233B - Final

5.6.12 Let X be a prevariety over an algebraically closed field k, and let $P \in X$ be a (closed) point of X. Let $D = \text{Spec } k[x]/(x^2)$ be the "double point". Show that the tangent space $T_{X,P}$ to X at P can be canonically identified with the set of morphisms $D \to X$ that map the unique point of D to P.

Proof. Let $f : D \to X$ be a morphism mapping $(x) \in D$ to $P \in X$. Because morphisms of schemes correspond to homomorphisms of ringed spaces, we have a map on the stalk, $f^* : \mathcal{O}_{X,P} \to k[x]/(x^2)$, that sends the maximal ideal $\mathfrak{m}_P$ to (x). Write $f^*(g) = \alpha(g) + \beta(g)x \in k[x]/(x^2)$ so that $f^*(g) = \beta(g) \in (x)$ for $g \in \mathfrak{m}_P$. We can then use f^* to build a functional $\varphi : \mathfrak{m}_P \to k$ by $\varphi(g) = \beta(g)$. Now take $g, h \in \mathfrak{m}_P$. We then have

$$f^*(gh) = f^*(g)f^*(h)$$

$$\iff \beta(gh)x = \beta(g)\beta(h)x^2$$

$$\iff \beta(gh) = 0,$$

so $\mathfrak{m}_P^2 \subseteq \ker \beta$ and we can consider φ as a functional $\mathfrak{m}_P/\mathfrak{m}_P^2 \to k$, an element of the tangent space at P. In short, we have constructed a map $\Phi : \text{Hom}(\mathcal{O}_{X,P}, k[x]/(x^2)) \to \text{Hom}(\mathfrak{m}_P/\mathfrak{m}_P^2, k) \cong T_{X,P}$ that sends $[g \mapsto \alpha(g) + \beta(g)x]$ to $[g + \mathfrak{m}_P^2 \mapsto \beta(g)]$.

One (I) should show that this assignment is injective.

On the other hand, suppose we have functional $\varphi \in \text{Hom}(\mathfrak{m}_P/\mathfrak{m}_P^2, k) \cong T_{X,P}$. Our goal is to use φ to build a morphism $D \to X$ mapping (x) to P. Since the stalk $\mathcal{O}_{X,P}$ is a local ring with maximal ideal $\mathfrak{m}_P$, we can write $\mathcal{O}_{X,P} = k \oplus \mathfrak{m}_P$ and uniquely define a map $f^* : \mathcal{O}_{X,P} \to k[x]/(x^2)$ by specifying what it does on the components of this decomposition. Define $f^* : \mathcal{O}_{X,P} \to k[x]/(x^2)$ by $f^*(g) = 0$ if $g \in k$ and $f^*(g) = \varphi(g + \mathfrak{m}_P^2)x$ if $g \in \mathfrak{m}_P$. Furthermore, this assignment is inverse to Φ. $\square$

5.6.13 Let X be an affine variety, let Y be a closed subscheme of X defined by the ideal $I \subset A(X)$, and let $\tilde{X}$ be the blow-up of X at I. Show that:

(i) $\tilde{X} = \text{Proj}(\bigoplus_{d \geq 0} I^d)$, where $I^0 := A(X)$.

(ii) The projection map $\tilde{X} \to X$ is the morphism induced by the ring homomorphism $I^0 \to \bigoplus_{d \geq 0} I^d$.

(iii) The exceptional divisor of the blow-up, i.e. the fiber $Y \times_X \tilde{X}$ of the blow-up $\tilde{X} \to X$ over Y, is isomorphic to $\text{Proj}(\bigoplus_{d \geq 0} I^d/I^{d+1})$.

Proof. $\square$

6.7.3 Let $X \subset \mathbb{P}^n$ scheme with Hilbert polynomial χ. Define the arithmetic genus of X to be $g(X) = (-1)^{\dim X} \cdot (\chi(0) - 1)$.

(i) Show that $g(\mathbb{P}^n) = 0$.

Proof. We follow the lead of Example 6.1.2 from Gathmann's notes. The coordinate ring of $\mathbb{P}^n$ is $k[x_0, \dots, x_n]$. The corresponding Hilbert function $h_{\mathbb{P}^n}(d)$ then counts the number of monomials in $k[x_0, \dots, x_n]$ of degree d, so we have

$$h_{\mathbb{P}^n}(d) = \binom{d+n}{n}.$$

This Hilbert function is a polynomial in d, so it coincides with the corresponding Hilbert function $\chi_{\mathbb{P}^n}$ and the genus of $\mathbb{P}^n$ is given by

$$\begin{aligned} g(\mathbb{P}^n) &= (-1)^{\dim \mathbb{P}^n} \left(\binom{n+0}{n} - 1 \right) \\ &= (-1)^n \cdot 0 \\ &= 0. \end{aligned}$$

□

(ii) If X is a hypersurface of degree d in $\mathbb{P}^n$, show that $g(X) = \binom{d-1}{n}$. In particular, if $C \subset \mathbb{P}^2$ is a plane curve of degree d, then $g(C) = \frac{1}{2}(d-1)(d-2)$.

Proof. Now we follow example 6.1.8(iii). Since the coordinate ring of X is given by $k[x_0, \dots, x_n]/(f) = k[x_0, \dots, x_n]/(f \cdot k[x_0, \dots, x_n])$ for some polynomial f we have that

$$\begin{aligned} h_X(t) &= \dim_k (k[x_0, \dots, x_n]/(f \cdot k[x_0, \dots, x_n]))^{(t)} \\ &= \dim_k k[x_0, \dots, x_n]^{(t)} - \dim_k k[x_0, \dots, x_n]^{(t - \deg f)} \\ &= \binom{t+n}{n} - \binom{t-d+n}{n}. \end{aligned}$$

This is again a polynomial in t, so the Hilbert function and Hilbert polynomial coincide. We then have

$$\begin{aligned} g(X) &= (-1)^{\dim X} \left(\binom{0+n}{n} - \binom{-d+n}{n} - 1 \right) \\ &= (-1)^{n-1} \cdot (-1)^n \binom{d-1}{n} \\ &= \binom{d-1}{n}. \end{aligned}$$

Moving from the first to the second line we used the lesser-known (to me at least) identity $\binom{m}{k} = (-1)^k \binom{k-m-1}{k}$.

If C is a plane curve in $\mathbb{P}^2$ then we simply substitute $n = 2$ into the above formula to obtain $g(C) = \frac{1}{2}(d-1)(d-2)$ as desired. □

(iii) Compute the arithmetic genus of the union of the three coordinate axes

$$Z(x_1x_2, x_1x_3, x_2x_3) \subset \mathbb{P}^3.$$

Solution. Let X be the union of the three coordinate axes in $\mathbb{P}^3$. From the definition of the Hilbert function we that $h_X(d)$ is the dimension of the degree d piece of the graded coordinate ring $k[x_0, \ldots, x_3]/(x_1x_2, x_1x_3, x_2x_3)$ – the number of degree d monomials divisible by at most one of x_1, x_2, x_3 and possibly divisible by x_0. These can look like $x_0^{d-k}x_i^k$ for $i = 1, 2, 3$ and $1 \leq k \leq d$ or x_0^d. There are $3d$ monomials in the former category and one in latter, so $h_X(d) = 3d + 1$ for sufficiently large d. We then have

$$\begin{aligned} g(X) &= (-1)^{\dim X}(3 \cdot 0 + 1 - 1) \\ &= 0. \end{aligned}$$

□

6.7.8 Let $C_1 = \{f_1 = 0\}$ and $C_2 = \{f_2 = 0\}$ be affine curves in $\mathbb{A}_k^2$, and let $P \in C_1 \cap C_2$ be a point. Show that the intersection multiplicity of C_1 and C_2 at P (i.e. the length of the component at P of the intersection scheme $C_1 \cap C_2$) is equal to the dimension of the vector space $\mathcal{O}_{\mathbb{A}^2,P}/(f_1, f_2)$ over k.

Proof. Write $C_1 = \text{Spec } k[x, y]/(f_1)$ and $C_2 = \text{Spec } k[x, y]/(f_2)$. The intersection scheme is then given by $C_1 \cap C_2 = \text{Spec } k[x, y]/(f_1, f_2)$. Essentially, all there is to show is that looking at the component at P of $C_1 \cap C_2$ corresponds to localizing $k[x, y]$ at P and then quotienting by (f_1, f_2).

The component of $C_1 \cap C_2$ at P corresponds to (equivalence classes of) quotients $\frac{f}{g}$ with $f, g \in k[x, y]/(f_1, f_2)$ where g does not vanish at P. But we obtain the same set by looking at quotients $\frac{f}{g} \in k[x, y]$ where g doesn't vanish at P, i.e. the stalk $\mathcal{O}_{\mathbb{A}^2,P}$, and then quotienting by (f_1, f_2), so the component of $C_1 \cap C_2$ at P is $\mathcal{O}_{\mathbb{A}^2,P}/(f_1, f_2)$.

Now the intersection multiplicity of C_1 and C_2 at P is defined to be the length of the component at P of the intersection scheme $C_1 \cap C_2$. We have just shown that this component is Spec $\mathcal{O}_{\mathbb{A}^2,P}/(f_1, f_2)$, so the length is the dimension over k of $\mathcal{O}_{\mathbb{A}^2,P}/(f_1, f_2)$. □

7.8.8 What is the line bundle on $\mathbb{P}^n \times \mathbb{P}^m$ leading to the Segre embedding $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ by the correspondence of lemma 7.5.14? What is the line bundle leading to the degree-d Veronese embedding $\mathbb{P}^n \to \mathbb{P}^N$?

Solution. The Segre embedding $S : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$, where $N = (n + 1)(m + 1) - 1$ is given by $S([x_0 : \ldots : x_n], [y_0 : \ldots : y_m]) = [x_iy_j]$ where $0 \leq i \leq m$ and $0 \leq j \leq n$. By lemma 7.5.14 we have that the corresponding line bundle is given by $\mathcal{L} = S^*\mathcal{O}_{\mathbb{P}^N}(1)$. By the discussion in example 7.2.12 we have that since S is given by homogeneous degree 2 polynomials, $S^*\mathcal{O}_{\mathbb{P}^N}(1) = \mathcal{O}_{\mathbb{P}^m \times \mathbb{P}^n}(2 \cdot 1) = \mathcal{O}_{\mathbb{P}^m \times \mathbb{P}^n}(2)$.

Let's briefly reiterate that discussion here for the sake of completeness.

Directly computing the pullback $S^*\mathcal{O}_{\mathbb{P}^N}(1)$ gives quotients of the form

$$\frac{f(x_0y_0, \ldots, x_my_n)}{g(x_0y_0, \ldots, x_my_n)}$$

where $\deg f - \deg g = 1$. But this isn't a sheaf of $\mathcal{O}_{\mathbb{P}^m\times\mathbb{P}^n}$ modules (and therefore not a line bundle) since multiplying a section like $x_0y_0 \in S^*\mathcal{O}_{\mathbb{P}^N}$ by the section $\frac{x_0}{y_0} \in \mathcal{O}_{\mathbb{P}^m\times\mathbb{P}^n}$ gives x_0^2, which is not of the form described by the pullback. The actual definition of the pullback sheaf is given by

$$S^*\mathcal{O}_{\mathbb{P}^N} = S^{-1}\mathcal{O}_{\mathbb{P}^N} \otimes_{S^{-1}\mathcal{O}_{\mathbb{P}^N}} \mathcal{O}_{\mathbb{P}^m\times\mathbb{P}^n},$$

which is exactly the set of quotients $\frac{f}{g}$ with $\deg f - \deg g = 2$ since $\deg S = 2$.

By the same reasoning, since the degree d Veronese embedding $V_d : \mathbb{P}^n \to \mathbb{P}^N$, unsurprisingly, has degree d, we have that the corresponding line bundle is given by $\mathcal{O}_{\mathbb{P}^n}(d)$. □

7.8.10 Let X be a smooth projective curve, and let $P \in X$ be a point. Show that there is a rational function on X that is regular everywhere except at P.

Proof. The Riemann-Roch theorem states that the dimension of the space of global sections of a divisor D, $h^0(D)$ satisfies

$$h^0(D) - h^0(K_X - D) = \deg D + 1 - g(X),$$

where K_X is the divisor class associated to the canonical bundle of X, ω_X. Let Q be a point on X not equal to P and let D be the divisor $D = kQ - P$ where k is strictly larger than the genus of X. We then have that $\deg D = k - 1$ and the Riemann-Roch theorem gives

$$\begin{aligned} h^0(D) - h^0(K_X - D) &= (k-1) + 1 - g(X) \\ &= k - g(X) \\ &> 0. \end{aligned}$$

Since $h^0(D)$ and $h^0(K_X - D)$ are both nonnegative, we must have $h^0(D) > 0$, so the space of sections with divisor class D is nonempty. But sections in this divisor class are regular everywhere except at P, so we have shown that there are functions with the desired property. □