Math 175 - Homework 4

1. The $n \times n$ determinant a_n is defined for $n \geq 1$ by

$$a_n = \begin{vmatrix} p & p-q & 0 & 0 & \cdots & 0 & 0 \\ q & p & p-q & 0 & \cdots & 0 & 0 \\ 0 & q & p & p-q & \cdots & 0 & 0 \\ 0 & 0 & q & p & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & p & p-q \\ 0 & 0 & 0 & 0 & \cdots & q & p \end{vmatrix},$$

   where p and q are distinct nonzero constants. Find a recurrence relation for a_n.

2. Find the number of ways to seat n penguin couples around a table in each of the following cases.

   (a) Male and female penguins alternate.

   (b) Every female penguin is next to her mate.

3. Let d_n be the number of bijections $f : [n] \to [n]$ so that $f(k) \neq k$ for all $k \in [n]$. Find a recurrence relation for d_n.

4. Let $1 \leq r \leq n$ and consider all r-element subsets of the set $\{1, 2, \ldots, n\}$. Each of these subsets has a largest element. Find the arithmetic mean of these largest elements.

5. A triangulation of an n-gon P_n, $n \geq 3$, is a subdivision of P_n into triangles by means of nonintersecting diagonals of P_n. Let $a_0 = 1$ and for $n \geq 1$, let t_n be the number of different triangulations of an $(n+2)$-gon. Find a recurrence relation for t_n.