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260A - Homework 3

Problem 1. Let $(b_1, b_2, \ldots)$ be a sequence of complex numbers such that $\sum_{n=1}^{\infty} b_n c_n$ is convergent for every $c = (c_1, c_2, \ldots) \in \ell^2$. Show that $b \in \ell^2$.

Proof. Consider the sequence of maps $T_n : \ell^2 \to \mathbb{C}$ that send $(c_1, \ldots)$ to $\sum_{j=1}^{n} b_j c_j$. Since each T_n is just a finite sum, we have that the T_n's form a sequence of bounded linear operators on ℓ^2. Furthermore, this sequence is pointwise bounded: given any $(c_1, c_2, \ldots) \in \ell^2$, since $\sum_{j=1}^{\infty} b_j c_j$ converges, we have that the sequence of partial sums $|T_n(c_1, c_2, \ldots)| = |\sum_{j=1}^{n} b_j c_j|$ is bounded. By the uniform boundedness principle, we have that

$$\sup_{n\in\mathbb{N},\ \|(c_1,c_2,\ldots)\|_2=1} |T_n(c_1, c_2, \ldots)| = \sup_{n\in\mathbb{N}} \|T_n\| = \sum_{j=1}^{\infty} |b_j| < \infty,$$

so $(b_1, b_2, \ldots) \in \ell^2$. $\square$

Problem 2. Let M be a measurable subset of $\mathbb{R}^n$ with finite positive measure. Prove that $L^q(M)$ is of the first category in $L^p(M)$ if $1 \le p < q \le \infty$.

Proof. Since M has finite measure, we have that $L^q(M) \subseteq L^p(M)$ whenever $1 \le p < q \le \infty$. Consider the injection $\iota : L^q(M) \to L^p(M)$ that simply sends $f \in L^p(M)$ to itself. By the generalized Hölder inequality we have that $\|\iota(f)\|_{L^p} = \|f\|_{L^p} \le \mu(M)^{1/r} \|f\|_{L^q}$, where $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. This shows that ι is bounded, and therefore continuous. Since $L^p(M)$ and $L^q(M)$ are Banach spaces, the open mapping theorem tells us that the image of ι is either surjective and open or of the first category in $L^p(M)$.

Our plan is to show that ι is *not* surjective, i.e. that $L^p(M) \setminus L^q(M)$ is nonempty. We'll do this by showing that $L^q(M) = L^p(M)$ would forbid the existence of subsets of M with arbitrarily small measure. Since sets of positive measure in $\mathbb{R}^n$ *do* contain sets of arbitrarily small measure, we'll conclude that $L^q(M) \neq L^p(M)$.

If the embedding $\iota : L^q(M) \to L^p(M)$ were surjective, then the open mapping theorem would imply that ι is actually a homeomorphism. In particular, its inverse, $\iota' : L^p(M) \to L^q(M)$ is a bounded operator. Let A be a subset of M with positive, finite measure and define the function

$$f_A(x) = \frac{1}{m(A)^{1/p}} \cdot \chi_A(x).$$

It's clear that $\|f_A\|_{L^p} = 1$ and that $\|f_A\|_{L^q} = \frac{1}{m(A)^{1/p-1/q}}$. Since ι' is bounded, we have

$$0 < \|f_A\|_{L^q} \le \|\iota'\| \cdot \|f_A\|_{L^p} \implies 0 < \frac{1}{\|\iota'\|^{\frac{pq}{q-p}}} \le m(A).$$

This puts a positive lower bound on the measure of subsets of M. But M, as a subset of $\mathbb{R}^n$ with positive measure, contains set of arbitrarily small measure. We conclude that $L^q(M)$ is of the first category of $L^p(M)$. $\square$

Problem 3. Let $(X, \mathcal{A}, \mu)$ be a finite measure space. Assume that E is a closed subspace of $L^2(X, \mu)$, and that E is contained in $L^\infty(X, \mu)$. Prove that E is finite dimensional.

Proof. By Hölder's inequality, the embedding $\iota : L^\infty(X) \to L^2(X)$, $f \mapsto f$ is continuous, i.e., $\|f\|_{L^2} \leq \mu(X)^{1/2} \cdot \|f\|_{L^\infty}$. When we restrict ι to E we obtain a continuous surjection from E to itself. By the open mapping theorem, $\iota : E \to E$ is a homeomorphism, so there is some positive $C > 0$ with $\|f\|_{L^\infty} \leq C \cdot \|f\|_{L^2}$ for any $f \in E$. Now let $e_1, \ldots, e_n$ be an orthonormal set in E and fix $a \in \mathbb{C}^n$. Then for all x in S_a, where S_a has μ-full measure in X, we have

$$\begin{aligned} |a_1 e_1(x) + \cdots + a_n e_n(x)|^2 &\leq \|a_1 e_1 + \cdots + a_n e_n\|_{L^\infty}^2 \\ &\leq C^2 \cdot \|a_1 e_1 + \cdots a_n e_n\|_{L^2}^2 \\ &= C^2 \cdot (|a_1|^2 + \cdots + |a_n|^2). \end{aligned}$$

We'd like to replace the a_i's with $\overline{e_i(x)}$'s, but here x depends on a. We accomplish this through a limiting process (Alec Fox showed me how to do this).

Let Q be a countable dense subset of $\mathbb{C}^n$. The intersection $S := \cap_{q \in Q} S_q$ has full measure in X. Now for any $a \in \mathbb{C}^n$, we can find a sequence $q^{(k)}$ in Q that limits to a. For any k and $x \in S$ we have by the above inequalities

$$\left| \sum_{j=1}^n b_j^{(k)} e_j(x) \right|^2 \leq C^2 \cdot \sum_{j=1}^n |b_j^{(k)}|^2.$$

Taking the limit $k \to \infty$ gives

$$\left| \sum_{j=1}^n a_j e_j(x) \right|^2 \leq C^2 \cdot \sum_{j=1}^n |a_j|^2.$$

Now for any $x \in S$, which is μ-almost all of X, we can substitute $a_j = \overline{e_j(x)}$ into the above inequality to obtain (by the orthonormality of the e_j's)

$$\sum_{j=1}^n |e_j(x)|^2 \leq C^2.$$

Integration gives

$$n = \sum_{j=1}^n \int_X |e_j(x)|^2 \, d\mu \leq \int_X C^2 \, d\mu = C^2 \cdot \mu(X) < \infty,$$

so E is finite-dimensional. $\square$

Problem 4. Let X be a locally compact and locally convex space.

(i) Let U be a compact neighborhood of the origin. Show that one can find $x_1, \ldots, x_n$ so that $U \subseteq \cup_{j=1}^n (x_j + \frac{1}{2}U)$, and thus, a finite dimensional space, M, with $U \subseteq M + \frac{1}{2}U$.

Proof. Cover U with dilates of itself: $U \subseteq \cup_{x\in U}(x + \frac{1}{2}U^\circ)$. This is indeed an open cover since U, as a compact neighborhood, has nonempty interior. By compactness, we can extract a finite subcover, based around the points $x_1, \dots, x_n$:

$$U \subseteq \bigcup_{j=1}^{n} x_j + \frac{1}{2}U^\circ \subseteq \bigcup_{j=1}^{n} x_j + \frac{1}{2}U.$$

Let M be the linear span of the x_j's, $M := \langle x_1, \dots, x_j\rangle$. Since M is the span of finitely many vectors, it is finite dimensional and we clearly have the inclusion

$$U \subseteq \bigcup_{j=1}^{n} x_j + \frac{1}{2}U \subseteq M + \frac{1}{2}U.$$

□

(ii) Prove that $U \subseteq M + \frac{1}{2^m}U$ for any m.

Proof. In the above construction, it would appear the our choice of finite dimensional space, M, depends on our choice of cover. An induction on m will show that it doesn't. The base case $m = 1$ follows from part (i). Now assume that $U \subseteq M + \frac{1}{2^m}U$. We dilate both sides of this inclusion to obtain

$$\frac{1}{2}U \subseteq \frac{1}{2}M + \frac{1}{2^{m+1}}U = M + \frac{1}{2^{m+1}}U.$$

By part (i) we then have

$$U \subseteq M + \frac{1}{2}U \subseteq M + \left(M + \frac{1}{2^{m+1}}U\right) = M + \frac{1}{2^{m+1}}U.$$

By induction, the proposition holds for all m. □

(iii) Prove that $U \subseteq \overline{M}$.

Proof. Take $x \notin \overline{M}$. Then there is some balanced neighborhood of the origin, V, with $x \notin M + V$. Since balanced neighborhoods are absorbing, we have that $U \subseteq \cup_{n=1}^{\infty} nV$. The compactness of U and the fact that this union is increasing (since V is balanced) tells us that $U \subseteq 2^N V$ for some large N. By part (ii) we have

$$\begin{aligned} U &\subseteq M + \frac{1}{2^N}U \\ &\subseteq M + \frac{1}{2^N}(2^N V) \\ &= M + V. \end{aligned}$$

Since $x \notin M + V$, we conclude that $x \notin U$. This shows that $U \subseteq \overline{M}$. □

(iv) Conclude that $\overline{M} = X = M$.

Proof. ∎

Problem 5. Let a_n, $n \in \mathbb{Z}$, be a sequence of complex numbers such that $a_n b_n$ is the sequence of Fourier coefficients of a continuous function on $\mathbb{R}/2\pi\mathbb{Z}$ when this is true for the sequence b_n, $n \in \mathbb{Z}$. Prove that there is a measure with Fourier coefficients a_n, $n \in \mathbb{Z}$.

Proof. Denote $\mathbb{R}/2\pi\mathbb{Z}$ by $\mathbb{T}$. The plan is to use the closed graph theorem and Riesz-Markov-Kakutani. Define the map $T : C(\mathbb{T}) \to C(\mathbb{T})$ that maps f to the continuous function with Fourier coefficients $a_n \widehat{f}(-n)$ (the reason for the negative sign will become clear). That T is well defined follows from the fact that the assignment of a continuous function to its Fourier coefficients is injective and from the hypothesis that $a_n \widehat{f}(-n)$ is indeed the set of Fourier coefficients of a continuous function.

That T is linear follows simply from the linearity of the integral. We'll use the closed graph theorem to show that T is continuous. Suppose that $f_j \to f$ in $C(\mathbb{T})$ and $Tf_j \to g$ in $C(\mathbb{T})$. Let's look at the Fourier coefficients of g.

$$\begin{aligned}
\widehat{g}(n) &= \frac{1}{2\pi} \int_{\mathbb{T}} g(x) e^{-inx} \, dx \\
&= \lim_{j\to\infty} \frac{1}{2\pi} \int_{\mathbb{T}} (Tf_j)(x) e^{-inx} \, dx \\
&= \lim_{j\to\infty} \widehat{Tf_j}(n) \\
&= a_n \cdot \lim_{j\to\infty} \widehat{f_j}(-n) \\
&= a_n \cdot \lim_{j\to\infty} \frac{1}{2\pi} \int_{\mathbb{T}} f_j(x) e^{inx} \, dx \\
&= a_n \cdot \frac{1}{2\pi} \int_{\mathbb{T}} f(x) e^{inx} \, dx \\
&= a_n \widehat{f}(-n) \\
&= \widehat{Tf}(n).
\end{aligned}$$

The movement of limits through integrals follows from the fact that convergence in $C(\mathbb{T})$ is uniform on a finite measure space. Again by the uniqueness of Fourier coefficients for continuous functions, we have that $Tf = g$, so by the closed graph theorem, T is continuous.

Now define the map $S : C(\mathbb{T}) \to \mathbb{C}$ by $Sf = (Tf)(0)$. Evaluation at zero is a continuous linear functional on $C(\mathbb{T})$ and the composition of continuous functions is continuous, so S is a continuous linear functional on $C(\mathbb{T})$. By the Riesz-Markov-Kakutani representation theorem, there is a unique regular Borel measure μ on $\mathbb{T}$ such that $Sf = \int_{\mathbb{T}} f \, d\mu$. This measure has the desired property since its

Fourier coefficients are given by

$$\begin{aligned}\int_{\mathbb{T}} e^{-inx}\ d\mu &= S(e^{-inx})\\ &= (a_n e^{inx})(0)\\ &= a_n.\end{aligned}$$

$\square$