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233B - Chapter 5 Exercises

Exercise 1. Find all closed points of the real affine plan $\mathbb{A}^2_{\mathbb{R}}$. What are their residue fields?

Solution. □

Exercise 2. Let $f(x,y) = y^2 - x^2 - x^3$. Describe the affine scheme $X = \text{Spec}R/(f)$ set theoretically for the following rings R.

(i) $R = \mathbb{C}[x,y]$

Solution. $y^2 = x^2 + x^3$. It's a self-intersecting elliptic curve. Spec is closed points and zero ideal (the curve itself). To show the latter, show that the polynomial is irreducible. Try to factor it (linear times quadratic). □

(ii) $R = \mathbb{C}[[x,y]]$

Solution. If a power series has a const term it's invertible. Poly factors here $(y - \sqrt{x^2+x^3})(y + \sqrt{x^2+x^3})$. Let $P_\pm = (y \pm \sqrt{x^2+x^3})$. These are prime ideals since they eliminate y upon quotienting, leaving you with $\mathbb{C}[[x]]$, an integral domain. $P_\pm$ both contain the maximal ideal $m = (x,y)$. □

(iii) $R = \mathbb{C}[x,y]_{(x,y)} \subseteq \mathbb{C}(x,y)$. The localization.

Solution. Kill all maximal ideals except (x,y). If $f = \frac{g_1}{h_1} \cdot \frac{g_2}{h_2}$, then $f \cdot h_1 \cdot h_2 = g_1 \cdot g_2$. Polynomials UFD and f is irreducible, so $f|g_1$ or $f|g_2$, in either cases the factorization is trivial. So the zero ideal is prime in the quotient. □

Exercise 3.

(i) X has infinitely many points, and $\dim X = 0$.

Solution. Need infinitely many maximal ideals but no prime ideals. Infinite product of fields. □

(ii) X has exactly one point and $\dim X = 1$.

Solution. Can't happen. Dim 1 gives a chain but only one point. □

(iii) X has two points, and $\dim X = 1$.

Solution. So one maximal ideal and one prime ideal containing it. Unique maximal ideal (local ring) and one prime ideal inside. $\mathbb{C}[[x]]$. Or a localization of $\mathbb{C}[x]$ at $x - \alpha$. □

(iv) $X = \operatorname{Spec}(R)$ with $R \subseteq \mathbb{C}[x]$, and $\dim X = 2$.

Solution. $\mathbb{Z}[x]$ has dimension 2. $(0) \supset (x) \supset (x, 2)$. Or $\mathbb{Q}[x, \pi]$ since π is transcendental. Or $\mathbb{Q}[t_1, \ldots, t_n]$ where the t_i are independent complex transcendentals over $\mathbb{Q}$.
Couldn't happen if R was a $\mathbb{C}$-subalgebra. It'd contain 1 and $\mathbb{C}$. $R \supseteq \mathbb{C}[f]$ where $f \in R$ (of minimal positive degree?). □

Problem 6. X =union of three coord lines in $\mathbb{C}^3$. $Y = \{(x, y) \in \mathbb{C}^2 : xy(x - y) = 0\}$ the union of three concurrent lines in $\mathbb{C}^2$. Are X and Y isomorphic as schemes? Use tangent spaces at origin.

Solution. □