Homework 2
Math 270A, Fall 2019, Prof. Roman Vershynin

Problem 1

The *total variation distance* between the distributions of random variables X and Y is defined as
$$d_{\mathrm{TV}}(X,Y) := \sup_{B\in\mathcal{B}} \Big|\mathbb{P}\left\{X\in B\right\} - \mathbb{P}\left\{Y\in B\right\}\Big|$$
where the supremum is over all Borel subsets $B\subset\mathbb{R}$.

(a). Show that $d_{\mathrm{TV}}(X,Y)$ is indeed a metric on the set of distributions (i.e. probability measures on the measurable space $(\mathbb{R},\mathcal{B})$).

(b). Suppose X and Y are integer-valued random variables. Prove that
$$d_{\mathrm{TV}}(X,Y) = \frac{1}{2}\sum_{k\in\mathbb{Z}} \Big|\mathbb{P}\left\{X=k\right\} - \mathbb{P}\left\{Y=k\right\}\Big|.$$

Problem 2

Let X and Y be independent random variables taking values in the positive integers and having the same distribution given by
$$\mathbb{P}\left\{X=n\right\} = \mathbb{P}\left\{Y=n\right\} = 2^{-n} \quad \text{for } n\in\mathbb{N}.$$
Find
$$\mathbb{P}\left\{X \text{ divides } Y\right\}.$$

Problem 3

A total of n bar magnets are placed end to end in a line with random independent orientations. Adjacent like poles repel, ends with opposite polarities join to form blocks. Find the expected number of blocks of joint magnets.

Problem 4

Let X and Y be random variables with mean 0, variance 1, and covariance $\mathrm{Cov}(X,Y)=\rho$. Show that
$$\mathbb{E}\max(X^2,Y^2) \le 1+\sqrt{1-\rho^2}.$$
(Hint: $\max(u,v)=\frac{1}{2}(u+v)+\frac{1}{2}|u-v|$. *Use this bound followed by Cauchy-Schwarz inequality.)*

Problem 5

A *median* of a random variable X is a number $m \in \mathbb{R}$ such that

$$\mathbb{P}\{X \leq m\} \geq \frac{1}{2} \quad \text{and} \quad \mathbb{P}\{X \geq m\} \geq \frac{1}{2}.$$

(a). Prove that a median exists for every random variable X.

(b). Show that the mean μ, median m, and variance σ^2 of a random variable X satisfy

$$|\mu - m| \leq C\sigma,$$

where C is an absolute constant.

Feel free to prove this e.g. for $C = 10$. The inequality actually holds for $C = 1$, although this could be harder to show.

Problem 6

Let $X_1, \ldots, X_n$ be independent, identically distributed random variables for which $\mathbb{E}[1/X_i] < \infty$. Consider the partial sums

$$S_m := X_1 + X_2 + \ldots + X_m.$$

Show that

$$\mathbb{E}[S_m/S_n] = m/n \quad \text{for all } m \leq n.$$

Problem 7

Suppose the joint density $f(x_1, \ldots, x_n)$ of random variables $X_1, \ldots, X_n$ can be represented as

$$f(x_1, \ldots, x_n) = f_1(x_1) \cdots f_n(x_n)$$

for some measurable functions $f_i : \mathbb{R} \to [0, \infty)$. Prove that $X_1, \ldots, X_n$ are independent.

(Note: f_i are not assumed to be probability densities.)

Problem 8

Let X, Y be independent random variables taking nonnegative values. Express the density (pdf) of XY in terms of the densities of X and Y.

Problem 9

Find an example of a discrete random variable with finite expectation and infinite variance.

Problem 10

Below is an explicit construction of Bernoulli independent random variables, which does not use Kolmogorov's extension theorem. Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\Omega = (0,1)$, $\mathcal{F}$ is the Borel sigma-algebra, and $\mathbb{P}$ is the Lebesgue measure. Let

$$X_n(\omega) := \begin{cases} 1 & \text{if } \lfloor 2^n \omega \rfloor \text{ is odd} \\ 0 & \text{if } \lfloor 2^n \omega \rfloor \text{ is even.} \end{cases}$$

Show that $X_1, X_2, \ldots$ are independent random variables with distribution $\mathrm{Ber}(1/2)$.