Continued Fractions 2

1. Let $c_1, c_2, \ldots, c_n$ be positive integers (except for maybe c_1). Define the sequences p_n, q_n by

$$\begin{array}{lll} p_{-1} = 0 & p_0 = 1 & p_n = c_n p_{n-1} + p_{n-2} \\ q_{-1} = 1 & q_0 = 0 & q_n = c_n q_{n-1} + q_{n-2}. \end{array}$$

   Prove that for any positive real number x,

$$[c_1; c_2, \ldots, c_n, x] = \frac{x p_{n-1} + p_{n-2}}{x q_{n-1} + q_{n-2}}.$$

2. Recall that the n-th convergent of $[c_1; c_2, \ldots]$ is defined to be $\frac{p_n}{q_n}$. Using the previous exercise, show that $[c_1; c_2, \ldots, c_n] = \frac{p_n}{q_n}$.

3. Find the continued fraction expansions of $\sqrt{2} - 1$ and $\frac{1}{\sqrt{3}}$. Compute the first three convergents.

4. Given that two irrational numbers have identical convergents $p_1/q_1, p_2/q_2, \ldots$, up to p_n/q_n, prove that their continued fraction expansions are identical up to c_n.

5. Evaluate the infinite continued fractions $[2; 1, 1, \ldots]$, $[2; 3, 1, 1, \ldots]$, and $[1; 3, 1, 2, 1, 2, \ldots]$. Assume for now that the notion of an infinite continued fraction makes sense.