Linear Dependence and Transformations

1. Determine if the vectors are linearly independent. Justify your answers.

   (a)
   $$\begin{bmatrix} -4 \\ 0 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 3 \\ 6 \end{bmatrix}$$

   (b)
   $$\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -4 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix}$$

2. Find the value(s) of h for which the vectors are linearly *dependent*. Justify your answers.

   (a)
   $$\begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}, \begin{bmatrix} -2 \\ -9 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ h \\ -9 \end{bmatrix}$$

   (b)
   $$\begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ h \end{bmatrix}$$

3. How many pivots columns must a 4×6 matrix have if its columns are linearly independent?

4. True or False? If the statement is false, provide a counterexample.

   (a) If $v_1, \ldots, v_4$ are in $\mathbb{R}^4$ and $v_3 = 2v_1 + v_2$, then $\{v_1, v_2, v_3, v_4\}$ is a linearly dependent set.

   (b) If v_1, v_2, v_3 are in $\mathbb{R}^3$ and v_3 is not a linear combination of v_1, v_2, then $\{v_1, v_2, v_3\}$ is a linearly independent set.

5. Let $A = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$, $u = \begin{bmatrix} 3 \\ 6 \\ -9 \end{bmatrix}$, and $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(x) = Ax$. Find $T(u)$ and $T(v)$.

6. Let $A = \begin{bmatrix} 1 & -3 & 2 \\ 3 & -8 & 8 \\ 0 & 1 & 2 \\ 1 & 0 & 8 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 6 \\ 3 \\ 10 \end{bmatrix}$. Define $T(x) = Ax$. Find a vector x whose image under T is b. Is this x unique?

7. Find all $x \in \mathbb{R}^4$ that are mapped into the zero vector by the transformation $x \mapsto Ax$ where $A = \begin{bmatrix} 3 & 2 & 10 & -6 \\ 1 & 0 & 2 & -4 \\ 0 & 1 & 2 & 3 \\ 1 & 4 & 10 & 8 \end{bmatrix}$.

8. Let $b = \begin{bmatrix} -1 \\ 3 \\ -1 \\ 4 \end{bmatrix}$, and let A be the matrix from the previous exercise. Is b in the range of the linear transformation $x \mapsto Ax$?

9. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ into $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and maps $v = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ into $\begin{bmatrix} 0 \\ 5 \end{bmatrix}$. Use the fact that T is linear to find the images under T of $-u$, $4v$, and $4v - u$.

10. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let $\{v_1, v_2, v_3\}$ be a linearly dependent set in $\mathbb{R}^n$. Explain why the set $\{T(v_1), T(v_2), T(v_2)\}$ is linearly dependent.