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260A - Homework 4

Problem 1. Let E and F be two Banach spaces, and let $T \in \mathcal{L}(E, F)$. Prove that $\mathrm{Im}(T)$ is closed if and only if there exists a constant $C > 0$ such that

$$\mathrm{dist}(x, \ker T) \leq C \cdot \|Tx\|, \quad \forall x \in E.$$

Proof. First suppose that the given inequality holds for some $C > 0$. Let Tx_n be a convergent sequence in the image of T. Then the sequence of $x_n + \ker T$'s converges in the quotient $E/\ker T$ by the given inequality. Since T is continuous, $\ker T$ is closed and the quotient $E/\ker T$ is complete. Thus, $x_n + \ker T$ converges to some $x + \ker T$. By continuity, Tx_n then converges to Tx, which is in the image of T. Thus, the image of T is closed.

Conversely, suppose that $\mathrm{Im}(T)$ is closed. □

Problem 5. Let X, Y, and Z be three Banach spaces with norms $\|\cdot\|_X$, $\|\cdot\|_Y$, and $\|\cdot\|_Z$. Assume that $X \subseteq Y$ with compact injection and that $Y \subseteq Z$ with continuous injection. Prove that for any $\epsilon > 0$ there exists $C_\epsilon \geq 0$ such that

$$\|u\|_Y \leq \epsilon \|u\|_X + C_\epsilon \|u\|_Z$$

for all $u \in X$.

Proof. Suppose the proposition were false: that for some ϵ and for every $C \geq 0$ there exists a u_C such that

$$\|u_C\|_Y > \epsilon \|u_C\|_X + C \|u_C\|_Z$$

for all $x \in X$. Set $C = n$ and let u_n be a sequence in X such that the above equality holds, i.e.

$$\|u_n\|_Y > \epsilon \|u_n\|_X + n \|u_n\|_Z. \tag{1}$$

We can assume without loss of generality that the sequence u_n has norm 1 in X, since replacing u_n with $\frac{u_n}{\|u_n\|_X}$ gives the same inequality after multiplying through by $\|u_n\|_X$. By the compactness of the injection of X into Y, we have that u_n has a convergent subsequence in Y. Without loss of generality, assume then that u_n converges in Y. Rearranging (1) gives

$$n\|u_n\|_Z < \|u_n\|_Y - \epsilon \|u_n\|_X \leq \|u_n\|_Y$$

$$\iff \|u_n\|_Z < \frac{1}{n} \|u_n\|_Y.$$

Since u_n converges in Y, the right-hand side of the above inequality must go to zero. Since Y continuously embeds into Z and u_n converges in Y, we must have that u_n converges to zero in both Y and Z. But then the left-hand side of (1) will tend to 0 and the right-hand side will tend to ϵ: a contradiction.

We conclude that the proposition is true.

In class we showed (using the Arzela-Ascoli theorem) that $C^1([0,1])$ compactly embeds into $C([0,1])$. We also have that $C([0,1])$ continuously embeds into $L^1([0,1])$ by $\int_0^1 |f| \, dx \leq \|f\|_\infty$. By the proposition we then have that for all $\epsilon > 0$ there is some C_ϵ with

$$\max_{x\in[0,1]} |f(x)| \leq \epsilon \cdot \max_{x\in[0,1]} |f'(x)| + C_\epsilon \|f\|_{L^1}$$

for all $f \in C^1([0,1])$. $\square$