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260A - Homework 4

Problem 1. Let E and F be two Banach spaces, and let $T \in \mathcal{L}(E, F)$. Prove that $\mathrm{Im}(T)$ is closed if and only if there exists a constant $C > 0$ such that

$$\mathrm{dist}(x, \ker T) \leq C \cdot \|Tx\|, \quad \forall x \in E.$$

Proof. First suppose that the given inequality holds for some $C > 0$. Let Tx_n be a convergent sequence in the image of T. Then the sequence of $x_n + \ker T$'s converges in the quotient $E/\ker T$ by the given inequality. Since T is continuous, $\ker T$ is closed and the quotient $E/\ker T$ is complete. Thus, $x_n + \ker T$ converges to some $x + \ker T$. By continuity, Tx_n then converges to Tx, which is in the image of T. Thus, the image of T is closed.

Conversely, suppose that $\mathrm{Im}(T)$ is closed. Then the image is a Banach space. By the first isomorphism theorem, T induces an isomorphism $\tilde{T} : E/\ker T \to \mathrm{Im}T$. By the open mapping theorem, $\tilde{T}$ is a homeomorphism, and the statement that $\tilde{T}$ is continuous is equivalent to the desired inequality. □

Problem 2. Prove that if H is a Hilbert space and B is a Banach space, then the space $\mathcal{L}_c(B, H)$ of compact operators $B \to H$ is the closure of the set of operators in $\mathcal{L}(B, H)$ which are of finite rank.

Proof. Suppose T is a compact operator $B \to H$. By the compactness of T, for any $n > 0$ we can find a finite covering of $\overline{T[B(0,1)]}$, the closure of the image of the unit ball in B, by balls of radius $\frac{1}{n}$. Say $\overline{T[B(0,1)]} \subseteq \cup_{j=1}^{M_n} B(y_j, \frac{1}{n})$ for some finite $M_n > 0$. Let P_n be the projection onto the vectors $y_1, \ldots, y_{M_n}$. Then $P_n T$ is clearly of finite rank as P_n has finite rank.

Now given any $x \in B(0,1) \subseteq B$, we can find a y_j with $\|Tx - y_j\|_H \leq \frac{1}{n}$. We use this to show that the P_nT's approximate T. We have

$$\begin{aligned}
\|P_nTx - Tx\|_H &\leq \|P_nTx - y_j\|_H + \|y_j - Tx\|_H \\
&\leq \|Tx - y_j\|_H + \|y_j - Tx\|_H \\
&\leq \frac{2}{n}.
\end{aligned}$$

The fact that $\|P_nTx - y_j\|_H \leq \|Tx - y_j\|_H$ follows from the fact that P_n projects onto the space spanned by the y_k's. Sending $n \to \infty$ shows that the finite rank P_nT's approximate T, so the compact operators $B \to H$ are in the closure of of the set of finite rank operators in $\mathcal{L}(B, H)$.

Conversely, suppose that T_n is a sequence of finite rank operators in $\mathcal{L}(B, H)$ that converges to $T \in \mathcal{L}(B, H)$. Choose N large so that $\|T_n - T\| < \epsilon$ for all $n > N$. Since finite-rank operators are compact, for any n we can cover $\overline{T_n[B(0,1)]}$ by finitely many ϵ-balls. Since $\|T_nx - Tx\|_H < \epsilon$ for any $x \in B(0,1)$, we have that the 2ϵ-balls with the same centers cover $\overline{T[B(0,1)]}$. Sine ϵ was arbitrary, this shows that the closed image of the unit ball under T is compact, so T is a compact operator. $\square$

Problem 3. Let B be a complex Banach space, $B \neq \{0\}$, and let $T \in \mathcal{L}(B, B)$. Prove the following.

(i) There exists a non-empty compact set $\mathrm{Spec}(T) \subseteq \mathbb{C}$, called the spectrum of T, such that the resolvent $R(z) = (T - zI)^{-1} \in \mathcal{L}(B, B)$ exists if and only if $z \notin \mathrm{Spec}(T)$.

Proof. $\square$

Problem 4. Let $E = L^p[0,1]$ with $1 \leq p < \infty$. Given $u \in E$, set

$$Tu(x) = \int_0^x u(t)\ dt.$$

(i) Prove that $T : E \to E$ is compact.

Proof. Fix $u \in L^p[0,1]$ and suppose that $x_n \to x$ in $[0,1]$. Since $|\chi_{[0,x_n]}(t)u(t)| \leq |u(t)| \in L^p[0,1]$ for all n, the dominated convergence theorem tells us that

$$Tu(x_n) = \int_0^1 \chi_{[0,x_n]}(t)u(t)\ dt \to \int_0^1 \chi_{[0,x]}(t)u(t)\ dt = Tu(x).$$

That is, T maps E into $C[0,1]$. Suppose we're given a bounded sequence $u_n \in L^p[0,1]$, i.e. $\|u_n\|_{L^p} \leq M < \infty$. We then have

$$\begin{aligned}|Tu_n(x)| &= \left|\int_0^1 \chi_{[0,x]}(t)u_n(t)\ dt\right| \\ &\leq \int_0^1 \chi_{[0,x]}(t)|u_n(t)|\ dt \\ &\leq x^{1/q} \cdot \|u\|_{L^p} \\ &\leq M,\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ (the inequality still holds if $p = 1$). Thus, the sequence of continuous functions Tu_n is uniformly bounded. Now fix $\epsilon > 0$. For any $x < y \in [0,1]$ and $p > 1$ we have

$$\begin{aligned}|Tu_n(x) - Tu_n(y)| &= \left|\int_0^1 \chi_{[x,y]}(t)u_n(t)\ dt\right| \\ &\leq \int_0^1 \chi_{[x,y]}(t)|u_n(t)|\ dt \\ &\leq |y-x|^{1/q} \cdot \|u_n\|_{L^p} \\ &\leq |y-x|^{1/q} \cdot M.\end{aligned}$$

We can choose $|y-x|$ small enough so that the above quantity is bounded by ϵ, which shows that the sequence of continuous functions Tu_n is equicontinuous (I'm not sure how get this to work for $p=1$). By the Arzela-Ascoli theorem we have that Tu_n has a uniformly convergent subsequence. Since uniform convergence implies L^p convergence, we have that Tu_n has a convergent subsequence in E, so T is a compact operator. $\square$

(ii) Compute the eigenvalues of T and the spectrum of T.

Solution. In our discussion of part (i) we showed that T maps $L^p[0,1]$ into $C[0,1]$. In particular, if $Tu = \lambda u$, then u must be continuous. But the fundamental theorem of calculus tells us that the integral of a continuous function is differentiable, so u is actually differentiable. Differentiating both sides of the eigenvalue equation gives $u = \lambda u'$. If $\lambda \neq 0$, then the solutions to this differential equation are of the form $u(x) = Ce^{x/\lambda}$, $C \in \mathbb{C}$. However, we must also have

$$\lambda u(0) = Tu(0) = \int_0^0 u(t)\ dt = 0,$$

so $u(0) = Ce^0 = 0$. But then C must be zero, which would force u to be identically zero. We conclude that there are no eigenvectors for $\lambda \neq 0$. If $\lambda = 0$ then any L^p function with vanishing integral is an eigenfunction with eigenvalue zero. $\square$

Problem 5. Let X, Y, and Z be three Banach spaces with norms $\|\cdot\|_X$, $\|\cdot\|_Y$, and $\|\cdot\|_Z$. Assume that $X \subseteq Y$ with compact injection and that $Y \subseteq Z$ with continuous injection. Prove that for any $\epsilon > 0$ there exists $C_\epsilon \geq 0$ such that

$$\|u\|_Y \leq \epsilon \|u\|_X + C_\epsilon \|u\|_Z$$

for all $u \in X$.

Proof. Suppose the proposition were false: that for some ϵ and for every $C \geq 0$ there exists a u_C such that

$$\|u_C\|_Y > \epsilon \|u_C\|_X + C\|u_C\|_Z$$

for all $x \in X$. Set $C = n$ and let u_n be a sequence in X such that the above equality holds, i.e.

$$\|u_n\|_Y > \epsilon \|u_n\|_X + n\|u_n\|_Z. \tag{1}$$

We can assume without loss of generality that the sequence u_n has norm 1 in X, since replacing u_n with $\frac{u_n}{\|u_n\|_X}$ gives the same inequality after multiplying through by $\|u_n\|_X$. By the compactness of the injection of X into Y, we have that u_n has a convergent subsequence in Y. Without loss of generality, assume then that u_n converges in Y. Rearranging (1) gives

$$n\|u_n\|_Z < \|u_n\|_Y - \epsilon\|u_n\|_X \leq \|u_n\|_Y$$

$$\iff \|u_n\|_Z < \frac{1}{n}\|u_n\|_Y.$$

Since u_n converges in Y, the right-hand side of the above inequality must go to zero. Since Y continuously embeds into Z and u_n converges in Y, we must have that u_n converges to zero in both Y and Z. But then the left-hand side of (1) will tend to 0 and the right-hand side will tend to ϵ: a contradiction. We conclude that the proposition is true.

In class we showed (using the Arzela-Ascoli theorem) that $C^1([0,1])$ compactly embeds into $C([0,1])$. We also have that $C([0,1])$ continuously embeds into $L^1([0,1])$ by $\int_0^1 |f| \, dx \leq \|f\|_\infty$. By the proposition we then have that for all $\epsilon > 0$ there is some C_ϵ with

$$\max_{x \in [0,1]} |f(x)| \leq \epsilon \cdot \max_{x \in [0,1]} |f'(x)| + C_\epsilon \|f\|_{L^1}$$

for all $f \in C^1([0,1])$. $\square$