Quiz 7

Student ID Number: Name ____________________
Math 3A, 6PM
Please justify all your answers December 6, 2018
Please also write your full name on the back

1. Is
$$\left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\-2\end{bmatrix} \right\}$$
an orthogonal basis for $\mathbb{R}^3$?

Solution. Yes it is. Call this set $\{v_1, v_2, v_3\}$.
$$\begin{aligned} v_1 \cdot v_2 &= 1(1) + 1(-1) + 1(0) = 0 \\ v_1 \cdot v_3 &= 1(1) + 1(1) + 1(-2) = 0 \\ v_2 \cdot v_3 &= 1(1) + (-1)1 + 0(-2) = 0 \end{aligned}$$
Since all the pairwise dot products are zero the set is orthogonal. An orthogonal set is linearly independent. Since we have three linearly independent vectors in $\mathbb{R}^3$, we have a basis. □

2. True or False? Explain.

   (a) Let $u, v \in \mathbb{R}^n$. If u is orthogonal to v then v is orthogonal to u.

   Solution. True. u is orthogonal to v means that $u \cdot v = 0$. Since $v \cdot u = u \cdot v = 0$, we have that v is orthogonal to u as well. □

   (b) Let A be an $n \times n$ matrix whose columns form an orthonormal basis for $\mathbb{R}^n$. Then $A^T A = I$.

   Solution. Say A has columns $v_1, \cdots, v_n$. Let's compute the product $A^T A$.
$$A^T A = \begin{bmatrix} — & v_1 & — \\ — & v_2 & — \\ & \vdots & \\ — & v_n & — \end{bmatrix} \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} v_1 \cdot v_1 & v_1 \cdot v_2 & \cdots & v_1 \cdot v_n \\ v_2 \cdot v_1 & v_2 \cdot v_2 & \cdots & v_2 \cdot v_n \\ \vdots & & \ddots & \vdots \\ v_n \cdot v_1 & v_n \cdot v_2 & \cdots & v_n \cdot v_n \end{bmatrix}.$$
   Since the columns of A are orthonormal we have that $v_i \cdot v_j = 0$ if $i \neq j$ (orthogonal) and $v_i \cdot v_i = 1$ (normal). Consequently, the diagonal entries in the above product are 1 and every other entry is zero. We then have that $A^T A = I$, the identity matrix. □