The LLL Algorithm

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Two lattices

- Recall that the **lattice**, L, generated by the linearly independent vectors $x_1, x_2, \ldots, x_n \in \mathbb{R}^n$ is the $\mathbb{Z}$-span of these vectors:

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- Consider the lattices, L and M, generated by the rows of the matrices X and Y, respectively.

$$X = \begin{bmatrix} -168 & 602 & 58 \\ 157 & -564 & -57 \\ 594 & -2134 & -219 \end{bmatrix}, \quad Y = \begin{bmatrix} -6 & 6 & -4 \\ 9 & 4 & 1 \\ -1 & 8 & 6 \end{bmatrix}.$$

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$$\begin{bmatrix} -168 \\ 602 \\ 58 \end{bmatrix}^T = 14 \begin{bmatrix} 4 \\ 2 \\ -9 \end{bmatrix}^T + 50 \begin{bmatrix} -1 \\ 8 \\ 6 \end{bmatrix}^T - 29 \begin{bmatrix} 6 \\ -6 \\ 4 \end{bmatrix}^T,$$

$$\begin{bmatrix} 157 \\ -564 \\ -57 \end{bmatrix}^T = -13 \begin{bmatrix} 4 \\ 2 \\ -9 \end{bmatrix}^T - 47 \begin{bmatrix} -1 \\ 8 \\ 6 \end{bmatrix}^T + 26 \begin{bmatrix} 6 \\ -6 \\ 4 \end{bmatrix}^T,$$

$$\begin{bmatrix} 594 \\ -2134 \\ -219 \end{bmatrix} = -49 \begin{bmatrix} 4 \\ 2 \\ -9 \end{bmatrix}^T - 178 \begin{bmatrix} -1 \\ 8 \\ 6 \end{bmatrix} + 102 \begin{bmatrix} 6 \\ -6 \\ 4 \end{bmatrix}^T.$$

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- In particular, we have the matrix equation

$$UY = X,$$

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- $\det U = -1$, so U^{-1} is an integer matrix as well. This gives us another matrix equation, $Y = U^{-1}X$.
- Since the entries of U^{-1} are integers, this equation expresses the rows of Y as integer linear combinations of the rows of X, so $M \subseteq L$.

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- Even though the rows of X and Y generate the same lattice, something about the Y-basis "feels" nicer.

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- Two qualities that make a basis desirable are:
  - Length: how long are the basis vectors?
  - Orthogonality: are the basis vectors nearly orthogonal to each other?

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- If $x = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$, $c_i \in \mathbb{Z}$, is in the lattice L generated by $v_1, \ldots, v_n$ then

$$|x|^2 = c_1^2 |v_1|^2 + c_2^2 |v_2|^2 + \cdots + c_n^2 |v_n|^2.$$

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- This completely solves the shortest vector problem (SVP) since

$$\arg\min_{x \in L} |x| = \arg\min_{x \in \{\pm v_1, \pm v_2, \ldots, \pm v_n\}} |x|.$$

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- If $y = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$, $c_i \in \mathbb{Z}$, is any vector in L then by the orthogonality of the v_i we have

$$|x-y|^2 = (t_1-c_1)^2|v_1|^2+(t_2-c_2)^2|v_2|^2+\cdots+(t_n-c_n)^2|v_n|^2.$$

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- If we take c_i to be the closest integer to t_i then we solve the closest vector problem (CVP).

Gram-Schmidt

Definition

Let $x_1, \ldots, x_m \in \mathbb{R}^n$ be a basis for a nonzero subspace, H. The **Gram-Schmidt process** produces an orthogonal basis for H:

$$
\begin{aligned}
x_1^* &= x_1 \\
x_2^* &= x_2 - \frac{x_2 \cdot x_1^*}{x_1^* \cdot x_1^*} x_1^* \\
x_3^* &= x_3 - \frac{x_3 \cdot x_1^*}{x_1^* \cdot x_1^*} x_1^* - \frac{x_3 \cdot x_2^*}{x_2^* \cdot x_2^*} x_2^* \\
&\vdots \\
x_m^* &= x_m - \frac{x_m \cdot x_1^*}{x_1^* \cdot x_1^*} x_1^* - \cdots - \frac{x_m \cdot x_{m-1}^*}{x_{m-1}^* \cdot x_{m-1}^*} x_{m-1}^*.
\end{aligned}
$$

We call $\{x_1^*, \ldots, x_m^*\}$ the **Gram-Schmidt orthogonalization (GSO)** of $\{x_1, \ldots, x_m\}$.