Dimension and Determinants

1. The vector x is in a subspace H with a basis $\mathcal{B} = \{b_1, b_2\}$. Find the $\mathcal{B}$-coordinate vector of x.

   (a) $b_1 = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$, $b_2 = \begin{bmatrix} -2 \\ -7 \\ 5 \end{bmatrix}$, $x = \begin{bmatrix} 2 \\ 9 \\ -7 \end{bmatrix}$.

   (b) $b_1 = \begin{bmatrix} -3 \\ 2 \\ -4 \end{bmatrix}$, $b_2 = \begin{bmatrix} 7 \\ -3 \\ 5 \end{bmatrix}$, $x = \begin{bmatrix} 5 \\ 0 \\ -2 \end{bmatrix}$.

2. We're given a matrix A and an echelon form of A. Find bases for Col(A) and Nul(A), and then state the dimension of these subspaces.

$$A = \begin{bmatrix} 2 & 4 & -5 & 2 & -3 \\ 3 & 6 & -8 & 3 & -5 \\ 0 & 0 & 9 & 0 & 9 \\ -3 & -6 & -7 & -3 & -10 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -5 & 1 & -4 \\ 0 & 0 & 5 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

3. Find a basis for the subspace spanned by the given vectors. What is the dimension of the subspace?

$$\begin{bmatrix} 1 \\ -1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ -7 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ -7 \\ 6 \\ -9 \end{bmatrix}.$$

4. Suppose a 4×7 matrix A has three pivot columns. Is Col(A) $= \mathbb{R}^3$? What is the dimension of Nul(A)? Explain

5. If the subspace of all solutions of $Ax = 0$ has a basis consisting of three vectors and if A is a 5×7 matrix, what is the rank of A?

6. Construct a 3×4 matrix with rank 1.

7. If a 9×8 matrix has rank 7, then what is the dimension of the solution space to $Ax = 0$?

8. Compute the determinant using the cofactor expansion in two different ways. Try to use as few computations as possible. For the 3×3s try the upward and downward product rule (sometimes called the rule of Sarrus).

(a) $\begin{vmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{vmatrix}$ (b) $\begin{vmatrix} 5 & -2 & 4 \\ 0 & 3 & -5 \\ 2 & -4 & 7 \end{vmatrix}$

(c) $\begin{vmatrix} 6 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 2 & 0 & 0 & 0 \\ 8 & 3 & 1 & 8 \end{vmatrix}$ (d) $\begin{vmatrix} 3 & 5 & -8 & 4 \\ 0 & -2 & 3 & -7 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 2 \end{vmatrix}$

9. Here you're given a matrix and one obtained from it by a row operation. How does this affect its determinant?

(a)
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \sim \begin{bmatrix} a & b \\ kc & kd \end{bmatrix}$$

(b)
$$\begin{bmatrix} 1 & 1 & 1 \\ -3 & 8 & -4 \\ 2 & -3 & 2 \end{bmatrix} \sim \begin{bmatrix} k & k & k \\ -3 & 8 & -4 \\ 2 & -3 & 2 \end{bmatrix}$$

(c)
$$\begin{bmatrix} a & b & c \\ 3 & 2 & 2 \\ 6 & 5 & 6 \end{bmatrix} \sim \begin{bmatrix} 3 & 2 & 2 \\ a & b & c \\ 6 & 5 & 6 \end{bmatrix}$$

10. Let $A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$. Write $5A$. Is $\det 5A = 5 \det A$?

11. What is the determinant of an elementary row replacement matrix?

12. What affect does adding two copies of one row to a different row have on the determinant of a matrix? What about columns?