Quiz 4

Student ID Number: Name ____________________

Math 180B, 3PM

Please justify all your answers May 16, 2019

Please also write your full name on the back

In these problems we'll show that if $d \in \mathbb{Z}$ is not a perfect cube then $\sqrt[3]{d}$ is irrational.

1. Suppose $\sqrt[3]{d} = a/b$ with a, b nonzero integers and d isn't a perfect cube. Let

$$A = \begin{pmatrix} 0 & 0 & d \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad v = \begin{pmatrix} a^2 \\ ab \\ b^2 \end{pmatrix}.$$

Compute the eigenvalues of A and show that v is an eigenvector of A. *Hint: When showing that v is an eigenvector, use the fact that $d = a^3/b^3$.*

2. Let $\ell \in \mathbb{Z}$ be such that $\ell < \sqrt[3]{d} < \ell + 1$. Show that

$$(A - \ell I)^r v = (\sqrt[3]{d} - \ell)^r v$$

for all $r \geq 1$. *Hint: Show it for $r = 1$ first using part (a).*

3. Prove that $(A - \ell I)^r v$ is a vector with integer entries for all $r \geq 1$. What can you say about the size of $\sqrt[3]{d} - \ell$? Take the limit as $r \to \infty$ to derive a contradiction, i.e. no such a and b exist.