Final Exam – Math 130B

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Instructions:

- You must show your work and clearly explain your line of reasoning.
- You need to solve FOUR of the five problems below. The maximum score is 100
- The exam is 2 hours, but you will automatically get extra half an hour for scanning and uploading your solution (and making sure that the file is readable. If not, then please rewrite).

GOOD LUCK!

1. Let $X, Y \sim U[0,1]$, be independent and let $Z = \max\{X, Y\}$.

   (a) (10 points) Calculate $\Pr[Z \leq a]$.

   (b) (10 points) Calculate the density function of Z.

   (c) (5 points) Calculate $Var(Z)$.

2. Video projector light bulbs are known to have a mean lifetime of $\mu = 100$ hours and standard deviation $\sigma = 75$ hours. The university uses the projectors for 9000 hours per semester.

   (a) (15 points) Use the central limit theorem to estimate the probability that 100 light bulbs will last the whole semester?

   (b) (10 points) Explain how to estimate the number of light bulbs necessary to have a 1% chance of running out of light bulbs before the semester ends. Don't actually do the whole computation.

3. Suppose that X is random variable with a moment generating function $M_X(t) = \left(\frac{1+e^{100t}}{2}\right)^n$.

   (a) (10 points) Find $\mathbb{E}[X]$.

   (b) (7 points) Find $\mathbb{E}[X^2]$ and then compute $Var(X)$.

   (c) (5 points) What is the probability that $X \geq 10\mathbb{E}[X]$? Use Chebyshev's inequality to upper bound this quantity.

   (d) (3 points) Do you see how to obtain a better bound than the one you obtained with Chebyshev's? if yes, then explain in words without computing.

4. Let X be any set, and let $\mathcal{F}$ be a collection of subsets of X, each of size exactly n (you can assume that n is sufficiently large if needed).

   (a) (15 points) Prove that if $|\mathcal{F}| \leq 2^{n-1} - 1$ then there exists a partition $X = X_1 \cup X_2$ such that for all $F \in \mathcal{F}$ we have that $F \cap X_1 \neq \emptyset$ and $F \cap X_2 \neq \emptyset$.

   (b) (10 points) Prove, using Chernoff's bounds, that if $|\mathcal{F}| \leq n^{100}$, then there exists a partition $X = X_1 \cup X_2$ such that for all $F \in \mathcal{F}$ we have $\left||F \cap X_1| - \frac{n}{2}\right| \leq C\sqrt{n \log n}$, for some fixed constant $C > 0$ that doesn't depend on n.

5. You and your friend are playing a game. You start by selecting a number X uniformly at random from $[0,1]$. Your friend picks numbers $Y_1, Y_2, \ldots$ uniformly in $[0,1]$ until they pick a number larger than $X/2$.

   (a) (10 points) Find the expected number of times, N, your friend needs to pick a number. *Hint: Condition on $X = x$.*

   (b) (10 points) Find the expected sum of your friend's numbers given that they had to pick N numbers.

   (c) (5 points) Find the expected sum of your friend's numbers.