

## 260A - Homework 1

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### Problem 1.

(i) Show that  $\ell^p$ ,  $1 \leq p \leq \infty$ , is a Banach space.

(ii) Prove that  $\ell^\infty = (\ell^1)^*$ , but  $(\ell^\infty)^* \neq \ell^1$ .

*Proof.* (i) Let  $a = (a^{(n)})$  and  $b = (b^{(n)})$  be in  $\ell^p$ ,  $1 < p < \infty$ . We have by Hölder's inequality for any complex  $\lambda$

$$\begin{aligned}
 \|a + \lambda b\|_p^p &= \sum_{n=1}^{\infty} |a^{(n)} + \lambda b^{(n)}|^p \\
 &= \sum_{n=1}^{\infty} |a^{(n)} + \lambda b^{(n)}| \cdot |a^{(n)} + \lambda b^{(n)}|^{p-1} \\
 &\leq \sum_{n=1}^{\infty} |a^{(n)}| \cdot |a^{(n)} + \lambda b^{(n)}|^{p-1} + |\lambda| \sum_{n=1}^{\infty} |b^{(n)}| \cdot |a^{(n)} + \lambda b^{(n)}|^{p-1} \\
 &\leq (\|a\|_p + |\lambda| \|b\|_p) \left( \sum_{n=1}^{\infty} |a^{(n)} + \lambda b^{(n)}|^{(p-1)\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \\
 &= (\|a\|_p + |\lambda| \|b\|_p) \|a + \lambda b\|_p^{p-1},
 \end{aligned}$$

Which shows that  $\|a + \lambda b\|_p \leq \|a\|_p + |\lambda| \|b\|_p < \infty$ . This shows both that  $\ell^p$ ,  $1 < p < \infty$ , is a vector space (as linear combinations of elements of  $\ell^p$  have finite  $p$ -norm) and that the  $p$ -norm satisfies the triangle inequality (take  $\lambda = 1$ ).

$\ell^1$  is a vector space and the  $\|\cdot\|_1$  norm satisfies the triangle inequality thanks to the triangle inequality on  $\mathbb{C}$ :

$$\begin{aligned}
 \|a + \lambda b\|_1 &= \sum_{n=1}^{\infty} |a^{(n)} + \lambda b^{(n)}| \\
 &\leq \sum_{n=1}^{\infty} |a^{(n)}| + |\lambda| \sum_{n=1}^{\infty} |b^{(n)}| \\
 &= \|a\|_1 + |\lambda| \|b\|_1.
 \end{aligned}$$

Similarly, for  $a, b \in \ell^\infty$  and  $\lambda \in \mathbb{C}$  we have

$$\|a + \lambda b\|_\infty = \sup_{n \geq 1} |a^{(n)} + \lambda b^{(n)}| \leq \sup_{n \geq 1} (|a^{(n)}| + |\lambda| |b^{(n)}|) \leq \sup_{n \geq 1} |a^{(n)}| + |\lambda| \sup_{n \geq 1} |b^{(n)}| = \|a\|_\infty + |\lambda| \|b\|_\infty.$$

We then have that  $\ell^p$  is a normed complex vector space. We now need to show completeness. First let's treat the case of  $p < \infty$ . Suppose that  $\{a_n\}$  is a Cauchy sequence in  $\ell^p$  (here  $a_i^{(j)}$  is the

$j$ -th entry in the  $i$ -th element of the sequence). Since this sequence is Cauchy we have that for any  $\epsilon > 0$  we can find  $N \in \mathbb{N}$  so that for all  $m, n > N$

$$\|a_m - a_n\|_p < \epsilon \iff \sum_{k=1}^{\infty} |a_m^{(k)} - a_n^{(k)}|^p < \epsilon^p.$$

Since each term in the above sum is nonnegative, we must have that  $|a_m^{(k)} - a_n^{(k)}| < \epsilon$  for each  $k$ . In particular, we have that for any fixed  $k$ ,  $\{a_n^{(k)}\}$  is a Cauchy sequence of complex numbers. Since  $\mathbb{C}$  is complete, we have that  $a_n^{(k)} \rightarrow a^{(k)} \in \mathbb{C}$  as  $n \rightarrow \infty$ .

Let  $a$  be the sequence of complex numbers whose  $k$ -th entry is built from our original Cauchy sequence by  $a^{(k)} = \lim_{n \rightarrow \infty} a_n^{(k)}$ . Our plan is to show that  $a_n \rightarrow a$  in  $\ell^p$  and that  $a$  is in  $\ell^p$ . Fix  $\epsilon > 0$ . Then for some  $N$  we have that  $\|a_m - a_n\|_p < \epsilon$  for all  $m, n > N$ . Our trick is to pass to a finite sum and then take limits in a particular order. For any  $L > 0$  and  $m, n$  sufficiently large we have

$$\sum_{k=0}^L |a_m^{(k)} - a_n^{(k)}|^p \leq \|a_m - a_n\|_p^p < \epsilon^p.$$

Now the right-hand side does not depend on  $m$ , so taking  $m \rightarrow \infty$  gives

$$\sum_{k=0}^L |a^{(k)} - a_n^{(k)}|^p < \epsilon^p.$$

Then we take  $L \rightarrow \infty$  which gives  $\|a - a_n\|_p < \epsilon$ , so  $a_n \rightarrow a$  in  $\ell^p$ . We can use this to show that  $a$  is in  $\ell^p$  since for all  $n$

$$\|a\|_p \leq \|a - a_n\|_p + \|a_n\|_p.$$

For  $n$  large enough the first term on the right is bounded by  $\epsilon$  and the second term is finite since each  $a_n$  is in  $\ell^p$ . Thus,  $\ell^p$  is complete, and therefore, a Banach space for  $1 \leq p < \infty$ .

Now let  $p = \infty$ . If  $\{a_n\}$  is a Cauchy sequence in  $\ell^\infty$  then for  $\epsilon > 0$  and  $m, n$  sufficiently large we have that  $\sup_{k \geq 0} |a_m^{(k)} - a_n^{(k)}| < \epsilon$ . Just like in the finite  $p$  case, this implies that for any fixed  $k$ ,  $\{a_n^{(k)}\}$  is a Cauchy sequence of complex numbers, so we can speak of the entrywise limit  $a$ . Also similar to the finite  $p$  case we have that for  $L$  large

$$\sup_{1 \leq k \leq L} |a_m^{(k)} - a_n^{(k)}| \leq \|a_m - a_n\|_\infty < \epsilon.$$

Sending  $m$  to infinity gives  $\sup_{1 \leq k \leq L} |a^{(k)} - a_n^{(k)}| < \epsilon$  and then sending  $L$  to infinity gives  $\|a - a_n\|_\infty \rightarrow 0$ . The argument used in the  $p < \infty$  case also shows that  $a \in \ell^\infty$ .

- (ii) First we'll show that  $(\ell^1)^* = \ell^\infty$  (i.e., they are isometrically isomorphic). Let  $\varphi : \ell^\infty \rightarrow (\ell^1)^*$  be the map that sends  $b \in \ell^\infty$  to  $T_b$ , where  $T_b(a) = \sum_{k=1}^{\infty} a^{(k)} b^{(k)}$ . That  $\varphi$  is linear is obvious. By Hölder's inequality we have that

$$|T_b(a)| \leq \sum_{k=1}^{\infty} |a^{(k)}| |b^{(k)}| \leq \|a\|_1 \cdot \|b\|_\infty,$$

This shows that  $T_b$  is bounded, and therefore continuous, so the image of  $\varphi$  indeed lives in  $(\ell^1)^*$ . In particular, this shows that  $\|\varphi(b)\| \leq \|b\|_\infty$  (so  $\varphi$  is a continuous map of vector spaces). To show that  $\varphi$  is an isometry, we need the reverse inequality.

Since  $\|b\|_\infty = \sup_{k \geq 1} |b^{(k)}|$ , for any  $\epsilon > 0$ , we can find a natural number  $N$  so that  $|b^{(N)}| > \|b\|_\infty - \epsilon$ . Consequently, if we let  $e_n$  be the sequence in  $\ell^1$  whose  $n$ -th entry is 1 and whose other entries are 0, we have that we can always find  $N$  so that  $|T_b(e_N)| = |b^{(N)}| > \|b\|_\infty - \epsilon$ . Since  $\epsilon$  was arbitrary and  $\|e_n\|_1 = 1$ , we have that  $\|T_b\|_\infty \geq \|b\|_\infty$ . Thus,  $\|\varphi(b)\| = \|b\|_\infty$  and  $\varphi$  is an isometry.

Since isometries are injective, it remains to show that  $\varphi$  is surjective. Let  $T$  be a functional in  $(\ell^1)^*$ . For any  $a \in \ell^1$  we have that  $a = \sum_{k=1}^\infty a^{(k)} e_k$  where  $\sum |a^{(k)}| < \infty$  and  $e_k$  is as it was above. Since  $a = \lim_{N \rightarrow \infty} \sum_{k=1}^N a^{(k)} e_k$ , continuity of  $T$  tells us that

$$T(a) = T\left(\sum_{k=1}^\infty a^{(k)} e_k\right) = \sum_{k=1}^\infty a^{(k)} T(e_k).$$

Since continuity is equivalent to boundedness, we have that  $|T(e_k)| < M < \infty$  for some  $M$ . Thus,  $T$  is the image of the bounded sequence sequence  $(T(e_1), T(e_2), \dots)$  under  $\varphi$ , so  $\varphi$  is surjective.  $\varphi$  is then a surjective isometry  $\ell^\infty \rightarrow (\ell^1)^*$ .

Now let's show that  $(\ell^\infty)^* \neq \ell^1$ . Let  $S$  be the subspace of  $\ell^\infty$  consisting of all convergent sequences and let  $T : S \rightarrow \mathbb{C}$  be the map that sends a convergent sequence to its limit.  $T$  is clearly linear and it's bounded since

$$|T(a)| = \left| \lim_{k \rightarrow \infty} a^{(k)} \right| \leq \limsup_{k \rightarrow \infty} |a^{(k)}| \leq \sup_{k \geq 1} |a^{(k)}| = \|a\|_\infty.$$

By the Hahn-Banach theorem,  $T$  extends to a continuous linear functional  $\tilde{T}$  on all of  $\ell^\infty$  that agrees with  $T$  on  $S$ .

If  $\tilde{T}(a)$  could be written  $\tilde{T}(a) = \sum_{k=1}^\infty a^{(k)} b^{(k)}$  for some  $b \in \ell^1$ , then for all  $n$  we would have  $b^{(n)} = \tilde{T}(e_n) = T(e_n) = 0$ . But then  $b$  would be the zero sequence and  $\tilde{T}$  is the zero functional, which is nonsense since  $\tilde{T}(1, 1, \dots) = T(1, 1, \dots) = 1$ . We conclude that  $\tilde{T}$  does not have the form required for  $(\ell^\infty)^* = \ell^1$ .

□

**Problem 2** Prove that if  $Z$  is a subspace of a normed linear space  $X$ , and  $y \in X$  has distance  $d$  from  $Z$ , then there exists  $\Lambda \in X^*$  such that  $\|\Lambda\| \leq 1$ ,  $\Lambda(y) = d$  and  $\Lambda(z) = 0$  for all  $z \in Z$ .

*Proof.* Consider the subspace  $Y = Z \oplus ky$  of  $X$ , where  $k$  the field over which  $X$  is defined. This sum is indeed direct since  $y$  is not in  $Z$ . Define the function  $f : Y \rightarrow \mathbb{R}$  by  $f(z + \alpha y) = \alpha d$ .  $f$  is linear since

$$\begin{aligned} f[\gamma(z + \alpha y) + (w + \beta y)] &= f[(w + \gamma z) + (\beta + \gamma\alpha)y] \\ &= (\beta + \gamma\alpha)d \\ &= \gamma f(z + \alpha y) + f(w + \beta y). \end{aligned}$$

We claim that  $|f(z + \alpha y)| \leq \|z + \alpha y\|$ . Intuitively, this is because  $|f(z + \alpha y)|$  is the distance from  $z + \alpha y$  to  $Z$ , which is at most  $\|z + \alpha y\|$ , since  $0 \in Z$ . Rigorously, since  $0 \in Z$  we have

$$\begin{aligned}
|f(z + \alpha y)| &= |\alpha \cdot d| \\
&= |\alpha| \cdot \inf_{w \in Z} \|y - w\| \\
&= \inf_{w \in Z} \|\alpha y + z - w\| \\
&\leq \|\alpha y + z - 0\| \\
&= \|\alpha y + z\|.
\end{aligned}$$

By the Hahn-Banach theorem,  $f$  extends to a continuous (as  $|f(x)| < \|x\|$  on  $Y$ ) linear function  $\Lambda$  on all of  $X$  that also satisfies  $|\Lambda(x)| \leq \|x\|$ . This gives  $\|\Lambda\| \leq 1$ . Furthermore, since  $\Lambda$  agrees with  $f$  on  $Y$ , we have that  $\Lambda(y) = f(y) = d$  and  $\Lambda(z) = f(z) = f(z + 0y) = 0$  for all  $z \in Z$ .  $\square$

**Problem 3.** Show that linear combinations of functions of the form

$$\mathbb{R} \ni t \mapsto \frac{1}{t - z}, \quad \text{Im}(z) \neq 0$$

are dense in the space of continuous functions on  $\mathbb{R}$  which tend to zero at infinity.

*Proof.* Let  $W$  be the set of linear combinations of functions of the given form. By the spanning criterion we have that the closure of  $W$  in  $C_{(0)}(\mathbb{R})$  is given by

$$\overline{W} = \bigcap_{\substack{T \in C_{(0)}(\mathbb{R})^* \\ T|_W = 0}} \ker T.$$

In other words, we want to show that intersection on the right is all of  $C_{(0)}(\mathbb{R})$ . Now by Riesz-Markov-Kakutani, we have that the dual space  $C_{(0)}(\mathbb{R})^*$  is the set of all complex Radon measures on  $\mathbb{R}$ . What we're trying to show is that for any  $\mu \in C_{(0)}(\mathbb{R})^*$ , if  $\int_{\mathbb{R}} \varphi d\mu = 0$  for all  $\varphi \in W$ , then  $\int_{\mathbb{R}} f d\mu = 0$  for all  $f \in C_{(0)}(\mathbb{R})$  (i.e.,  $\mu \equiv 0$ ).

Our plan is to use the Stone-Weierstrass theorem. Let  $\mu \in C_{(0)}(\mathbb{R})^*$  be such that for all complex  $z$  with  $\text{Im}(z) \neq 0$ ,  $f(z) = \int_{\mathbb{R}} \frac{1}{t - z} d\mu(t) = 0$ .  $f$  is finite for all real  $z$  since  $\text{Im}(z) \neq 0$  and  $\mu(\mathbb{R})$  is finite. Furthermore, by dominated convergence,  $f \in C^\infty(\mathbb{R})$  since  $\partial_z^n (\frac{1}{t - z}) = C_n \frac{1}{(t - z)^{n+1}}$  exists and is  $L^1(\mu(t))$ .  $\square$