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260A - Homework 2

Problem 1. Let $(e_n)_{n=1}^{\infty}$ be an orthonormal basis in the Hilbert space H. Let $T : H \to H$ be a linear continuous map such that

$$\sum_{n=1}^{\infty} \|Te_n\|^2$$

converges. Show that there is a sequence $(T_n)_{n=1}^{\infty}$ of linear continuous maps $H \to H$ such that $T_n(H)$ has a finite dimension and $\|T_n - T\| \to 0$ as $n \to \infty$.

Proof. Consider the projection T_m defined by

$$T_m(x) = \langle x, e_1\rangle Te_1 + \cdots + \langle x, e_m\rangle Te_m.$$

This function is continuous by an argument similar to the one used on Homework 1, where we showed that every finite dimensional subspace of a normed vector space admits a continuous projection (first we define projections onto the individual components on the space spanned by $e_1, \ldots, e_m$ and then extend these through Hahn-Banach).

The image of H under T_m has dimension at most m as the e_j's are linearly independent. Furthermore, we have by Cauchy-Schwarz

$$\begin{aligned} |T_n x - Tx|^2 &= \left| \sum_{j=n+1}^{\infty} \langle x, e_j\rangle Te_j \right|^2 \\ &\leq \|x\|^2 \cdot \sum_{j=n+1}^{\infty} \|Te_j\|^2. \end{aligned}$$

Since the sum $\sum_{j=1}^{\infty} \|Te_j\|^2$ converges, the tail (the last line in the above inequality) must go to zero as $n \to \infty$. We then have that $\|T_n - T\| \to 0$ as desired. $\square$

Problem 3. Let H be a separable infinite dimensional Hilbert space, and suppose that $e_1, e_2, \ldots$ is an orthonormal system in H. Let $f_1, \ldots$ be another orthonormal system which is complete.

(i) Prove that if $\sum_{n=1}^{\infty} \|e_n - f_n\|^2 < 1$ then $\{e_n\}$ is also a complete orthonormal system.

(ii) Suppose only that $\sum_{n=1}^{\infty} \|e_n - f_n\|^2 < \infty$. Prove that it is still true that $\{e_n\}$ is a complete orthonormal system.

Proof. (i) In order to show that the e_j's form a complete system, we'll show that if $\langle x, e_j\rangle = 0$ for all j then $x = 0$.

$\square$