Quiz 4

Student ID Number: Name ____________________

Math 140B, 5PM

Please justify all your answers February 15, 2019

Please also write your full name on the back

1. (a) Suppose that g is *continuous* at $x = 0$. Prove that $f(x) = xg(x)$ is differentiable at $x = 0$.

   Proof. We use the definition of the derivative.

   $$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{xg(x) - 0}{x} = \lim_{x\to 0} g(x) = g(0).$$

   The equality $\lim_{x\to 0} g(x) = g(0)$ follows from the continuity of g. □

   (b) Conversely, suppose that $f(0) = 0$ and f is differentiable at $x = 0$. Prove that there is a function g that is continuous at $x = 0$ and satisfies $f(x) = xg(x)$. *Hint: What should $g(0)$ be?*

   Proof. If $g(x)$ is going to satisfy $f(x) = xg(x)$ then we'll definitely have $g(x) = \frac{f(x)}{x}$ for $x \neq 0$. As it stands, this expression isn't defined at $x = 0$, but it works everywhere else. If g is to be continuous at zero we should be able to evaluate the limit at zero. Since $f(0) = 0$ we have

   $$\lim_{x\to 0} g(x) = \lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = f'(0).$$

   The last equality follows from the differentiability of f at zero. Since this limit exists, if we define g by

   $$g(x) = \begin{cases} \frac{f(x)}{x}, & \text{if } x \neq 0 \\ f'(0), & \text{if } x = 0 \end{cases},$$

   then g will be continuous at zero and $f(x) = xg(x)$. □

2. If f and g are differentiable on $[a, b]$ and $f'(x) = g'(x)$ for all $a < x < b$, show that $g(x) = f(x) + c$ for some constant c. Give a proof directly from the mean value theorem.

   Proof. Define the function $h(x) = f(x) - g(x)$. Since f and g are continuous and differentiable on $[a, b]$, so is h. We can then apply the mean value theorem to h. For any x, y satisfying $a \leq x < y \leq b$ we then have

   $$\frac{h(x) - h(y)}{x - y} = h'(z)$$

   for some $z \in (x, y)$. Now $h'(z) = (f - g)'(z) = f'(z) - g'(z) = 0$ by hypothesis. The right-hand side of the above equation is then zero, so multiplying both sides by $x - y$ shows that $h(x) - h(y) = 0$. Since this holds for any $a \leq x < y \leq b$, we have that h is constant on $[a, b]$. h is the difference between f and g, so f and g differ by a constant. □