More Continued Fractions Exercises

1. Let n be a positive integer. Come up with the continued fraction expansion for $\sqrt{n^2+1}$.

2. Let $\{c_n\}$ be the convergents of $\phi = [1; 1, 1, \ldots]$. Prove that for $n \geq 1$, we have $\frac{F_{n+1}}{F_n} = c_{n-1}$, i.e. $p_n = F_{n+2}$ and $q_n = F_{n+1}$. Conclude that $\phi = \lim_{n\to\infty} \frac{F_{n+1}}{F_n}$.

3. A continued fraction $[a_0; a_1, a_2, \ldots]$ where the a_n are all positive *real* numbers for $n \geq 1$ is called unary. In this problem we'll prove that a unary continued fraction converges if and only if $\sum a_n = \infty$.

   (a) Prove that $q_n \leq \prod_{k=1}^{n}(1 + a_k)$.

   (b) Prove that if the unary continued fraction converges, then $\sum a_n = \infty$

   (c) Prove that

   $$q_{2n} \geq 1 + a_1(a_2 + a_4 + \cdots + a_{2n}), \quad q_{2n-1} \geq a_1 + a_3 + \cdots + a_{2n-1},$$

   where the first inequality holds for $n \geq 1$ and the second for $n \geq 2$.

   (d) Prove that if $\sum a_n = \infty$, then the unary continued fraction converges.