Homework 2 (Due Friday, March 15, 2019)

Please turn in solutions to any 4 of the following 5 problems.

Problem 1. Let H be a complex separable Hilbert space and let T_1, $T_2 \in \mathcal{L}(H, H)$ be Hilbert-Schmidt operators. Let $T = T_2 T_1$. Show that

$$\operatorname{Tr} T = \sum (T e_j, e_j)$$

exists if e_j is an orthonormal basis for H, and prove that the sum is independent of the choice of basis.

Problem 2. When $a(x, \xi) \in \mathcal{S}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$, let us consider the Weyl quantization of a,

$$a^w(x, D_x)u(x) = \frac{1}{(2\pi)^n} \iint e^{i(x-y)\xi} a\left(\frac{x+y}{2}, \xi\right) u(y)\, dy\, d\xi,$$

acting on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$.

- Show that $a^w(x, D_x)$ is symmetric on $\mathcal{S}(\mathbb{R}^n)$,
$$(a^w(x, D_x)u, v)_{L^2} = (u, a^w(x, D_x)v)_{L^2}, \quad u, v \in \mathcal{S}(\mathbb{R}^n),$$
precisely when a is real-valued.
- Suppose now that $a = a(\xi)$ is real-valued and depends only on the momentum variable ξ. Show that $a^w(D_x)$ is unitarily equivalent to a multiplication operator by $a(\xi)$. What is the spectrum of $a^w(D_x)$?

Problem 3. Let us consider the Sturm-Liouville operator

$$P = -\frac{d}{dt}\left(p(t)\frac{d}{dt}\right) + q(t),$$

where $p \in C^1(\mathbb{R})$, $p > 0$, and $q \in C(\mathbb{R})$, $q \geq 0$. Show that the operator P, equipped with the domain $C_0^\infty(\mathbb{R})$, is essentially selfadjoint on $L^2(\mathbb{R})$.

Hint. Show that it suffices to verify that the range $(P+1)(C_0^\infty(\mathbb{R}))$ is dense in $L^2(\mathbb{R})$ and establish this property.

Problem 4. Let T be a closed densely defined operator in a complex separable Hilbert space. Show that the operators T^*T and TT^* are self-adjoint, when equipped with their natural domains of definition.

Problem 5. Let

$$P = -\Delta + V,$$

on $L^2(\mathbb{R}^n)$, where the potential $V \in C(\mathbb{R}^n; \mathbb{R})$ is bounded from below. Let us equip P with the domain $\mathcal{D}(P) = C_0^\infty(\mathbb{R}^n)$, so that P becomes essentially self-adjoint. Assume that $V(x) \to +\infty$ as $|x| \to \infty$. Show that the spectrum of the closure $\overline{P}$ is discrete, consisting of isolated real eigenvalues of finite multiplicity, accumulating at $+\infty$ only. Show also that there is an orthonormal basis of $L^2(\mathbb{R}^n)$, consisting of eigenfunctions of $\overline{P}$.