## Homework 1 (Due Friday, February 22, 2019)

**Problem 1.** Let us define the *Sobolev space* $H^s(\mathbb{R}^n)$, $s \geq 0$, to be the set of all functions $u \in L^2(\mathbb{R}^n)$ such that

$$\|u\|^2_{H^s} = \frac{1}{(2\pi)^n} \int |\widehat{u}(\xi)|^2 (1 + |\xi|^2)^s \, d\xi < \infty. \tag{0.1}$$

- Show that $H^s(\mathbb{R}^n)$ is a Hilbert space when equipped with the scalar product

$$(u, v)_{H^s} = \frac{1}{(2\pi)^n} \int \widehat{u}(\xi)\overline{\widehat{v}(\xi)}(1 + |\xi|^2)^s \, d\xi.$$

- When $K \subset \mathbb{R}^n$ is compact, we put

$$H^s(K) = \{u \in H^s(\mathbb{R}^n);\ \mathrm{supp}\ (u) \subset K\}.$$

  This is a closed linear subspace of $H^s(\mathbb{R}^n)$ and hence also a Hilbert space. Show that the inclusion map $H^s(K) \to H^t(\mathbb{R}^n)$ is compact, if $s > t \geq 0$. *Hint.* Let $u_j \in H^s(K)$ be a bounded sequence. Show first that the sequence of smooth functions $\widehat{u}_j \in C^\infty(\mathbb{R}^n)$ is uniformly bounded and equicontinuous on each compact subset of $\mathbb{R}^n$.

**Problem 2.** Let $B_1$ and $B_2$ be Banach spaces and let $T \in \mathcal{L}(B_1, B_2)$. Prove that if $T$ is compact, then $\|Tu_n\|_{B_2} \to 0$ for every sequence $u_n \in B_1$ such that $u_n \to 0$ in the weak topology $\sigma(B_1, B_1^*)$. Prove the converse when $B_1$ is reflexive and $B_1^*$ is separable.

**Problem 3.** Let $H$ be a complex separable Hilbert space. An operator $T \in \mathcal{L}(H, H)$ is called a *Hilbert-Schmidt* operator if for some orthonormal basis $\{e_j\}$ of $H$, we have

$$\sum \|Te_j\|^2 < \infty. \tag{0.2}$$

- Show that if $T$ satisfies (0.2) for one orthonormal basis, then it satisfies (0.2) for every orthonormal basis, and the sum in (0.2) is independent of the choice of the basis. The square root $\|T\|_{\mathrm{HS}}$ of this sum is called the Hilbert-Schmidt norm of $T$.
- Show that the operator norm of $T$ does not exceed the Hilbert-Schmidt norm.
- Show that if $T$ is of Hilbert-Schmidt class, then so is $T^*$ and $\|T\|_{\mathrm{HS}} = \|T^*\|_{\mathrm{HS}}$.
- Show that every Hilbert-Schmidt operator is compact.
- Show that if $T$ is a Hilbert-Schmidt operator, and $S \in \mathcal{L}(H, H)$ then $ST$ is Hilbert-Schmidt and

$$\|ST\|_{\mathrm{HS}} \leq \|S\| \, \|T\|_{\mathrm{HS}}.$$

- Let $K \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Prove that if $f \in L^2(\mathbb{R}^n)$, then
$$\mathcal{K}f(x) = \int K(x,y) f(y)\, dy$$
exists for almost every $x$, and that $\mathcal{K}$ is a Hilbert-Schmidt operator from $L^2(\mathbb{R}^n)$ to itself, with the Hilbert-Schmidt norm equal to the norm of $K$ in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Prove that every Hilbert-Schmidt operator on $L^2(\mathbb{R}^n)$ is of this form.

**Problem 4**. Let $K$ be a compact self-adjoint operator on a Hilbert space $H$, and assume that $K \geq 0$. Let $\lambda_1 \geq \lambda_2 \geq \ldots$ be the sequence of non-zero eigenvalues of $K$, repeated according to their multiplicity and arranged in a decreasing order. Prove the Courant-Fischer minimax formula,
$$\lambda_k = \min_{codim V = k-1} \max_{u \in V,\, \|u\| \leq 1} (Ku, u),$$
where $V$ varies over the set of linear subspaces of $H$ of codimension $k-1$.

**Problem 5**. Let $f \in C(\mathbb{R}/2\pi\mathbb{Z})$ be such that $f(\theta_0) = 0$ for some $\theta_0 \in \mathbb{R}/2\pi\mathbb{Z}$. Show that the associated Toeplitz operator, $\mathrm{Top}(f)$, is *n*ot Fredholm on the Hardy space $H^2 \subset L^2(\mathbb{R}/2\pi\mathbb{Z})$.

Hint. Assume first that $f$ vanishes on a non-empty open set.