1. For what values of C are the following functions probability density functions?

(a) $f(x) = C[x(1-x)]^{-1/2}$, $x \in (0,1)$.

(b) $f(x) = C\exp(-x - e^{-x})$, $x \in \mathbb{R}$.

(c) $f(x) = C/(1+x^2)$, $x \in \mathbb{R}$.

1b) need $\int_{\mathbb{R}} f(x)\,dx = 1$

$$1 = \int_{-\infty}^{\infty} C\exp(-x - e^{-x})\,dx$$

$u = e^{-x} \Rightarrow du = -e^{-x}dx$

$e^{-(-\infty)} = e^{\infty} = \infty$

$e^{-\infty} = 0 = \lim_{t \to +\infty} e^{-t}$

$$= C\int_{-\infty}^{\infty} e^{-x} e^{-e^{-x}}\,dx$$

$$= -C\int_{\infty}^{0} e^{-u}\,du$$

$$= C\int_{0}^{\infty} e^{-u}\,du = -Ce^{-u}\Big|_0^{\infty} = C$$

$$\Rightarrow C = 1$$

on an exam expect integrals of polynomials or e^x, etc

2. For what values of α is $E[|X|^\alpha]$ finite if X has the density f. (This is called the α-th moment of X).

(a) $f(x) = e^{-x}$, $x \geq 0$.

(b) $f(x) = 1/[\pi(1+x^2)]$, $x \in \mathbb{R}$.

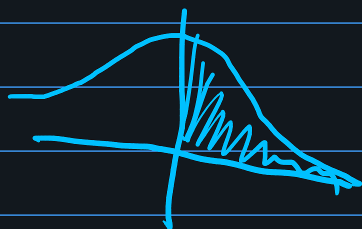
b) want α : $\mathbb{E}[|X|^\alpha] < \infty$

recall: if X has density $f(x)$, then

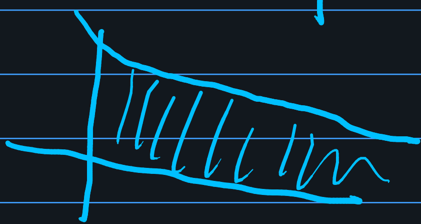
$$\mathbb{E}[g(X)] = \int_{\mathbb{R}} g(x) f(x)\,dx$$

$$\mathbb{E}[|X|^\alpha] = \int_{-\infty}^{\infty} \frac{|x|^\alpha}{\pi(1+x^2)}\,dx$$

$$= \int_{-\infty}^{0} \frac{|x|^\alpha}{\pi(1+x^2)}\,dx + \int_{0}^{\infty} \frac{|x|^\alpha}{\pi(1+x^2)}$$



$$= 2\int_{0}^{\infty} \frac{|x|^\alpha}{\pi(1+x^2)}\,dx$$



at least need $\lim_{x\to\infty} \frac{|x|^\alpha}{\pi(1+x^2)} = 0$

can't have $\alpha \geq 2$

since if $\alpha > 2$, numerator has higer degree

& $\lim_{x\to\infty} \frac{|x|^\alpha}{1+x^2} = \infty$

if $\alpha = 2$, then $\frac{|x|^2}{1+x^2} \to 1$

say $\alpha < 2$

$\int \frac{x^\alpha}{1+x^2} dx$ Claim: need $2-\alpha > 1$

$\Rightarrow 1 > \alpha$

do u substitution.

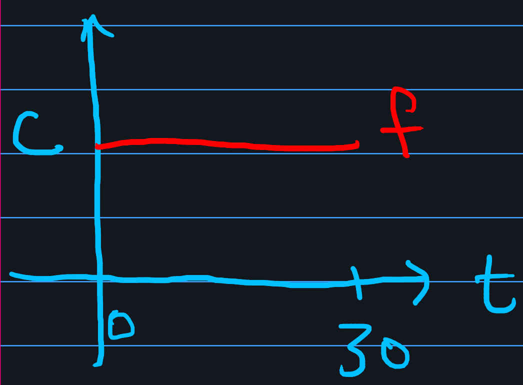
3. You arrive at a bus stop at 10 AM knowing that the bus will arrive at some time uniformly distributed between 10 and 10:30.

   (a) What is the probability that you will have to wait longer than 10 minutes?

   (b) If at 10:15 the bus has not yet arrived, what is the probability that you will have to wait at least an additional 10 minutes?

a) first: find the density of X: the waiting time (in minutes)

$X \sim \text{Unif}(0, 30)$. call the density f



$$\int_0^{30} f(x)dx = 1$$

$$\| $$

$$\int_0^{30} c\, dx = cx\Big|_0^{30} = 30c = 1$$

$$\Rightarrow c = 1/30$$

$$\Rightarrow f(x) = 1/30$$

$$\Pr[\text{wait longer than } 10] = \Pr[X \geq 10]$$

$$= \int_{10}^{30} f(x)\,dx = \int_{10}^{30} \frac{1}{30}\,dx = \frac{30-10}{30}$$

$$= 2/3$$

b) $\Pr[\text{Wait} \geq 10 \text{ more mins} \mid \text{hasn't arrived by } 15]$

$$= \Pr[X \geq 25 \mid X \geq 15]$$

$$= \frac{\Pr[X \geq 25, X \geq 15]}{\Pr[X \geq 15]} = \frac{\Pr[X \geq 25]}{\Pr[X \geq 15]}$$

$$= \frac{\int_{25}^{30} \frac{1}{30}\,dx}{\int_{15}^{30} \frac{1}{30}\,dx} = \frac{5/30}{15/30} = 1/3$$

4 & 5 : next time