

## 0.1 Spring 2016

1. Assume  $f \in L^1[0, 1]$ . Compute

$$\lim_{k \rightarrow \infty} \int_{[0,1]} |f|^{1/k} dx.$$

*Solution.* Let's split this integral into three regions.

$$\int_{[0,1]} |f|^{1/k} dx = \int_{f=0} |f|^{1/k} dx + \int_{0 < |f| \leq 1} |f|^{1/k} dx + \int_{|f| > 1} |f|^{1/k} dx.$$

The integral over the first region is clearly zero for all  $k$ . On the second region we have that  $|f|^{1/k} \leq 1$  for all  $k$ . Since the interval  $[0, 1]$  has finite measure, we have that the constant function 1 is in  $L^1(\{x : 0 < |f| \leq 1\})$ , so the dominated convergence theorem says that the integral over the second region goes to  $m(\{0 < |f| \leq 1\})$ . Similarly, on the third region we have that  $|f|^{1/k} \leq |f|$ , which is in  $L^1$ , so the dominated convergence theorem says that the third integral goes to  $m(\{|f| > 1\})$ . Combining these, we have that

$$\lim_{k \rightarrow \infty} \int_{[0,1]} |f|^{1/k} dx = m(\{|f| > 0\}).$$

□

2. Let  $\{f_n\}$  be a sequence of measurable functions on  $[0, 1]$  and  $0 \leq f_n \leq 1$  a.e. Assume that

$$\lim_{n \rightarrow \infty} \int_{[0,1]} f_n g dx = \int_{[0,1]} f g dx$$

for some  $f \in L^1[0, 1]$  and any  $g \in C[0, 1]$ . Prove that  $0 \leq f \leq 1$  a.e.

*Solution.* Since  $f \in L^1[0, 1]$ , by the Lebesgue differentiation theorem we have that

$$\frac{1}{m(E)} \int_E f(t) dt \rightarrow f(x) \tag{1}$$

as  $E$  shrinks to  $x$  for almost all  $x$ . Furthermore, since  $0 \leq f_n \leq 1$  we also have that

$$\frac{1}{m(E)} \int_E f_n(t) dt \rightarrow f_n(x) \in [0, 1]$$

as  $E$  shrink to  $x$  for almost all  $x$ . Intuitively, we'd like to replace the integral of  $f$  in (1) with a limit of integrals of  $f_n$ .

We claim that the function  $g$  in the given hypothesis can be replaced with the indicator function of an interval  $\chi_I$ . To see this, let  $g_m$  be a sequence of continuous functions with  $g_m \rightarrow \chi_I$  in  $L^1$  and  $0 \leq \chi_I \leq 1$ . By extracting a subsequence we can assume that  $g_m \rightarrow \chi_I$  a.e. as well. We then have

$$\int_0^1 |f_n \chi_I - f \chi_I| \leq \int_0^1 |f_n \chi_I - f_n g_m| + \int_0^1 |f_n g_m - f g_m| + \int_0^1 |f g_m - f \chi_I|.$$

Since  $\|f_n\|_{L^\infty} \leq 1$ , we have that the first integral on the RHS can be made small uniformly in  $n$  by picking  $m$  large. The second integral goes to zero as  $n \rightarrow \infty$  by hypothesis since  $g_m$  is continuous. The third integral can be made small for  $m$  large by dominated convergence since  $|fg_m| \leq |f| \in L^1$ .

For almost all  $x$ , if  $I_k$  is a sequence of intervals shrinking to  $x$  then

$$\begin{aligned} \frac{1}{m(I_k)} \int_{I_k} f \, dx &= \frac{1}{m(I_k)} \int f \chi_{I_k} \, dx \\ &= \lim_{n \rightarrow \infty} \frac{1}{m(I_k)} \int f_n \chi_{I_k} \, dx. \end{aligned}$$

Since  $0 \leq f_n \leq 1$ , the RHS is in  $[0, 1]$  for almost all  $x$ . By the Lebesgue differentiation theorem we then have that  $0 \leq f \leq 1$  a.e.  $\square$

3. Let  $f, g \in L^2(\mathbb{R}, \mathcal{M}_L, \mu_L)$ . Show that  $f * g$  is a continuous function on  $\mathbb{R}$  vanishing at infinity, that is,  $f * g \in C(R)$  and  $\lim_{|x| \rightarrow \infty} (f * g)(x) = 0$ .

*Proof.* For any  $h$  we have by Hölder's inequality

$$|(f * g)(x + h) - (f * g)(x)| = \left| \int f(t)[g(x + h - t) - g(x - t)] \, dt \right| \quad (2)$$

$$\leq \|f\|_{L^2} \cdot \|g_h - g\|_{L^2}, \quad (3)$$

where  $F_h(x) = F(x + h)$  for any function  $F$ . Now for any  $\epsilon > 0$  we can find  $\varphi \in C_0(\mathbb{R})$  with  $\|g - \varphi\|_{L^2} = \|g_h - \varphi_h\|_{L^2} < \epsilon$ . By the triangle inequality we then have

$$\begin{aligned} \|g_h - g\|_{L^2} &\leq \|g_h - \varphi_h\|_{L^2} + \|\varphi_h - \varphi\|_{L^2} + \|\varphi - g\|_{L^2} \\ &< \|\varphi_h - \varphi\|_{L^2} + 2\epsilon. \end{aligned}$$

Suppose that  $\varphi$  has support contained in the compact set  $K$ . If we pick  $h$  small enough then we can guarantee that  $\varphi_h - \varphi$  is supported on a set with measure at most  $2 \cdot m(K)$ . Now since  $\varphi$  is continuous with compact support, it is uniformly continuous, so we can choose  $h$  small enough that  $|\varphi_h(x) - \varphi(x)| = |\varphi(x + h) - \varphi(x)| < \epsilon$  for all  $x$  in the support of  $\varphi_h - \varphi$ . For such  $h$  we have

$$\|\varphi_h - \varphi\|_{L^2} \leq \epsilon \cdot (2 \cdot m(K))^{1/2},$$

so (2) can be made arbitrarily small, which shows that  $f * g$  is continuous.

First we claim that if  $\varphi$  and  $\psi$  are continuous with compact support then  $\varphi * \psi$  vanishes at infinity. By definition we have that

$$(\varphi * \psi)(x) = \int \varphi(t)\psi(x - t) \, dt.$$

The product  $\varphi(t)\psi(x-t)$  is nonzero only if  $t$  is in the support of  $\varphi$  and  $x-t$  is in the support of  $\psi$ . If pick  $x$  large enough then supports of  $t \mapsto \varphi(t)$  and  $t \mapsto \psi(x-t)$  are disjoint, so this integral is zero.

Let  $f_n$  and  $g_n$  be sequences in  $C_0(\mathbb{R})$  converging in  $L^2$  to  $f$  and  $g$ , respectively. We then have

$$\begin{aligned} |(f * g)(x) - (f_n * g_n)(x)| &\leq |(f * g)(x) - (f_n * g)(x)| + |(f_n * g)(x) - (f_n * g_n)(x)| \\ &\leq \|g\|_{L^2} \cdot \|f - f_n\|_{L^2} + \|f_n\|_{L^2} \cdot \|g - g_n\|_{L^2}. \end{aligned}$$

Since  $f_n \rightarrow f$  and  $g_n \rightarrow g$  in  $L^2$ , we have that  $f_n * g_n$  converges uniformly to  $f * g$ . Since  $f_n * g_n$  vanishes at infinity, we must then have that  $f * g$  vanishes at infinity.  $\square$

4. Let  $(X, \mathcal{A}, \mu)$  be a finite measure space, and let  $p_1 \in (1, \infty]$ . Let  $\{f_n\}$  be a uniformly bounded sequence in  $L^{p_1}(X, \mathcal{A}, \mu)$ . Suppose  $f = \lim_{n \rightarrow \infty} f_n$  exists  $\mu$ -a.e. Prove that  $f \in L^p(X, \mathcal{A}, \mu)$  for all  $p \in [1, p_1]$  and  $f_n \rightarrow f$  in  $L^p(X, \mathcal{A}, \mu)$  for all  $p \in [1, p_1]$ .

*Proof.* Suppose that  $\|f_n\|_{L^{p_1}} \leq M$  for all  $n$ . First we claim that the  $f_n$  are in  $L^p(X, \mathcal{A}, \mu)$  for all  $p \in [1, p_1]$ . In fact, they are uniformly bounded:

$$\begin{aligned} \int_X |f_n|^p &= \int_{|f_n| < 1} |f_n|^p + \int_{|f_n| \geq 1} |f_n|^p \\ &\leq \int_{|f_n| < 1} 1 + \int_{|f_n| \geq 1} |f_n|^{p_1} \\ &\leq \mu(\{f \leq 1\}) + M^{p_1}. \end{aligned} \tag{4}$$

Since  $(X, \mathcal{A}, \mu)$  is a finite measure space, this quantity is finite, so  $f_n \in L^p(X, \mathcal{A}, \mu)$  for all  $n$  and  $p \in [1, p_1]$ . We can then use the fact that  $f_n \rightarrow f$  a.e. and Fatou's lemma to show that  $f \in L^p(X, \mathcal{A}, \mu)$  for  $p \in [1, p_1]$ :

$$\int_X |f|^p \leq \liminf_{n \rightarrow \infty} \int_X |f_n|^p < \infty,$$

where the finiteness follows from the  $L^p$  uniform-boundedness of the  $f_n$ .

To establish convergence in  $L^p$ ,  $p \in [1, p_1)$  our plan is to use the Vitali convergence theorem. The family  $f_n$  is tight over  $X$  since  $X$  is a finite measure space and we're given that  $f_n \rightarrow f$  a.e., so it only remains to show that the  $f_n$ 's are uniformly integrable. To this end, let  $E$  be any measurable subset of  $X$ . Since  $f_n$  is in  $L^{p_1}$ , we have that  $|f_n|^p \in L^{p_1/p}$ . If we let  $q$  be the Hölder conjugate to  $p_1/p$  then we have

$$\begin{aligned} \int_E |f_n|^p &= \int_X |f_n|^p \cdot \chi_E \\ &\leq \| |f_n|^p \|_{L^{p_1/p}} \cdot \|\chi_E\|_{L^q} \\ &\leq M^{p_1^2/p} \cdot \mu(E)^{1/q}. \end{aligned}$$

If we choose  $E$  so that  $\mu(E)^{1/q} < \epsilon \cdot M^{-p_1^2/p}$ , then we'll have that  $\int_E |f_n|^p < \epsilon$ , so the  $f_n$ 's are uniformly integrable. By the Vitali convergence theorem we have that  $f_n \rightarrow f$  in  $L^p$  for  $p \in [1, p_1]$ .  $\square$

5. Let  $(X, \mathcal{A}, \mu)$  be a measure space, and let  $f : X \rightarrow [0, \infty)$  be  $\mathcal{A}$ -measurable. Consider the measure space  $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu_L)$ , where  $\mathcal{B}_{\mathbb{R}}$  is the Borel  $\sigma$ -algebra on  $\mathbb{R}$  and  $\mu_L$  is the Lebesgue measure, and form the product measure space  $(X \times \mathbb{R}, \sigma(\mathcal{A} \times \mathcal{B}_{\mathbb{R}}), \mu \times \mu_L)$ . Define  $E \subset X \times \mathbb{R}$  by  $(x, y) \in E \iff y \in [0, f(x))$ . Prove that  $E \in \sigma(\mathcal{A} \times \mathcal{B}_{\mathbb{R}})$  and  $(\mu \times \mu_L)(E) = \int_X f \, d\mu$ .

*Proof.* A function is measurable if it pulls measurable sets back to measurable sets. The plan is then to write  $E$  as a union and/or intersection of preimages of measurable sets under measurable functions. The function  $F(x, y) = f(x)$  is measurable since

$$F^{-1}((-\infty, \alpha]) = \{(x, y) : f(x) \leq \alpha\} = \{x : f(x) \leq \alpha\} \times \mathbb{R} \in \sigma(\mathcal{A} \times \mathcal{B}_{\mathbb{R}}),$$

as  $f$  is  $\mu$ -measurable. We also clearly have that the function  $G(x, y) = y$  is measurable. Now consider the function  $H(x, y) = y - f(x)$ .  $H$  is measurable as it is the difference of the measurable functions  $G$  and  $F$ . We then have that  $E$  is measurable through the following decomposition

$$\begin{aligned} E &= \{(x, y) : 0 \leq y < f(x)\} \\ &= \{(x, y) : y \geq 0\} \cap \{(x, y) : y < f(x)\} \\ &= G^{-1}([0, \infty)) \cap H^{-1}((-\infty, 0)). \end{aligned}$$

If  $\{f > 0\}$  is  $\sigma$ -finite we can use Tonelli's theorem to say

$$\begin{aligned} (\mu \times \mu_L)(E) &= \int_{X \times \mathbb{R}} \chi_E(x, y) \, d(\mu \times \mu_L) \\ &= \int_X \int_{\mathbb{R}} \chi_E(x, y) \, d\mu_L d\mu \\ &= \int_X \int_{\mathbb{R}} \chi_{[0, f(x))}(y) \, dy d\mu \\ &= \int_X f(x) \, d\mu. \end{aligned}$$

On the other hand, suppose that  $\{f > 0\}$  is not  $\sigma$ -finite. We claim that  $\int_X f \, d\mu = +\infty$ . Indeed, since we can decompose this set into a countable union,

$$\{f > 0\} = \bigcup_{m=1}^{\infty} \left\{ \frac{1}{m+1} < f \leq \frac{1}{m} \right\} \cup \bigcup_{n=1}^{\infty} \{n < f \leq n+1\}, \quad (5)$$

we must have that one of these sets has infinite measure. We need to show that  $(\mu \times \mu_L)(E) = +\infty$  too. For any  $\alpha, \beta > 0$  we have that if  $\alpha \leq f(x) < \beta$  then the product set

$$\{x : \alpha \leq f(x) < \beta\} \times \{y : 0 \leq y < \alpha\}$$

is contained in  $E$ . This product set has measure  $\alpha \cdot \mu_L\{\alpha \leq f < \beta\}$ , so by monotonicity we have that

$$\alpha \cdot \mu_L\{\alpha \leq f < \beta\} \leq (\mu \times \mu_L)(E)$$

for all  $\alpha, \beta > 0$ . But by the decomposition (5), we have that some set of the form  $\{\alpha \leq f(x) < \beta\}$  must have infinite measure, so we must have  $(\mu \times \mu_L)(E) = +\infty$ .  $\square$

6. Let  $f \in L^1(\mathbb{R})$  and let  $a_1, \dots, a_k \in \mathbb{R}$  and  $b_1, \dots, b_k \in \mathbb{R} \setminus \{0\}$ . Assume that the quotients  $\frac{a_j}{b_j}$  are all distinct. Determine

$$\lim_{t \rightarrow \infty} \int \left| \sum_{j=1}^k f(b_j x + t a_j) \right| dx.$$

*Solution.* Let  $\varphi \in C_0(\mathbb{R})$  be such that  $\|f - \varphi\|_{L^1} < \epsilon$ . Our plan is to compute the desired limit with  $\varphi$  in place of  $f$  and then argue that the difference can be made small. We have

$$\int \left| \sum_{j=1}^k \varphi(b_j x + t a_j) \right| dx = \int \left| \sum_{j=1}^k \varphi \left[ b_j \left( x + \frac{a_j}{b_j} t \right) \right] \right| dx$$

Now  $\varphi(b_j x + t a_j)$  is  $\varphi$  stretched horizontally by a factor of  $b_j$  and shifted over  $a_j/b_j$ . Since the support of  $\varphi$  is compact and the  $a_j/b_j$  are distinct, the supports of these transformations are disjoint for sufficiently large  $t$ . When these supports are disjoint we then have

$$\begin{aligned} \int \left| \sum_{j=1}^k \varphi(b_j x + t a_j) \right| dx &= \int \sum_{j=1}^k |\varphi(b_j x + t a_j)| dx \\ &= \|\varphi\|_{L^1} \cdot \sum_{j=1}^k \frac{1}{b_j}. \end{aligned}$$

That we can approximate the desired sum for  $f \in L^1$  follows from the reverse triangle inequality.

$$\begin{aligned} \left| \int \left| \sum_{j=1}^k f(b_j x + t a_j) \right| dx - \int \left| \sum_{j=1}^k \varphi(b_j x + t a_j) \right| dx \right| &\leq \sum_{j=1}^k \int |f(b_j x + t a_j) - \varphi(b_j x + t a_j)| dx \\ &= \epsilon \cdot \sum_{j=1}^k \frac{1}{b_j}. \end{aligned}$$

$\square$

## 0.2 Fall 2015

1. Let  $E$  be a measurable subset of  $[0, 2\pi]$ . Assume that  $f \in C(\mathbb{R})$  is 1-periodic, i.e.  $f(x+1) = f(x)$ . Compute

$$\lim_{n \rightarrow \infty} \int_E f(nx) dx.$$

*Solution.* We rewrite the integral over  $E$  as an integral over  $\mathbb{R}$  against the indicator function of  $E$ :

$$\int_E f(nx) dx = \int f(nx) \chi_E(x) dx.$$

Now let  $\varphi \in C_0^\infty(\mathbb{R})$ . Since  $f \in C(\mathbb{R})$  is 1-periodic, it has a 1-periodic continuous primitive  $F$  with  $F' = f$ . By the chain rule we have  $[\frac{1}{n}F(nx)]' = f(nx)$ . Integration by parts gives

$$\int f(nx) \varphi(x) dx = -\frac{1}{n} \int F(nx) \varphi'(x) dx.$$

$F(nx)$  is bounded since  $F$  is 1-periodic and  $\varphi \in C_0^\infty(\mathbb{R})$ , so it's integrable. We then have

$$\begin{aligned} \left| \int f(nx) \varphi(x) dx \right| &\leq \frac{1}{n} \|F\|_\infty \cdot \|\varphi'\|_{L^1} \\ &\rightarrow 0. \end{aligned}$$

Since  $E$  is a measurable subset of  $[0, 2\pi]$ , it has finite measure and  $\chi_E \in L^1(\mathbb{R})$ . We can then find  $\varphi \in C_0^\infty(\mathbb{R})$  with  $\|\chi_E - \varphi\|_{L^1} < \epsilon$ . Since  $f$  is continuous and 1-periodic, it is bounded and we have

$$\begin{aligned} \left| \int f(nx) \chi_E(x) dx - \int f(nx) \varphi(x) dx \right| &\leq \|f\|_\infty \cdot \|\chi_E - \varphi\|_{L^1} \\ &\leq \|f\|_\infty \cdot \epsilon. \end{aligned}$$

Since  $\int f(nx) \varphi(x) dx \rightarrow 0$ , we must have  $\int_E f(nx) \rightarrow 0$ . □

2. Suppose  $f \in L^1[0, 1]$  and assume that there exists  $C > 0$  such that for all measurable subsets  $E \subset [0, 1]$  we have

$$\int_E |f(x)| dx \leq C \mu(E)^{1/2}.$$

Show that  $f \in L^p[0, 1]$  for  $1 \leq p < 2$ . Show that the statement fails for  $p = 2$  by giving a counterexample.

*Proof.* We have that

$$|f(x)|^p - 1 \leq \sum_{n=1}^{\infty} \chi_{\{|f|^p \geq n\}}(x) \leq |f(x)|^p.$$

Since  $[0, 1]$  is a finite measure space, integrating through this inequality shows that  $f \in L^p[0, 1]$  if and only if the series

$$\sum_{n=1}^{\infty} \mu\{|f(x)|^p \geq n\} = \sum_{n=1}^{\infty} \mu\{|f(x)| \geq n^{1/p}\}.$$

converges. By Chebyshev's inequality and the given hypotheses we have

$$n^{1/p} \mu\{|f| \geq n^{1/p}\} \leq \int_{\{|f| \geq n^{1/p}\}} |f| dx \leq C \mu\{|f| \geq n^{1/p}\}^{1/2}.$$

Dividing through by  $n^{1/p} \mu\{|f| \geq n^{1/p}\}^{1/2}$  and squaring gives

$$\sum_{n=1}^{\infty} \mu\{|f(x)| \geq n^{1/p}\} \leq \sum_{n=1}^{\infty} \frac{C^2}{n^{2/p}},$$

which converges for all  $p \in [1, 2)$ .

□

3. Show that a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^+$  is measurable if and only if  $E = \{(x, y) : 0 \leq y \leq f(x)\}$  is a measurable set of  $\mathbb{R}^{n+1}$ .

*Proof.* Suppose  $f$  is measurable. Then the function  $F(x, y) = f(x)$  is a measurable function  $\mathbb{R}^{n+1} \rightarrow \mathbb{R}$ . Since  $G(x, y) = y$  is also measurable,  $H(x, y) = y - f(x)$  is measurable as the difference of measurable functions. We can then write  $E$  as the intersection of two measurable sets:

$$E = G^{-1}([0, \infty)) \cap H^{-1}((-\infty, 0]).$$

Thus,  $E$  is measurable if  $f$  is measurable.

Conversely, suppose that  $E$  is a measurable set. Then for any  $\alpha \geq 0$  the set  $A \cap G^{-1}(\alpha) = F^{-1}[[\alpha, \infty))$ . This shows that  $F$ , and therefore  $f$ , is measurable. □

4. Let  $f \in L^1(\mathbb{R})$  and set

$$f_h(x) = \frac{1}{2h} \int_{x-h}^{x+h} f(t) dt, \quad h > 0.$$

Show that  $f_h \in L^1(\mathbb{R})$  and  $f_h \rightarrow f$  in  $L^1(\mathbb{R})$ .

*Proof.* Let's integrate  $f_h$ . By Tonelli we have

$$\begin{aligned} \int |f_h(x)| dx &= \frac{1}{2h} \int \left| \int f(t) \chi_{[x-h, x+h]}(t) dt \right| dx \\ &\leq \frac{1}{2h} \int \int |f(t)| \chi_{[t-h, t+h]}(x) dx dt \\ &= \|f\|_{L^1}. \end{aligned} \tag{6}$$

Since  $f \in L^1(\mathbb{R})$ , we have that this quantity is finite and  $f_h \in L^1(\mathbb{R})$ .

Now since  $f \in L^1(\mathbb{R})$ ,  $f_h \rightarrow f$  a.e. by the Lebesgue differentiation theorem. By Fatou's lemma and (6), we have for any sequence  $h_n \rightarrow 0$

$$\begin{aligned} \int |f| dx &\leq \liminf_{n \rightarrow \infty} \int |f_{h_n}| dx \\ &\leq \int |f| dx, \end{aligned}$$

so  $\liminf_{n \rightarrow \infty} \int |f_{h_n}| = \int |f|$ . By the triangle inequality we have  $|f_{h_n}| + |f| - |f - f_{h_n}| \geq 0$ . Since  $|f_{h_n}| + |f| - |f - f_{h_n}|$  converges to  $2|f|$  a.e., another application of Fatou's lemma gives

$$2 \int |f| \, dx \leq \liminf_{n \rightarrow \infty} \int (|f_{h_n}| + |f| - |f - f_{h_n}|) \, dx$$

$$\iff \limsup_{n \rightarrow \infty} \int |f - f_{h_n}| \, dx \leq 0.$$

We then have  $\int |f - f_{h_n}| \rightarrow 0$ , so  $f_{h_n} \rightarrow f$  in  $L^1$  for any  $h_n \rightarrow 0$ . □

5. Let  $(X, \mathcal{A}, \mu)$  be a measure space and let  $f_k : X \rightarrow \mathbb{R}$  be a sequence of measurable functions satisfying the following:

$$\int_X |f_k|^2 \, d\mu \leq 2015, \quad \text{for all } k,$$

and

$$\int_X f_j f_k \, d\mu = 0, \quad \text{for all } j \neq k.$$

Prove that for all  $\beta > 3/2$ ,

$$\lim_{n \rightarrow \infty} \frac{1}{n^\beta} \sum_{k=1}^{n^2} f_k(x) = 0, \quad \text{for a.a. } x \in X.$$

*Proof.* Let's compute the  $L^2$  norm of the sum

$$\begin{aligned} \left\| \frac{1}{n^\beta} \sum_{j=1}^{n^2} f_j \right\|_{L^2}^2 &= \frac{1}{n^{2\beta}} \left( \sum_{j=1}^{n^2} f_j, \sum_{k=1}^{n^2} f_k \right) \\ &= \frac{1}{n^{2\beta}} \sum_{j=1}^{n^2} \sum_{k=1}^{n^2} (f_j, f_k) \\ &= \frac{1}{n^{2\beta}} \sum_{j=1}^{n^2} \|f_j\|_{L^2}^2 \\ &\leq \frac{2015}{n^{2\beta-2}}. \end{aligned}$$

Now if  $\beta > 3/2$ ,  $2\beta - 2 > 1$ , so the above quantity is summable in  $n$ . Summability and wanting to show that something holds for almost all  $x$  leads us to think Borel-Cantelli might be useful.

For any fixed  $\epsilon > 0$ , Chebyshev gives us

$$\begin{aligned} \mu \left\{ x : \left| \frac{1}{n^\beta} \sum_{j=1}^{n^2} f_j \right|^2 \geq \epsilon \right\} &\leq \frac{1}{\epsilon^2} \int_X \left( \frac{1}{n^\beta} \sum_{j=1}^{n^2} f_j \right)^2 \, dx \\ &\leq \frac{2015}{\epsilon^2 n^{2\beta-2}}. \end{aligned}$$

If we call the set on the LHS  $A_n$ , then we have  $\sum \mu(A_n) < \infty$ . By Borel-Cantelli we have  $\mu(\limsup_{n \rightarrow \infty} A_n) = 0$ , i.e., the set of  $x$  that belong to infinitely many  $A_n$  has measure zero, so the sum is zero for almost all  $x$ . □



6. Let  $A, B \subseteq \mathbb{R}^n$  be Lebesgue measurable sets and assume that for every  $x \in \mathbb{Q}^n$  there exists a null set  $N_x$  such that

$$A + x \subset B \cup N_x.$$

Show that if  $A$  is not a null set then the complement of  $B$  in  $\mathbb{R}^n$  is a null set.

*Proof.* Suppose  $A$  has positive measure. Since  $\mathbb{Q}$  is countable and the countable union of null sets is null, we have that  $A + \mathbb{Q} \subset B \cup N$  for some null set  $N$ . If  $A + \mathbb{Q}$  missed a set of positive measure, then the complement of  $B$  would contain a set of positive measure. Let's show that this cannot happen.

Suppose  $E$  is a set of positive measure with  $E \cap (A + \mathbb{Q}) = \emptyset$ . Define the function  $f$  by the convolution

$$f(x) = \int_{\mathbb{R}^n} \chi_A(x - y) \chi_E(y) dy.$$

If we choose  $x = q \in \mathbb{Q}^n$ , then the integrand is nonzero if and only if  $y \in E \cap (A + q) = \emptyset$ , so  $f(q) = 0$ . But the convolution is continuous if we take  $E$  to have finite measure and  $\mathbb{Q}^n$  is dense, so we must have  $f \equiv 0$ . But by Tonelli we have

$$\begin{aligned} \int_{\mathbb{R}^n \times \mathbb{R}^n} \chi_A(x - y) \chi_E(y) d(\mu_x \times \mu_y) &= \int \int \chi_A(x - y) \chi_E(y) dx dy \\ &= m(A)m(E). \end{aligned}$$

Since  $A$  is not null and  $E$  is assumed to have positive measure, this must be positive, contradicting  $f \equiv 0$ . We conclude that  $A + \mathbb{Q}$  is null.  $\square$

### 0.3 Spring 2015

1. Show that if  $f \in L^4(\mathbb{R})$  then

$$\lim_{c \rightarrow 1} \int_{\mathbb{R}} |f(cx) - f(x)|^4 dx = 0.$$

*Proof.* Suppose  $\varphi$  is continuous with compact support. Then  $\varphi(cx)$  converges to  $\varphi(x)$  uniformly, and since the support of  $\varphi$  is compact, we have that the desired limit holds with  $\varphi$  in place of  $f$ .

Now let  $\varphi \in C_0(\mathbb{R})$  be such that  $\|f - \varphi\|_{L^4} < \epsilon$ . Since  $|a + b|^p \leq 2^p(|a|^p + |b|^p)$  for all  $p > 0$  we have

$$\begin{aligned} \int |f(cx) - f(x)|^4 dx &= \int |f(cx) - \varphi(cx) + \varphi(cx) - \varphi(x) + \varphi(x) - f(x)|^4 dx \\ &\leq 2^4 \int |f(cx) - \varphi(cx)|^4 dx \\ &\quad + 2^8 \int |\varphi(cx) - \varphi(x)|^4 dx + 2^8 \int |\varphi(x) - f(x)|^4 dx. \end{aligned}$$

The first and third integrals are small since  $\|f - \varphi\|_{L^4} < \epsilon$  and the second integral can be made small as  $c \rightarrow 1$  since  $\varphi(cx) \rightarrow \varphi(x)$  uniformly on a compact set.  $\square$

2. Let  $f_n : (0, \infty) \rightarrow \mathbb{R}$ , be a sequence of Lebesgue measurable functions such that  $f_n \rightarrow f$  a.e. as  $n \rightarrow \infty$ . Assume that there exists  $g : (0, \infty) \rightarrow \mathbb{R}$  such that  $|f_n| \leq g$  for all  $n$  and  $g \in L^1(0, a)$  for all  $0 < a < \infty$ . Assume furthermore that

$$\int_1^\infty |f_n(\sqrt{x})| dx \leq C,$$

for all  $n$  and for some constant  $C > 0$ . Show that  $f_n \in L^1(0, \infty)$ ,  $f \in L^1(0, \infty)$  and  $f_n \rightarrow f$  in  $L^1(0, \infty)$  as  $n \rightarrow \infty$ .

*Proof.* First let's show that  $f_n \in L^1(0, \infty)$  for all  $n$ . Write

$$\int_0^\infty |f_n| dx = \int_0^1 |f_n| dx + \int_1^\infty |f_n| dx. \quad (7)$$

For the first integral, since  $|f_n| \leq g$  and  $g \in L^1(0, 1)$  we have

$$\int_0^1 |f_n| dx \leq \int_0^1 g dx < \infty.$$

For the second integral in (7) we use the hypothesis about  $f_n(\sqrt{x})$ .

$$\begin{aligned} C &\geq \int_1^\infty |f_n(\sqrt{x})| dx \\ &= 2 \int_1^\infty t |f_n(t)| dt \\ &\geq \int_1^\infty |f_n(t)| dt. \end{aligned}$$

Both integrals in (7) are then finite, so  $f_n \in L^1(0, \infty)$ . In fact, we actually have that the  $f_n$  are uniformly bounded in  $L^1(0, \infty)$  by  $\int_0^1 g dx + C$ . Since  $f_n \rightarrow f$  a.e. we can apply Fatou's lemma to show that  $f \in L^1(0, \infty)$ :

$$\begin{aligned} \int_0^\infty |f| dx &\leq \liminf_{n \rightarrow \infty} \int_0^\infty |f_n| dx \\ &\leq \int_0^1 g dx + C \\ &< \infty. \end{aligned}$$

Finally, since  $|f - f_n| \rightarrow 0$  a.e. and  $|f - f_n| \leq |f| + g \in L^1(0, \infty)$ , we can apply the dominated convergence theorem to show that  $f_n \rightarrow f$  in  $L^1(0, \infty)$ .  $\square$

3. Assume that  $f \in C^1(0, 1)$  and

$$\int_0^1 x |f'|^p dx < +\infty$$

for some  $p > 2$ . Show that  $\lim_{x \rightarrow 0^+} f(x)$  exists.

*Proof.* Let  $x_n \rightarrow 0$  and say the integral in the problem statement has value  $C < \infty$ . If  $q$  is such that  $\frac{1}{p} + \frac{1}{q} = 1$ , we have by Hölder's inequality

$$\begin{aligned}
|f(x_n) - f(x_m)| &= \int_{x_m}^{x_n} f'(x) \, dx \\
&\leq \int_{x_m}^{x_n} |f'(x)| \, dx \\
&= \int_0^1 x^{1/p} |f'(x)| x^{-1/p} \chi_{[x_m, x_n]}(x) \, dx \\
&\leq \left( \int_0^1 x |f'(x)|^p \, dx \right)^{1/p} \cdot \left( \int_{x_m}^{x_n} x^{-q/p} \, dx \right)^{1/q}.
\end{aligned}$$

Since  $p > 2$ , we have that  $q < 2$ , so the last line above becomes

$$|f(x_n) - f(x_m)| \leq C \cdot \frac{x^{1-q/p}}{1-q/p} \Big|_{x_m}^{x_n}.$$

Since  $q < 2$ , we have that  $1 - \frac{q}{p} > 0$ , so as  $x_m, x_n \rightarrow 0$ , this expression goes to zero. Thus, the sequence  $f(x_n)$  is Cauchy, so  $\lim_{x \rightarrow 0} f(x)$  exists.  $\square$

4. Suppose that  $E \subset [0, 1]^2$  is measurable. Denote

$$E_x = \{y \in [0, 1] : (x, y) \in E\}, \quad E_y = \{x \in [0, 1] : (x, y) \in E\}.$$

Show that if  $m(E_x) = 0$  for almost all  $x \in [0, \frac{1}{2}]$ , then

$$m(\{y \in [0, 1] : m(E_y) = 1\}) \leq \frac{1}{2}.$$

*Proof.*  $E$  is contained in the unit square, which has finite measure. By Tonelli's theorem we then have

$$\begin{aligned}
m(E) &= \int \chi_E(x, y) \, d(\mu_x \times \mu_y) \\
&= \int_0^1 \int_0^1 \chi_E(x, y) \, dy dx \\
&= \int_0^1 m(E_y) \, dy = \int_0^1 m(E_x) \, dx \\
&= \int_{1/2}^1 m(E_x) \, dx \\
&\leq \frac{1}{2}.
\end{aligned}$$

This gives us

$$\begin{aligned}
m(\{y \in [0, 1] : m(E_y) = 1\}) &= \int_{\{y \in [0, 1] : m(E_y) = 1\}} m(E_y) \, dy \\
&\leq \int_0^1 m(E_y) \, dy \\
&\leq \frac{1}{2}.
\end{aligned}$$

□

5. Let  $f \in L^p(\mathbb{R})$ ,  $1 < p < \infty$ , and let  $\alpha > 1 - \frac{1}{p}$ . Show that the series

$$\sum_{n=1}^{\infty} \int_n^{n+n^{-\alpha}} |f(x+y)| \, dy$$

converges for a.e.  $x \in \mathbb{R}$ .

*Proof.* Our strategy is to show that the sum, as a function of  $x$ , is locally integrable, and therefore finite almost everywhere. To this end, let  $k$  be an arbitrary integer. Since the above integrands are nonnegative, the monotone convergence theorem will let us interchange the sum with integrals. By Tonelli we will interchange the integrals.

$$\begin{aligned}
\int_k^{k+1} \sum_{n=1}^{\infty} \int_n^{n+n^{-\alpha}} |f(x+y)| \, dy dx &= \sum_{n=1}^{\infty} \int_k^{k+1} \int_n^{n+n^{-\alpha}} |f(x+y)| \, dy dx \\
&= \sum_{n=1}^{\infty} \int_k^{k+1} \int |f(y)| \cdot \chi_{[n+x, n+n^{-\alpha}+x]}(y) \, dy dx \\
&= \sum_{n=1}^{\infty} \int \int_k^{k+1} |f(y)| \cdot \chi_{[y-n-n^{-\alpha}, y-n]}(x) \, dx dy.
\end{aligned}$$

Let's compute the values of  $y$  for which  $[y-n-n^{-\alpha}, y-n] \cap [k, k+1]$  is nonzero. We need  $k < y-n$ , so  $k+n < y$ . We also need  $y-n-n^{-\alpha} < k+1$ , so  $y < k+n+n^{-\alpha}+1$ . This gives us

$$\begin{aligned}
\int_k^{k+1} \sum_{n=1}^{\infty} \int_n^{n+n^{-\alpha}} |f(x+y)| \, dy dx &= \sum_{n=1}^{\infty} \int_{k+n}^{k+n+n^{-\alpha}+1} \int |f(y)| \chi_{[y-n-n^{-\alpha}, y-n]}(x) \, dx dy \\
&= \sum_{n=1}^{\infty} n^{-\alpha} \int_{k+n}^{k+n+n^{-\alpha}+1} |f(y)| \, dy.
\end{aligned}$$

Our plan is to use Hölder's inequality with respect to the counting measure on the sequences  $n^{-\alpha}$  and  $\int_{k+n}^{k+n+n^{-\alpha}+1} |f(y)| \, dy$ . Since  $\alpha$  is given to be larger than the Hölder conjugate of  $p$ , we have that  $n^{-\alpha}$  is in  $\ell^q$ . We also have

$$\sum_{n=1}^{\infty} \left( \int_{k+n}^{k+n+n^{-\alpha}+1} |f(y)| \, dy \right)^p$$

□

6. Suppose  $E \subset \mathbb{R}$  is measurable and  $E = E + \frac{1}{n}$  for every natural number  $n \geq 1$ . Show that either  $m(E) = 0$  or  $m(E^c) = 0$ .

*Proof.* By induction we can see that  $E = E + \mathbb{Q}$ . Suppose  $E$  isn't null and  $E = E + \mathbb{Q}$  misses a set  $A$  of positive finite measure. Consider the convolution

$$f(x) = \int_{\mathbb{R}} \chi_E(x-y) \chi_A(y) dy.$$

Since  $E + \mathbb{Q} \cap A$  is empty, if  $x \in \mathbb{Q}$  then  $f(x) = 0$ . Furthermore, since  $A$  has finite measure and  $E$  is in  $L^\infty(\mathbb{R})$ , the convolution is continuous. Since  $\mathbb{Q}$  is dense and  $f$ , a continuous function vanishes on  $\mathbb{Q}$ , we must have  $f \equiv 0$ . But by Tonelli we have that  $\int f(x) dx = m(E)m(A)$ , which is positive. We conclude that  $E$  cannot miss a set of positive measure if it isn't null.  $\square$

## 0.4 Fall 2014

1. Let  $A$  be the collection of all subsets of  $\mathbb{R}$  that consist of exactly 5 points. Find the  $\sigma$ -algebra of sets generated by  $A$ .

*Solution.* By intersecting five element sets with exactly one point in common we can obtain all singleton subsets of  $\mathbb{R}$ . We claim that the  $\sigma$ -algebra generated by the singleton sets, which will be the  $\sigma$ -algebra generated by  $A$ , consists of all countable or co-countable subsets of  $\mathbb{R}$ .

Call the  $\sigma$ -algebra consisting of all countable or co-countable sets  $\mathcal{A}$ . Since  $\mathcal{A}$  contains all singletons, we clearly have  $\sigma(A) \subseteq \mathcal{A}$ . Conversely, let  $S \in \mathcal{A}$ . If  $S$  is countable, then it is a countable union of singletons, so  $S \in \sigma(A)$ . On the other hand, if  $S$  is co-countable, then its complement is in  $\sigma(A)$ . Since  $\sigma(A)$  is closed under taking complements, this puts  $S$  in  $\sigma(A)$  as well. We conclude that  $\sigma(A) = \mathcal{A}$ .  $\square$

2. Assume that  $f \in L^1(0,1)$  is a non-negative real-valued function satisfying  $\int_{[0,1]} f(x) dx = 1$ . Show that

$$\int_{[0,1]} \frac{1}{f(x)} dx \geq 1.$$

*Proof.* Since  $f \in L^1$  and  $f \geq 0$ , we have that  $\sqrt{f} \in L^2$ . We then have by Hölder's inequality

$$\begin{aligned}
1 &= \int 1 \, dx \\
&= \int \frac{\sqrt{f}}{\sqrt{f}} \, dx \\
&\leq \left\| \sqrt{f} \right\|_{L^2} \cdot \left\| 1/\sqrt{f} \right\|_{L^2} \\
&= \sqrt{\|f\|_{L^1}} \cdot \sqrt{\|1/f\|_{L^1}} \\
&= \sqrt{\|1/f\|_{L^1}}.
\end{aligned}$$

□

3. Denote

$$E = \left\{ x \in [0, 1] : \text{there exist infinitely many } p, q \in \mathbb{N} \text{ such that } |x - \frac{p}{q}| \leq \frac{1}{q^3} \right\}.$$

Show that  $m(E) = 0$ .

*Proof.* Let  $E_{p,q} = \{x \in [0, 1] : |x - p/q| \leq 1/q^3\}$  where  $p, q$  range over  $\mathbb{N}$ . Note that since we're confined to  $[0, 1]$ , these sets are empty for  $p > q$  for any fixed  $q$ . We also have that  $m(E_{p,q}) \leq \frac{2}{q^3}$ . We can then sum (using Tonelli to sum over  $p$  and  $q$  individually)

$$\begin{aligned}
\sum_{p,q \in \mathbb{N}} m(E_{p,q}) &= \sum_{q \in \mathbb{N}} \sum_{0 \leq p < q} m(E_{p,q}) \\
&\leq \sum_{q \in \mathbb{N}} q \cdot \frac{2}{q^3} \\
&= \frac{\pi^2}{3}.
\end{aligned}$$

Since this sum is finite, by Borel-Cantelli we must have that  $m(\limsup E_{p,q}) = 0$ .  $\limsup E_{p,q}$  is the set of  $x \in [0, 1]$  belonging to infinitely many  $E_{p,q}$ , which is exactly the definition of  $E$ . □

4. Assume that  $\eta \in L^1(\mathbb{R})$  is a non-negative function satisfying  $\int_{\mathbb{R}} \eta \, dx = 1$ . Show that for any  $f \in L^1(\mathbb{R})$ ,

$$\|f * \eta\|_{L^1} \leq \|f\|_{L^1}.$$

*Proof.* We use Tonelli's theorem

$$\begin{aligned}
 \int |(f * \eta)(x)| \, dx &\leq \int \int |f(x-y)|\eta(y) \, dy dx \\
 &= \int \int |f(x-y)|\eta(y) \, dx dy \\
 &= \|f\|_{L^1} \int \eta(y) \, dy \\
 &= \|f\|_{L^1} .
 \end{aligned}$$

□

5. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be continuous and periodic with period one. Prove that

$$\lim_{n \rightarrow \infty} \int_0^1 f(nx) \cos^2(2\pi x) \, dx = \frac{1}{2} \int_0^1 f(x) \, dx .$$

*Proof.*

□