HOMEWORK 2
MATH 270B, WINTER 2020, PROF. ROMAN VERSHYNIN

PROBLEM 1 (SCHEFFÉ'S THEOREM)

(a) (For continuous distributions) Prove that if probability density functions of X_n converge to probability density function of X pointwise, then X_n converges to X weakly.

(b) (For discrete distributions) Prove that if probability mass functions of X_n converge to probability mass function of X pointwise, then X_n converges to X weakly.

(c) (No converse) In general, weak convergence does not imply pointwise convergence of probability density functions. Show this by example.

PROBLEM 2 (WEAK LIMIT OF NORMAL RANDOM VARIABLES)

Consider normal random variables $X_n \sim N(\mu_n, \sigma_n^2)$. Assume X_n converge weakly to some random variable X. Prove that $X \sim N(\mu, \sigma^2)$ where $\mu = \lim \mu_n$ and $\sigma^2 = \lim s_n^2$ (and both limits exist).

PROBLEM 3 (NO CONVERGENCE IN PROBABILITY IN CLT)

Let $X_1, X_2, \ldots$ be independent Rademacher random variables[1] Let $S_n = X_1 + \cdots + X_n$.

(a) Prove that the sequence $(S_n/\sqrt{n})$ is unbounded almost surely.

(b) Prove that $(S_n/\sqrt{n})$ does not converge in probability.

PROBLEM 4 (NON-SUMMABLE VARIANCES YIELD CLT)

Let $X_1, X_2, \ldots$ be independent random variables such that there exists $M > 0$ so that $|X_i| \le M$ almost surely for all i. Show that if $\sum_i \mathrm{Var}(X_i) = \infty$ then the sum $S_n = X_1 + \cdots + X_n$ satisfies

$$\frac{S_n - \mathbb{E}\, S_n}{\sqrt{\mathrm{Var}(S_n)}} \to N(0,1) \quad \text{weakly.}$$

[1] A Rademacher random variable takes values $-1, 1$ with probability $1/2$ each.

Problem 5 (Lyapunov's CLT)

Let $X_1, X_2, \dots$ be independent random variables with zero means and unit variances. (Do not assume that X_i have the same distribution though.) Assume that

$$\sup_i \mathbb{E}\,|X_i|^{2+\delta} < \infty$$

for some $\delta > 0$. Prove that

$$\frac{X_1 + \cdots + X_n}{\sqrt{n}} \to N(0,1) \quad \text{weakly.}$$

Problem 6 (Bayes formula)

Let (Ω, Σ, P) be a probability space and $\mathcal{F} \subset \Sigma$ be a sub-σ-algebra. Consider two events $E \in \Sigma$ and $F \in \mathcal{F}$.

(a) Check that

$$P(F|E) = \frac{\mathbb{E}\left[P(E|\mathcal{F})\mathbf{1}_F\right]}{\mathbb{E}\,P(E|\mathcal{F})}.$$

(b) Specialize this equation to the case where $\mathcal{F}$ is generated by a partition $\Omega = F_1 \sqcup \cdots \sqcup F_n$, i.e. $\mathcal{F} = \sigma(F_1, \dots, F_n)$. Deduce Bayes formula in its familiar form:

$$P(F_i|E) = \frac{P(E|F_i)\,P(F_i)}{\sum_i P(E|F_i)\,P(F_i)}.$$

Problem 7 (Conditional Cauchy-Schwarz inequality)

Show that

$$(\mathbb{E}[XY|\mathcal{F}])^2 \le \mathbb{E}[X^2|\mathcal{F}] \cdot \mathbb{E}[Y^2|\mathcal{F}]$$

almost surely.

Problem 8 (Law of total variance)

Define conditional variance of X by

$$\mathrm{Var}(X|\mathcal{F}) := \mathbb{E}[X^2|\mathcal{F}] - (\mathbb{E}[X|\mathcal{F}])^2.$$

Show that

$$\mathrm{Var}(X) = \mathbb{E}(\mathrm{Var}(X|\mathcal{F})) + \mathrm{Var}(\mathbb{E}[X|\mathcal{F}]).$$

Problem 9 (Conditioning always reduces second moment)

Let $Y := \mathbb{E}[X|\mathcal{F}]$. Show that if $\mathbb{E}(Y^2) = \mathbb{E}(X^2)$ then $X = Y$ a.s.