Homework 4 (Due Friday, December 7, 2018)

Please turn in solutions to any 3 of the following 5 problems.

Problem 1. Let E and F be two Banach spaces, and let $T \in \mathcal{L}(E, F)$. Prove that $\mathrm{Im}(T)$ is closed if and only if there exists a constant $C > 0$ such that

$$\mathrm{dist}(x, \mathrm{Ker}(T)) \leq C\|Tx\|, \quad \forall x \in E.$$

Problem 2. Prove that if H is a Hilbert space and B is a Banach space, then the space $\mathcal{L}_c(B, H)$ of compact operators $B \to H$ is the closure of the set of operators in $\mathcal{L}(B, H)$ which are of finite rank. Here we recall that the rank of an operator $T \in \mathcal{L}(B, H)$ is the dimension of the range of T.

Problem 3. Let B be a complex Banach space, $B \neq \{0\}$, and let $T \in \mathcal{L}(B, B)$. Prove the following:

(i) There exists a non-empty compact set $\mathrm{Spec}(T) \subset \mathbb{C}$, called the spectrum of T, such that the resolvent $R(z) := (T - zI)^{-1} \in \mathcal{L}(B, B)$ exists if and only if $z \notin \mathrm{Spec}(P)$.

(ii) The resolvent $R(z)$ is holomorphic in the complement of $\mathrm{Spec}(T)$, with the various equivalent definitions of holomorphy in Problem 6, Homework 3.

(iii) We have $|z| \leq \|T\|$ when $z \in \mathrm{Spec}(T)$.

Problem 4. Let $E = L^p(0, 1)$ with $1 \leq p < \infty$. Given $u \in E$, set

$$Tu(x) = \int_0^x u(t)dt.$$

(i) Prove that $T : E \to E$ is compact.

(ii) Compute the eigenvalues of T and the spectrum of T.

(iii) Give an explicit formula for $(T - \lambda I)^{-1}$ when $\lambda \notin \mathrm{Spec}(T)$.

Problem 5. Let X, Y and Z be three Banach spaces with norms $\|\cdot\|_X$, $\|\cdot\|_Y$, and $\|\cdot\|_Z$. Assume that $X \subset Y$ with compact injection and that $Y \subset Z$ with continuous injection. Prove that for any $\varepsilon > 0$ there exists $C_\varepsilon \geq 0$ such that

$$\|u\|_Y \leq \varepsilon\|u\|_X + C_\varepsilon\|u\|_Z,$$

for all $u \in X$.

Application: Prove that for any $\varepsilon > 0$ there exists $C_\varepsilon \geq 0$ such that

$$\max_{[0,1]} |u| \leq \varepsilon \max_{[0,1]} |u'| + C_\varepsilon\|u\|_{L^1}, \quad u \in C^1([0, 1]).$$