a good approximation for only small changes in bond yields
- Bonds with greater convexity have more curvature in the price-yield relationship and the duration rule is a less accurate approximation of interest rate sensitivity

BKM 16.2

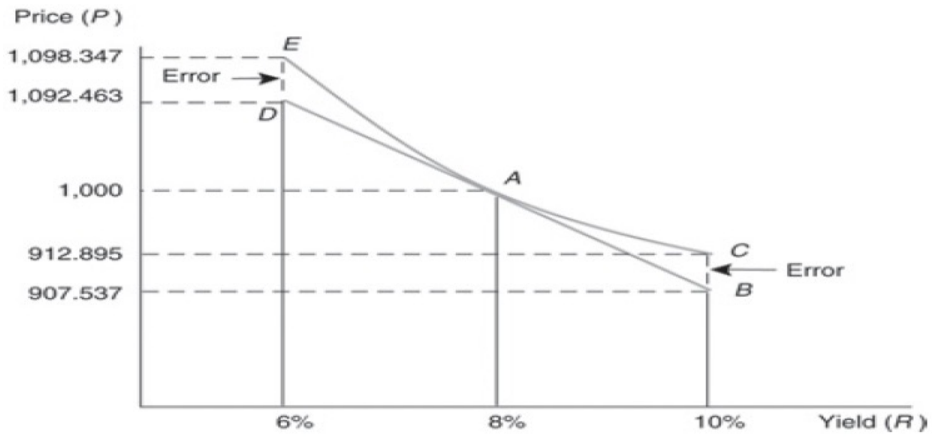
Duration Approximation with Convexity

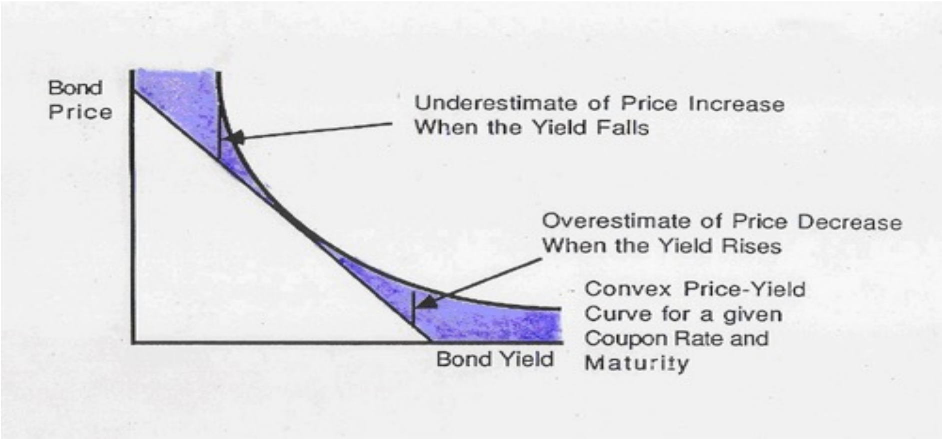


- The (approximate) price change calculated based on the duration rule is always lower than the actual price changes
- It overestimates price decrease for interest rate increases
- It underestimates price increase for interest rate decreases

BKM 16.2

Approximation Error





- When yield increases (and thus price would decrease), duration will decrease gradually. Thus the estimated price change using the original duration overestimates the loss
- When yield decreases (and thus price would increase), duration will increase gradually. Thus the estimated price change using the original duration underestimates the gain

BKM 16.2

Convexity Formula

- Convexity of the price-yield relationship can be calculated by:

$$Convexity = \frac{1}{P \times (1+y)^2} \sum_{t=1}^{n} \left[\frac{CF_t}{(1+y)^t} (t^2 + t) \right]$$

- We can the correct for convexity by making the following adjustment:

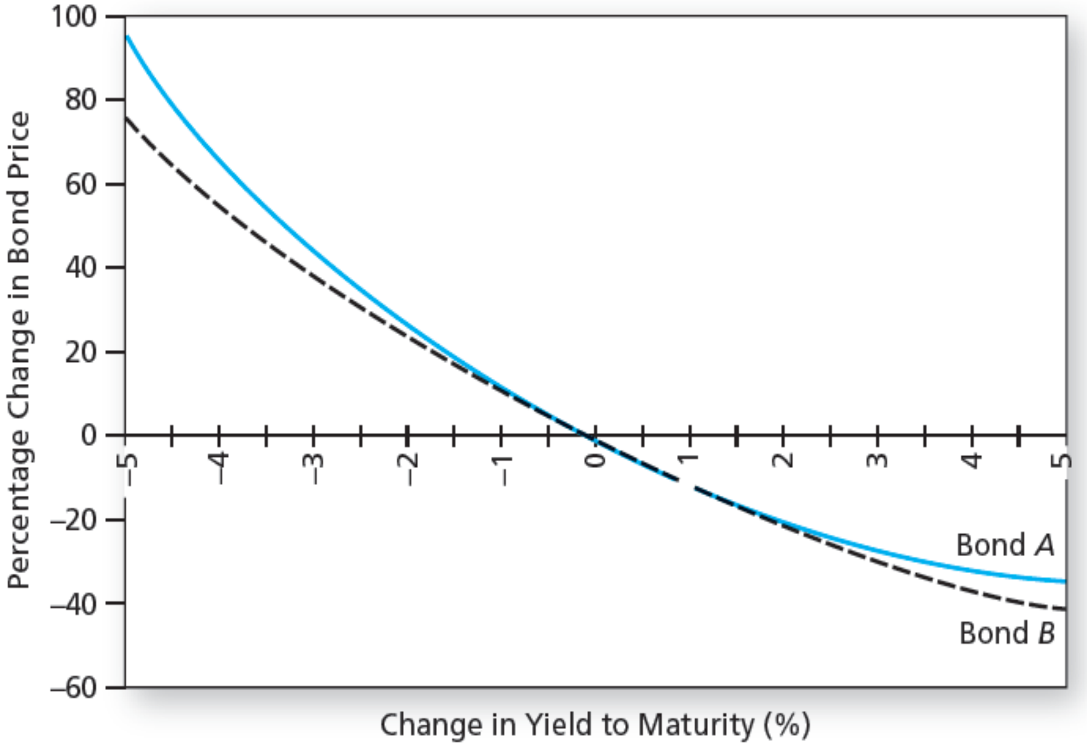
$$\frac{\Delta P}{P} = -D^* \, \Delta y \; + \; ½ \, \text{Convexity} \, (\Delta y)^2$$

- See file “L2 Duration and Convexity” for an example of how convexity is calculated

- We use this formula for larger interest rate changes. However note that even this convexity adjustment is an approximation (though much more accurate)

BKM 16.2

Why do Investors Like Convexity?



- Compare these two bonds
- Duration rule approximates Bond B more accurately than Bond A
- Bonds with greater convexity (A) gain more in price when yields fall than they lose in price when yields rise compared to less convex bonds (B)
- Due to this asymmetry, A typically trades at a premium to B (all else being equal)

BKM 16.2

Portfolio Duration

- We can think of a coupon bond as a portfolio of zero-coupon bonds
  - Then the duration of this coupon bond is just the weighted average of durations of the component zeros
- Similarly, a portfolio of bonds can be viewed as an aggregate bond
  - In this case, the CF of this aggregate bond is the sum of CF's of its individual component bonds
- So duration of this portfolio of bonds should be the weighted average of the durations of its components:

$$D_{port} = \sum_{i=1}^{N} D_i * w_i$$

  - The weights are based on the relative value of each bond in the portfolio

$$w_i = P_i / [\sum_{i=1}^{N} P_i] = P_i / P_{port}$$

BKM 16.3

UNSW | AGSM Business School

Asset-Liability Matching

- Say we have a known future liability to meet
  - One way to make sure we have sufficient funds to cover the future liability is to buy a zero-coupon bond that pays exactly the amount of the liability when it is due
  - Most of the time this zero-coupon bond will be difficult to find in the market

- Instead we may have to buy coupon bonds (whose maturity may match up to when the liability is due). Now we are exposed to interest rate risk:
  - **Higher** interest rates are beneficial if we have taken on **reinvestment risk**, since you can reinvest at better rates
  - **Lower** interest rates are beneficial if we have taken on **liquidity risk** (or *price risk* as labelled in BKM), since we can sell bonds at a higher price

BKM 16.3

Immunisation

- As we saw, interest rate risk on bond portfolios has two off-setting risk components as interest rates increase:
  - **Liquidity (Price) risk** – bond prices fall as interest rates increase
  - **Re-investment risk** – coupons are re-invested at higher rates as interest rates increase
- We can construct a bond portfolio that offsets these effects:
  - We **immunize** a portfolio by matching the interest rate risk of assets and liabilities
  - **Immunization** is a method of controlling interest rate risk which is widely used by pension funds, insurance companies, and banks
  - Liquidity (price) risk and reinvestment rate risk exactly cancel out
- **Result:** The value of assets will track the value of liabilities whether rates rise or fall

BKM 16.3

Immunisation

- **We immunize by matching the duration of our assets and liabilities**
- If we set the duration of a bond portfolio equal to a targeted horizon (equal to our liability), then the bond portfolio's value at the target horizon is fixed
  - Reinvestment risk and liquidity (price) risk will offset each other
- **Example:** Insurance company has a liability in 5 years of a $10,000 guaranteed investment contract (note: $10,000 in todays dollars) with a fixed interest rate of 8%. The liability is worth $14,693.28 in 5 years
  - To fund this liability, the insurance company purchases 10 $1,000 6-year bonds with 8% coupon (this investment is the off-setting asset)
  - The current market interest rate is 8%
  - Regardless of whether the interest rate were to fall to 7% or increase to 9%, the asset value in 5 years covers the liability of $14,693.28. **Why?**
  - **Because duration of the asset and liability are matched**

BKM 16.3

Immunisation

Payment Number	Years Remaining until Obligation	Accumulated Value of Invested Payment		
A. Rates remain at 8%				
1	4	$800 \times (1.08)^4$	=	1,088.39
2	3	$800 \times (1.08)^3$	=	1,007.77
3	2	$800 \times (1.08)^2$	=	933.12
4	1	$800 \times (1.08)^1$	=	864.00
5	0	$800 \times (1.08)^0$	=	800.00
Sale of bond	0	10,800/1.08	=	10,000.00
				14,693.28
B. Rates fall to 7%				
1	4	$800 \times (1.07)^4$	=	1,048.64
2	3	$800 \times (1.07)^3$	=	980.03
3	2	$800 \times (1.07)^2$	=	915.92
4	1	$800 \times (1.07)^1$	=	856.00
5	0	$800 \times (1.07)^0$	=	800.00
Sale of bond	0	10,800/1.07	=	10,093.46
				14,694.05
C. Rates increase to 9%				
1	4	$800 \times (1.09)^4$	=	1,129.27
2	3	$800 \times (1.09)^3$	=	1,036.02
3	2	$800 \times (1.09)^2$	=	950.48
4	1	$800 \times (1.09)^1$	=	872.00
5	0	$800 \times (1.09)^0$	=	800.00
Sale of bond	0	10,800/1.09	=	9,908.26
				14,696.02

- If we match duration of assets and liabilities, the value changes on both sides (due to interest rate changes) will offset each other

Re-investment risk

Offset by Liquidity (price) risk

Offset by Re-investment risk

Liquidity (Price) risk

BKM 16.3

Immunisation Procedure

- Given a liability with duration D_L and two bonds with durations D_A and D_B
  - Choose portfolio weights X_A and (1 - X_A) such that asset portfolio duration D_P is:

$$D_P = X_A \cdot D_A + (1 - X_A)D_B = D_L$$

- Chose the size of the investment so that the PV of the asset portfolio is equal to the PV of the liability

- If we invested some fraction X_A in bond A and the rest, (1 – X_A), in bond B, we can derive X_A using some simple algebra as follows:

$$D_P = X_A \cdot D_A + (1 - X_A)D_B = D_L$$

$$X_A \cdot D_A + D_B - X_A \cdot D_B = D_L$$

$$X_A = \frac{D_L - D_B}{D_A - D_B}$$

BKM 16.3

Immunisation Example

- X Corp has to pay \$1 million in 10 years
- It has an asset portfolio of two annual bonds:

Bonds	Coupon	Maturity	Price
Bond A	6%	30 yrs	\$691.79
Bond B	11%	10 yrs	1128.35

- Bond A has liquidity and reinvestment risk, Bond B has reinvestment risk
- The first step is to derive the YTM and durations for the bonds (see file "L2 Immunization" Worksheet: Annual Coupon Bonds)
  - $y_A = y_B$ = 9%, and D_A = 11.88 and D_B = 6.75
- What is the PV of the liability at the market interest rate of 9% =\$422,411
  - The amount we invest in the two bonds today - $V_A + V_B$ - must equal the PV of the liability. Therefore: $V_A + V_B$ = PV(\$1m@9%) = \$422,411

BKM 16.3

UNSW | AGSM Business School

Immunisation Example

- Duration matching
  - $(X_A)11.88 + (X_B)6.75 = 10$ or $(X_A)11.88 + (1 - X_A)6.75 = 10$
  - $X_A = (D_L - D_B) / (D_A - D_B)$
  - $X_A = .6337$ and $X_B = .3663$ (ie invest 63.4% in A and 36.6% in B)
  - V_A = $267,694 and V_B = $154,717 ($267,694 + $154,717 = $422,411)
- Buy 387 (267,694/691.79) of Bond A and 137 (154,717/1128.35) of Bond B
- The table below from file "L2 Immunization" shows that the liability is covered at different interest rates:

Interest Rate	8%	9%	10%
Bond A Price	774.84	691.79	622.92
Bond A Value	299,832	267,694	241,045
Bond B Price	1,201.30	1,128.35	1,061.45
Bond B Value	164,719	154,717	145,543
PV of Asset	464,552	422,411	386,588
PV of Obligation	463,193	422,411	385,543
Asset Value Year 10	1,002,933	1,000,000	1,002,710
Obligation Year 10	1,000,000	1,000,000	1,000,000
Difference	**2,932.57**	**0.00**	**2,709.69**

BKM 16.3

Rebalancing & Dynamic Immunisation

- Notice that in the previous example, the asset values are close but don't always exactly match the liability value if interest rates change?
- The portfolio weights are chosen to equalize the durations given the yield and maturities that we see today
  - These parameters will change constantly
  - To keep the portfolio immune to interest rate changes, we must constantly recalculate the appropriate weights and adjust our portfolio accordingly
- So immunization is a dynamic process
- For an example of rebalancing, see BKM Examples 16.4 and 16.5 on pg 520-521

BKM 16.3

Next Lecture

- **Markowitz Portfolio Theory**

Key Concepts

- Measuring expected return and risk
- Risk and risk aversion
- Preference and utility – utility function and indifference curves
- Forming portfolios
- Diversification

Readings

- BKM 5 Risk, Return and the Historical Record: 5.4 – 5.5
- BKM 6 Capital Allocation to Risky Assets: 6.1, 6.5
- BKM 7 Optimal Risky Portfolios: 7.1 – 7.2, 7.4, Appendix B