Pell's Equation 1

1. Recall Pell's equation is given by
$$x^2 - dy^2 = N,$$
where d and N are integers. Show that if $d < 0$ then this equation has only finitely many solutions.

2. Let d be a squarefree integer. Recall that $\mathbb{Z}[\sqrt{d}] = \{a + b\sqrt{d} : a, b \in \mathbb{Z}\}$. Define the conjugate of $z = a + b\sqrt{d}$ to be $\overline{z} = a - b\sqrt{d}$ and the function $N : \mathbb{Z}[\sqrt{d}] \to \mathbb{Z}$ (the norm) on $z = a + b\sqrt{d}$ by
$$N(z) = z\overline{z} = a^2 - db^2.$$
   (a) Show that $a_1 + b_1\sqrt{d} = a_2 + b_2\sqrt{d}$ if and only if $a_1 = a_2$ and $b_1 = b_2$.

   (b) Show that the norm is multiplicative, i.e. $N(zw) = N(z)N(w)$ for all $z, w \in \mathbb{Z}[\sqrt{d}]$.

   (c) Show that solutions to Pell's equation correspond to elements of $\mathbb{Z}[\sqrt{d}]$ with norm 1.

   (d) Show that $\alpha \in \mathbb{Z}[\sqrt{d}]$ is a unit if and only if it has norm ± 1.

3. Prove Brahmagupta's composition rule: if x_1, x_2, y_1, y_2 satisfy
$$x_1^2 - dy_1^2 = a, \quad x_2^2 - dy_2^2 = b,$$
then the composition
$$(x_1, y_1) \cdot (x_2, y_2) := (x_1x_2 + dy_1y_2, x_1y_2 + y_1x_2)$$
is a solution to
$$x_3^2 - dy_3^2 = ab.$$

4. Let U^+ be the set of all $\alpha \in \mathbb{Z}[\sqrt{d}]$, $d > 0$ squarefree, with $N(\alpha) = 1$. Prove that U^+ is an infinite abelian group under multiplication. Furthermore, prove that it is generated by ϵ and -1, where ϵ is the smallest nontrivial element with $N(\epsilon) = 1$ and $\epsilon > 1$.

5. (a) Let α be an irrational number and n a positive integer. Show that there exist $p \in \mathbb{Z}$ and $q \in \{1, 2, \ldots, n\}$ such that $|\alpha - p/q| < \frac{1}{(n+1)q}$.

(b) If α is a real number show that there are infinitely many pairs of positive integers (p, q) satisfying $|\alpha - \frac{p}{q}| < \frac{1}{q^2}$.

(c) Using the previous two parts of this exercise, conclude that Pell's equation has a non-trivial solution.