Homework 1 (Due Friday, October 26, 2018)

Problem 1. Let $a = (a_n)_{n=1}^{\infty}$ be a sequence of complex numbers. We set

$$l^\infty = \{a : \|a\|_\infty = \sup_n |a_n| < \infty\},$$
$$l^p = \left\{a : \|a\|_p = \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{1/p} < \infty\right\}, \quad 1 \le p < \infty.$$

(i) Show that l^p, $1 \le p \le \infty$, are Banach spaces.

(ii) Prove that $l^\infty = (l^1)^*$ but $(l^\infty)^* \neq l^1$. More precisely, for the last part, show that there exists a linear continuous form on l^∞ which is *not* of the form

$$l^\infty \ni a \mapsto \sum_{j=1}^{\infty} a_j b_j,$$

where $b = (b_j)_{j=1}^{\infty} \in l^1$.

Problem 2. Prove that if Z is a subspace of a normed linear space X, and $y \in X$ has distance d from Z, then there exists $\Lambda \in X^*$ such that $\|\Lambda\| \le 1$, $\Lambda(y) = d$ and $\Lambda(z) = 0$ for all $z \in Z$.

Note. The distance referred to above is $d = \inf\{\|z - y\| : z \in Z\}$.

Problem 3. Show that linear combinations of functions of the form

$$\mathbb{R} \ni t \mapsto \frac{1}{t - z}, \quad \operatorname{Im} z \neq 0,$$

are dense in the space of continuous functions on $\mathbb{R}$ which tend to 0 at infinity. (Here we equip the latter space with the uniform norm.)

Problem 4. Let V be a complex vector space and let f_j, $0 \le j \le N$, be linear forms on V such that

$$\cap_{j=1}^{N} \operatorname{Ker} f_j \subset \operatorname{Ker} f_0.$$

Show that f_0 is a linear combination of the f_j's, $1 \le j \le N$.

Problem 5. Let X be a Banach space such that X^* is separable. Prove that X is separable.

Problem 6. Show that the closure in $L^2(\mathbb{R})$ of the set of functions of the form

$$p(x)e^{-x^2}, \quad x \in \mathbb{R},$$

where p is a complex polynomial on $\mathbb{R}$, is equal to all of $L^2(\mathbb{R})$.

Hint. Use the Fourier transformation.

Problem 7. Let $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ be 2π–periodic, so that $f(x + 2\pi) = f(x)$, $x \in \mathbb{R}$. Show that linear combinations of the translates $f(x - a)$, $a \in \mathbb{R}$, are dense in $L^1(0, 2\pi)$ if and only if each Fourier coefficient of f is $\neq 0$.

Problem 8. Let E_1 be a finite-dimensional subspace of the normed space E. Show that there exists a continuous projection $P : E \to E_1$, i.e., a continuous linear map $P : E \to E$ such that $P^2 = P$ and the range of P is equal to E_1.