4. In this problem we show that for every k there exists a constant C_k such that every graph on n vertices with minimum degree $\delta \geq C_k \log n$ must contain a cycle of length divisible by k.

   (a) Fix k numbered colors (color 1, color 2, etc.). Randomly color each vertex with one of these colors. Show that there exists a coloring for which each vertex colored i has a neighbor colored $i+1$ (mod k). *Hint:* $1 - x \leq e^{-x}$.

   (b) Now orient the edges between vertices colored i and $i + i$ so that they point from the one colored i to the one colored $i+1$. Delete the rest of the edges. Show that the resulting graph must contain a directed cycle. Argue that this cycle has length divisible by k.

Pr[our coloring has this property]

wts

= 1 − Pr[doesn't have this property] > 0

¬(∀ vertices colored i, ∃ neighbor colored $i+1$)

(∃ vertex colored i, none of its neighbors colored $i+1$)

Fix some vertex v. Say it's colored i.

Pr[none of v's neighbors colored $i+1$]

v has $d(v)$ neighbors ($d(v) \geq \delta$)

$$= \left(\frac{k-1}{k}\right)^{d(v)} = \left(1 - \frac{1}{k}\right)^{d(v)}$$

Pr[some vertex has none of its neighbors .....]

$$\leq n\left(1-\frac{1}{k}\right)^{d(v)} \leq ne^{-d(v)/k} \quad (*)$$

$1 - x \leq e^{-x}$

$$d(v) \geq \delta \geq C_k \log n$$

$$\Rightarrow -d(v) \leq -C_k \log n$$

$$(*) \leq n e^{-\frac{C_k \log n}{k}}$$

$$= n \left(e^{\log n}\right)^{-\frac{C_k}{k}} = n^{1-\frac{C_k}{k}}$$

$$\Rightarrow \Pr[\text{random coloring has this property}] \geq 1 - \underbrace{n^{1-\frac{C_k}{k}}}$$

> 0

wts

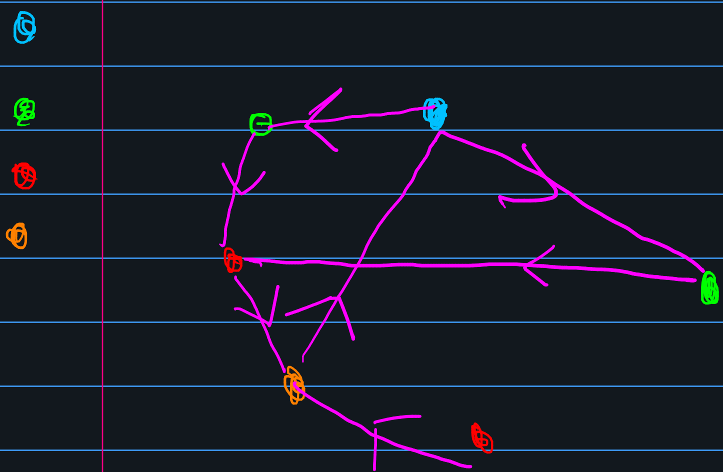
$$\boxed{\begin{aligned} \text{need } 1 - \frac{C_k}{k} &< 0 \\ 1 &< \frac{C_k}{k} \\ k &< C_k \end{aligned}}$$

Let C_k be any integer $> k$, e.g. $C_k = k+1$

Claim:

If a directed graph has minimal out deg. ≥ 1

$\Rightarrow$ must have a directed cycle



- Start at any vertex.
- Walk along one of its out edges (you can do this since each vertex has out degree ≥ 1)
- You can do this infinitely
- There are only finitely many vertices, so you must revisit a vertex,
- to revisit a vertex in this way, you must walk along a directed cycle.

$\boxtimes$ (claim)

$X = \#\, t$-cliques , $Y = \#\, t$-indep sets

$$E[X] = \binom{n}{t} \left(\frac{1}{2}\right)^{\binom{t}{2}} \leq \frac{n^t}{t!}$$

$$\shortparallel$$

$$E[Y]$$