Riemann Integration 1

1. Let f be a continuous function on $[a, b]$. Show that f is Riemann integrable on $[a, b]$.

2. Let $f(x) = \sin \frac{1}{x}$ if $x \neq 0$ and set $f(0) = 0$. Show that f is Riemann integrable on $[0, 1]$.

3. (Hard) Let $g(x) = \sin(\csc(1/x))$ where $\csc \frac{1}{x}$ is defined and zero otherwise. Show that g is Riemann integrable on $[0, 1]$. Hint: mimic the previous problem near every point where $\csc \frac{1}{x}$ is undefined.

4. Suppose f is integrable on $[a, b]$ and $c \in \mathbb{R}$. Define g on $[a + c, b + c]$ by $g(x) = f(x - c)$. Show that g is integrable and $\int_{a+c}^{b+c} g(x) = \int_a^b f(x)$. This is called the *translation invariance* of the integral.

5. If f is Riemann integrable on $[a, b]$, show that for any real number c, $F(x) = c + \int_a^x f(t)\,dt$ is Lipschitz, i.e. there exists some constant M such that $|F(x) - F(y)| \leq M|x - y|$ for all x, y in $[a, b]$.

6. Thomae's function (sometimes called the raindrop function) is defined by

$$T(x) = \begin{cases} 1, & \text{if } x = 0 \\ 1/n, & \text{if } x = m/n \in \mathbb{Q} \setminus \{0\} \text{ is in lowest terms with } n > 0 \\ 0, & \text{if } x \notin \mathbb{Q} \end{cases}.$$

   (a) Show that T has countably many discontinuities in $[0, 1]$.

   (b) Show that $L(T, P) = 0$ for any partition P of $[0, 1]$.

   (c) Let $\epsilon > 0$ and consider the set of points $D_{\epsilon/2} = \{x \in [0, 1] : T(x) \geq \epsilon > 2\}$. How big is $D_{\epsilon/2}$?

   (d) Explain how to construct a partition P_ϵ of $[0, 1]$ so that $U(T, P_\epsilon) < \epsilon$. Conclude that T is Riemann integrable on $[0, 1]$ and compute the integral.