Liam Hardiman *November 9, 2018*

260A - Homework 2

Problem 1. Let $(e_n)_{n=1}^{\infty}$ be an orthonormal basis in the Hilbert space H. Let $T : H \to H$ be a linear continuous map such that

$$\sum_{n=1}^{\infty} \|Te_n\|^2$$

converges. Show that there is a sequence $(T_n)_{n=1}^{\infty}$ of linear continuous maps $H \to H$ such that $T_n(H)$ has a finite dimension and $\|T_n - T\| \to 0$ as $n \to \infty$.

Proof. Consider the projection P_m defined by

$$P_m(x) = \langle x, e_1 \rangle e_1 + \cdots + \langle x, e_m \rangle e_m.$$

This function is continuous by an argument similar to the one used on Homework 1, where we showed that every finite dimensional subspace of a normed vector space admits a continuous projection (first we define projections onto the individual components on the space spanned by $e_1, \ldots, e_m$ and then extend these through Hahn-Banach). $\square$