1. A random variable X has density

$$f(x) = \begin{cases} c(x+\sqrt{x}) & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(*i*) Determine c (that is, for which value of c the function f is indeed a density function?).

(*ii*) Compute $\mathbb{E}[1/X]$.

(*iii*) Compute the density function of $Y = X^2$.

(*iv*) Compute $Var(X)$.

(*v*) Suppose that 10 independent trials are being performed, each of which is distributed as X^2 (so the output of each trial is some number). What is the probability that in exactly 3 of the trials we will obtain numbers smaller than $1/4$?

iii) Start by computing the CDF of X^2, $G(t)$.

$$G(t) = \Pr[X^2 \le t]$$

$$= \Pr[-\sqrt{t} \le X \le \sqrt{t}]$$

$$= \Pr[X \le \sqrt{t}] \text{ since } X \in [0,1].$$

$$= F(\sqrt{t}),$$

where $F(t) = \Pr[X \le t]$ is the CDF of X.

density of X^2, $g(t) = G(t)'$

$$= [F(\sqrt{t})]' = \frac{F'(\sqrt{t})}{2\sqrt{t}} = \frac{f(\sqrt{t})}{2\sqrt{t}}$$

$$= \frac{1}{2\sqrt{t}} c\left(\sqrt{t} + \sqrt{\sqrt{t}}\right) = \boxed{\frac{c}{2}\left(1 + t^{-1/4}\right)}$$

(v) Suppose that 10 independent trials are being performed, each of which is distributed as X^2 (so the output of each trial is some number). What is the probability that in exactly 3 of the trials we will obtain numbers smaller than $1/4$?

binomial. $\text{Binom}(p, n)$

prob of success

\# of trials $= 10$

a trial is a success if $X^2 \leq 1/4$

$$p = \Pr[X^2 \leq 1/4]$$
$$= \Pr[-1/2 \leq X \leq 1/2]$$
$$= \Pr[X \leq 1/2] \quad \text{since } X \geq 0$$
$$= \int_0^{1/2} f(x)\,dx$$
$$= c\int_0^{1/2} x + \sqrt{x}\,dx = c\left[\tfrac{1}{2}x^2 + \tfrac{2}{3}x^{3/2}\right]_0^{1/2}$$
$$= c\left[\tfrac{1}{8} + \tfrac{2}{3}\cdot\left(\tfrac{1}{2}\right)^{3/2}\right]$$

plug in from part (i)

$\Pr[3 \text{ successes in } 10 \text{ trials}]$

$$= \boxed{\binom{10}{3} p^3 (1-p)^7} \quad \text{where } p =$$

if it asked for ≥ 3 successes

$$\sum_{k=3}^{10}\binom{10}{k}p^k(1-p)^{10-k}$$

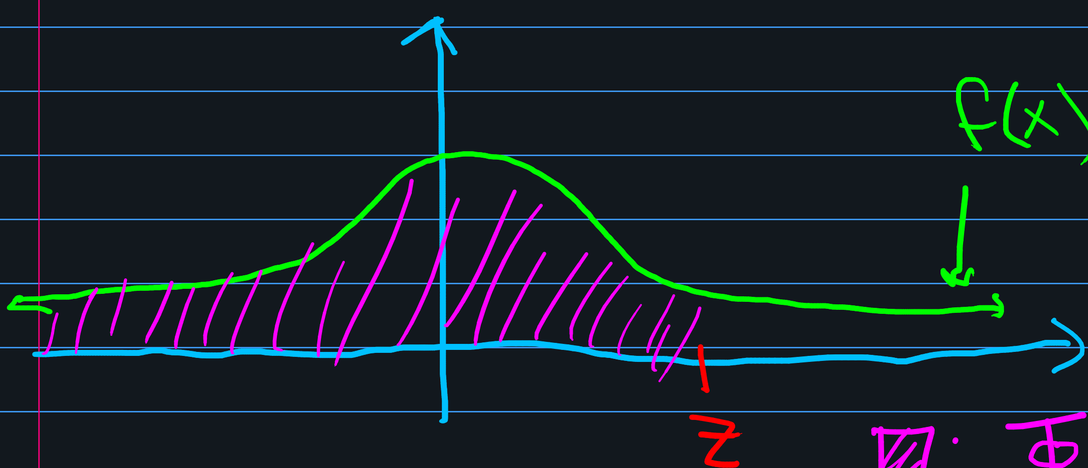
$$=\sum_{k=3}^{10}\Pr[k\text{ successes}]$$

$$=1-\sum_{k=0}^{2}\Pr[k\text{ successes}]$$

$$\Phi(z)=\Pr[N\leq z]\text{, where}$$

$$N\sim\mathcal{N}(0,1)$$

normal, mean 0, var 1



$$f(x)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

$$\Phi(z)=\int_{-\infty}^{z}f(t)\,dt$$

Table 5.1 Area $\Phi(x)$ Under the Standard Normal Curve to the Left of X.

X	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

$\Pr[N \leq 1.71]$

$\approx .9564$

say you have

$X \sim N(\mu, \sigma^2)$

$\Pr[X \leq t]$

$$z = \frac{t-\mu}{\sigma}$$

then look up $\Pr[N \leq z]$

$$\frac{X-\mu}{\sigma} \sim N(0,1)$$

2. Let $X \sim Poisson(1)$, $Y \sim Geo(2/3)$, and suppose that $N \sim N(\mu, \sigma^2)$, where

$$\mu = 9 \cdot \Pr[Y \geq 3 \mid Y \geq 1] \text{ and } \sigma^2 = 4\mathbb{E}[X].$$

Calculate

$$\Pr[-1 \leq N \leq 2].$$

$-$ $+$

$$Z_+ = \frac{2-\mu}{\sigma}$$

$$Z_- = \frac{-1-\mu}{\sigma}$$

$$\Phi(Z_+) - \Phi(Z_-)$$

look up on table

$$\mu = 9 \cdot \Pr[Y \geq 3 \mid Y \geq 1]$$
$$= 9 \cdot \Pr[Y \geq 3]$$

since $Y \geq 1$ always

$$= 9(1 - \Pr[Y \leq 2])$$
$$= 9(1 - (1-(1-p)^2))$$
$$= 9\left(1 - \frac{2}{3}\right)^2 = 1$$

X is memoryless if

$$\Pr[X \geq a+b \mid X \geq a] = \Pr[X \geq b]$$