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260B - Homework 1

Problem 1. Define the Sobolev space $H^s(\mathbb{R}^d)$, $s \geq 0$ to be the set of all functions $u \in L^2(\mathbb{R}^d)$ such that

$$\|u\|_{H^s}^2 = \frac{1}{(2\pi)^d} \int |\widehat{u}(\xi)|^2 (1 + |\xi|^2)^s \, d\xi < \infty.$$

(a) Show that $H^s(\mathbb{R}^d)$ is a Hlibert space when equipped with the scalar product

$$(u, v)_{H^s} = \frac{1}{(2\pi)^d} \int \widehat{u}(\xi)\overline{\widehat{v}(\xi)}(1 + |\xi|^2)^s \, d\xi.$$

Proof. Denote $\langle \xi \rangle := (1 + |\xi|^2)^{1/2}$ (apparently this is sometimes called the "Japanese bracket" of ξ).

It's clear that the alleged inner product is linear, conjugate symmetric, and positive definite (since the Fourier transform is an isometry from L^2 to itself). That it is well-defined follows from Hölder's inequality:

$$\begin{aligned}
|(u, v)| &\leq \frac{1}{(2\pi)^d} \int |\widehat{u}(\xi)||\widehat{v}(\xi)| \cdot \langle x \rangle^{2s} \, d\xi \\
&= \frac{1}{(2\pi)^d} \int (|\widehat{u}(\xi)| \cdot \langle \xi \rangle^s) \cdot (|\widehat{v}(\xi)| \cdot \langle \xi \rangle^s) \, d\xi \\
&\leq \frac{1}{(2\pi)^d} \|\widehat{u}(\xi) \cdot \langle \xi \rangle^s\|_{L^2} \cdot \|\widehat{v}(\xi) \cdot \langle \xi \rangle^s\|_{L^2} \\
&= \|u\|_{H^s} \cdot \|v\|_{H^s} \\
&< \infty.
\end{aligned}$$

The interesting part is showing that this space is complete with respect to this norm. Suppose that u_n is a Cauchy sequence in $H^s(\mathbb{R}^d)$. Then for $\epsilon > 0$ and m, n sufficiently large we have

$$\begin{aligned}
\epsilon &\geq \|u_n - u_m\|_{H^s}^2 \\
&= \frac{1}{(2\pi)^d} \int |\widehat{u_n - u_m}(\xi)|^2 \cdot \langle \xi \rangle^{2s} \, d\xi \\
&= \frac{1}{(2\pi)^d} \int |\widehat{u_n}(\xi) \cdot \langle \xi \rangle^s - \widehat{u_m}(\xi) \cdot \langle \xi \rangle^s|^2 \, d\xi.
\end{aligned}$$

So the sequence $\widehat{u_n}(\xi) \cdot \langle \xi \rangle^s$ is Cauchy in L^2. Since $L^2(\mathbb{R}^d)$ is complete, $\widehat{u_n}(\xi) \cdot \langle \xi \rangle^s$ converges to some $v \in L^2(\mathbb{R}^d)$. By Hölder's inequality $v(\xi) \cdot \langle \xi \rangle^{-s}$ is also in $L^2(\mathbb{R}^d)$, so it has a well-defined inverse Fourier transform.

We claim that u_n converges to $\mathcal{F}^{-1}(v(\xi) \cdot \langle \xi \rangle^{-s})$ in $H^s(\mathbb{R}^d)$. It was designed for this purpose after all.

$$\begin{aligned}
\|u_n - \mathcal{F}^{-1}(v(\xi) \cdot \langle \xi \rangle^{-s})\|_{H^s}^2 &= \frac{1}{(2\pi)^d} \int |\widehat{u_n}(\xi) - v(\xi) \cdot \langle \xi \rangle^{-s}|^2 \cdot \langle \xi \rangle^{2s} \, d\xi \\
&= \frac{1}{(2\pi)^d} \int |\widehat{u_n}(\xi) \cdot \langle \xi \rangle^{s} - v(\xi)|^2 \, d\xi \\
&\to 0.
\end{aligned}$$

That $\mathcal{F}^{-1}(v(\xi) \cdot \langle \xi \rangle^{-s})$ is in $H^s(\mathbb{R}^d)$ follows immediately from v being in $L^2(\mathbb{R}^d)$. Thus, $H^s(\mathbb{R}^d)$ is complete. □

(b) When $K \subseteq \mathbb{R}^d$ is compact we define

$$H^s(K) = \{u \in H^s(\mathbb{R}^d) : \operatorname{supp}(u) \subseteq K\}.$$

Show that $H^s(K)$ is a closed linear subspace of $H^s(\mathbb{R}^d)$, and hence also a Hilbert space. Show that the inclusion map $H^s(K) \to H^t(\mathbb{R}^d)$ is compact if $s > t \geq 0$.

Proof. □