260B - Homework 1

Problem 1. Define the Sobolev space $H^s(\mathbb{R}^d)$, $s \geq 0$ to be the set of all functions $u \in L^2(\mathbb{R}^d)$ such that

$$\|u\|_{H^s}^2 = \frac{1}{(2\pi)^d} \int |\widehat{u}(\xi)|^2 (1 + |\xi|^2)^s \, d\xi < \infty.$$

(a) Show that $H^s(\mathbb{R}^d)$ is a Hlibert space when equipped with the scalar product

$$(u, v)_{H^s} = \frac{1}{(2\pi)^d} \int \widehat{u}(\xi) \overline{\widehat{v}(\xi)} (1 + |\xi|^2)^s \, d\xi.$$

Proof. Denote $\langle \xi \rangle := (1 + |\xi|^2)^{1/2}$ (apparently this is sometimes called the "Japanese bracket" of ξ).

It's clear that the alleged inner product is linear, conjugate symmetric, and positive definite (since the Fourier transform is an isometry from L^2 to itself). That it is well-defined follows from Hölder's inequality:

$$\begin{aligned}
|(u, v)| &\leq \frac{1}{(2\pi)^d} \int |\widehat{u}(\xi)||\widehat{v}(\xi)| \cdot \langle x \rangle^{2s} \, d\xi \\
&= \frac{1}{(2\pi)^d} \int (|\widehat{u}(\xi)| \cdot \langle \xi \rangle^s) \cdot (|\widehat{v}(\xi)| \cdot \langle \xi \rangle^s) \, d\xi \\
&\leq \frac{1}{(2\pi)^d} \|\widehat{u}(\xi) \cdot \langle \xi \rangle^s\|_{L^2} \cdot \|\widehat{v}(\xi) \cdot \langle \xi \rangle^s\|_{L^2} \\
&= \|u\|_{H^s} \cdot \|v\|_{H^s} \\
&< \infty.
\end{aligned}$$

The interesting part is showing that this space is complete with respect to this norm. Suppose that u_n is a Cauchy sequence in $H^s(\mathbb{R}^d)$. Then for $\epsilon > 0$ and m, n sufficiently large we have

$$\begin{aligned}
\epsilon &\geq \|u_n - u_m\|_{H^s}^2 \\
&= \frac{1}{(2\pi)^d} \int |\widehat{u_n - u_m}(\xi)|^2 \cdot \langle \xi \rangle^{2s} \, d\xi \\
&= \frac{1}{(2\pi)^d} \int |\widehat{u_n}(\xi) \cdot \langle \xi \rangle^s - \widehat{u_m}(\xi) \cdot \langle \xi \rangle^s|^2 \, d\xi.
\end{aligned}$$

So the sequence $\widehat{u_n}(\xi) \cdot \langle \xi \rangle^s$ is Cauchy in L^2. Since $L^2(\mathbb{R}^d)$ is complete, $\widehat{u_n}(\xi) \cdot \langle \xi \rangle^s$ converges to some $v \in L^2(\mathbb{R}^d)$. By Hölder's inequality $v(\xi) \cdot \langle \xi \rangle^{-s}$ is also in $L^2(\mathbb{R}^d)$, so it has a well-defined inverse Fourier transform.

We claim that u_n converges to $\mathcal{F}^{-1}(v(\xi)\cdot\langle\xi\rangle^{-s})$ in $H^s(\mathbb{R}^d)$. It was designed for this purpose after all.

$$\begin{aligned}\|u_n - \mathcal{F}^{-1}(v(\xi)\cdot\langle\xi\rangle^{-s})\|_{H^s}^2 &= \frac{1}{(2\pi)^d}\int |\widehat{u_n}(\xi) - v(\xi)\cdot\langle\xi\rangle^{-s}|^2\cdot\langle\xi\rangle^{2s}\ d\xi \\ &= \frac{1}{(2\pi)^d}\int |\widehat{u_n}(\xi)\cdot\langle\xi\rangle^{s} - v(\xi)|^2\ d\xi \\ &\to 0.\end{aligned}$$

That $\mathcal{F}^{-1}(v(\xi)\cdot\langle\xi\rangle^{-s})$ is in $H^s(\mathbb{R}^d)$ follows immediately from v being in $L^2(\mathbb{R}^d)$. Thus, $H^s(\mathbb{R}^d)$ is complete. □

(b) When $K\subseteq\mathbb{R}^d$ is compact we define

$$H^s(K) = \{u\in H^s(\mathbb{R}^d) : \operatorname{supp}(u)\subseteq K\}.$$

Show that $H^s(K)$ is a closed linear subspace of $H^s(\mathbb{R}^d)$, and hence also a Hilbert space. Show that the inclusion map $H^s(K)\to H^t(\mathbb{R}^d)$ is compact if $s>t\geq 0$.

Proof. Let u_n be a convergent sequence in $H^s(K)$. By part (a) we know that u_n converges to some u in $H^s(\mathbb{R}^d)$ (and in $L^2(\mathbb{R}^d)$). To show that u indeed lives in $H^s(K)$, we need to show that its support is contained in K. If u's support *wasn't* contained in K then it would have nonzero integral outside of K just like all of the u_n's. Let's do a computation.

$$\begin{aligned}\int_{\mathbb{R}^d\setminus K}|u(x)|^2\ dx &\leq \int_{\mathbb{R}^d\setminus K}|u(x)-u_n(x)|^2\ dx + \int_{\mathbb{R}^d\setminus K}|u_n(x)|^2\ dx \\ &= \int_{\mathbb{R}^d\setminus K}|u(x)-u_n(x)|^2\ dx.\end{aligned}$$

Taking the limit on both sides and using the fact that u_n converges to u in L^2 shows that u isn't supported outside of K, so u lives in $H^s(K)$ and the space is closed.

Now to show that the inclusion $H^s(K)\to H^t(\mathbb{R}^d)$ is compact for $s>t\geq 0$. To this end, let $u_j\in H^s(K)$ be a bounded sequence, say with $\|u_j\|_{H^s(K)}\leq 1$. We claim that the $\widehat{u_j}$'s are smooth. To see this, we expand the exponential into its power series.

$$\begin{aligned}\widehat{u_j}(\xi) &= \int_K u(x)e^{-ix\cdot\xi}\ dx \\ &= \int_K u(x)\left(\sum_{n=0}^{\infty}\frac{(-ix\cdot\xi)^n}{n!}\right)dx \\ &= \sum_{n=0}^{\infty}\int_K u(x)\frac{(-ix\cdot\xi)^n}{n!}\ dx.\end{aligned}$$

The interchange of summation and integration is justified since K is compact and the power series of the exponential converges uniformly on compact sets. The $x \cdot \xi$ in the integrand can be expanded to show that the above sum is a series of polynomials. The theory of power series then shows that since the Fourier transform is defined everywhere and is given by this power series, it is smooth.

Our plan is to apply the Arzela-Ascoli theorem to the sequence $\widehat{u_j}$. Let's show that this sequence is uniformly bounded. We use Parseval's theorem and the fact that the u_j's are compactly supported.

$$\begin{aligned}
|\widehat{u_j}(\xi)| &= \left| \int_{\mathbb{R}^d} u_j(x) e^{-ix\cdot\xi} \, dx \right| \\
&= \left| \int_K u_j(x) e^{-ix\cdot\xi} \, dx \right| \\
&\leq \|u_j\|_{L^2(K)} \cdot \|e^{-ix\cdot\xi}\|_{L^2(K)} \\
&= C_K \|\widehat{u_j}\|_{L^2(\mathbb{R}^d)} \\
&\leq C_K \|u_j\|_{H^s(K)}.
\end{aligned}$$

Since $\|u_j\|_{H^s(K)} \leq 1$, the Fourier transforms are uniformly bounded. The same argument shows that the partial derivatives of the $\widehat{u_j}$'s are uniformly bounded, which means that the $\widehat{u_j}$'s are Lipschitz continuous with the same Lipschitz constant. Consequently, the $\widehat{u_j}$'s are equicontinuous on compact subsets of $\mathbb{R}^d$.

By the Arzela-Ascoli theorem, $\widehat{u_j}$ has a uniformly convergent subsequence on every compact subset of $\mathbb{R}^d$. Let F_k be the closed ball in $\mathbb{R}^d$ with radius k. We get a uniformly convergent subsequence on F_1 and from this we can extract a further subsequence that converges uniformly on F_2, and so on. Taking the diagonal entries from these subsequences gives a subsequence, $\widehat{u_{j_k}}$, that converges *pointwise* on $\mathbb{R}^d$.

Finally, we'll show that the corresponding subsequence u_{j_k} converges in $H^t(\mathbb{R}^d)$. □

Problem 2. Let B_1 and B_2 be Banach spaces and let $T \in \mathcal{L}(B_1, B_2)$. Prove that if T is compact then $\|Tu_n\|_{B_2} \to 0$ for every sequence $u_n \in B_1$ such that $u_n \to 0$ in the weak topology $\sigma(B_1, B_1^*)$. prove the converse when B_1 is reflexive and B_1^* is separable.

Proof. Our plan is to show that any subsequence of Tu_n has a further subsequence converging to zero. To this end, let Tu_{n_j} be a subsequence of Tu_n. Since $u_n \rightharpoonup 0$, we also have that $u_{n_j} \rightharpoonup 0$. By the uniform boundedness principle, u_{n_j} is strongly bounded. Since T is compact, Tu_{n_j} has a strongly convergent subsequence, $Tu_{n_{j_k}}$. This strongly convergent subsequence is also weakly convergent and we can compute its limit. For any continuous linear functional $\eta \in B_2^*$ we have by the weak convergence of u_n to zero

$$\langle Tu_{n_{j_k}}, \eta\rangle_2 = \langle u_{n_{j_k}}, T^*\eta\rangle_1 \to 0.$$

So $Tu_{n_{j_k}} \rightharpoonup 0$. Since $Tu_{n_{j_k}}$ converges weakly *and* strongly, the limits must be the same. We conclude that $Tu_{n_{j_k}} \to 0$ strongly. Thus, any subsequence of Tu_n contains a further subsequence strongly converging to zero, so $Tu_n \to 0$.

Conversely suppose that B_1 is reflexive, B_1^* is separable, and that for every sequence $u_n \in B_1$ with $u_n \rightharpoonup 0$ we also have that $Tu_n \to 0$ for some bounded operator T. Since B_1^* is separable, we have by Banach-Alaoglu that the unit ball in $\sigma(B_1^{**}, B_1^*)$ is compactly metrizable. But B_1 is reflexive, so $B_1^{**} \cong B_1$ and the unit ball in B_1 is weakly compact.

Let $\{u_n\}$ be a sequence in B_1 with $\|u_n\| \leq 1$ for all n. Sequential compactness is equivalent to compactness in metric spaces, and since the unit ball in B_1 is compactly metrizable by the above paragraph, u_n has a subsequence, u_{n_k}, that converges weakly to some u, i.e. $(u_{n_k} - u) \rightharpoonup 0$. By hypothesis we then have that $T(u_{n_k} - u) \to 0$, so Tu_{n_k} converges strongly to Tu. We have then shown that Tu_n has a strongly convergent subsequence, so T is compact. □

Problem 3. Let H be a complex separable Hilbert space. An operator $T \in \mathcal{L}(H, H)$ is called a Hilbert-Schmidt operator if for some orthonormal basis $\{e_j\}$ of H we have

$$\sum \|Te_j\|^2 < \infty.$$

(a) Show that if T satisfies the above inequality for some orthonormal basis then it satisfies it for every orthonormal basis and the sum is independent of the choice of basis. Define $\|T\|_{HS}$ to be the square root of this sum.

Proof. Let f_i be another orthonormal basis. By Parseval's theorem we have

$$\begin{aligned}\sum_i \|Tf_i\|^2 &= \sum_i \sum_j |\langle Tf_i, e_j\rangle|^2 \\ &= \sum_i \sum_j |\langle T^*e_j, f_i\rangle|^2 \\ &= \sum_j \|T^*e_j\|^2.\end{aligned}$$

Switching the order of summation is justified by Tonelli's theorem as each term is nonnegative. It looks like we can switch between orthonormal bases at the cost of switching from T to its adjoint, T^*. But repeating the above calculation with $f_i = e_i$ shows that $\sum \|T^*e_j\|^2 = \sum \|Te_j\|^2$, so we have that the sum is independent of orthonormal basis. □

(b) Show that the operator norm of T does not exceed the Hilbert-Schmidt norm.

Proof. Suppose $x \in H$ has norm 1. Write $x = \sum_j \langle x, e_j \rangle e_j$. By Hölder's inequality

$$\|Tx\|^2 = \left\| \sum_j \langle x, e_j \rangle T e_j \right\|^2 \leq \sum_j |\langle x, e_j \rangle|^2 \cdot \sum_j \|Te_j\|^2 = \sum_j \|Te_j\|^2.$$

The right-most expression is the Hilbert-Schmidt norm. Since this holds for all unit x, we have that the operator norm is bounded by the Hilbert-Schmidt norm. □

(c) Show that if T is of Hilbert-Schmidt class, then so is T^* and $\|T\|_{HS} = \|T^*\|_{HS}$.

Proof. We proved this when proving part (a). □

(d) Show that every Hilbert-Schmidt operator is compact.

Proof. Our plan is to write the Hilbert-Schmidt operator, T, as a limit of finite rank (and therefore compact) operators. Fix an orthonormal basis e_j. Define T_N by

$$T_N e_j = \begin{cases} Te_j, & \text{if } 1 \leq j \leq N \\ 0, & \text{else} \end{cases}. \tag{1}$$

Let's show that $\|T_N - T\| \to 0$ in the operator norm. For any $x \in H$ we have

$$\begin{aligned} \|(T_N - T)x\|^2 &= \left\| (T_N - T) \sum_j \langle x, e_j \rangle e_j \right\|^2 \\ &= \left\| \sum_{j=N+1}^{\infty} \langle x, e_j \rangle T e_j \right\|^2 \\ &\leq \sum_{j=N+1}^{\infty} |\langle x, e_j \rangle|^2 \cdot \sum_{j=N+1}^{\infty} \|Te_j\|^2 \\ &\leq \|x\|^2 \cdot \sum_{j=N+1}^{\infty} \|Te_j\|^2 \end{aligned}$$

Since T is Hilbert-Schmidt, the last sum here is the tail of a convergent series, so it vanishes as $N \to \infty$. Since the set of compact operators is closed, we have that T is compact. □

(e) Show that if T is a Hilbert-Schmidt operator and $S \in \mathcal{L}(H, H)$ then ST is Hilbert-Schmidt and

$$\|ST\|_{HS} \leq \|S\| \cdot \|T\|_{HS}.$$

Proof. A short computation.

$$\|ST\|_{HS}^2 = \sum_j \|STe_j\|^2 \leq \|S\|^2 \cdot \sum_j \|Te_j\|^2 = \|S\|^2 \cdot \|T\|_{HS}^2.$$

□

(f) Let $K \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Prove that if $f \in L^2(\mathbb{R}^n)$, then

$$\mathcal{K}f(x) = \int K(x,y)f(y)\ dy$$

exists for almost every x, and that $\mathcal{K}$ is a Hilbert-Schmidt operator from $L^2(\mathbb{R}^n)$ to itself, with Hilbert-Schmidt norm equal to the norm of K in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Prove that every Hilbert-Schmidt operator on $L^2(\mathbb{R}^n)$ is of this form.

Proof. That $\mathcal{K}f$ exists a.e. and is in $L^2(\mathbb{R}^n)$ follows from Fubini-Tonelli: integrating $|K(x,y)|$ with respect to y gives an integrable function in x.

$$\begin{aligned}\|\mathcal{K}f\|_{L^2}^2 &= \int \left| \int K(x,y)f(y)\ dy \right|^2 dx \\ &\leq \|f\|_{L^2}^2 \cdot \|K(x,y)\|_{L^2}^2.\end{aligned}$$

Let e_j be an orthonormal basis for $L^2(\mathbb{R}^n)$. The theory of tensor products shows that $\overline{e_j(x)}e_k(y)$ is an orthonormal basis for $L^2(\mathbb{R}^n \times \mathbb{R}^n)$, so by Parseval,

$$\begin{aligned}\|\mathcal{K}\|_{HS}^2 &= \sum_j \|\mathcal{K}e_j\|_{L^2}^2 \\ &= \sum_j \sum_k |\langle \mathcal{K}e_j, e_k\rangle|^2 \\ &= \sum_{j,k} \left| \int \int K(x,y)e_j(y)\ dy\ \overline{e_k(x)}\ dx \right|^2 \\ &= \sum_{j,k} \left| \int \int K(x,y)e_j(y)\overline{e_k(x)}\ dydx \right|^2 \\ &= \sum_{j,k} \langle K, \overline{e_j}e_k\rangle \\ &= \|K\|_{L^2}.\end{aligned}$$

Let T be a Hilbert-Schmidt operator on L^2. The idea is to approximate T by operators of the desired form and then show that the limit also has the desired form, much in the same way we proved part (d). Fix an orthonormal basis e_j for $L^2(\mathbb{R}^n)$ and define T_N by (1) on the previous page. For any $f \in L^2(\mathbb{R}^n)$ we have

$$\begin{aligned}(T_Nf)(x) = \sum_{j=1}^N \langle f, e_j\rangle Te_j &= \sum_{j=1}^N \left[\int f(y)\overline{e_j}(y)\ dy \right] (Te_j)(x) \\ &= \int \left[\sum_{j=1}^N (Te_j)(x)\overline{e_j}(y) \right] f(y)\ dy.\end{aligned}$$

This motivates us to define $K_N(x,y) := \sum_{j=1}^{N}(Te_j)(x)\overline{e_j}(y)$. That $T_N \to T$ in the operator norm follows from our discussion of part (d). It remains to show that T_N converges to an operator of the desired form. $\square$

Problem 4. Let K be a compact self-adjoint operator on a Hilbert space H, and assume that $K \geq 0$. Let $\lambda_1 \geq \lambda_2 \geq \dots$ be the sequence of non-zero eigenvalues of K, repeated according to their multiplicity and arranged in a decreasing order. Prove the Courant-Fischer minimax formula

$$\lambda_k = \min_{codim\ V=k-1} \max_{u\in V, \|u\|\leq 1} (Ku, u).$$

Proof. By the spectral theorem for compact self-adjoint operators, we have an orthonormal basis of eigenvectors, e_j, of K. Let E_k be the span of $e_1, \dots, e_k$. Then E_k

$$\begin{aligned}
\sup_{u\in E_k, \|u\|\leq 1} (Ku, u) &= \sup_{u\in E_k, \|u\|\leq 1} \sup_{j=1}^{k} \lambda_j |(u, e_j)|^2 \\
&\geq \sup_{u\in E_k, \|u\|\leq 1} \lambda_k \sum_{j=1}^{k} |(u, e_j)|^2 \\
&= \lambda_{k+1}.
\end{aligned}$$

This supremum is actually realized by setting $u = e_k$. Now suppose that V has codimension $k-1$. We can split H as $H = V \oplus V^\perp = E_k \oplus E_k^\perp$. Since E_k has dimension k and $V^\perp$ has dimension $k-1$, we must have that E_k and V overlap. Taking the infimum over all such subspaces gives

$$\inf_{codim\ V=k-1} \sup_{u\in V, \|u\|\leq 1} (Ku, u) \geq \lambda_k.$$

Picking $V = E_{k-1}^\perp$ realizes this lower bound, so we have equality above. $\square$

Problem 5. Let $f \in C(\mathbb{R}/2\pi\mathbb{Z})$ be such that $f(\theta_0) = 0$ for some $\theta_0 \in \mathbb{R}/2\pi\mathbb{Z}$. Show that the associated Toeplitz operator, T_f is not Fredholm on the Hardy space $H^2 \subseteq L^2(\mathbb{R}/2\pi\mathbb{Z})$.

Proof. Following Nicolas Raymond's approach, we first consider the case where f vanishes in a neighborhood of θ_0. Suppose f is zero on the interval $(\theta_0 - r, \theta_0 + r)$ for some $r > 0$. Finitely many translates of this interval fill up the entirety of the circle, so if we let M_f be the operator that multiplies by f and let S_r be the operator that translates a function forward r units we have that $(S_r M_f)^n = 0$ for some n.

S_r is Fredholm, so if T_f were also Fredholm, then the product $(S_r T_f)^n$ would be as well. This means that $(S_r T_f)^n$ would have an inverse modulo a compact operator, i.e. there would exist some $T : H^2 \to H^2$ and a compact K such that

$$T(S_r T_f)^n = I_{H^2} + K.$$

We claim that $(S_rT_f)^n$ is compact. Once we show this, subtracting K from both sides of the above equation will show that the identity is a compact operator on H^2. By Riesz' theorem this is a contradiction because H^2 isn't finite dimensional.

Let P be the projection from L^2 onto H^2. Then $T_f = PM_f$. We fiddle around with commutators and use the fact that S_r and P commute with one another

$$\begin{aligned}(S_rT_f)^n &= (PS_rM_f)^n \\ &= P(S_rM_fP)^{n-1}S_rM_f.\end{aligned}$$

$\square$