1 Question 1

Following Numerical Recipes, we will do a Von Neumann stability analysis on the leapfrog scheme for the advection equation:

$$\frac{f(t+dt,x)-f(t-dt,x)}{2dt} = -v\frac{f(t,x+dx)-f(t,x-dx)}{2dx}$$
$$f(t+dt,x) = f(t-dt,x) - a\left[f(t,x+dx)-f(t,x-dx)\right]$$

Substituting a complex exponential solution of the form: $f(x,t) = \xi^t \exp(ikx)$ gives:

$$\xi^{t+dt}e^{ikx} = \xi^{t-dt}e^{ikx} - a\,\xi^t e^{ikx}(e^{ikdx} - e^{-ikdx})$$

Dividing both sides by $\xi^t e^{ikx}$:

$$\xi^{dt} = \xi^{-dt} - a(e^{ikdx} - e^{-ikdx})$$
$$\xi^{dt} = \xi^{-dt} - 2ia\sin(kdx)$$

Multiplying by ξ^{dt} and rearranging:

$$(\xi^{dt})^2 + 2\,\xi^{dt} ia\sin(kdx) - 1 = 0$$

$$\Rightarrow \xi^{dt} = \frac{-2ia\sin(kdx) \pm \sqrt{(2ia\sin(kdx))^2+4}}{2} = -ia\sin(kdx) \pm \sqrt{1-a^2\sin^2(kdx)}$$

If $a > 1$:

$$\Rightarrow 1 - a^2\sin^2(kdx) < 0$$
$$\Rightarrow \xi^{dt} = -ia\sin(kdx) \pm i\sqrt{a^2\sin^2(kdx)-1} = i\left(a\sin(kdx) \pm \sqrt{a^2\sin^2(kdx)-1}\right)$$
$$|\xi^{dt}|^2 = \left(a\sin(kdx) \pm \sqrt{a^2\sin^2(kdx)-1}\right)^2$$
$$= a^2\sin^2(kdx) + 1 - a^2\sin^2(kdx) \pm a\sin(kdx)\sqrt{a^2\sin^2(kdx)-1}$$
$$= 1 \pm a\sin(kdx)\sqrt{a^2\sin^2(kdx)-1} = \left(|\xi(k)|^2\right)^{dt} \neq 1$$
$$\Rightarrow |\xi(k)|^2 \neq 1$$

If $a \leq 1$:

$$\Rightarrow 1 - a^2\sin^2(kdx) \geq 0$$
$$\Rightarrow \xi^{dt} = -ia\sin(kdx) \pm \sqrt{1-a^2\sin^2(kdx)}$$
$$|\xi^{dt}|^2 = a^2\sin^2(kdx) + 1 - a^2\sin^2(kdx) = 1$$
$$\Rightarrow |\xi(k)|^2 = 1$$

Since energy is proportional to $|\xi^t|^2$ (the amplitude squared of the assumed solution), energy is conserved only when $|\xi|^2 = 1$, since $1^t = 1$ for all t. Therefore energy is conserved only when the CFL condition is met: $a = v\frac{dt}{dx} \leq 1$.

2 Question 2

a)

For the relaxation method, each iteration replaces the updates the new potential as the charge density plus the average of its neighbour's old potentials. If we set $V[0,0] = \rho[0,0] = 1$, then:

$$V[0,0] = \rho[0,0] + \frac{1}{4}(V[1,0] + V[-1,0] + V[0,1] + V[0,-1])$$

$$\Rightarrow V[1,0] + V[-1,0] + V[0,1] + V[0,-1] = 0$$

But $V[1,0] = V[-1,0] = V[0,1] = V[0,-1]$:

$$\Rightarrow V[1,0] = 0$$

Since the potential at $V[1,0]$ lacks the radial symmetry like $V[0,0]$ to solve for it exactly, we will need to use a program to update the potential in iterations. Here is the code used to do so:

```
n=100
V=np.zeros([n,n])
rho=np.zeros([n,n])
rho[0,0]=1
niter=n*20

for i in range(niter):
    Vavg=1/4*(np.roll(V,1,0)+np.roll(V,-1,0)+np.roll(V,1,1)+np.roll(V,-1,1))
    Vnew=rho+Vavg
    V=Vnew
    V=V+(1-V.max())  # offset potential to keep V[0,0]=1
    if i%10==0:
        plt.clf()
        plt.imshow(np.fft.fftshift(V),cmap='Reds')  # fftshift to plot negative indices
        plt.colorbar()
        plt.pause(0.2)
    if i%50==0:
        print('After '+str(i)+' iterations',
              '\nV[1,0] =',V[1,0],
              '\nV[2,0] =',V[2,0],
              '\nV[5,0] =',V[5,0])
```

b)

Since the convolution operation is linear, it has some matrix representation. We can therefore use the conjugate gradient method to solve the matrix equation Ax = b where A is the convolution operator with the Green's function kernel acting on the charge density distribution x that gives the potential b. To solve the matrix equation using the conjugate gradient method, we don't need an explicit form for this matrix since we can just use `scipy.signal.convolve2d` any time A acts on some x.