Mean Value Theorem and Taylor's Theorem

1. Let f be differentiable on an open interval I and suppose that $f'(x)$ is nonzero for all x in I. Show that f is one-to-one on I.

2. Let f be differentiable on an open interval I and suppose that $|f'(x)| < M$ for some positive number M. Prove that f is Lipschitz on I with Lipschitz constant M, i.e.

$$|f(x) - f(y)| \leq M|x - y| \text{ for all } x, y \in I.$$

3. Suppose that f is differentiable on an open interval I and suppose that $f'(x) \neq 1$ for all $x \in I$. Prove that f has at most one fixed point on I. A fixed point is a point y such that $f(y) = y$.

4. Suppose f is differentiable on an open interval I and suppose that $[a, b]$ is a closed interval contained in I with $f'(a) < 0 < f'(b)$.

   (a) Show that there exist points c and d with $a < c < d < b$ such that $f(c) < f(a)$ and $f(d) < f(b)$.

   (b) Show that f attains its minimum value on $[a, b]$ at an interior point (i.e. not at a or b).

   (c) Conclude that $f'(x_0) = 0$ for some x_0 in $[a, b]$. Why can't we just use the intermediate value theorem?

   (d) Deduce Darboux's theorem: if $f'(a) < L < f'(b)$ then $f'(x_0) = L$ for some x_0 in (a, b).

5. Find the Taylor series representations for each of the following functions. For precisely what values of x is each series representation valid?

   (a) $x \cos x^2$

   (b) $\frac{x}{(1+4x^2)^2}$

   (c) $\log(1 + x^2)$

6. Find an example or explain why no such example exists.

   (a) An infinitely differentiable function $g(x)$ on all of $\mathbb{R}$ with a Taylor series that converges to $g(x)$ only for $x \in (-1, 1)$.

   (b) An infinitely differentiable function $h(x)$ with the same Taylor series as that of $\sin x$ but such that $h(x) \neq \sin x$ for all $x \neq 0$.

   (c) An infinitely differentiable function $f(x)$ on $\mathbb{R}$ with a Taylor series that converges to $f(x)$ if and only if $x \leq 0$.