Quiz 5

Student ID Number: Name ____________________

Math 140B, 5PM

Please justify all your answers February 21, 2019

Please also write your full name on the back

1. Suppose that f is differentiable on an open interval I containing the point b and that $f'(b) < 0$. Show there are numbers a and c with $a < b < c$ such that $f(a) > f(b) > f(c)$.

   Proof. Note that $f'(b) < 0$ does *not* imply that f is decreasing in a neighborhood of b. If f were *continuously* differentiable then this would be true. You saw an example in lecture of a differentiable, but not continuously differentiable function with positive derivative at a point that isn't increasing on any neighborhood of that point (we'll review it here too after these quiz questions). Let's use the definition of the derivative. We have that

$$\lim_{x\to b} \frac{f(x)-f(b)}{x-b} = f'(b) < 0.$$

   Consequently, for all x sufficiently close to b we have that $\frac{f(x)-f(b)}{x-b} < 0$. Let $a < b$ be close enough to b so that $\frac{f(a)-f(b)}{a-b} < 0$. Since $a < b$, multiplying through by $a-b$ gives $f(a)-f(b) > 0$, so $f(a) > f(b)$. Similarly, let $c > b$ be close enough to b so that $\frac{f(c)-f(a)}{c-a} < 0$. Since $c > a$, multiplying through by $c-a$ gives $f(c)-f(a) < 0$, so $f(c) < f(a)$. We've then found our $a < b < c$ with $f(a) > f(b) > f(c)$. $\square$

2. Find the Taylor polynomial of degree 3 centered at zero, $P_3(x)$, of $f(x) = \sinh x = \frac{1}{2}(e^x - e^{-x})$. Find an upper bound for the remainder, $|f(x) - P_3(x)|$, at $x = 1$.

   Proof. The third degree Taylor polynomial is given by

$$P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3.$$

   Let's compute the derivatives.

$$f'(x) = \frac{1}{2}(e^x + e^{-x}) = \cosh x, \quad f''(x) = \frac{1}{2}(e^x - e^{-x}) = \sinh x,$$
$$f^{(3)}(x) = \frac{1}{2}(e^x + e^{-x}) = \cosh x$$
$$f'(0) = \cosh 0 = 1, \quad f''(0) = \sinh 0 = 0, \quad f^{(3)}(0) = \cosh 0 = 1$$

   So our polynomial is $P_3(x) = x + \frac{1}{3!}x^3$. Let $r > 1$. By Taylor's theorem we have

$$|f(1) - P_3(1)| = \left|\frac{f^{(4)}(y)}{4!}1^4\right|$$

   for some $y \in (-r, r)$. Now $f^{(4)}(x) = \sinh x$. On $(-r, r)$ we have

$$\left|\frac{f^{(4)}(y)}{4!}1^4\right| = \frac{1}{4!} \cdot \frac{|e^y - e^{-y}|}{2} \leq \frac{1}{4!}e^r.$$

   Taking the limit $r \to 1^+$ shows that $|f(1) - P_3(1)| \leq \frac{1}{4!}e$. $\square$

Weird But Important Example: Define the function g by

$$g(x) = \begin{cases} -(x + 2x^2 \sin \frac{1}{x}), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}.$$

We have by standard differentiation rules that

$$g'(x) = -(1 + 4x \sin \frac{1}{x} - 2 \cos \frac{1}{x})$$

for $x \neq 0$. This thing is undefined at $x = 0$, so to compute the derivative there we take the limit

$$\lim_{x \to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \to 0} \frac{g(x)}{x} = \lim_{x \to 0} -(1 + 2x \sin \frac{1}{x}) = -1.$$

So g is differentiable everywhere and the derivative is

$$g'(x) = \begin{cases} -(1 + 4x \sin \frac{1}{x} - 2 \cos \frac{1}{x}), & \text{if } x \neq 0 \\ -1, & \text{if } x = 0 \end{cases}.$$

Looks innocent enough, but the derivative isn't continuous at zero. The $4x \sin \frac{1}{x}$ part behaves fine near zero, but $-2 \cos \frac{1}{x}$ oscillates wildly and $\lim_{x \to 0} -2 \cos \frac{1}{x}$ doesn't exist. This bad behavior of the derivative will show that g isn't decreasing in any neighborhood of zero even though $g'(0) < 0$.

Consider the sequence $x_n = \frac{1}{2\pi n}$ which decreases monotonically to zero. We have that $g'(x_n) = 1$. By the first quiz problem, for any $n > 1$ we can find $(a_n, b_n) \subseteq (x_{n+1}, x_{n-1})$ with $a_n < x_n < b_n$ and $g(a_n) < g(x_n) < g(b_n)$. Since we can find points x_n arbitrarily close to zero, we conclude that g isn't decreasing on any neighborhood of zero.