More Continued Fractions Exercises

1. Let n be a positive integer. Come up with the continued fraction expansion for $\sqrt{n^2+1}$.

2. Let $\{c_n\}$ be the convergents of $\phi = [1; 1, 1, \ldots]$. Prove that for $n \geq 1$, we have $\frac{F_{n+1}}{F_n} = c_n$, i.e. $p_n = F_{n+1}$ and $q_n = F_n$, where F_n is the n-th Fibonacci number. Conclude that $\phi = \lim_{n\to\infty} \frac{F_{n+1}}{F_n}$. Recall that the Fibonacci numbers are defined by the following recurrence relation:

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-2} + F_{n-1}.$$

3. A continued fraction $[a_1; a_2, a_3, \ldots]$ where the a_n are all positive *real* numbers for $n \geq 1$ is called unary. In this problem we'll prove that a unary continued fraction converges if and only if $\sum a_n = \infty$.

   (a) Prove that $q_n \leq \prod_{k=1}^{n}(1 + a_k)$.

   (b) Prove that if the unary continued fraction converges, then $\sum a_n = \infty$

   (c) Prove that

   $$q_{2n+1} \geq 1 + a_2(a_3 + a_5 + \cdots + a_{2n+1}), \quad q_{2n} \geq a_2 + a_4 + \cdots + a_{2n},$$

   for $n \geq 1$.

   (d) Prove that if $\sum a_n = \infty$, then the unary continued fraction converges.