Spring 2016

1. Assume $f \in L^1[0,1]$. Compute
$$\lim_{k\to\infty} \int_{[0,1]} |f|^{1/k}\ dx.$$

Solution. Let's split this integral into three regions.
$$\int_{[0,1]} |f|^{1/k}\ dx = \int_{f=0} |f|^{1/k}\ dx + \int_{0<|f|\le 1} |f|^{1/k}\ dx + \int_{|f|>1} |f|^{1/k}\ dx.$$
The integral over the first region is clearly zero for all k. On the second region we have that $|f|^{1/k} \le 1$ for all k. Since the interval $[0,1]$ has finite measure, we have that the constant function 1 is in $L^1(\{x : 0 < |f| \le 1\})$, so the dominated convergence theorem says that the integral over the second region goes to $m(\{0 < |f| \le 1\})$. Similarly, on the third region we have that $|f|^{1/k} \le |f|$, which is in L^1, so the dominated convergence theorem says that the third integral goes to $m(\{|f| > 1\})$. Combining these, we have that
$$\lim_{k\to\infty} \int_{[0,1]} |f|^{1/k}\ dx = m(\{|f| > 0\}).$$
□

2. Let $\{f_n\}$ be a sequence of measurable functions on $[0,1]$ and $0 \le f_n \le 1$ a.e. Assume that
$$\lim_{n\to\infty} \int_{[0,1]} f_n g\ dx = \int_{[0,1]} fg\ dx$$
for some $f \in L^1[0,1]$ and any $g \in C[0,1]$. Prove that $0 \le f \le 1$ a.e.

Solution. Since $f \in L^1[0,1]$, by the Lebesgue differentiation theorem we have that
$$\frac{1}{m(E)} \int_E f(t)\ dt \to f(x) \tag{1}$$
as E shrinks to x for almost all x. Furthermore, since $0 \le f_n \le 1$ we also have that
$$\frac{1}{m(E)} \int_E f_n(t)\ dt \to f_n(x) \in [0,1]$$
as E shrink to x for almost all x. Intuitively, we'd like to replace the integral of f in (1) with a limit of integrals of f_n.

We claim that the function g in the given hypothesis can be replaced with the indicator function of an interval χ_I. To see this, let g_m be a sequence of continuous functions with $g_m \to \chi_I$ in L^1 and $0 \le \chi_I \le 1$. By extracting a subsequence we can assume that $g_m \to \chi_I$ a.e. as well. We then have
$$\int_0^1 |f_n\chi_I - f\chi_i| \le \int_0^1 |f_n\chi_I - f_n g_m| + \int_0^1 |f_n g_m - f g_m| + \int_0^1 |f g_m - f\chi_i|.$$

Since $\|f_n\|_{L^\infty} \leq 1$, we have that the first integral on the RHS can be made small uniformly in n by picking m large. The second integral goes to zero as $n \to \infty$ by hypothesis since g_m is continuous. The third integral can be made small for m large by dominated convergence since $|fg_m| \leq |f| \in L^1$.

For almost all x, if I_k is a sequence of intervals shrinking to x then

$$\begin{aligned}\frac{1}{m(I_k)} \int_{I_k} f \, dx &= \frac{1}{m(I_k)} \int f \chi_{I_k} \, dx \\ &= \lim_{n\to\infty} \frac{1}{m(I_k)} \int f_n \chi_{I_k} \, dx.\end{aligned}$$

Since $0 \leq f_n \leq 1$, the RHS is in $[0,1]$ for almost all x. By the Lebesgue differentiation theorem we then have that $0 \leq f \leq 1$ a.e. $\square$

3. Let $f, g \in L^2(\mathbb{R}, \mathcal{M}_L, \mu_L)$. Show that $f * g$ is a continuous function on $\mathbb{R}$ vanishing at infinity, that is, $f * g \in C(R)$ and $\lim_{|x|\to\infty}(f * g)(x) = 0$.

Proof. For any h we have by Hölder's inequality

$$|(f * g)(x+h) - (f * g)(x)| = \left| \int f(t)[g(x+h-t) - g(x-t)] \, dt \right| \tag{2}$$

$$\leq \|f\|_{L^2} \cdot \|g_h - g\|_{L^2}, \tag{3}$$

where $F_h(x) = F(x+h)$ for any function F. Now for any $\epsilon > 0$ we can find $\varphi \in C_0(\mathbb{R})$ with $\|g - \varphi\|_{L^2} = \|g_h - \varphi_h\|_{L^2} < \epsilon$. By the triangle inequality we then have

$$\begin{aligned}\|g_h - g\|_{L^2} &\leq \|g_h - \varphi_h\|_{L^2} + \|\varphi_h - \varphi\|_{L^2} + \|\varphi - g\|_{L^2} \\ &< \|\varphi_h - \varphi\|_{L^2} + 2\epsilon.\end{aligned}$$

Suppose that φ has support contained in the compact set K. If we pick h small enough then we can guarantee that $\varphi_h - \varphi$ is supported on a set with measure at most $2 \cdot m(K)$. Now since φ is continuous with compact support, it is uniformly continuous, so we can choose h small enough that $|\varphi_h(x) - \varphi(x)| = |\varphi(x+h) - \varphi(x)| < \epsilon$ for all x in the support of $\varphi_h - \varphi$. For such h we have

$$\|\varphi_h - \varphi\|_{L^2} \leq \epsilon \cdot (2 \cdot m(K))^{1/2},$$

so (2) can be made arbitrarily small, which shows that $f * g$ is continuous.

First we claim that if φ and ψ are continuous with compact support then $\varphi * \psi$ vanishes at infinity. By definition we have that

$$(\varphi * \psi)(x) = \int \varphi(t)\psi(x-t) \, dt.$$

The product $\varphi(t)\psi(x-t)$ is nonzero only if t is in the support of φ and $x-t$ is in the support of φ. If pick x large enough then supports of $t \mapsto \varphi(t)$ and $t \mapsto \psi(x-t)$ are disjoint, so this integral is zero.

Let f_n and g_n be sequences in $C_0(\mathbb{R})$ converging in L^2 to f and g, respectively. We then have

$$\begin{aligned}|(f * g)(x) - (f_n * g_n)(x)| &\leq |(f * g)(x) - (f_n * g)(x)| + |(f_n * g)(x) - (f_n * g_n)(x)| \\ &\leq \|g\|_{L^2} \cdot \|f - f_n\|_{L^2} + \|f_n\|_{L^2} \cdot \|g - g_n\|_{L^2}.\end{aligned}$$

Since $f_n \to f$ and $g_n \to g$ in L^2, we have that $f_n * g_n$ converges uniformly to $f * g$. Since $f_n * g_n$ vanishes at infinity, we must then have that $f * g$ vanishes at infinity. $\square$