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233A - Final

1.4.6 Let Y be a subspace of a topological space X. Show that Y is irreducible if and only if the closure of Y in X is irreducible.

Proof. First suppose that Y is irreducible. If $\overline{Y}$ (the closure of Y in X) were reducible, then we could write $\overline{Y} = \tilde{F}_1 \cup \tilde{F}_2$, where $\tilde{F}_1$ and $\tilde{F}_2$ are nonempty (relatively) closed subsets of $\overline{Y}$. In particular, this means that we can write $\overline{Y} \subseteq F_1 \cup F_2$, where F_1 and F_2 are closed in X and Y is not entirely contained in either F_1 or F_2. If Y is contained in say F_1, then $\overline{Y} \subseteq \overline{F}_1 = F_1$, which contradicts the reducibility of $\overline{Y}$, so Y isn't contained in F_1. By symmetry, Y is not contained in F_2 either. But we have

$$Y \subseteq \overline{Y} \subseteq F_1 \cup F_2.$$

This shows that Y is contained in the union of closed (in X) subsets, but is contained in neither set individually, contradicting the irreducibility of Y. We conclude that $\overline{Y}$ is also irreducible.

Conversely, suppose that $\overline{Y}$ is irreducible but Y is reducible. Then $Y \subseteq F_1 \cup F_2$, where F_1 and F_2 are closed in X and Y is contained in neither F_1 nor F_2. When we take the closure of both sides of this inclusion we get

$$\overline{Y} \subseteq \overline{F_1 \cup F_2} = \overline{F_1} \cup \overline{F_2} = F_1 \cup F_2.$$

Since $\overline{Y}$ is irreducible, it must be contained in F_1 or F_2, say F_1. But then $Y \subseteq \overline{Y} \subseteq F_1$, contradicting our assumption about Y not being contained in F_1. We conclude that Y is irreducible. □

2.6.13 Let X and Y be prevarieties with affine open covers $\{U_i\}$ and $\{V_j\}$, respectively. Construct the product prevariety $X \times Y$ by gluing the affine varieties $U_i \times V_j$ together. Moreover, show that there are projection morphisms $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ satisfying the usual universal property for products.

Proof. The affine varieties $U_i \times V_j$ (as the product of two affine varieties is an affine variety) form a finite affine open cover for $X \times Y$ as a topological space. The idea now is to glue the sets $U_i \times V_j$ and $U_k \times V_l$ along the identity morphism on the intersection $(U_i \cap U_k) \times (V_j \cap V_l)$. Let $f_{ijkl} : U_i \times V_j \to U_k \times V_l$ be the identity morphism on the intersection. Then we clearly have that $f_{ijkl} = (f_{klij})^{-1}$ and the cocycle condition holds on triple intersections.

Let's show that $X \times Y$ is irreducible. Suppose that $X \times Y = F_1 \cup F_2$ where F_1 and F_2 are closed, no-one properly containing $X \times Y$. For any fixed $y \in Y$, the map $\iota_y : X \to X \times Y$ that sends x to (x, y) is continuous. Consequently, the preimage, $\iota_y^{-1}(F_i)$ is closed in X for $i = 1, 2$ and all y. Since the arbitrary intersection of closed sets is closed, we have that the covering of $X \times Y$ by closed sets induces a covering of X by closed sets. But X is irreducible, so this covering must be trivial. We conclude that $X \times Y$ is irreducible. So far we have that $X \times Y$ has an affine open covering, and is irreducible.

We build the ring of regular functions on $X \times Y$ locally. Say U is open in $X \times Y$ and contains $x \in U_i \times V_j$ for some i, j. We say that a function f is regular on U if its restriction to $U \cap U_i \times V_j$ is regular when considered as a function on the variety $U_i \times V_j$. The sheaf properties of the rings of functions on $U_i \times V_j$ are inherited.

Let's show that our projection maps, π_X and π_Y are indeed morphisms. That they are continuous is clear. Say, $U \subseteq X$ is open and $f : U \to k$ is a regular function on X. Take $P \in \pi_X^{-1}(U)$ and write $P = (x, y)$ where $x \in X$ and $y \in Y$. The pullback, π_X^* behaves as follows:

$$(\pi_X^* f)(P) = f \circ \pi_X(P) = f(x).$$

Since f is regular, this shows that $\pi_X^* f$ is regular. Since π_X pulls regular functions back to regular functions, it is a morphism. The same holds for π_Y.

Finally, let's show that our projections satisfy the universal property of products. Suppose we're given a prevariety Z and morphisms $f : Z \to X$ and $f : Z \to Y$. We need to show that there is a unique morphism $h : Z \to X \times Y$ that makes the left diagram commute.

$$\begin{tikzcd}
Z \arrow[dr, dotted, "h"] \arrow[drr, bend left, "g"] \arrow[dd, bend right, "f"'] & & \\
& X \times Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \\
& X &
\end{tikzcd}
\qquad
\begin{tikzcd}
Z_{i,j} \arrow[dr, dotted, "h_{i,j}"] \arrow[drr, bend left, "g|_{Z_{i,j}}"] \arrow[dd, bend right, "f_{Z_{i,j}}"'] & & \\
& U_i \times V_j \arrow[r, "\pi_Y|_{U_i \times V_j}"] \arrow[d, "\pi_X|_{U_i \times V_j}"] & V_j \\
& U_i &
\end{tikzcd}$$

Define $h(z) = (f(z), g(z))$. This map clearly makes the diagram commute, at least set-theoretically. It remains to show that h is a morphism and that it is unique. The idea is to pass to the universal property of the product varieties $U_i \times V_j$. Since the morphisms f and g are continuous, for any $U_i \times V_j$ we have that $f^{-1}(U_i) \cap g^{-1}(V_j)$ is an affine open set in Z. But then this set can be covered by an affine variety, say $Z_{i,j}$. The restrictions of f and g to $Z_{i,j}$ induce a unique map $h_{i,j} : Z_{i,j} \to U_i \times V_j$ by the universal property of products of affine varieties, shown in the diagram on the right. The $Z_{i,j}$ cover Z, so the $h_{i,j}$ weave together to agree with h. Since each $h_{i,j}$ is a unique morphism, we have that h is a unique morphism too. $\square$

3.5.5 Let V be the vector space over k of homogeneous degree-2 polynomials in three variables x_0, x_1, x_2 and let $\mathbb{P}(V) \cong \mathbb{P}^5$ be its projectivization.

(i) Show that the space of conics in $\mathbb{P}^2$ can be identified with an open subset U of $\mathbb{P}^5$. What geometric objects can be associated to the points in $\mathbb{P}^5 \setminus U$?

Proof. A conic in $\mathbb{P}^2$ is determined by a homogeneous quadratic equation

$$f(x, y, z) = ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0.$$

Equivalently, we can represent this equation with the matrix equation

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0. \tag{1}$$

This matrix is symmetric, so we can change coordinates so that the above matrix is diagonal, giving the equation $AX^2 + BY^2 + CZ^2 = 0$, for some A, B, C. If two of A, B, C are zero, say $A = B = 0$, then the conic $CZ^2 = CZ \cdot Z$ is reducible. If just $A = 0$, then $BY^2 + CZ^2 = (\sqrt{B}Y + \sqrt{-C}Z)(\sqrt{B}Y - \sqrt{-C}Z)$ is again reducible. These correspond to degenerate conics. If one or two of A, B, C are zero, then the diagonal matrix with entries A, B, C has determinant zero. But the determinant is invariant under coordinate changes, so the matrix in equation (1) also has determinant zero.

Suppose that none of A, B, C are zero. If $AX^2 + BY^2 + CZ^2$ were reducible, we could write

$$AX^2 + BY^2 + CZ^2 = (\sqrt{A}X + g(Y, Z))(\sqrt{A}X + h(Y, Z)).$$

Multiplying this out shows that $g(Y, Z) + h(Y, Z) = 0$ and $g(Y, Z)h(Y, Z) = BY^2 + CZ^2$. This would imply that

$$-g(Y, Z)^2 = (\sqrt{B}Y + \sqrt{-C}Z)(\sqrt{B}Y - \sqrt{-C}Z).$$

$k[Y, Z]$ is a unique factorization domain, but the left-hand side of this equation is a square and the right-hand side isn't (under the modest assumption that the characteristic of k is not 2). This shows that $AX^2+BY^2+CZ^2$ is irreducible, and therefore corresponds to a non-degenerate conic. By a similar argument used in the degerate case, this shows implies that the determinant of the matrix in (1) is nonzero.

We have shown that our conic is non-degenerate if and only if the determinant of the matrix in (1) is non-vanishing. The determinant is a polynomial in the in the coefficients $(a : b : c : d : e : f) \in \mathbb{P}^5$, so the non-vanishing locus, and therefore the set of non-degenerate conics, corresponds to an open set in $\mathbb{P}^5$. □

(ii) Show that it is a linear condition in $\mathbb{P}^5$ for the conics to pass through a given point in $\mathbb{P}^2$. If $P \in \mathbb{P}^2$ is a point, show that there is a linear subspace $L \subseteq \mathbb{P}^5$ such that the conics passing through P are exactly those in $U \cap L$. What happens in $\mathbb{P}^5 \setminus U$?

Proof. Suppose that the conic determined by $f(x, y, z) = ax^2 + bxy + cy^2 + dxz + eyz + fz^2$ passes through the point $(x_0, y_0, z_0) \in \mathbb{P}^2$. Then the coefficients a, b, c, d, e, f satisfy the linear equation

$$ax_0^2 + bx_0y_0 + cy_0^2 + dx_0z_0 + ey_0z_0 + fz_0^2 = 0.$$

Call the set of all coefficients satisfying the above equation L. Since L is the solution set to a homogeneous linear equation, it is a linear subspace of $\mathbb{P}^5$. As the set U corresponds to the non-degenerate conics in $\mathbb{P}^2$, we have that $U \cap L$ corresponds to the non-degenerate conics passing through $(x_0 : y_0 : z_0)$. On the other hand, $(\mathbb{P}^5 \setminus U) \cap L$ corresponds to the degenerate conics passing through $(x_0 : y_0 : z_0)$. □

(iii) Prove that there is a unique conic through any five points in $\mathbb{P}^2$, as long as no three of them lie on a line. What happens if three of them do lie on a line?

Proof. Given five points in $\mathbb{P}^2$ in general linear position, that a conic $ax^2+bxy+cy^2+dxz+eyz+fz^2$ should pass through these points gives us five linear equations in a, b, c, d, e, f by part (ii). At least one coefficient must be nonzero, so we can actually reduce this to five homogeneous equations in five unknowns. Solutions to this system correspond to the intersection of five hyperplanes: one per equation. Suppose that this system doesn't have a unique solution. Then the intersection of these hyperplanes must contain a line, ℓ.

Say $f(x, y, z)$, $g(x, y, z)$ are two distinct points on this line, which correspond to distinct conics through our five points. Two distinct points on a line completely determine it, so ℓ can be parameterized as $\ell : [s \cdot f(x, y, z) + t \cdot g(x, y, z)]$, $[s : t] \in \mathbb{P}^1$. Any point on this line is a conic through the five points.

I'm not quite sure how to draw a contradiction from here. I know that given any point in $\mathbb{P}^2$ we can choose s and t so that $s \cdot f + t \cdot g = 0$ passes through it. Given two of our five points, say P and Q, we can then force $s \cdot f + t \cdot g$ to pass through a point collinear to P and Q, say R. I think the idea is that after changing coordinates so that the line containing P, Q, R is the z coordinate axis, we'll have that $s \cdot f(x, y, 0) + t \cdot g(x, y, 0)$ will vanish at three points. This will somehow force $s \cdot f(x, y, 0) + t \cdot g(x, y, 0)$ to be identically zero and $f = z \cdot g$. This in turn will force g to be linear and our conic $s \cdot f + t \cdot g = 0$ to be the union of two lines. Since we have five points and two lines, three points must lie on one line, contradicting the hypothesis that our points are in general position. I'm not quite sure how to work out the details here. □

4.6.10 Let $X \subseteq \mathbb{A}^n$ be an affine variety, and let $Y_1, Y_2 \subsetneq X$ be irreducible, closed subsets, no-one contained in the other. Let $\tilde{X}$ be the blow-up of X at the (possibly non-radical) ideal $I(Y_1) + I(Y_2)$. Then the strict transforms of Y_1 and Y_2 on $\tilde{X}$ are disjoint.

Proof. Say $I(Y_1) = \langle f_0, \ldots, f_r \rangle$ and $I(Y_2) = \langle g_0, \ldots, g_s \rangle$. Then $I(Y_1) + I(Y_2) = \langle f_0, \ldots, f_r, g_0, \ldots, g_s \rangle$. To construct the blowup of X, $\tilde{X}$, we let $U = X \setminus (Y_1 \cap Y_2)$ and define the map $\phi : U \to \mathbb{P}^{r+s}$ by

$$\phi(P) = [f_0(P) : \cdots : f_r(P) : g_0(P) : \cdots : g_s(P)]. \tag{2}$$

This map is well-defined since the f_i and g_j do not vanish simultaneously on the complement of $Y_1 \cap Y_2$.

Now the strict transform of Y_1, $\tilde{Y_1}$ is the closure of the set $\{(P, \phi(P)) : P \in Y_1 \cap U\}$ in $\mathbb{A}^n \times \mathbb{P}^{r+s}$. Ditto for $\tilde{Y_2}$, where P comes from $Y_2 \cap U$. We might not have any earthly idea what these closures look like, but we do know that the strict transforms satisfy these containments.

$$\tilde{Y_1} \subseteq Z(f_i(P) - x_i,\ y_j g_k(P) - y_k g_j(P),\ f_i(P)) = Z(x_i,\ y_j g_k(P) - y_k g_j(P),\ f_i(P))$$

$$\tilde{Y_2} \subseteq Z(g_i(P) - y_i,\ x_j f_k(P) - x_k f_j(P),\ g_i(P)) = Z(y_i,\ x_j f_k(P) - x_k f_j(P),\ g_i(P))$$

Let's explain why these containments hold. For $\tilde{Y_1}$, $f_i(P)$ must certainly vanish for $0 \leq i \leq r$ since this condition merely says that P must come from Y_1. That $f_i(P) - x_i$ must vanish comes from (2). Finally, that $y_j g_k(P) - y_k g_j(P)$ must vanish for $0 \leq j, k \leq s$ comes from the fact that $\tilde{Y} \subseteq \tilde{X}$ and this holds in $\tilde{X}$. That $f_i(P)$ and $f_i(P) - x_i$ both vanish implies that x_i must vanish. The same argument applied to $\tilde{Y_2}$ shows that the y_j must vanish. Any P that forces the x_i and y_j to vanish simultaneously leads to a contradiction since we can't allow all coordinates of a projective point to vanish. We conclude that there is no such P. Since these sets containing the strict transforms are disjoint, we must have that the strict transforms themselves are disjoint. $\square$