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233 - Homework 1

1.4.1 Let $X_1, X_2 \subset \mathbb{A}^n$ be algebraic sets. Show that

(i) $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$.

(ii) $I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)}$.

Proof. (i) Suppose $f \in k[x_1, \ldots, x_n]$ vanishes on $X_1 \cup X_2$. Then it must vanish on X_1 and X_2, so $I(X_1 \cup X_2) \subseteq I(X_1) \cap I(X_2)$. Conversely, suppose that f vanishes on X_1 and X_2. Then it vanishes on their union as well, so $I(X_1 \cup X_2) \supseteq I(X_1) \cap I(X_2)$, and we're done.

(ii) Since X_1 and X_2 are algebraic sets, we have that $X_1 = Z(J_1)$ and $X_2 = Z(J_2)$ for some ideals $J_1, J_2 \subseteq k[x_1, \ldots, k_n]$. By Hilbert's Nullstellensatz we have that

$$\sqrt{I(X_1) + I(X_2)} = \sqrt{I(Z(J_1)) + I(Z(J_2))} = \sqrt{\sqrt{J_1} + \sqrt{J_2}}.$$

Now $Z(J_i) = Z(\sqrt{J_i})$, so we can take J_1 and J_2 to be radical, which gives

$$\sqrt{I(X_1) + I(X_2)} = \sqrt{J_1 + J_2}.$$

On the other hand, we have, again by Nullstellensatz

$$I(X_1) \cap I(X_2) = I(Z(J_1)) \cap I(Z(J_2)) = I(Z(J_1 + J_2)) = \sqrt{J_1 + J_2},$$

and we're done.

□

1.4.2 Let $X \subseteq \mathbb{A}^3$ be the union of the three coordinate axes. Determine generators for the ideal $I(X)$. Show that $I(X)$ cannot be generated by fewer than three elements, although X has codimension 2 in $\mathbb{A}^3$.

Proof. The z-axis is the set of points where $x = y = 0$. In order for a polynomial, p, to vanish here we need $p(0, 0, z) = 0$ for all z. This tells us that p can contain no constant term and that any monomial divisible by z must also be divisible by x or y. Thus, any monomial vanishing on the z axis must be divisible by x or y. The same argument shows that any monomial vanishing on the x-axis must be divisible by y or z and any monomial vanishing on the y axis must be divisible by x or z. By problem 1.4.1, we're interested in the ideal $(x, y) \cap (y, z) \cap (x, z)$.
Going piece by piece we have

$$I = (x, y) \cap (y, z) \cap (x, z) = (x, y) \cap (xy, z) = (xy, xz, yz).$$

Now we show that this ideal cannot be generated by fewer than three elements of $k[x, y, z]$. It's clearly not generated by a single element because xy, xz, and yz don't have a common factor. Suppose that $I = (f, g)$. □