1. Every day, Alice comes to the bus stop exactly at 7am. She takes the first bus that arrives. The arrival of the first bus is an exponential random variable with expectation 20 minutes. Also, every working day, and independently, Bob comes to the same bus stop at a random time, uniformly distributed between 7 and 7:30am.

   (a) (10 points) What is the probability that tomorrow Alice will wait for more than 30 minutes?

   (b) (15 points) Assume day-to-day independence. Consider Bob late if he comes after 7:20. What is the probability that Bob will be late on 2 or more working days among the next 10 working days?

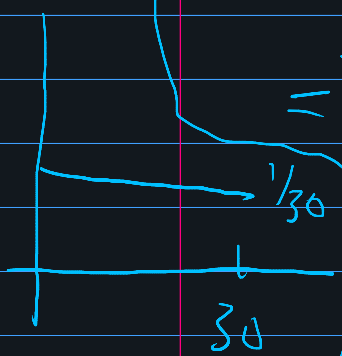
   (c) (15 points) What is the probability that Alice and Bob will meet at the station tomorrow?

a) Let X be the bus arrival time.

$$\Pr[X \geq 30] = \int_{30}^{\infty} \frac{1}{20} e^{-x/20}\,dx$$

$$= -e^{-x/20}\Big|_{30}^{\infty} = e^{-3/2}$$

b) let Y = Bob's arrival time.



$$\Pr[\text{Bob is late today}]$$

$$= \Pr[Y \geq 20] = \int_{20}^{30} \frac{1}{30}\,dy = \frac{1}{30}(30-20)$$

$$= 1/3$$

∴ # of late days in next 10 days is a binomial RV. $n=10$, $p=1/3$

$$\Pr[\geq 2 \text{ successes in } 10 \text{ trials}]$$

$$= \sum_{k=2}^{10} \binom{10}{k} \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{10-k}$$

$\square$

c) They meet iff Bob gets there before the bus, ie, iff $Y \leq X$. We want

$\Pr[Y \leq X]$

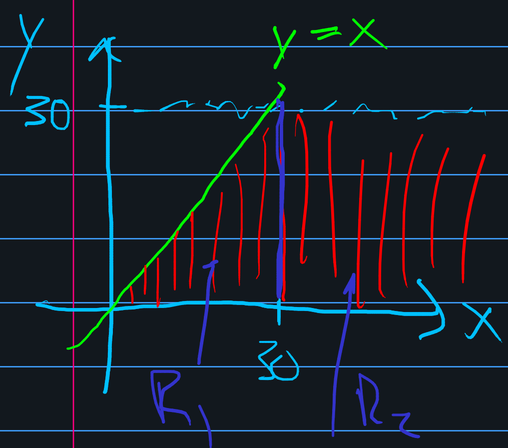
$$= \int_{y \leq x} \underbrace{f_{X,Y}(x,y)}_{\text{joint density}} \, dy\,dx$$

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

$$= \frac{1}{20} e^{-x/20} \cdot \frac{1}{30}$$

$0 \leq x < \infty$
$0 \leq y \leq 30$

$$\int_{y \leq x} \frac{1}{20 \cdot 30} e^{-x/20} \, dy\,dx = \int_0^{30} \int_y^{\infty} \frac{1}{30} \cdot \frac{1}{20} e^{-x/20} \, dx\,dy$$



= region of interest

$$P = \int_{R_1} + \int_{R_2}$$

$$\int_{R_1} = \int_0^{30}\int_0^{x} \frac{1}{30}\cdot\frac{1}{20} e^{-x/20}\, dydx$$

$$\int_{R_2} = \int_{30}^{\infty}\int_0^{30} \frac{1}{30}\cdot\frac{1}{20} e^{-x/20}\, dydx$$

2. Let X, Y, and Z be independent random variables, each of which is uniformly distributed on the interval $[0, 1]$.

(a) (15 points) Find the joint density function of XY and Z^2, and compute $\Pr[XY < Z^2]$.

(b) (15 points) Compute $Var(XY + Z)$.

a) Let $U = XY$, $V = Z^2$. want

$$f_{U,V}(u,v) = f_U(u)\, f_V(v)$$

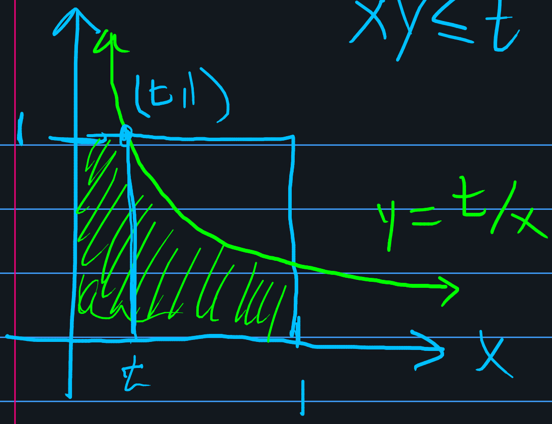
Since $U \perp V$ because $Z \perp X, Y$

$$f_U : \Pr[XY \le t] = \int_{xy \le t} f_{X,Y}(x,y)\, dydx$$

$$= \int_{xy \le t} f_X(x) f_Y(y)\, dydx = \int_{\substack{xy \le t \\ x,y \in [0,1]^2}} dydx = (*)$$

Y

$$xy \le t \iff y \le t/x$$



$$\Rightarrow |X| = \int_0^t \int_0^1 dy\,dx + \int_t^1 \int_0^{t/x} dy\,dx$$

$$= t + \int_t^1 t/x \, dx$$

$$= t + t \log x \Big|_t^1 = t - t\log t$$

$$\Rightarrow \underline{f_U(u)} = \frac{d}{du} \Pr[XY \le u]$$

$$= \frac{d}{du}\left(u - u \log u\right)$$

$$= 1 - u \cdot \frac{1}{u} - \log u = -\log u$$

$$= \underline{\log(1/u)}$$

$f_V(v)$: $\Pr[Z^2 \le v] = \Pr[-\sqrt{v} \le Z \le \sqrt{v}]$

$$= \Pr[Z \le \sqrt{v}] = \sqrt{v} \quad \text{for } v \in [0,1]$$

$$\Rightarrow f_V(v) = \frac{d}{dv} \Pr[Z^2 \le v] = \frac{d}{dv}\sqrt{v} = \underline{\frac{1}{2\sqrt{v}}}$$

$$\Rightarrow f_{U,V}(u,v) = \frac{1}{2\sqrt{v}} \log(1/u)$$

$$\uparrow\ U \perp\!\!\!\perp V$$

b) $\mathrm{Var}[XY + Z]$

$$= \mathrm{Var}[XY] + \mathrm{Var}[Z] + \underbrace{\cancel{2\mathrm{Cov}(XY, Z)}}_{0}$$

since $Z \perp\!\!\!\perp XY$

$\mathrm{Var}[Z] = 1/12$ $\qquad$ $\mathrm{Var}[\mathrm{Unif}[a,b]] = \frac{1}{12}(b-a)^2$

$$\mathrm{Var}[XY] = \mathbb{E}[(XY)^2] - \mathbb{E}[XY]^2$$

$$= \mathbb{E}[X^2Y^2] - \mathbb{E}[XY]^2$$

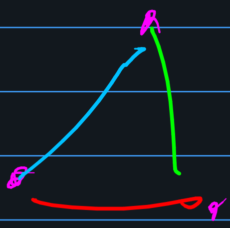
$X \perp\!\!\!\perp Y$ $\rightarrow$ $= \mathbb{E}[X^2]\,\mathbb{E}[Y^2] - \underbrace{\mathbb{E}[X]}_{\frac{1}{2}}{}^2\,\underbrace{\mathbb{E}[Y]}_{\frac{1}{2}}{}^2$

$$\boxed{\mathbb{E}[f(X)g(Y)] = \mathbb{E}[f(X)]\,\mathbb{E}[g(Y)]}$$

$$\underbrace{\mathrm{Var}[Y]}_{\frac{1}{12}} = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2$$

$$= \mathbb{E}[Y^2] - \frac{1}{4}$$

3. (30 points) Let K_n be the *complete graph* on n vertices; that is, its edge-set consists of all $\binom{n}{2}$ possible unordered pairs of vertices. Suppose that some coloring of the edge-set of K_n is given. A triangle is called *rainbow* if it has at most one edge from each color. Show that there exists a coloring of the edges of K_n using three colors with at least $\binom{n}{3}\frac{2}{9}$ rainbow triangles.



Color K_n randomly, let $X = \#\{\text{rainbow triangles}\}$.

$$\mathbb{E}[X] = (\#\text{triangles}) \cdot \Pr[\text{a given triangle is rainbow}]$$

Pf: for each triangle T, let X_T be the indicator

$$X_T = \begin{cases} 1 & \text{if } T \text{ rainbow} \\ 0 & \text{else} \end{cases}$$

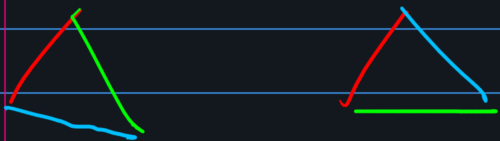
$$X = \sum_{T \text{ triangle}} X_T \Rightarrow \mathbb{E}[X] = \sum_T E[X_T]$$

$$= \sum_T \Pr[T \text{ rainbow}]$$

$$= (\#\text{triangles}) \Pr[T \text{ rainbow}]$$

$\#$ triangles: $\binom{n}{3}$ since any 3 vertices determine a triangle because we're in K_n

$\Pr[\text{rainbow}] = \frac{6}{27} = \frac{2}{9}$



$\Rightarrow E[X] = \binom{n}{3} \cdot \frac{2}{9}$

$\Rightarrow$ there is a coloring w/ $\binom{n}{3}\frac{2}{9}$ $\geq$ rainbow triangles.