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233A - Final

1.4.6 Let Y be a subspace of a topological space X. Show that Y is irreducible if and only if the closure of Y in X is irreducible.

Proof. First suppose that Y is irreducible. If $\overline{Y}$ (the closure of Y in X) were reducible, then we could write $\overline{Y} = \tilde{F_1} \cup \tilde{F_2}$, where $\tilde{F_1}$ and $\tilde{F_2}$ are nonempty (relatively) closed subsets of $\overline{Y}$. In particular, this means that we can write $\overline{Y} \subseteq F_1 \cup F_2$, where F_1 and F_2 are closed in X and Y is not entirely contained in either F_1 or F_2. If Y is contained in say F_1, then $\overline{Y} \subseteq \overline{F}_1 = F_1$, which contradicts the reducibility of $\overline{Y}$, so Y isn't contained in F_1. By symmetry, Y is not contained in F_2 either. But we have

$$Y \subseteq \overline{Y} \subseteq F_1 \cup F_2.$$

This shows that Y is contained in the union of closed (in X) subsets, but is contained in neither set individually, contradicting the irreducibility of Y. We conclude that $\overline{Y}$ is also irreducible.

Conversely, suppose that $\overline{Y}$ is irreducible but Y is reducible. Then $Y \subseteq F_1 \cup F_2$, where F_1 and F_2 are closed in X and Y is contained in neither F_1 nor F_2. When we take the closure of both sides of this inclusion we get

$$\overline{Y} \subseteq \overline{F_1 \cup F_2} = \overline{F_1} \cup \overline{F_2} = F_1 \cup F_2.$$

Since $\overline{Y}$ is irreducible, it must be contained in F_1 or F_2, say F_1. But then $Y \subseteq \overline{Y} \subseteq F_1$, contradicting our assumption about Y not being contained in F_1. We conclude that Y is irreducible. □

2.6.13 Let X and Y be prevarieties with affine open covers $\{U_i\}$ and $\{V_j\}$, respectively. Construct the product prevariety $X \times Y$ by gluing the affine varieties $U_i \times V_j$ together. Moreover, show that there are projection morphisms $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ satisfying the usual universal property for products.

Proof. The affine varieties $U_i \times V_j$ (as the product of two affine varieties is an affine variety) form a finite affine open cover for $X \times Y$ as a topological space. The idea now is to glue the sets $U_i \times V_j$ and $U_k \times V_l$ along the identity morphism on the intersection $(U_i \cap U_k) \times (V_j \cap V_l)$. Let $f_{ijkl} : U_i \times V_j \to U_k \times V_l$ be the identity morphism on the intersection. Then we clearly have that $f_{ijkl} = (f_{klij})^{-1}$ and the cocycle condition holds on triple intersections.

Let $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ be the usual set-theoretic projection maps. As maps of topological spaces they are certainly continuous. We just have to show that they are morphisms. □

3.5.5 Let V be the vector space over k of homogeneous degree-2 polynomials in three variables x_0, x_1, x_2 and let $\mathbb{P}(V) \cong \mathbb{P}^5$ be its projectivization.

(i) Show that the space of conics in $\mathbb{P}^2$ can be identified with an open subset U of $\mathbb{P}^5$. What geometric objects can be associated to the points in $\mathbb{P}^5 \setminus U$?

(ii) Show that it is a linear condition in $\mathbb{P}^5$ for the conics to pass through a given point in $\mathbb{P}^2$. If $P \in \mathbb{P}^2$ is a point, show that there is a linear subspace $L \subseteq \mathbb{P}^5$ such that the conics passing through P are exactly those in $U \cap L$. What happens in $\mathbb{P}^5 \setminus U$?

(iii) Prove that there is a unique conic through any five points in $\mathbb{P}^2$, as long as no three of them lie on a line. What happens if three of them do lie on a line?

4.6.10 Let $X \subseteq \mathbb{A}^n$ be an affine variety, and let $Y_1, Y_2 \subsetneq X$ be irreducible, closed subsets, no-one contained in the other. Let $\tilde{X}$ be the blow-up of X at the (possibly non-radical) ideal $I(Y_1) + I(Y_2)$. Then the strict transforms of Y_1 and Y_2 on $\tilde{X}$ are disjoint.