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260A - Homework 3

Problem 1. Let $(b_1, b_2, \ldots)$ be a sequence of complex numbers such that $\sum_{n=1}^{\infty} b_n c_n$ is convergent for every $c = (c_1, c_2, \ldots) \in \ell^2$. Show that $b \in \ell^2$.

Proof. Consider the sequence of maps $T_n : \ell^2 \to \mathbb{C}$ that send $(c_1, \ldots)$ to $\sum_{j=1}^{n} b_j c_j$. Since each T_n is just a finite sum, we have that the T_n's form a sequence of bounded linear operators on ℓ^2. Furthermore, this sequence is pointwise bounded: given any $(c_1, c_2, \ldots) \in \ell^2$, since $\sum_{j=1}^{\infty} b_j c_j$ converges, we have that the sequence of partial sums $|T_n(c_1, c_2, \ldots)| = |\sum_{j=1}^{n} b_j c_j|$ is bounded. By the uniform boundedness principle, we have that

$$\sup_{n\in\mathbb{N},\ \|(c_1,c_2,\ldots)\|_2=1} |T_n(c_1, c_2, \ldots)| = \sup_{n\in\mathbb{N}} \|T_n\| = \sum_{j=1}^{\infty} |b_j| < \infty,$$

so $(b_1, b_2, \ldots) \in \ell^2$. $\square$

Problem 2. Let M be a measurable subset of $\mathbb{R}^n$ with finite positive measure. Prove that $L^q(M)$ is of the first category in $L^p(M)$ if $1 \le p < q \le \infty$.

Proof. Since M has finite measure, we have that $L^q(M) \subseteq L^p(M)$ whenever $1 \le p < q \le \infty$. Consider the injection $\iota : L^q(M) \to L^p(M)$ that simply sends $f \in L^p(M)$ to itself. By the generalized Hölder inequality we have that $\|\iota(f)\|_{L^p} = \|f\|_{L^p} \le \mu(M)^{1/r} \|f\|_{L^q}$, where $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. This shows that ι is bounded, and therefore continuous. Since $L^p(M)$ and $L^q(M)$ are Banach spaces, the open mapping theorem tells us that the image of ι is either surjective and open or of the first category in $L^p(M)$.

Our plan is to show that ι is *not* surjective, i.e. that $L^p(M) \setminus L^q(M)$ is nonempty. We'll do this by showing that $L^q(M) = L^p(M)$ would forbid the existence of subsets of M with arbitrarily small measure. Since sets of positive measure in $\mathbb{R}^n$ *do* contain sets of arbitrarily small measure, we'll conclude that $L^q(M) \neq L^p(M)$.

If the embedding $\iota : L^q(M) \to L^p(M)$ were surjective, then the open mapping theorem would imply that ι is actually a homeomorphism. In particular, its inverse, $\iota' : L^p(M) \to L^q(M)$ is a bounded operator. Let A be a subset of M with positive, finite measure and define the function

$$f_A(x) = \frac{1}{m(A)^{1/p}} \cdot \chi_A(x).$$

It's clear that $\|f_A\|_{L^p} = 1$ and that $\|f_A\|_{L^q} = \frac{1}{m(A)^{1/p-1/q}}$. Since ι' is bounded, we have

$$0 < \|f_A\|_{L^q} \le \|\iota'\| \cdot \|f_A\|_{L^p} \implies 0 < \frac{1}{\|\iota'\|^{\frac{pq}{q-p}}} \le m(A).$$

This puts a positive lower bound on the measure of subsets of M. But M, as a subset of $\mathbb{R}^n$ with positive measure, contains set of arbitrarily small measure. We conclude that $L^q(M)$ is of the first category of $L^p(M)$. $\square$

Problem 3. Let $(X, \mathcal{A}, \mu)$ be a finite measure space. Assume that E is a closed subspace of $L^2(X, \mu)$, and that E is contained in $L^\infty(X, \mu)$. Prove that E is finite dimensional.

Proof. By Hölder's inequality, the embedding $\iota : L^\infty(X) \to L^2(X)$, $f \mapsto f$ is continuous, i.e., $\|f\|_{L^\infty} \leq C \cdot \|f\|_{L^2}$ for some finite $C > 0$. Let $e_1, \dots e_n$ be an orthonormal set in E. $\square$