Final Review

1. Suppose that $a < b$. Prove the following

   (a) If f is continuous and nonnegative on $[a, b]$ and $g : [a, b] \to [a, b]$ is continuous and increasing on $[a, b]$, then
   $$F(x) = \int_a^{g(x)} f(t)\ dt$$
   is increasing on $[a, b]$.

   (b) If f and g are differentiable on $[a, b]$, if f' and g' are Riemann integrable on $[a, b]$, and if $f(a) = 0$ but g is never zero on $[a, b]$, then
   $$f(x) = \int_a^x g(t) \left(\frac{f(t)}{g(t)}\right)'\ dt + \int_a^x \frac{f(t)g'(t)}{g(t)} dt$$
   for all $x \in [a, b]$. (It's not as bad as it looks).

   (c) If f and g are differentiable on $[a, b]$ and if f' and g' are Riemann integrable on $[a, b]$ then
   $$\int_a^b f'(x)g(x)\ dx + \int_a^b f(x)g'(x) = 0$$
   if and only if $f(a)g(a) = f(b)g(b)$.

2. Let (f_n) be a sequence of integrable functions on $[a, b]$ and suppose $f_n \to f$ uniformly on $[a, b]$. Prove f is integrable and
   $$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n.$$

3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x - 1 & \text{if } x \in \mathbb{Q} \\ 1 - x & \text{if } x \notin \mathbb{Q} \end{cases}.$$

Is f continuous at $x = 0$?

4. Use Taylor's theorem to show that

$$0 < x - \log(1 + x) < \frac{1}{2}x^2$$

for all $x > 0$.

5. True or False. There is a sequence of continuous functions $f_n : [-1, 1] \to \mathbb{R}$ converging uniformly to f, given by

$$f(x) = \begin{cases} -1 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}.$$

6. Given $f(x) = |x|$ show that there is a sequence of polynomials $p_n(x)$ with $p_n(0) = 0$ for all n that converges to f uniformly on $[-1, 1]$.

7. Let f be a continuously differentiable function on $[a, b]$. Show that there exists a sequence of polynomials p_n with $p_n \to f$ uniformly and $p_n' \to f'$ uniformly on $[a, b]$.

8. Consider the sequence $\{f_n\}$ of functions defined on $[0, \pi]$ by $f_n(x) = \sin^n(x)$. Show that $\{f_n\}$ converges pointwise. Find its pointwise limit. Does f_n converge uniformly?

9. Let $f_n(x) = \frac{1}{n}x^n$. Show that f_n converges uniformly on $[0, 1]$ but f_n' does not.