260B - Homework 1

Problem 1. Define the Sobolev space $H^s(\mathbb{R}^d)$, $s \geq 0$ to be the set of all functions $u \in L^2(\mathbb{R}^d)$ such that

$$\|u\|_{H^s}^2 = \frac{1}{(2\pi)^d} \int |\widehat{u}(\xi)|^2 (1+|\xi|^2)^s \, d\xi < \infty.$$

(a) Show that $H^s(\mathbb{R}^d)$ is a Hlibert space when equipped with the scalar product

$$(u,v)_{H^s} = \frac{1}{(2\pi)^d} \int \widehat{u}(\xi)\overline{\widehat{v}(\xi)}(1+|\xi|^2)^s \, d\xi.$$

Proof. Denote $\langle \xi \rangle := (1+|\xi|^2)^{1/2}$ (apparently this is sometimes called the "Japanese bracket" of ξ).

It's clear that the alleged inner product is linear, conjugate symmetric, and positive definite (since the Fourier transform is an isometry from L^2 to itself). That it is well-defined follows from Hölder's inequality:

$$\begin{aligned}
|(u,v)| &\leq \frac{1}{(2\pi)^d} \int |\widehat{u}(\xi)||\widehat{v}(\xi)| \cdot \langle x \rangle^{2s} \, d\xi \\
&= \frac{1}{(2\pi)^d} \int (|\widehat{u}(\xi)| \cdot \langle \xi \rangle^s) \cdot (|\widehat{v}(\xi)| \cdot \langle \xi \rangle^s) \, d\xi \\
&\leq \frac{1}{(2\pi)^d} \|\widehat{u}(\xi) \cdot \langle \xi \rangle^s\|_{L^2} \cdot \|\widehat{v}(\xi) \cdot \langle \xi \rangle^s\|_{L^2} \\
&= \|u\|_{H^s} \cdot \|v\|_{H^s} \\
&< \infty.
\end{aligned}$$

The interesting part is showing that this space is complete with respect to this norm. Suppose that u_n is a Cauchy sequence in $H^s(\mathbb{R}^d)$. Then for $\epsilon > 0$ and m, n sufficiently large we have

$$\begin{aligned}
\epsilon &\geq \|u_n - u_m\|_{H^s}^2 \\
&= \frac{1}{(2\pi)^d} \int |\widehat{u_n - u_m}(\xi)|^2 \cdot \langle \xi \rangle^{2s} \, d\xi \\
&= \frac{1}{(2\pi)^d} \int |\widehat{u_n}(\xi) \cdot \langle \xi \rangle^s - \widehat{u_m}(\xi) \cdot \langle \xi \rangle^s|^2 \, d\xi.
\end{aligned}$$

So the sequence $\widehat{u_n}(\xi) \cdot \langle \xi \rangle^s$ is Cauchy in L^2. Since $L^2(\mathbb{R}^d)$ is complete, $\widehat{u_n}(\xi) \cdot \langle \xi \rangle^s$ converges to some $v \in L^2(\mathbb{R}^d)$. By Hölder's inequality $v(\xi) \cdot \langle \xi \rangle^{-s}$ is also in $L^2(\mathbb{R}^d)$, so it has a well-defined inverse Fourier transform.

We claim that u_n converges to $\mathcal{F}^{-1}(v(\xi)\cdot\langle\xi\rangle^{-s})$ in $H^s(\mathbb{R}^d)$. It was designed for this purpose after all.

$$\begin{aligned}\|u_n - \mathcal{F}^{-1}(v(\xi)\cdot\langle\xi\rangle^{-s})\|_{H^s}^2 &= \frac{1}{(2\pi)^d}\int |\widehat{u_n}(\xi) - v(\xi)\cdot\langle\xi\rangle^{-s}|^2\cdot\langle\xi\rangle^{2s}\ d\xi \\ &= \frac{1}{(2\pi)^d}\int |\widehat{u_n}(\xi)\cdot\langle\xi\rangle^{s} - v(\xi)|^2\ d\xi \\ &\to 0.\end{aligned}$$

That $\mathcal{F}^{-1}(v(\xi)\cdot\langle\xi\rangle^{-s})$ is in $H^s(\mathbb{R}^d)$ follows immediately from v being in $L^2(\mathbb{R}^d)$. Thus, $H^s(\mathbb{R}^d)$ is complete. □

(b) When $K\subseteq\mathbb{R}^d$ is compact we define

$$H^s(K) = \{u\in H^s(\mathbb{R}^d) : \operatorname{supp}(u)\subseteq K\}.$$

Show that $H^s(K)$ is a closed linear subspace of $H^s(\mathbb{R}^d)$, and hence also a Hilbert space. Show that the inclusion map $H^s(K)\to H^t(\mathbb{R}^d)$ is compact if $s>t\geq 0$.

Proof. Let u_n be a convergent sequence in $H^s(K)$. By part (a) we know that u_n converges to some u in $H^s(\mathbb{R}^d)$ (and in $L^2(\mathbb{R}^d)$). To show that u indeed lives in $H^s(K)$, we need to show that its support is contained in K. If u's support *wasn't* contained in K then it would have nonzero integral outside of K just like all of the u_n's. Let's do a computation.

$$\begin{aligned}\int_{\mathbb{R}^d\setminus K}|u(x)|^2\ dx &\leq \int_{\mathbb{R}^d\setminus K}|u(x)-u_n(x)|^2\ dx + \int_{\mathbb{R}^d\setminus K}|u_n(x)|^2\ dx \\ &= \int_{\mathbb{R}^d\setminus K}|u(x)-u_n(x)|^2\ dx.\end{aligned}$$

Taking the limit on both sides and using the fact that u_n converges to u in L^2 shows that u isn't supported outside of K, so u lives in $H^s(K)$ and the space is closed.

Now to show that the inclusion $H^s(K)\to H^t(\mathbb{R}^d)$ is compact for $s>t\geq 0$. To this end, let $u_j\in H^s(K)$ be a bounded sequence, say with $\|u_j\|_{H^s(K)}\leq 1$. We claim that the $\widehat{u_j}$'s are smooth. To see this, we expand the exponential into its power series.

$$\begin{aligned}\widehat{u_j}(\xi) &= \int_K u(x)e^{-ix\cdot\xi}\ dx \\ &= \int_K u(x)\left(\sum_{n=0}^{\infty}\frac{(-ix\cdot\xi)^n}{n!}\right)dx \\ &= \sum_{n=0}^{\infty}\int_K u(x)\frac{(-ix\cdot\xi)^n}{n!}\ dx.\end{aligned}$$

The interchange of summation and integration is justified since K is compact and the power series of the exponential converges uniformly on compact sets. The $x \cdot \xi$ in the integrand can be expanded to show that the above sum is a series of polynomials. The theory of power series then shows that since the Fourier transform is defined everywhere and is given by this power series, it is smooth.

Our plan is to apply the Arzela-Ascoli theorem to the sequence $\widehat{u_j}$. Let's show that this sequence is uniformly bounded. We use Parseval's theorem and the fact that the u_j's are compactly supported.

$$\begin{aligned}
|\widehat{u_j}(\xi)| &= \left| \int_{\mathbb{R}^d} u_j(x) e^{-ix\cdot\xi} \, dx \right| \\
&= \left| \int_K u_j(x) e^{-ix\cdot\xi} \, dx \right| \\
&\leq \|u_j\|_{L^2(K)} \cdot \|e^{-ix\cdot\xi}\|_{L^2(K)} \\
&= C_K \|\widehat{u_j}\|_{L^2(\mathbb{R}^d)} \\
&\leq C_K \|u_j\|_{H^s(K)}.
\end{aligned}$$

Since $\|u_j\|_{H^s(K)} \leq 1$, the Fourier transforms are uniformly bounded. The same argument shows that the partial derivatives of the $\widehat{u_j}$'s are uniformly bounded, which means that the $\widehat{u_j}$'s are Lipschitz continuous with the same Lipschitz constant. Consequently, the $\widehat{u_j}$'s are equicontinuous on compact subsets of $\mathbb{R}^d$.

By the Arzela-Ascoli theorem, $\widehat{u_j}$ has a uniformly convergent subsequence on every compact subset of $\mathbb{R}^d$. Let F_k be the closed ball in $\mathbb{R}^d$ with radius k. We get a uniformly convergent subsequence on F_1 and from this we can extract a further subsequence that converges uniformly on F_2, and so on. Taking the diagonal entries from these subsequences gives a subsequence, $\widehat{u_{j_k}}$, that converges *pointwise* on $\mathbb{R}^d$.

Finally, we'll show that the corresponding subsequence u_{j_k} converges in $H^t(\mathbb{R}^d)$. $\square$

Problem 2. Let B_1 and B_2 be Banach spaces and let $T \in \mathcal{L}(B_1, B_2)$. Prove that if T is compact then $\|Tu_n\|_{B_2} \to 0$ for every sequence $u_n \in B_1$ such that $u_n \to 0$ in the weak topology $\sigma(B_1, B_1^*)$. prove the converse when B_1 is reflexive and B_1^* is separable.

Proof. Our plan is to show that any subsequence of Tu_n has a further subsequence converging to zero. To this end, let Tu_{n_j} be a subsequence of Tu_n. Since $u_n \rightharpoonup 0$, we also have that $u_{n_j} \rightharpoonup 0$. By the uniform boundedness principle, u_{n_j} is strongly bounded. Since T is compact, Tu_{n_j} has a strongly convergent subsequence, $Tu_{n_{j_k}}$. This strongly convergent subsequence is also weakly convergent and we can compute its limit. For any continuous linear functional $\eta \in B_2^*$ we have by the weak convergence of u_n to zero

$$\langle Tu_{n_{j_k}}, \eta\rangle_2 = \langle u_{n_{j_k}}, T^*\eta\rangle_1 \to 0.$$

So $Tu_{n_{j_k}} \rightharpoonup 0$. Since $Tu_{n_{j_k}}$ converges weakly *and* strongly, the limits must be the same. We conclude that $Tu_{n_{j_k}} \to 0$ strongly. Thus, any subsequence of Tu_n contains a further subsequence strongly converging to zero, so $Tu_n \to 0$.

Conversely suppose that B_1 is reflexive, B_1^* is separable, and that for every sequence $u_n \in B_1$ with $u_n \rightharpoonup 0$ we also have that $Tu_n \to 0$ for some bounded operator T. Since B_1^* is separable, we have by Banach-Alaoglu that the unit ball in $\sigma(B_1^{**}, B_1^*)$ is compactly metrizable. But B_1 is reflexive, so $B_1^{**} \cong B_1$ and the unit ball in B_1 is weakly compact.

Let $\{u_n\}$ be a sequence in B_1 with $\|u_n\| \leq 1$ for all n. Sequential compactness is equivalent to compactness in metric spaces, and since the unit ball in B_1 is compactly metrizable by the above paragraph, u_n has a subsequence, u_{n_k}, that converges weakly to some u, i.e. $(u_{n_k} - u) \rightharpoonup 0$. By hypothesis we then have that $T(u_{n_k} - u) \to 0$, so Tu_{n_k} converges strongly to Tu. We have then shown that Tu_n has a strongly convergent subsequence, so T is compact. □

Problem 3. Let H be a complex separable Hilbert space. An operator $T \in \mathcal{L}(H, H)$ is called a Hilbert-Schmidt operator if for some orthonormal basis $\{e_j\}$ of H we have

$$\sum \|Te_j\|^2 < \infty.$$

(a) Show that if T satisfies the above inequality for some orthonormal basis then it satisfies it for every orthonormal basis and the sum is independent of the choice of basis. Define $\|T\|_{HS}$ to be the square root of this sum.

Proof. Let f_i be another orthonormal basis. By Parseval's theorem we have

$$\begin{aligned}\sum_i \|Tf_i\|^2 &= \sum_i \sum_j |\langle Tf_i, e_j\rangle|^2 \\ &= \sum_i \sum_j |\langle T^*e_j, f_i\rangle|^2 \\ &= \sum_j \|T^*e_j\|^2.\end{aligned}$$

Switching the order of summation is justified by Tonelli's theorem as each term is nonnegative. It looks like we can switch between orthonormal bases at the cost of switching from T to its adjoint, T^*. But repeating the above calculation with $f_i = e_i$ shows that $\sum \|T^*e_j\|^2 = \sum \|Te_j\|^2$, so we have that the sum is independent of orthonormal basis. □

(b) Show that the operator norm of T does not exceed the Hilbert-Schmidt norm.

Proof. $\square$