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233A - Final

1.4.6 Let Y be a subspace of a topological space X. Show that Y is irreducible if and only if the closure of Y in X is irreducible.

Proof. First suppose that Y is irreducible. If $\overline{Y}$ (the closure of Y in X) were reducible, then we could write $\overline{Y} = \tilde{F}_1 \cup \tilde{F}_2$, where $\tilde{F}_1$ and $\tilde{F}_2$ are nonempty (relatively) closed subsets of $\overline{Y}$. In particular, this means that we can write $\overline{Y} \subseteq F_1 \cup F_2$, where F_1 and F_2 are closed in X and Y is not entirely contained in either F_1 or F_2. If Y is contained in say F_1, then $\overline{Y} \subseteq \overline{F}_1 = F_1$, which contradicts the reducibility of $\overline{Y}$, so Y isn't contained in F_1. By symmetry, Y is not contained in F_2 either. But we have

$$Y \subseteq \overline{Y} \subseteq F_1 \cup F_2.$$

This shows that Y is contained in the union of closed (in X) subsets, but is contained in neither set individually, contradicting the irreducibility of Y. We conclude that $\overline{Y}$ is also irreducible.

Conversely, suppose that $\overline{Y}$ is irreducible but Y is reducible. Then $Y \subseteq F_1 \cup F_2$, where F_1 and F_2 are closed in X and Y is contained in neither F_1 nor F_2. When we take the closure of both sides of this inclusion we get

$$\overline{Y} \subseteq \overline{F_1 \cup F_2} = \overline{F_1} \cup \overline{F_2} = F_1 \cup F_2.$$

Since $\overline{Y}$ is irreducible, it must be contained in F_1 or F_2, say F_1. But then $Y \subseteq \overline{Y} \subseteq F_1$, contradicting our assumption about Y not being contained in F_1. We conclude that Y is irreducible. □

2.6.13 Let X and Y be prevarieties with affine open covers $\{U_i\}$ and $\{V_j\}$, respectively. Construct the product prevariety $X \times Y$ by gluing the affine varieties $U_i \times V_j$ together. Moreover, show that there are projection morphisms $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ satisfying the usual universal property for products.

Proof. The affine varieties $U_i \times V_j$ (as the product of two affine varieties is an affine variety) form a finite affine open cover for $X \times Y$ as a topological space. The idea now is to glue the sets $U_i \times V_j$ and $U_k \times V_l$ along the identity morphism on the intersection $(U_i \cap U_k) \times (V_j \cap V_l)$. Let $f_{ijkl} : U_i \times V_j \to U_k \times V_l$ be the identity morphism on the intersection. Then we clearly have that $f_{ijkl} = (f_{klij})^{-1}$ and the cocycle condition holds on triple intersections.

Let's show that $X \times Y$ is irreducible. Suppose that $X \times Y = F_1 \cup F_2$ where F_1 and F_2 are closed, no-one properly containing $X \times Y$. For any fixed $y \in Y$, the map $\iota_y : X \to X \times Y$ that sends x to (x, y) is continuous. Consequently, the preimage, $\iota_y^{-1}(F_i)$ is closed in X for $i = 1, 2$ and all y. Since the arbitrary intersection of closed sets is closed, we have that the covering of $X \times Y$ by closed sets induces a covering of X by closed sets. But X is irreducible, so this covering must be trivial. We conclude that $X \times Y$ is irreducible. So far we have that $X \times Y$ has an affine open covering, and is irreducible.

We build the ring of regular functions on $X \times Y$ locally. Say U is open in $X \times Y$ and contains $x \in U_i \times V_j$ for some i, j. We say that a function f is regular on U if its restriction to $U \cap U_i \times V_j$ is regular when considered as a function on the variety $U_i \times V_j$. The sheaf properties of the rings of functions on $U_i \times V_j$ are inherited.

Let's show that our projection maps, π_X and π_Y are indeed morphisms. That they are continuous is clear. Say, $U \subseteq X$ is open and $f : U \to k$ is a regular function on X. Take $P \in \pi_X^{-1}(U)$ and write $P = (x, y)$ where $x \in X$ and $y \in Y$. The pullback, π_X^* behaves as follows:

$$(\pi_X^* f)(P) = f \circ \pi_X(P) = f(x).$$

Since f is regular, this shows that $\pi_X^* f$ is regular. Since π_X pulls regular functions back to regular functions, it is a morphism. The same holds for π_Y.

Finally, let's show that our projections satisfy the universal property of products. Suppose we're given a prevariety Z and morphisms $f : Z \to X$ and $f : Z \to Y$. We need to show that there is a unique morphism $h : Z \to X \times Y$ that makes the left diagram commute.

$$\begin{tikzcd}
Z \arrow[dr, dotted, "h"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\
& X \times Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \\
& X &
\end{tikzcd}
\qquad
\begin{tikzcd}
Z_{i,j} \arrow[dr, dotted, "h_{i,j}"] \arrow[drr, bend left, "g|_{Z_{i,j}}"] \arrow[ddr, bend right, "f_{Z_{i,j}}"'] & & \\
& U_i \times V_j \arrow[r, "\pi_Y|_{U_i \times V_j}"] \arrow[d, "\pi_X|_{U_i \times V_j}"] & V_j \\
& U_i &
\end{tikzcd}$$

Define $h(z) = (f(z), g(z))$. This map clearly makes the diagram commute, at least set-theoretically. It remains to show that h is a morphism and that it is unique. The idea is to pass to the universal property of the product varieties $U_i \times V_j$. Since the morphisms f and g are continuous, for any $U_i \times V_j$ we have that $f^{-1}(U_i) \cap g^{-1}(V_j)$ is an affine open set in Z. But then this set can be covered by an affine variety, say $Z_{i,j}$. The restrictions of f and g to $Z_{i,j}$ induce a unique map $h_{i,j} : Z_{i,j} \to U_i \times V_j$ by the universal property of products of affine varieties, shown in the diagram on the right. The $Z_{i,j}$ cover Z, so the $h_{i,j}$ weave together to agree with h. Since each $h_{i,j}$ is a unique morphism, we have that h is a unique morphism too. $\square$

3.5.5 Let V be the vector space over k of homogeneous degree-2 polynomials in three variables x_0, x_1, x_2 and let $\mathbb{P}(V) \cong \mathbb{P}^5$ be its projectivization.

(i) Show that the space of conics in $\mathbb{P}^2$ can be identified with an open subset U of $\mathbb{P}^5$. What geometric objects can be associated to the points in $\mathbb{P}^5 \setminus U$?

Proof. A conic in $\mathbb{P}^2$ is determined by a homogeneous quadratic equation

$$f(x, y, z) = ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0.$$

Equivalently, we can represent this equation with the matrix equation

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0. \tag{1}$$

This matrix is symmetric, so we can change coordinates so that the above matrix is diagonal, giving the equation $AX^2 + BY^2 + CZ^2 = 0$, for some A, B, C. If two of A, B, C are zero, say $A = B = 0$, then the conic $CZ^2 = CZ \cdot Z$ is reducible. If just $A = 0$, then $BY^2 + CZ^2 = (\sqrt{B}Y + \sqrt{-C}Z)(\sqrt{B}Y - \sqrt{-C}Z)$ is again reducible. These correspond to degenerate conics. If one or two of A, B, C are zero, then the diagonal matrix with entries A, B, C has determinant zero. But the determinant is invariant under coordinate changes, so the matrix in equation (1) also has determinant zero.

Suppose that none of A, B, C are zero. If $AX^2 + BY^2 + CZ^2$ were reducible, we could write

$$AX^2 + BY^2 + CZ^2 = (\sqrt{A}X + g(Y, Z))(\sqrt{A}X + h(Y, Z)).$$

Multiplying this out shows that $g(Y, Z) + h(Y, Z) = 0$ and $g(Y, Z)h(Y, Z) = BY^2 + CZ^2$. This would imply that

$$-g(Y, Z)^2 = (\sqrt{B}Y + \sqrt{-C}Z)(\sqrt{B}Y - \sqrt{-C}Z).$$

$k[Y, Z]$ is a unique factorization domain, but the left-hand side of this equation is a square and the right-hand side isn't (under the modest assumption that the characteristic of k is not 2). This shows that $AX^2 + BY^2 + CZ^2$ is irreducible, and therefore corresponds to a non-degenerate conic. By a similar argument used in the degerate case, this shows implies that the determinant of the matrix in (1) is nonzero.

We have shown that our conic is non-degenerate if and only if the determinant of the matrix in (1) is non-vanishing. The determinant is a polynomial in the in the coefficients $(a : b : c : d : e : f) \in \mathbb{P}^5$, so the non-vanishing locus, and therefore the set of non-degenerate conics, corresponds to an open set in $\mathbb{P}^5$. □

(ii) Show that it is a linear condition in $\mathbb{P}^5$ for the conics to pass through a given point in $\mathbb{P}^2$. If $P \in \mathbb{P}^2$ is a point, show that there is a linear subspace $L \subseteq \mathbb{P}^5$ such that the conics passing through P are exactly those in $U \cap L$. What happens in $\mathbb{P}^5 \setminus U$?

Proof. Suppose that the conic determined by $f(x, y, z) = ax^2 + bxy + cy^2 + dxz + eyz + fz^2$ passes through the point $(x_0, y_0, z_0) \in \mathbb{P}^2$. Then the coefficients a, b, c, d, e, f satisfy the linear equation

$$ax_0^2 + bx_0y_0 + cy_0^2 + dx_0z_0 + ey_0z_0 + fz_0^2 = 0.$$

Call the set of all coefficients satisfying the above equation L. Since L is the solution set to a homogeneous linear equation, it is a linear subspace of $\mathbb{P}^5$. As the set U corresponds to the non-degenerate conics in $\mathbb{P}^2$, we have that $U \cap L$ corresponds to the non-degenerate conics passing through $(x_0 : y_0 : z_0)$. On the other hand, $(\mathbb{P}^5 \setminus U) \cap L$ corresponds to the degenerate conics passing through $(x_0 : y_0 : z_0)$. $\square$

(iii) Prove that there is a unique conic through any five points in $\mathbb{P}^2$, as long as no three of them lie on a line. What happens if three of them do lie on a line?

Proof. $\square$

4.6.10 Let $X \subseteq \mathbb{A}^n$ be an affine variety, and let $Y_1, Y_2 \subsetneq X$ be irreducible, closed subsets, no-one contained in the other. Let $\tilde{X}$ be the blow-up of X at the (possibly non-radical) ideal $I(Y_1) + I(Y_2)$. Then the strict transforms of Y_1 and Y_2 on $\tilde{X}$ are disjoint.