HOMEWORK 5 - CS323

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Problem 1.

Express the following polynomial in the correct form for evaluation by Horner's method:

$$p(t) = 5t^3 - 3t^2 + 7t - 2$$

$p(t) = 5t^3 - 3t^2 + 7t - 2$
$= t(5t^2 - 3t + 7) - 2$
$= t(t(5t - 3) + 7) - 2$

Problem 2.

How many multiplications are required to evaluate a polynomial $p(t)$ of degree $n-1$ at a given point t

a. Represented in the monomial basis?
b. Represented in the Lagrange basis?
c. Represented in the Newton basis?

a.
$n^3 + (n-1)$ multiplications $\implies$ the polynomial coefficients can be solved in n^3, then using horners method to evaluate requires $n-1$ multiplications. If coefficients are known, only $n-1$.

b.
$(2(n-2)+1)(n-1)$ multiplications $\implies$ no need to find the coefficients, thus for $n-1$ terms, each li we have $2(n-2)$ plus 1 for the division. This computational cost is proportional to only n muliplications for each $li(x)$ or $O(n^2)$ total

c.
$n^2 + (n-1)$ multiplications $\implies$ the polynomial coefficients can be solved in n^2 multiplications using forward substitution, then using horners method to evaluate requires $n-1$ multiplications. If coefficients are known, onlt $n-1$.

Problem 3.

a. Determine the polynomial interpolant to the data

t	1	2	3	4
$p(t)$	11	29	65	125

using the monomial basis.

b. Determine the Lagrange polynomial interpolant to the same data and show that the resulting polynomial is equivalent to that obtained in part (a).

c. Compute the Newton polynomial interpolant to the same data using each of the three methods discussed in class (triangular matrix, incremental interpolation, and divided differences) and show that each produces the same result as the previous two methods.

a.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} * \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 29 \\ 65 \\ 125 \end{bmatrix}$$

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$$a = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

$P_3(t) = 5 + 2t + 3t^2 + t^3$

b.

$*note\,x\,should\,be\,t\,in\,the\,following..$

$P_3(t) = (11)\frac{(x-2)(x-3)(x-4)}{(1-2)(1-3)(1-4)} + (29)\frac{(x-1)(x-3)(x-4)}{(2-1)(2-3)(2-4)} + (65)\frac{(x-1)(x-2)(x-4)}{(3-1)(3-2)(3-4)} + (125)\frac{(x-1)(x-2)(x-3)}{(4-1)(4-2)(4-3)}$

$P_3(t) = 38t^3 - 217t^2 + 363t - 162 - 36t^3 + 223t^2 - 359t + 172$

$P_3(t) = 5 + 2t + 3t^2 + t^3$

c.

Triangular Matrix Method:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 \\ 1 & 3 & 6 & 6 \end{bmatrix} * \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 29 \\ 65 \\ 125 \end{bmatrix}$$

$$a = \begin{bmatrix} 11 \\ 18 \\ 9 \\ 1 \end{bmatrix}$$

$P_3(t) = 11 + 18(t-1) + 9(t-1)(t-2) + (t-1)(t-2)(t-3)$

$P_3(t) = 5 + 2t + 3t^2 + t^3$

Incremental Interpolation Method:

$P_0 = 11$

$P_1 = 11 + 18(t-1)$

$P_2 = 11 + 18(t-1) + 9(t-1)(t-2)$

$P_3 = 11 + 18(t-1) + 9(t-1)(t-2) + 1(t-1)(t-2)(t-3)$

$P_3(t) = 5 + 2t + 3t^2 + t^3$

Divided Differences Method:

$f[t_0] = p(t_0) = 11$

$f[t_1] = p(t_1) = 29$

$f[t_2] = p(t_2) = 65$

$f[t_3] = p(t_3) = 125$

$f[t_0, t_1] = 18$

$f[t_1, t_2] = 36$

$f[t_2, t_3] = 60$

$f[t_0, t_1, t_2] = 9$

$f[t_1, t_2, t_3] = 12$

$f[t_0, t_1, t_2, t_3] = 1$

$P_3(t) = f[t_0] + f[t_0, t_1](t-t_0) + f[t_0, t_1, t_2](t-t_0)(t-t_1) + f[t_0, t_1, t_2, t_3](t-t_0)(t-t_1)(t-t_2)$

$P_3(t) = 11 + 18(t-1) + 9(t-1)(t-2) + (t-1)(t-2)(t-3)$

$P_3(t) = 5 + 2t + 3t^2 + t^3$

Problem 4.

a. For a given set of data points $t_1, \ldots, t_n$, define the function $\pi(t)$ by

$$\pi(t) = (t - t_1)(t - t_2)\ldots(t - t_n)$$

Show that

$$\pi'(t_j) = (t_j - t_1)\ldots(t_j - t_{j-1})(t_j - t_{j+1})\ldots(t_j - t_n)$$

Hint: If $f_1, \ldots, f_n$ are functions of a variable t, then

$$\frac{d}{dt}\prod_{i=1}^{n} f_i = \sum_{i=1}^{n} f_i' \prod_{j\neq i} f_j$$

b. Use the result of part (a) to show that the j^{th} Lagrange basis function can be expressed as

$$l_j(t) = \frac{\pi(t)}{(t - t_j)\pi'(t_j)}$$

a.
$\pi(t) = (t - t_1)(t - t_2)...(t - t_n) = \prod_{i=1}^{n}(t - t_i)$

By definition we know $\frac{d}{dt}\prod_{i=1}^{n}\pi(t) = \sum_{i=1}^{n}\pi(t)'\prod_{j\neq i}\pi(t)$

So, $\frac{d}{dt}\prod_{i=1}^{n}(t-t_i) = \sum_{i=1}^{n}(1)\prod_{j\neq i}(t-t_j) = (t-t_1)...(t-t_{i-1})(t-t_{i+1})...(t-t_n)$

Therfore, $\pi'(t_j) = (t_j - t_1)...(t_j - t_{j-1})(t_j - t_{j+1})...(t_j - t_n)$

b.
$\frac{\pi(t)}{(t-t_j)\pi'(t_j)} = \frac{\prod_{i=1}^{n}(t-t_i)}{(t-t_j)\prod_{j\neq i}^{n}(t_i-t_j)} = \frac{\prod_{i=1}^{n}(t-t_i)}{(t-t_j)\prod_{j\neq i}^{n}(t_i-t_j)} = \frac{(t-t_j)\prod_{j\neq i}^{n}(t-t_i)}{(t-t_j)\prod_{j\neq i}^{n}(t_i-t_j)} = \frac{\prod_{j\neq i}^{n}(t-t_i)}{\prod_{j\neq i}^{n}(t_i-t_j)} = \prod_{j\neq i}^{n}\frac{(t-t_i)}{(t_i-t_j)}$