Math 175 - Homework 8

1. Find the number of n-digit quaternary sequences (each digit is 0, 1, 2, or 3) that contain an odd number of 0's, an even number of 1's, and at least one 3.

2. For $m, n \in \mathbb{N}$, let $\mathcal{M}$ be the set of all $m \times n$ matrices whose entries are zeros or ones. Let

$$\mathcal{M}_r = \{M \in \mathcal{M} : M \text{ has at least one zero row}\}$$

and

$$\mathcal{M}_c = \{M \in \mathcal{M} : M \text{ has at least one zero column}\}.$$

Show that the number of matrices in $(\mathcal{M} \setminus \mathcal{M}_r) \cap \mathcal{M}_c$ is given by

$$\sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} (2^{n-i} - 1)^m.$$

3. Let $S_1, S_2, \ldots, S_{50}$ be subsets of a finite set S such that any subset has more than half of the number of elements of S. Prove that there exists a subset of S with at most 5 elements that has nonempty intersection with each of the 50 subsets. *Hint: let $S = \{s_1, \ldots, s_n\}$ and for $s \in S$ define $d(i)$ to be the number of subsets among $S_1, \ldots, S_{50}$ that contain s_i. Consider the sum*

$$d(1) + d(2) + \cdots + d(n).$$

4. Let $G = (V, E)$ be a finite graph and let $c(G)$ be the number of connected components of G.

   (a) Show that for any edge $e \in E$, $c(G) \geq c(G - e) - 1$, where $G - e$ is the graph obtained by deleting the edge e from G.

   (b) Show that $c(G) + |E| \geq |V|$. *Hint: Induction.*

5. In the last homework we defined an isomorphism of graphs. If G is a graph, an *automorphism* of G is an isomorphism from G to itself.

   (a) Let $\mathrm{Aut}(G)$ be the set of automorphisms of G. Show that $\mathrm{Aut}(G)$ is a group under composition.

   (b) A graph whose vertices are labeled but whose edges are not is called a *labeled graph*. Let $\mathcal{G}_n$ be the set of labeled graphs with vertex set $V = \{v_1, \ldots, v_n\}$. Draw a picture of all the graphs in $\mathcal{G}_3$.

   (c) Show that the number of distinct labellings of a given unlabeled graph G on n vertices is $n!/|\mathrm{Aut}(G)|$. Deduce that

$$\sum_{G \in \mathcal{G}_n} \frac{n!}{|\mathrm{Aut}(G)|} = 2^{\binom{n}{2}}.$$

   (d) Deduce further that the number of unlabeled graphs on n vertices is at least

$$\left\lceil \frac{2^{\binom{n}{2}}}{n!} \right\rceil.$$