Homework 2 (Due Friday, November 9, 2018)

Problem 1. Let $(e_n)_{n=1}^{\infty}$ be an orthonormal basis in the Hilbert space H. Let $T : H \to H$ be a linear continuous map such that

$$\sum_{n=1}^{\infty} \|Te_n\|^2$$

converges. Show that there is a sequence $(T_n)_{n=1}^{\infty}$ of linear continuous maps $H \to H$ such that $T_n(H)$ has a finite dimension and $\|T_n - T\| \to 0$ as $n \to \infty$.

Problem 2. Suppose that V is a linear space with the norm $\|\cdot\|$ which satisfies the parallelogram identity for all $u, v \in V$. Show that

$$\langle u, v\rangle = \frac{1}{4}(\|u+v\|^2 - \|u-v\|^2 + i\|u+iv\|^2 - i\|u-iv\|^2)$$

is a scalar product on V.

Problem 3. Let H be a separable infinite dimensional Hilbert space, and suppose that $e_1, e_2, \dots$ is an orthonormal system in H. Let $f_1, f_2, \dots$ be another orthonormal system which is complete.

(i) Prove that if $\sum_{n=1}^{\infty} \|e_n - f_n\|^2 < 1$ then $\{e_n\}$ is also complete orthonormal system.

(ii) Suppose only that $\sum_{n=1}^{\infty} \|e_n - f_n\|^2 < \infty$. Prove that it is still true that $\{e_n\}$ is a complete orthonormal system.

Problem 4. When B_1 and B_2 are Banach spaces, we say that a linear operator $T : B_1 \to B_2$ is compact if for any bounded sequence (x_n) in B_1, the sequence (Tx_n) has a convergent subsequence. Show that if T is compact then $\mathrm{Im}T$ has a dense countable subset.

Problem 5. Let H be a Hilbert space and let $U : H \to H$ be unitary, so that $UU^* = U^*U = 1$. Here $U^* : H \to H$ is the adjoint of U defined by

$$\langle Ux, y\rangle = \langle x, U^*y\rangle, \quad \text{for all } x, y \in H.$$

(i) Show that

$$H = \mathrm{Ker}(I-U) \oplus \overline{\mathrm{Im}(I-U)},$$

where the direct sum is orthogonal.

(ii) Let P be the orthogonal projection onto $\mathrm{Ker}(I-U)$ and let us set

$$S_n = \frac{1}{n}\sum_{j=0}^{n-1} U^j.$$

Show that $S_n x \to Px$ for all $x \in H$, as $n \to \infty$. (This is the von Neumann mean ergodic theorem.)

Problem 6. Let us introduce a space $\mathcal{B}$ defined as follows,

$$\mathcal{B} = \left\{ u : \mathbb{C} \to \mathbb{C}, \quad u \text{ is holomorphic and } \int_{\mathbf{C}} |u(z)|^2 e^{-|z|^2} \, L(dz) < \infty \right\}.$$

Here $L(dz)$ is the Lebesgue measure in $\mathbb{C}$. Show that $\mathcal{B}$ becomes a Hilbert space when equipped with the scalar product

$$\langle u, v \rangle = \int_{\mathbb{C}} u(z)\overline{v(z)} e^{-|z|^2} \, L(dz).$$

Problem 7.

(i) Let S be a unitary operator on a complex Hilbert space. Prove that for every complex number $|\lambda| < 1$ the operator $S - \lambda I$ is invertible. Here I denotes the identity operator.

(ii) For a fixed vector v in the Hilbert space and all $\lambda \in \mathbb{C}$, $|\lambda| < 1$, we define

$$h(\lambda) = \langle (S + \lambda I)(S - \lambda I)^{-1} v, v \rangle.$$

Show that $\operatorname{Re} h$ is a positive harmonic function.

Problem 8. Let H be the Hilbert space $L^2(\mathbb{R})$ and define $U : H \to H$ by

$$Uf(x) = f(x - 1).$$

Show that U has no (non-zero) eigenvectors.