

0.1 Spring 2016

1. Assume $f \in L^1[0, 1]$. Compute

$$\lim_{k \rightarrow \infty} \int_{[0,1]} |f|^{1/k} dx.$$

Solution. Let's split this integral into three regions.

$$\int_{[0,1]} |f|^{1/k} dx = \int_{f=0} |f|^{1/k} dx + \int_{0 < |f| \leq 1} |f|^{1/k} dx + \int_{|f| > 1} |f|^{1/k} dx.$$

The integral over the first region is clearly zero for all k . On the second region we have that $|f|^{1/k} \leq 1$ for all k . Since the interval $[0, 1]$ has finite measure, we have that the constant function 1 is in $L^1(\{x : 0 < |f| \leq 1\})$, so the dominated convergence theorem says that the integral over the second region goes to $m(\{0 < |f| \leq 1\})$. Similarly, on the third region we have that $|f|^{1/k} \leq |f|$, which is in L^1 , so the dominated convergence theorem says that the third integral goes to $m(\{|f| > 1\})$. Combining these, we have that

$$\lim_{k \rightarrow \infty} \int_{[0,1]} |f|^{1/k} dx = m(\{|f| > 0\}).$$

□

2. Let $\{f_n\}$ be a sequence of measurable functions on $[0, 1]$ and $0 \leq f_n \leq 1$ a.e. Assume that

$$\lim_{n \rightarrow \infty} \int_{[0,1]} f_n g dx = \int_{[0,1]} f g dx$$

for some $f \in L^1[0, 1]$ and any $g \in C[0, 1]$. Prove that $0 \leq f \leq 1$ a.e.

Solution. Since $f \in L^1[0, 1]$, by the Lebesgue differentiation theorem we have that

$$\frac{1}{m(E)} \int_E f(t) dt \rightarrow f(x) \tag{1}$$

as E shrinks to x for almost all x . Furthermore, since $0 \leq f_n \leq 1$ we also have that

$$\frac{1}{m(E)} \int_E f_n(t) dt \rightarrow f_n(x) \in [0, 1]$$

as E shrink to x for almost all x . Intuitively, we'd like to replace the integral of f in (1) with a limit of integrals of f_n .

We claim that the function g in the given hypothesis can be replaced with the indicator function of an interval χ_I . To see this, let g_m be a sequence of continuous functions with $g_m \rightarrow \chi_I$ in L^1 and $0 \leq \chi_I \leq 1$. By extracting a subsequence we can assume that $g_m \rightarrow \chi_I$ a.e. as well. We then have

$$\int_0^1 |f_n \chi_I - f \chi_I| \leq \int_0^1 |f_n \chi_I - f_n g_m| + \int_0^1 |f_n g_m - f g_m| + \int_0^1 |f g_m - f \chi_I|.$$

Since $\|f_n\|_{L^\infty} \leq 1$, we have that the first integral on the RHS can be made small uniformly in n by picking m large. The second integral goes to zero as $n \rightarrow \infty$ by hypothesis since g_m is continuous. The third integral can be made small for m large by dominated convergence since $|fg_m| \leq |f| \in L^1$.

For almost all x , if I_k is a sequence of intervals shrinking to x then

$$\begin{aligned} \frac{1}{m(I_k)} \int_{I_k} f \, dx &= \frac{1}{m(I_k)} \int f \chi_{I_k} \, dx \\ &= \lim_{n \rightarrow \infty} \frac{1}{m(I_k)} \int f_n \chi_{I_k} \, dx. \end{aligned}$$

Since $0 \leq f_n \leq 1$, the RHS is in $[0, 1]$ for almost all x . By the Lebesgue differentiation theorem we then have that $0 \leq f \leq 1$ a.e. \square

3. Let $f, g \in L^2(\mathbb{R}, \mathcal{M}_L, \mu_L)$. Show that $f * g$ is a continuous function on \mathbb{R} vanishing at infinity, that is, $f * g \in C(R)$ and $\lim_{|x| \rightarrow \infty} (f * g)(x) = 0$.

Proof. For any h we have by Hölder's inequality

$$|(f * g)(x + h) - (f * g)(x)| = \left| \int f(t)[g(x + h - t) - g(x - t)] \, dt \right| \quad (2)$$

$$\leq \|f\|_{L^2} \cdot \|g_h - g\|_{L^2}, \quad (3)$$

where $F_h(x) = F(x + h)$ for any function F . Now for any $\epsilon > 0$ we can find $\varphi \in C_0(\mathbb{R})$ with $\|g - \varphi\|_{L^2} = \|g_h - \varphi_h\|_{L^2} < \epsilon$. By the triangle inequality we then have

$$\begin{aligned} \|g_h - g\|_{L^2} &\leq \|g_h - \varphi_h\|_{L^2} + \|\varphi_h - \varphi\|_{L^2} + \|\varphi - g\|_{L^2} \\ &< \|\varphi_h - \varphi\|_{L^2} + 2\epsilon. \end{aligned}$$

Suppose that φ has support contained in the compact set K . If we pick h small enough then we can guarantee that $\varphi_h - \varphi$ is supported on a set with measure at most $2 \cdot m(K)$. Now since φ is continuous with compact support, it is uniformly continuous, so we can choose h small enough that $|\varphi_h(x) - \varphi(x)| = |\varphi(x + h) - \varphi(x)| < \epsilon$ for all x in the support of $\varphi_h - \varphi$. For such h we have

$$\|\varphi_h - \varphi\|_{L^2} \leq \epsilon \cdot (2 \cdot m(K))^{1/2},$$

so (2) can be made arbitrarily small, which shows that $f * g$ is continuous.

First we claim that if φ and ψ are continuous with compact support then $\varphi * \psi$ vanishes at infinity. By definition we have that

$$(\varphi * \psi)(x) = \int \varphi(t)\psi(x - t) \, dt.$$

The product $\varphi(t)\psi(x-t)$ is nonzero only if t is in the support of φ and $x-t$ is in the support of ψ . If pick x large enough then supports of $t \mapsto \varphi(t)$ and $t \mapsto \psi(x-t)$ are disjoint, so this integral is zero.

Let f_n and g_n be sequences in $C_0(\mathbb{R})$ converging in L^2 to f and g , respectively. We then have

$$\begin{aligned} |(f * g)(x) - (f_n * g_n)(x)| &\leq |(f * g)(x) - (f_n * g)(x)| + |(f_n * g)(x) - (f_n * g_n)(x)| \\ &\leq \|g\|_{L^2} \cdot \|f - f_n\|_{L^2} + \|f_n\|_{L^2} \cdot \|g - g_n\|_{L^2}. \end{aligned}$$

Since $f_n \rightarrow f$ and $g_n \rightarrow g$ in L^2 , we have that $f_n * g_n$ converges uniformly to $f * g$. Since $f_n * g_n$ vanishes at infinity, we must then have that $f * g$ vanishes at infinity. \square

4. Let (X, \mathcal{A}, μ) be a finite measure space, and let $p_1 \in (1, \infty]$. Let $\{f_n\}$ be a uniformly bounded sequence in $L^{p_1}(X, \mathcal{A}, \mu)$. Suppose $f = \lim_{n \rightarrow \infty} f_n$ exists μ -a.e. Prove that $f \in L^p(X, \mathcal{A}, \mu)$ for all $p \in [1, p_1]$ and $f_n \rightarrow f$ in $L^p(X, \mathcal{A}, \mu)$ for all $p \in [1, p_1]$.

Proof. Suppose that $\|f_n\|_{L^{p_1}} \leq M$ for all n . First we claim that the f_n are in $L^p(X, \mathcal{A}, \mu)$ for all $p \in [1, p_1]$. In fact, they are uniformly bounded:

$$\begin{aligned} \int_X |f_n|^p &= \int_{|f_n| < 1} |f_n|^p + \int_{|f_n| \geq 1} |f_n|^p \\ &\leq \int_{|f_n| < 1} 1 + \int_{|f_n| \geq 1} |f_n|^{p_1} \\ &\leq \mu(\{f \leq 1\}) + M^{p_1}. \end{aligned} \tag{4}$$

Since (X, \mathcal{A}, μ) is a finite measure space, this quantity is finite, so $f_n \in L^p(X, \mathcal{A}, \mu)$ for all n and $p \in [1, p_1]$. We can then use the fact that $f_n \rightarrow f$ a.e. and Fatou's lemma to show that $f \in L^p(X, \mathcal{A}, \mu)$ for $p \in [1, p_1]$:

$$\int_X |f|^p \leq \liminf_{n \rightarrow \infty} \int_X |f_n|^p < \infty,$$

where the finiteness follows from the L^p uniform-boundedness of the f_n .

To establish convergence in L^p , $p \in [1, p_1)$ our plan is to use the Vitali convergence theorem. The family f_n is tight over X since X is a finite measure space and we're given that $f_n \rightarrow f$ a.e., so it only remains to show that the f_n 's are uniformly integrable. To this end, let E be any measurable subset of X . Since f_n is in L^{p_1} , we have that $|f_n|^p \in L^{p_1/p}$. If we let q be the Hölder conjugate to p_1/p then we have

$$\begin{aligned} \int_E |f_n|^p &= \int_X |f_n|^p \cdot \chi_E \\ &\leq \| |f_n|^p \|_{L^{p_1/p}} \cdot \|\chi_E\|_{L^q} \\ &\leq M^{p_1^2/p} \cdot \mu(E)^{1/q}. \end{aligned}$$

If we choose E so that $\mu(E)^{1/q} < \epsilon \cdot M^{-p_1^2/p}$, then we'll have that $\int_E |f_n|^p < \epsilon$, so the f_n 's are uniformly integrable. By the Vitali convergence theorem we have that $f_n \rightarrow f$ in L^p for $p \in [1, p_1]$. \square

5. Let (X, \mathcal{A}, μ) be a measure space, and let $f : X \rightarrow [0, \infty)$ be \mathcal{A} -measurable. Consider the measure space $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu_L)$, where $\mathcal{B}_{\mathbb{R}}$ is the Borel σ -algebra on \mathbb{R} and μ_L is the Lebesgue measure, and form the product measure space $(X \times \mathbb{R}, \sigma(\mathcal{A} \times \mathcal{B}_{\mathbb{R}}), \mu \times \mu_L)$. Define $E \subset X \times \mathbb{R}$ by $(x, y) \in E \iff y \in [0, f(x))$. Prove that $E \in \sigma(\mathcal{A} \times \mathcal{B}_{\mathbb{R}})$ and $(\mu \times \mu_L)(E) = \int_X f \, d\mu$.

Proof. A function is measurable if it pulls measurable sets back to measurable sets. The plan is then to write E as a union and/or intersection of preimages of measurable sets under measurable functions. The function $F(x, y) = f(x)$ is measurable since

$$F^{-1}((-\infty, \alpha]) = \{(x, y) : f(x) \leq \alpha\} = \{x : f(x) \leq \alpha\} \times \mathbb{R} \in \sigma(\mathcal{A} \times \mathcal{B}_{\mathbb{R}}),$$

as f is μ -measurable. We also clearly have that the function $G(x, y) = y$ is measurable. Now consider the function $H(x, y) = y - f(x)$. H is measurable as it is the difference of the measurable functions G and F . We then have that E is measurable through the following decomposition

$$\begin{aligned} E &= \{(x, y) : 0 \leq y < f(x)\} \\ &= \{(x, y) : y \geq 0\} \cap \{(x, y) : y < f(x)\} \\ &= G^{-1}([0, \infty)) \cap H^{-1}((-\infty, 0]). \end{aligned}$$

If $\{f > 0\}$ is σ -finite we can use Tonelli's theorem to say

$$\begin{aligned} (\mu \times \mu_L)(E) &= \int_{X \times \mathbb{R}} \chi_E(x, y) \, d(\mu \times \mu_L) \\ &= \int_X \int_{\mathbb{R}} \chi_E(x, y) \, d\mu_L d\mu \\ &= \int_X \int_{\mathbb{R}} \chi_{[0, f(x))}(y) \, dy d\mu \\ &= \int_X f(x) \, d\mu. \end{aligned}$$

On the other hand, suppose that $\{f > 0\}$ is not σ -finite. We claim that $\int_X f \, d\mu = +\infty$. Indeed, since we can decompose this set into a countable union,

$$\{f > 0\} = \bigcup_{m=1}^{\infty} \left\{ \frac{1}{m+1} < f \leq \frac{1}{m} \right\} \cup \bigcup_{n=1}^{\infty} \{n < f \leq n+1\}, \quad (5)$$

we must have that one of these sets has infinite measure. We need to show that $(\mu \times \mu_L)(E) = +\infty$ too. For any $\alpha, \beta > 0$ we have that if $\alpha \leq f(x) < \beta$ then the product set

$$\{x : \alpha \leq f(x) < \beta\} \times \{y : 0 \leq y < \alpha\}$$

is contained in E . This product set has measure $\alpha \cdot \mu_L\{\alpha \leq f < \beta\}$, so by monotonicity we have that

$$\alpha \cdot \mu_L\{\alpha \leq f < \beta\} \leq (\mu \times \mu_L)(E)$$

for all $\alpha, \beta > 0$. But by the decomposition (5), we have that some set of the form $\{\alpha \leq f(x) < \beta\}$ must have infinite measure, so we must have $(\mu \times \mu_L)(E) = +\infty$. \square

6. Let $f \in L^1(\mathbb{R})$ and let $a_1, \dots, a_k \in \mathbb{R}$ and $b_1, \dots, b_k \in \mathbb{R} \setminus \{0\}$. Assume that the quotients $\frac{a_j}{b_j}$ are all distinct. Determine

$$\lim_{t \rightarrow \infty} \int \left| \sum_{j=1}^k f(b_j x + t a_j) \right| dx.$$

Solution. Let $\varphi \in C_0(\mathbb{R})$ be such that $\|f - \varphi\|_{L^1} < \epsilon$. Our plan is to compute the desired limit with φ in place of f and then argue that the difference can be made small. We have

$$\int \left| \sum_{j=1}^k \varphi(b_j x + t a_j) \right| dx = \int \left| \sum_{j=1}^k \varphi \left[b_j \left(x + \frac{a_j}{b_j} t \right) \right] \right| dx$$

Now $\varphi(b_j x + t a_j)$ is φ stretched horizontally by a factor of b_j and shifted over a_j/b_j . Since the support of φ is compact and the a_j/b_j are distinct, the supports of these transformations are disjoint for sufficiently large t . When these supports are disjoint we then have

$$\begin{aligned} \int \left| \sum_{j=1}^k \varphi(b_j x + t a_j) \right| dx &= \int \sum_{j=1}^k |\varphi(b_j x + t a_j)| dx \\ &= \|\varphi\|_{L^1} \cdot \sum_{j=1}^k \frac{1}{b_j}. \end{aligned}$$

That we can approximate the desired sum for $f \in L^1$ follows from the reverse triangle inequality.

$$\begin{aligned} \left| \int \left| \sum_{j=1}^k f(b_j x + t a_j) \right| dx - \int \left| \sum_{j=1}^k \varphi(b_j x + t a_j) \right| dx \right| &\leq \sum_{j=1}^k \int |f(b_j x + t a_j) - \varphi(b_j x + t a_j)| dx \\ &= \epsilon \cdot \sum_{j=1}^k \frac{1}{b_j}. \end{aligned}$$

\square

0.2 Fall 2015

1. Let E be a measurable subset of $[0, 2\pi]$. Assume that $f \in C(\mathbb{R})$ is 1-periodic, i.e. $f(x+1) = f(x)$. Compute

$$\lim_{n \rightarrow \infty} \int_E f(nx) dx.$$

Solution. We rewrite the integral over E as an integral over \mathbb{R} against the indicator function of E :

$$\int_E f(nx) dx = \int f(nx)\chi_E(x) dx.$$

Now let $\varphi \in C_0^\infty(\mathbb{R})$. Since $f \in C(\mathbb{R})$ is 1-periodic, it has a 1-periodic continuous primitive F with $F' = f$. By the chain rule we have $[\frac{1}{n}F(nx)]' = f(nx)$. Integration by parts gives

$$\int f(nx)\varphi(x) dx = -\frac{1}{n} \int F(nx)\varphi'(x) dx.$$

$F(nx)$ is bounded since F is 1-periodic and $\varphi \in C_0^\infty(\mathbb{R})$, so it's integrable. We then have

$$\begin{aligned} \left| \int f(nx)\varphi(x) dx \right| &\leq \frac{1}{n} \|F\|_\infty \cdot \|\varphi'\|_{L^1} \\ &\rightarrow 0. \end{aligned}$$

Since E is a measurable subset of $[0, 2\pi]$, it has finite measure and $\chi_E \in L^1(\mathbb{R})$. We can then find $\varphi \in C_0^\infty(\mathbb{R})$ with $\|\chi_E - \varphi\|_{L^1} < \epsilon$. Since f is continuous and 1-periodic, it is bounded and we have

$$\begin{aligned} \left| \int f(nx)\chi_E(x) dx - \int f(nx)\varphi(x) dx \right| &\leq \|f\|_\infty \cdot \|\chi_E - \varphi\|_{L^1} \\ &\leq \|f\|_\infty \cdot \epsilon. \end{aligned}$$

Since $\int f(nx)\varphi(x) dx \rightarrow 0$, we must have $\int_E f(nx) \rightarrow 0$. □

2. Suppose $f \in L^1[0, 1]$ and assume that there exists $C > 0$ such that for all measurable subsets $E \subset [0, 1]$ we have

$$\int_E |f(x)| dx \leq C\mu(E)^{1/2}.$$

Show that $f \in L^p[0, 1]$ for $1 \leq p < 2$. Show that the statement fails for $p = 2$ by giving a counterexample.

Proof. We have that

$$|f(x)|^p - 1 \leq \sum_{n=1}^{\infty} \chi_{\{|f|^p \geq n\}}(x) \leq |f(x)|^p.$$

Since $[0, 1]$ is a finite measure space, integrating through this inequality shows that $f \in L^p[0, 1]$ if and only if the series

$$\sum_{n=1}^{\infty} \mu\{|f(x)|^p \geq n\} = \sum_{n=1}^{\infty} \mu\{|f(x)| \geq n^{1/p}\}.$$

converges. By Chebyshev's inequality and the given hypotheses we have

$$n^{1/p} \mu\{|f| \geq n^{1/p}\} \leq \int_{\{|f| \geq n^{1/p}\}} |f| dx \leq C\mu\{|f| \geq n^{1/p}\}^{1/2}.$$

Dividing through by $n^{1/p} \mu\{|f| \geq n^{1/p}\}^{1/2}$ and squaring gives

$$\sum_{n=1}^{\infty} \mu\{|f(x)| \geq n^{1/p}\} \leq \sum_{n=1}^{\infty} \frac{C^2}{n^{2/p}},$$

which converges for all $p \in [1, 2)$.

□

3. Show that a function $f : \mathbb{R}^n \rightarrow \mathbb{R}^+$ is measurable if and only if $E = \{(x, y) : 0 \leq y \leq f(x)\}$ is a measurable set of \mathbb{R}^{n+1} .

Proof. Suppose f is measurable. Then the function $F(x, y) = f(x)$ is a measurable function $\mathbb{R}^{n+1} \rightarrow \mathbb{R}$. Since $G(x, y) = y$ is also measurable, $H(x, y) = y - f(x)$ is measurable as the difference of measurable functions. We can then write E as the intersection of two measurable sets:

$$E = G^{-1}([0, \infty)) \cap H^{-1}((-\infty, 0]).$$

Thus, E is measurable if f is measurable.

Conversely, suppose that E is a measurable set. Then for any $\alpha \geq 0$ the set $A \cap G^{-1}(\alpha) = F^{-1}[[\alpha, \infty))$. This shows that F , and therefore f , is measurable. □

4. Let $f \in L^1(\mathbb{R})$ and set

$$f_h(x) = \frac{1}{2h} \int_{x-h}^{x+h} f(t) dt, \quad h > 0.$$

Show that $f_h \in L^1(\mathbb{R})$ and $f_h \rightarrow f$ in $L^1(\mathbb{R})$.

Proof. Let's integrate f_h . By Tonelli we have

$$\begin{aligned} \int |f_h(x)| dx &= \frac{1}{2h} \int \left| \int f(t) \chi_{[x-h, x+h]}(t) dt \right| dx \\ &\leq \frac{1}{2h} \int \int |f(t)| \chi_{[t-h, t+h]}(x) dx dt \\ &= \|f\|_{L^1}. \end{aligned} \tag{6}$$

Since $f \in L^1(\mathbb{R})$, we have that this quantity is finite and $f_h \in L^1(\mathbb{R})$.

Now since $f \in L^1(\mathbb{R})$, $f_h \rightarrow f$ a.e. by the Lebesgue differentiation theorem. By Fatou's lemma and (6), we have for any sequence $h_n \rightarrow 0$

$$\begin{aligned} \int |f| dx &\leq \liminf_{n \rightarrow \infty} \int |f_{h_n}| dx \\ &\leq \int |f| dx, \end{aligned}$$

so $\liminf_{n \rightarrow \infty} \int |f_{h_n}| = \int |f|$. By the triangle inequality we have $|f_{h_n}| + |f| - |f - f_{h_n}| \geq 0$. Since $|f_{h_n}| + |f| - |f - f_{h_n}|$ converges to $2|f|$ a.e., another application of Fatou's lemma gives

$$\begin{aligned} 2 \int |f| \, dx &\leq \liminf_{n \rightarrow \infty} \int (|f_{h_n}| + |f| - |f - f_{h_n}|) \, dx \\ &\iff \limsup_{n \rightarrow \infty} \int |f - f_{h_n}| \, dx \leq 0. \end{aligned}$$

We then have $\int |f - f_{h_n}| \rightarrow 0$, so $f_{h_n} \rightarrow f$ in L^1 for any $h_n \rightarrow 0$. □

5. Let (X, \mathcal{A}, μ) be a measure space and let $f_k : X \rightarrow \mathbb{R}$ be a sequence of measurable functions satisfying the following:

$$\int_X |f_k|^2 \, d\mu \leq 2015, \quad \text{for all } k,$$

and

$$\int_X f_j f_k \, d\mu = 0, \quad \text{for all } j \neq k.$$

Prove that for all $\beta > 3/2$,

$$\lim_{n \rightarrow \infty} \frac{1}{n^\beta} \sum_{k=1}^{n^2} f_k(x) = 0, \quad \text{for a.a. } x \in X.$$

Proof. Let's compute the L^2 norm of the sum

$$\begin{aligned} \left\| \frac{1}{n^\beta} \sum_{j=1}^{n^2} f_j \right\|_{L^2}^2 &= \frac{1}{n^{2\beta}} \left(\sum_{j=1}^{n^2} f_j, \sum_{k=1}^{n^2} f_k \right) \\ &= \frac{1}{n^{2\beta}} \sum_{j=1}^{n^2} \sum_{k=1}^{n^2} (f_j, f_k) \\ &= \frac{1}{n^{2\beta}} \sum_{j=1}^{n^2} \|f_j\|_{L^2}^2 \\ &\leq \frac{2015}{n^{2\beta-2}}. \end{aligned}$$

Now if $\beta > 3/2$, $2\beta - 2 > 1$, so the above quantity is summable in n . Summability and wanting to show that something holds for almost all x leads us to think Borel-Cantelli might be useful. For any fixed $\epsilon > 0$, Chebyshev gives us

$$\begin{aligned} \mu \left\{ x : \left| \frac{1}{n^\beta} \sum_{j=1}^{n^2} f_j \right|^2 \geq \epsilon \right\} &\leq \frac{1}{\epsilon^2} \int_X \left(\frac{1}{n^\beta} \sum_{j=1}^{n^2} f_j \right)^2 \, dx \\ &\leq \frac{2015}{\epsilon^2 n^{2\beta-2}}. \end{aligned}$$

If we call the set on the LHS A_n , then we have $\sum \mu(A_n) < \infty$. By Borel-Cantelli we have $\mu(\limsup_{n \rightarrow \infty} A_n) = 0$, i.e., the set of x that belong to infinitely many A_n has measure zero, so the sum is zero for almost all x . □

0.3 Spring 2015

1. Show that if $f \in L^4(\mathbb{R})$ then

$$\lim_{c \rightarrow 1} \int_{\mathbb{R}} |f(cx) - f(x)|^4 dx = 0.$$

Proof. Suppose φ is continuous with compact support. Then $\varphi(cx)$ converges to $\varphi(x)$ uniformly, and since the support of φ is compact, we have that the desired limit holds with φ in place of f .

Now let $\varphi \in C_0(\mathbb{R})$ be such that $\|f - \varphi\|_{L^4} < \epsilon$. Since $|a + b|^p \leq 2^p(|a|^p + |b|^p)$ for all $p > 0$ we have

$$\begin{aligned} \int |f(cx) - f(x)|^4 dx &= \int |f(cx) - \varphi(cx) + \varphi(cx) - \varphi(x) + \varphi(x) - f(x)|^4 dx \\ &\leq 2^4 \int |f(cx) - \varphi(cx)|^4 dx \\ &\quad + 2^8 \int |\varphi(cx) - \varphi(x)|^4 dx + 2^8 \int |\varphi(x) - f(x)|^4 dx. \end{aligned}$$

The first and third integrals are small since $\|f - \varphi\|_{L^4} < \epsilon$ and the second integral can be made small as $c \rightarrow 1$ since $\varphi(cx) \rightarrow \varphi(x)$ uniformly on a compact set. \square

2. Let $f_n : (0, \infty) \rightarrow \mathbb{R}$, be a sequence of Lebesgue measurable functions such that $f_n \rightarrow f$ a.e. as $n \rightarrow \infty$. Assume that there exists $g : (0, \infty) \rightarrow \mathbb{R}$ such that $|f_n| \leq g$ for all n and $g \in L^1(0, a)$ for all $0 < a < \infty$. Assume furthermore that

$$\int_1^\infty |f_n(\sqrt{x})| dx \leq C,$$

for all n and for some constant $C > 0$. Show that $f_n \in L^1(0, \infty)$, $f \in L^1(0, \infty)$ and $f_n \rightarrow f$ in $L^1(0, \infty)$ as $n \rightarrow \infty$.

Proof. First let's show that $f_n \in L^1(0, \infty)$ for all n . Write

$$\int_0^\infty |f_n| dx = \int_0^1 |f_n| dx + \int_1^\infty |f_n| dx. \quad (7)$$

For the first integral, since $|f_n| \leq g$ and $g \in L^1(0, 1)$ we have

$$\int_0^1 |f_n| dx \leq \int_0^1 g dx < \infty.$$

For the second integral in (7) we use the hypothesis about $f_n(\sqrt{x})$.

$$\begin{aligned} C &\geq \int_1^\infty |f_n(\sqrt{x})| dx \\ &= 2 \int_1^\infty t |f_n(t)| dt \\ &\geq \int_1^\infty |f_n(t)| dt. \end{aligned}$$

Both integrals in (7) are then finite, so $f_n \in L^1(0, \infty)$. In fact, we actually have that the f_n are uniformly bounded in $L^1(0, \infty)$ by $\int_0^1 g \, dx + C$. Since $f_n \rightarrow f$ a.e. we can apply Fatou's lemma to show that $f \in L^1(0, \infty)$:

$$\begin{aligned} \int_0^\infty |f| \, dx &\leq \liminf_{n \rightarrow \infty} \int_0^\infty |f_n| \, dx \\ &\leq \int_0^1 g \, dx + C \\ &< \infty. \end{aligned}$$

Finally, since $|f - f_n| \rightarrow 0$ a.e. and $|f - f_n| \leq |f| + g \in L^1(0, \infty)$, we can apply the dominated convergence theorem to show that $f_n \rightarrow f$ in $L^1(0, \infty)$. \square

3. Assume that $f \in C^1(0, 1)$ and

$$\int_0^1 x |f'|^p \, dx < +\infty$$

for some $p > 2$. Show that $\lim_{x \rightarrow 0^+} f(x)$ exists.

Proof. Let $x_n \rightarrow 0$. We have

$$\begin{aligned} |f(x_n) - f(x_m)| &= \left| \int_{x_m}^{x_n} f'(x) \, dx \right| \\ &\leq \int_{x_m}^{x_n} |f'(x)| \, dx \\ &= \int_0^1 x^{1/p} |f'(x)| x^{-1/p} \chi_{[x_m, x_n]}(x) \, dx \end{aligned}$$

\square