Braid Group Cryptography

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- We form a group by taking all possible words in S. The inverse of a word w is formed by writing the symbols in w in reverse order and replacing each x_j appearing in w by x_j^{-1}. The group operation is concatenation of words.

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Equivalently, G is the quotient of the free group on S by the normal closure of R. We say G is **finitely presented** if S and R are finite sets.

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- Any group with infinitely many generators, e.g. $\mathbb{Z}^{\oplus \mathbb{Z}}$
- There are finitely generated groups that are not finitely related, e.g. the wreath product of $\mathbb{Z}$ with itself.

The Word Problem

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Output **yes**.

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In 1955 Pyotr Novikov showed that there are finitely presented groups in which the word problem is **undecidable** - it is provably impossible to construct an algorithm that always outputs the correct answer.

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Discrete Logarithm Problem in H

input: *Elements g, h of H such that $h \in \langle g \rangle$*
output: *An integer k such that $g^k = h$*

The Braid Group

Definition

The braid group on n strands, B_n is defined by the presentation

$$\begin{aligned} B_n = \langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \mid \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j,\ |i-j| = 1; \\ \sigma_i\sigma_j = \sigma_j\sigma_i,\ |i-j| > 1\rangle. \end{aligned}$$

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There is, however, a more geometric understanding of the braid group.

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- Arrange two sets of n items in vertical columns on opposite sides of the page. Fasten one end of a string to each item on the left side of the page. To each item on the right side attach the other end of one string. This connection is a **braid**.

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- Two connections that can be made to look the same by tightening the strings are considered the same braid.
- Composing two braids consists of drawing them next to one another, gluing the points in the middle, and connecting the strands.

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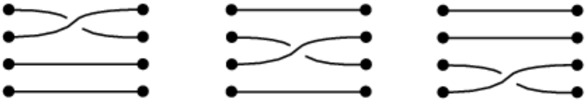

σ_1 σ_2 σ_3

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σ_1^{-1}

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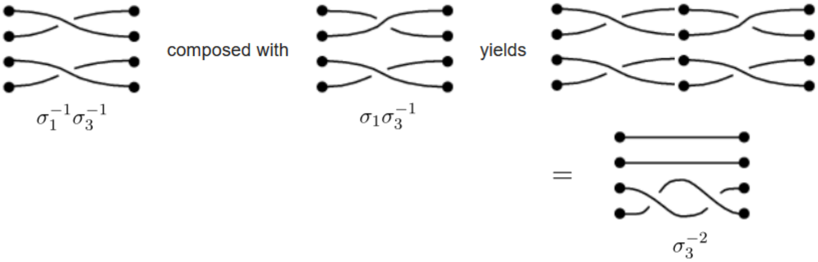

Braid Group Facts

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- The $n-1$ transpositions $(i, i+1)$ in the symmetric group S_n obey the braid relations and generate S_n. Consequently, there is a surjective homomorphism $\rho : B_n \to S_n$ that sends σ_i to $(i, i+1)$.

Special Braids

Definition (Permutation Braid)

To each permutation $\tau = b_1 b_2 \cdots b_n \in S_n$, associate an n-braid A by connecting the right i-th point to the left b_i-th point with positive crossings (the strand from i to b_i passes under the one from j to b_j if $i < j$). A braid of this form is called a **permutation braid** or **canonical factor**. The set of all such braids is denoted $\tilde{\Sigma}_n$.

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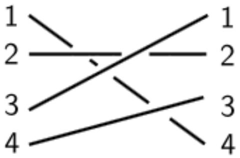

Figure: The braid $A \in \tilde{\Sigma}_4$ corresponding to $\pi = 4213 \in S_4$.

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Figure: The fundamental braid $\Delta_4 \in B_4$ corresponding to the permutation $\Omega_4 = 4321$.

Left-Canonical Form for Braids

Theorem (Elrifai and Morton '94)

For any $W \in B_n$ there is a unique representation called the **left-canonical form** *given by*

$$W = \Delta^u A_1 A_2 \cdots A_p, \quad u \in \mathbb{Z}, A_i \in \tilde{\Sigma}_n \setminus \{e, \Delta\},$$

where $A_i A_{i+1}$ is left-weighted for $1 \leq i \leq p-1$. We call p the **canonical length** *of W.*

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Note that the correspondence between a permutation $\pi \in S_n$ to its canonical factor $A \in B_n$ is a right inverse of the homomorphism $\rho : B_n \to S_n$, so the cardinality of $\tilde{\Sigma}_n$ is $n!$.

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Theorem (Ko et al. 2000 using Epstein et al. '92)

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2. *Let $U = \Delta^u A_1 \cdots A_p$ and $V = \Delta^v B_1 \cdots B_q$ be the left-canonical forms of two n-braids. Then we can compute the left canonical form of UV in time $O(pqn \log n)$.*

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These two theorems show that the word problem in B_n is efficiently solvable, that is, we can efficiently differentiate between any two given elements of B_n.

Diffie-Hellman with Braids: Anshel, Anshel, Goldfeld '99

1. Alice and Bob publicly agree on subgroups B_n, $A = \langle a_1, \ldots, a_k \rangle$ and $B = \langle b_1, \ldots, b_m \rangle$.

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5. Alice multiplies on the left by x^{-1}, obtaining $x^{-1}y^{-1}xy$. Bob multiplies on the left by y^{-1} and inverts, $(y^{-1}x^{-1}yx)^{-1} = x^{-1}y^{-1}xy$. Alice and Bob now share $[x, y]$.

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- This looks a lot like the classical Diffie-Hellman problem: find $g^{ab} \pmod{p}$ from g^a, g^b, g, and p. One way to solve the classical DHP is by solving the discrete logarithm problem in $\mathbb{Z}/p\mathbb{Z}$.

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- How about solving the analog of the discrete log problem in B_n: the (simultaneous) conjugacy search problem?

Complications with the Conjugacy Problem

- In the discrete log case, $g^a \equiv g^b \pmod{p}$ if and only if $a \cong b \pmod{p-1}$. In the conjugacy case, if $a_i^y = a_i^{y'}$ for all i then $y = c_a y$ for some c_a in the centralizer of A:

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$$\begin{aligned} y^{-1} a_i y = (y')^{-1} a_i y' &\iff y' y^{-1} a_i y (y')^{-1} = a_i \\ &\iff y' y^{-1} \in C_A. \end{aligned}$$

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- That exponents are only defined mod $p-1$ in the discrete log case doesn't matter much. On the other hand, suppose $y' = c_a y$ and $x' = c_b x$ for $c_a \in C_A$ and $c_b \in C_B$. Then

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- If $x' \in A$ and $y' \in B$, then $c_b \in A$ and $c_a \in B$, so we can move the c_a's and c_b's around to obtain $[x, y] = [x', y']$.

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- If $x' \in A$ and $y' \in B$, then $c_b \in A$ and $c_a \in B$, so we can move the c_a's and c_b's around to obtain $[x, y] = [x', y']$.
- So solving the simultaneous conjugacy problem doesn't seem to be enough for Eve to compute $[x, y]$. She needs to find conjugating elements that lie in the public subgroups A and B.

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The Membership Search Problem in G

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input: *elements* $x, a_1, \ldots, a_k \in G$

output: *an expression (if it exists) of* x *as a word in* $a_1, \ldots, a_k$.

Shpilrain and Ushakov ('06) point out that the corresponding decision problem is unsolvable in B_n for $n \geq 6$ since such groups contain subgroups isomorphic to $F_2 \times F_2$, where the membership decision problem is unsolvable due to Mihailova ('58).