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260B - Homework 2

Problem 1. Let H be a complex separable Hilbert space and let $T_1, T_2 \in \mathcal{L}(H, H)$ be Hilbert-Schmidt operators. Let $T = T_2 T_1$. Show that

$$\text{tr } T = \sum (Te_j, e_j)$$

exists if e_j is an orthonormal basis for H, and prov that the sum is independent of the choice of basis.

Proof. That the trace exists for any fixed orthonormal basis e_j follows from Cauchy-Schwartz and Hölder's inequality.

$$\begin{aligned}
|\text{tr } T| &= \left| \sum (Te_j, e_j) \right| \\
&\leq \sum |(T_1 e_j, T_2^* e_j)| \\
&\leq \sum \|T_1 e_j\| \cdot \|T_2^* e_j\| \\
&\leq \left(\sum \|T_1 e_j\|^2 \right)^{1/2} \left(\sum \|T_2^* e_j\|^2 \right)^{1/2} \\
&= \|T_1\|_{HS} \cdot \|T_2\|_{HS} < \infty.
\end{aligned}$$

On the last line we used the fact that $\|T_2^*\|_{HS} = \|T_2\|_{HS}$. Since the Hilbert-Schmidt norm is independent of the choice of basis, we have that the trace exists regardless of choice of basis.

It remains to show that the actual value of the trace is basis-independent. Let e_j and f_k be two orthonormal bases of H. We use the parallelogram identity and the fact that the Hilbert-Schmidt norm is independent of basis

$$\begin{aligned}
2\|T_1\|_{HS}^2 + 2\|T_2\|_{HS}^2 &= \|T_1 + T_2\|_{HS}^2 + \|T_1 - T_2\|_{HS}^2 \\
&= \left(\|T_1\|_{HS}^2 + \sum_j (T_1 e_j, T_2 e_j) + \sum_j (T_2 e_j, T_1 e_j) + \|T_2\|_{HS}^2 \right) \\
&+ \left(\|T_1\|_{HS}^2 - \sum_k (T_1 f_k, T_2 f_k) - \sum_j (T_2 f_k, T_1 f_k) + \|T_2\|_{HS}^2 \right).
\end{aligned}$$

After canceling the $\|T_1\|_{HS}$ and $\|T_2\|_{HS}$ terms from both sides, we see that the real parts of $\sum (Te_j, e_j)$ and $\sum (Tf_k, f_k)$ are equal. The same argument shows that their imaginary parts are equal as well, so the trace is basis-independent.

$\square$

Problem 2. When $a(x,\xi) \in \mathcal{S}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$, let us consider the Weyl quantization of a,

$$a^w(x, D_x)u(x) = \frac{1}{(2\pi)^n} \int\int e^{i(x-y)\xi} a\left(\frac{x+y}{2}, \xi\right) u(y)\ dyd\xi,$$

acting on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$.

(a) Show that $a^w(x, D_x)$ is symmetric on $\mathcal{S}(\mathbb{R}^n)$,

$$(a^w(x, D_x)u, v)_{L^2} = (u, a^w(x, D_x)v)_{L^2}, \quad u, v \in \mathcal{S}(\mathbb{R}^n)$$

precisely when a is real-valued.

Proof. We have that

$$\begin{aligned}
(a^w(x, D_x)u, v)_{L^2} &= \frac{1}{(2\pi)^n} \int\int\int e^{i(x-y)\xi} a\left(\frac{x+y}{2}, \xi\right) u(y)\overline{v(x)}\ dyd\xi dx \\
&= \frac{1}{(2\pi)^n} \int\int\int \overline{e^{i(y-x)\xi} \overline{a}\left(\frac{x+y}{2}, \xi\right) \overline{u(y)}v(x)}\ dyd\xi dx.
\end{aligned}$$

Now a, u, an v are all Schwartz functions in their respective variables, so their product is absolutely integrable. We can then use Fubini to switch the order of integration.

$$\begin{aligned}
(a^w(x, D_x)u, v) &= \frac{1}{(2\pi)^n} \int\int\int \overline{e^{i(y-x)\xi} \overline{a}\left(\frac{x+y}{2}, \xi\right) \overline{u(y)}v(x)}\ dxd\xi dy \\
&= \frac{1}{(2\pi)^n} \int u(y) \overline{\int\int e^{i(y-x)\xi} \overline{a}\left(\frac{x+y}{2} \xi\right) v(x)\ dxd\xi dy} \\
&= (u, \overline{a}^w(x, D_x)v).
\end{aligned}$$

If a is real valued then we clearly have that $a^w(x, D_x)$ is symmetric. Conversely, if $a^w(x, D_x)$ is symmetric, then we must have that $a^w(x, D_x)u = \overline{a}^w(x, D_x)u$ for all $u \in \mathcal{S}(\mathbb{R})$. Let's compute the difference:

$$\begin{aligned}
[a^w(x, D_x) - \overline{a}^w(x, D_x)]u &= \frac{1}{(2\pi)^n} \int\int e^{i(x-y)\xi} \left[a\left(\frac{x+y}{2}, \xi\right) - \overline{a}\left(\frac{x+y}{2}, \xi\right)\right] u(x)\ dyd\xi dx \\
&= 0.
\end{aligned}$$

I want to say that this is the Fourier transform of another Fourier transform and then invoke the injectivity of the Fourier transform on L^2. That would then force the above integrand to vanish identically for all $u \in \mathcal{S}(\mathbb{R})$, which would mean that $a = \overline{a}$, so a would be real-valued. I'm not quite sure how to fill in the details. $\square$

(b) Suppose now that $a = a(\xi)$ is real-valued and depends only on the momentum variable ξ. Show that $a^w(D_x)$ is unitarily equivalent to a multiplication operator by $a(\xi)$. What is the spectrum of $a^w(D_x)$.

Proof. Let's compute $a^w(D_x)u$ for $u \in \mathcal{S}(\mathbb{R})$. We have

$$\begin{aligned} a^w(D_x)u &= \frac{1}{(2\pi)^n} \int\int e^{i(x-y)\xi} a(\xi) u(y)\ dyd\xi \\ &= \frac{1}{(2\pi)^n} \int e^{ix\xi} a(\xi) \int e^{-iy\xi} u(y)\ dyd\xi \\ &= \frac{1}{(2\pi)^n} \int e^{ix\xi} a(\xi) \widehat{u}(\xi)\ d\xi \\ &= \mathcal{F}^{-1}(a \cdot \widehat{u}) \\ &= (\mathcal{F}^{-1} M_a \mathcal{F})u, \end{aligned}$$

where M_a is the multiplication by a operator. Since the Fourier transform is a unitary isomorphism $\mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$, we have that $a^w(D_x)$ is unitarily equivalent to M_a.
Now let's compute the spectrum. Since $a^w(D_x)$ is unitarily equivalent to multiplication by $a(\xi)$, it suffices to compute the spectrum of M_a. The resolvent set $\rho(M_a)$ is the set of complex λ such that $(M_a - \lambda)$ is bijective onto $L^2(\mathbb{R})$. Now for any $\varphi \in \mathcal{S}(\mathbb{R})$, we have that

$$(M_a - \lambda)\varphi = a\varphi - \lambda\varphi.$$

Since $a \in \mathcal{S}(\mathbb{R})$, we have that $a\varphi$ is in $\mathcal{S}(\mathbb{R})$ as well, as is $\lambda\varphi$. Then the image of $(M_a - \lambda)$ is contained in $\mathcal{S}(\mathbb{R})$, so this operator cannot possibly biject onto $L^2(\mathbb{R})$ for any λ. The resolvent set is then empty, so the spectrum of M_a, and therefore $a^w(D_x)$, is all of $\mathbb{C}$. □

Problem 3. Let us consider the Sturm-Liouville operator

$$P = -\frac{d}{dt}\left(p(t)\frac{d}{dt}\right) + q(t),$$

where $p \in C^1(\mathbb{R})$, $p > 0$, and $q \in C(\mathbb{R})$, $q \geq 0$. Show that the operator P equipped with the domain $C_0^\infty(\mathbb{R})$, is essentially self-adjoint on $L^2(\mathbb{R})$.

Proof. P is self-adjoint if and only if its closure, $\overline{P}$, is equal to its adjoint, P^*. We will show this by showing that $P + I$, equipped with the domain of P, is essentially self-adjoint. We claim that it suffices to show that the image of C_0^∞ under $P + 1$ is dense in $L^2(\mathbb{R})$. We will prove this claim after we have shown the density.

We have the splitting $L^2(\mathbb{R}) = \ker(P + I) \oplus \overline{\operatorname{Im}(P + I)}$. If we can show that $P + I$ is injective then we will have shown that its image is dense in $L^2(\mathbb{R})$. Suppose that for $u \in L^2(\mathbb{R})$ we have that $(P+I)u = 0$ in the distributional sense. Then the real and imaginary parts of $(P + I)u$ vanish, so we can restrict our attention to real valued functions $u \in L^2(\mathbb{R})$.

Since u satisfies a second-order ODE it must have $C^2(\mathbb{R})$ regularity (why?). We claim that if u is in both $C^2(\mathbb{R})$ and $L^2(\mathbb{R})$, and it satisfies $(P + I)u \geq 0$ then $u \geq 0$. To see this, suppose that $u(x_0) < 0$

for some x_0. If this point is a local minimum of u then $u'(x_0) = 0$ and $u''(x_0) > 0$. From the definition of our operator P and the fact that p and q are nonnegative we have

$$-p(x_0)u''(x_0) = (P+1)u(x_0) - (q(x_0)+1)u(x_0) > 0.$$

But this would force $u''(x_0)$ to be negative, contradicting the assumption that x_0 is a local minimum of u. We have then shown that if $u(x_0) < 0$ then x_0 cannot be a local minimum.

Now let $R > |x_0|$ and let x_1 be such that $u(x_1) = \min_{|x|\le R} u(x) < 0$. By our above discussion, x_1 must lie on the boundary of $[-R, R]$, or else u would have a local minimum at a point where it assumes a negative value. Suppose $x_1 = R$ and let $x > R$. If $u(x) > u(R)$ then u would have need to have a local minimum on $[R, x]$, again contradicting our above discussion. But then $u(x) \le u(R) \le u(x_0) < 0$. This shows that u is away from zero for x sufficiently large, contradicting the assumption that $u \in L^2(\mathbb{R})$. A similar contradiction arises if $x_0 = -R$. We conclude that u cannot assume negative values if $(P+I)u \ge 0$.

If $(P+I)u = 0$ then $(P+I)u \ge 0$ and $(P+I)(-u) \ge 0$. By our previous discussion we have $u \ge 0$ and $-u \ge 0$, so $u = 0$. We have then shown that $(P+I)$ is injective, so its image must be dense. $\square$

Problem 4. Let T be a closed densely defined operator on a complex separable Hilbert space. Show that the operators T^*T and TT^* are self-adjoint, when equipped with their natural domains of definition.

Proof. We'll show that T^*T is self-adjoint. Applying this result to T^* will show that TT^* is self-adjoint too.

We have that $V(G(T))^\perp = G(T^*)$ where $V : H \times H \to H \times H$ sends (u, v) to $(v, -u)$ and $G(T)$ is the graph of T. Showing that T is self-adjoint amounts to showing that $G(T) = V(G(T))^\perp$. This orthogonality statement says that $H \oplus H = G(T^*) \oplus V(G(T))$. So for any $w \in H$ there are unique $u \in \mathcal{D}(T^*)$ and $v \in \mathcal{D}(T)$ with

$$H \oplus H \ni (0, w) = (u, T^*u) + V(v, Tv) = (u, T^*u) + (Tv, -v). \tag{1}$$

Looking at each component, we have that $Tv = -u$ and $T^*u = w + v$. Substitution gives $T^*Tv = -(w + v)$, so v is in $\mathcal{D}(T^*T)$. Since w was arbitrary, we also have that $T^*T + I$ is surjective. Suppose $(T^*T + I)x = (T^*T + I)y$. By the uniqueness of u and v in (1), we must then have that $x = y$.

Since $T^*T + I$ is bijective, it has an inverse $(T^*T + I)^{-1} : H \to \mathcal{D}(T^*T)$. Given $x \in H$ we can write $x = (T^*T + I)y$ for some $y \in \mathcal{D}(T^*T)$ by surjectivity. Since T is closed we also have that $T^{**} = T$. In the following computation the parentheses denote inner products of elements in H.

$$\begin{aligned}((T^*T+I)^{-1}x,x) &= (y,(T^*T+I)y) \\ &= (Ty,Ty)+\|y\|^2 \\ &= ((T^*T+I)y,y) \\ &= (x,(T^*T+I)^{-1}x).\end{aligned}$$

Thus, (T^*T+I) is symmetric with domain H, so it is self-adjoint. Now let's look at the graph of T^*T+I. Since $(T^*T+I)^{-1}$ is self-adjoint we have

$$\begin{aligned}V(G(T^*T+I))^\perp &= G(-(T^*T+I)^{-1})^\perp \\ &= V(G(-(T^*T+I)^{-1})) \\ &= G(T^*T+I),\end{aligned}$$

so T^*T+I is self-adjoint. We claim that this forces T^*T to be self-adjoint as well. Clearly T^*T is symmetric if and only if T^*T+I is symmetric, and $\mathcal{D}(T^*T)=\mathcal{D}((T^*T)^*)$ if and only if $\mathcal{D}(T^*T+I)=\mathcal{D}((T^*T+I)^*)$ follows from the Riesz representation theorem. $\square$