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260A - Homework 1

Problem 1.

(i) Show that ℓ^p, $1 \le p \le \infty$, is a Banach space.

(ii) Prove that $\ell^\infty = (\ell^1)^*$, but $(\ell^\infty)^* \neq \ell^1$.

Proof. (i) Let $a = (a^{(n)})$ and $b = (b^{(n)})$ be in ℓ^p, $1 < p < \infty$. We have by Hölder's inequality for any complex λ

$$\begin{aligned}
\|a+\lambda b\|_p^p &= \sum_{n=1}^\infty |a^{(n)} + \lambda b^{(n)}|^p \\
&= \sum_{n=1}^\infty |a^{(n)} + \lambda b^{(n)}| \cdot |a^{(n)} + \lambda b^{(n)}|^{p-1} \\
&\le \sum_{n=1}^\infty |a^{(n)}| \cdot |a^{(n)} + \lambda b^{(n)}|^{p-1} + |\lambda| \sum_{n=1}^\infty |b^{(n)}| \cdot |a^{(n)} + \lambda b^{(n)}|^{p-1} \\
&\le (\|a\|_p + |\lambda|\, \|b\|_p) \left(\sum_{n=1}^\infty |a^{(n)} + \lambda b^{(n)}|^{(p-1)\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \\
&= (\|a\|_p + |\lambda|\, \|b\|_p)\, \|a+\lambda b\|_p^{p-1},
\end{aligned}$$

Which shows that $\|a+\lambda b\|_p \le \|a\|_p + |\lambda|\, \|b\|_p < \infty$. This shows both that ℓ^p, $1 < p < \infty$, is a vector space (as linear combinations of elements of ℓ^p have finite p-norm) and that the p-norm satisfies the triangle inequality (take $\lambda = 1$).

ℓ^1 is a vector space and the $\|\cdot\|_1$ norm satisfies the triangle inequality thanks to the triangle inequality on $\mathbb{C}$:

$$\begin{aligned}
\|a+\lambda b\|_1 &= \sum_{n=1}^\infty |a^{(n)} + \lambda b^{(n)}| \\
&\le \sum_{n=1}^\infty |a^{(n)}| + |\lambda| \sum_{n=1}^\infty |b^{(n)}| \\
&= \|a\|_p + |\lambda|\, \|b\|_p.
\end{aligned}$$

Similarly, for $a, b \in \ell^\infty$ and $\lambda \in \mathbb{C}$ we have

$$\|a+\lambda b\|_\infty = \sup_{n\ge1} |a^{(n)} + \lambda b^{(n)}| \le \sup_{n\ge1}(|a^{(n)}| + |\lambda||b^{(n)}|) \le \sup_{n\ge1} |a^{(n)}| + |\lambda| \sup_{n\ge1} |b^{(n)}| = \|a\|_\infty + |\lambda|\, \|b\|_\infty.$$

We then have that ℓ^p is a normed complex vector space. We now need to show completeness. First let's treat the case of $p < \infty$. Suppose that $\{a_n\}$ is a Cauchy sequence in ℓ^p (here $a_i^{(j)}$ is the

j-th entry in the i-th element of the sequence). Since this sequence is Cauchy we have that for any $\epsilon > 0$ we can find $N \in \mathbb{N}$ so that for all $m, n > N$

$$\|a_m - a_n\|_p < \epsilon \iff \sum_{k=1}^{\infty} |a_m^{(k)} - a_n^{(k)}|^p < \epsilon^p.$$

Since each term in the above sum is nonnegative, we must have that $|a_m^{(k)} - a_n^{(k)}| < \epsilon$ for each k. In particular, we have that for any fixed k, $\{a_n^{(k)}\}$ is a Cauchy sequence of complex numbers. Since $\mathbb{C}$ is complete, we have that $a_n^{(k)} \to a^{(k)} \in \mathbb{C}$ as $n \to \infty$.

Let a be the sequence of complex numbers whose k-th entry is built from our original Cauchy sequence by $a^{(k)} = \lim_{n\to\infty} a_n^{(k)}$. Our plan is to show that $a_n \to a$ in ℓ^p and that a is in ℓ^p. Fix $\epsilon > 0$. Then for some N we have that $\|a_m - a_n\|_p < \epsilon$ for all $m, n > N$. Our trick is to pass to a finite sum and then take limits in a particular order. For any $L > 0$ and m, n sufficiently large we have

$$\sum_{k=0}^{L} |a_m^{(k)} - a_n^{(k)}|^p \leq \|a_m - a_n\|_p^p < \epsilon^p.$$

Now the right-hand side does not depend on m, so taking $m \to \infty$ gives

$$\sum_{k=0}^{L} |a^{(k)} - a_n^{(k)}|^p < \epsilon^p.$$

Then we take $L \to \infty$ which gives $\|a - a_n\|_p < \epsilon$, so $a_n \to a$ in ℓ^p. We can use this to show that a is in ℓ^p since for all n

$$\|a\|_p \leq \|a - a_n\|_p + \|a_n\|_p.$$

For n large enough the first term on the right is bounded by ϵ and the second term is finite since each a_n is in ℓ^p. Thus, ℓ^p is complete, and therefore, a Banach space for $1 \leq p < \infty$.

Now let $p = \infty$. If $\{a_n\}$ is a Cauchy sequence in ℓ^∞ then for $\epsilon > 0$ and m, n sufficiently large we have that $\sup_{k>0} |a_m^{(k)} - a_n^{(k)}| < \epsilon$. Just like in the finite p case, this implies that for any fixed k, $\{a_n^{(k)}\}$ is a Cauchy sequence of complex numbers, so we can speak of the entrywise limit a. Also similar to the finite p case we have that for L large

$$\sup_{1\leq k\leq L} |a_m^{(k)} - a_n^{(k)}| \leq \|a_m - a_n\|_\infty < \epsilon.$$

Sending m to infinity gives $\sup_{1\leq k\leq L} |a^{(k)} - a_n^{(k)}| < \epsilon$ and then sending L to infinity gives $\|a - a_n\|_\infty \to 0$. The argument used in the $p < \infty$ case also shows that $a \in \ell^\infty$.

(ii) First we'll show that $(\ell^1)^* = \ell^\infty$ (i.e., they are isometrically isomorphic). Let $\varphi : \ell^\infty \to (\ell^1)^*$ be the map that sends $b \in \ell^\infty$ to T_b, where $T_b(a) = \sum_{k=1}^{\infty} a^{(k)} b^{(k)}$. That φ is linear is obvious. By Hölder's inequality we have that

$$|T_b(a)| \leq \sum_{k=1}^{\infty} \left|a^{(k)}\right| \left|b^{(k)}\right| \leq \|a\|_1 \cdot \|b\|_\infty,$$

This shows that T_b is bounded, and therefore continuous, so the image of φ indeed lives in $(\ell^1)^*$. In particular, this shows that $\|\varphi(b)\| \leq \|b\|_\infty$ (so φ is a continuous map of vector spaces). To show that φ is an isometry, we need the reverse inequality.

Since $\|b\|_\infty = \sup_{k\geq 1} |b^{(k)}|$, for any $\epsilon > 0$, we can find a natural number N so that $|b^{(N)}| > \|b\|_\infty - \epsilon$. Consequently, if we let e_n be the sequence in ℓ^1 whose n-th entry is 1 and whose other entries are 0, we have that we can always find N so that $|T_b(e_N)| = |b^{(N)}| > \|b\|_\infty - \epsilon$. Since ϵ was arbitrary and $\|e_n\|_1 = 1$, we have that $\|T_b\|_\infty \geq \|b\|_\infty$. Thus, $\|\varphi(b)\| = \|b\|_\infty$ and φ is an isometry.

Since isometries are injective, it remains to show that φ is surjective. Let T be a functional in $(\ell^1)^*$. For any $a \in \ell^1$ we have that $a = \sum_{k=1}^\infty a^{(k)} e_k$ where $\sum |a^{(k)}| < \infty$ and e_k is as it was above. Since $a = \lim_{N\to\infty} \sum_{k=1}^N a^{(k)} e_k$, continuity of T tells us that

$$T(a) = T\left(\sum_{k=1}^\infty a^{(k)} e_k\right) = \sum_{k=1}^\infty a^{(k)} T(e_k).$$

Since continuity is equivalent to boundedness, we have that $|T(e_k)| < M < \infty$ for some M. Thus, T is the image of the bounded sequence sequence $(T(e_1), T(e_2), \ldots)$ under φ, so φ is surjective. φ is then a surjective isometry $\ell^\infty \to (\ell^1)^*$.

Now let's show that $(\ell^\infty)^* \neq \ell^1$. Let S be the subspace of ℓ^∞ consisting of all convergent sequences and let $T : S \to \mathbb{C}$ be the map that sends a convergent sequence to its limit. T is clearly linear and it's bounded since

$$|T(a)| = |\lim_{k\to\infty} a^{(k)}| \leq \limsup_{k\to\infty} |a^{(k)}| \leq \sup_{k\geq 1} |a^{(k)}| = \|a\|_\infty.$$

By the Hahn-Banach theorem, T extends to a continuous linear functional $\tilde{T}$ on all of ℓ^∞ that agrees with T on S.

If $\tilde{T}(a)$ could be written $\tilde{T}(a) = \sum_{k=1}^\infty a^{(k)} b^{(k)}$ for some $b \in \ell^1$, then for all n we would have $b^{(n)} = \tilde{T}(e_n) = T(e_n) = 0$. But then b would be the zero sequence and $\tilde{T}$ is the zero functional, which is nonsense since $\tilde{T}(1, 1, \ldots) = T(1, 1, \ldots) = 1$. We conclude that $\tilde{T}$ does not have the form required for $(\ell^\infty)^* = \ell^1$.

□

Problem 2 Prove that if Z is a subspace of a normed linear space X, and $y \in X$ has distance d from Z, then there exists $\Lambda \in X^*$ such that $\|\Lambda\| \leq 1$, $\Lambda(y) = d$ and $\Lambda(z) = 0$ for all $z \in Z$.

Proof. Consider the subspace $Y = Z \oplus ky$ of X, where k the field over which X is defined. This sum is indeed direct since y is not in Z. Define the function $f : Y \to \mathbb{R}$ by $f(z + \alpha y) = \alpha d$. f is linear since

$$\begin{aligned} f[\gamma(z + \alpha y) + (w + \beta y)] &= f[(w + \gamma z) + (\beta + \gamma\alpha) y] \\ &= (\beta + \gamma\alpha) d \\ &= \gamma f(z + \alpha y) + f(w + \beta y). \end{aligned}$$

We claim that $|f(z+\alpha y)| \leq \|z+\alpha y\|$. Intuitively, this is because $|f(z+\alpha y)|$ is the distance from $z+\alpha y$ to Z, which is at most $\|z+\alpha y\|$, since $0 \in Z$. Rigorously, since $0 \in Z$ we have

$$\begin{aligned}
|f(z+\alpha y)| &= |\alpha \cdot d| \\
&= |\alpha| \cdot \inf_{w \in Z} \|y - w\| \\
&= \inf_{w \in Z} \|\alpha y + z - w\| \\
&\leq \|\alpha y + z - 0\| \\
&= \|\alpha y + z\|.
\end{aligned}$$

By the Hahn-Banach theorem, f extends to a continuous (as $|f(x)| < \|x\|$ on Y) linear function Λ on all of X that also satisfies $|\Lambda(x)| \leq \|x\|$. This gives $\|\Lambda\| \leq 1$. Furthermore, since Λ agrees with f on Y, we have that $\Lambda(y) = f(y) = d$ and $\Lambda(z) = f(z) = f(z+0y) = 0$ for all $z \in Z$. □

Problem 3. Show that linear combinations of functions of the form

$$\mathbb{R} \ni t \mapsto \frac{1}{t-z}, \quad \operatorname{Im}(z) \neq 0$$

are dense in the space of continuous functions on $\mathbb{R}$ which tend to zero at infinity.

Proof. Let W be the the set of linear combinations of functions of the given form. We'd like to apply Stone-Weierstrass, but unfortunately, W isn't a suba-algebra of $C_{(0)}(\mathbb{R})$ since it isn't closed under multiplication. Our plan is to make ourselves a sub-algebra.
By the spanning criterion we have that the closure of W in $C_{(0)}(\mathbb{R})$ is given by

$$\overline{W} = \bigcap_{\substack{T \in C_{(0)}(\mathbb{R})^* \\ T|_W = 0}} \ker T.$$

Now by Riesz-Markov-Kakutani, we have that the dual space, $C_{(0)}(\mathbb{R})^*$, is the set of all complex Radon measures on $\mathbb{R}$. It then suffices to show that for any $\mu \in C_{(0)}(\mathbb{R})^*$ that satisfies $\int_{\mathbb{R}} \varphi \; d\mu = 0$ for all $\varphi \in W$, then $\int_{\mathbb{R}} f \; d\mu = 0$ for all $f \in C_{(0)}(\mathbb{R})$.

Let μ be a measure such that $\int \varphi \; d\mu$ for all φ in W and let $f(z) = \int_{\mathbb{R}} \frac{1}{t+z} d\mu(t)$ for $\operatorname{Im}(z) \neq 0$. By hypothesis, f is identically zero. By dominated convergence, f is infinitely differentiable with $f^{(n)}(z) = C_n \int_{\mathbb{R}} \frac{1}{(t+z)^{n+1}} d\mu(t) = 0$ for some constant C_n dependent on n.

Now the set, $\mathcal{A}$, of all linear combinations of functions of the form $t \mapsto \frac{1}{(t+z)^n}$ is an algebra of continuous functions that separates points and vanishes nowhere. By Stone-Weierstrass, their uniform closure is all of $C_{(0)}(\mathbb{R})$. Since any function in $C_{(0)}(\mathbb{R})$ can be uniformly approximated by an element of $\mathcal{A}$ and $\mu(\mathbb{R})$ is finite, we have that $\int \psi d\mu = 0$ for any continuous function ψ. By the spanning criterion, the closure of W is all of $C_{(0)}(\mathbb{R})$. □

Problem 4. Let V be a complex vector space and let f_j, $0 \le j \le N$, be linear forms on V such that

$$\bigcap_{j=1}^{N} \ker f_j \subseteq \ker f_0.$$

Show that f_0 is a linear combination of the f_j's, $1 \le j \le N$.

Proof. (This is lemma 3.9 in Rudin's *Functional Analysis.*) In order to apply any result related to Hahn-Banach, we need to be working with a normed vector space, which V needn't be. Our plan is to map into $\mathbb{C}^n$, which clearly is a normed space. We'll apply Hahn-Banach there and use that to help us back in V. Define $f : V \to \mathbb{C}^n$ by $f(x) = (f_1(x), \ldots, f_N(x))$. Now define the linear functional $T : f(V) \to \mathbb{C}$ by $T(f(x)) = f_0(x)$.

First we need to show that T is well-defined. Suppose $f(x) = f(y)$. Then $f_j(x) = f_j(y)$ for $j = 1, \ldots, N$. In this case, $x - y$ is in the kernel of each f_j, so by hypothesis, it's in the kernel of f_0 too, so $T(f(x)) = T(f(y))$. Any linear functional on the finite dimensional space $\mathbb{C}^N$ is continuous, so T is a linear continuous functional on $f(V)$. By Hahn-Banach, we can extend T to a linear functional, $\tilde{T}$, on all of $\mathbb{C}^N$.

Now any continuous linear functional on $\mathbb{C}^N$ has the form

$$\tilde{T}(z_1, \ldots, z_N) = \alpha_1 z_1 + \cdots + \alpha_N z_N$$

for some complex numbers $\alpha_1, \ldots, \alpha_N$. This representation gives us exactly what we need. For any $x \in V$ we have

$$\begin{aligned} f_0(x) &= \tilde{T}(f(x)) \\ &= \tilde{T}(f_1(x), \ldots, f_N(x)) \\ &= \alpha_1 f_1(x) + \cdots \alpha_N f_N(x), \end{aligned}$$

so f_0 is a linear combination of the f_j's. □

Problem 5. Let X be a Banach space such that X^* is separable. Prove that X is separable.

Proof. Let T_n be a countable and dense subset of X^*. For each n we can find an x_n in X so that $\frac{1}{2}\|T_n\| \le |T_n x_n| \le \|T_n\|$ and $\|x_n\| = 1$. We claim that the rational span of the x_n's, Y, is a countable dense subset of X.

Suppose not. Then we can find an open neighborhood in X disjoint from $\overline{Y}$. By the geometric form of Hahn-Banach, we can find a closed affine hyperplane separating $\overline{Y}$ and this neighborhood (since linear subspaces and their complements are convex). That is, we can find $T \in X^*$ that vanishes on $\overline{Y}$ but is not identically zero. Now by the density of the T_n's, we can find a sequence T_{n_j} that limits to T in X^*. Now let's look at the norms of the T_{n_j}'s

$$\begin{aligned}
\frac{1}{2}\|T_{n_j}\| &\leq |T_{n_j}x_{n_j}| \\
&\leq |T_{n_j}x_{n_j} - Tx_{n_j}| + |Tx_{n_j}| \\
&= |T_{n_j}x_{n_j} - Tx_{n_j}| \\
&\leq \|T_{n_j} - T\|,
\end{aligned}$$

which goes to zero by construction. But then T would be the zero functional - a contradiction. We conclude that $\overline{Y} = X$ and X is separable. $\square$

Problem 6. Show that the closure in $L^2(\mathbb{R})$ of the set of functions of the form

$$p(x)e^{-x^2}, \quad x \in \mathbb{R},$$

where p is a complex polynomial on $\mathbb{R}$, is equal to all of $L^2(\mathbb{R})$.

Proof. $\square$