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260B - Homework 2

Problem 1. Let H be a complex separable Hilbert space and let $T_1, T_2 \in \mathcal{L}(H,H)$ be Hilbert-Schmidt operators. Let $T = T_2T_1$. Show that

$$\text{tr } T = \sum (Te_j, e_j)$$

exists if e_j is an orthonormal basis for H, and prov that the sum is independent of the choice of basis.

Proof. That the trace exists for any fixed orthonormal basis e_j follows from Cauchy-Schwartz and Hölder's inequality.

$$\begin{aligned}
|\text{tr } T| &= \left|\sum (Te_j, e_j)\right| \\
&\leq \sum |(T_1e_j, T_2^*e_j)| \\
&\leq \sum \|T_1e_j\| \cdot \|T_2^*e_j\| \\
&\leq \left(\sum \|T_1e_j\|^2\right)^{1/2} \left(\sum \|T_2^*e_j\|^2\right)^{1/2} \\
&= \|T_1\|_{HS} \cdot \|T_2\|_{HS} < \infty.
\end{aligned}$$

On the last line we used the fact that $\|T_2^*\|_{HS} = \|T_2\|_{HS}$. Since the Hilbert-Schmidt norm is independent of the choice of basis, we have that the trace exists regardless of choice of basis.

It remains to show that the actual value of the trace is basis-independent. Let e_j and f_k be two orthonormal bases for H. By Parseval we have

$$\begin{aligned}
\sum (Te_j, e_j) &= \sum_j \left(\sum_k (Te_j, f_k)f_k, e_j\right) \\
&= \sum_j \sum_k (Te_j, f_k)(f_k, e_j)
\end{aligned}$$

and

$$\begin{aligned}
\sum (Tf_k, f_k) &= \sum_k \left(\sum_j (f_k, e_j)Te_j, f_k\right) \\
&= \sum_k \sum_j (f_k, e_j)(Te_j, f_k).
\end{aligned}$$

Now if we can reverse the order of summation we will be done. □