260B - Homework 1

Problem 1. Define the Sobolev space $H^s(\mathbb{R}^d)$, $s \geq 0$ to be the set of all functions $u \in L^2(\mathbb{R}^d)$ such that

$$\|u\|_{H^s}^2 = \frac{1}{(2\pi)^d} \int |\widehat{u}(\xi)|^2 (1+|\xi|^2)^s \, d\xi < \infty.$$

(a) Show that $H^s(\mathbb{R}^d)$ is a Hlibert space when equipped with the scalar product

$$(u,v)_{H^s} = \frac{1}{(2\pi)^d} \int \widehat{u}(\xi)\overline{\widehat{v}(\xi)}(1+|\xi|^2)^s \, d\xi.$$

Proof. Denote $\langle \xi \rangle := (1+|\xi|^2)^{1/2}$ (apparently this is sometimes called the "Japanese bracket" of ξ).

It's clear that the alleged inner product is linear, conjugate symmetric, and positive definite (since the Fourier transform is an isometry from L^2 to itself). That it is well-defined follows from Hölder's inequality:

$$\begin{aligned}
|(u,v)| &\leq \frac{1}{(2\pi)^d} \int |\widehat{u}(\xi)||\widehat{v}(\xi)| \cdot \langle x \rangle^{2s} \, d\xi \\
&= \frac{1}{(2\pi)^d} \int (|\widehat{u}(\xi)| \cdot \langle \xi \rangle^s) \cdot (|\widehat{v}(\xi)| \cdot \langle \xi \rangle^s) \, d\xi \\
&\leq \frac{1}{(2\pi)^d} \|\widehat{u}(\xi) \cdot \langle \xi \rangle^s\|_{L^2} \cdot \|\widehat{v}(\xi) \cdot \langle \xi \rangle^s\|_{L^2} \\
&= \|u\|_{H^s} \cdot \|v\|_{H^s} \\
&< \infty.
\end{aligned}$$

The interesting part is showing that this space is complete with respect to this norm. Suppose that u_n is a Cauchy sequence in $H^s(\mathbb{R}^d)$. Then for $\epsilon > 0$ and m, n sufficiently large we have

$$\begin{aligned}
\epsilon &\geq \|u_n - u_m\|_{H^s}^2 \\
&= \frac{1}{(2\pi)^d} \int |\widehat{u_n - u_m}(\xi)|^2 \cdot \langle \xi \rangle^{2s} \, d\xi \\
&= \frac{1}{(2\pi)^d} \int |\widehat{u_n}(\xi) \cdot \langle \xi \rangle^s - \widehat{u_m}(\xi) \cdot \langle \xi \rangle^s|^2 \, d\xi.
\end{aligned}$$

So the sequence $\widehat{u_n}(\xi) \cdot \langle \xi \rangle^s$ is Cauchy in L^2. Since $L^2(\mathbb{R}^d)$ is complete, $\widehat{u_n}(\xi) \cdot \langle \xi \rangle^s$ converges to some $v \in L^2(\mathbb{R}^d)$. By Hölder's inequality $v(\xi) \cdot \langle \xi \rangle^{-s}$ is also in $L^2(\mathbb{R}^d)$, so it has a well-defined inverse Fourier transform.

We claim that u_n converges to $\mathcal{F}^{-1}(v(\xi)\cdot\langle\xi\rangle^{-s})$ in $H^s(\mathbb{R}^d)$. It was designed for this purpose after all.

$$\begin{aligned}
\|u_n - \mathcal{F}^{-1}(v(\xi)\cdot\langle\xi\rangle^{-s})\|_{H^s}^2 &= \frac{1}{(2\pi)^d}\int |\widehat{u_n}(\xi) - v(\xi)\cdot\langle\xi\rangle^{-s}|^2\cdot\langle\xi\rangle^{2s}\ d\xi \\
&= \frac{1}{(2\pi)^d}\int |\widehat{u_n}(\xi)\cdot\langle\xi\rangle^{s} - v(\xi)|^2\ d\xi \\
&\to 0.
\end{aligned}$$

That $\mathcal{F}^{-1}(v(\xi)\cdot\langle\xi\rangle^{-s})$ is in $H^s(\mathbb{R}^d)$ follows immediately from v being in $L^2(\mathbb{R}^d)$. Thus, $H^s(\mathbb{R}^d)$ is complete. □

(b) When $K\subseteq\mathbb{R}^d$ is compact we define

$$H^s(K) = \{u\in H^s(\mathbb{R}^d) : \operatorname{supp}(u)\subseteq K\}.$$

Show that $H^s(K)$ is a closed linear subspace of $H^s(\mathbb{R}^d)$, and hence also a Hilbert space. Show that the inclusion map $H^s(K)\to H^t(\mathbb{R}^d)$ is compact if $s>t\geq 0$.

Proof. Let u_n be a convergent sequence in $H^s(K)$. By part (a) we know that u_n converges to some u in $H^S(\mathbb{R}^d)$ (and in $L^2(\mathbb{R}^d)$). To show that u indeed lives in $H^s(K)$, we need to show that its support is contained in K. If u's support *wasn't* contained in K then it would have nonzero integral outside of K just like all of the u_n's. Let's do a computation.

$$\begin{aligned}
\int_{\mathbb{R}^d\setminus K}|u(x)|^2\ dx &\leq \int_{\mathbb{R}^d\setminus K}|u(x)-u_n(x)|^2\ dx + \int_{\mathbb{R}^d\setminus K}|u_n(x)|^2\ dx \\
&= \int_{\mathbb{R}^d\setminus K}|u(x)-u_n(x)|^2\ dx.
\end{aligned}$$

Taking the limit on both sides and using the fact that u_n converges to u in L^2 shows that u isn't supported outside of K, so u lives in $H^s(K)$ and the space is closed.

Now to show that the inclusion $H^s(K)\to H^t(\mathbb{R}^d)$ is compact for $s>t\geq 0$. To this end, let $u_j\in H^s(K)$ be a bounded sequence. We claim that the $\widehat{u_j}$'s are smooth. To see this, we expand the exponential into its power series.

$$\begin{aligned}
\widehat{u_j}(\xi) &= \int_K u(x)e^{-ix\cdot\xi}\ dx \\
&= \int_K u(x)\left(\sum_{n=0}^{\infty}\frac{(-ix\cdot\xi)^n}{n!}\right)dx \\
&= \sum_{n=0}^{\infty}\int_K u(x)\frac{(-ix\cdot\xi)^n}{n!}\ dx.
\end{aligned}$$

The interchange of summation and integration is justified since K is compact and the power series of the exponential converges uniformly on compact sets. □