271A - Homework 4

Problem 1.

Let B_t be a Brownian motion. Show that

$$Y_t = (1+t)B_{(1+t)^{-1}} - B_1$$

is a Brownian motion on $[0,\infty)$.

Proof. We have $Y_0 = B_1 - B_1 = 0$. Since $t \mapsto \frac{1}{1+t}$ is continuous and $t \mapsto B_t$ is a.s. continuous, we have that $t \mapsto (1+t)B_{(1+t)^{-1}} - B_1$ is a.s. continuous. We also have by the linearity of expectation that $\mathbb{E}[Y_t] = 0$ for all t. Let $s < t$ and consider the increment $Y_t - Y_s$. Since $\frac{1}{1+t} < \frac{1}{1+s}$ we have

$$\begin{aligned} Y_t - Y_s &= (1+t)B_{(1+t)^{-1}} - (1+s)B_{(1+s)^{-1}} \\ &= (1+t)B_{(1+t)^{-1}} - (1+s)B_{(1+s)^{-1}} + (1+s)B_{(1+t)^{-1}} - (1+s)B_{(1+t)^{-1}} \\ &= (t-s)B_{(1+t)^{-1}} - (1+s)[B_{(1+s)^{-1}} - B_{(1+t)^{-1}}]. \end{aligned}$$

Since B is a Brownian motion, the increments $B_{(1+t)^{-1}}$ and $B_{(1+s)^{-1}} - B_{(1+t)^{-1}}$ are independent. Since the above is a sum of independent Gaussians, the increment $Y_t - Y_s$ is also Gaussian. The linearity of expectation gives $\mathbb{E}[Y_t - Y_s] = 0$ and the variance is given by

$$\begin{aligned} \mathrm{Var}[Y_t - Y_s] &= (t-s)^2 \cdot \mathrm{Var}[B_{(1+t)^{-1}}] + (1+s)^2 \cdot \mathrm{Var}[B_{(1+s)^{-1}} - B_{(1+t)^{-1}}] \\ &= \frac{(t-s)^2}{1+t} + (1+s)^2 \left(\frac{1}{1+s} - \frac{1}{1+t} \right) \\ &= t - s. \end{aligned}$$

It remains to show that the increments of Y are independent. To this end, let $0 \le s_1 < t_1 \le \cdots \le s_n < t_n$. Since the increments $Y_{t_i} - Y_{s_i}$ are Gaussian, it suffices to show that they are uncorrelated. Suppose $i < j$ we then have

$$\begin{aligned} &\mathrm{Cov}(Y_{t_i} - Y_{s_i}, Y_{t_j} - Y_{s_j}) \\ &= (1+t_i)(1+t_j)\left(\frac{1}{1+t_i} \wedge \frac{1}{1+t_j} \right) - (1+t_i)(1+s_j)\left(\frac{1}{1+t_i} \wedge \frac{1}{1+s_j} \right) \\ &- (1+s_i)(1+t_j)\left(\frac{1}{1+s_i} \wedge \frac{1}{1+t_j} \right) + (1+s_i)(1+s_j)\left(\frac{1}{1+s_i} \wedge \frac{1}{1+s_j} \right) \\ &= 0. \end{aligned}$$

THus, the increments are independent and we conclude that Y is a Brownian motion. ☐

Problem 2. Consider the random walk

$$S_n = \sum_{i=1}^{n} \zeta_i$$

for $\mathbb{P}[\zeta_i = 1] = p$ and $\mathbb{P}[\zeta_i = -1] = 1 - p$ and the ζ_i are independent. Given λ, find γ so that

$$\exp(\gamma S_n - \lambda n)$$

is a martingale with respect to the filtration generated by the ζ_i's.

Solution. Let $M_n = \exp(\gamma S_n - \lambda n)$ and let $\mathcal{F}_n$ be the filtration generated by $\zeta_1, \ldots, \zeta_n$. We compute the conditional expectation

$$\begin{aligned}\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] &= \mathbb{E}[\exp(\gamma S_n - \lambda n) \cdot \exp(\gamma \zeta_{n+1} - \lambda) \mid \mathcal{F}_n] \\ &= M_n \cdot \mathbb{E}[\exp(\gamma \zeta_{n+1} - \lambda)].\end{aligned}$$

Here we've used that M_n is $\mathcal{F}_n$-measurable and ζ_{n+1} is independent of $\mathcal{F}_n$. In order for M to be a martingale, we need $\mathbb{E}[\exp(\gamma \zeta_{n+1} - \lambda)] = 1$. Let's make it happen.

$$\begin{aligned}&\mathbb{E}[\exp(\gamma \zeta_{n+1} - \lambda)] = 1 \\ \iff\ & pe^{\gamma - \lambda} + (1-p)e^{-\gamma - \lambda} = 1 \\ \iff\ & pe^{2\gamma} - e^{\lambda} e^{\gamma} + (1-p) = 0 \\ \iff\ & e^{\gamma} = \frac{e^{\lambda} \pm \sqrt{e^{2\lambda} - 4p(1-p)}}{2p} \\ \iff\ & \gamma = \log \frac{e^{\lambda} \pm \sqrt{e^{2\lambda} - 4p(1-p)}}{2p}.\end{aligned}$$

Of course, this expression makes sense only when the numerator is positive and $e^{2\lambda} \geq 4p(1-p)$. $\square$

Problem 3. Assume that $X_n \to X$ in probability and $X_n \to Y$ a.s. Show that $X = Y$ a.s.

Proof. First, we claim that since $X_n \to X$ in probability, every subsequence of X_n contains a further subsequence converging to X a.s. Since the a.s. limit is unique, we can conclude that $X = Y$ a.s.

To prove our claim, we fix a subsequence X_{n_k}. Since $X_n \to X$ in probability, there is some n_{k_1} so that for all n at least n_{k_1},

$$\mathbb{P}[|X_n - X| > 2^{-1}] \leq 2^{-1}.$$

Inductively, suppose that we've constructed $n_{k_1} < \cdots < n_{k_m}$ so that

$$\mathbb{P}[|X_{n_{k_i}} - X| > 2^{-i}] \leq 2^{-i}$$

for $i = 1, \dots, m$. Again by convergence in probability, we can choose $n_{k_{m+1}} > n_{k_m}$ so that

$$\mathbb{P}[|X_{n_{k_{m+1}}} - X| > 2^{-(m+1)}] \leq 2^{-(m+1)}.$$

If we let A_m be the event $\{|X_{n_{k_m}} - X| > 2^{-m}\}$, then $\mathbb{P}[A_m]$ is summable by construction. By Borel-Cantelli, $\mathbb{P}[\limsup A_m] = 0$. Consequently, we have

$$\mathbb{P}\left[\bigcup_{\ell \in \mathbb{N}} \bigcap_{m \geq \ell} \{|X_{n_{k_m}} - X| > 2^{-m}\}\right] = 1,$$

so $X_{n_{k_m}} \to X$ a.s. □

Problem 4.

(a.) Show that a continuous stochastic process with non-zero and finite 2-variation has infinite 1-variation.

Proof. Let X_t be a continuous stochastic process with nonzero finite quadratic variation. We then have for any partition $\Pi = \{0 = t_0 < t_1 < \cdots < t_n = t\}$,

$$\begin{aligned} 0 &< \sum_{k=1}^{n} |X_{t_k} - X_{t_{k-1}}|^2 \\ &\leq \max_{k \leq n} |X_{t_k} - X_{t_{k-1}}| \cdot \sum_{k=1}^{n} |X_{t_k} - X_{t_{k-1}}| \\ &\leq \max_{|s-t| < \|\Pi\|} |X_s - X_t| \cdot \sum_{k=1}^{n} |X_{t_k} - X_{t_{k-1}}|. \end{aligned}$$

By continuity, we have $\lim_{\|\Pi\| \to 0} \max_{|s-t|<\|\Pi\|} |X_s - X_t| = 0$, so dividing the above inequality through by $\max_{|s-t|<\|\Pi\|} |X_s - X_t|$ and sending $\|\Pi\| \to 0$ establishes that the total variation of X must be infinite. □

(b.) Show that a continuous process with finite total variation has zero quadratic variation.

Proof. With X defined as in part (a.), the same reasoning gives

$$\sum_{k=1}^{n} |X_{t_k} - X_{t_{k-1}}|^2 \leq \max_{|s-t|<\|\Pi\|} |X_s - X_t| \cdot \sum_{k=1}^{n} |X_{t_k} - X_{t_{k-1}}|.$$

As we send $\|\Pi\| \to 0$, the left-hand side approaches the quadratic variation. The max term on the right goes to zero by continuity and the sum on the right approaches the total variation, which is assumed to be finite. Consequently, the right-hand side limits to zero, so the quadratic variation of X is zero. □

(c.) Show that the total variation of a Brownian motion is infinite a.s.

Proof. We showed in class that for a Brownian motion B, $L_t^{(2)}(B) = t$ a.s. By part (a.), we must have that $L_t^{(1)}(B)$ is infinite a.s. $\square$

Problem 5. Consider the compound Poisson process

$$X_t = \sum_{i=1}^{N_t} Y_i,$$

with N_t Poisson with parameter λ and independent of the Y_i which are centered iid with variance σ^2.

(a.) Show that X_s and $X_t - X_s$ are independent.

Proof. Suppose $s < t$. We then have that

$$X_s = \sum_{i=1}^{N_s} Y_i, \qquad X_t - X_s = \sum_{i=N_s+1}^{N_t} Y_i.$$

These sums are over disjoint indices and N has independent increments. Since the Y_i's are independent, we conclude that X_s and $X_t - X_s$ are sums of mutually independent random variables, and are hence independent. $\square$

(b.) Find the quadratic variation, $\langle X \rangle_t$, associated with the process.

Solution. Fix $\Pi = \{0 = t_0 < t_1 < \cdots < t_n = t\}$. We then have

$$V_t^{(2)}(\Pi, X) = \mathbb{E}\left[\sum_{i=1}^{n} |X_{t_i} - X_{t_{i-1}}|^2\right] = \sum_{i=1}^{n} \mathbb{E}[X_{t_i}^2 - 2X_{t_i}X_{t_{i-1}} + X_{t_{i-1}}^2]. \tag{1}$$

We showed on the last homework assignment that $\mathbb{E}[X_t^2] = \lambda t \sigma^2$. Let's compute the cross term. Since the Y_i's are independent, we have

$$\begin{aligned}
\mathbb{E}[X_{t_i}X_{t_{i-1}}] &= \mathbb{E}\left[\left(\sum_{j=1}^{N_{t_i}} Y_j\right)\left(\sum_{k=1}^{N_{t_{i-1}}} Y_k\right)\right] \\
&= \mathbb{E}\left[\sum_{j=1}^{N_{t_{i-1}}} Y_j^2\right] \\
&= \lambda t_{i-1}\sigma^2.
\end{aligned}$$

Returning to (1), we then have

$$\begin{aligned}
V_t^{(2)}(\Pi, X) &= \lambda\sigma^2 \sum_{i=1}^{n} |t_i - t_{i-1}| \\
&= \lambda\sigma^2 t.
\end{aligned}$$

Since this quantity is independent of the partition Π, we conclude that $\langle X \rangle_t = \lambda\sigma^2 t$. $\square$

(c.) Compute $\mathbb{E}[X_t^2 - \langle X \rangle_t]$.

Solution. By the linearity of expectation we have

$$\begin{aligned}
\mathbb{E}[X_t^2 - \langle X \rangle_t] &= \mathbb{E}[X_t^2] - \mathbb{E}[\langle X \rangle_t] \\
&= \lambda\sigma^2 t - \lambda\sigma^2 t \\
&= 0.
\end{aligned}$$

□