Matrix Inversion

1. Find the inverses of the matrices (i) by using the formula for the inverse of a 2 by 2 matrix and (ii) by row reduction.

   (a) $\begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & -4 \\ 7 & -8 \end{bmatrix}$

2. Solve the system of equations using matrix inversion.

$$\begin{array}{rcrcl} 2x & + & y & = & 1 \\ -3x & + & 2y & = & 0 \end{array}$$

3. True or False? Explain.

   (a) A product of invertible $n \times n$ matrices is invertible, and the inverse of the product is the product of the inverses.

   (b) If A is invertible, then so is A^{-1} and the inverse of A^{-1} is A.

   (c) If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ad = bc$ then A is not invertible.

   (d) If A can be row reduced to the identity matrix, then A must be invertible.

   (e) If A is invertible, then elementary row operations that reduce A to the identity matrix also reduce A^{-1} to the identity matrix.

4. Let A be an invertible $n \times n$ matrix, and let B be an $n \times p$ matrix. Show that the equation $AX = B$ has a unique solution $A^{-1}B$.

5. Suppose A and B are $n \times n$, B is invertible, and AB is invertible. Show that A is invertible.

6. Explain why the columns of an $n \times n$ matrix span $\mathbb{R}^n$ when A is invertible.

7. Find the inverses of the matrices by row reduction.

   (a)

$$\begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

   (b)

$$\begin{bmatrix} 0 & -3 & -2 \\ 1 & -4 & -2 \\ -3 & 4 & 1 \end{bmatrix}$$

8. Determine whether the matrices are invertible using as few computations as possible.

   (a)
$$\begin{bmatrix} 5 & 7 \\ -3 & -6 \end{bmatrix}$$

   (b)
$$\begin{bmatrix} -7 & 0 & 4 \\ 3 & 0 & -1 \\ 2 & 0 & 9 \end{bmatrix}$$

   (c)
$$\begin{bmatrix} -1 & -3 & 0 & 1 \\ 3 & 5 & 8 & -3 \\ -2 & -6 & 3 & 2 \\ 0 & -1 & 2 & 1 \end{bmatrix}$$

9. An $m \times n$ matrix is lower triangular if all of its entries lying above the main diagonal are zero. When is a square lower triangular matrix invertible?

10. Can a square matrix with two identical columns be invertible?

11. If A is invertible, then the columns of A^{-1} are linearly independent. Why?

12. If the equation $Gx = y$ has more than one solution for some $y \in \mathbb{R}^n$, can the columns of G span $\mathbb{R}^n$?

13. Explain why the columns of A^2 span $\mathbb{R}^n$ whenever the columns of A are linearly independent.

14. If A is an $n \times n$ matrix and the transformation $x \mapsto Ax$ is one-to-one, is it necessarily also onto? What if $x \mapsto Ax$ is onto. Then is it one-to-one too?