

## 260A - Homework 2

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**Problem 1.** Let  $(e_n)_{n=1}^\infty$  be an orthonormal basis in the Hilbert space  $H$ . Let  $T : H \rightarrow H$  be a linear continuous map such that

$$\sum_{n=1}^{\infty} \|Te_n\|^2$$

converges. Show that there is a sequence  $(T_n)_{n=1}^\infty$  of linear continuous maps  $H \rightarrow H$  such that  $T_n(H)$  has a finite dimension and  $\|T_n - T\| \rightarrow 0$  as  $n \rightarrow \infty$ .

*Proof.* Consider the projection  $T_m$  defined by

$$T_m(x) = \langle x, e_1 \rangle Te_1 + \cdots + \langle x, e_m \rangle Te_m.$$

This function is continuous by an argument similar to the one used on Homework 1, where we showed that every finite dimensional subspace of a normed vector space admits a continuous projection (first we define projections onto the individual components on the space spanned by  $e_1, \dots, e_m$  and then extend these through Hahn-Banach).

The image of  $H$  under  $T_m$  has dimension at most  $m$  as the  $e_j$ 's are linearly independent. Furthermore, we have by Cauchy-Schwarz

$$\begin{aligned} |T_n x - T x|^2 &= \left| \sum_{j=n+1}^{\infty} \langle x, e_j \rangle Te_j \right|^2 \\ &\leq \|x\|^2 \cdot \sum_{j=n+1}^{\infty} \|Te_j\|^2. \end{aligned}$$

Since the sum  $\sum_{j=1}^{\infty} \|Te_j\|^2$  converges, the tail (the last line in the above inequality) must go to zero as  $n \rightarrow \infty$ . We then have that  $\|T_n - T\| \rightarrow 0$  as desired.  $\square$

**Problem 3.** Let  $H$  be a separable infinite dimensional Hilbert space, and suppose that  $e_1, e_2, \dots$  is an orthonormal system in  $H$ . Let  $f_1, \dots$  be another orthonormal system which is complete.

- (i) Prove that if  $\sum_{n=1}^{\infty} \|e_n - f_n\|^2 < 1$  then  $\{e_n\}$  is also a complete orthonormal system.
- (ii) Suppose only that  $\sum_{n=1}^{\infty} \|e_n - f_n\|^2 < \infty$ . Prove that it is still true that  $\{e_n\}$  is a complete orthonormal system.

*Proof.* (i) In order to show that the  $e_j$ 's form a complete system, we'll show that if  $\langle x, e_j \rangle = 0$  for all

$j$  then  $x = 0$ . If this is the case then we have by Cauchy-Schwarz

$$\begin{aligned}
\|x\|^2 &= \left\| \sum_{j=1}^{\infty} \langle x, f_j \rangle f_j \right\|^2 \\
&= \left\| \sum_{j=1}^{\infty} \langle x, f_j - e_j \rangle f_j + \langle x, e_j \rangle f_j \right\|^2 \\
&\leq \|x\|^2 \cdot \sum_{j=1}^{\infty} \|f_j - e_j\|^2 \\
&< \|x\|^2.
\end{aligned}$$

This is a contradiction unless  $x = 0$ , so we conclude that the  $e_j$ 's are complete.

- (ii) (This one was tricky. This solution is in Halmos's book on Hilbert space problems). If the given sum is to converge, then we can choose  $N$  large enough so that  $\sum_{j=N}^{\infty} \|e_j - f_j\|^2 < 1$ . Now define the operator  $T : H \rightarrow H$  by

$$Tf_j = \begin{cases} f_j & \text{if } j < N \\ e_j & \text{if } j \geq N \end{cases}.$$

We have the following bound for any  $x \in H$

$$\begin{aligned}
\|x - Tx\|^2 &= \left\| \sum_{j=1}^{\infty} \langle x, f_j \rangle f_j - Tx \right\|^2 \\
&= \left\| \sum_{j=N}^{\infty} \langle x, f_j \rangle (f_j - e_j) \right\|^2 \\
&\leq \|x\|^2 \cdot \sum_{j=N}^{\infty} \|f_j - e_j\|^2 \\
&\leq \|x\|^2.
\end{aligned}$$

In particular, we have that the operator  $I - T$  has norm less than 1. We claim this means that  $T$  is invertible. In general, if an operator  $A$  satisfies  $\|A\| < 1$ , then  $I - A$  is invertible where it is defined. The bound on  $A$  tells us that  $\sum \|A\|^n$  is finite, so the operator  $\sum A^n$  exists. Furthermore we have

$$(1 - A) \cdot \sum_{n=0}^{\infty} A^n = \sum_{n=0}^{\infty} A^n - \sum_{n=1}^{\infty} A^n = I.$$

Returning to the problem at hand, we have that  $T$  is invertible on the span of the  $f_j$ 's. Since this span is dense, by continuity we have that  $T$ 's inverse extends to an operator on all of  $H$ . By invertibility we have that the  $Tf_j$ 's span all of  $H$ . But  $\{Tf_j\} = \{f_1, \dots, f_{N-1}, e_N, e_{N+1}, \dots\}$ .

In particular, we have that the orthogonal complement to the span of  $\{e_N, \dots\}$  has dimension  $N - 1$ . But  $e_1, \dots, e_{N-1}$  are  $N - 1$  linearly independent vectors outside of the span of  $\{e_N, \dots\}$ . We conclude that the  $e_j$ 's span  $H$ .

□

**Problem 4.** Let  $T : B_1 \rightarrow B_2$  be a compact operator where  $B_1$  and  $B_2$  are Banach spaces. Show that if  $T$  is compact then  $\text{Im}T$  has a dense countable subset.

*Proof.* Write  $B_1 = \cup_{n \in \mathbb{N}} B(0, n)$ . That is,  $B_1$  is a union of balls centered at the origin with natural radius. From this we have that  $\text{Im}T = \cup_{n \in \mathbb{N}} T[B(0, n)]$ . If we can show that  $T[B(0, n)]$  is separable for each  $n$  then we're done since the countable union of separable spaces is separable.

Our plan is to show that  $T[B(0, n)]$  is precompact for each  $n$ . Then we'll show that precompact spaces are separable and we'll be finished. That  $T[B(0, n)]$  is precompact follows from the compactness of  $T$ . If we let  $y_k = Tx_k$  be a sequence in  $T[B(0, n)]$ , then the compactness of  $T$  says that  $y_k$  has a convergent subsequence.

Suppose  $A$  is a precompact subset of a metric space  $(X, d)$ . For each  $m$  the collection  $\{B(x, \frac{1}{m})\}_{x \in A}$  forms an open cover for the closure of  $A$ . Since the closure of  $A$  is compact, we need to take only finitely many of these balls to cover  $\overline{A}$ . As  $m$  ranges over the natural numbers, the centers of these finite coverings form a countable dense subset of  $A$ , so  $A$  is separable.

The image of  $T$  is the union  $\cup_{n \in \mathbb{N}} T[B(0, n)]$ . Since each  $T[B(0, n)]$  is precompact, it is separable. Since the countable union of separable sets is separable, we have that the image of  $T$  is separable. □

**Problem 5.** Let  $H$  be a Hilbert space and let  $U : H \rightarrow H$  be unitary so that  $UU^* = U^*U = 1$ .

(i) Show that

$$H = \ker(1 - U) \oplus \overline{\text{Im}(I - U)},$$

where the direct sum is orthogonal.

(ii) Let  $P$  be the orthogonal projection onto  $\ker(1 - U)$  and let us set

$$S_n = \frac{1}{n} \sum_{j=0}^{n-1} U^j.$$

Show that  $S_n x \rightarrow Px$  for all  $x \in H$  as  $n \rightarrow \infty$ .

*Proof.* (i) Suppose  $y$  is in  $\overline{\text{Im}(I - U)}$  and  $z$  is in  $\ker(1 - U)$ . Then there is a sequence of  $y_n \in \text{Im}(1 - U)$  with  $y_n = (1 - U)x_n$  for some sequence  $x_n \in H$  and  $y_n \rightarrow y$ . By the continuity of the inner product

we have

$$\begin{aligned}
\langle y, z \rangle &= \lim_{n \rightarrow \infty} \langle (1 - U)x_n, z \rangle \\
&= \lim_{n \rightarrow \infty} \langle (U^* - 1)Ux_n, z \rangle \\
&= \lim_{n \rightarrow \infty} \langle Ux_n, (U - 1)z \rangle \\
&= 0.
\end{aligned}$$

This shows that  $\ker(1 - U) \subseteq \overline{\operatorname{Im}(1 - U)}^\perp$ . Suppose conversely that for some  $z \in H$  we have that  $\langle y, z \rangle = 0$  for all  $y \in \overline{\operatorname{Im}(1 - U)}$ . We then have

$$\begin{aligned}
\|(1 - U)z\|^2 &= \langle (1 - U)z, (1 - U)z \rangle \\
&= \langle (1 - U^*)(1 - U)z, z \rangle \\
&= \langle (1 - U)(1 - U^*)z, z \rangle \\
&= 0.
\end{aligned}$$

This shows that  $\overline{\operatorname{Im}(1 - U)}^\perp \subseteq \ker(1 - U)$ . Finally, we know that any Hilbert space splits as a direct sum of a closed subspace and its orthogonal complement, so  $H = \ker(1 - U) \oplus \overline{\operatorname{Im}(1 - U)}$ .

- (ii) Take  $y \in H$ . By part (i) of this exercise, we can write  $y = \lim_{m \rightarrow \infty} (y_0 + y_m)$ , where  $y_0$  is in the kernel of  $1 - U$  and  $y_m = (1 - U)x_m$  for some sequence  $x_m$  in  $H$ .  $P$ , the orthogonal projection onto the kernel of  $1 - U$  will send  $y$  to  $y_0$ . We then have

$$\begin{aligned}
\|S_n y - P y\| &= \lim_{m \rightarrow \infty} \|S_n(y_0 + y_m) - y_0\| \\
&= \lim_{m \rightarrow \infty} \left\| \left( \frac{1}{n} \sum_{j=0}^{n-1} U^j \right) (y_0 + y_m) - y_0 \right\| \\
&= \lim_{m \rightarrow \infty} \left\| \left( \frac{1}{n} \sum_{j=0}^{n-1} U^j \right) (1 - U)x_m \right\| \\
&= \lim_{m \rightarrow \infty} \frac{1}{n} \cdot \left\| \sum_{j=0}^{n-1} (U^j - U^{j+1})x_m \right\| \\
&= \lim_{m \rightarrow \infty} \frac{1}{n} \|(1 - U^n)x_m\| \\
&\leq \lim_{m \rightarrow \infty} \frac{2}{n} \|x_m\|.
\end{aligned}$$

The last line follows from the fact that  $U$  is unitary, so  $\|U^n x\| = \|x\|$ . As  $n \rightarrow \infty$ , this quantity goes to zero as desired.

□

**Problem 6.** Define the space  $\mathcal{B}$  by

$$\mathcal{B} = \left\{ u : \mathbb{C} \rightarrow \mathbb{C} : u \text{ is holomorphic and } \int_{\mathbb{C}} |u(z)|^2 e^{-|z|^2} L(dz) < \infty \right\},$$

where  $L(dz)$  is the Lebesgue measure in  $\mathbb{C}$ . Show that  $\mathcal{B}$  becomes a Hilbert space when equipped with the scalar product

$$\langle u, v \rangle = \int_{\mathbb{C}} u(z) \overline{v(z)} e^{-|z|^2} L(dz).$$

*Proof.* That  $\mathcal{B}$  is a complex vector space follows immediately from the triangle inequality on  $\mathbb{C}$ . That the proposed scalar product is finite follows from Cauchy-Schwarz:

$$\begin{aligned} |\langle u, v \rangle| &= \left| \int_{\mathbb{C}} u(z) \overline{v(z)} e^{-|z|^2} L(dz) \right| \\ &\leq \int_{\mathbb{C}} (|u(z)| e^{-|z|^2/2}) \cdot (|v(z)| e^{-|z|^2/2}) L(dz) \\ &\leq \left( \int_{\mathbb{C}} |u(z)|^2 e^{-|z|^2} L(dz) \right)^{1/2} \cdot \left( \int_{\mathbb{C}} |v(z)|^2 e^{-|z|^2} L(dz) \right)^{1/2} \\ &< \infty. \end{aligned}$$

That the scalar product is sesquilinear is obvious. It remains to show that  $\mathcal{B}$  is complete with respect to the norm induced by this inner product. Suppose  $u_n$  is a Cauchy sequence in  $\mathcal{B}$ . Then for a given  $\epsilon > 0$  we have that

$$\int_{\mathbb{C}} |u_m(z) - u_n(z)|^2 e^{-|z|^2} L(dz) < \epsilon$$

for  $m$  and  $n$  sufficiently large. In this case we also have that for any  $R > 0$

$$\begin{aligned} \epsilon &> \int_{\mathbb{C}} |u_m(z) - u_n(z)|^2 e^{-|z|^2} L(dz) \\ &\geq \int_{|z| \leq R} |u_m(z) - u_n(z)|^2 e^{-|z|^2} L(dz) \\ &\geq e^{-R^2} \int_{|z| \leq R} |u_m(z) - u_n(z)|^2 L(dz). \end{aligned}$$

Now take any  $w \in B(0, R)$  and  $r < \text{dist}(w, \partial B(0, R))$ . By Cauchy's integral formula we have

$$\epsilon \geq e^{-R^2} \cdot \pi r^2 \cdot |u_m(z) - u_n(z)|^2.$$

In particular, this shows that  $u_n$  is uniformly Cauchy on the closed disk of radius  $R$ . Since any compact set in  $\mathbb{C}$  lies inside  $\overline{D(0, R)}$  for some  $R$ , we have that  $u_n$  converges uniformly on compact subsets of  $\mathbb{C}$ . Now the uniform limit of holomorphic functions on a compact set is holomorphic. By the uniqueness of analytic continuation, we have that  $u_n$  converges pointwise to an entire function,  $u$ .

Now we need to show that  $u$  is in  $\mathcal{B}$ . This follows from the uniform convergence of  $u_n$  to  $u$  on compact subsets and the fact that  $u_n \rightarrow u$  in  $L^2$  on all compact subsets.  $\square$

**Problem 7.**

- (i) Let  $S$  be a unitary operator on a complex Hilbert space. Prove that for every complex number  $|\lambda| < 1$  the operator  $S - \lambda I$  is invertible.
- (ii) For a fixed vector  $v$  in the Hilbert space and all  $\lambda \in \mathbb{C}$ ,  $|\lambda| < 1$ , we define

$$h(\lambda) = \langle (S + \lambda I)(S - \lambda I)^{-1}v, v \rangle.$$

Show that  $\operatorname{Re}(h)$  is a positive harmonic function.

*Proof.* (i) Since  $S$  is unitary we have that  $S - \lambda I = S(1 - \lambda S^*)$ . Furthermore, we have that  $\sum_{j=0}^{\infty} \|\lambda S^*\|^j = \sum_{j=0}^{\infty} |\lambda|^j < \infty$ . This shows that  $\sum_{j=0}^{\infty} (\lambda S^*)^j$  converges to a linear operator: the inverse of  $1 - \lambda S^*$ . This tells us that  $(S - \lambda I)^{-1} = S^*(1 - \lambda S^*)^{-1}$  is well-defined.

- (ii) Check out this computation.

$$\begin{aligned} h(\lambda) &= \langle (S + \lambda I)(S - \lambda I)^{-1}v, v \rangle \\ &= \left\langle (S + \lambda I)S^* \sum_{n=0}^{\infty} \lambda^n (S^*)^n v, v \right\rangle \\ &= \left\langle \sum_{n=0}^{\infty} [\lambda^n (S^*)^n + \lambda^{n+1} (S^*)^{n+1}] v, v \right\rangle \\ &= \langle v, v \rangle + 2 \sum_{n=1}^{\infty} \langle (S^*)^n v, v \rangle \lambda^n. \end{aligned}$$

By Cauchy-Schwarz and the fact that  $S$  is unitary we have that  $|\langle (S^*)^n v, v \rangle| \leq \|v\|^2$ , so the above sum converges for all  $|\lambda| < 1$  and  $h$  is holomorphic on the unit disk and its real part is harmonic.  $\square$

**Problem 8.** Let  $H$  be the Hilbert space  $L^2(\mathbb{R})$  and define  $U : H \rightarrow H$  by

$$Uf(x) = f(x - 1).$$

Show that  $U$  has no nonzero eigenvectors.

*Proof.* If  $Uf = \lambda f$  then we would have

$$\int |\lambda|^2 |f(x)|^2 dx = \int |f(x - 1)|^2 dx = \int |f(x)|^2 dx,$$

which would imply that  $|\lambda| = 1$ . This tells us that for all  $n \in \mathbb{Z}$

$$\int_n^{n+1} |f(x)|^2 dx = \int_n^{n+1} |f(x - 1)|^2 dx = \cdots = \int_n^{n+1} |f(x - n)|^2 dx = \int_0^1 |f(x)|^2 dx.$$

But this is a problem, since by the monotone convergence theorem we have

$$\int |f|^2 \, dx = \sum_{n \in \mathbb{Z}} \int_n^{n+1} |f(x)|^2 \, dx = \sum_{n \in \mathbb{Z}} \int_0^1 |f(x)|^2 \, dx$$

The only way this can be finite is if  $\int_0^1 |f(x)|^2 \, dx = 0$ , i.e.  $f = 0$  a.e..

□