Math 180B - Sums of Divisors

1. Recall that a function $f : \mathbb{N} \to \mathbb{C}$ is multiplicative if $f(mn) = f(m)f(n)$ for all $(m, n) = 1$. Prove that if f is multiplicative and not identically zero then $f(1) = 1$.

2. Let $\sigma(x) = \sum_{d|x} d$. Find the smallest positive integer n so that $\sigma(x) = n$ has no solution.

3. Prove that the number of positive irreducible fractions ≤ 1 with denominator $\leq n$ is $\phi(1) + \phi(2) + \cdots + \phi(n)$.

4. Prove that $\sum_{d|n} f(d) = \sum_{d|n} f(n/d)$.

5. Define the function $\mu : \mathbb{N} \to \{-1, 0, 1\}$ by

$$\mu(n) = \begin{cases} 1, & \text{if } n \text{ is square-free and has an even number of prime divisors} \\ -1, & \text{if } n \text{ is square-free and has an odd number of prime divisors} \\ 0, & \text{if } n \text{ has a squared prime factor} \end{cases}.$$

   (a) Prove that $\mu(n)$ is multiplicative and

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n > 1 \end{cases}.$$

   (b) Prove that if $F(n) = \sum_{d|n} f(d)$ for every positive integer n, then $f(n) = \sum_{d|n} \mu(d)F(n/d)$.

   (c) Prove the converse to (b): if $f(n) = \sum_{d|n} \mu(d)F(n/d)$, then $F(n) = \sum_{d|n} f(d)$.