Real Analysis Qualifying Exams

0.1 Spring 2016

1. Assume $f \in L^1[0,1]$. Compute
$$\lim_{k\to\infty} \int_{[0,1]} |f|^{1/k}\ dx.$$

Solution. Let's split this integral into three regions.
$$\int_{[0,1]} |f|^{1/k}\ dx = \int_{f=0} |f|^{1/k}\ dx + \int_{0<|f|\le 1} |f|^{1/k}\ dx + \int_{|f|>1} |f|^{1/k}\ dx.$$
The integral over the first region is clearly zero for all k. On the second region we have that $|f|^{1/k} \le 1$ for all k. Since the interval $[0,1]$ has finite measure, we have that the constant function 1 is in $L^1(\{x : 0 < |f| \le 1\})$, so the dominated convergence theorem says that the integral over the second region goes to $m(\{0 < |f| \le 1\})$. Similarly, on the third region we have that $|f|^{1/k} \le |f|$, which is in L^1, so the dominated convergence theorem says that the third integral goes to $m(\{|f| > 1\})$. Combining these, we have that
$$\lim_{k\to\infty} \int_{[0,1]} |f|^{1/k}\ dx = m(\{|f| > 0\}).$$
□

2. Let $\{f_n\}$ be a sequence of measurable functions on $[0,1]$ and $0 \le f_n \le 1$ a.e. Assume that
$$\lim_{n\to\infty} \int_{[0,1]} f_n g\ dx = \int_{[0,1]} fg\ dx$$
for some $f \in L^1[0,1]$ and any $g \in C[0,1]$. Prove that $0 \le f \le 1$ a.e.

Solution. Since $f \in L^1[0,1]$, by the Lebesgue differentiation theorem we have that
$$\frac{1}{m(E)} \int_E f(t)\ dt \to f(x) \tag{1}$$
as E shrinks to x for almost all x. Furthermore, since $0 \le f_n \le 1$ we also have that
$$\frac{1}{m(E)} \int_E f_n(t)\ dt \to f_n(x) \in [0,1]$$
as E shrink to x for almost all x. Intuitively, we'd like to replace the integral of f in (1) with a limit of integrals of f_n.

We claim that the function g in the given hypothesis can be replaced with the indicator function of an interval χ_I. To see this, let g_m be a sequence of continuous functions with $g_m \to \chi_I$ in L^1 and $0 \le \chi_I \le 1$. By extracting a subsequence we can assume that $g_m \to \chi_I$ a.e. as well. We then have
$$\int_0^1 |f_n\chi_I - f\chi_i| \le \int_0^1 |f_n\chi_I - f_n g_m| + \int_0^1 |f_n g_m - f g_m| + \int_0^1 |f g_m - f\chi_i|.$$

Since $\|f_n\|_{L^\infty} \leq 1$, we have that the first integral on the RHS can be made small uniformly in n by picking m large. The second integral goes to zero as $n \to \infty$ by hypothesis since g_m is continuous. The third integral can be made small for m large by dominated convergence since $|fg_m| \leq |f| \in L^1$.

For almost all x, if I_k is a sequence of intervals shrinking to x then

$$\begin{aligned}\frac{1}{m(I_k)} \int_{I_k} f \, dx &= \frac{1}{m(I_k)} \int f\chi_{I_k} \, dx \\ &= \lim_{n\to\infty} \frac{1}{m(I_k)} \int f_n \chi_{I_k} \, dx.\end{aligned}$$

Since $0 \leq f_n \leq 1$, the RHS is in $[0, 1]$ for almost all x. By the Lebesgue differentiation theorem we then have that $0 \leq f \leq 1$ a.e. $\square$

3. Let $f, g \in L^2(\mathbb{R}, \mathcal{M}_L, \mu_L)$. Show that $f * g$ is a continuous function on $\mathbb{R}$ vanishing at infinity, that is, $f * g \in C(R)$ and $\lim_{|x|\to\infty}(f * g)(x) = 0$.

Proof. For any h we have by Hölder's inequality

$$|(f*g)(x+h) - (f*g)(x)| = \left|\int f(t)[g(x+h-t) - g(x-t)] \, dt\right| \tag{2}$$

$$\leq \|f\|_{L^2} \cdot \|g_h - g\|_{L^2}, \tag{3}$$

where $F_h(x) = F(x+h)$ for any function F. Now for any $\epsilon > 0$ we can find $\varphi \in C_0(\mathbb{R})$ with $\|g - \varphi\|_{L^2} = \|g_h - \varphi_h\|_{L^2} < \epsilon$. By the triangle inequality we then have

$$\begin{aligned}\|g_h - g\|_{L^2} &\leq \|g_h - \varphi_h\|_{L^2} + \|\varphi_h - \varphi\|_{L^2} + \|\varphi - g\|_{L^2} \\ &< \|\varphi_h - \varphi\|_{L^2} + 2\epsilon.\end{aligned}$$

Suppose that φ has support contained in the compact set K. If we pick h small enough then we can guarantee that $\varphi_h - \varphi$ is supported on a set with measure at most $2 \cdot m(K)$. Now since φ is continuous with compact support, it is uniformly continuous, so we can choose h small enough that $|\varphi_h(x) - \varphi(x)| = |\varphi(x+h) - \varphi(x)| < \epsilon$ for all x in the support of $\varphi_h - \varphi$. For such h we have

$$\|\varphi_h - \varphi\|_{L^2} \leq \epsilon \cdot (2 \cdot m(K))^{1/2},$$

so (2) can be made arbitrarily small, which shows that $f * g$ is continuous.

First we claim that if φ and ψ are continuous with compact support then $\varphi * \psi$ vanishes at infinity. By definition we have that

$$(\varphi * \psi)(x) = \int \varphi(t)\psi(x-t) \, dt.$$

The product $\varphi(t)\psi(x-t)$ is nonzero only if t is in the support of φ and $x-t$ is in the support of φ. If pick x large enough then supports of $t \mapsto \varphi(t)$ and $t \mapsto \psi(x-t)$ are disjoint, so this integral is zero.

Let f_n and g_n be sequences in $C_0(\mathbb{R})$ converging in L^2 to f and g, respectively. We then have

$$\begin{aligned}|(f * g)(x) - (f_n * g_n)(x)| &\leq |(f * g)(x) - (f_n * g)(x)| + |(f_n * g)(x) - (f_n * g_n)(x)| \\ &\leq \|g\|_{L^2} \cdot \|f - f_n\|_{L^2} + \|f_n\|_{L^2} \cdot \|g - g_n\|_{L^2}\,.\end{aligned}$$

Since $f_n \to f$ and $g_n \to g$ in L^2, we have that $f_n * g_n$ converges uniformly to $f * g$. Since $f_n * g_n$ vanishes at infinity, we must then have that $f * g$ vanishes at infinity. $\square$

4. Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and let $p_1 \in (1, \infty]$. Let $\{f_n\}$ be a uniformly bounded sequence in $L^{p_1}(X, \mathcal{A}, \mu)$. Suppose $f = \lim_{n\to\infty} f_n$ exists μ-a.e. Prove that $f \in L^p(X, \mathcal{A}, \mu)$ for all $p \in [1, p_1]$ and $f_n \to f$ in $L^p(X, \mathcal{A}, \mu)$ for all $p \in [1, p_1)$.

Proof. Suppose that $\|f_n\|_{L^{p_1}} \leq M$ for all n. First we claim that the f_n are in $L^p(X, \mathcal{A}, \mu)$ for all $p \in [1, p_1]$. In fact, they are uniformly bounded:

$$\begin{aligned}\int_X |f_n|^p &= \int_{|f_n|<1} |f_n|^p + \int_{|f_n|\geq 1} |f_n|^p \\ &\leq \int_{|f_n|<1} 1 + \int_{|f_n|\geq 1} |f_n|^{p_1} \\ &\leq \mu(\{f \leq 1\}) + M^{p_1}.\end{aligned} \tag{4}$$

Since $(X, \mathcal{A}, \mu)$ is a finite measure space, this quantity is finite, so $f_n \in L^p(X, \mathcal{A}, \mu)$ for all n and $p \in [1, p_1]$. We can then use the fact that $f_n \to f$ a.e. and Fatou's lemma to show that $f \in L^p(X, \mathcal{A}, \mu)$ for $p \in [1, p_1]$:

$$\int_X |f|^p \leq \liminf_{n\to\infty} \int_X |f_n|^p < \infty,$$

where the finiteness follows from the L^p uniform-boundedness of the f_n.

To establish convergence in L^p, $p \in [1, p_1)$ our plan is to use the Vitali convergence theorem. The family f_n is tight over X since X is a finite measure space and we're given that $f_n \to f$ a.e., so it only remains to show that the f_n's are uniformly integrable. To this end, let E be any measurable subset of X. Since f_n is in L^{p_1}, we have that $|f_n|^p \in L^{p_1/p}$. If we let q be the Hölder conjugate to p_1/p then we have

$$\begin{aligned}\int_E |f_n|^p &= \int_X |f_n|^p \cdot \chi_E \\ &\leq \||f_n|^p\|_{L^{p_1/p}} \cdot \|\chi_E\|_{L^q} \\ &\leq M^{p_1^2/p} \cdot \mu(E)^{1/q}.\end{aligned}$$

If we choose E so that $\mu(E)^{1/q} < \epsilon \cdot M^{-p_1^2/p}$, then we'll have that $\int_E |f_n|^p < \epsilon$, so the f_n's are uniformly integrable. By the Vitali convergence theorem we have that $f_n \to f$ in L^p for $p \in [1, p_1]$. $\square$

5. Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f : X \to [0, \infty)$ be $\mathcal{A}$-measurable. Consider the measure space $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu_L)$, where $\mathcal{B}_{\mathbb{R}}$ is the Borel σ-algebra on $\mathbb{R}$ and μ_L is the Lebesgue measure, and form the product measure space $(X \times \mathbb{R}, \sigma(\mathcal{A} \times \mathcal{B}_{\mathbb{R}}), \mu \times \mu_L)$. Define $E \subset X \times R$ by $(x, y) \in E \iff y \in [0, f(x))$. Prove that $E \in \sigma(\mathcal{A} \times \mathcal{B}_{\mathbb{R}})$ and $(\mu \times \mu_L)(E) = \int_X f \, d\mu$.

Proof. A function is measurable if it pulls measurable sets back to measurable sets. The plan is then to write E is a union and/or intersection of preimages of measurable sets under measurable functions. The function $F(x, y) = f(x)$ is measurable since

$$F^{-1}[(-\infty, \alpha]) = \{(x, y) : f(x) \leq \alpha\} = \{x : f(x) \leq \alpha\} \times \mathbb{R} \in \sigma(\mathcal{A} \times \mathcal{B}_{\mathbb{R}}),$$

as f is μ-measurable. We also clearly have that the function $G(x, y) = y$ is measurable. Now consider the function $H(x, y) = y - f(x)$. H is measurable as it is the difference of the measurable functions G and F. We then have that E is measurable through the following decomposition

$$\begin{aligned} E &= \{(x, y) : 0 \leq y < f(x)\} \\ &= \{(x, y) : y \geq 0\} \cap \{(x, y) : y < f(x)\} \\ &= G^{-1}[[0, \infty)] \cap H^{-1}[(-\infty, 0)]. \end{aligned}$$

If $\{f > 0\}$ is σ-finite we can use Tonelli's theorem to say

$$\begin{aligned} (\mu \times \mu_L)(E) &= \int_{X \times \mathbb{R}} \chi_E(x, y) \; d(\mu \times \mu_L) \\ &= \int_X \int_{\mathbb{R}} \chi_E(x, y) \; d\mu_L d\mu \\ &= \int_X \int_{\mathbb{R}} \chi_{[0, f(x))}(y) \; dy d\mu \\ &= \int_X f(x) \; d\mu. \end{aligned}$$

On the other hand, suppose that $\{f > 0\}$ is note σ-finite. We claim that $\int_X f \, d\mu = +\infty$. Indeed, since we can decompose this set into a countable union,

$$\{f > 0\} = \bigcup_{m=1}^{\infty} \{\frac{1}{m+1} < f \leq \frac{1}{m}\} \cup \bigcup_{n=1}^{\infty} \{n < f \leq n + 1\}, \tag{5}$$

we must have that one of these sets has infinite measure. We need to show that $(\mu \times \mu_L)(E) = +\infty$ too. For any $\alpha, \beta > 0$ we have that if $\alpha \leq f(x) < \beta$ then the product set

$$\{x : \alpha \leq f(x) < \beta\} \times \{y : 0 \leq \alpha\}$$

is contained in E. This product set has measure $\alpha \cdot \mu_L\{\alpha \leq f < \beta\}$, so by monotonicity we have that

$$\alpha \cdot \mu_L\{\alpha \leq f < \beta\} \leq (\mu \times \mu_L)(E)$$

for all $\alpha, \beta > 0$. But by the decomposition (5), we have that some set of the form $\{\alpha \leq f(x) < \beta\}$ must have infinite measure, so we must have $(\mu \times \mu_L)(E) = +\infty$. $\square$

6. Let $f \in L^1(\mathbb{R})$ and let $a_1, \ldots, a_k \in \mathbb{R}$ and $b_1, \ldots, b_k \in \mathbb{R} \setminus \{0\}$. Assume that the quotients $\frac{a_j}{b_j}$ are all distinct. Determine

$$\lim_{t\to\infty} \int \left| \sum_{j=1}^{k} f(b_j x + t a_j) \right| dx.$$

Solution. Let $\varphi \in C_0(\mathbb{R})$ be such that $\|f - \varphi\|_{L^1} < \epsilon$. Our plan is to compute the desired limit with φ in place of f and then argue that the difference can be made small. We have

$$\int \left| \sum_{j=1}^{k} \varphi(b_j x + t a_j) \right| dx = \int \left| \sum_{j=1}^{k} \varphi \left[b_j \left(x + \frac{a_j}{b_j} t \right) \right] \right| dx$$

Now $\varphi(b_j x + t a_j)$ is φ stretched horizontally by a factor of b_j and shifted over a_j/b_j. Since the support of φ is compact and the a_j/b_j are distinct, the supports of these transformations are disjoint for sufficiently large t. When these supports are disjoint we then have

$$\int \left| \sum_{j=1}^{k} \varphi(b_j x + t a_j) \right| dx = \int \sum_{j=1}^{k} |\varphi(b_j x + t a_j)| \, dx$$

$$= \|\varphi\|_{L^1} \cdot \sum_{j=1}^{k} \frac{1}{b_j}.$$

That we can approximate the desired sum for $f \in L^1$ follows from the reverse triangle inequality.

$$\left| \int \left| \sum_{j=1}^{k} f(b_j x + t a_j) \right| dx - \int \left| \sum_{j=1}^{k} \varphi(b_j x + t a_j) \right| dx \right| \leq \sum_{j=1}^{k} \int |f(b_j x + t a_j) - \varphi(b_j x + t a_j)| \, dx$$

$$= \epsilon \cdot \sum_{j=1}^{k} \frac{1}{b_k}.$$

$\square$

0.2 Fall 2015

1. Let E be a measurable subset of $[0, 2\pi]$. Assume that $f \in C(\mathbb{R})$ is 1-periodic, i.e. $f(x+1) = f(x)$. Compute

$$\lim_{n\to\infty} \int_E f(nx) \, dx.$$

Solution. We rewrite the integral over E as an integral over $\mathbb{R}$ against the indicator function of E:

$$\int_E f(nx)\ dx = \int f(nx)\chi_E(x)\ dx.$$

Now let $\varphi \in C_0^\infty(\mathbb{R})$. Since $f \in C(\mathbb{R})$ is 1-periodic, it has a 1-periodic continuous primitive F with $F' = f$. By the chain rule we have $[\frac{1}{n}F(nx)]' = f(nx)$. Integration by parts gives

$$\int f(nx)\varphi(x)\ dx = -\frac{1}{n}\int F(nx)\varphi'(x)\ dx.$$

$F(nx)$ is bounded since F is 1-periodic and $\varphi \in C_0^\infty(\mathbb{R})$, so it's integrable. We then have

$$\begin{aligned}\left|\int f(nx)\varphi(x)\ dx\right| &\leq \frac{1}{n}\|F\|_\infty \cdot \|\varphi'\|_{L^1}\\ &\to 0.\end{aligned}$$

Since E is a measurable subset of $[0, 2\pi]$, it has finite measure and $\chi_E \in L^1(\mathbb{R})$. We can then find $\varphi \in C_0^\infty(\mathbb{R})$ with $\|\chi_E - \varphi\|_{L^1} < \epsilon$. Since f is continuous and 1-periodic, it is bounded and we have

$$\begin{aligned}\left|\int f(nx)\chi_E(x)\ dx - \int f(nx)\varphi(x)\ dx\right| &\leq \|f\|_\infty \cdot \|\chi_E - \varphi\|_{L^1}\\ &\leq \|f\|_\infty \cdot \epsilon.\end{aligned}$$

Since $\int f(nx)\varphi(x)\ dx \to 0$, we must have $\int_E f(nx) \to 0$. □

2. Suppose $f \in L^1[0,1]$ and assume that there exists $C > 0$ such that for all measurable subsets $E \subset [0,1]$ we have

$$\int_E |f(x)|\ dx \leq C\mu(E)^{1/2}.$$

Show that $f \in L^p[0,1]$ for $1 \leq p < 2$. Show that the statement fails for $p = 2$ by giving a counterexample.

Proof. We have that

$$|f(x)|^p - 1 \leq \sum_{n=1}^\infty \chi_{\{|f|^p \geq n\}}(x) \leq |f(x)|^p.$$

Since $[0,1]$ is a finite measure space, integrating through this inequality shows that $f \in L^p[0,1]$ if and only if the series

$$\sum_{n=1}^\infty \mu\{|f(x)|^p \geq n\} = \sum_{n=1}^\infty \mu\{|f(x)| \geq n^{1/p}\}.$$

converges. By Chebyshev's inequality and the given hypotheses we have

$$n^{1/p}\mu\{|f| \geq n^{1/p}\} \leq \int_{\{|f| \geq n^{1/p}\}} |f|\ dx \leq C\mu\{|f| \geq n^{1/p}\}^{1/2}.$$

Dividing through by $n^{1/p}\mu\{|f| \geq n^{1/p}\}^{1/2}$ and squaring gives

$$\sum_{n=1}^{\infty} \mu\{|f(x)| \geq n^{1/p}\} \leq \sum_{n=1}^{\infty} \frac{C^2}{n^{2/p}},$$

which converges for all $p \in [1, 2)$.

$\square$

3. Show that a function $f : \mathbb{R}^n \to \mathbb{R}^+$ is measurable if and only if $E = \{(x, y) : 0 \leq y \leq f(x)\}$ is a measurable set of $\mathbb{R}^{n+1}$.

Proof. Suppose f is measurable. Then the function $F(x, y) = f(x)$ is a measurable function $\mathbb{R}^{n+1} \to \mathbb{R}$. Since $G(x, y) = y$ is also measurable, $H(x, y) = y - f(x)$ is measurable as the difference of measurable functions. We can then write E as the intersection of two measurable sets:

$$E = G^{-1}[[0, \infty)] \cap H^{-1}[(-\infty, 0]].$$

Thus, E is measurable if f is measurable.

Conversely, suppose that E is a measurable set. Then for any $\alpha \geq 0$ the set $A \cap G^{-1}(\alpha) = F^{-1}[[\alpha, \infty)]$. This shows that F, and therefore f, is measurable. $\square$

4. Let $f \in L^1(\mathbb{R})$ and set

$$f_h(x) = \frac{1}{2h} \int_{x-h}^{x+h} f(t)\ dt, \quad h > 0.$$

Show that $f_h \in L^1(\mathbb{R})$ and $f_h \to f$ in $L^1(\mathbb{R})$.

Proof. Let's integrate f_h. By Tonelli we have

$$\begin{aligned} \int |f_h(x)|\ dx &= \frac{1}{2h} \int \left| \int f(t)\chi_{[x-h,x+h]}(t)\ dt \right| dx \\ &\leq \frac{1}{2h} \int \int |f(t)|\chi_{[t-h,t+h]}(x)\ dxdt \qquad (6) \\ &= \|f\|_{L^1}. \end{aligned}$$

Since $f \in L^1(\mathbb{R})$, we have that this quantity is finite and $f_h \in L^1(\mathbb{R})$.

Now since $f \in L^1(\mathbb{R})$, $f_h \to f$ a.e. by the Lebesgue differentiation theorem. By Fatou's lemma and (6), we have for any sequence $h_n \to 0$

$$\begin{aligned} \int |f|\ dx &\leq \liminf_{n\to\infty} \int |f_{h_n}|\ dx \\ &\leq \int |f|\ dx, \end{aligned}$$

so $\liminf_{n\to\infty} \int |f_{h_n}| = \int |f|$. By the triangle inequality we have $|f_{h_n}| + |f| - |f - f_{h_n}| \geq 0$. Since $|f_{h_n}| + |f| - |f - f_{h_n}|$ converges to $2|f|$ a.e., another application of Fatou's lemma gives

$$2\int |f| \; dx \leq \liminf_{n\to\infty} \int (|f_{h_n}| + |f| - |f - f_{h_n}|) \; dx$$

$$\iff \limsup_{n\to\infty} \int |f - f_{h_n}| \; dx \leq 0.$$

We then have $\int |f - f_{h_n}| \to 0$, so $f_{h_n} \to f$ in L^1 for any $h_n \to 0$. $\square$

5. Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f_k : X \to \mathbb{R}$ be a sequence of measurable functions satisfying the following:

$$\int_X |f_k|^2 \; d\mu \leq 2015, \quad \text{for all } k,$$

and

$$\int_X f_j f_k \; d\mu = 0, \quad \text{for all } j \neq k.$$

Prove that for all $\beta > 3/2$,

$$\lim_{n\to\infty} \frac{1}{n^\beta} \sum_{k=1}^{n^2} f_k(x) = 0, \quad \text{for a.a. } x \in X.$$

Proof. Let's compute the L^2 norm of the sum

$$\begin{aligned}
\left\| \frac{1}{n^\beta} \sum_{j=1}^{n^2} f_j \right\|_{L^2}^2 &= \frac{1}{n^{2\beta}} \left(\sum_{j=1}^{n^2} f_j, \sum_{k=1}^{n^2} f_k \right) \\
&= \frac{1}{n^{2\beta}} \sum_{j=1}^{n^2} \sum_{k=1}^{n^2} (f_j, f_k) \\
&= \frac{1}{n^{2\beta}} \sum_{j=1}^{n^2} \|f_j\|_{L^2}^2 \\
&\leq \frac{2015}{n^{2\beta-2}}.
\end{aligned}$$

Now if $\beta > 3/2$, $2\beta - 2 > 1$, so the above quantity is summable in n. Summability and wanting to show that something holds for almost all x leads us to think Borel-Cantelli might be useful. For any fixed $\epsilon > 0$, Chebyshev gives us

$$\begin{aligned}
\mu \left\{ x : \left| \frac{1}{n^\beta} \sum_{j=1}^{n^2} f_j \right|^2 \geq \epsilon \right\} &\leq \frac{1}{\epsilon^2} \int_X \left(\frac{1}{n^\beta} \sum_{j=1}^{n^2} f_j \right)^2 dx \\
&\leq \frac{2015}{\epsilon^2 n^{2\beta-2}}.
\end{aligned}$$

If we call the set on the LHS A_n, then we have $\sum \mu(A_n) < \infty$. By Borel-Cantelli we have $\mu(\limsup_{n\to\infty} A_n) = 0$, i.e., the set of x that belong to infinitely many A_n has measure zero, so the sum is zero for almost all x. $\square$

6. Let $A, B \subseteq \mathbb{R}^n$ be Lebesgue measurable sets and assume that for every $x \in \mathbb{Q}^n$ there exists a null set N_x such that
$$A + x \subset B \cup N_x.$$
Show that if A is not a null set then the complement of B in $\mathbb{R}^n$ is a null set.

Proof. Suppose A has positive measure. Since $\mathbb{Q}$ is countable and the countable union of null sets is null, we have that $A + \mathbb{Q} \subset B \cup N$ for some null set N. If $A + \mathbb{Q}$ missed a set of positive measure, then the complement of B would contain a set of positive measure. Let's show that this cannot happen.

Suppose E is a set of positive measure with $E \cap (A + \mathbb{Q}) = \emptyset$. Define the function f by the convolution
$$f(x) = \int_{\mathbb{R}^n} \chi_A(x-y)\chi_E(y)\ dy.$$
If we choose $x = q \in \mathbb{Q}^n$, then the integrand is nonzero if and only if $y \in E \cap (A + q) = \emptyset$, so $f(q) = 0$. But the convolution is continuous if we take E to have finite measure and $\mathbb{Q}^n$ is dense, so we must have $f \equiv 0$. But by Tonelli we have
$$\begin{aligned} \int_{\mathbb{R}^n \times \mathbb{R}^n} \chi_A(x-y)\chi_E(y)\ d(\mu_x \times \mu_y) &= \int\int \chi_A(x-y)\chi_E(y)\ dxdy \\ &= m(A)m(E). \end{aligned}$$
Since A is not null and E is assumed to have positive measure, this must be positive, contradicting $f \equiv 0$. We conclude that $A + \mathbb{Q}$ is null. $\square$

0.3 Spring 2015

1. Show that if $f \in L^4(\mathbb{R})$ then
$$\lim_{c \to 1} \int_{\mathbb{R}} |f(cx) - f(x)|^4\ dx = 0.$$

Proof. Suppose φ is continuous with compact support. Then $\varphi(cx)$ converges to $\varphi(x)$ uniformly, and since the support of φ is compact, we have that the desired limit holds with φ in place of f.

Now let $\varphi \in C_0(\mathbb{R})$ be such that $\|f - \varphi\|_{L^4} < \epsilon$. Since $|a + b|^p \leq 2^p(|a|^p + |b|^p)$ for all $p > 0$ we have
$$\begin{aligned} \int |f(cx) - f(x)|^4\ dx &= \int |f(cx) - \varphi(cx) + \varphi(cx) - \varphi(x) + \varphi(x) - f(x)|^4\ dx \\ &\leq 2^4 \int |f(cx) - \varphi(cx)|^4\ dx \\ &+ 2^8 \int |\varphi(cx) - \varphi(x)|^4\ dx + 2^8 \int |\varphi(x) - f(x)|^4\ dx. \end{aligned}$$

The first and third integrals are small since $\|f-\varphi\|_{L^4} < \epsilon$ and the second integral can be made small as $c \to 1$ since $\varphi(cx) \to \varphi(x)$ uniformly on a compact set. $\square$

2. Let $f_n : (0,\infty) \to \mathbb{R}$, be a sequence of Lebesgue measurable functions such that $f_n \to f$ a.e. as $n \to \infty$. Assume that there exists $g : (0,\infty) \to \mathbb{R}$ such that $|f_n| \le g$ for all n and $g \in L^1(0,a)$ for all $0 < a < \infty$. Assume furthermore that

$$\int_1^\infty |f_n(\sqrt{x})|\, dx \le C,$$

for all n and for some constant $C > 0$. Show that $f_n \in L^1(0,\infty)$, $f \in L^1(0,\infty)$ and $f_n \to f$ in $L^1(0,\infty)$ as $n \to \infty$.

Proof. First let's show that $f_n \in L^1(0,\infty)$ for all n. Write

$$\int_0^\infty |f_n|\, dx = \int_0^1 |f_n|\, dx + \int_1^\infty |f_n|\, dx. \tag{7}$$

For the first integral, since $|f_n| \le g$ and $g \in L^1(0,1)$ we have

$$\int_0^1 |f_n|\, dx \le \int_0^1 g\, dx < \infty.$$

For the second integral in (7) we use the hypothesis about $f_n(\sqrt{x})$.

$$\begin{aligned} C &\ge \int_1^\infty |f_n(\sqrt{x})|\, dx \\ &= 2\int_1^\infty t|f_n(t)|\, dt \\ &\ge \int_1^\infty |f_n(t)|\, dt. \end{aligned}$$

Both integrals in (7) are then finite, so $f_n \in L^1(0,\infty)$. In fact, we actually have that the f_n are uniformly bounded in $L^1(0,\infty)$ by $\int_0^1 g\, dx + C$. Since $f_n \to f$ a.e. we can apply Fatou's lemma to show that $f \in L^1(0,\infty)$:

$$\begin{aligned} \int_0^\infty |f|\, dx &\le \liminf_{n\to\infty} \int_0^\infty |f_n|\, dx \\ &\le \int_0^1 g\, dx + C \\ &< \infty. \end{aligned}$$

Our plan is to use the Vitali convergence theorem to show that $f_n \to f$ in $L^1(0,\infty)$. We are given that $f_n \to f$ a.e., which implies that $f_n \to f$ in measure. Since $|f - f_n| \le |f| + g$, we have that

$f_n \to f$ in $L^1(0,a)$ for any a by the dominated convergence theorem, so the f_n's are uniformly integrable. To establish tightness, note that for any $t > 1$ we have

$$\int_t^\infty |f_n(x)|\ dx = \int_{t^2}^\infty \frac{|f_n(\sqrt{x})|}{2\sqrt{x}}\ dx$$

$$\leq \frac{C}{2t},$$

which goes to zero as $t \to \infty$. By the Vitali convergence theorem we have that $f_n \to f$ in $L^1(0,\infty)$. □

3. Assume that $f \in C^1(0,1)$ and

$$\int_0^1 x|f'|^p\ dx < +\infty$$

for some $p > 2$. Show that $\lim_{x \to 0^+} f(x)$ exists.

Proof. Let $x_n \to 0$ and say the integral in the problem statement has value $C < \infty$. If q is such that $\frac{1}{p} + \frac{1}{q} = 1$, we have by Hölder's inequality

$$|f(x_n) - f(x_m)| = \left| \int_{x_m}^{x_n} f'(x)\ dx \right|$$

$$\leq \int_{x_m}^{x_n} |f'(x)|\ dx$$

$$= \int_0^1 x^{1/p} |f'(x)| x^{-1/p} \chi_{[x_m,x_n]}(x)\ dx$$

$$\leq \left(\int_0^1 x|f'(x)|^p\ dx \right)^{1/p} \cdot \left(\int_{x_m}^{x_n} x^{-q/p}\ dx \right)^{1/q}.$$

Since $p > 2$, we have that $q < 2$, so the last line above becomes

$$|f(x_n) - f(x_m)| \leq C \cdot \left. \frac{x^{1-q/p}}{1-q/p} \right|_{x_m}^{x_n}.$$

Since $q < 2$, we have that $1 - \frac{q}{p} > 0$, so as $x_m, x_n \to 0$, this expression goes to zero. Thus, the sequence $f(x_n)$ is Cauchy, so $\lim_{x \to 0} f(x)$ exists. □

4. Suppose that $E \subset [0,1]^2$ is measurable. Denote

$$E_x = \{y \in [0,1] : (x,y) \in E\}, \quad E_y = \{x \in [0,1] : (x,y) \in E\}.$$

Show that if $m(E_x) = 0$ for almost all $x \in [0, \frac{1}{2}]$, then

$$m(\{y \in [0,1] : m(E_y) = 1\}) \leq \frac{1}{2}.$$

Proof. E is contained in the unit square, which has finite measure. By Tonelli's theorem we then have

$$\begin{aligned}
m(E) &= \int \chi_E(x,y)\ d(\mu_x \times \mu_y) \\
&= \int_0^1 \int_0^1 \chi_E(x,y)\ dydx \\
&= \int_0^1 m(E_y)\ dy = \int_0^1 m(E_x)\ dx \\
&= \int_{1/2}^1 m(E_x)\ dx \\
&\leq \frac{1}{2}.
\end{aligned}$$

This gives us

$$\begin{aligned}
m(\{y \in [0,1] : m(E_y) = 1\}) &= \int_{\{y\in[0,1]:m(E_y)=1\}} m(E_y)\ dy \\
&\leq \int_0^1 m(E_y)\ dy \\
&\leq \frac{1}{2}.
\end{aligned}$$

□

5. Let $f \in L^p(\mathbb{R})$, $1 < p < \infty$, and let $\alpha > 1 - \frac{1}{p}$. Show that the series

$$\sum_{n=1}^{\infty} \int_n^{n+n^{-\alpha}} |f(x+y)|\ dy$$

converges for a.e. $x \in \mathbb{R}$.

Proof. Our strategy is to show that the sum, as a function of x, is locally integrable, and therefore finite almost everywhere. To this end, let k be an arbitrary integer. Since the above integrands are nonnegative, the monotone convergence theorem will let us interchange the sum with integrals. By Tonelli we will interchange the integrals.

$$\begin{aligned}
\int_k^{k+1} \sum_{n=1}^{\infty} \int_n^{n+n^{-\alpha}} |f(x+y)|\ dydx &= \sum_{n=1}^{\infty} \int_k^{k+1} \int_n^{n+n^{-\alpha}} |f(x+y)|\ dydx \\
&= \sum_{n=1}^{\infty} \int_k^{k+1} \int |f(y)| \cdot \chi_{[n+x,n+n^{-\alpha}+x]}(y)\ dydx \\
&= \sum_{n=1}^{\infty} \int \int_k^{k+1} |f(y)| \cdot \chi_{[y-n-n^{-\alpha},y-n]}(x)\ dxdy.
\end{aligned}$$

Let's compute the values of y for which $[y-n-n^{-\alpha}, y-n] \cap [k, k+1]$ is nonzero. We need $k < y-n$, so $k+n < y$. We also need $y-n-n^{-\alpha} < k+1$, so $y < k+n+n^{-\alpha}+1$. This gives us

$$\int_k^{k+1} \sum_{n=1}^{\infty} \int_n^{n+n^{-\alpha}} |f(x+y)| \, dydx = \sum_{n=1}^{\infty} \int_{k+n}^{k+n+n^{-\alpha}+1} \int |f(y)| \chi_{[y-n-n^{-\alpha}, y-n]}(x) \, dxdy$$

$$= \sum_{n=1}^{\infty} n^{-\alpha} \int_{k+n}^{k+n+n^{-\alpha}+1} |f(y)| \, dy.$$

Our plan is to use Hölder's inequality with respect to the counting measure on the sequences $n^{-\alpha}$ and $\int_{k+n}^{k+n+n^{-\alpha}+1} |f(y)| \, dy$. Since α is given to be larger than the Hölder conjugate of p, we have that $n^{-\alpha}$ is in ℓ^q. We also have

$$\sum_{n=1}^{\infty} \left(\int_{k+n}^{k+n+n^{-\alpha}+1} |f(y)| \, dy \right)^p$$

□

6. Suppose $E \subset \mathbb{R}$ is measurable and $E = E + \frac{1}{n}$ for every natural number $n \geq 1$. Show that either $m(E) = 0$ or $m(E^c) = 0$.

Proof. By induction we can see that $E = E + \mathbb{Q}$. Suppose E isn't null and $E = E + \mathbb{Q}$ misses a set A of positive finite measure. Consider the consider the convolution

$$f(x) = \int_{\mathbb{R}} \chi_E(x-y) \chi_A(y) \, dy.$$

Since $E + \mathbb{Q} \cap A$ is empty, if $x \in \mathbb{Q}$ then $f(x) = 0$. Furthermore, since A has finite measure and E is in $L^{\infty}(\mathbb{R})$, the convolution is continuous. Since $\mathbb{Q}$ is dense and f, a continuous function vanishes on $\mathbb{Q}$, we must have $f \equiv 0$. But by Tonelli we have that $\int f(x) \, dx = m(E)m(A)$, which is positie. We conclude that E cannot miss a set of positive measure if it isn't null. □

0.4 Fall 2014

1. Let A be the collection of all subsets of $\mathbb{R}$ that consist of exactly 5 points. Find the σ-algebra of sets generated by A.

Solution. By intersecting five element sets with exactly one point in common we can obtain all singleton subsets of $\mathbb{R}$. We claim that the σ-algebra generated by the singleton sets, which will be the σ-algebra generated by A, consists of all countable or co-countable subsets of $\mathbb{R}$.

Call the σ-algebra consisting of all countable or co-countable sets $\mathcal{A}$. Since $\mathcal{A}$ contains all singletons, we clearly have $\sigma(A) \subseteq \mathcal{A}$. Conversely, let $S \in \mathcal{A}$. If S is countable, then it is a countable union of singletons, so $S \in \sigma(A)$. On the other hand, if S is co-countable, then its complement is in $\sigma(A)$. Since $\sigma(A)$ is closed under taking complements, this puts S in $\sigma(A)$ as well. We conclude that $\sigma(A) = \mathcal{A}$. □

2. Assume that $f \in L^1(0,1)$ is a non-negative real-valued function satisfying $\int_{[0,1]} f(x)\, dx = 1$. Show that

$$\int_{[0,1]} \frac{1}{f(x)}\, dx \geq 1.$$

Proof. Since $f \in L^1$ and $f \geq 0$, we have that $\sqrt{f} \in L^2$. We then have by Hölder's inequality

$$\begin{aligned}
1 &= \int 1\, dx \\
&= \int \frac{\sqrt{f}}{\sqrt{f}}\, dx \\
&\leq \left\|\sqrt{f}\right\|_{L^2} \cdot \left\|1/\sqrt{f}\right\|_{L^2} \\
&= \sqrt{\|f\|_{L^1}} \cdot \sqrt{\|1/f\|_{L^1}} \\
&= \sqrt{\|1/f\|_{L^1}}.
\end{aligned}$$

$\square$

3. Denote

$$E = \left\{ x \in [0,1] : \text{there exist infinitely many } p, q \in \mathbb{N} \text{ such that } |x - \frac{p}{q}| \leq \frac{1}{q^3} \right\}.$$

Show that $m(E) = 0$.

Proof. Let $E_{p,q} = \{x \in [0,1] : |x - p/q| \leq 1/q^3\}$ where p, q range over $\mathbb{N}$. Note that since we're confined to $[0,1]$, these sets are empty for $p > q$ for any fixed q. We also have that $m(E_{p,q}) \leq \frac{2}{q^3}$. We can then sum (using Tonelli to sum over p and q individually)

$$\begin{aligned}
\sum_{p,q \in \mathbb{N}} m(E_{p,q}) &= \sum_{q \in \mathbb{N}} \sum_{0 \leq p < q} m(E_{p,q}) \\
&\leq \sum_{q \in \mathbb{N}} q \cdot \frac{2}{q^3} \\
&= \frac{\pi^2}{3}.
\end{aligned}$$

Since this sum is finite, by Borel-Cantelli we must have that $m(\limsup E_{p,q}) = 0$. $\limsup E_{p,q}$ is the set of $x \in [0,1]$ belonging to infinitely many $E_{p,q}$, which is exactly the definition of E. $\square$

4. Assume that $\eta \in L^1(\mathbb{R})$ is a non-negative function satisfying $\int_{\mathbb{R}} \eta\, dx = 1$. Show that for any $f \in L^1(\mathbb{R})$,

$$\|f * \eta\|_{L^1} \leq \|f\|_{L^1}.$$

Proof. We use Tonelli's theorem

$$\begin{aligned}
\int |(f * \eta)(x)| \; dx &\leq \int \int |f(x-y)| \eta(y) \; dy dx \\
&= \int \int |f(x-y)| \eta(y) \; dx dy \\
&= \|f\|_{L^1} \int \eta(y) \; dy \\
&= \|f\|_{L^1} .
\end{aligned}$$

□

5. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and periodic with period one. Prove that

$$\lim_{n \to \infty} \int_0^1 f(nx) \cos^2(2\pi x) \; dx = \frac{1}{2} \int_0^1 f(x) \; dx.$$

Proof. □