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# 218A - Homework 1

**1-1** Let $X$ be the set of all points $(x, y) \in \mathbb{R}^2$ such that $y = \pm 1$, and let $M$ be the quotient of $X$ by the equivalence relation generated by $(x, -1) \sim (x, 1)$ for all $x \neq 0$. Show that $M$ is locally Euclidean and second-countable, but not Hausdorff.

*Proof.* There is a bijective correspondence between the nonzero real numbers and the elements of $M$ that are not the images of $(0, 1)$ and $(0, -1)$ under the quotient map. Let $0_+$ be the image of $(0, 1)$ under the quotient map and let $0_-$ be the image of $(0, -1)$. Denote by $[r]$ the equivalence class $\{(1, r), (1, -r)\}$, $r \neq 0$.

To show that $M$ is locally Euclidean it suffices to show that each of the subspaces $M \setminus \{0_+\}$ and $M \setminus \{0_-\}$ is homeomorphic to $\mathbb{R}$. Let $f_+ : M \setminus \{0_-\}$ be defined by

$$f_+([x]) = \begin{cases} x & \text{, if } x \neq 0 \\ 0 & \text{, if } [x] = 0_+ \end{cases}$$

$f_+$ is clearly a bijection of sets and continuity follows from the fact that the inverse image of an interval $(a, b)$ is given by

$$(f_+)^-[(a, b)] = \begin{cases} \{[x] : x \in (a, b)\} & \text{, if } 0 \notin (a, b) \\ \{[x] : x \in (a, 0)\} \cup 0_+ \cup \{[x] : x \in (0, b)\} & \text{, otherwise} \end{cases}$$

which is open in $M$ in either case. Continuity of the inverse follows from the same reasoning, where we consider a basis for the topology on $M$ given by sets of the form $\{[x] : x \in (a, b),\ 0 \notin (a, b)\}$ and $\{[x] : x \in (a, 0) \cup (0, a)\} \cup 0_+$.

The basis for the topology on $M$ considered above can be made countable by taking $a$ and $b$ to be rational, so $M$ is second-countable. Finally $M$ is not Hausdorff because $0_+$ and $0_-$ are not separable by open sets. Specifically, any open neighborhood of $0_+$ would contain a set of the form $\{[x] : x \in (-a, 0) \cup (a, 0)\}$ for some $a$, which intersects the neighborhood of $0_-$ given by $\{[x] : x \in (-a, 0) \cup (a, 0)\} \cup \{0_-\}$. ☐

**1-6** Let $M$ be a nonempty topological manifold of dimension $n \geq 1$. If $M$ has a smooth structure, show that it has uncountably many distinct ones.

*Proof.* Since $M$ is a manifold, for any $x \in M$ we can find an open neighborhood $U$ of $x$ and a homeomorphism $\phi : U \to \mathbb{B}^n$ and $\phi(x) = 0$. ☐