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271A - Homework 1

1. (a) Let Δ_1, $\Delta_2, \ldots$ be independent random variables with mean 0 and variance 1. Let $X_1 = \Delta_1$ and for $n = 1, 2, \ldots$ let $X_{n+1} = X_n + \Delta_{n+1} f_n(X_1, \ldots, X_n)$ for f_n given bounded deterministic functions. Show that $\{X_n\}$ is a martingale (specify the filtration).

Solution. Let $\{\mathcal{F}_n\}$ be the filtration given by $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. We have that $\mathbb{E}[|X_1|] = \mathbb{E}[|\Delta_1|] < \infty$, since Δ_1 has finite mean. Suppose now that $\mathbb{E}[|X_n|] < \infty$ for all $n \leq k$ for some k. We then have

$$\begin{aligned}
\mathbb{E}[|X_{k+1}|] &= \mathbb{E}[|X_k + \Delta_{k+1} f_k(X_1, \ldots, X_k)|] \\
&\leq \mathbb{E}[|X_k|] + \|f_k\|_{L^\infty} \cdot \mathbb{E}[|\Delta_{k+1}|] \\
&< \infty.
\end{aligned}$$

By induction, each X_n is integrable. Since we're dealing with a discrete stochastic process, it suffices to check the martingale property on consecutive variable-filtration pairs, $\mathbb{E}[X_{n+1}|\mathcal{F}_n]$. Here's a computation.

$$\begin{aligned}
\mathbb{E}[X_{n+1}|\mathcal{F}_n] &= \mathbb{E}[X_{n+1} - X_n + X_n|\mathcal{F}_n] \\
&= \mathbb{E}[\Delta_{n+1} f_n(X_1, \ldots, X_n)|\mathcal{F}_n] + X_n \\
&= \mathbb{E}[\Delta_{n+1}|\mathcal{F}_n] \cdot f_n(X_1, \ldots, X_n) + X_n \quad (f_n(X_1, \ldots, X_n) \text{is } \mathcal{F}_n \text{ measurable}) \\
&= \mathbb{E}[\Delta_{n+1}] \cdot f_n(X_1, \ldots, X_n) + X_n \quad (\Delta_{n+1} \text{ is independent of } \mathcal{F}_n) \\
&= X_n.
\end{aligned}$$

Thus, $\{X_n\}$ is a martingale adapted to the filtration $\{\mathcal{F}_n\}$. □

(b) Let $Y_1, \ldots$ be independent random variables with mean 0 and variance σ^2. Let $X_n = (\sum_{k=1}^n Y_k)^2 - n\sigma^2$ and show that $\{X_n\}$ is a martingale.

Solution. Let $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. Let's verify that the variables X_n are integrable.

$$\begin{aligned}
\mathbb{E}[|X_n|] &\leq \sum_{k=1}^n \mathbb{E}[Y_k^2] + 2 \sum_{1 \leq i < j \leq n} \mathbb{E}[Y_i Y_j] + n\sigma^2 \\
&= 2n\sigma^2 \quad (\text{since } Y_i \text{ and } Y_j \text{ are independent for } i \neq j).
\end{aligned}$$

Great. Now let's show that $\mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n$.

$$\begin{aligned}
\mathbb{E}[X_{n+1}|\mathcal{F}_n] &= \mathbb{E}[X_{n+1} - X_n + X_n|\mathcal{F}_n] \\
&= \mathbb{E}\left[\left(\sum_{k=1}^{n+1} Y_k\right)^2 - \left(\sum_{k=1}^{n} Y_k\right)^2 \middle| \mathcal{F}_n\right] - \sigma^2 + X_n \\
&= \mathbb{E}\left[Y_{n+1}\left(2\sum_{k=1}^{n} Y_k + Y_{n+1}\right) \middle| \mathcal{F}_n\right] - \sigma^2 + X_n \\
&= \mathbb{E}[Y_{n+1}^2|\mathcal{F}_n] + 2\sum_{k=1}^{n} \mathbb{E}[Y_{n+1}Y_k|\mathcal{F}_n] - \sigma^2 + X_n \\
&= \sigma^2 + 0 - \sigma^2 + X_n \quad \text{(since the } Y_k\text{'s are independent)} \\
&= X_n.
\end{aligned}$$

□

2. (a) Show that if $X_n \to X$ in L^p, $p \geq 1$, then

$$X_n \to X \text{ in probability.}$$

Solution. Suppose that the random variables X_n are defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. First, we claim that if $X_n \to X$ in L^1. This follows from Hölder's inequality and the finiteness of $\mathbb{P}(\Omega)$.

$$\begin{aligned}
\int_\Omega |X_n - X| \, d\mathbb{P} &\leq \|X_n - X\|_{L^p} \cdot \mathbb{P}(\Omega)^{1/q} \\
&\to 0,
\end{aligned}$$

where q is the Hölder conjugate of p. Now suppose that X_n didn't converge to X in probability. Then for some $\epsilon > 0$, there are infinitely many n such that $\mathbb{P}[E_n > \epsilon] > \epsilon$, where E_n is the event $E_n = \{|X_n - X| > \epsilon\}$. Check this out

$$\begin{aligned}
\int_\Omega |X_n - X| \, d\mathbb{P} &\geq \int_{E_n} |X_n - X| \, d\mathbb{P} \\
&\int_{E_n} \epsilon \, d\mathbb{P} \\
&= \epsilon^2.
\end{aligned}$$

Then X_n *doesn't* converge to X in L^1. We conclude that $X_n \to X$ in probability. □

(b) Construct an example with a sequence X_n of random variables that converges in L^p, but not almost surely.

Solution. Consider the typewriter sequence $f_{n,k}$ given by $f_{n,k}(x) = \chi_{[k2^{-n},(k+1)2^{-n}]}(x)$, where $n = 1, 2, \dots$ and $k = 0, 1, \dots 2^n - 1$. Since $f_{n,k}$ is supported on a set of measure 2^{-n}, $f_{n,k} \to 0$ in L^1. But $f_{n,k}(x)$ doesn't converge for any x, since for any fixed n, $f_{n,k}(x) = 1$ for some k. Consequently, $f_{n,k}$ doesn't converge almost surely. □

3. Prove that $B^2(t) - t$ is a martingale, where $B(t)$ is a standard Brownian motion.

Proof. Define the filtration $\mathcal{F}_t = \sigma(B_s : s \leq t)$. Let's verify that $B_t^2 - t$ is integrable.

$$\begin{aligned}\mathbb{E}[|B_t^2 - t|] &\leq \mathbb{E}[B_t^2] + t \\ &= 2t.\end{aligned}$$

Here we've used the fact that $B_t - B_s \sim \mathcal{N}(0, t-s)$. Check this out.

$$\begin{aligned}\mathbb{E}[B_t^2 - t|\mathcal{F}_s] &= \mathbb{E}[(B_t - B_s + B_s)^2 - t|\mathcal{F}_s] \\ &= \mathbb{E}[(B_t - B_s)^2|\mathcal{F}_s] + 2B_s\mathbb{E}[B_t - B_s|\mathcal{F}_s] + B_s^2 - t \\ &= \mathbb{E}[(B_t - B_s)^2] + 2B_s\mathbb{E}[B_t - B_s] + B_s^2 - t \quad \text{(since increments are independent)} \\ &= (t - s) + 0 + B_s^2 - t \\ &= B_s^2 - s.\end{aligned}$$

We conclude that $B_t^2 - t$ is a martingale. □

4. Let W_t be a standard n-dimensional Brownian motion and fix $t_0 \geq 0$. Prove that

$$\tilde{W}(t) = U[W(t_0 + t) - W(t_0)] : \ t \geq 0,$$

is a standard n-dimensional Brownian motion for U an orthogonal matrix.

Proof. Intuitively, re-centering coordinates and rotating a Brownian motion should yield another Brownian motion. $\tilde{W}$ starts at zero.

$$\begin{aligned}\tilde{W}(0) &= U[W(t_0) - W(t_0)] \\ &= U(0) \\ &= 0.\end{aligned}$$

U is a linear transformation, so $x \mapsto Ux$ is continuous. Since $t \mapsto W(t)$ is almost surely continuous, $t \mapsto U[W(t_0 + t) - W(t_0)]$ is almost surely continuous. Fix $t_1 \leq t_2 \leq \cdots \leq t_m$. Since $\{W(t_k) - W(t_{k-1}) : k = 1, \dots, m\}$ are independent and $x \mapsto Ux$ is a bijection,

$\{\tilde{W}(t_k) - \tilde{W}(t_{k-1})\} = \{U[W(t_k + t_0) - W(t_{k-1} + t_0)\}$ are also independent.

Let $\tilde{W}^{(1)}, \ldots, \tilde{W}^{(n)}$ be the components of $\tilde{W}$. Since $W(t + t_0) - W(t_0)$ is a standard Brownian motion, the components of the increments $\tilde{W}(t) - \tilde{W}(s)$ are given by

$$\tilde{W}^{(i)}(t) - \tilde{W}^{(i)}(s) = u_{i,1}\xi_1 + u_{i,2}\xi_2 + \cdots + u_{i,n}\xi_n,$$

where $u_{i,j}$ is the i, j-th entry of U and each ξ_j has distribution $\mathcal{N}(0, t - s)$. Since the components of W are independent, so are the ξ_k's. Because U is an orthogonal matrix, $u_{i,1}^2 + \cdots + u_{i,n}^2 = 1$ and we have

$$\begin{aligned}\xi_j &\sim u_{i,1}\mathcal{N}(0, t - s) + u_{i,2}\mathcal{N}(0, t - s) + \cdots + u_{i,n}\mathcal{N}(0, t - s) \\ &= \mathcal{N}(0, (u_{i,1}^2 + \cdots + u_{i,n}^2)(t - s)) \\ &= \mathcal{N}(0, t - s).\end{aligned}$$

We conclude that $\tilde{W}$ is a Brownian motion. $\square$