The LLL Algorithm

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1. Motivation
2. Gram-Schmidt
3. Basis Reduction
4. The LLL algorithm

Two lattices

- Recall that the **lattice**, L, generated by the linearly independent vectors $x_1, x_2, \ldots, x_n \in \mathbb{R}^n$ is the $\mathbb{Z}$-span of these vectors:

$$L = \{c_1 x_1 + c_2 x_2 + \cdots + c_n x_n : c_i \in \mathbb{Z}, 1 \leq i \leq n\}.$$

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- Consider the lattices, L and M, generated by the rows of the matrices X and Y, respectively.

$$X = \begin{bmatrix} -168 & 602 & 58 \\ 157 & -564 & -57 \\ 594 & -2134 & -219 \end{bmatrix}, \quad Y = \begin{bmatrix} -6 & 6 & -4 \\ 9 & 4 & 1 \\ -1 & 8 & 6 \end{bmatrix}.$$

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$$\begin{bmatrix} -168 \\ 602 \\ 58 \end{bmatrix}^T = 14 \begin{bmatrix} 4 \\ 2 \\ -9 \end{bmatrix}^T + 50 \begin{bmatrix} -1 \\ 8 \\ 6 \end{bmatrix}^T - 29 \begin{bmatrix} 6 \\ -6 \\ 4 \end{bmatrix}^T,$$

$$\begin{bmatrix} 157 \\ -564 \\ -57 \end{bmatrix}^T = -13 \begin{bmatrix} 4 \\ 2 \\ -9 \end{bmatrix}^T - 47 \begin{bmatrix} -1 \\ 8 \\ 6 \end{bmatrix}^T + 26 \begin{bmatrix} 6 \\ -6 \\ 4 \end{bmatrix}^T,$$

$$\begin{bmatrix} 594 \\ -2134 \\ -219 \end{bmatrix} = -49 \begin{bmatrix} 4 \\ 2 \\ -9 \end{bmatrix}^T - 178 \begin{bmatrix} -1 \\ 8 \\ 6 \end{bmatrix} + 102 \begin{bmatrix} 6 \\ -6 \\ 4 \end{bmatrix}^T.$$

Two lattices

- In particular, we have the matrix equation

$$UY = X,$$

$$\begin{bmatrix} 14 & 50 & -29 \\ -13 & -47 & 27 \\ -49 & -178 & 102 \end{bmatrix} \begin{bmatrix} 4 & 2 & -9 \\ -1 & 8 & -6 \\ 6 & -6 & 4 \end{bmatrix} = \begin{bmatrix} -168 & 602 & 58 \\ 157 & -564 & -57 \\ 594 & -2134 & -219 \end{bmatrix}.$$

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- $\det U = -1$, so U^{-1} is an integer matrix as well. This gives us another matrix equation, $Y = U^{-1}X$.
- Since the entries of U^{-1} are integers, this equation expresses the rows of Y as integer linear combinations of the rows of X, so $M \subseteq L$.

Two lattices

- Even though the rows of X and Y generate the same lattice, something about the Y-basis "feels" nicer.

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- Two qualities that make a basis desirable are:
  - Length: how long are the basis vectors?
  - Orthogonality: are the basis vectors nearly orthogonal to each other?

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- If $x = c_1x_1 + c_2x_2 + \cdots + c_nx_n$, $c_i \in \mathbb{Z}$, is in the lattice L generated by $x_1, \ldots, x_n$ then

$$|x|^2 = c_1^2|x_1|^2 + c_2^2|x_2|^2 + \cdots + c_n^2|x_n|^2.$$

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- This completely solves the shortest vector problem (SVP) since

$$\arg\min_{x \in L} |x| = \arg\min_{x \in \{\pm x_1, \pm x_2, \ldots, \pm x_n\}} |x|.$$

What makes a basis "nice"?

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$$x = t_1 x_1 + t_2 x_2 + \cdots + t_n x_n,$$

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- If $y = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$, $c_i \in \mathbb{Z}$, is any vector in L then by the orthogonality of the x_i we have

$$|x - y|^2 = (t_1 - c_1)^2 |x_1|^2 + (t_2 - c_2)^2 |x_2|^2 + \cdots + (t_n - c_n)^2 |x_n|^2.$$

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- If we take c_i to be the closest integer to t_i then we solve the closest vector problem (CVP).

Measuring orthogonality

Definition

Let $x_1, \ldots, x_n$ be a basis for the lattice $L \subset \mathbb{R}^n$. We define the determinant of L, $\det L$ to be the volume of the n-dimensional parallelepiped with sides defined by $x_1, \ldots, x_n$:

$$\det L = |\det X|,$$

where the rows of X are the basis vectors $x_1, \ldots, x_n$.

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Theorem

The determinant of L is independent of basis.

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Theorem (Hadamard's Inequality)

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- If the basis vectors are closer to being orthogonal, then Hadamard's inequality is closer to an equality.

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Gram-Schmidt

Definition

Let $x_1, \ldots, x_m \in \mathbb{R}^n$ be a basis for a nonzero subspace, H. The **Gram-Schmidt process** produces an orthogonal basis for H:

$$\begin{aligned} x_1^* &= x_1 \\ x_2^* &= x_2 - \mu_{2,1} x_1^* \\ x_3^* &= x_3 - \mu_{3,1} x_1^* - \mu_{3,2} x_2^* \\ &\vdots \\ x_m^* &= x_m - \mu_{m,1} x_1^* - \cdots - \mu_{m,m-1} x_{m-1}^*, \end{aligned}$$

where $\mu_{i,j} = \frac{x_i \cdot x_j^*}{x_j^* \cdot x_j^*}$. We call $\{x_1^*, \ldots, x_m^*\}$ the **Gram-Schmidt orthogonalization (GSO)** of $\{x_1, \ldots, x_m\}$.

Gram-Schmidt

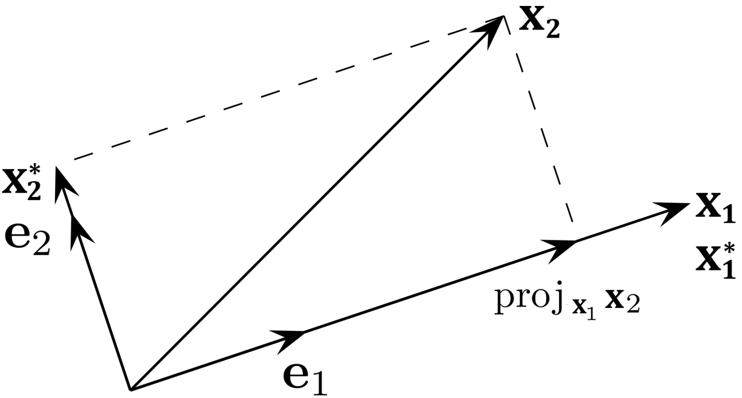

Figure: The first step of the Gram-Schmidt process. Image modified from `https://en.wikipedia.org/wiki/Gram-Schmidt_process`

Gram-Schmidt

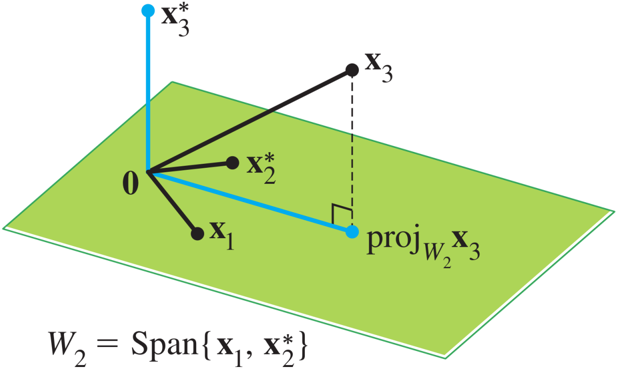

Figure: The second step of the Gram-Schmidt process. Image modified from D. Lay, S. Lay, and J. McDonald. *Linear Algebra and its Applications*. Fifth Edition. 2016.

A useful estimate

Proposition

Let $x_1, x_2, \ldots, x_n$ be a basis for the lattice $L \subset \mathbb{R}^n$ and let $x_1^*, x_2^*, \ldots, x_n^*$ be its Gram-Schmidt orthogonalization. For any nonzero $y \in L$ we have

$$|y| \geq \min\{|x_1^*|, |x_2^*|, \ldots, |x_n^*|\}.$$

That is, any nonzero lattice vector is at least as long as the shortest vector in the Gram-Schmidt orthogonalization.

A useful estimate

Proof

- We can write

$$y = \sum_{i=1}^{n} c_i x_i, \quad c_i \in \mathbb{Z}.$$

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- Since $y \neq 0$, at lease one c_i is nonzero. Let k be the largest index with $c_k \neq 0$.

$$y = \sum_{i=1}^{k} \sum_{j=1}^{i} c_i \mu_{ij} x_j^* = \sum_{j=1}^{k} \left(\sum_{i=j}^{k} c_i \mu_{ij} \right) x_j^*$$

$$= c_k x_k^* + \sum_{j=1}^{k-1} \nu_j x_j^*,$$

for some real ν_j.

A useful estimate

Proof contd...

- Take the norm-squared on both sides.

$$
\begin{aligned}
|y|^2 &= \left| c_k x_k^* + \sum_{j=1}^{k-1} \nu_j x_j^* \right|^2 \\
&= c_k^2 |x_k^*|^2 + \sum_{j=1}^{k-1} \nu_j^2 |x_j^*|^2 \\
&\geq |x_k^*|^2 \\
&\geq \min\{|x_1^*|^2, |x_2^*|^2, \ldots, |x_n^*|^2\}.
\end{aligned}
$$

$\square$

A useful equality

Proposition

If $x_1, \ldots, x_n$ is a basis for the lattice $L \subset \mathbb{R}^n$ and $x_1^*, \ldots, x_n^*$ is its GSO then

$$\det L = \prod_{i=1}^{n} |x_i^*|.$$

Proof

- We have that $\det L = \det X$, where the rows of X are the basis vectors $x_1, \ldots, x_n$.

A useful equality

Proof contd...

- By the definition of the GSO we have $X = MX^*$ where the rows of X^* are the vectors $x_1^*, \ldots, x_n^*$ and M consists of the GSO coefficients:

$$M = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ \mu_{2,1} & 1 & 0 & \cdots & 0 & 0 \\ \mu_{3,1} & \mu_{3,2} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{n,1} & \mu_{n,2} & \mu_{n3} & \cdots & \mu_{n,n-1} & 1 \end{bmatrix}.$$

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- Since M has determinant 1 we have

$$\det L = |\det X| = |\det M| |\det X^*| = \prod_{i=1}^{n} |x_i^*|.$$

Gram-Schmidt

- Given a basis $x_1, \ldots, x_n$ for a lattice $L \subset \mathbb{R}^n$, the GSO vectors $x_1^*, \ldots, x_n^*$ need not live in L since the coefficients $\frac{x_i \cdot x_j^*}{x_j^* \cdot x_j^*}$ need not be integers.

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- Can we salvage the Gram-Schmidt process and come up with a (nearly) orthogonal basis for L?

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Basis reduction

Definition

Let α, $\frac{1}{4} < \alpha < 1$ be a real number. Let $x_1, \ldots, x_n$ be a basis for the lattice $L \subset \mathbb{R}^n$ and let $x_1^*, \ldots, x_n^*$ be its Gram-Schmidt orthogonalization. We say that the basis $x_1, \ldots, x_n$ is α-**reduced** if

1. (size condition) $|\mu_{ij}| \leq \frac{1}{2}$ for all $i \leq j < i \leq n$,
2. (Lovász condition) $|x_i^*|^2 \geq (\alpha - \mu_{i,i-1}^2)|x_{i-1}^*|^2$ for $2 \leq i \leq n$.

Size condition

- $x_2 - \mu_{2,1} x_1$ is orthogonal to x_1, but might not be in the lattice spanned by $x_1, x_2, \ldots, x_n$.

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- The size condition, $|\mu_{ij}| \leq \frac{1}{2}$, then says that $\lceil \mu_{ij} \rfloor = 0$: x_i is already nearly orthogonal to x_j.

Lovász condition

- Assuming the size condition is met, the Lovász condition, $|x_i^*|^2 \geq (\alpha - \mu_{i,i-1}^2)|x_{i-1}^*|^2$ for all $i \geq 2$, says that x_i^* isn't isn't "too much" shorter than x_{i-1}.

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- Rearranging gives $|x_i^* + \mu_{i,i-1}x_{i-1}^*|^2 \geq \alpha|x_{i-1}^*|^2$. This says

$$|\text{Projection of } x_i \text{ onto Span}\{x_1, \ldots, x_{i-2}\}| \geq \alpha|\text{Projection of } x_{i-1} \text{ onto Span}\{x_1, \ldots, x_{i-2}\}|.$$

Shortness

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- Repeatedly applying this to $x_1^* = x_1$ gives

$$|x_1|^2 \leq \beta|x_2^*|^2 \leq \beta^2|x_3^*|^2 \leq \cdots \leq \beta^{n-1}|x_n^*|^2.$$

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- If we let y be any nonzero vector in the lattice spanned by $x_1, \ldots, x_n$ we then have

$$|y| \geq \min\{|x_1^*|, \ldots, |x_n^*|\} \geq \beta^{-(n-1)/2}|x_1|.$$

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$$|y| \geq \min\{|x_1^*|, \ldots, |x_n^*|\} \geq \beta^{-(n-1)/2}|x_1|.$$

- This gives a bound on the first vector in an α-reduced basis in terms of the shortest nonzero vector y in L:

$$|x_1| \leq \beta^{(n-1)/2}|y|.$$

Orthogonality

- If $x_1, \ldots, x_n$ is α-reduced, the Lovász condition gives us

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- Repeated application gives $|x_j^*|^2 \leq \beta^{i-j}|x_i^*|^2$.
- Writing x_i in terms of the GSO, $x_1^*, \ldots, x_n^*$ and applying the above inequality gives

$$|x_i|^2 \leq \beta^{i-1}|x_i^*|^2.$$

Orthogonality

- Multiplying this inequality by itself for $1 \leq i \leq n$ gives

$$\prod_{i=1}^{n} |x_i|^2 \leq \beta^{n(n-1)/2} \prod_{i=1}^{n} |x_i^*|^2 = \beta^{n(n-1)/2} (\det L)^2.$$

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- Taking the square root and using Hadamard's inequality we have

$$\det L \leq \prod_{i=1}^{n} |x_i| \leq \beta^{n(n-1)/4} \det L.$$

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Can we find a reduced basis?

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- Reduced bases are nice. Their vectors are short and nearly orthogonal.
- Does every lattice admit a reduced basis? If it does, can we compute it efficiently?

The LLL algorithm

Algorithm 1 The LLL Algorithm

Input: A basis $\{x_1, \ldots, x_n\}$ of the lattice $L \subset \mathbb{R}^n$ and a reduction parameter $\alpha \in (\frac{1}{4}, 1)$.

Output: An α-reduced basis $\{y_1, \ldots, y_n\}$ of the lattice L.

Copy $x_1, \ldots, x_n$ into $y_1, \ldots, y_n$.
Set $k \leftarrow 2$
while $k \leq n$ **do**
    for $j = k-1, k-2, \ldots, 2, 1$ **do**
        Set $y_k \leftarrow y_k - \lceil \mu_{k,j} \rfloor y_j$.
    if $|y_k^*|^2 \geq (\alpha - \mu_{k,k-1}^2)|y_{k-1}^*|^2$ **then**
        Set $k \leftarrow k+1$.
    else
        Swap y_{k-1} and y_k.
        Set $k \leftarrow \max(k-1, 2)$.
return $\{y_1, \ldots, y_n\}$.

Why swap?

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$$|y_k^*|^2 < (\alpha - \mu_{k,k-1}^2)|y_{k-1}^*|^2.$$

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- Since $\det L_{k-1} = |y_1^*| \cdots |y_{k-1}^*|$, replacing y_{k-1}^* with the shorter y_k^* will decrease $\det L_{k-1}$.
- The swapping step attempts to order the vectors y_i so that the determinants of the sublattices, $\det L_i$, are minimized.