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233B - Final

5.6.12 Let X be a prevariety over an algebraically closed field k, and let $P \in X$ be a (closed) point of X. Let $D = \operatorname{Spec} k[x]/(x^2)$ be the "double point". Show that the tangent space $T_{X,P}$ to X at P can be canonically identified with the set of morphisms $D \to X$ that map the unique point of D to P.

Proof. Let $f : D \to X$ be a morphism mapping $(x) \in D$ to $P \in X$. Because morphisms of schemes correspond to homomorphisms of ringed spaces, we have a map on the stalk, $f^* : \mathcal{O}_{X,P} \to k[x]/(x^2)$, that sends the maximal ideal $\mathfrak{m}_P$ to (x). Write $f^*(g) = \alpha(g) + \beta(g)x \in k[x]/(x^2)$ so that $fhg) = \beta(g) \in (x)$ for $g \in \mathfrak{m}_P$. We can then use f^* to build a functional $\varphi : \mathfrak{m}_P \to k$ by $\varphi(g) = \beta(g)$. Now take $g, h \in \mathfrak{m}_P$. We then have

$$f^*(gh) = f^*(g)f^*(h)$$

$$\iff \beta(gh)x = \beta(g)\beta(h)x^2$$

$$\iff \beta(gh) = 0,$$

so $\mathfrak{m}_P^2 \subseteq \ker \beta$ and we can consider β as a functional $\mathfrak{m}_P/\mathfrak{m}_P^2 \to k$, an element of the tangent space at P.

Let's show that this assignment is injective. Suppose that $f_1^*, f_2^* : \mathcal{O}_{X,P} \to k[x]/(x^2)$ give rise to the same functional $\mathfrak{m}_P/\mathfrak{m}_P^2$.

On the other hand, suppose we have functional $\varphi \in \operatorname{Hom}(\mathfrak{m}_{\mathfrak{P}}/\mathfrak{m}_P^2, k) \cong T_{X,P}$. Our goal is to use φ to build a morphism $D \to X$ mapping (x) to P. □

5.6.13 Let X be an affine variety, let Y be a closed subscheme of X defined by the ideal $I \subset A(X)$, and let $\tilde{X}$ be the blow-up of X at I. Show that:

(i) $\tilde{X} = \operatorname{Proj}(\bigoplus_{d \geq 0} I^d)$, where $I^{(0)} := A(X)$.

(ii) The projection map $\tilde{X} \to X$ is the morphism induced by the ring homomorphism $I^{(0)} \to \bigoplus_{d \geq 0} I^{(d)}$.

(iii) The exceptional divisor of the blow-up, i.e. the fiber $Y \times_X \tilde{X}$ of the blow-up $\tilde{X} \to X$ over Y, is isomorphic to $\operatorname{Proj}(\bigoplus_{d \geq 0} I^{(d)}/I^{(d+1)}$.

Proof. □

6.7.3 Let $X \subset \mathbb{P}^n$ scheme with Hilbert polynomial χ. Define the arithmetic genus of X to be $g(X) = (-1)^{\dim X} \cdot (\chi(0) - 1)$.

(i) Show that $g(\mathbb{P}^n) = 0$.

(ii) If X is a hypersurface of degree d in $\mathbb{P}^n$, show that $g(X) = \binom{d-1}{n}$. In particular, if $C \subset \mathbb{P}^2$ is a plane curve of degree d, then $g(C) = \frac{1}{2}(d-1)(d-2)$.

(iii) Compute the arithmetic genus of the union of the three coordinate axes

$$Z(x_1x_2, x_1x_3, x_2x_3) \subset \mathbb{P}^3.$$

6.7.8 Let $C_1 = \{f_1 = 0\}$ and $C_2 = \{f_2 = 0\}$ be affine curves in $\mathbb{A}^2_k$, and let $P \in C_1 \cap C_2$ be a point. Show that the intersection multiplicity of C_1 and C_2 at P (i.e. the length of the component at P of the intersection scheme $C_1 \cap C_2$) is equal to the dimension of the vector space $\mathcal{O}_{\mathbb{A}^2,P}/(f_1, f_2)$ over k.

7.8.8 What is the line bundle on $\mathbb{P}^n \times \mathbb{P}^m$ leading to the Segre embedding $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ by the correspondence of ... What is the line bundle leading to the degree-d Veronese embedding $\mathbb{P}^n \to \mathbb{P}^N$?

7.8.10 Let X be a smooth projective curve, and let $P \in X$ be a point. Show that there is a rational function on X that is regular everywhere except at P.