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271B - Homework 5

Problem 1. Consider

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t, \quad X_0 = 1, \tag{1}$$

with $\mu(x) = x + a$, $\sigma(x) = 4x$. Assuming $X_t > 0$, find dY_t when $Y_t = \sqrt{X_t}$. Can you find Y_t?

Solution. Let $g(t, x) = \sqrt{x}$ so that $Y_t = g(t, X_t)$. By Itô's lemma we have

$$\begin{aligned} dY_t &= \frac{1}{2}X_t^{-1/2}dX_t - \frac{1}{8}X_t^{-3/2}(dX_t)^2 \\ &= \frac{1}{2}X_t^{-1/2}[(X_t + a)dt] - \frac{1}{8}X_t^{-3/2}(16X_t^2 dt) \\ &= \frac{a - 3Y_t^2}{2Y_t}dt + 2Y_t dB_t. \end{aligned}$$

Using dY_t to find Y_t proved difficult. Finding X_t and taking the square root worked however. We use the stochastic version of integrating factors. Define the function

$$F_t = \exp\left(\frac{1}{2}\int_0^t 16 \ ds - \int_0^t 4 \ dB_s\right) = \exp(8t - 4B_t).$$

From this we get

$$dF_t = 16F_t dt - 4F_t dB_t.$$

We apply the multivariable Itô lemma to obtain (after some tedious algebra)

$$\begin{aligned} d(X_t F_t) &= X_t dF_t + F_t dX_t + d\langle X_t, F_t\rangle \\ &= F_t(a + X_t)dt. \end{aligned}$$

Setting $Z_t = X_t F_t$, we obtain the linear DE

$$\frac{dZ_t}{dt} - Z_t = aF_t.$$

We multiply through by e^{-t} to obtain

$$\frac{d}{dt}(Z_t e^{-t}) = ae^{7t - 4B_t}.$$

Integrating through and substituting X_t back in gives

$$X_t = \frac{1 + \int_0^t \exp(7s - 4B_s)ds}{\exp(7t - 4B_t)}.$$

Taking the square root gives Y_t.

$$Y_t = \left(\frac{1 + \int_0^t \exp(7s - 4B_s)ds}{\exp(7t - 4B_t)}\right)^{1/2}.$$

□

Problem 2. Let X_t be as in (1) but with $\mu(x) = 2x$ and $\sigma(x) = x^a$ and and $Y_t = X_t^b$. Find b so that $\langle Y \rangle_t$ is linear in t.

Solution. By Itô's lemma we have

$$dY_t = bX^{b-1}dX_t + \frac{1}{2}b(b-1)X^{b-2}(dX_t)^2$$

$$= f(X_t)dt + bX_t^{a+b-1}dB_t,$$

for some function f. Consequently we have

$$d\langle Y \rangle_t = (bX_t^{a+b-1})^2 dt.$$

From this we compute the quadratic variation:

$$\langle Y \rangle_t = \int_0^t d\langle Y \rangle_t = b^2 \int_0^t X_s^{2a+2b-2} ds.$$

Setting $b = 1 - a$ makes the exponent in the integrand zero, which makes $\langle Y \rangle_t$ linear in t. □

Problem 3. Let

$$dX_t = \sqrt{1 + X_t}\, dB_t, \quad X_0 = 0.$$

Find $\mathbb{E}[X_t]$ and $\mathbb{E}[X_t^2]$.

Solution. □