*Liam Hardiman* *October 23, 2019*

# 271A- Homework 2

1. Use Jensen's inequality to show that for $p \geq 1$.

$$\|\mathbb{E}[X|\mathcal{G}]\|_{L^p} \leq \|X\|_{L^p}.$$

*Proof.* Let's look at the $p$-norm to the $p$-th power.

$$\begin{aligned}
\|\mathbb{E}[X|\mathcal{G}]\|_{L^p}^p &= \int |\mathbb{E}[X|\mathcal{G}]|^p \, d\mathbb{P} \\
&\leq \int \mathbb{E}[|X|^p|\mathcal{G}] \, d\mathbb{P} \quad \text{(by Jensen's inequality)} \\
&= \int |X|^p \, d\mathbb{P} \quad \text{(by definition of conditional expectation)} \\
&= \|X\|_{L^p}^p.
\end{aligned}$$

Taking the $p$-th root of both sides establishes the claim. □

2. Let $(X_n : n \in \mathbb{N})$ be a sequence of independent random variables, each exponentially distributed:

$$\mathbb{P}[X_n > x] = e^{-x}, \quad x \geq 0.$$

(a) A random variable $\tau$ has the lack of memory property if

$$\mathbb{P}[\tau > a + b \mid \tau > a] = \mathbb{P}[\tau > b].$$

Show that a random variable has the memoryless property if and only if it is exponentially distributed.

*Proof.* Suppose $\tau$ is exponentially distributed, i.e.

$$\mathbb{P}[\tau > x] = \begin{cases} e^{-\lambda x}, & \text{if } x \geq 0, \\ 1, & \text{if } x < 0. \end{cases}$$

By the definition of conditional probability we have

$$\mathbb{P}[\tau > a + b \mid \tau > a] = \frac{\mathbb{P}[(\tau > a + b) \wedge (\tau > a)]}{\mathbb{P}[\tau > a]}$$

Now if $b \geq 0$, $\mathbb{P}[(\tau > a+b) \wedge (\tau > a)] = \mathbb{P}[\tau > a+b]$. This gives

$$\begin{aligned}
\mathbb{P}[\tau > a+b \mid \tau > a] &= \frac{\mathbb{P}[\tau > a+b]}{\mathbb{P}[\tau > a]} \\
&= \frac{e^{-\lambda(a+b)}}{e^{-\lambda a}} \\
&= e^{-\lambda b} \\
&= \mathbb{P}[\tau > b].
\end{aligned}$$

On the other hand, if $b < 0$, $\mathbb{P}[(\tau > a+b) \wedge (t > a)] = \mathbb{P}[\tau > a]$, which gives

$$\begin{aligned}
\mathbb{P}[\tau > a+b \mid \tau > a] &= \frac{\mathbb{P}[\tau > a]}{\mathbb{P}[\tau > a]} \\
&= 1 \\
&= \mathbb{P}[\tau > b].
\end{aligned}$$

Conversely, suppose that $\tau$ is memoryless. If $b$ is positive then we have

$$\mathbb{P}[\tau > a+b] = \mathbb{P}[\tau > a] \cdot \mathbb{P}[\tau > b].$$

If we let $F(x) = \mathbb{P}[\tau > x]$, then $F$ satisfies the exponential property:

$$F(a+b) = F(a)F(b).$$

Setting $a = b$, we have $F(2a) = F(a)^2$. Inductively, we obtain $F(na) = F(a)^n$ for any positive integer $n$. Taking the $n$-th root of both sides gives $F(a/n) = F(a)^{1/n}$. Combining these gives $F(\frac{m}{n}a) = F(a)^{m/n}$. For any rational $\frac{m}{n}$.

Since any real number is a limit of rational numbers, we obtain $F(ra) = F(a)^r$ for any real $r$ by continuity. Since this holds for any $a \geq 0$, we can set $a = 1$ to obtain $F(r) = F(1)^r$, so $F$, the distribution of $\tau$, is exponential. $\square$

(b) Compute the expectation and variance of $X_n$. Let $Y = X_n + X_{n+1}$. Find the correlation coefficient between $Y$ and $X_n$. Find $\mathbb{E}[Y|X_{n+1}]$.

*Solution.* Let's compute the first two moments.

$$\begin{aligned}
\mathbb{E}[X_n] &= \int_0^\infty e^{-x}\, dx \\
&= 1.
\end{aligned}$$

$$
\begin{aligned}
\mathbb{E}[X_n^2] &= 2\int_0^\infty xe^{-x}\ dx \\
&= 2.
\end{aligned}
$$

We then have $\mathbb{E}[X_n] = 1$ and $\mathrm{Var}[X_n] = \mathbb{E}[X_n^2] - \mathbb{E}[X_n]^2 = 1$.

Let $Y = X_n + X_{n+1}$. We'll need the mean and variance of $Y$ to compute the correlation coefficient.

$$\mathbb{E}[Y] = \mathbb{E}[X_n] + \mathbb{E}[X_{n+1}] = 2.$$

Since $X_n$ and $X_{n+1}$ are independent, we also have

$$\mathrm{Var}[Y] = \mathrm{Var}[X_n] + \mathrm{Var}[X_{n+1}] = 2.$$

Now let's compute the correlation coefficient.

$$
\begin{aligned}
\rho_{Y,X_{n+1}} &= \frac{\mathbb{E}[(Y-\mu_Y)(X_{n+1}-\mu_{n+1})]}{\sigma_Y\sigma_{n+1}} \\
&= \frac{\mathbb{E}[YX_{n+1}] - \mu_{n+1}\mathbb{E}[Y] - \mu_Y\mathbb{E}[X_{n+1}] + \mu_Y\mu_{n+1}}{\sqrt{2}} \\
&= \frac{\mathbb{E}[X_nX_{n+1}] + \mathbb{E}[X_{n+1}^2] - 2 - 2 + 2}{\sqrt{2}} \\
&= \frac{1}{\sqrt{2}}.
\end{aligned}
$$

Now for the conditional expectation. Since conditional expectation is linear and $X_n$ and $X_{n+1}$ are independent,

$$
\begin{aligned}
\mathbb{E}[Y|X_{n+1}] &= \mathbb{E}[X_n|X_{n+1}] + \mathbb{E}[X_{n+1}|X_{n+1}] \\
&= \mathbb{E}[X_n] + X_{n+1} \\
&= 1 + X_{n+1}.
\end{aligned}
$$

□

(c) Show that

$$\mathbb{P}[X_n > \alpha\log n \text{ for infinitely many } n] = \begin{cases} 0 & \text{for } \alpha > 1, \\ 1 & \text{else} \end{cases}.$$

*Proof.* Let $E_n$ be the event $E_n = \{X_n > \alpha\log n\}$. Let's sum these events

$$
\begin{aligned}
\sum_{n=1}^\infty \mathbb{P}[E_n] &= \sum_{n=1}^\infty e^{-\alpha\log n} \\
&= \sum_{n=1}^\infty \frac{1}{n^\alpha}.
\end{aligned}
$$

This is summable if and only if $\alpha > 1$. By Borel-Cantelli, we have

$$\mathbb{P}[\limsup E_n] = \mathbb{P}[X_n > \alpha \log n \text{ for infinitely many } n] = 0.$$

Since the $X_n$'s are given to be independent, if the above sum diverges, $\mathbb{P}[\limsup E_n] = 1$. $\square$