Spring 2016

1. Assume $f \in L^1[0,1]$. Compute
$$\lim_{k\to\infty}\int_{[0,1]} |f|^{1/k}\ dx.$$

Solution. Let's split this integral into three regions.
$$\int_{[0,1]} |f|^{1/k}\ dx = \int_{f=0} |f|^{1/k}\ dx + \int_{0<|f|\le 1} |f|^{1/k}\ dx + \int_{|f|>1} |f|^{1/k}\ dx.$$
The integral over the first region is clearly zero for all k. On the second region we have that $|f|^{1/k} \le 1$ for all k. Since the interval $[0,1]$ has finite measure, we have that the constant function 1 is in $L^1(\{x : 0 < |f| \le 1\})$, so the dominated convergence theorem says that the integral over the second region goes to $m(\{0 < |f| \le 1\})$. Similarly, on the third region we have that $|f|^{1/k} \le |f|$, which is in L^1, so the dominated convergence theorem says that the third integral goes to $m(\{|f| > 1\})$. Combining these, we have that
$$\lim_{k\to\infty}\int_{[0,1]} |f|^{1/k}\ dx = m(\{|f| > 0\}).$$
□

2. Let $\{f_n\}$ be a sequence of measurable functions on $[0,1]$ and $0 \le f_n \le 1$ a.e. Assume that
$$\lim_{n\to\infty}\int_{[0,1]} f_n g\ dx = \int_{[0,1]} fg\ dx$$
for some $f \in L^1[0,1]$ and any $g \in C[0,1]$. Prove that $0 \le f \le 1$ a.e.

Solution. Since $f \in L^1[0,1]$, by the Lebesgue differentiation theorem we have that
$$\frac{1}{m(E)}\int_E f(t)\ dt \to f(x) \tag{1}$$
as E shrinks to x for almost all x. Furthermore, since $0 \le f_n \le 1$ we also have that
$$\frac{1}{m(E)}\int_E f_n(t)\ dt \to f_n(x) \in [0,1]$$
as E shrink to x for almost all x. Intuitively, we'd like to replace the integral of f in (1) with a limit of integrals of f_n.

We claim that the function g in the given hypothesis can be replaced with the indicator function of an interval χ_I. To see this, let g_m be a sequence of continuous functions with $g_m \to \chi_I$ in L^1 and $0 \le \chi_I \le 1$. By extracting a subsequence we can assume that $g_m \to \chi_I$ a.e. as well. We then have
$$\int_0^1 |f_n\chi_I - f\chi_i| \le \int_0^1 |f_n\chi_I - f_n g_m| + \int_0^1 |f_n g_m - f g_m| + \int_0^1 |f g_m - f\chi_i|.$$

Since $\|f_n\|_{L^\infty} \le 1$, we have that the first integral on the RHS can be made small uniformly in n by picking m large. The second integral goes to zero as $n \to \infty$ by hypothesis since g_m is continuous. The third integral can be made small for m large by dominated convergence since $|fg_m| \le |f| \in L^1$.

For almost all x, if I_k is a sequence of intervals shrinking to x then

$$\begin{aligned} \frac{1}{m(I_k)} \int_{I_k} f \, dx &= \frac{1}{m(I_k)} \int f\chi_{I_k} \, dx \\ &= \lim_{n\to\infty} \frac{1}{m(I_k)} \int f_n \chi_{I_k} \, dx. \end{aligned}$$

Since $0 \le f_n \le 1$, the RHS is in $[0,1]$ for almost all x. By the Lebesgue differentiation theorem we then have that $0 \le f \le 1$ a.e. $\square$

3. Let $f, g \in L^2(\mathbb{R}, \mathcal{M}_L, \mu_L)$. Show that $f * g$ is a continuous function on $\mathbb{R}$ vanishing at infinity, that is, $f * g \in C(R)$ and $\lim_{|x|\to\infty}(f * g)(x) = 0$.

Proof. For any h we have by Hölder's inequality

$$|(f*g)(x+h) - (f*g)(x)| = \left| \int f(t)[g(x+h-t) - g(x-t)] \, dt \right| \tag{2}$$

$$\le \|f\|_{L^2} \cdot \|g_h - g\|_{L^2}, \tag{3}$$

where $F_h(x) = F(x+h)$ for any function F. Now for any $\epsilon > 0$ we can find $\varphi \in C_0(\mathbb{R})$ with $\|g - \varphi\|_{L^2} = \|g_h - \varphi_h\|_{L^2} < \epsilon$. By the triangle inequality we then have

$$\begin{aligned} \|g_h - g\|_{L^2} &\le \|g_h - \varphi_h\|_{L^2} + \|\varphi_h - \varphi\|_{L^2} + \|\varphi - g\|_{L^2} \\ &< \|\varphi_h - \varphi\|_{L^2} + 2\epsilon. \end{aligned}$$

Suppose that φ has support contained in the compact set K. If we pick h small enough then we can guarantee that $\varphi_h - \varphi$ is supported on a set with measure at most $2 \cdot m(K)$. Now since φ is continuous with compact support, it is uniformly continuous, so we can choose h small enough that $|\varphi_h(x) - \varphi(x)| = |\varphi(x+h) - \varphi(x)| < \epsilon$ for all x in the support of $\varphi_h - \varphi$. For such h we have

$$\|\varphi_h - \varphi\|_{L^2} \le \epsilon \cdot (2 \cdot m(K))^{1/2},$$

so (2) can be made arbitrarily small, which shows that $f * g$ is continuous.

First we claim that if φ and ψ are continuous with compact support then $\varphi * \psi$ vanishes at infinity. By definition we have that

$$(\varphi * \psi)(x) = \int \varphi(t)\psi(x-t) \, dt.$$

The product $\varphi(t)\psi(x-t)$ is nonzero only if t is in the support of φ and $x-t$ is in the support of φ. If pick x large enough then supports of $t \mapsto \varphi(t)$ and $t \mapsto \psi(x-t)$ are disjoint, so this integral is zero.

Let f_n and g_n be sequences in $C_0(\mathbb{R})$ converging in L^2 to f and g, respectively. We then have

$$\begin{aligned}|(f * g)(x) - (f_n * g_n)(x)| &\leq |(f * g)(x) - (f_n * g)(x)| + |(f_n * g)(x) - (f_n * g_n)(x)| \\ &\leq \|g\|_{L^2} \cdot \|f - f_n\|_{L^2} + \|f_n\|_{L^2} \cdot \|g - g_n\|_{L^2}\,.\end{aligned}$$

Since $f_n \to f$ and $g_n \to g$ in L^2, we have that $f_n * g_n$ converges uniformly to $f * g$. Since $f_n * g_n$ vanishes at infinity, we must then have that $f * g$ vanishes at infinity. $\square$

4. Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and let $p_1 \in (1, \infty]$. Let $\{f_n\}$ be a uniformly bounded sequence in $L^{p_1}(X, \mathcal{A}, \mu)$. Suppose $f = \lim_{n\to\infty} f_n$ exists μ-a.e. Prove that $f \in L^p(X, \mathcal{A}, \mu)$ for all $p \in [1, p_1]$ and $f_n \to f$ in $L^p(X, \mathcal{A}, \mu)$ for all $p \in [1, p_1)$.

Proof. Suppose that $\|f_n\|_{L^{p_1}} \leq M$ for all n. First we claim that the f_n are in $L^p(X, \mathcal{A}, \mu)$ for all $p \in [1, p_1]$. In fact, they are uniformly bounded:

$$\begin{aligned}\int_X |f_n|^p &= \int_{|f_n|<1} |f_n|^p + \int_{|f_n|\geq 1} |f_n|^p \\ &\leq \int_{|f_n|<1} 1 + \int_{|f_n|\geq 1} |f_n|^{p_1} \\ &\leq \mu(\{f \leq 1\}) + M^{1/p_1}.\end{aligned}$$

Since $(X, \mathcal{A}, \mu)$ is a finite measure space, this quantity is finite, so $f_n \in L^p(X, \mathcal{A}, \mu)$ for all n and $p \in [1, p_1]$. We can then use the fact that $f_n \to f$ a.e. and Fatou's lemma to show that $f \in L^p(X, \mathcal{A}, \mu)$ for $p \in [1, p_1]$:

$$\int_X |f|^p \leq \liminf_{n\to\infty} \int_X |f_n|^p < \infty,$$

where the finiteness follows from the L^p uniform-boundedness of the f_n.

To establish convergence in L^p, $p \in [1, p_1)$ our plan is to use the Vitali convergence theorem. The family f_n is tight over X since X is a finite measure space and we're given that $f_n \to f$ a.e., so it only remains to show that the f_n's are uniformly integrable. Intuitively, since the f_n's are in L^p, the measure of the set $\{f_n \geq N\}$ should shrink as N grows. Now since $p < p_1$, if $N > 1$ then

$$|f_n|^p \chi_{\{|f_n|\geq N\}} N^{p_1 - p} \leq |f_n|^{p_1}.$$

If we integrate both sides over any measurable set E we have

$$\int_{E \cap \{|f_n|\geq N\}} |f_n|^p \leq \frac{M}{N^{p_1 - p}}.$$

On the complement we have

$$\int_{E\cap\{|f_n|<N\}} |f_n|^p \leq N^p \cdot \mu(E).$$

Putting these together, we have that

$$\begin{aligned}\int_E |f_n|^p &= \int_{E\cap\{|f_n|\geq N\}} |f_n|^p + \int_{E\cap\{|f_n|<N\}} |f_n|^p \\ &\leq \frac{M}{R^{p_1-p}} + R^p \cdot \mu(E).\end{aligned}$$

If we choose R so that $M/R^{p_1-p} < \epsilon/2$ and E so that $R^p \cdot \mu(E) < \epsilon/2$ then we'll have that $\int_E |f_n|^p < \epsilon$ for any E of sufficiently small measure, so the f_n's are uniformly integrable. By the Vitali convergence theorem we have that $f_n \to f$ in L^p for $p \in [1, p_1)$. $\square$

5. Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f : X \to [0, \infty)$ be $\mathcal{A}$-measurable. Consider the measure space $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu_L)$, where $\mathcal{B}_{\mathbb{R}}$ is the Borel σ-algebra on $\mathbb{R}$ and μ_L is the Lebesgue measure, and form the product measure space $(X \times \mathbb{R}, \sigma(\mathcal{A} \times \mathcal{B}_{\mathbb{R}}), \mu \times \mu_L)$. Define $E \subset X \times R$ by $(x, y) \in E \iff y \in [0, f(x))$. Prove that $E \in \sigma(\mathcal{A} \times \mathcal{B}_{\mathbb{R}})$ and $(\mu \times \mu_L)(E) = \int_X f \; d\mu$.

Proof. A function is measurable if it pulls measurable sets back to measurable sets. The plan is then to write E is a union and/or intersection of preimages of measurable sets under measurable functions. The function $F(x, y) = f(x)$ is measurable since

$$F^{-1}[(-\infty, \alpha]) = \{(x, y) : f(x) \leq \alpha\} = \{x : f(x) \leq \alpha\} \times \mathbb{R} \in \sigma(\mathcal{A} \times \mathcal{B}_{\mathbb{R}}),$$

as f is μ-measurable. We also clearly have that the function $G(x, y) = y$ is measurable. Now consider the function $H(x, y) = y - f(x)$. H is measurable as it is the difference of the measurable functions G and F. We then have that E is measurable through the following decomposition

$$\begin{aligned}E &= \{(x, y) : 0 \leq y < f(x)\} \\ &= \{(x, y) : y \geq 0\} \cap \{(x, y) : y < f(x)\} \\ &= G^{-1}[[0, \infty)] \cap H^{-1}[(-\infty, 0)].\end{aligned}$$

If $\{f > 0\}$ is σ-finite we can use Tonelli's theorem to say

$$\begin{aligned}(\mu \times \mu_L)(E) &= \int_{X\times\mathbb{R}} \chi_E(x, y) \; d(\mu \times \mu_L) \\ &= \int_X \int_{\mathbb{R}} \chi_E(x, y) \; d\mu_L d\mu \\ &= \int_X \int_{\mathbb{R}} \chi_{[0,f(x))}(y) \; dy d\mu \\ &= \int_X f(x) \; d\mu.\end{aligned}$$

On the other hand, suppose that $\{f > 0\}$ is note σ-finite. We claim that $\int_X f \, d\mu = +\infty$. Indeed, since we can decompose this set into a countable union,

$$\{f > 0\} = \bigcup_{m=1}^{\infty} \{\frac{1}{m+1} < f \leq \frac{1}{m}\} \cup \bigcup_{n=1}^{\infty} \{n < f \leq n+1\}, \tag{4}$$

we must have that one of these sets has infinite measure. We need to show that $(\mu \times \mu_L)(E) = +\infty$ too. For any $\alpha, \beta > 0$ we have that if $\alpha \leq f(x) < \beta$ then the product set

$$\{x : \alpha \leq f(x) < \beta\} \times \{y : 0 \leq \alpha\}$$

is contained in E. This product set has measure $\alpha \cdot \mu_L\{\alpha \leq f < \beta\}$, so by monotonicity we have that

$$\alpha \cdot \mu_L\{\alpha \leq f < \beta\} \leq (\mu \times \mu_L)(E)$$

for all $\alpha, \beta > 0$. But by the decomposition (4), we have that some set of the form $\{\alpha \leq f(x) < \beta\}$ must have infinite measure, so we must have $(\mu \times \mu_L)(E) = +\infty$. □

6. Let $f \in L^1(\mathbb{R})$ and let $a_1, \ldots, a_k \in \mathbb{R}$ and $b_1, \ldots, b_k \in \mathbb{R} \setminus \{0\}$. Assume that the quotients $\frac{a_j}{b_j}$ are all distinct. Determine

$$\lim_{t \to \infty} \int \left| \sum_{j=1}^{k} f(b_j x + t a_j) \right| dx.$$

Solution. Let $\varphi \in C_0(\mathbb{R})$ be such that $\|f - \varphi\|_{L^1} < \epsilon$. Our plan is to compute the desired limit with φ in place of f and then argue that the difference can be made small. We have

$$\int \left| \sum_{j=1}^{k} \varphi(b_j x + t a_j) \right| dx = \int \left| \sum_{j=1}^{k} \varphi \left[b_j \left(x + \frac{a_j}{b_j} t \right) \right] \right| dx$$

Now $\varphi(b_j x + t a_j)$ is φ stretched horizontally by a factor of b_j and shifted over a_j/b_j. Since the support of φ is compact and the a_j/b_j are distinct, the supports of these transformations are disjoint for sufficiently large t. When these supports are disjoint we then have

$$\int \left| \sum_{j=1}^{k} \varphi(b_j x + t a_j) \right| dx = \int \sum_{j=1}^{k} |\varphi(b_j x + t a_j)| \, dx$$

$$= \|\varphi\|_{L^1} \cdot \sum_{j=1}^{k} \frac{1}{b_j}.$$

That we can approximate the desired sum for $f \in L^1$ follows from the reverse triangle inequality.

$$\left| \int \left| \sum_{j=1}^{k} f(b_j x + t a_j) \right| dx - \int \left| \sum_{j=1}^{k} \varphi(b_j x + t a_j) \right| dx \right| \leq \sum_{j=1}^{k} \int |f(b_j x + t a_j) - \varphi(b_j x + t a_j)| \, dx$$

$$= \epsilon \cdot \sum_{j=1}^{k} \frac{1}{b_k}.$$

□

Fall 2015

1. Let E be a measurable subset of $[0, 2\pi]$. Assume that $f \in C(\mathbb{R})$ is 1-periodic, i.e. $f(x+1) = f(x)$. Compute
$$\lim_{n\to\infty} \int_E f(nx)\ dx.$$

   Solution. We rewrite the integral over E as an integral over $\mathbb{R}$ against the indicator function of E:
$$\int_E f(nx)\ dx = \int f(nx)\chi_E(x)\ dx.$$
Now let $\varphi \in C_0^\infty(\mathbb{R})$. Since $f \in C(\mathbb{R})$ is 1-periodic, it has a 1-periodic continuous primitive F with $F' = f$. By the chain rule we have $[\frac{1}{n}F(nx)]' = f(nx)$. Integration by parts gives
$$\int f(nx)\varphi(x)\ dx = -\frac{1}{n}\int F(nx)\varphi'(x)\ dx.$$
$F(nx)$ is bounded since F is 1-periodic and $\varphi \in C_0^\infty(\mathbb{R})$, so it's integrable. We then have
$$\begin{aligned}\left|\int f(nx)\varphi(x)\ dx\right| &\le \frac{1}{n}\|F\|_\infty \cdot \left\|\varphi'\right\|_{L^1} \\ &\to 0.\end{aligned}$$
Since E is a measurable subset of $[0, 2\pi]$, it has finite measure and $\chi_E \in L^1(\mathbb{R})$. We can then find $\varphi \in C_0^\infty(\mathbb{R})$ with $\|\chi_E - \varphi\|_{L^1} < \epsilon$. Since f is continuous and 1-periodic, it is bounded and we have
$$\begin{aligned}\left|\int f(nx)\chi_E(x)\ dx - \int f(nx)\varphi(x)\ dx\right| &\le \|f\|_\infty \cdot \|\chi_E - \varphi\|_{L^1} \\ &\le \|f\|_\infty \cdot \epsilon.\end{aligned}$$
Since $\int f(nx)\varphi(x)\ dx \to 0$, we must have $\int_E f(nx) \to 0$. □

2. Suppose $f \in L^1[0, 1]$ and assume that there exists $C > 0$ such that for all measurable subsets $E \subset [0, 1]$ we have
$$\int_E |f(x)|\ dx \le C\mu(E)^{1/2}.$$
Show that $f \in L^p[0, 1]$ for $1 \le p < 2$. Show that the statement fails for $p = 2$ by giving a counterexample.

   Proof. We have that
$$|f(x)|^p - 1 \le \sum_{n=1}^{\infty} \chi_{\{|f|^p \ge n\}}(x) \le |f(x)|^p.$$
Since $[0, 1]$ is a finite measure space, integrating through this inequality shows that $f \in L^p[0, 1]$ if and only if the series
$$\sum_{n=1}^{\infty} \mu\{|f(x)|^p \ge n\} = \sum_{n=1}^{\infty} \mu\{|f(x)| \ge n^{1/p}\}.$$

converges. By Chebyshev's inequality and the given hypotheses we have

$$n^{1/p}\mu\{|f| \geq n^{1/p}\} \leq \int_{\{|f|\geq n^{1/p}\}} |f| \, dx \leq C\mu\{|f| \geq n^{1/p}\}^{1/2}.$$

Dividing through by $n^{1/p}\mu\{|f| \geq n^{1/p}\}^{1/2}$ and squaring gives

$$\sum_{n=1}^{\infty} \mu\{|f(x)| \geq n^{1/p}\} \leq \sum_{n=1}^{\infty} \frac{C^2}{n^{2/p}},$$

which converges for all $p \in [1, 2)$.

$\square$

3. Show that a function $f : \mathbb{R}^n \to \mathbb{R}^+$ is measurable if and only if $E = \{(x, y) : 0 \leq y \leq f(x)\}$ is a measurable set of $\mathbb{R}^{n+1}$.

Proof. Suppose f is measurable. Then the function $F(x, y) = f(x)$ is a measurable function $\mathbb{R}^{n+1} \to \mathbb{R}$. Since $G(x, y) = y$ is also measurable, $H(x, y) = y - f(x)$ is measurable as the difference of measurable functions. We can then write E as the intersection of two measurable sets:

$$E = G^{-1}[[0, \infty)] \cap H^{-1}[(-\infty, 0]].$$

Thus, E is measurable if f is measurable.

Conversely, suppose that E is a measurable set. Then for any $\alpha \geq 0$ the set $A \cap G^{-1}(\alpha) = F^{-1}[[\alpha, \infty)]$. This shows that F, and therefore f, is measurable. $\square$

4. Let $f \in L^1(\mathbb{R})$ and set

$$f_h(x) = \frac{1}{2h} \int_{x-h}^{x+h} f(t) \, dt, \quad h > 0.$$

Show that $f_h \in L^1(\mathbb{R})$ and $f_h \to f$ in $L^1(\mathbb{R})$.

Proof. Let's integrate f_h. By Tonelli we have

$$\begin{aligned} \int |f_h(x)| \, dx &= \frac{1}{2h} \int \left| \int f(t)\chi_{[x-h,x+h]}(t) \, dt \right| dx \\ &\leq \frac{1}{2h} \int \int |f(t)|\chi_{[t-h,t+h]}(x) \, dxdt \\ &= \|f\|_{L^1}. \end{aligned} \tag{5}$$

Since $f \in L^1(\mathbb{R})$, we have that this quantity is finite and $f_h \in L^1(\mathbb{R})$.

Now since $f \in L^1(\mathbb{R})$, $f_h \to f$ a.e. by the Lebesgue differentiation theorem. By Fatou's lemma and (5), we have for any sequence $h_n \to 0$

$$\int |f|\, dx \leq \liminf_{n\to\infty} \int |f_{h_n}|\, dx$$
$$\leq \int |f|\, dx,$$

so $\liminf_{n\to\infty} \int |f_{h_n}| = \int |f|$. By the triangle inequality we have $|f_{h_n}| + |f| - |f - f_{h_n}| \geq 0$. Since $|f_{h_n}| + |f| - |f - f_{h_n}|$ converges to $2|f|$ a.e., another application of Fatou's lemma gives

$$2\int |f|\, dx \leq \liminf_{n\to\infty} \int (|f_{h_n}| + |f| - |f - f_{h_n}|)\, dx$$
$$\iff \limsup_{n\to\infty} \int |f - f_{h_n}|\, dx \leq 0.$$

We then have $\int |f - f_{h_n}| \to 0$, so $f_{h_n} \to f$ in L^1 for any $h_n \to 0$. □

5. Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f_k : X \to \mathbb{R}$ be a sequence of measurable functions satisfying the following:

$$\int_X |f_k|^2\, d\mu \leq 2015, \quad \text{for all } k,$$

and

$$\int_X f_j f_k\, d\mu = 0, \quad \text{for all } j \neq k.$$

Prove that for all $\beta > 3/2$,

$$\lim_{n\to\infty} \frac{1}{n^\beta} \sum_{k=1}^{n^2} f_k(x) = 0, \quad \text{for a.a. } x \in X.$$

Proof. Let's compute the L^2 norm of the sum

$$\left\| \frac{1}{n^\beta} \sum_{j=1}^{n^2} f_j \right\|_{L^2}^2 = \frac{1}{n^{2\beta}} \left(\sum_{j=1}^{n^2} f_j, \sum_{k=1}^{n^2} f_k \right)$$
$$= \frac{1}{n^{2\beta}} \sum_{j=1}^{n^2} \sum_{k=1}^{n^2} (f_j, f_k)$$
$$= \frac{1}{n^{2\beta}} \sum_{j=1}^{n^2} \|f_j\|_{L^2}^2$$
$$\leq \frac{2015}{n^{2\beta-2}}.$$

Now if $\beta > 3/2$, $2\beta - 2 > 1$, so the above quantity is summable in n. Summability and wanting to show that something holds for almost all x leads us to thing Borel-Cantelli might be useful.

For any fixed $\epsilon > 0$, Chebyshev gives us

$$\mu\left\{x : \left|\frac{1}{n^\beta}\sum_{j=1}^{n^2} f_j\right|^2 \geq \epsilon\right\} \leq \frac{1}{\epsilon^2}\int_X \left(\frac{1}{n^\beta}\sum_{j=1}^{n^2} f_j\right)^2 dx$$

$$\leq \frac{2015}{\epsilon^2 n^{2\beta-2}}.$$

If we call the set on the LHS A_n, then we have $\sum \mu(A_n) < \infty$. By Borel-Cantelli we have $\mu(\limsup_{n\to\infty} A_n) = 0$, i.e., the set of x that belong to infinitely many A_n has measure zero, so the sum is zero for almost all x. □