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260A - Homework 1

Problem 1.

(i) Show that ℓ^p, $1 \le p \le \infty$, is a Banach space.

(ii) Prove that $\ell^\infty = (\ell^1)^*$, but $(\ell^\infty)^* \ne \ell^1$.

Proof. (i) Let $a = (a^{(n)})$ and $b = (b^{(n)})$ be in ℓ^p, $1 < p < \infty$. We have by Hölder's inequality for any complex λ

$$\begin{aligned}
\|a+\lambda b\|_p^p &= \sum_{n=1}^\infty |a^{(n)} + \lambda b^{(n)}|^p \\
&= \sum_{n=1}^\infty |a^{(n)} + \lambda b^{(n)}| \cdot |a^{(n)} + \lambda b^{(n)}|^{p-1} \\
&\le \sum_{n=1}^\infty |a^{(n)}| \cdot |a^{(n)} + \lambda b^{(n)}|^{p-1} + |\lambda| \sum_{n=1}^\infty |b^{(n)}| \cdot |a^{(n)} + \lambda b^{(n)}|^{p-1} \\
&\le \left(\|a\|_p + |\lambda|\,\|b\|_p\right) \left(\sum_{n=1}^\infty |a^{(n)} + \lambda b^{(n)}|^{(p-1)\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \\
&= \left(\|a\|_p + |\lambda|\,\|b\|_p\right) \|a+\lambda b\|_p^{p-1},
\end{aligned}$$

Which shows that $\|a+\lambda b\|_p \le \|a\|_p + |\lambda|\,\|b\|_p < \infty$. This shows both that ℓ^p, $1 < p < \infty$, is a vector space (as linear combinations of elements of ℓ^p have finite p-norm) and that the p-norm satisfies the triangle inequality (take $\lambda = 1$).
ℓ^1 is a vector space and the $\|\cdot\|_1$ norm satisfies the triangle inequality thanks to the triangle inequality on $\mathbb{C}$:

$$\begin{aligned}
\|a+\lambda b\|_1 &= \sum_{n=1}^\infty |a^{(n)} + \lambda b^{(n)}| \\
&\le \sum_{n=1}^\infty |a^{(n)}| + |\lambda| \sum_{n=1}^\infty |b^{(n)}| \\
&= \|a\|_p + |\lambda|\,\|b\|_p .
\end{aligned}$$

Similarly, for $a, b \in \ell^\infty$ and $\lambda \in \mathbb{C}$ we have

$$\|a+\lambda b\|_\infty = \sup_{n\ge1} |a^{(n)} + \lambda b^{(n)}| \le \sup_{n\ge1}(|a^{(n)}| + |\lambda||b^{(n)}|) \le \sup_{n\ge1} |a^{(n)}| + |\lambda| \sup_{n\ge1} |b^{(n)}| = \|a\|_\infty + |\lambda|\,\|b\|_\infty .$$

We then have that ℓ^p is a normed complex vector space. We now need to show completeness. First let's treat the case of $p < \infty$. Suppose that $\{a_n\}$ is a Cauchy sequence in ℓ^p (here $a_i^{(j)}$ is the

j-th entry in the i-th element of the sequence). Since this sequence is Cauchy we have that for any $\epsilon > 0$ we can find $N \in \mathbb{N}$ so that for all $m, n > N$

$$\|a_m - a_n\|_p < \epsilon \iff \sum_{k=1}^{\infty} |a_m^{(k)} - a_n^{(k)}|^p < \epsilon^p.$$

Since each term in the above sum is nonnegative, we must have that $|a_m^{(k)} - a_n^{(k)}| < \epsilon$ for each k. In particular, we have that for any fixed k, $\{a_n^{(k)}\}$ is a Cauchy sequence of complex numbers. Since $\mathbb{C}$ is complete, we have that $a_n^{(k)} \to a^{(k)} \in \mathbb{C}$ as $n \to \infty$.

Let a be the sequence of complex numbers whose k-th entry is built from our original Cauchy sequence by $a^{(k)} = \lim_{n\to\infty} a_n^{(k)}$. Our plan is to show that $a_n \to a$ in ℓ^p and that a is in ℓ^p. Fix $\epsilon > 0$. Then for some N we have that $\|a_m - a_n\|_p < \epsilon$ for all $m, n > N$. Our trick is to pass to a finite sum and then take limits in a particular order. For any $L > 0$ and m, n sufficiently large we have

$$\sum_{k=0}^{L} |a_m^{(k)} - a_n^{(k)}|^p \leq \|a_m - a_n\|_p^p < \epsilon^p.$$

Now the right-hand side does not depend on m, so taking $m \to \infty$ gives

$$\sum_{k=0}^{L} |a^{(k)} - a_n^{(k)}|^p < \epsilon^p.$$

Then we take $L \to \infty$ which gives $\|a - a_n\|_p < \epsilon$, so $a_n \to a$ in ℓ^p. We can use this to show that a is in ℓ^p since for all n

$$\|a\|_p \leq \|a - a_n\|_p + \|a_n\|_p .$$

For n large enough the first term on the right is bounded by ϵ and the second term is finite since each a_n is in ℓ^p. Thus, ℓ^p is complete, and therefore, a Banach space for $1 \leq p < \infty$.

Now let $p = \infty$. If $\{a_n\}$ is a Cauchy sequence in ℓ^∞ then for $\epsilon > 0$ and m, n sufficiently large we have that $\sup_{k>0} |a_m^{(k)} - a_n^{(k)}| < \epsilon$. Just like in the finite p case, this implies that for any fixed k, $\{a_n^{(k)}\}$ is a Cauchy sequence of complex numbers, so we can speak of the entrywise limit a. Also similar to the finite p case we have that for L large

$$\sup_{1\leq k\leq L} |a_m^{(k)} - a_n^{(k)}| \leq \|a_m - a_n\|_\infty < \epsilon.$$

Sending m to infinity gives $\sup_{1\leq k\leq L} |a^{(k)} - a_n^{(k)}| < \epsilon$ and then sending L to infinity gives $\|a - a_n\|_\infty \to 0$. The argument used in the $p < \infty$ case also shows that $a \in \ell^\infty$.

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