Algebra Qualifying Exam - Fall 2018

1. Let G be a group of size 42.

   (a) Prove that G has a subgroup H of order 6 and any two such subgroups are conjugate in G.

   Proof. By Sylow's theorems, G has a subgroup P of order 2 and a subgroup Q of order 3. Since P and Q have relatively prime orders, $P \cap Q$ is trivial and $|PQ| = 2 \cdot 3 = 6$. □

   (b) Deduce that $G = H \ltimes N$, where N is a normal subgroup of order 7.

2. Let p be a prime. Show that for any Sylow p-subgroup $H \subset GL_n(\mathbb{F}_p)$ there exists a basis in the vector space $V = \mathbb{F}_p^n$ such that H consists of $\mathbb{F}_p$-linear maps given, in that basis, by an upper-triangular matrix with 1 on the diagonal.

3. For a group G, let $G_1 = G$ and let $G_{n+1} := [G, G_n]$. We say that G is nilpotent if $G_N = 1$ for some N. Prove that if G is a p-group, i.e. $|G| = p^r$ for some prime p, then G is nilpotent.

4. Let $R \subset \mathbb{Q}$ be the subring in the field of rational numbers, given by the fractions $\frac{a}{b}$ with $a \in \mathbb{Z}$ and $b = 2^k 3^l$ with $k, l \geq 0$. Describe the ideals of R. Is R a PID?

5. Let $S = \{a + bi : a, b \in \mathbb{Z}\} \subset \mathbb{C}$ be the ring of Gaussian integers.

   (a) Show that S is a Euclidean domain.

   (b) Find a decomposition of $a = 11 \in S$ into a product of irreducibles in S.

   (c) Find a decomposition of $b = 13 \in S$ into a product of irreducibles in S.

6. Let V be a nonzero finite-dimensional vector space over the complex numbers.

   (a) If S and T are commuting linear operators on V, prove that each eigenspace of S is mapped into itself by T.

   (b) Let $A_1, \ldots, A_k$ be finitely many linear operators on V that commute pairwise. Prove that they have a common eigenvector in V.

   (c) If V has dimension n, show that there exists a nested sequence of subspaces

   $$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V,$$

   where each V_j has dimension j and is mapped into itself by each of the operators $A_1, \ldots, A_k$.

7. Let A be an $n \times n$ matrix with complex coefficients and assume that every eigenvalue λ of A satisfies $Im(\lambda) > 0$. Consider the $(2n) \times (2n)$ matrix

$$B = \begin{bmatrix} A & 0 \\ 0 & \overline{A} \end{bmatrix}.$$

Find the invariant factors of B in terms of invariant factors of A and prove that B is similar to a real-valued matrix.

8. Find the Galois group of $x^6 - 2$ over $\mathbb{Q}$ and over $\mathbb{F}_5$.

9. Let K/F be a finite Galois algebraic extension with no proper intermediate fields. Prove that $[K : F]$ is prime.

10. Let Q denote the quaternion group, i.e.

$$Q = \{\pm 1, \pm i, \pm j, \pm k\}$$

with $i^2 = j^2 = k^2 = ijk = -1$, $-1 \in Z(Q)$ and $(-1)^2 = 1$.

  (a) Classify the conjugacy classes of Q.

  (b) Construct the character table of Q.