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271A - Homework 5

Problem 1. Consider the space $\mathbb{R}^d$ and the usual $\|\cdot\|_2$ metric. Show explicitly that a probability measure $\mathbb{P}$ on the measurable space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is uniquely determined by

$$F(x_1, \ldots, x_d) = \mathbb{P}[y : y_1 \leq x_1, \ldots, y_d \leq x_d].$$

Proof. Our strategy is to use the π-λ theorem. Suppose $\mathbb{P}_1$ and $\mathbb{P}_2$ are two probability measures that agree on sets of the form $\{y_1 \leq x_1, \ldots, y_d \leq x_d)$. Define the collection Π by

$$\Pi = \big\{\{y : y_1 \leq x_1, \ldots, y_d \leq x_d\} : x \in \mathbb{R}^d\big\}.$$

That is, Π consists of all products of rays. As our notation suggests, Π is a π-system since it is clearly nonempty and the intersection of any two products of rays is again a product of rays. We also define the collection Λ to be the sets in $\sigma(\Pi)$, the σ-algebra generated by Π, on which $\mathbb{P}_1$ and $\mathbb{P}_2$ agree:

$$\Lambda = \{E \in \sigma(\Pi) : \mathbb{P}_1(E) = \mathbb{P}_2(E)\}.$$

This collection is indeed well-defined since every set in Π is a Borel set, so $\sigma(\Pi) \subseteq \mathcal{B}(\mathbb{R}^d)$ and each $E \in \sigma(\Pi)$ is $\mathbb{P}_1$ and $\mathbb{P}_2$ measurable. We again claim that our notation makes sense and that Λ is a λ-system. Let's verify this claim.

- $\mathbb{R}^d \in \Lambda$: We can write $\mathbb{R}^d$ as a union of ray-products, $\mathbb{R}^d = \bigcup_{n=1}^{\infty}(-\infty, n]^d$, so $\mathbb{R}^d$ is indeed in $\sigma(\Pi)$. That $\mathbb{P}_1[\mathbb{R}^d] = \mathbb{P}_2[\mathbb{R}^d]$ follows from the fact that $\mathbb{P}_1$ and $\mathbb{P}_2$ are probability measures.
- Closure under complements: Let $E \in \Lambda$. Since $\mathbb{P}_1$ and $\mathbb{P}_2$ are probability measures, we can write
$$\mathbb{P}_1[E^c] = 1 - \mathbb{P}_1[E] = 1 - \mathbb{P}_2[E] = \mathbb{P}_2[E^c].$$
- Closure under countable disjoint union: Let $\{E_n\}_{n=1}^{\infty}$ be a countable disjoint family in Λ. By countable additivity of measure, we have
$$\mathbb{P}_1(\cup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mathbb{P}_1[E_n] = \sum_{n=1}^{\infty} \mathbb{P}_2[E_n] = \mathbb{P}_2(\cup_{n=1}^{\infty} E_n).$$

Hence, Λ is a λ-system. By the π-λ theorem, $\sigma(\Pi) \subseteq \Lambda$. By construction, we also have $\Lambda \subseteq \sigma(\Pi)$, so $\Lambda = \sigma(\Pi)$. But the products of rays generate all of $\mathcal{B}(\mathbb{R}^d)$, so we must have $\mathbb{P}_1 = \mathbb{P}_2$ on all of $\mathcal{B}(\mathbb{R}^d)$. $\square$

Problem 2. Show that if a set A of continuous paths on $[0, 1]$ is equicontinuous at each point in $[0, 1]$ then the set is uniformly equicontinuous.

Proof. Suppose the family is not uniformly equicontinuous but still equicontinuous at each point in $[0, 1]$. Then there is some $\epsilon > 0$ such that for all $n \in \mathbb{N}$, there is some $f_n \in A$ and $x_n, y_n \in [0, 1]$ so that $|x_n - y_n| < 1/n$ but

$$|f_n(x_n) - f_n(y_n)| \geq 2\epsilon. \tag{1}$$

Since $[0,1]$ is compact, there is a subsequence n_k so that $x_{n_k} \to x^*$. Since $|x_{n_k} - y_{n_k}| < 1/n_k$, we have that $y_{n_k} \to x^*$ as well.

Now by equicontinuity, there is some $\delta > 0$ so that $|x^* - x| < \delta$ implies that $|f(x^*) - f(x)| < \epsilon$ for all $f \in A$. Choose K large enough so that both $|x_{n_k} - x^*|$ and $|y_{n_k} - x^*|$ are both less than δ for all $k > K$. For $k > K$ we then have

$$\begin{aligned} |f(x_{n_k}) - f(y_{n_k})| &\leq |f(x_{n_k}) - f(x^*)| + |f(x^*) - f(y_{n_k})| \\ &< \epsilon + \epsilon \\ &= 2\epsilon. \end{aligned}$$

But this contradicts (1), so we conclude that A must be uniformly equicontinuous. □

Problem 3. Let ζ_i, $i = 1, 2, \ldots$ be iid with finite first, second, and fourth moments, and consider the random walk $S_n = \sum_{i=1}^n \zeta_i$. Define the process Y_t by interpolating S_n as follows:

$$Y_t = \sum_{i=1}^{\lfloor t \rfloor} \zeta_i + (t - \lfloor t \rfloor)\zeta_{i+1}.$$

By properly rescaling and normalizing, construct a family of processes that converges in distribution to a standard Brownian motion.

Solution. Suppose $\mathrm{Var}[\zeta_i] = \sigma^2$. Define the sequence of processes $X_t^{(m)}$ by

$$X_t^{(m)} = \frac{1}{\sigma\sqrt{m}} Y_{mt}.$$

We showed in class that $X^{(m)}$ converges in distribution to a standard Brownian motion and we'll recap some of the details here.

We need to show that the family of processes $X^{(m)}$ is tight and that its finite dimensional distributions converge weakly to those of Brownian motion. □

Problem 4. Suppose $\{X_n\}_{n=1}^\infty$ is a sequence of random variables taking values in a metric space (S_1, ρ_1) and converging in distribution to X. Suppose (S_2, ρ_2) is another metric space, and $\phi : S_1 \to S_2$ is continuous. Show that $Y_n = \phi(X_n)$ converges in distribution to $Y = \phi(X)$.

Proof. Since $X_n \to X$ in distribution, we have that for all bounded continuous $f : S_1 \to \mathbb{R}$,

$$\mathbb{E}[f(X_n)] \to \mathbb{E}[f(X)]. \tag{2}$$

Now let g be any bounded continuous function $g : S_2 \to \mathbb{R}$. Since ϕ is continuous, so is $g \circ \phi$, and since g is bounded, so is $g \circ \phi$. The composition $g \circ \phi$ is then a bounded continuous function $S_1 \to \mathbb{R}$, so by (2), we have

$$\mathbb{E}[g(Y_n)] = \mathbb{E}[(g \circ \phi)(X_n)] \to \mathbb{E}[(g \circ \phi)(X)] = \mathbb{E}[g(Y)].$$

□

Problem 5. Consider the space $C[0,1]$ of continuous functions on $[0,1]$ with the supremum metric and associated norm. Show that this metric space is separable and complete. Show that a probability measure on $(C[0,1], \mathcal{B}(C[0,1]))$ is tight.

Proof. The polynomials with rational coefficients form a countable dense subset of $C[0,1]$ by the Weierstrass approximation theorem. To show completeness, suppose suppose f_n is a Cauchy (in our metric) sequence of continuous functions on $[0,1]$. f_n is then pointwise Cauchy, so there is a pointwise limit $f : [0,1] \to \mathbb{R}$. It remains to show that f is continuous. To this end, let x be an arbitrary point in $[0,1]$ and let $\epsilon > 0$ be arbitrary. We then have

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)|.$$

Since f_n converges to f in the uniform norm, we can make the first and third terms small for all n sufficiently large. If n is sufficiently large, the continuity of f_n gives us a $\delta > 0$ so that $|x - y| < \delta$ implies $|f_n(x) - f_n(y)| < \epsilon$. For n large and δ chosen in this way, the above quantity can be made small, so f is continuous and our space is complete.

Let $\mathbb{P}$ be a probability measure on $(C[0,1], \mathcal{B}(C[0,1]))$. We want to show that the family $A = \{\mathbb{P}\}$ is tight. By Prokhorov's theorem, A is tight if and only if every sequence in A has a weakly convergent subsequence. But every sequence in A is the constant sequence $(\mathbb{P}, \mathbb{P}, \ldots)$, and hence weakly convergent. We conclude that the singleton $\{\mathbb{P}\}$ is tight. (This seems too easy. Am I misunderstanding Prokhorov's theorem?) $\square$

Problem 6. Let X_t, $0 < t < 2^N$ be a stochastic process. Define the Haar detail coefficients by

$$d_n(j) = \frac{1}{\sqrt{2^n}} \int_{\mathbb{R}} \psi(t/2^n - j) X(t)\, dt, \quad n = 1, 2, \ldots, N, \quad j = 1, 2, \ldots, 2^{N-n},$$

with the mother wavelet defined by

$$\psi(t) = \begin{cases} -1 & \text{if } -1 \leq t < -1/2 \\ 1 & \text{if } -1/2 \leq t < 0 \\ 0 & \text{otherwise} \end{cases}.$$

The scale spectrum of X relative to the Haar wavelet basis is the sequence S_j defined by

$$S_n = \frac{1}{2^{N-n}} \sum_{j=1}^{2^{N-n}} d_n(j)^2, \quad n = 1, 2, \ldots, N.$$

Assume that X is a centered, continuous, Gaussian process, starting at the origin, with homogeneous increments and covariance function

$$\mathbb{E}[X_t X_s] = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}),$$

for some parameter $H \in (0,1)$. Compute $\mathbb{E}[S_n]$.

Solution. Suppose X is a process defined on the triplet $(\Omega, \mathcal{F}, \mathbb{P})$. By the linearity of expectation we have

$$\mathbb{E}[S_n] = \frac{1}{2^{N-n}} \sum_{j=1}^{2^{N-n}} \mathbb{E}[d_n(j)^2].$$

Let's compute $\mathbb{E}[d_n(j)^2]$. By Fubini we have

$$\begin{aligned}
\mathbb{E}[d_n(j)^2] &= \int_\Omega \left(\frac{1}{\sqrt{2^n}} \int_\mathbb{R} \psi(t/2^n - j) X_t(\omega)\ dt \right)^2 d\mathbb{P} \\
&= \frac{1}{2^n} \int_\mathbb{R} \int_\mathbb{R} \psi(s/2^n - j)\psi(t/2^n - j) \int_\Omega X_s(\omega) X_t(\omega)\ d\mathbb{P}\ dtds \\
&= \frac{1}{2^n} \int_\mathbb{R} \int_\mathbb{R} \psi(s/2^n - j)\psi(t/2^n - j) \mathbb{E}[X_s X_t]\ dtds \\
&= \frac{1}{2^{n+1}} \int_\mathbb{R} \int_\mathbb{R} \psi(s/2^n - j)\psi(t/2^n - j)(t^{2H} + s^{2H} - |t-s|^{2H})\ dtds.
\end{aligned}$$

Now since $\int \psi(s/2^n - j)\ ds = 0$ and t^{2H} does not depend on s, the integral of the t^{2H} term above vanishes. For the same reason, the integral of the s^{2H} term vanishes and we're left with

$$\mathbb{E}[d_n(j)^2] = -\frac{1}{2^{n+1}} \int_\mathbb{R} \int_\mathbb{R} \psi(s/2^n - j)\psi(t/2^n - j)|t-s|^{2H}\ dtds.$$

Now let's substitute $s' = \frac{s-2^n j}{2^n}$ and $t' = \frac{t-2^n j}{2^n}$ to get (after renaming s' to s and t' to t)

$$\mathbb{E}[d_n(j)^2] = -2^{n(2H+1)-1} \int_\mathbb{R} \int_\mathbb{R} \psi(s)\psi(t)|t-s|^{2H}\ dtds.$$

The function $(s,t) \mapsto |t-s|^{2H}$ is symmetric about the line $s = t$ and its level sets $|t-s|^{2H} = c$ are pairs of parallel lines that themselves run parallel to the line $s = t$. The product $\psi(s)\psi(t)$ is nonnegative in the squares $[-1,-1/2) \times [-1,-1/2)$ and $(-1/2, 0] \times (-1/2, 0]$ of the s-t plane. Conversely, the product is negative in the squares $[-1,-1/2) \times (-1/2, 0]$ and $(-1/2, 0] \times [-1,-1/2)$. In terms of the diagram in Figure 1, our integral becomes

$$\begin{aligned}
\mathbb{E}[d_n(j)^2] &= -2^{n(2H+2)-1} \left(4 \int_{R_1} |t-s|^{2H}\ dA - 2 \int_{R_2} |t-s|^{2H}\ dA \right) \\
&= -2^{n(2H+2)-1} \left(4 \int_{-1/2}^{0} \int_{s}^{0} (t-s)^{2H}\ dA - 2 \int_{-1}^{-1/2} \int_{-1/2}^{0} (t-s)^{2H}\ dA \right) \\
&= \frac{(1-2^{-2H})2^{n(2H+2)}}{(2H+1)(2H+2)}.
\end{aligned}$$

Note that this quantity is independent of j. Consequently, we have

$$\mathbb{E}[S_n] = \frac{(1-2^{-2H})2^{n(2H+2)}}{(2H+1)(2H+2)}.$$

$\square$

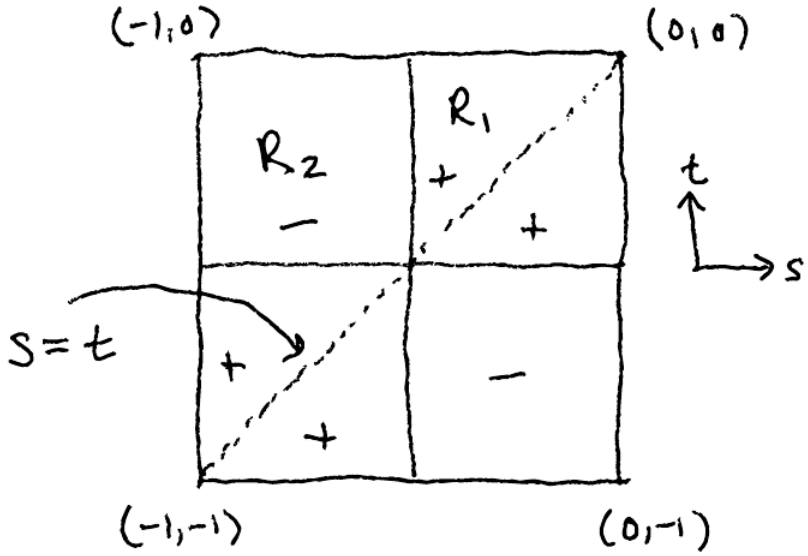

Figure 1: The integral of $|t-s|^{2H}$ over the regions marked with a $+$ are equal. The same goes for those regions marked with a $-$.