271A - Homework 6

Problem 1. Consider the discrete time process $X_n = a + bn + \zeta_n$ with ζ_n, $n = 0, \pm 1, \pm 2, \ldots$ being iid centered with variance σ^2 and a, b constants. Define

$$W_n = (2q+1)^{-1} \sum_{j=-q}^{q} X_{n+j}.$$

Compute the autocovariance function of $W_n : \gamma(n,m) = \mathrm{Cov}(W_n, W_m)$ and the autocorrelation function $\rho(n,m) = \mathrm{Corr}(W_n, W_m)$. Consider $Y_n = W_n - W_{n-1}$ and compute the autocovariance and autocorrelation functions for this process. Are either of these processes stationary?

Solution. We start by computing the covariance of X_i and X_j.

$$\mathrm{Cov}(X_i, X_j) = \mathrm{Cov}(a + bi + \zeta_i, a + bj + \zeta_j) = \mathrm{Cov}(\zeta_i, \zeta_j) = \delta_{i,j}\sigma^2,$$

where $\delta_{i,j}$ is the Kronecker δ. Let's compute the variance while we're at it.

$$\mathrm{Var}[W_n] = \frac{1}{(2q+1)^2} \sum_{i=-q}^{q} \mathrm{Var}[X_{n_j}] = \frac{\sigma^2}{2q+1}.$$

Now we can compute the autocovariance of W_m and W_n.

$$\begin{aligned}
\gamma_W(m,n) &= \frac{1}{(2q+1)^2} \mathrm{Cov}\left(\sum_{i=-q}^{q} X_{m+i}, \sum_{j=-q}^{q} X_{n+j} \right) \\
&= \frac{1}{(2q+1)^2} \sum_{-q \le i,j \le q} \mathrm{Cov}(X_{m+i}, X_{n+j}) \\
&= \frac{\sigma^2}{(2q+1)^2} \sum_{-q \le i,j \le q} \delta_{m+i,n+j} \\
&= \frac{\sigma^2}{(2q+1)^2} \#\{-q \le i,j \le q : i - j = n - m\}.
\end{aligned}$$

For any fixed i, $i - j = n - m$ if and only if $j = i - (n-m)$. Such a j exists if and only if

$$-q \le i - (n-m) \le q \iff (n-m) - q \le i \le (n-m) + q.$$

We then need the size of the intersection $[-q,q] \cap [(n-m)-q, (n-m)+q]$. Since $[(n-m)-q, (n-m)+q]$ is simply $[-q,q]$ shifted over by $(n-m)$, their intersection has size $(2q+1) - |n-m|$ if $|n-m| \le 2q+1$ and zero otherwise. We then have

$$\gamma_W(m,n) = \begin{cases} 0, & \text{if } |n-m| > 2q+1 \\ \frac{\sigma^2}{(2q+1)^2}[(2q+1) - |n-m|], & \text{else.} \end{cases}$$

Now for the autocorrelation

$$\rho_W(m,n) = \frac{\mathrm{Cov}(W_m, W_n)}{\sqrt{\mathrm{Var}[W_m]\cdot \mathrm{Var}[W_n]}} = \begin{cases} 0, & \text{if } |n-m| > 2q+1 \\ \frac{1}{2q+1}[(2q+1) - |n-m|], & \text{else.} \end{cases}$$

Now let's take care of Y_n. By definition we have

$$Y_n = W_n - W_{n-1} = \frac{1}{2q+1}\left(\sum_{i=-q}^{q} X_{n+i} - \sum_{j=-q}^{q} X_{n-1+j}\right) = \frac{1}{2q+1}(X_{n+q} - X_{n-1-q}).$$

We'll need the variance

$$\mathrm{Var}[Y_n] = \frac{1}{(2q+1)^2}(\mathrm{Var}[X_{n+q}] + \mathrm{Var}[X_{n-1-q}]) = \frac{2\sigma^2}{(2q+1)^2}.$$

First the autocovariance.

$$\begin{aligned} \gamma_Y(m,n) &= \frac{1}{(2q+1)^2}\mathrm{Cov}(X_{m+q} - X_{m-1-q}, X_{n+q} - X_{n-1-q}) \\ &= \frac{\sigma^2}{(2q+1)^2}(\delta_{m+q,n+q} - \delta_{m+q,n-1-q} - \delta_{m-1-q,n+q} + \delta_{m-1-q,n-1-q}) \\ &= \frac{\sigma^2}{(2q+1)^2}(2\delta_{m,n} - \delta_{|n-m|,2q+1}). \end{aligned}$$

And finally, the autocorrelation.

$$\rho_Y(m,n) = \frac{\mathrm{Cov}(Y_m, Y_n)}{\sqrt{\mathrm{Var}[Y_m]\cdot \mathrm{Var}[Y_n]}} = \frac{1}{2}(2\delta_{m,n} - \delta_{|n-m|,2q+1}).$$

Let's check for stationarity. For any $\Delta \in \mathbb{Z}$ we have

$$W_{n+\Delta} = \frac{1}{2q+1}\sum_{i=-q}^{q}(a + b(n+\Delta+i) + \zeta_{n+i+\Delta}) = a + b(n+\Delta) + \frac{1}{2q+1}\sum_{i=-q}^{q}\zeta_{n+i+\Delta}.$$

On the other hand,

$$W_n = a + bn + \frac{1}{2q+1}\sum_{i=-q}^{q}\zeta_{n+i}.$$

Since the deterministic parts of these sums differ while the random parts are equal in distribution, we have that W is not stationary. Y_n, however, is stationary since

$$Y_{n+\Delta} = \frac{1}{2q+1}(X_{n+\Delta+q} - X_{n+\Delta-1-q}) = b + \frac{\zeta_{n+\Delta+q} - \zeta_{n+\Delta-1-q}}{2q+1} \overset{D}{=} Y_n.$$

$\square$

Problem 2. For $H \in (0,1)$ and B^H fractional Brownian motion and $t_0 \in (0,\infty)$ show that

$$\limsup_{t\to t_0}\left|\frac{B_t^H - B_{t_0}^H}{t - t_0}\right| = \infty$$

with probability one.

Proof. By the stationarity of the increments of fractional Brownian motion we have

$$\mathbb{P}\left[\limsup_{t\to t_0}\left|\frac{B_t^H - B_{t_0}^H}{t-t_0}\right| = \infty\right] = \mathbb{P}\left[\limsup_{t\to 0}\left|\frac{B_t^H}{t}\right| = \infty\right].$$

By self-similarity, we have $B_t^H \overset{D}{=} t^H B_1^H$. Since $H \in (0,1)$ and B_1^H is finite, we have

$$\mathbb{P}\left[\limsup_{t\to t_0}\left|\frac{B_t^H - B_{t_0}^H}{t-t_0}\right| = \infty\right] = \mathbb{P}\left[\limsup_{t\to 0}|t^{H-1}B_1^H| = \infty\right] = 1.$$

$\square$

Problem 3. Consider Lévy's method for simulating Brownian motion on the interval $[0,1]$. Let B_t be the limit process and $B_t^{(n)}$ be the process after depth iteration n in the procedure. Find a bound for $\mathbb{E}[|B_t - B_t^{(n)}|^2]$.

Solution. Let $S_k^{(n)}$ denote the k-th element in the n-th level of the Schauder basis:

$$S_k^{(n)}(t) = \begin{cases} 2^{-\frac{n+1}{2}}(1 + 2^{n+1}(x - k2^{-k})), & \text{if } k2^{-k} - 2^{-(n+1)} \leq x < k2^{-k} \\ 2^{-\frac{n+1}{2}}(1 - 2^{n+1}(x - k2^{-k})), & \text{if } k2^{-k} \leq x < k2^{-k} + 2^{-(n+1)}, \end{cases}$$

where $n = 0, 1, 2, \ldots$ and $k \in \{1, 3, \ldots, 2^n - 1\} =: I(n)$. In Lévy's construction, we define

$$B_t^{(N)} = \sum_{n\leq N,\ k\in I(n)} S_k^{(n)}(t)\cdot \zeta_k^{(n)}$$

and argue that $B_t^{(N)}$ converges uniformly to a standard Brownian motion almost surely. Since the $\zeta_k^{(n)}$'s are independent, centered, and have unit variance, we have

$$\begin{aligned}\mathbb{E}[|B_t - B_t^{(N)}|^2] &= \mathbb{E}\left[\left|\sum_{n>N,\ k\in I(n)} S_k^{(n)}(t)\zeta_k^{(n)}\right|^2\right] \\ &= \sum_{n>N,\ k\in I(n)} (S_k^{(n)}(t))^2.\end{aligned}$$

Within any fixed level n, the Schauder basis elements $S_j^{(n)}$ and $S_k^{(n)}$ have disjoint support for $j \neq k$. We then have

$$\mathbb{E}[|B_t - B_t^{(N)}|^2] \leq \sum_{n>N}\left\|S_k^{(n)}\right\|_{L^\infty}^2 = \sum_{n>N} 2^{-(n+1)} = 2^{-(N+1)}.$$

$\square$

Problem 5. Let $\{X^{(m)}\}_{m=1}^\infty$ be a sequence of continuous stochastic processes on $t \in [0,\infty)$ satisfying

(i) $X_0^{(m)} = x_0$, with x_0 deterministic.

(ii) $\sup_{m\geq 1}\mathbb{E}[|X_t^{(m)} - X_s^{(m)}|^\alpha] \leq C_T|t-s|^{1+\beta}$ for all $T > 0$ and $0 \leq s,t \leq T$

for some global positive constants α, β and some C_T depending on T. show that the induced probability measures on $(C[0,\infty), \mathcal{B}(C[0,\infty)))$ form a tight sequence.

Proof. The induced probability measures P_n form a tight sequence if and only if the following two conditions hold

$$\lim_{\lambda\to\infty} \sup_{n\geq 1} P_n[\omega : |\omega(0)| > \lambda] = 0 \tag{1}$$

$$\lim_{\delta\to 0} \sup_{n\geq 1} P_n[\omega : m^T(\omega,\delta) > \epsilon] = 0; \quad \forall T > 0, \epsilon > 0. \tag{2}$$

Since $X_0^{(n)} = x_0$ for all n, $P_n[\omega : |\omega(0)| > \lambda] = 0$ for all $\lambda \geq |x_0|$, so the first condition holds. Now by Kolmogorov's continuity theorem, $\mathbb{E}|X_t^{(m)} - X_s^{(m)}|^\alpha \leq C_T|t-s|^{1+\beta}$ for all m implies that each $X^{(m)}$ has a continuous modification that is locally Hölder continuous with exponent γ for every $\gamma \in (0, \beta/\alpha)$. That is,

$$\mathbb{P}\left[\omega : \sup_{0\leq|t-s|\leq h(\omega)} \frac{|X_t^{(n)} - X_s^{(n)}|}{|t-s|^\gamma} \leq K_m\right] = 1.$$

for all m and for some a.s. positive $h(\omega)$ and some $K_m > 0$. The idea is to use the a.s. Hölder continuity of the paths $X^{(m)}(\omega)$ to force the modulus of continuity $m^T(\omega,\delta)$ to be small in δ with probability small in δ. □