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271C - Homework 1

Problem 1. Find the solution to

$$dX_t^{(1)} = X_t^{(2)} dt + \sigma^{(1)} dB_t^{(1)}$$

$$dX_t^{(2)} = X_t^{(1)} dt + \sigma^{(2)} dB_t^{(2)}$$

$$X_0^{(1)} = 1, \quad X_0^{(2)} = 0,$$

with B correlated Brownian motions: $d\langle B^{(1)}, B^{(2)}\rangle_t = \rho$ and $\sigma^{(j)}$ constants.

Solution. We write the SDE in matrix-vector form

$$dX_t = AX_t\ dt + \Sigma\ dB_t, \quad X_0 = [1\ 0]^T, \tag{1}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sigma^{(1)} & 0 \\ 0 & \sigma^{(2)} \end{bmatrix}.$$

We multiply (1) through by the integrating factor $\exp(-At)$.

$$e^{-At} dX_t = e^{-At} AX_t\ dt + e^{-At}\Sigma\ dB_t. \tag{2}$$

Now by Itô we have

$$d(e^{-At}X_t) = -Ae^{-At}X_t\ dt + e^{-At}\ dX_t.$$

Substituting this into (2) and integrating gives

$$e^{-At}X_t - X_0 = \int_0^t e^{-As}\Sigma\ dB_s.$$

From this we deduce

$$X_t = e^{At}X_0 + \int_0^t e^{A(t-s)}\Sigma\ dB_s.$$

□

Problem 2. Consider a geometric Brownian motion X solving

$$X_t = x_0 + \mu \int_0^t X_s\ ds + \sigma \int_0^t X_s\ dB_s,$$

with $x_0 > 0$. Show that we have a strong solution satisfying:

(i) if $\mu < \sigma^2/2$ then $\lim_{t\to\infty} X_t = 0$, $\sup_{0\le t<\infty} X_t < \infty$ a.s.

(ii) if $\mu > \sigma^2/2$ then $\inf_{0\le t<\infty} X_t > 0$, $\lim_{t\to\infty} X_t = \infty$ a.s.

(iii) if $\mu = \sigma^2/2$ then $\inf_{0\le t<\infty} X_t = 0$, $\sup_{0\le t<\infty} X_t = \infty$ a.s.

Proof. Differentiating the given integral equation gives

$$dX_t = \mu X_t \ dt + \sigma X_t \ dB_t.$$

By Itô's lemma we have

$$\begin{aligned} d(\log X_t) &= \frac{1}{X_t} \ dX_t - \frac{1}{2} \cdot \frac{1}{X_t^2} \ d\langle X \rangle_t \\ &= \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma \ dB_t \end{aligned}$$

Integrating through gives

$$X_t = x_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma B_t\right].$$

Since

$$|\mu x| + |\sigma x| \le (|\mu| + |\sigma|)|x|$$

and

$$|\mu x - \mu y| + |\sigma x - \sigma y| \le (|\mu| + |\sigma|)|x - y|,$$

our SDE has a unique strong solution. Now the law of the iterated logarithm states that

$$\limsup_{t\to\infty} \frac{B_t}{\sqrt{2t \log\log t}} = 1 \quad \text{and} \quad \liminf_{t\to\infty} \frac{B_t}{\sqrt{2t \log\log t}} = -1 \text{ a.s.}$$

If $\mu \neq \sigma^2/2$, then the $(\mu - \sigma^2/2)t$ term will dominate the σB_t term asymptotically. If $\mu < \sigma^2/2$ then the exponential will tend to zero a.s. Continuity then implies that $\sup_{0\le t<\infty} X_t < \infty$ a.s. If $\mu > \sigma^2/2$ then the exponential will tend to infinity a.s. Continuity then implies that $\inf_{0\le t<\infty} X_t > 0$. Finally, if $\mu = \sigma^2/2$, then since B_t almost surely attains arbitrarily large positive and negative values, $\inf_{0\le t<\infty} X_t = 0$ and $\sup_{0\le t<\infty} X_t = \infty$. $\square$

Problem 3. Let $u(t,x)$ be the smooth solution of the terminal PDE

$$\begin{cases} \partial_t u(t,x) + \frac{1}{2}\sigma^2(x)\partial_x^2 u(t,x) + \mu(x)\partial_x u(t,x) + c(x)u(t,x) = 0 \\ u(T,x) = h(x) \end{cases}, \quad t < T, \ x \in \mathbb{R},$$

with μ, σ, c, h bounded and smooth. Let X_t be the Itô process defined by

$$X_t = x + \int_0^t \mu(X_s) \ ds + \int_0^t \sigma(X_s) \ dB_s.$$

Show that

$$Y_t = \exp\left[\int_0^t c(X_s) \ ds\right] u(t, X_t)$$

is a martingale. Deduce that

$$u(t, X_t) = E\left\{\exp\left[\int_t^T c(X_s) \ ds\right] h(X_T) \mid \mathcal{F}_t\right\}.$$

Proof. By Itô we have

$$
\begin{aligned}
dY_t &= e^{\int_0^t c(X_s)ds}\left[(c(X_t)u(t,X_t)+\partial_t u(t,X_t))dt+\partial_x u(t,x_t)\ dX_t+\frac{1}{2}\partial_x^2 u(t,X_t)\ d\langle X\rangle_t\right]\\
&= e^{\int_0^t c(X_s)ds}\left[\left(c(X_t)u(t,X_t)+\partial_t u(t,X_t)+\mu(X_t)\partial_x u(t,X_t)+\frac{1}{2}\sigma^2(X_t)\partial_x^2 u(t,X_t)\right)dt\right.\\
&\left.+\sigma(X_t)\partial_x u(t,X_t)\ dB_t\right]\\
&= e^{\int_0^t c(X_s)ds}\sigma(X_t)\partial_x u(t,X_t)\ dB_t.
\end{aligned}
$$

Since σ and c are bounded and smooth and u is smooth, the quantity in front of the dB_t is in class I, so Y_t is a martingale.

Now since Y_t is a martingale we have

$$
\begin{aligned}
E\left\{\exp\left[\int_t^T c(X_s)\ ds\right]h(X_T)\mid \mathcal{F}_t\right\} &= E\left\{\exp\left[\int_0^T c(X_s)\ ds\right]\exp\left[-\int_0^t c(X_s)\ ds\right]u(T,x)\mid\mathcal{F}_t\right\}\\
&= \exp\left[-\int_0^t c(X_s)\ ds\right]E\left\{\exp\left[\int_0^T c(X_s)\ ds\right]u(T,x)\mid\mathcal{F}_t\right\}\\
&= \exp\left[-\int_0^t c(X_s)\ ds\right]\exp\left[\int_0^t c(X_s)\ ds\right]u(t,x)\\
&= u(t,x).
\end{aligned}
$$

□

Øksendal 5.15 Consider the nonlinear SDE

$$dX_t = rX_t(K-X_t)\ dt+\beta X_t\ dB_t;\quad X_0=x>0. \tag{3}$$

Verify that

$$X_t=\frac{\exp[(rK-\frac{1}{2}\beta^2)t+\beta B_t]}{x^{-1}+r\int_0^t\exp[(rK-\frac{1}{2}\beta^2)s+\beta B_s]\ ds};\quad t\geq 0 \tag{4}$$

is the unique strong solution of (3).

Proof. We follow Øksendall 5.16, which we proved on our final exam last quarter. Define the integrating factor $F_t=\exp(\frac{1}{2}\beta^2 t-\beta B_t)$ and set $Y_t=F_tX_t$. By exercise 5.16 we have

$$dY_t=d(F_tX_t)=F_t[rX_t(K-X_t)]\ dt=F_t\left[r\frac{Y_t}{F_t}\left(K-\frac{Y_t}{F_t}\right)\right]\ dt.$$

Dividing through by $-Y_t^2$ we get

$$-\frac{dY_t}{Y_t^2}=\left(-\frac{rK}{Y_t}+\frac{r}{F_t}\right)dt.$$

Since dY_t has no martingale term, we have

$$d\left(\frac{1}{Y_t}\right) = -\frac{1}{Y_t^2}\ dY_t = \left(-\frac{rK}{Y_t} + \frac{r}{F_t}\right) dt.$$

This is a linear ODE in $\frac{1}{Y_t}$, so it has a unique strong solution. Since $X_t = Y_t/F_t$, we have a unique strong solution for X_t as well. It remains to show that the solution is indeed (4).

To deal with the rK factor, we introduce the integrating factor e^{rKt}.

$$\begin{aligned} d\left(\frac{e^{rKt}}{Y_t}\right) &= \frac{rKe^{rKt}}{Y_t}\ dt - \frac{e^{rKt}}{Y_t^2}\ dY_t \\ &= \frac{re^{rKt}}{F_t}\ dt. \end{aligned}$$

From this we deduce that

$$\frac{e^{rKt}}{Y_t} = \frac{1}{Y_0} + r\int_0^t \exp\left[\left(rk - \frac{1}{2}\beta^2\right)t + \beta B_s\right]\ ds.$$

Using $X_t = Y_t/F_t$ gives (4). $\square$