

# Real Analysis Qualifying Exams

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Spring 2016

1. Assume  $f \in L^1[0, 1]$ . Compute

$$\lim_{k \rightarrow \infty} \int_{[0,1]} |f|^{1/k} dx.$$

*Solution.* Let's split this integral into three regions.

$$\int_{[0,1]} |f|^{1/k} dx = \int_{f=0} |f|^{1/k} dx + \int_{0 < |f| \leq 1} |f|^{1/k} dx + \int_{|f| > 1} |f|^{1/k} dx.$$

The integral over the first region is clearly zero for all  $k$ . On the second region we have that  $|f|^{1/k} \leq 1$  for all  $k$ . Since the interval  $[0, 1]$  has finite measure, we have that the constant function 1 is in  $L^1(\{x : 0 < |f| \leq 1\})$ , so the dominated convergence theorem says that the integral over the second region goes to  $m(\{0 < |f| \leq 1\})$ . Similarly, on the third region we have that  $|f|^{1/k} \leq |f|$ , which is in  $L^1$ , so the dominated convergence theorem says that the third integral goes to  $m(\{|f| > 1\})$ . Combining these, we have that

$$\lim_{k \rightarrow \infty} \int_{[0,1]} |f|^{1/k} dx = m(\{|f| > 0\}).$$

□

2. Let  $\{f_n\}$  be a sequence of measurable functions on  $[0, 1]$  and  $0 \leq f_n \leq 1$  a.e. Assume that

$$\lim_{n \rightarrow \infty} \int_{[0,1]} f_n g dx = \int_{[0,1]} f g dx$$

for some  $f \in L^1[0, 1]$  and any  $g \in C[0, 1]$ . Prove that  $0 \leq f \leq 1$  a.e.

*Solution.* Since  $f \in L^1[0, 1]$ , by the Lebesgue differentiation theorem we have that

$$\frac{1}{m(E)} \int_E f(t) dt \rightarrow f(x) \tag{1}$$

as  $E$  shrinks to  $x$  for almost all  $x$ . Furthermore, since  $0 \leq f_n \leq 1$  we also have that

$$\frac{1}{m(E)} \int_E f_n(t) dt \rightarrow f_n(x) \in [0, 1]$$

as  $E$  shrink to  $x$  for almost all  $x$ . Intuitively, we'd like to replace the integral of  $f$  in (1) with a limit of integrals of  $f_n$ .

We claim that the function  $g$  in the given hypothesis can be replaced with the indicator function of an interval  $\chi_I$ . To see this, let  $g_m$  be a sequence of continuous functions with  $g_m \rightarrow \chi_I$  in  $L^1$  and  $0 \leq \chi_I \leq 1$ . By extracting a subsequence we can assume that  $g_m \rightarrow \chi_I$  a.e. as well. We then have

$$\int_0^1 |f_n \chi_I - f \chi_I| \leq \int_0^1 |f_n \chi_I - f_n g_m| + \int_0^1 |f_n g_m - f g_m| + \int_0^1 |f g_m - f \chi_I|.$$

Since  $\|f_n\|_{L^\infty} \leq 1$ , we have that the first integral on the RHS can be made small uniformly in  $n$  by picking  $m$  large. The second integral goes to zero as  $n \rightarrow \infty$  by hypothesis since  $g_m$  is continuous. The third integral can be made small for  $m$  large by dominated convergence since  $|fg_m| \leq |f| \in L^1$ .

For almost all  $x$ , if  $I_k$  is a sequence of intervals shrinking to  $x$  then

$$\begin{aligned} \frac{1}{m(I_k)} \int_{I_k} f \, dx &= \frac{1}{m(I_k)} \int f \chi_{I_k} \, dx \\ &= \lim_{n \rightarrow \infty} \frac{1}{m(I_k)} \int f_n \chi_{I_k} \, dx. \end{aligned}$$

Since  $0 \leq f_n \leq 1$ , the RHS is in  $[0, 1]$  for almost all  $x$ . By the Lebesgue differentiation theorem we then have that  $0 \leq f \leq 1$  a.e.  $\square$

3. Let  $f, g \in L^2(\mathbb{R}, \mathcal{M}_L, \mu_L)$ . Show that  $f * g$  is a continuous function on  $\mathbb{R}$  vanishing at infinity, that is,  $f * g \in C(R)$  and  $\lim_{|x| \rightarrow \infty} (f * g)(x) = 0$ .

*Proof.* For any  $h$  we have by Hölder's inequality

$$|(f * g)(x + h) - (f * g)(x)| = \left| \int f(t)[g(x + h - t) - g(x - t)] \, dt \right| \quad (2)$$

$$\leq \|f\|_{L^2} \cdot \|g_h - g\|_{L^2}, \quad (3)$$

where  $F_h(x) = F(x + h)$  for any function  $F$ . Now for any  $\epsilon > 0$  we can find  $\varphi \in C_0(\mathbb{R})$  with  $\|g - \varphi\|_{L^2} = \|g_h - \varphi_h\|_{L^2} < \epsilon$ . By the triangle inequality we then have

$$\begin{aligned} \|g_h - g\|_{L^2} &\leq \|g_h - \varphi_h\|_{L^2} + \|\varphi_h - \varphi\|_{L^2} + \|\varphi - g\|_{L^2} \\ &< \|\varphi_h - \varphi\|_{L^2} + 2\epsilon. \end{aligned}$$

Suppose that  $\varphi$  has support contained in the compact set  $K$ . If we pick  $h$  small enough then we can guarantee that  $\varphi_h - \varphi$  is supported on a set with measure at most  $2 \cdot m(K)$ . Now since  $\varphi$  is continuous with compact support, it is uniformly continuous, so we can choose  $h$  small enough that  $|\varphi_h(x) - \varphi(x)| = |\varphi(x + h) - \varphi(x)| < \epsilon$  for all  $x$  in the support of  $\varphi_h - \varphi$ . For such  $h$  we have

$$\|\varphi_h - \varphi\|_{L^2} \leq \epsilon \cdot (2 \cdot m(K))^{1/2},$$

so (2) can be made arbitrarily small, which shows that  $f * g$  is continuous.

First we claim that if  $\varphi$  and  $\psi$  are continuous with compact support then  $\varphi * \psi$  vanishes at infinity. By definition we have that

$$(\varphi * \psi)(x) = \int \varphi(t)\psi(x - t) \, dt.$$

The product  $\varphi(t)\psi(x-t)$  is nonzero only if  $t$  is in the support of  $\varphi$  and  $x-t$  is in the support of  $\psi$ . If pick  $x$  large enough then supports of  $t \mapsto \varphi(t)$  and  $t \mapsto \psi(x-t)$  are disjoint, so this integral is zero.

Let  $f_n$  and  $g_n$  be sequences in  $C_0(\mathbb{R})$  converging in  $L^2$  to  $f$  and  $g$ , respectively. We then have

$$\begin{aligned} |(f * g)(x) - (f_n * g_n)(x)| &\leq |(f * g)(x) - (f_n * g)(x)| + |(f_n * g)(x) - (f_n * g_n)(x)| \\ &\leq \|g\|_{L^2} \cdot \|f - f_n\|_{L^2} + \|f_n\|_{L^2} \cdot \|g - g_n\|_{L^2}. \end{aligned}$$

Since  $f_n \rightarrow f$  and  $g_n \rightarrow g$  in  $L^2$ , we have that  $f_n * g_n$  converges uniformly to  $f * g$ . Since  $f_n * g_n$  vanishes at infinity, we must then have that  $f * g$  vanishes at infinity.  $\square$

4. Let  $(X, \mathcal{A}, \mu)$  be a finite measure space, and let  $p_1 \in (1, \infty]$ . Let  $\{f_n\}$  be a uniformly bounded sequence in  $L^{p_1}(X, \mathcal{A}, \mu)$ . Suppose  $f = \lim_{n \rightarrow \infty} f_n$  exists  $\mu$ -a.e. Prove that  $f \in L^p(X, \mathcal{A}, \mu)$  for all  $p \in [1, p_1]$  and  $f_n \rightarrow f$  in  $L^p(X, \mathcal{A}, \mu)$  for all  $p \in [1, p_1)$ .

*Proof.* Suppose that  $\|f_n\|_{L^{p_1}} \leq M$  for all  $n$ . First we claim that the  $f_n$  are in  $L^p(X, \mathcal{A}, \mu)$  for all  $p \in [1, p_1]$ . In fact, they are uniformly bounded:

$$\begin{aligned} \int_X |f_n|^p &= \int_{|f_n| < 1} |f_n|^p + \int_{|f_n| \geq 1} |f_n|^p \\ &\leq \int_{|f_n| < 1} 1 + \int_{|f_n| \geq 1} |f_n|^{p_1} \\ &\leq \mu(\{f \leq 1\}) + M^{1/p_1}. \end{aligned}$$

Since  $(X, \mathcal{A}, \mu)$  is a finite measure space, this quantity is finite, so  $f_n \in L^p(X, \mathcal{A}, \mu)$  for all  $n$  and  $p \in [1, p_1]$ . We can then use the fact that  $f_n \rightarrow f$  a.e. and Fatou's lemma to show that  $f \in L^p(X, \mathcal{A}, \mu)$  for  $p \in [1, p_1]$ :

$$\int_X |f|^p \leq \liminf_{n \rightarrow \infty} \int_X |f_n|^p < \infty,$$

where the finiteness follows from the  $L^p$  uniform-boundedness of the  $f_n$ .

To establish convergence in  $L^p$ ,  $p \in [1, p_1)$  our plan is to use the Vitali convergence theorem. The family  $f_n$  is tight over  $X$  since  $X$  is a finite measure space and we're given that  $f_n \rightarrow f$  a.e., so it only remains to show that the  $f_n$ 's are uniformly integrable. Intuitively, since the  $f_n$ 's are in  $L^p$ , the measure of the set  $\{f_n \geq N\}$  should shrink as  $N$  grows. Now since  $p < p_1$ , if  $N > 1$  then

$$|f_n|^p \chi_{\{|f_n| \geq N\}} N^{p_1 - p} \leq |f_n|^{p_1}.$$

If we integrate both sides over any measurable set  $E$  we have

$$\int_{E \cap \{|f_n| \geq N\}} |f_n|^p \leq \frac{M}{N^{p_1 - p}}.$$

On the complement we have

$$\int_{E \cap \{|f_n| < N\}} |f_n|^p \leq N^p \cdot \mu(E).$$

Putting these together, we have that

$$\begin{aligned} \int_E |f_n|^p &= \int_{E \cap \{|f_n| \geq N\}} |f_n|^p + \int_{E \cap \{|f_n| < N\}} |f_n|^p \\ &\leq \frac{M}{R^{p_1-p}} + R^p \cdot \mu(E). \end{aligned}$$

If we choose  $R$  so that  $M/R^{p_1-p} < \epsilon/2$  and  $E$  so that  $R^p \cdot \mu(E) < \epsilon/2$  then we'll have that  $\int_E |f_n|^p < \epsilon$  for any  $E$  of sufficiently small measure, so the  $f_n$ 's are uniformly integrable. By the Vitali convergence theorem we have that  $f_n \rightarrow f$  in  $L^p$  for  $p \in [1, p_1)$ .  $\square$

5. Let  $(X, \mathcal{A}, \mu)$  be a measure space, and let  $f : X \rightarrow [0, \infty)$  be  $\mathcal{A}$ -measurable. Consider the measure space  $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu_L)$ , where  $\mathcal{B}_{\mathbb{R}}$  is the Borel  $\sigma$ -algebra on  $\mathbb{R}$  and  $\mu_L$  is the Lebesgue measure, and form the product measure space  $(X \times \mathbb{R}, \sigma(\mathcal{A} \times \mathcal{B}_{\mathbb{R}}), \mu \times \mu_L)$ . Define  $E \subset X \times \mathbb{R}$  by  $(x, y) \in E \iff y \in [0, f(x))$ . Prove that  $E \in \sigma(\mathcal{A} \times \mathcal{B}_{\mathbb{R}})$  and  $(\mu \times \mu_L)(E) = \int_X f \, d\mu$ .

*Proof.* A function is measurable if it pulls measurable sets back to measurable sets. The plan is then to write  $E$  as a union and/or intersection of preimages of measurable sets under measurable functions. The function  $F(x, y) = f(x)$  is measurable since

$$F^{-1}((-\infty, \alpha]) = \{(x, y) : f(x) \leq \alpha\} = \{x : f(x) \leq \alpha\} \times \mathbb{R} \in \sigma(\mathcal{A} \times \mathcal{B}_{\mathbb{R}}),$$

as  $f$  is  $\mu$ -measurable. We also clearly have that the function  $G(x, y) = y$  is measurable. Now consider the function  $H(x, y) = y - f(x)$ .  $H$  is measurable as it is the difference of the measurable functions  $G$  and  $F$ . We then have that  $E$  is measurable through the following decomposition

$$\begin{aligned} E &= \{(x, y) : 0 \leq y < f(x)\} \\ &= \{(x, y) : y \geq 0\} \cap \{(x, y) : y < f(x)\} \\ &= G^{-1}([0, \infty)) \cap H^{-1}((-\infty, 0)). \end{aligned}$$

If  $\{f > 0\}$  is  $\sigma$ -finite we can use Tonelli's theorem to say

$$\begin{aligned} (\mu \times \mu_L)(E) &= \int_{X \times \mathbb{R}} \chi_E(x, y) \, d(\mu \times \mu_L) \\ &= \int_X \int_{\mathbb{R}} \chi_E(x, y) \, d\mu_L d\mu \\ &= \int_X \int_{\mathbb{R}} \chi_{[0, f(x))}(y) \, dy d\mu \\ &= \int_X f(x) \, d\mu. \end{aligned}$$

On the other hand, suppose that  $\{f > 0\}$  is not  $\sigma$ -finite. We claim that  $\int_X f \, d\mu = +\infty$ . Indeed, since we can decompose this set into a countable union,

$$\{f > 0\} = \bigcup_{m=1}^{\infty} \left\{ \frac{1}{m+1} < f \leq \frac{1}{m} \right\} \cup \bigcup_{n=1}^{\infty} \{n < f \leq n+1\}, \quad (4)$$

we must have that one of these sets has infinite measure. We need to show that  $(\mu \times \mu_L)(E) = +\infty$  too. For any  $\alpha, \beta > 0$  we have that if  $\alpha \leq f(x) < \beta$  then the product set

$$\{x : \alpha \leq f(x) < \beta\} \times \{y : 0 \leq \alpha\}$$

is contained in  $E$ . This product set has measure  $\alpha \cdot \mu_L\{\alpha \leq f < \beta\}$ , so by monotonicity we have that

$$\alpha \cdot \mu_L\{\alpha \leq f < \beta\} \leq (\mu \times \mu_L)(E)$$

for all  $\alpha, \beta > 0$ . But by the decomposition (4), we have that some set of the form  $\{\alpha \leq f(x) < \beta\}$  must have infinite measure, so we must have  $(\mu \times \mu_L)(E) = +\infty$ .  $\square$

6. Let  $f \in L^1(\mathbb{R})$  and let  $a_1, \dots, a_k \in \mathbb{R}$  and  $b_1, \dots, b_k \in \mathbb{R} \setminus \{0\}$ . Assume that the quotients  $\frac{a_j}{b_j}$  are all distinct. Determine

$$\lim_{t \rightarrow \infty} \int \left| \sum_{j=1}^k f(b_j x + t a_j) \right| dx.$$

*Solution.* Let  $\varphi \in C_0(\mathbb{R})$  be such that  $\|f - \varphi\|_{L^1} < \epsilon$ . Our plan is to compute the desired limit with  $\varphi$  in place of  $f$  and then argue that the difference can be made small. We have

$$\int \left| \sum_{j=1}^k \varphi(b_j x + t a_j) \right| dx = \int \left| \sum_{j=1}^k \varphi \left[ b_j \left( x + \frac{a_j}{b_j} t \right) \right] \right| dx$$

Now  $\varphi(b_j x + t a_j)$  is  $\varphi$  stretched horizontally by a factor of  $b_j$  and shifted over  $a_j/b_j$ . Since the support of  $\varphi$  is compact and the  $a_j/b_j$  are distinct, the supports of these transformations are disjoint for sufficiently large  $t$ . When these supports are disjoint we then have

$$\begin{aligned} \int \left| \sum_{j=1}^k \varphi(b_j x + t a_j) \right| dx &= \int \sum_{j=1}^k |\varphi(b_j x + t a_j)| dx \\ &= \|\varphi\|_{L^1} \cdot \sum_{j=1}^k \frac{1}{b_j}. \end{aligned}$$

That we can approximate the desired sum for  $f \in L^1$  follows from the reverse triangle inequality.

$$\begin{aligned} \left| \int \left| \sum_{j=1}^k f(b_j x + t a_j) \right| dx - \int \left| \sum_{j=1}^k \varphi(b_j x + t a_j) \right| dx \right| &\leq \sum_{j=1}^k \int |f(b_j x + t a_j) - \varphi(b_j x + t a_j)| dx \\ &= \epsilon \cdot \sum_{j=1}^k \frac{1}{b_j}. \end{aligned}$$

$\square$

## Fall 2015

1. Let  $E$  be a measurable subset of  $[0, 2\pi]$ . Assume that  $f \in C(\mathbb{R})$  is 1-periodic, i.e.  $f(x+1) = f(x)$ . Compute

$$\lim_{n \rightarrow \infty} \int_E f(nx) \, dx.$$

*Solution.* We rewrite the integral over  $E$  as an integral over  $\mathbb{R}$  against the indicator function of  $E$ :

$$\int_E f(nx) \, dx = \int f(nx) \chi_E(x) \, dx.$$

Now let  $\varphi \in C_0^\infty(\mathbb{R})$ . Since  $f \in C(\mathbb{R})$  is 1-periodic, it has a 1-periodic continuous primitive  $F$  with  $F' = f$ . By the chain rule we have  $[\frac{1}{n}F(nx)]' = f(nx)$ . Integration by parts gives

$$\int f(nx) \varphi(x) \, dx = -\frac{1}{n} \int F(nx) \varphi'(x) \, dx.$$

$F(nx)$  is bounded since  $F$  is 1-periodic and  $\varphi \in C_0^\infty(\mathbb{R})$ , so it's integrable. We then have

$$\begin{aligned} \left| \int f(nx) \varphi(x) \, dx \right| &\leq \frac{1}{n} \|F\|_\infty \cdot \|\varphi'\|_{L^1} \\ &\rightarrow 0. \end{aligned}$$

Since  $E$  is a measurable subset of  $[0, 2\pi]$ , it has finite measure and  $\chi_E \in L^1(\mathbb{R})$ . We can then find  $\varphi \in C_0^\infty(\mathbb{R})$  with  $\|\chi_E - \varphi\|_{L^1} < \epsilon$ . Since  $f$  is continuous and 1-periodic, it is bounded and we have

$$\begin{aligned} \left| \int f(nx) \chi_E(x) \, dx - \int f(nx) \varphi(x) \, dx \right| &\leq \|f\|_\infty \cdot \|\chi_E - \varphi\|_{L^1} \\ &\leq \|f\|_\infty \cdot \epsilon. \end{aligned}$$

Since  $\int f(nx) \varphi(x) \, dx \rightarrow 0$ , we must have  $\int_E f(nx) \rightarrow 0$ . □

2. Suppose  $f \in L^1[0, 1]$  and assume that there exists  $C > 0$  such that for all measurable subsets  $E \subset [0, 1]$  we have

$$\int_E |f(x)| \, dx \leq C \mu(E)^{1/2}.$$

Show that  $f \in L^p[0, 1]$  for  $1 \leq p < 2$ . Show that the statement fails for  $p = 2$  by giving a counterexample.

*Proof.* We have that

$$|f(x)|^p - 1 \leq \sum_{n=1}^{\infty} \chi_{\{|f|^p \geq n\}}(x) \leq |f(x)|^p.$$

Since  $[0, 1]$  is a finite measure space, integrating through this inequality shows that  $f \in L^p[0, 1]$  if and only if the series

$$\sum_{n=1}^{\infty} \mu\{|f(x)|^p \geq n\} = \sum_{n=1}^{\infty} \mu\{|f(x)| \geq n^{1/p}\}.$$

converges. By Chebyshev's inequality and the given hypotheses we have

$$n^{1/p} \mu\{|f| \geq n^{1/p}\} \leq \int_{\{|f| \geq n^{1/p}\}} |f| \, dx \leq C \mu\{|f| \geq n^{1/p}\}^{1/2}.$$

Dividing through by  $n^{1/p} \mu\{|f| \geq n^{1/p}\}^{1/2}$  and squaring gives

$$\sum_{n=1}^{\infty} \mu\{|f(x)| \geq n^{1/p}\} \leq \sum_{n=1}^{\infty} \frac{C^2}{n^{2/p}},$$

which converges for all  $p \in [1, 2)$ . □