Secret Sharing

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November 19, 2018

What is Secret Sharing?

- Eleven scientists are working on a secret project. They wish to lock up the documents in a cabinet so that the cabinet can be opened if and only if six or more of the scientists are present. What is the smallest number of locks needed? What is the smallest number of keys to the locks each scientist must carry?[1]

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- $\binom{11}{6} = 462$ locks and $\binom{10}{5} = 252$ keys.

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What is Secret Sharing?

The goal of secret sharing is dividing a secret S into n pieces, called *shares*, so that no fewer than $k \leq n$ shares are sufficient for reassembling S. This is called a (k, n)-**threshold scheme**.

Why is Secret Sharing?

"Threshold schemes are ideally suited to applications in which a group of mutually suspicious individuals with conflicting interests must cooperate." – Adi Shamir

Example Application

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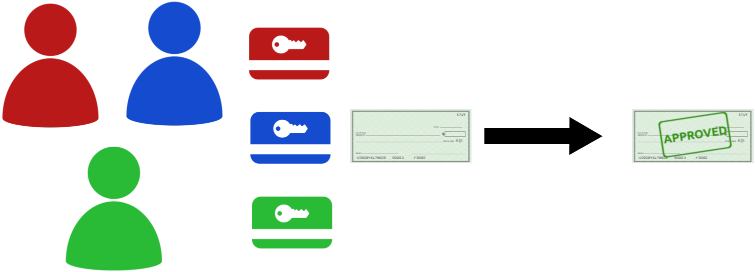

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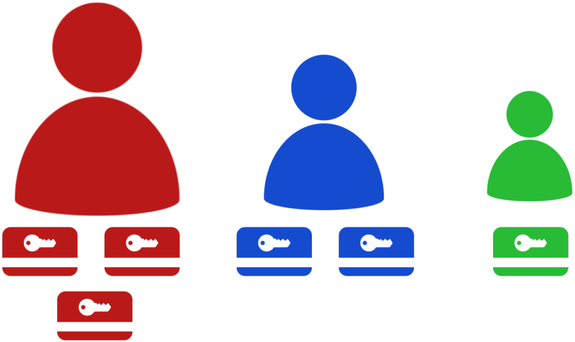

How is Secret Sharing? - Shamir's Method (1979)[2]

- Say we want to set up a (k, n)-threshold scheme

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$$p(x) = S + a_1 x + \cdots + a_{k-1} x^{k-1}.$$

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- Issue to person i, $1 \leq i \leq n$, the share $D_i = (i, p(i))$.

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$$\begin{array}{ccccccccc}
S & + & a_1 \cdot 1 & + & \cdots & + & a_{k-1}(1)^{k-1} & = & p(1) \\
S & + & a_2 \cdot 2 & + & \cdots & + & a_{k-1}(2)^{k-1} & = & p(2) \\
 & & & & & & \vdots & & \\
S & + & a_1 \cdot m & + & \cdots & + & a_{k-1}(m)^{k-1} & = & p(m)
\end{array}$$

- Linear algebra tells us we get a solution if and only if we have at least k equations (shares)

Downside to Shamir's Scheme

Each share is an element of $\mathbb{Z}/p\mathbb{Z}$, just like the secret. n shares means n times the storage.

Information Dispersal - Rabin (1989)[3]

- Split a file F into n pieces so that any $k \leq n$ pieces can reconstruct F

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- Not perfectly secret

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- Break the file into 8 byte blocks

$$\begin{aligned} F &= (b_1, \ldots, b_8), (b_9, \ldots, b_{16}), \ldots, (b_{793}, \ldots, b_{800}) \\ &= f_1, f_2, \ldots, f_{100} \end{aligned}$$

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$$\text{File block} \;\Big\|\; \underbrace{b_1 \mid b_2 \mid b_3 \mid b_4 \mid b_5 \mid b_6 \mid b_7 \mid b_8}_{f_1} \;\Big\|\; \underbrace{b_9 \mid \cdots}$$

$$j\text{-th Share Vector} \;\Big\|\; \underbrace{a_{j1} \mid a_{j2} \mid a_{j3} \mid a_{j4} \mid a_{j5} \mid a_{j6} \mid a_{j7} \mid a_{j8}}_{a_j} \;\Big\|\; \underbrace{a_{j1} \mid \cdots}$$

$$j\text{-th Share Compressed File} \;\Big\|\; F_{j1} = a_j \cdot f_1 \pmod{p} \;\Big\|\; \cdots$$

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- This gives a compressed file $F_j = F_{j1}, \ldots, F_{j(100)}$. Let the j-th share be $S_j = (a_j, F_j)$, $1 \leq j \leq 15$.

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- If we're given 8 shares we get this matrix equation

$$\begin{bmatrix} - & a_1 & - \\ & \vdots & \\ - & a_8 & - \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_8 \end{bmatrix} = \begin{bmatrix} F_{1,1} \\ \vdots \\ F_{8,1} \end{bmatrix}.$$

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- With 8 shares, this matrix is invertible and we can recover the first block. The same matrix recovers all blocks

Secret Sharing with Short Shares - Krawczyk (1994)[4]

- Let's set up a (k, n)-threshold scheme

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- Encrypt S with some secure cipher using key K, $E = Enc(S, K)$

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- Use Shamir's secret sharing to split K into n pieces, each with size $|K|$

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