# Algebra Qualifying Exam - Fall 2018

1. Let $G$ be a group of size 42.

   (a) Prove that $G$ has a subgroup $H$ of order 6 and any two such subgroups are conjugate in $G$.

   (b) Deduce that $G = H \ltimes N$, where $N$ is a normal subgroup of order 7.

2. Let $p$ be a prime. Show that for any Sylow $p$-subgroup $H \subset GL_n(\mathbb{F}_p)$ there exists a basis in the vector space $V = \mathbb{F}_p^n$ such that $H$ consists of $\mathbb{F}_p$-linear maps given, in that basis, by an upper-triangular matrix with 1 on the diagonal.

3. For a group $G$, let $G_1 = G$ and let $G_{n+1} := [G, G_n]$. We say that $G$ is nilpotent if $G_N = 1$ for some $N$. Prove that if $G$ is a $p$-group, i.e. $|G| = p^r$ for some prime $p$, then $G$ is nilpotent.

4. Let $R \subset \mathbb{Q}$ be the subring in the field of rational numbers, given by the fractions $\frac{a}{b}$ with $a \in \mathbb{Z}$ and $b = 2^k 3^l$ with $k, l \geq 0$. Describe the ideals of $R$. Is $R$ a PID?

5. Let $S = \{a + bi : a, b \in \mathbb{Z}\} \subset \mathbb{C}$ be the ring of Gaussian integers.

   (a) Show that $S$ is a Euclidean domain.

   (b) Find a decomposition of $a = 11 \in S$ into a product of irreducibles in $S$.

   (c) Find a decomposition of $b = 13 \in S$ into a product of irreducibles in $S$.

6. Let $V$ be a nonzero finite-dimensional vector space over the complex numbers.

   (a) If $S$ and $T$ are commuting linear operators on $V$, prove that each eigenspace of $S$ is mapped into itself by $T$.

   (b) Let $A_1, \dots, A_k$ be finitely many linear operators on $V$ that commute pairwise. Prove that they have a common eigenvector in $V$.

   (c) If $V$ has dimension $n$, show that there exists a nested sequence of subspaces

   $$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V,$$

   where each $V_j$ has dimension $j$ and is mapped into itself by each of the operators $A_1, \dots, A_k$.

7. Let $A$ be an $n \times n$ matrix with complex coefficients and assume that every eigenvalue $\lambda$ of $A$ satisfies $Im(\lambda) > 0$. Consider the $(2n) \times (2n)$ matrix

$$B = \begin{bmatrix} A & 0 \\ 0 & \overline{A} \end{bmatrix}.$$

Find the invariant factors of $B$ in terms of invariant factors of $A$ and prove that $B$ is similar to a real-valued matrix.

8. Find the Galois group of $x^6 - 2$ over $\mathbb{Q}$ and over $\mathbb{F}_5$.

9. Let $K/F$ be a finite Galois algebraic extension with no proper intermediate fields. Prove that $[K : F]$ is prime.

10. Let $Q$ denote the quaternion group, i.e.

$$Q = \{\pm 1, \pm i, \pm j, \pm k\}$$

with $i^2 = j^2 = k^2 = ijk = -1$, $-1 \in Z(Q)$ and $(-1)^2 = 1$.

(a) Classify the conjugacy classes of $Q$.

(b) Construct the character table of $Q$.