0.1 Spring 2018

1. Let $f \in L^2(\mathbb{R})$ and let $(\alpha_n)_{n=1}^\infty$, $(\beta_n)_{n=1}^\infty$ be sequences of real numbers such that $\alpha_n \neq 0$ for all n and $\sum_{n=1}^\infty \frac{|\beta_n|}{|\alpha_n|^{1/2}} < \infty$. Show that $\lim_{n\to\infty} \beta_n f(\alpha_n x) = 0$ for almost all $x \in \mathbb{R}$.

   Proof. It suffices to show that the function $F(x) = \sum_{n=1}^\infty \beta_n f(\alpha_n x)$ is locally integrable. To this end, let K be any compact subset of $\mathbb{R}$. We integrate over K.

   $$\left| \int_K \sum_{n=1}^\infty \beta_n f(\alpha_n x)\ dx \right| \leq \int_K \sum_{n=1}^\infty |\beta_n f(\alpha_n x)|\ dx.$$

   Since each term in the sum is nonnegative, we can interchange summation and integration by Tonelli's theorem, then do a change of variables and apply Hölder's inequality.

   $$\begin{aligned} \left| \int_K \sum_{n=1}^\infty \beta_n f(\alpha_n x)\ dx \right| &\leq \sum_{n=1}^\infty \int |\beta_n f(x) \cdot \frac{1}{\alpha_n} \chi_{\alpha_n K}(x)|\ dx \\ &\leq \|f\|_{L^2} \cdot \sqrt{m(K)} \cdot \sum_{n=1}^\infty |\beta_n|/|\alpha_n|^{1/2} \\ &< \infty. \end{aligned}$$

   Since $F(x)$ is integrable we must have that the tail of the series, $\beta_n f(\alpha_n x)$ must go to zero as $n \to \infty$ for almost all x. □

2. Let $g_k, g \in L^1(\mathbb{R})$ and assume that $g_k \to g$ in $L^1(\mathbb{R})$ as $k \to \infty$. Let $(\alpha_k)_{k=1}^\infty$ be a bounded sequence in $\mathbb{R}$, and let $f_k(x) := g_k(x + \alpha_k)$, $k = 1, 2, \ldots$. Prove that the sequence of functions $(f_k)_{k=1}^\infty$ has a subsequence that converges in $L^1(\mathbb{R})$ almost everywhere and in measure.

   Proof. Convergence in L^1 implies subsequential a.e. convergence. a.e. convergence implies convergence in measure, so it suffices to show that f_k has a subsequence that converges in $L^1(\mathbb{R})$. Since α_k is a bounded sequence, it has a convergent subsequence by Bolzano-Weierstrass. For notational clarity, let's just assume that α_k itself converges to $\alpha \in \mathbb{R}$. The plan is to use a standard approximation argument to show that $g_k(x + \alpha_k)$ converges in L^1 to $g(x + \alpha)$.

   Observe that

   $$\|g(x+\alpha) - g_k(x+\alpha_k)\|_{L^1} \leq \|g(x+\alpha) - g(x+\alpha_k)\|_{L^1} + \|g(x+\alpha_k) - g_k(x+\alpha_k)\|_{L^1}.$$

   Now since $g_k \to g$ in L^1, given $\epsilon > 0$ the second integral on the RHS is less than ϵ for k sufficiently large. Furthermore, a standard approximation by compactly supported continuous functions shows that the first integral can be made less than ϵ for k large. □

3. Let $f, g \in L^p(\mathbb{R}^n)$ for some $1 < p < \infty$. Assume that for every $0 < t < \infty$,

$$m(\{x \in \mathbb{R}^n : |g(x)| > t\}) \leq \frac{1}{t} \int_{\{x \in \mathbb{R}^n : |g(x)| > t\}} |f(x)| \, dx,$$

where m is the Lebesgue measure. Show that

$$\|g\|_{L^p(\mathbb{R}^n)} \leq p' \|f\|_{L^p(\mathbb{R}^n)}, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

Proof. By Tonelli's theorem we have for any $L^1(X)$ function φ,

$$\begin{aligned}
\int_0^\infty m(\{x : |\varphi(x)| > t\}) \, dt &= \int_0^\infty \int_X \chi_{\{x:\ |\varphi(x)| > t\}}(x) \, dx dt \\
&= \int_X \int_0^\infty \chi_{\{t:\ |\varphi(x)| > t\}}(t) \, dt dx \\
&= \int_X |\varphi(x)| \, dx.
\end{aligned}$$

In the context of this problem we then have, again by Tonelli

$$\begin{aligned}
\int_{\mathbb{R}^n} |g(x)|^p \, dx &= \int_0^\infty m(\{x : |g(x)| > t^{1/p}\}) \, dt \\
&\leq \int_0^\infty t^{-1/p} \int_{\{x:\ |f(x)| > t^{1/p}\}} |f(x)| \, dx dt \\
&= \int_{\mathbb{R}^n} \int_0^{|f(x)|^p} t^{-1/p} |f(x)| \, dt dx \\
&= p' \int_{\mathbb{R}^n} |f(x)|^{p/p'} \cdot |f(x)| \, dx \\
&= p' \int_{\mathbb{R}^n} |f(x)|^p \, dx.
\end{aligned}$$

The desired result follows after taking the p-th root of both sides and using $(p')^{1/p} \leq p'$ as $p > 1$. ☐

4. Let $f \in L^1(\mathbb{R})$ be such that

$$\int_E f(y) \, dy = 0$$

for all Lebesgue measurable sets $E \subset \mathbb{R}$ with $m(E) = \pi$. Prove or disprove that $f(x) = 0$ for almost all $x \in \mathbb{R}$.

Proof. Let $F(x) = \int_0^x f(t) \, dt$. Since $f \in L^1(\mathbb{R})$, the Lebesgue differentiation theorem says that for almost all $x \in \mathbb{R}$,

$$\begin{aligned}
\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} &= \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t) \, dt \\
&= f(x).
\end{aligned}$$

Thus, F is differentiable almost everywhere with a.e. derivative f. We also have that F is π-periodic since for all $x \in \mathbb{R}$

$$\begin{aligned} F(x+\pi) - F(x) &= \int_x^{x+\pi} f(t)\ dt \\ &= 0. \end{aligned}$$

Intuitively, since F is (a.e.) differentiable and π-periodic, its derivative should be π-periodic too. The Lebesgue differentiation theorem confirms this intuition: for almost all $x \in \mathbb{R}$ we have

$$\begin{aligned} f(x+\pi) - f(x) &= \lim_{h \to 0} \frac{1}{h} \int_{x+\pi}^{x+\pi+h} f(t)\ dt - \frac{1}{h} \int_x^{x+h} f(t)\ dt \\ &= \lim_{h \to 0} \frac{1}{h} (F(x+\pi+h) - F(x+\pi) - F(x+h) + F(x)) \\ &= 0. \end{aligned}$$

Since f is equal to a periodic function almost everywhere and is integrable over $\mathbb{R}$, it must be zero almost everywhere. □

5. Let μ be a positive finite Borel measure on $[0,1]$ and let $\varphi : [0,1] \to [0,1]$ be continuous. Assume that $\mu(\varphi^{-1}(E)) = 0$ for every Borel set $E \subset [0,1]$ with $\mu(E) = 0$. Show that there is a Borel measurable function $f : [0,1] \to [0,\infty)$ such that

$$\int_0^1 g(\varphi(x))\ d\mu(x) = \int_0^1 g(x) f(x)\ d\mu(x)$$

for all continuous $g : [0,1] \to \mathbb{R}$.

Proof. Consider the map that sends a Borel set $E \subset [0,1]$ to $\mu(\varphi^{-1}(E))$ (called the *pushforward* of μ along φ, $\varphi_* \mu$). One easily checks that this map indeed defines a Borel measure on $[0,1]$. The condition that $\mu(\varphi^{-1}(E)) = 0$ for all E with $\mu(E) = 0$ says that $\varphi_* \mu$ is absolutely continuous with respect to μ. By the Radon-Nikodym theorem, there is then a Borel-measurable function f with $d(\varphi_* \mu) = f d\mu$.

We claim that if g is continuous, then $\int_0^1 g\ d(\varphi_* \mu) = \int g \circ \varphi\ d\mu$. □

0.2 Spring 2016

1. Assume $f \in L^1[0,1]$. Compute

$$\lim_{k \to \infty} \int_{[0,1]} |f|^{1/k}\ dx.$$

Solution. Let's split this integral into three regions.

$$\int_{[0,1]} |f|^{1/k}\ dx = \int_{f=0} |f|^{1/k}\ dx + \int_{0<|f|\le 1} |f|^{1/k}\ dx + \int_{|f|>1} |f|^{1/k}\ dx.$$

The integral over the first region is clearly zero for all k. On the second region we have that $|f|^{1/k} \le 1$ for all k. Since the interval $[0,1]$ has finite measure, we have that the constant function 1 is in $L^1(\{x : 0 < |f| \le 1\})$, so the dominated convergence theorem says that the integral over the second region goes to $m(\{0 < |f| \le 1\})$. Similarly, on the third region we have that $|f|^{1/k} \le |f|$, which is in L^1, so the dominated convergence theorem says that the third integral goes to $m(\{|f| > 1\})$. Combining these, we have that

$$\lim_{k\to\infty} \int_{[0,1]} |f|^{1/k}\ dx = m(\{|f| > 0\}).$$

□

2. Let $\{f_n\}$ be a sequence of measurable functions on $[0,1]$ and $0 \le f_n \le 1$ a.e. Assume that

$$\lim_{n\to\infty} \int_{[0,1]} f_n g\ dx = \int_{[0,1]} fg\ dx$$

for some $f \in L^1[0,1]$ and any $g \in C[0,1]$. Prove that $0 \le f \le 1$ a.e.

Solution. Since $f \in L^1[0,1]$, by the Lebesgue differentiation theorem we have that

$$\frac{1}{m(E)} \int_E f(t)\ dt \to f(x) \tag{1}$$

as E shrinks to x for almost all x. Furthermore, since $0 \le f_n \le 1$ we also have that

$$\frac{1}{m(E)} \int_E f_n(t)\ dt \to f_n(x) \in [0,1]$$

as E shrink to x for almost all x. Intuitively, we'd like to replace the integral of f in (1) with a limit of integrals of f_n.

We claim that the function g in the given hypothesis can be replaced with the indicator function of an interval χ_I. To see this, let g_m be a sequence of continuous functions with $g_m \to \chi_I$ in L^1 and $0 \le \chi_I \le 1$. By extracting a subsequence we can assume that $g_m \to \chi_I$ a.e. as well. We then have

$$\int_0^1 |f_n\chi_I - f\chi_i| \le \int_0^1 |f_n\chi_I - f_n g_m| + \int_0^1 |f_n g_m - f g_m| + \int_0^1 |f g_m - f\chi_i|.$$

Since $\|f_n\|_{L^\infty} \le 1$, we have that the first integral on the RHS can be made small uniformly in n by picking m large. The second integral goes to zero as $n \to \infty$ by hypothesis since g_m is continuous. The third integral can be made small for m large by dominated convergence since $|f g_m| \le |f| \in L^1$.

For almost all x, if I_k is a sequence of intervals shrinking to x then

$$\frac{1}{m(I_k)} \int_{I_k} f \, dx = \frac{1}{m(I_k)} \int f \chi_{I_k} \, dx$$

$$= \lim_{n \to \infty} \frac{1}{m(I_k)} \int f_n \chi_{I_k} \, dx.$$

Since $0 \le f_n \le 1$, the RHS is in $[0, 1]$ for almost all x. By the Lebesgue differentiation theorem we then have that $0 \le f \le 1$ a.e. □

3. Let $f, g \in L^2(\mathbb{R}, \mathcal{M}_L, \mu_L)$. Show that $f * g$ is a continuous function on $\mathbb{R}$ vanishing at infinity, that is, $f * g \in C(R)$ and $\lim_{|x| \to \infty} (f * g)(x) = 0$.

Proof. For any h we have by Hölder's inequality

$$|(f * g)(x + h) - (f * g)(x)| = \left| \int f(t)[g(x + h - t) - g(x - t)] \, dt \right| \tag{2}$$

$$\le \|f\|_{L^2} \cdot \|g_h - g\|_{L^2}, \tag{3}$$

where $F_h(x) = F(x + h)$ for any function F. Now for any $\epsilon > 0$ we can find $\varphi \in C_0(\mathbb{R})$ with $\|g - \varphi\|_{L^2} = \|g_h - \varphi_h\|_{L^2} < \epsilon$. By the triangle inequality we then have

$$\|g_h - g\|_{L^2} \le \|g_h - \varphi_h\|_{L^2} + \|\varphi_h - \varphi\|_{L^2} + \|\varphi - g\|_{L^2}$$

$$< \|\varphi_h - \varphi\|_{L^2} + 2\epsilon.$$

Suppose that φ has support contained in the compact set K. If we pick h small enough then we can guarantee that $\varphi_h - \varphi$ is supported on a set with measure at most $2 \cdot m(K)$. Now since φ is continuous with compact support, it is uniformly continuous, so we can choose h small enough that $|\varphi_h(x) - \varphi(x)| = |\varphi(x + h) - \varphi(x)| < \epsilon$ for all x in the support of $\varphi_h - \varphi$. For such h we have

$$\|\varphi_h - \varphi\|_{L^2} \le \epsilon \cdot (2 \cdot m(K))^{1/2},$$

so (2) can be made arbitrarily small, which shows that $f * g$ is continuous.

First we claim that if φ and ψ are continuous with compact support then $\varphi * \psi$ vanishes at infinity. By definition we have that

$$(\varphi * \psi)(x) = \int \varphi(t)\psi(x - t) \, dt.$$

The product $\varphi(t)\psi(x-t)$ is nonzero only if t is in the support of φ and $x-t$ is in the support of φ. If pick x large enough then supports of $t \mapsto \varphi(t)$ and $t \mapsto \psi(x-t)$ are disjoint, so this integral is zero.

Let f_n and g_n be sequences in $C_0(\mathbb{R})$ converging in L^2 to f and g, respectively. We then have

$$|(f*g)(x)-(f_n*g_n)(x)| \le |(f*g)(x)-(f_n*g)(x)|+|(f_n*g)(x)-(f_n*g_n)(x)|$$

$$\le \|g\|_{L^2}\cdot\|f-f_n\|_{L^2}+\|f_n\|_{L^2}\cdot\|g-g_n\|_{L^2}.$$

Since $f_n\to f$ and $g_n\to g$ in L^2, we have that f_n*g_n converges uniformly to $f*g$. Since f_n*g_n vanishes at infinity, we must then have that $f*g$ vanishes at infinity. □

4. Let $(X,\mathcal{A},\mu)$ be a finite measure space, and let $p_1\in(1,\infty]$. Let $\{f_n\}$ be a uniformly bounded sequence in $L^{p_1}(X,\mathcal{A},\mu)$. Suppose $f=\lim_{n\to\infty}f_n$ exists μ-a.e. Prove that $f\in L^p(X,\mathcal{A},\mu)$ for all $p\in[1,p_1]$ and $f_n\to f$ in $L^p(X,\mathcal{A},\mu)$ for all $p\in[1,p_1)$.

Proof. Suppose that $\|f_n\|_{L^{p_1}}\le M$ for all n. First we claim that the f_n are in $L^p(X,\mathcal{A},\mu)$ for all $p\in[1,p_1]$. In fact, they are uniformly bounded:

$$\begin{aligned}\int_X|f_n|^p &= \int_{|f_n|<1}|f_n|^p+\int_{|f_n|\ge1}|f_n|^p\\ &\le \int_{|f_n|<1}1+\int_{|f_n|\ge1}|f_n|^{p_1} \qquad (4)\\ &\le \mu(\{f\le1\})+M^{p_1}.\end{aligned}$$

Since $(X,\mathcal{A},\mu)$ is a finite measure space, this quantity is finite, so $f_n\in L^p(X,\mathcal{A},\mu)$ for all n and $p\in[1,p_1]$. We can then use the fact that $f_n\to f$ a.e. and Fatou's lemma to show that $f\in L^p(X,\mathcal{A},\mu)$ for $p\in[1,p_1]$:

$$\int_X|f|^p\le\liminf_{n\to\infty}\int_X|f_n|^p<\infty,$$

where the finiteness follows from the L^p uniform-boundedness of the f_n.

To establish convergence in L^p, $p\in[1,p_1)$ our plan is to use the Vitali convergence theorem. The family f_n is tight over X since X is a finite measure space and we're given that $f_n\to f$ a.e., so it only remains to show that the f_n's are uniformly integrable. To this end, let E be any measurable subset of X. Since f_n is in L^{p_1}, we have that $|f_n|^p\in L^{p_1/p}$. If we let q be the Hölder conjugate to p_1/p then we have

$$\begin{aligned}\int_E|f_n|^p &= \int_X|f_n|^p\cdot\chi_E\\ &\le \||f_n|^p\|_{L^{p_1/p}}\cdot\|\chi_E\|_{L^q}\\ &\le M^{p_1^2/p}\cdot\mu(E)^{1/q}.\end{aligned}$$

If we choose E so that $\mu(E)^{1/q}<\epsilon\cdot M^{-p_1^2/p}$, then we'll have that $\int_E|f_n|^p<\epsilon$, so the f_n's are uniformly integrable. By the Vitali convergence theorem we have that $f_n\to f$ in L^p for $p\in[1,p_1]$. □

5. Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f : X \to [0, \infty)$ be $\mathcal{A}$-measurable. Consider the measure space $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu_L)$, where $\mathcal{B}_{\mathbb{R}}$ is the Borel σ-algebra on $\mathbb{R}$ and μ_L is the Lebesgue measure, and form the product measure space $(X \times \mathbb{R}, \sigma(\mathcal{A} \times \mathcal{B}_{\mathbb{R}}), \mu \times \mu_L)$. Define $E \subset X \times R$ by $(x, y) \in E \iff y \in [0, f(x))$. Prove that $E \in \sigma(\mathcal{A} \times \mathcal{B}_{\mathbb{R}})$ and $(\mu \times \mu_L)(E) = \int_X f \, d\mu$.

Proof. A function is measurable if it pulls measurable sets back to measurable sets. The plan is then to write E is a union and/or intersection of preimages of measurable sets under measurable functions. The function $F(x, y) = f(x)$ is measurable since

$$F^{-1}[(-\infty, \alpha]) = \{(x, y) : f(x) \le \alpha\} = \{x : f(x) \le \alpha\} \times \mathbb{R} \in \sigma(\mathcal{A} \times \mathcal{B}_{\mathbb{R}}),$$

as f is μ-measurable. We also clearly have that the function $G(x, y) = y$ is measurable. Now consider the function $H(x, y) = y - f(x)$. H is measurable as it is the difference of the measurable functions G and F. We then have that E is measurable through the following decomposition

$$\begin{aligned} E &= \{(x, y) : 0 \le y < f(x)\} \\ &= \{(x, y) : y \ge 0\} \cap \{(x, y) : y < f(x)\} \\ &= G^{-1}[[0, \infty)] \cap H^{-1}[(-\infty, 0)]. \end{aligned}$$

If $\{f > 0\}$ is σ-finite we can use Tonelli's theorem to say

$$\begin{aligned} (\mu \times \mu_L)(E) &= \int_{X \times \mathbb{R}} \chi_E(x, y) \, d(\mu \times \mu_L) \\ &= \int_X \int_{\mathbb{R}} \chi_E(x, y) \, d\mu_L d\mu \\ &= \int_X \int_{\mathbb{R}} \chi_{[0, f(x))}(y) \, dy d\mu \\ &= \int_X f(x) \, d\mu. \end{aligned}$$

On the other hand, suppose that $\{f > 0\}$ is note σ-finite. We claim that $\int_X f \, d\mu = +\infty$. Indeed, since we can decompose this set into a countable union,

$$\{f > 0\} = \bigcup_{m=1}^{\infty} \{\frac{1}{m+1} < f \le \frac{1}{m}\} \cup \bigcup_{n=1}^{\infty} \{n < f \le n + 1\}, \tag{5}$$

we must have that one of these sets has infinite measure. We need to show that $(\mu \times \mu_L)(E) = +\infty$ too. For any $\alpha, \beta > 0$ we have that if $\alpha \le f(x) < \beta$ then the product set

$$\{x : \alpha \le f(x) < \beta\} \times \{y : 0 \le \alpha\}$$

is contained in E. This product set has measure $\alpha \cdot \mu_L\{\alpha \le f < \beta\}$, so by monotonicity we have that

$$\alpha \cdot \mu_L\{\alpha \le f < \beta\} \le (\mu \times \mu_L)(E)$$

for all $\alpha, \beta > 0$. But by the decomposition (5), we have that some set of the form $\{\alpha \le f(x) < \beta\}$ must have infinite measure, so we must have $(\mu \times \mu_L)(E) = +\infty$. □

6. Let $f \in L^1(\mathbb{R})$ and let $a_1, \dots, a_k \in \mathbb{R}$ and $b_1, \dots, b_k \in \mathbb{R} \setminus \{0\}$. Assume that the quotients $\frac{a_j}{b_j}$ are all distinct. Determine

$$\lim_{t \to \infty} \int \left| \sum_{j=1}^{k} f(b_j x + t a_j) \right| dx.$$

Solution. Let $\varphi \in C_0(\mathbb{R})$ be such that $\|f - \varphi\|_{L^1} < \epsilon$. Our plan is to compute the desired limit with φ in place of f and then argue that the difference can be made small. We have

$$\int \left| \sum_{j=1}^{k} \varphi(b_j x + t a_j) \right| dx = \int \left| \sum_{j=1}^{k} \varphi \left[b_j \left(x + \frac{a_j}{b_j} t \right) \right] \right| dx$$

Now $\varphi(b_j x + t a_j)$ is φ stretched horizontally by a factor of b_j and shifted over a_j/b_j. Since the support of φ is compact and the a_j/b_j are distinct, the supports of these transformations are disjoint for sufficiently large t. When these supports are disjoint we then have

$$\begin{aligned} \int \left| \sum_{j=1}^{k} \varphi(b_j x + t a_j) \right| dx &= \int \sum_{j=1}^{k} |\varphi(b_j x + t a_j)| \, dx \\ &= \|\varphi\|_{L^1} \cdot \sum_{j=1}^{k} \frac{1}{b_j}. \end{aligned}$$

That we can approximate the desired sum for $f \in L^1$ follows from the reverse triangle inequality.

$$\begin{aligned} \left| \int \left| \sum_{j=1}^{k} f(b_j x + t a_j) \right| dx - \int \left| \sum_{j=1}^{k} \varphi(b_j x + t a_j) \right| dx \right| &\le \sum_{j=1}^{k} \int |f(b_j x + t a_j) - \varphi(b_j x + t a_j)| \, dx \\ &= \epsilon \cdot \sum_{j=1}^{k} \frac{1}{b_k}. \end{aligned}$$

□

0.3 Fall 2015

1. Let E be a measurable subset of $[0, 2\pi]$. Assume that $f \in C(\mathbb{R})$ is 1-periodic, i.e. $f(x+1) = f(x)$. Compute

$$\lim_{n \to \infty} \int_E f(nx) \, dx.$$

Solution. We rewrite the integral over E as an integral over $\mathbb{R}$ against the indicator function of E:

$$\int_E f(nx) \, dx = \int f(nx) \chi_E(x) \, dx.$$

Now let $\varphi \in C_0^\infty(\mathbb{R})$. Since $f \in C(\mathbb{R})$ is 1-periodic, it has a 1-periodic continuous primitive F with $F' = f$. By the chain rule we have $[\frac{1}{n}F(nx)]' = f(nx)$. Integration by parts gives

$$\int f(nx)\varphi(x)\, dx = -\frac{1}{n}\int F(nx)\varphi'(x)\, dx.$$

$F(nx)$ is bounded since F is 1-periodic and $\varphi \in C_0^\infty(\mathbb{R})$, so it's integrable. We then have

$$\begin{aligned}\left|\int f(nx)\varphi(x)\, dx\right| &\leq \frac{1}{n}\|F\|_\infty \cdot \|\varphi'\|_{L^1} \\ &\to 0.\end{aligned}$$

Since E is a measurable subset of $[0, 2\pi]$, it has finite measure and $\chi_E \in L^1(\mathbb{R})$. We can then find $\varphi \in C_0^\infty(\mathbb{R})$ with $\|\chi_E - \varphi\|_{L^1} < \epsilon$. Since f is continuous and 1-periodic, it is bounded and we have

$$\begin{aligned}\left|\int f(nx)\chi_E(x)\, dx - \int f(nx)\varphi(x)\, dx\right| &\leq \|f\|_\infty \cdot \|\chi_E - \varphi\|_{L^1} \\ &\leq \|f\|_\infty \cdot \epsilon.\end{aligned}$$

Since $\int f(nx)\varphi(x)\, dx \to 0$, we must have $\int_E f(nx) \to 0$. $\square$

2. Suppose $f \in L^1[0,1]$ and assume that there exists $C > 0$ such that for all measurable subsets $E \subset [0,1]$ we have

$$\int_E |f(x)|\, dx \leq C\mu(E)^{1/2}.$$

Show that $f \in L^p[0,1]$ for $1 \leq p < 2$. Show that the statement fails for $p = 2$ by giving a counterexample.

Proof. We have that

$$|f(x)|^p - 1 \leq \sum_{n=1}^\infty \chi_{\{|f|^p \geq n\}}(x) \leq |f(x)|^p.$$

Since $[0,1]$ is a finite measure space, integrating through this inequality shows that $f \in L^p[0,1]$ if and only if the series

$$\sum_{n=1}^\infty \mu\{|f(x)|^p \geq n\} = \sum_{n=1}^\infty \mu\{|f(x)| \geq n^{1/p}\}.$$

converges. By Chebyshev's inequality and the given hypotheses we have

$$n^{1/p}\mu\{|f| \geq n^{1/p}\} \leq \int_{\{|f| \geq n^{1/p}\}} |f|\, dx \leq C\mu\{|f| \geq n^{1/p}\}^{1/2}.$$

Dividing through by $n^{1/p}\mu\{|f| \geq n^{1/p}\}^{1/2}$ and squaring gives

$$\sum_{n=1}^\infty \mu\{|f(x)| \geq n^{1/p}\} \leq \sum_{n=1}^\infty \frac{C^2}{n^{2/p}},$$

which converges for all $p \in [1, 2)$.

$\square$

3. Show that a function $f : \mathbb{R}^n \to \mathbb{R}^+$ is measurable if and only if $E = \{(x, y) : 0 \leq y \leq f(x)\}$ is a measurable set of $\mathbb{R}^{n+1}$.

Proof. Suppose f is measurable. Then the function $F(x, y) = f(x)$ is a measurable function $\mathbb{R}^{n+1} \to \mathbb{R}$. Since $G(x, y) = y$ is also measurable, $H(x, y) = y - f(x)$ is measurable as the difference of measurable functions. We can then write E as the intersection of two measurable sets:

$$E = G^{-1}[[0, \infty)] \cap H^{-1}[(-\infty, 0]].$$

Thus, E is measurable if f is measurable.

Conversely, suppose that E is a measurable set. Then for any $\alpha \geq 0$ the set $A \cap G^{-1}(\alpha) = F^{-1}[[\alpha, \infty)]$. This shows that F, and therefore f, is measurable. □

4. Let $f \in L^1(\mathbb{R})$ and set

$$f_h(x) = \frac{1}{2h} \int_{x-h}^{x+h} f(t)\ dt, \quad h > 0.$$

Show that $f_h \in L^1(\mathbb{R})$ and $f_h \to f$ in $L^1(\mathbb{R})$.

Proof. Let's integrate f_h. By Tonelli we have

$$\begin{aligned} \int |f_h(x)|\ dx &= \frac{1}{2h} \int \left| \int f(t) \chi_{[x-h, x+h]}(t)\ dt \right| dx \\ &\leq \frac{1}{2h} \int \int |f(t)| \chi_{[t-h, t+h]}(x)\ dx dt \\ &= \|f\|_{L^1}. \end{aligned} \tag{6}$$

Since $f \in L^1(\mathbb{R})$, we have that this quantity is finite and $f_h \in L^1(\mathbb{R})$.

Now since $f \in L^1(\mathbb{R})$, $f_h \to f$ a.e. by the Lebesgue differentiation theorem. By Fatou's lemma and (6), we have for any sequence $h_n \to 0$

$$\begin{aligned} \int |f|\ dx &\leq \liminf_{n\to\infty} \int |f_{h_n}|\ dx \\ &\leq \int |f|\ dx, \end{aligned}$$

so $\liminf_{n\to\infty} \int |f_{h_n}| = \int |f|$. By the triangle inequality we have $|f_{h_n}| + |f| - |f - f_{h_n}| \geq 0$. Since $|f_{h_n}| + |f| - |f - f_{h_n}|$ converges to $2|f|$ a.e., another application of Fatou's lemma gives

$$\begin{aligned} 2 \int |f|\ dx &\leq \liminf_{n\to\infty} \int (|f_{h_n}| + |f| - |f - f_{h_n}|)\ dx \\ \iff \limsup_{n\to\infty} \int |f - f_{h_n}|\ dx &\leq 0. \end{aligned}$$

We then have $\int |f - f_{h_n}| \to 0$, so $f_{h_n} \to f$ in L^1 for any $h_n \to 0$. □

5. Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f_k : X \to \mathbb{R}$ be a sequence of measurable functions satisfying the following:

$$\int_X |f_k|^2 \, d\mu \leq 2015, \quad \text{for all } k,$$

and

$$\int_X f_j f_k \, d\mu = 0, \quad \text{for all } j \neq k.$$

Prove that for all $\beta > 3/2$,

$$\lim_{n\to\infty} \frac{1}{n^\beta} \sum_{k=1}^{n^2} f_k(x) = 0, \quad \text{for a.a. } x \in X.$$

Proof. Let's compute the L^2 norm of the sum

$$\begin{aligned}
\left\| \frac{1}{n^\beta} \sum_{j=1}^{n^2} f_j \right\|_{L^2}^2 &= \frac{1}{n^{2\beta}} \left(\sum_{j=1}^{n^2} f_j, \sum_{k=1}^{n^2} f_k \right) \\
&= \frac{1}{n^{2\beta}} \sum_{j=1}^{n^2} \sum_{k=1}^{n^2} (f_j, f_k) \\
&= \frac{1}{n^{2\beta}} \sum_{j=1}^{n^2} \|f_j\|_{L^2}^2 \\
&\leq \frac{2015}{n^{2\beta-2}}.
\end{aligned}$$

Now if $\beta > 3/2$, $2\beta - 2 > 1$, so the above quantity is summable in n. Summability and wanting to show that something holds for almost all x leads us to think Borel-Cantelli might be useful. For any fixed $\epsilon > 0$, Chebyshev gives us

$$\begin{aligned}
\mu \left\{ x : \left| \frac{1}{n^\beta} \sum_{j=1}^{n^2} f_j \right|^2 \geq \epsilon \right\} &\leq \frac{1}{\epsilon^2} \int_X \left(\frac{1}{n^\beta} \sum_{j=1}^{n^2} f_j \right)^2 dx \\
&\leq \frac{2015}{\epsilon^2 n^{2\beta-2}}.
\end{aligned}$$

If we call the set on the LHS A_n, then we have $\sum \mu(A_n) < \infty$. By Borel-Cantelli we have $\mu(\limsup_{n\to\infty} A_n) = 0$, i.e., the set of x that belong to infinitely many A_n has measure zero, so the sum is zero for almost all x. $\square$

6. Let $A, B \subseteq \mathbb{R}^n$ be Lebesgue measurable sets and assume that for every $x \in \mathbb{Q}^n$ there exists a null set N_x such that

$$A + x \subset B \cup N_x.$$

Show that if A is not a null set then the complement of B in $\mathbb{R}^n$ is a null set.

Proof. Suppose A has positive measure. Since $\mathbb{Q}$ is countable and the countable union of null sets is null, we have that $A + \mathbb{Q} \subset B \cup N$ for some null set N. If $A + \mathbb{Q}$ missed a set of positive measure, then the complement of B would contain a set of positive measure. Let's show that this cannot happen.

Suppose E is a set of positive measure with $E \cap (A + \mathbb{Q}) = \emptyset$. Define the function f by the convolution

$$f(x) = \int_{\mathbb{R}^n} \chi_A(x-y)\chi_E(y)\ dy.$$

If we choose $x = q \in \mathbb{Q}^n$, then the integrand is nonzero if and only if $y \in E \cap (A + q) = \emptyset$, so $f(q) = 0$. But the convolution is continuous if we take E to have finite measure and $\mathbb{Q}^n$ is dense, so we must have $f \equiv 0$. But by Tonelli we have

$$\begin{aligned}\int_{\mathbb{R}^n \times \mathbb{R}^n} \chi_A(x-y)\chi_E(y)\ d(\mu_x \times \mu_y) &= \int \int \chi_A(x-y)\chi_E(y)\ dxdy \\ &= m(A)m(E).\end{aligned}$$

Since A is not null and E is assumed to have positive measure, this must be positive, contradicting $f \equiv 0$. We conclude that $A + \mathbb{Q}$ is null. ☐

0.4 Spring 2015

1. Show that if $f \in L^4(\mathbb{R})$ then

$$\lim_{c \to 1} \int_{\mathbb{R}} |f(cx) - f(x)|^4\ dx = 0.$$

Proof. Suppose φ is continuous with compact support. Then $\varphi(cx)$ converges to $\varphi(x)$ uniformly, and since the support of φ is compact, we have that the desired limit holds with φ in place of f.

Now let $\varphi \in C_0(\mathbb{R})$ be such that $\|f - \varphi\|_{L^4} < \epsilon$. Since $|a + b|^p \le 2^p(|a|^p + |b|^p)$ for all $p > 0$ we have

$$\begin{aligned}\int |f(cx) - f(x)|^4\ dx &= \int |f(cx) - \varphi(cx) + \varphi(cx) - \varphi(x) + \varphi(x) - f(x)|^4\ dx \\ &\le 2^4 \int |f(cx) - \varphi(cx)|^4\ dx \\ &+ 2^8 \int |\varphi(cx) - \varphi(x)|^4\ dx + 2^8 \int |\varphi(x) - f(x)|^4\ dx.\end{aligned}$$

The first and third integrals are small since $\|f - \varphi\|_{L^4} < \epsilon$ and the second integral can be made small as $c \to 1$ since $\varphi(cx) \to \varphi(x)$ uniformly on a compact set. ☐

2. Let $f_n : (0,\infty) \to \mathbb{R}$, be a sequence of Lebesgue measurable functions such that $f_n \to f$ a.e. as $n \to \infty$. Assume that there exists $g : (0,\infty) \to \mathbb{R}$ such that $|f_n| \le g$ for all n and $g \in L^1(0,a)$ for all $0 < a < \infty$. Assume furthermore that

$$\int_1^\infty |f_n(\sqrt{x})| \, dx \le C,$$

for all n and for some constant $C > 0$. Show that $f_n \in L^1(0,\infty)$, $f \in L^1(0,\infty)$ and $f_n \to f$ in $L^1(0,\infty)$ as $n \to \infty$.

Proof. First let's show that $f_n \in L^1(0,\infty)$ for all n. Write

$$\int_0^\infty |f_n| \, dx = \int_0^1 |f_n| \, dx + \int_1^\infty |f_n| \, dx. \tag{7}$$

For the first integral, since $|f_n| \le g$ and $g \in L^1(0,1)$ we have

$$\int_0^1 |f_n| \, dx \le \int_0^1 g \, dx < \infty.$$

For the second integral in (7) we use the hypothesis about $f_n(\sqrt{x})$.

$$\begin{aligned} C &\ge \int_1^\infty |f_n(\sqrt{x})| \, dx \\ &= 2\int_1^\infty t|f_n(t)| \, dt \\ &\ge \int_1^\infty |f_n(t)| \, dt. \end{aligned}$$

Both integrals in (7) are then finite, so $f_n \in L^1(0,\infty)$. In fact, we actually have that the f_n are uniformly bounded in $L^1(0,\infty)$ by $\int_0^1 g \, dx + C$. Since $f_n \to f$ a.e. we can apply Fatou's lemma to show that $f \in L^1(0,\infty)$:

$$\begin{aligned} \int_0^\infty |f| \, dx &\le \liminf_{n\to\infty} \int_0^\infty |f_n| \, dx \\ &\le \int_0^1 g \, dx + C \\ &< \infty. \end{aligned}$$

Our plan is to use the Vitali convergence theorem to show that $f_n \to f$ in $L^1(0,\infty)$. We are given that $f_n \to f$ a.e., which implies that $f_n \to f$ in measure. Since $|f - f_n| \le |f| + g$, we have that $f_n \to f$ in $L^1(0,a)$ for any a by the dominated convergence theorem, so the f_n's are uniformly integrable. To establish tightness, note that for any $t > 1$ we have

$$\begin{aligned} \int_t^\infty |f_n(x)| \, dx &= \int_{t^2}^\infty \frac{|f_n(\sqrt{x})|}{2\sqrt{x}} \, dx \\ &\le \frac{C}{2t}, \end{aligned}$$

which goes to zero as $t \to \infty$. By the Vitali convergence theorem we have that $f_n \to f$ in $L^1(0,\infty)$. $\square$

3. Assume that $f \in C^1(0,1)$ and
$$\int_0^1 x|f'|^p \, dx < +\infty$$
for some $p > 2$. Show that $\lim_{x\to 0^+} f(x)$ exists.

Proof. Let $x_n \to 0$ and say the integral in the problem statement has value $C < \infty$. If q is such that $\frac{1}{p} + \frac{1}{q} = 1$, we have by Hölder's inequality

$$\begin{aligned}
|f(x_n) - f(x_m)| &= \left| \int_{x_m}^{x_n} f'(x) \, dx \right| \\
&\leq \int_{x_m}^{x_n} |f'(x)| \, dx \\
&= \int_0^1 x^{1/p}|f'(x)|x^{-1/p}\chi_{[x_m,x_n]}(x) \, dx \\
&\leq \left(\int_0^1 x|f'(x)|^p \, dx \right)^{1/p} \cdot \left(\int_{x_m}^{x_n} x^{-q/p} \, dx \right)^{1/q}.
\end{aligned}$$

Since $p > 2$, we have that $q < 2$, so the last line above becomes

$$|f(x_n) - f(x_m)| \leq C \cdot \left. \frac{x^{1-q/p}}{1 - q/p} \right|_{x_m}^{x_n}.$$

Since $q < 2$, we have that $1 - \frac{q}{p} > 0$, so as $x_m, x_n \to 0$, this expression goes to zero. Thus, the sequence $f(x_n)$ is Cauchy, so $\lim_{x\to 0} f(x)$ exists. $\square$

4. Suppose that $E \subset [0,1]^2$ is measurable. Denote

$$E_x = \{y \in [0,1] : (x,y) \in E\}, \quad E_y = \{x \in [0,1] : (x,y) \in E\}.$$

Show that if $m(E_x) = 0$ for almost all $x \in [0, \frac{1}{2}]$, then

$$m(\{y \in [0,1] : m(E_y) = 1\}) \leq \frac{1}{2}.$$

Proof. E is contained in the unit square, which has finite measure. By Tonelli's theorem we then have

$$\begin{aligned}
m(E) &= \int \chi_E(x,y)\ d(\mu_x \times \mu_y) \\
&= \int_0^1 \int_0^1 \chi_E(x,y)\ dydx \\
&= \int_0^1 m(E_y)\ dy = \int_0^1 m(E_x)\ dx \\
&= \int_{1/2}^1 m(E_x)\ dx \\
&\le \frac{1}{2}.
\end{aligned}$$

This gives us

$$\begin{aligned}
m(\{y \in [0,1] : m(E_y) = 1\}) &= \int_{\{y\in[0,1]:m(E_y)=1\}} m(E_y)\ dy \\
&\le \int_0^1 m(E_y)\ dy \\
&\le \frac{1}{2}.
\end{aligned}$$

□

5. Let $f \in L^p(\mathbb{R})$, $1 < p < \infty$, and let $\alpha > 1 - \frac{1}{p}$. Show that the series

$$\sum_{n=1}^{\infty} \int_n^{n+n^{-\alpha}} |f(x+y)|\ dy$$

converges for a.e. $x \in \mathbb{R}$.

Proof. Our strategy is to show that the sum, as a function of x, is locally integrable, and therefore finite almost everywhere. To this end, let k be an arbitrary integer. Since the above integrands are nonnegative, the monotone convergence theorem will let us interchange the sum with integrals. By Tonelli we will interchange the integrals.

$$\begin{aligned}
\int_k^{k+1} \sum_{n=1}^{\infty} \int_n^{n+n^{-\alpha}} |f(x+y)|\ dydx &= \sum_{n=1}^{\infty} \int_k^{k+1} \int_n^{n+n^{-\alpha}} |f(x+y)|\ dydx \\
&= \sum_{n=1}^{\infty} \int_k^{k+1} \int |f(y)| \cdot \chi_{[n+x,n+n^{-\alpha}+x]}(y)\ dydx \\
&= \sum_{n=1}^{\infty} \int \int_k^{k+1} |f(y)| \cdot \chi_{[y-n-n^{-\alpha},y-n]}(x)\ dxdy.
\end{aligned}$$

Let's compute the values of y for which $[y-n-n^{-\alpha}, y-n] \cap [k, k+1]$ is nonzero. We need $k < y-n$, so $k+n < y$. We also need $y-n-n^{-\alpha} < k+1$, so $y < k+n+n^{-\alpha}+1$. This gives us

$$\int_k^{k+1} \sum_{n=1}^{\infty} \int_n^{n+n^{-\alpha}} |f(x+y)| \; dydx = \sum_{n=1}^{\infty} \int_{k+n}^{k+n+n^{-\alpha}+1} \int |f(y)| \chi_{[y-n-n^{-\alpha}, y-n]}(x) \; dxdy$$

$$= \sum_{n=1}^{\infty} n^{-\alpha} \int_{k+n}^{k+n+n^{-\alpha}+1} |f(y)| \; dy.$$

Our plan is to use Hölder's inequality with respect to the counting measure on the sequences $n^{-\alpha}$ and $\int_{k+n}^{k+n+n^{-\alpha}+1} |f(y)| \; dy$. Since α is given to be larger than the Hölder conjugate of p, we have that $n^{-\alpha}$ is in ℓ^q. We also have

$$\sum_{n=1}^{\infty} \left(\int_{k+n}^{k+n+n^{-\alpha}+1} |f(y)| \; dy \right)^p$$

$\square$

6. Suppose $E \subset \mathbb{R}$ is measurable and $E = E + \frac{1}{n}$ for every natural number $n \geq 1$. Show that either $m(E) = 0$ or $m(E^c) = 0$.

Proof. By induction we can see that $E = E + \mathbb{Q}$. Suppose E isn't null and $E = E + \mathbb{Q}$ misses a set A of positive finite measure. Consider the consider the convolution

$$f(x) = \int_{\mathbb{R}} \chi_E(x-y)\chi_A(y) \; dy.$$

Since $E + \mathbb{Q} \cap A$ is empty, if $x \in \mathbb{Q}$ then $f(x) = 0$. Furthermore, since A has finite measure and E is in $L^{\infty}(\mathbb{R})$, the convolution is continuous. Since $\mathbb{Q}$ is dense and f, a continuous function vanishes on $\mathbb{Q}$, we must have $f \equiv 0$. But by Tonelli we have that $\int f(x) \; dx = m(E)m(A)$, which is positie. We conclude that E cannot miss a set of positive measure if it isn't null. $\square$

0.5 Fall 2014

1. Let A be the collection of all subsets of $\mathbb{R}$ that consist of exactly 5 points. Find the σ-algebra of sets generated by A.

Solution. By intersecting five element sets with exactly one point in common we can obtain all singleton subsets of $\mathbb{R}$. We claim that the σ-algebra generated by the singleton sets, which will be the σ-algebra generated by A, consists of all countable or co-countable subsets of $\mathbb{R}$.

Call the σ-algebra consisting of all countable or co-countable sets $\mathcal{A}$. Since $\mathcal{A}$ contains all singletons, we clearly have $\sigma(A) \subseteq \mathcal{A}$. Conversely, let $S \in \mathcal{A}$. If S is countable, then it is a countable union of singletons, so $S \in \sigma(A)$. On the other hand, if S is co-countable, then its complement is in $\sigma(A)$. Since $\sigma(A)$ is closed under taking complements, this puts S in $\sigma(A)$ as well. We conclude that $\sigma(A) = \mathcal{A}$. $\square$

2. Assume that $f \in L^1(0,1)$ is a non-negative real-valued function satisfying $\int_{[0,1]} f(x)\,dx = 1$. Show that

$$\int_{[0,1]} \frac{1}{f(x)}\,dx \geq 1.$$

Proof. Since $f \in L^1$ and $f \geq 0$, we have that $\sqrt{f} \in L^2$. We then have by Hölder's inequality

$$\begin{aligned}
1 &= \int 1\,dx \\
&= \int \frac{\sqrt{f}}{\sqrt{f}}\,dx \\
&\leq \left\|\sqrt{f}\right\|_{L^2} \cdot \left\|1/\sqrt{f}\right\|_{L^2} \\
&= \sqrt{\|f\|_{L^1}} \cdot \sqrt{\|1/f\|_{L^1}} \\
&= \sqrt{\|1/f\|_{L^1}}.
\end{aligned}$$

□

3. Denote

$$E = \left\{ x \in [0,1] : \text{there exist infinitely many } p, q \in \mathbb{N} \text{ such that } |x - \frac{p}{q}| \leq \frac{1}{q^3} \right\}.$$

Show that $m(E) = 0$.

Proof. Let $E_{p,q} = \{x \in [0,1] : |x - p/q| \leq 1/q^3\}$ where p, q range over $\mathbb{N}$. Note that since we're confined to $[0,1]$, these sets are empty for $p > q$ for any fixed q. We also have that $m(E_{p,q}) \leq \frac{2}{q^3}$. We can then sum (using Tonelli to sum over p and q individually)

$$\begin{aligned}
\sum_{p,q\in\mathbb{N}} m(E_{p,q}) &= \sum_{q\in\mathbb{N}} \sum_{0\leq p<q} m(E_{p,q}) \\
&\leq \sum_{q\in\mathbb{N}} q \cdot \frac{2}{q^3} \\
&= \frac{\pi^2}{3}.
\end{aligned}$$

Since this sum is finite, by Borel-Cantelli we must have that $m(\limsup E_{p,q}) = 0$. $\limsup E_{p,q}$ is the set of $x \in [0,1]$ belonging to infinitely many $E_{p,q}$, which is exactly the definition of E. □

4. Assume that $\eta \in L^1(\mathbb{R})$ is a non-negative function satisfying $\int_{\mathbb{R}} \eta\,dx = 1$. Show that for any $f \in L^1(\mathbb{R})$,

$$\|f * \eta\|_{L^1} \leq \|f\|_{L^1}.$$

Proof. We use Tonelli's theorem

$$\begin{aligned}\int |(f*\eta)(x)|\ dx &\leq \int\int |f(x-y)|\eta(y)\ dydx \\ &= \int\int |f(x-y)|\eta(y)\ dxdy \\ &= \|f\|_{L^1}\int \eta(y)\ dy \\ &= \|f\|_{L^1}\,.\end{aligned}$$

□

5. Let $f:\mathbb{R}\to\mathbb{R}$ be continuous and periodic with period one. Prove that

$$\lim_{n\to\infty}\int_0^1 f(nx)\cos^2(2\pi x)\ dx = \frac{1}{2}\int_0^1 f(x)\ dx.$$

Proof. The idea is to replace the cosine with the characteristic function of an interval $[a,b]$ and show that

$$\lim_{n\to\infty}\int_0^1 f(nx)\chi_{[a,b]}(x)\ dx = (b-a)\int_0^1 f(x)\ dx.$$

Since the step functions are dense in L^1, we can then apply an approximation argument to show that

$$\lim_{n\to\infty}\int_0^1 f(nx)\cos^2(2\pi x)\ dx = ()$$

□

0.6 Spring 2014

1. Let A be a subset of $\mathbb{R}$ of positive Lebesgue measure. Prove that there exist $k,n\in\mathbb{N}$ and $x,y\in A$ with $|x-y| = k/2^n$.

Proof. The main idea is to show that the difference set $A-A$ contains a neighborhood of the origin. Since the set of dyads, $D=\{k/2^n : k\in\mathbb{Z}, n\in\mathbb{N}\}$, is dense in $\mathbb{R}$, it must intersect the interval inside $A-A$.

Let's show that $A-A$ contains an interval. If we assume that A has positive *finite* measure (just intersect A with $[-N,N]$ for sufficiently large N), then the function $\varphi=\chi_A*\chi_{-A}$ is continuous as the convolution of an L^∞ function with an L^1 function. We see that

$$\varphi(0)=\int_{\mathbb{R}}\chi_A(t)\chi_{-A}(0-t)\ dt = m(A).$$

Since $m(A) > 0$ and φ is continuous, we have that φ is positive on some neighborhood of the origin, say $(-\delta, \delta)$. We claim that $(-\delta, \delta)$ is in $A - A$. If $\varphi(x) > 0$, then the integrand $\chi_A(t)\chi_{-A}(x-t)$ must be nonzero for some t. Then $t \in A$ and $x - t \in -A$, so $x = t + (x - t) \in A - A$.

Now let's show that D is dense in $\mathbb{R}$. Given any $x \in \mathbb{R}$ and any $\epsilon > 0$ we'll show that there is a dyad in the ϵ-neighborhood of x. Choose n such that $\frac{1}{2^n} < \epsilon$ and k such that $k \leq x \cdot 2^n \leq k + 1$. Then $k/2^n$ is in the ϵ-neighborhood of x. ☐

2. Either prove or give a counterexample: If a sequence of functions f_n on a measure space (X, μ) satisfies $\int_X |f_n| \, d\mu \leq \frac{1}{n^2}$, then $f_n \to 0$ μ-a.e.

   Solution. This is true. By the monotone convergence theorem we have that $\int \sum |f_n| = \sum \int |f_n|$. By hypothesis, the second sum is finite, so $\sum |f_n|$ is integrable, and therefore finite a.e.. If this sum is finite a.e. then $|f_n(x)| \to 0$ for almost all x. ☐

3. Let $f \in L^4[a, b]$ and let $F(x) = \int_a^x f(x) \, dx$. Show that $\lim_{h\to 0} \frac{F(x+h)-F(x)}{h^{3/4}} = 0$ for all $x \in (a, b)$.

   Proof. We have that
$$|F(x + h) - F(x)| \leq \int_a^b |f(t)|\chi_{[x,x+h]}(t) \, dt.$$
   The trick here is that we can square the indicator function at no cost. By Hölder we then have
$$\begin{aligned} |F(x + h) - F(x)| &\leq \left\|f \cdot \chi_{[x,x+h]}\right\|_{L^4} \cdot \left\|\chi_{[x,x+h]}\right\|_{L^{4/3}} \\ &= \left\|f \cdot \chi_{[x,x+h]}\right\|_{L^4} \cdot h^{3/4}. \end{aligned}$$
   Now by the absolute continuity of the integral, we can choose h small enough that the first factor on the last line above is small. The result then follows. ☐

4. Assume $f, g \in L^2(\mathbb{R})$. Define
$$A(x) = \int_{\mathbb{R}} f(x - y)g(y) \, dy.$$
   Show that $A(x) \in C(\mathbb{R})$ and
$$\lim_{|x|\to\infty} A(x) = 0.$$
   Proof. By Hölder's inequality we can see that $|A(x)| \leq \|f\|_{L^2} \|g\|_{L^2}$ for all x. As for continuity, we have
$$|A(x + h) - A(x)| \leq \int |[f(x + h - y) - f(x - y)]g(y)| \, dy.$$
   The idea is to approximate f by a continuous function with compact support. ☐

5. Is it possible for a continuous function $f : [0, 1] \to \mathbb{R}$ to have

(a) Infinitely many strict local minima?

Solution. Yes. For example, let $f(x) = x \sin \frac{1}{x}$. f is continuous as $\lim_{x \to 0} f(x) = 0$ since $\sin \frac{1}{x}$ is bounded near the origin. Any local minimum of $\sin \frac{1}{x}$, of which there are infinitely many accumulating at the origin, is a local minimum of f as well. □

(b) Uncountably many strict local minima?

6. Let A be the collection of functions $f \in L^1(X, \mu)$ such that $\|f\|_{L^1} = 1$ and $\int_X f \, d\mu = 0$. Prove that for every $g \in L^\infty(X, \mu)$,

$$\sup_{f \in A} \int_X fg \, d\mu = \frac{1}{2}(\text{ess sup } g - \text{ess inf } g).$$

Proof. □