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270B - Homework 3

Problem 1. Let $X_1, X_2, \ldots$ be independent random variables with means μ_i and finite variances σ_i^2. Consider the sums $S_n = X_1 + \cdots + X_n$. Find sequences of real numbers (b_i) and (c_i) such that $S_n^2 + b_n S_n + c_n$ is a martingale with respect to the σ-algebras generated by $X_1, \ldots, X_n$.

Solution. Let's start by centering the sum: define the random variable $M_n = S_n - \sum_{i=1}^n \mu_i$. Since the X_i's are independent, we have $\mathrm{Var}[M_n] = \sum_{i=1}^n \sigma_i^2$. We claim that

$$V_n = M_n^2 - \sum_{i=1}^n \sigma_i^2 = \left(S_n - \sum_{i=1}^n \mu_i\right)^2 - \sum_{i=1}^n \sigma_i^2$$

is a martingale with respect to the filtration $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. Let's start the computation.

$$\begin{aligned}
\mathbb{E}[V_{n+1}|\mathcal{F}_n] &= \mathbb{E}[S_{n+1}^2] - 2\left(\sum_{i=1}^{n+1}\mu_i\right)\mathbb{E}[S_{n+1}|\mathcal{F}_n] + \left(\sum_{i=1}^{n+1}\mu_i\right)^2 - \sum_{i=1}^{n+1}\sigma_i^2 \\
&= S_n^2 + 2S_n\mu_{n+1} + \mathbb{E}[X_{n+1}^2] - 2\left(\sum_{i=1}^{n+1}\mu_i\right)(S_n + \mu_{n+1}) + \left(\sum_{i=1}^{n+1}\mu_i\right)^2 - \sum_{i=1}^{n+1}\sigma_i^2 \\
&= S_n^2 - 2\left(\sum_{i=1}^{n}\mu_i\right)S_n + \mathbb{E}[X_{n+1}^2] - 2\mu_{n+1}^2 + \left(\sum_{i=1}^{n}\mu_i\right)^2 + \mu_{n+1}^2 - \sum_{i=1}^{n+1}\sigma_i^2 \\
&= S_n^2 - 2\left(\sum_{i=1}^{n}\mu_i\right)S_n + \left(\sum_{i=1}^{n}\mu_i\right)^2 - \sum_{i=1}^{n}\sigma_i^2 \\
&= V_n.
\end{aligned}$$

Here we've used the fact that S_n is $\mathcal{F}_n$-measurable and X_{n+1} is independent of $\mathcal{F}_n$. The sequences we want are then

$$b_n = -2\sum_{i=1}^n \mu_i, \qquad c_n = \left(\sum_{i=1}^n \mu_i\right)^2 - \sum_{i=1}^n \sigma_i^2$$

□