HomeWork 2 Math 271B, Winter 2020.

1. Let S, T and $T_n,\ n = 1, 2, \cdots$ be stropping times.

   Show that $T \wedge S, \quad T \vee S, \quad T + S, \quad \sup_n T_n$ are also stopping times.

2. Let X_t be an adapted and continuous stochastic process, and define

$$T_\Gamma = \inf\{t \geq 0 \mid X_t \in \Gamma\}$$

   for Γ a closed set. Show that T_Γ is a stopping time.

3. Show that if X_t is martingale with respect to some filtration then it is also a martingale with respect to the filtration generated by itself.

4. Let a, b be deterministic and f, g of Class I. Show that if

$$a + \int_0^T f_s d\beta_s = b + \int_0^T g_s d\beta_s,$$

   (β is Brownian motion as usual) then $a = b$ and $f = g$ a.a. for $(t, \omega) \in (0, T) \times \Omega$.

5. Assume that X_t is of Class I and continuous in mean square on $[0, T]$, that is for $t \in [0, T]$

$$\mathbb{E}[X_t^2] < \infty,$$
$$\lim_{s \to t} \mathbb{E}[(X_t - X_s)^2] = 0.$$

   Define

$$\phi_t^{(n)} = \sum_j X_{t_{j-1}^{(n)}} \chi_{[t_{j-1}^{(n)}, t_j^{(n)})}(t), \quad t_j^{(n)} = j2^{-n}.$$

   Show that for $0 \leq t \leq T$:

$$\int_0^t X_s d\beta_s = \lim_{n \to \infty} \int_0^t \phi_s^{(n)} d\beta_s,$$

   limit in $L^2(\mathbb{P})$.

6. Let X_t be a deterministic continuous function and

$$Y_t = \int_0^t X_s d\beta_s.$$

   Deduce the law of the process Y.