Spring 2016

1. Assume $f \in L^1[0,1]$. Compute
$$\lim_{k\to\infty} \int_{[0,1]} |f|^{1/k}\ dx.$$

Solution. Let's split this integral into three regions.
$$\int_{[0,1]} |f|^{1/k}\ dx = \int_{f=0} |f|^{1/k}\ dx + \int_{0<|f|\le 1} |f|^{1/k}\ dx + \int_{|f|>1} |f|^{1/k}\ dx.$$
The integral over the first region is clearly zero for all k. On the second region we have that $|f|^{1/k} \le 1$ for all k. Since the interval $[0,1]$ has finite measure, we have that the constant function 1 is in $L^1(\{x : 0 < |f| \le 1\})$, so the dominated convergence theorem says that the integral over the second region goes to $m(\{0 < |f| \le 1\})$. Similarly, on the third region we have that $|f|^{1/k} \le |f|$, which is in L^1, so the dominated convergence theorem says that the third integral goes to $m(\{|f| > 1\})$. Combining these, we have that
$$\lim_{k\to\infty} \int_{[0,1]} |f|^{1/k}\ dx = m(\{|f| > 0\}).$$
□

2. Let $\{f_n\}$ be a sequence of measurable functions on $[0,1]$ and $0 \le f_n \le 1$ a.e. Assume that
$$\lim_{n\to\infty} \int_{[0,1]} f_n g\ dx = \int_{[0,1]} fg\ dx$$
for some $f \in L^1[0,1]$ and any $g \in C[0,1]$. Prove that $0 \le f \le 1$ a.e.

Solution. Since $f \in L^1[0,1]$, by the Lebesgue differentiation theorem we have that
$$\frac{1}{m(E)} \int_E f(t)\ dt \to f(x) \tag{1}$$
as E shrinks to x for almost all x. Furthermore, since $0 \le f_n \le 1$ we also have that
$$\frac{1}{m(E)} \int_E f_n(t)\ dt \to f_n(x) \in [0,1]$$
as E shrink to x for almost all x. Intuitively, we'd like to replace the integral of f in (1) with a limit of integrals of f_n.

We claim that the function g in the given hypothesis can be replaced with the indicator function of an interval χ_I. To see this, let g_m be a sequence of continuous functions with $g_m \to \chi_I$ in L^1 and $0 \le \chi_I \le 1$. By extracting a subsequence we can assume that $g_m \to \chi_I$ a.e. as well. We then have
$$\int_0^1 |f_n\chi_I - f\chi_i| \le \int_0^1 |f_n\chi_I - f_n g_m| + \int_0^1 |f_n g_m - f g_m| + \int_0^1 |f g_m - f\chi_i|.$$

Since $\|f_n\|_{L^\infty} \le 1$, we have that the first integral on the RHS can be made small uniformly in n by picking m large. The second integral goes to zero as $n \to \infty$ by hypothesis since g_m is continuous. The third integral can be made small for m large by dominated convergence since $|fg_m| \le |f| \in L^1$.

For almost all x, if I_k is a sequence of intervals shrinking to x then

$$\frac{1}{m(I_k)} \int_{I_k} f \, dx = \frac{1}{m(I_k)} \int f \chi_{I_k} \, dx$$
$$= \lim_{n\to\infty} \frac{1}{m(I_k)} \int f_n \chi_{I_k} \, dx.$$

Since $0 \le f_n \le 1$, the RHS is in $[0,1]$ for almost all x. By the Lebesgue differentiation theorem we then have that $0 \le f \le 1$ a.e. $\square$

3. Let $f, g \in L^2(\mathbb{R}, \mathcal{M}_L, \mu_L)$. Show that $f * g$ is a continuous function on $\mathbb{R}$ vanishing at infinity, that is, $f * g \in C(R)$ and $\lim_{|x|\to\infty}(f*g)(x) = 0$.

Proof. For any h we have by Hölder's inequality

$$|(f*g)(x+h) - (f*g)(x)| = \left| \int f(t)[g(x+h-t) - g(x-t)] \, dt \right| \tag{2}$$
$$\le \|f\|_{L^2} \cdot \|g_h - g\|_{L^2}, \tag{3}$$

where $F_h(x) = F(x+h)$ for any function F. Now for any $\epsilon > 0$ we can find $\varphi \in C_0(\mathbb{R})$ with $\|g - \varphi\|_{L^2} = \|g_h - \varphi_h\|_{L^2} < \epsilon$. By the triangle inequality we then have

$$\|g_h - g\|_{L^2} \le \|g_h - \varphi_h\|_{L^2} + \|\varphi_h - \varphi\|_{L^2} + \|\varphi - g\|_{L^2}$$
$$< \|\varphi_h - \varphi\|_{L^2} + 2\epsilon.$$

Suppose that φ has support contained in the compact set K. If we pick h small enough then we can guarantee that $\varphi_h - \varphi$ is supported on a set with measure at most $2 \cdot m(K)$. Now since φ is continuous with compact support, it is uniformly continuous, so we can choose h small enough that $|\varphi_h(x) - \varphi(x)| = |\varphi(x+h) - \varphi(x)| < \epsilon$ for all x in the support of $\varphi_h - \varphi$. For such h we have

$$\|\varphi_h - \varphi\|_{L^2} \le \epsilon \cdot (2 \cdot m(K))^{1/2},$$

so (2) can be made arbitrarily small, which shows that $f * g$ is continuous.

First we claim that if φ and ψ are continuous with compact support then $\varphi * \psi$ vanishes at infinity. By definition we have that

$$(\varphi * \psi)(x) = \int \varphi(t)\psi(x-t) \, dt.$$

The product $\varphi(t)\psi(x-t)$ is nonzero only if t is in the support of φ and $x-t$ is in the support of φ. If pick x large enough then supports of $t \mapsto \varphi(t)$ and $t \mapsto \psi(x-t)$ are disjoint, so this integral is zero.

Let f_n and g_n be sequences in $C_0(\mathbb{R})$ converging in L^2 to f and g, respectively. We then have

$$\begin{aligned}|(f*g)(x)-(f_n*g_n)(x)| &\le |(f*g)(x)-(f_n*g)(x)|+|(f_n*g)(x)-(f_n*g_n)(x)| \\ &\le \|g\|_{L^2}\cdot\|f-f_n\|_{L^2}+\|f_n\|_{L^2}\cdot\|g-g_n\|_{L^2}.\end{aligned}$$

Since $f_n \to f$ and $g_n \to g$ in L^2, we have that $f_n * g_n$ converges uniformly to $f*g$. Since $f_n * g_n$ vanishes at infinity, we must then have that $f*g$ vanishes at infinity. □

4. Let $(X,\mathcal{A},\mu)$ be a finite measure space, and let $p_1 \in (1,\infty]$. Let $\{f_n\}$ be a uniformly bounded sequence in $L^{p_1}(X,\mathcal{A},\mu)$. Suppose $f=\lim_{n\to\infty} f_n$ exists μ-a.e. Prove that $f \in L^p(X,\mathcal{A},\mu)$ for all $p \in [1,p_1]$ and $f_n \to f$ in $L^p(X,\mathcal{A},\mu)$ for all $p \in [1,p_1)$.

Proof. Suppose that $\|f_n\|_{L^{p_1}} \le M$ for all n. First we claim that the f_n are in $L^p(X,\mathcal{A},\mu)$ for all $p \in [1,p_1]$. In fact, they are uniformly bounded:

$$\begin{aligned}\int_X |f_n|^p &= \int_{|f_n|<1} |f_n|^p + \int_{|f_n|\ge 1} |f_n|^p \\ &\le \int_{|f_n|<1} 1 + \int_{|f_n|\ge 1} |f_n|^{p_1} \\ &\le \mu(\{f \le 1\}) + M^{1/p_1}.\end{aligned}$$

Since $(X,\mathcal{A},\mu)$ is a finite measure space, this quantity is finite, so $f_n \in L^p(X,\mathcal{A},\mu)$ for all n and $p \in [1,p_1]$. We can then use the fact that $f_n \to f$ a.e. and Fatou's lemma to show that $f \in L^p(X,\mathcal{A},\mu)$ for $p \in [1,p_1]$:

$$\int_X |f|^p \le \liminf_{n\to\infty} \int_X |f_n|^p < \infty,$$

where the finiteness follows from the L^p uniform-boundedness of the f_n.

To establish convergence in L^p, $p \in [1,p_1)$ our plan is to use the Vitali convergence theorem. The family f_n is tight over X since X is a finite measure space and we're given that $f_n \to f$ a.e., so it only remains to show that the f_n's are uniformly integrable. Intuitively, since the f_n's are in L^p, the measure of the set $\{f_n \ge N\}$ should shrink as N grows. Now since $p < p_1$, if $N > 1$ then

$$|f_n|^p \chi_{\{|f_n|\ge N\}} N^{p_1-p} \le |f_n|^{p_1}.$$

If we integrate both sides over any measurable set E we have

$$\int_{E\cap\{|f_n|\ge N\}} |f_n|^p \le \frac{M}{N^{p_1-p}}.$$

On the complement we have
$$\int_{E\cap\{|f_n|<N\}} |f_n|^p \leq N^p \cdot \mu(E).$$
Putting these together, we have that
$$\int_E |f_n|^p = \int_{E\cap\{|f_n|\geq N\}} |f_n|^p + \int_{E\cap\{|f_n|<N\}} |f_n|^p$$
$$\leq \frac{M}{R^{p_1-p}} + R^p \cdot \mu(E).$$
If we choose R so that $M/R^{p_1-p} < \epsilon/2$ and E so that $R^p \cdot \mu(E) < \epsilon/2$ then we'll have that $\int_E |f_n|^p < \epsilon$ for any E of sufficiently small measure, so the f_n's are uniformly integrable. By the Vitali convergence theorem we have that $f_n \to f$ in L^p for $p \in [1, p_1)$. $\square$

5. Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f : X \to [0, \infty)$ be $\mathcal{A}$-measurable. Consider the measure space $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu_L)$, where $\mathcal{B}_{\mathbb{R}}$ is the Borel σ-algebra on $\mathbb{R}$ and μ_L is the Lebesgue measure, and form the product measure space $(X \times \mathbb{R}, \sigma(\mathcal{A} \times \mathcal{B}_{\mathbb{R}}), \mu \times \mu_L)$. Define $E \subset X \times R$ by $(x, y) \in E \iff y \in [0, f(x))$. Prove that $E \in \sigma(\mathcal{A} \times \mathcal{B}_{\mathbb{R}})$ and $(\mu \times \mu_L)(E) = \int_X f \; d\mu$.

Proof. A function is measurable if it pulls measurable sets back to measurable sets. The plan is then to write E is a union and/or intersection of preimages of measurable sets under measurable functions. The function $F(x, y) = f(x)$ is measurable since
$$F^{-1}[(-\infty, \alpha]) = \{(x, y) : f(x) \leq \alpha\} = \{x : f(x) \leq \alpha\} \times \mathbb{R} \in \sigma(\mathcal{A} \times \mathcal{B}_{\mathbb{R}}),$$
as f is μ-measurable. We also clearly have that the function $G(x, y) = y$ is measurable. Now consider the function $H(x, y) = y - f(x)$. H is measurable as it is the difference of the measurable functions G and F. We then have that E is measurable through the following decomposition
$$E = \{(x, y) : 0 \leq y < f(x)\}$$
$$= \{(x, y) : y \geq 0\} \cap \{(x, y) : y < f(x)\}$$
$$= G^{-1}[[0, \infty)] \cap H^{-1}[(-\infty, 0)].$$
If $\{f > 0\}$ is σ-finite we can use Tonelli's theorem to say
$$(\mu \times \mu_L)(E) = \int_{X\times\mathbb{R}} \chi_E(x, y) \; d(\mu \times \mu_L)$$
$$= \int_X \int_{\mathbb{R}} \chi_E(x, y) \; d\mu_L d\mu$$
$$= \int_X \int_{\mathbb{R}} \chi_{[0,f(x))}(y) \; dy d\mu$$
$$= \int_X f(x) \; d\mu.$$

On the other hand, suppose that $\{f > 0\}$ is note σ-finite. We claim that $\int_X f \, d\mu = +\infty$. Indeed, since we can decompose this set into a countable union,

$$\{f > 0\} = \bigcup_{m=1}^{\infty} \{\frac{1}{m+1} < f \leq \frac{1}{m}\} \cup \bigcup_{n=1}^{\infty} \{n < f \leq n+1\}, \tag{4}$$

we must have that one of these sets has infinite measure. We need to show that $(\mu \times \mu_L)(E) = +\infty$ too. For any $\alpha, \beta > 0$ we have that if $\alpha \leq f(x) < \beta$ then the product set

$$\{x : \alpha \leq f(x) < \beta\} \times \{y : 0 \leq \alpha\}$$

is contained in E. This product set has measure $\alpha \cdot \mu_L\{\alpha \leq f < \beta\}$, so by monotonicity we have that

$$\alpha \cdot \mu_L\{\alpha \leq f < \beta\} \leq (\mu \times \mu_L)(E)$$

for all $\alpha, \beta > 0$. But by the decomposition (4), we have that some set of the form $\{\alpha \leq f(x) < \beta\}$ must have infinite measure, so we must have $(\mu \times \mu_L)(E) = +\infty$. □

6. Let $f \in L^1(\mathbb{R})$ and let $a_1, \ldots, a_k \in \mathbb{R}$ and $b_1, \ldots, b_k \in \mathbb{R} \setminus \{0\}$. Assume that the quotients $\frac{a_j}{b_j}$ are all distinct. Determine

$$\lim_{t \to \infty} \int \left| \sum_{j=1}^{k} f(b_j x + t a_j) \right| dx.$$

Solution. Let $\varphi \in C_0(\mathbb{R})$ be such that $\|f - \varphi\|_{L^1} < \epsilon$. Our plan is to compute the desired limit with φ in place of f and then argue that the difference can be made small. We have

$$\int \left| \sum_{j=1}^{k} \varphi(b_j x + t a_j) \right| dx = \int \left| \sum_{j=1}^{k} \varphi \left[b_j \left(x + \frac{a_j}{b_j} t \right) \right] \right| dx$$

Now $\varphi(b_j x + t a_j)$ is φ stretched horizontally by a factor of b_j and shifted over a_j/b_j. Since the support of φ is compact and the a_j/b_j are distinct, the supports of these transformations are disjoint for sufficiently large t. When these supports are disjoint we then have

$$\begin{aligned}\int \left| \sum_{j=1}^{k} \varphi(b_j x + t a_j) \right| dx &= \int \sum_{j=1}^{k} |\varphi(b_j x + t a_j)| \, dx \\ &= \|\varphi\|_{L^1} \cdot \sum_{j=1}^{k} \frac{1}{b_j}.\end{aligned}$$

That we can approximate the desired sum for $f \in L^1$ follows from the reverse triangle inequality.

$$\begin{aligned}\left| \int \left| \sum_{j=1}^{k} f(b_j x + t a_j) \right| dx - \int \left| \sum_{j=1}^{k} \varphi(b_j x + t a_j) \right| dx \right| &\leq \sum_{j=1}^{k} \int |f(b_j x + t a_j) - \varphi(b_j x + t a_j)| \, dx \\ &= \epsilon \cdot \sum_{j=1}^{k} \frac{1}{b_k}.\end{aligned}$$

□