Homework 3 (Due Monday, November 26, 2018)

Please turn in solutions to any 5 of the following 6 problems.

Problem 1. Let $b = (b_1, b_2, \dots)$ be a sequence of complex numbers such that $\sum_{n=1}^{\infty} b_n c_n$ is convergent for every $c = (c_1, c_2, \dots) \in l^2$. Show that $b \in l^2$.

Problem 2. Let M be a measurable subset of $\mathbb{R}^n$ with finite positive measure. Prove that $L^q(M)$ is of the first category in $L^p(M)$ if $1 \leq p < q \leq \infty$.

Problem 3. Let $(X, \mathcal{A}, \mu)$ be a finite measure space. Assume that E is a closed subspace of $L^2(X, \mu)$, and that E is contained in $L^\infty(X, \mu)$. Prove that E is finite dimensional.

Problem 4. The purpose of this exercise is to prove that every locally compact, locally convex space X is finite dimensional.

(i) Let U be a compact neighborhood of 0. Show that one can find $x_1, \dots, x_n$ so that $U \subset \cup_{j=1}^{n} (x_j + \frac{1}{2}U)$, and thus, a finite dimensional space, M, with $U \subset M + \frac{1}{2}U$.
(ii) Prove that $U \subset M + \frac{1}{2^m}U$ for any m.
(iii) Prove that $U \subset \overline{M}$.
(iv) Conclude that $\overline{M} = X = M$.

Problem 5. Let a_n, $n \in \mathbb{Z}$, be a sequence of complex numbers such that $a_n b_n$, $n \in \mathbb{Z}$, is the sequence of Fourier coefficients of a continuous function on $\mathbb{R}/2\pi\mathbb{Z}$ when this is true for the sequence b_n, $n \in \mathbb{Z}$. Prove that there is a measure with Fourier coefficients a_n, $n \in \mathbb{Z}$.

Problem 6. Let B be a complex Banach space and let F be a function $\Omega \to \mathcal{L}(B, B)$, where Ω is an open set in $\mathbb{C}$. Assume that the function $z \mapsto \langle F(z)u, v\rangle$ is holomorphic in Ω for arbitrary $u \in B$ and $v \in B^*$. Prove that the limit

$$F'(z) = \lim_{w \to 0} \frac{F(z+w) - F(z)}{w}$$

exists in the operator norm for every $z \in \Omega$ and that

$$\langle F'(z)u, v\rangle = \frac{d}{dz}\langle F(z)u, v\rangle, \quad z \in \Omega,\ u \in B,\ v \in B^*.$$