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260A - Homework 2

Problem 1. Let $(e_n)_{n=1}^\infty$ be an orthonormal basis in the Hilbert space H. Let $T : H \to H$ be a linear continuous map such that

$$\sum_{n=1}^\infty \|Te_n\|^2$$

converges. Show that there is a sequence $(T_n)_{n=1}^\infty$ of linear continuous maps $H \to H$ such that $T_n(H)$ has a finite dimension and $\|T_n - T\| \to 0$ as $n \to \infty$.

Proof. Consider the projection T_m defined by

$$T_m(x) = \langle x, e_1\rangle Te_1 + \cdots + \langle x, e_m\rangle Te_m.$$

This function is continuous by an argument similar to the one used on Homework 1, where we showed that every finite dimensional subspace of a normed vector space admits a continuous projection (first we define projections onto the individual components on the space spanned by $e_1, \ldots, e_m$ and then extend these through Hahn-Banach).

The image of H under T_m has dimension at most m as the e_j's are linearly independent. Furthermore, we have by Cauchy-Schwarz

$$\begin{aligned} |T_n x - Tx|^2 &= \left| \sum_{j=n+1}^\infty \langle x, e_j\rangle Te_j \right|^2 \\ &\leq \|x\|^2 \cdot \sum_{j=n+1}^\infty \|Te_j\|^2. \end{aligned}$$

Since the sum $\sum_{j=1}^\infty \|Te_j\|^2$ converges, the tail (the last line in the above inequality) must go to zero as $n \to \infty$. We then have that $\|T_n - T\| \to 0$ as desired. $\square$

Problem 3. Let H be a separable infinite dimensional Hilbert space, and suppose that $e_1, e_2, \ldots$ is an orthonormal system in H. Let $f_1, \ldots$ be another orthonormal system which is complete.

(i) Prove that if $\sum_{n=1}^\infty \|e_n - f_n\|^2 < 1$ then $\{e_n\}$ is also a complete orthonormal system.

(ii) Suppose only that $\sum_{n=1}^\infty \|e_n - f_n\|^2 < \infty$. Prove that it is still true that $\{e_n\}$ is a complete orthonormal system.

Proof. (i) In order to show that the e_j's form a complete system, we'll show that if $\langle x, e_j\rangle = 0$ for all

j then $x = 0$. If this is the case then we have by Cauchy-Schwarz

$$
\begin{aligned}
\|x\|^2 &= \left\| \sum_{j=1}^{\infty} \langle x, f_j \rangle f_j \right\|^2 \\
&= \left\| \sum_{j=1}^{\infty} \langle x, f_j - e_j \rangle f_j + \langle x, e_j \rangle f_j \right\|^2 \\
&\leq \|x\|^2 \cdot \sum_{j=1}^{\infty} \|f_j - e_j\|^2 \\
&< \|x\|^2.
\end{aligned}
$$

This is a contradiction unless $x = 0$, so we conclude that the e_j's are complete.

(ii) (This one was tricky. This solution is in Halmos's book on Hilbert space problems). If the given sum is to converge, then we can choose N large enough so that $\sum_{j=N}^{\infty} \|e_j - f_j\|^2 < 1$. Now define the operator $T : H \to H$ by

$$
Tf_j = \begin{cases} f_j & \text{if } j < N \\ e_j & \text{if } j \geq N \end{cases}.
$$

We have the following bound for any $x \in H$

$$
\begin{aligned}
\|x - Tx\|^2 &= \left\| \sum_{j=1}^{\infty} \langle x, f_j \rangle f_j - Tx \right\|^2 \\
&= \left\| \sum_{j=N}^{\infty} \langle x, f_j \rangle (f_j - e_j) \right\|^2 \\
&\leq \|x\|^2 \cdot \sum_{j=N}^{\infty} \|f_j - e_j\|^2 \\
&\leq \|x\|^2.
\end{aligned}
$$

In particular, we have that the operator $I - T$ has norm less than 1. We claim this means that T is invertible. In general, if an operator A satisfies $\|A\| < 1$, then $I - A$ is invertible where it is defined. The bound on A tells us that $\sum \|A\|^n$ is finite, so the operator $\sum A^n$ exists. Furthermore we have

$$
(1 - A) \cdot \sum_{n=0}^{\infty} A^n = \sum_{n=0}^{\infty} A^n - \sum_{n=1}^{\infty} A^n = I.
$$

Returning to the problem at hand, we have that T is invertible on the span of the f_j's. Since this span is dense, by continuity we have that T's inverse extends to an operator on all of H. By invertibility we have that the Tf_j's span all of H. But $\{Tf_j\} = \{f_1, \ldots, f_{N-1}, e_N, e_{N+1}, \ldots\}$.

In particular, we have that the orthogonal complement to the span of $\{e_N, \ldots, \}$ has dimension $N-1$. But $e_1, \ldots, e_{N-1}$ are $N-1$ linearly independent vectors outside of the span of $\{e_N, \ldots\}$. We conclude that the e_j's span H.

□

Problem 4. Let $T : B_1 \to B_2$ be a compact operator where B_1 and B_2 are Banach spaces. Show that if T is compact then ImT has a dense countable subset.

Proof. Write $B_1 = \cup_{n\in\mathbb{N}} B(0, n)$. That is, B_1 is a union of balls centered at the origin with natural radius. From this we have that $\mathrm{Im}T = \cup_{n\in\mathbb{N}} T[B(0, n)]$. If we can show that $T[B(0, n)]$ is separable for each n then we're done since the countable union of separable spaces is separable.

Our plan is to show that $T[B(0, n)]$ is precompact for each n. Then we'll show that precompact spaces are separable and we'll be finished. That $T[B(0, n)]$ is precompact follows from the compactness of T. If we let $y_k = Tx_k$ be a sequence in $T[B(0, n)]$, then the compactness of T says that y_k has a convergent subsequence.

Suppose A is a precompact subset of a metric space (X, d). For each m the collection $\{B(x, \frac{1}{m})\}_{x\in A}$ forms an open cover for the closure of A. Since the closure of A is compact, we need to take only finitely many of these balls to cover $\overline{A}$. As m ranges over the natural numbers, the centers of these finite coverings form a countable dense subset of A, so A is separable.

The image of T is the union $\cup_{n\in\mathbb{N}} T[B(0, n)]$. Since each $T[B(0, n)]$ is precompact, it is separable. Since the countable union of separable sets is separable, we have that the image of T is separable. □

Problem 5. Let H be a Hilbert space and let $U : H \to H$ be unitary so that $UU^* = U^*U = 1$.

(i) Show that
$$H = \ker(1 - U) \oplus \overline{\mathrm{Im}\,(I - U)},$$
where the direct sum is orthogonal.

(ii) Let P be the orthogonal projection onto $\ker(1 - U)$ and let us set
$$S_n = \frac{1}{n} \sum_{j=0}^{n-1} U^j.$$
Show that $S_n x \to Px$ for all $x \in H$ as $n \to \infty$.

Proof. (i) Suppose y is in $\overline{\mathrm{Im}\,(1 - U)}$ and z is in $\ker(1-U)$. Then there is a sequence of $y_n \in \mathrm{Im}\,(1 - U)$ with $y_n = (1-U)x_n$ for some sequence $x_n \in H$ and $y_n \to y$. By the continuity of the inner product we have

$$\begin{aligned}\langle y, z\rangle &= \lim_{n\to\infty} \langle (1-U)x_n, z\rangle \\ &= \lim_{n\to\infty} \langle (U^*-1)Ux_n, z\rangle \\ &= \lim_{n\to\infty} \langle Ux_n, (U-1)z\rangle \\ &= 0.\end{aligned}$$

This shows that $\ker(1-U) \subseteq \overline{\operatorname{Im}(1-U)}^{\perp}$. Suppose conversely that for some $z \in H$ we have that $\langle y, z\rangle = 0$ for all $y \in \overline{\operatorname{Im}(1-U)}$. We then have

$$\begin{aligned}\|(1-U)z\|^2 &= \langle (1-U)z, (1-U)z\rangle \\ &= \langle (1-U^*)(1-U)z, z\rangle \\ &= \langle (1-U)(1-U^*)z, z\rangle \\ &= 0.\end{aligned}$$

This shows that $\overline{\operatorname{Im}(1-U)}^{\perp} \subseteq \ker(1-U)$. Finally, we know that any Hilbert space splits as a direct sum of a closed subspace and its orthogonal complement, so $H = \ker(1-U) \oplus \overline{\operatorname{Im}(1-U)}$.

(ii) Take $y \in H$. By part (i) of this exercise, we can write $y = \lim_{m\to\infty}(y_0 + y_m)$, where y_0 is in the kernel of $1-U$ and $y_m = (1-U)x_m$ for some sequence x_m in H. P, the orthogonal projection onto the kernel of $1-U$ will send y to y_0. We then have

$$\begin{aligned}\|S_n y - Py\| &= \lim_{m\to\infty} \|S_n(y_0+y_m) - y_0\| \\ &= \lim_{m\to\infty} \left\| \left(\frac{1}{n}\sum_{j=0}^{n-1} U^j\right)(y_0+y_m) - y_0 \right\| \\ &= \lim_{m\to\infty} \left\| \left(\frac{1}{n}\sum_{j=0}^{n-1} U^j\right)(1-U)x_m \right\| \\ &= \lim_{m\to\infty} \frac{1}{n} \cdot \left\| \sum_{j=0}^{n-1} (U^j - U^{j+1})x_m \right\| \\ &= \lim_{m\to\infty} \frac{1}{n} \|(1-U^n)x_m\| \\ &\le \lim_{m\to\infty} \frac{2}{n} \|x_m\|.\end{aligned}$$

The last line follows from the fact that U is unitary, so $\|U^n x\| = \|x\|$. As $n \to \infty$, this quantity goes to zero as desired.

□

Problem 6. Define the space $\mathcal{B}$ by

$$\mathcal{B} = \left\{ u : \mathbb{C} \to \mathbb{C} : u \text{ is holomorphic and } \int_{\mathbb{C}} |u(z)|^2 e^{-|z|^2} L(dz) < \infty \right\},$$

where $L(dz)$ is the Lebesgue measure in $\mathbb{C}$. Show that $\mathcal{B}$ becomes a Hilbert space when equipped with the scalar product

$$\langle u, v \rangle = \int_{\mathbb{C}} u(z)\overline{v(z)} e^{-|z|^2} L(dz).$$

Proof. That $\mathcal{B}$ is a complex vector space follows immediately from the triangle inequality on $\mathbb{C}$. That the proposed scalar product is finite follows from Cauchy-Schwarz:

$$\begin{aligned}
|\langle u, v \rangle| &= \left| \int_{\mathbb{C}} u(z)\overline{v(z)} e^{-|z|^2} L(dz) \right| \\
&\leq \int_{\mathbb{C}} (|u(z)| e^{-|z|^2/2}) \cdot (|v(z)| e^{-|z|^2/2}) L(dz) \\
&\leq \left(\int_{\mathbb{C}} |u(z)|^2 e^{-|z|^2} L(dz) \right)^{1/2} \cdot \left(\int_{\mathbb{C}} |v(z)|^2 e^{-|z|^2} L(dz) \right)^{1/2} \\
&< \infty.
\end{aligned}$$

That the scalar product is sesquilinear is obvious. It remains to show that $\mathcal{B}$ is complete with respect to the norm induced by this inner product. □