HomeWork 3 Math 271A, Fall 2019.

1. Show that the conditions of Kolmogorov's extension/consistency theorem is satisfied for the finite dimensional distributions associated with the Brownian motion paths.

2. Let $\mathbf{B}_t$ be a two-dimensional Brownian motion and
$$D_\rho = \{\mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x}| < \rho\}.$$
Compute $\mathbb{P}(\mathbf{B}_t \in D_\rho)$.

3. Let B_t be Brownian motion. Show that

   a. $Y_t = B_T - B_{T-t}$ is Brownian motion on $[0, T]$.

   b. Show that $Y_t' = -B_t$ is also Brownian motion.

   c. Show that
$$Y_t'' = \begin{cases} tB_{1/t}; & 0 < t < \infty \\ 0; & t = 0 \end{cases}$$
   is Brownian motion.

   For part c. you may use that:
$$\lim_{t \to \infty} \frac{B_t}{t} = 0 \quad a.s.$$

4. For τ_i being independent exponentially distributed random variables with parameter λ. Let $S_0 = 0$ and $S_n = \sum_{i=1}^n \tau_i, n \geq 1$. We may then think of S_n as the time at which the n'th customer arrives in a queue, and S_n the inter arrival times. Define as in class
$$N_t = \max(n \geq 0; S_n \leq t).$$

   a. Show that the process N_t is right continuous.

   b. Show that N_t is a Poisson random variable with parameter λt.

5. For $Y_i, i \geq 1$ being independent centered random variables with variance σ^2 construct the compound Poisson process as in class by
$$X_t = \sum_{i=1}^{N_t} Y_i.$$

   a. Compute the mean and variance of X_t.

   b. Compute the characteristic function for X_t.

   c. Show that the sequence of random variables $Z_n = X_n/\sqrt{\lambda n}, n = 1, 2, 3...$ converges weakly to a normal random variable, what is its variance?

6. Let $X_t, t \geq 0$ be a a Gaussian process with zero mean and with covariance $\mathbb{E}[X_t X_{t+s}] = \exp(-\alpha|s|)$ (this is the Ornstein-Uhlenbeck process). Show that X has a version with continuous paths.