Math 175 - Final Exam Review Problems

1. Calculate the following sum (that is, find an explicit formula with at most two summands):
$$\sum_{k=3}^{n} \binom{n}{k}\binom{k}{3}.$$

2. Stanford beats Washington at soccer by the score 8-6. Joe didn't watch the game but he knows the game was never tied except for 0-0 at the beginning. How many arrangements of the goals (that is, sequences of length 14 with 8 S's and 6 W's) exist?

3. Let a_n be the number of sequences of length n over the alphabet $\{0, 1, 2\}$ for which no two consecutive digits are even. Find an explicit expression for a_n.

4. Calculate the following sum
$$\sum_{k=0}^{n} \binom{2n+1}{k}.$$

5. In how many non-negative integers smaller than 10000 does each of the digits 2, 5, 8 appear at least once?

6. Suppose that we color the squares of a 6×16 chessboard using 3 colors. Show that there exists a rectangle with all of its corners the same color.

7. Show that if a graph G on n vertices has more than $n^2/4$ edges, then it must contain a triangle.

8. How many labeled trees with n vertices have exactly $n/2$ leaves? *Hint: Look at Prüfer codes.*

9. Let A be an $n \times n$ matrix such that all the entries are nonnegative. Moreover, assume that for each i and for each j we have
$$\sum_{k=1}^{n} a_{ik} = 1, \quad \text{and} \quad \sum_{k=1}^{n} a_{kj} = 1.$$
That is, the sum of each row and each column is 1. Show that there is a permutation $\pi : [n] \to [n]$ for which all the entries $a_{i\pi(i)}$ are positive. *Hint: Define a bipartite graph whose parts are copies of $[n]$ with appropriate edges.*

10. Let a_n be the number of nonnegative integer solutions to $x_1 + \cdots + x_{10} = n$ for which $x_1, \ldots, x_5$ are even and $x_6, \ldots, x_{10}$ are odd. Find the generating function of the sequence (a_n) and calculate a_{2020}.

11. Let a_n be the number of subsets $A \subseteq [n]$ for which the following restriction holds: for every $i \leq n$, if $i \in A$ then at least one of $i-1$, $i+1$ is in A as well. For example, if $5 \in A$ then at least one of 4 and 6 is in A. Find a recurrence relation for a_n. *Hint: write A as a binary vector.*

12. Show that there exists an integer k for which the number 7^k ends with the digits 0001 (i.e. $7^k - 1$ is divisible by 10000).

13. How many nonnegative integers solutions are there to $x_1 + x_2 + x_3 = 70$ if

    (a) $x_1, x_2, x_3 \leq 30$?

    (b) $x_2, x_3 \geq 15$ and $x_1 \geq 10$?

    (c) $x_1 \leq 5$, $x_2 \leq 40$ and $x_3 \leq 40$?

14. Consider the grid $\mathbb{Z} \times \mathbb{Z}$. Suppose that Fred the mosquito is standing at $(0,0)$ and wants to reach (n,k). At each time step, Fred can move either one step up or one step to the right. In how many ways can Fred reach (n,k)? What if Fred wants to reach $(100,70)$ but is not allowed to pass through $(60,50)$?

15. Let $m \geq n$. Show that the number of surjective functions $f : [m] \to [n]$ is precisely

$$\sum_{k=0}^{n-1} (-1)^k \binom{n}{k} (n-k)^m.$$

    When $m = n$ prove the identity

$$n! = \sum_{k=0}^{n-1} (-1)^k \binom{n}{k} (n-k)^n.$$

16. How many positive integers less than 1000 have no factor between 1 and 10?

17. Let A be an $n \times n$ matrix with entries $1, 2, \ldots, n^2$ (each number appears exactly once in A). Count the number of matrices B with the same property for which no row of B is identical to any of the rows of A.

18. Find a generating function for a_k in each case.

    (a) a_k is the number of solutions to $x_1 + x_2 + x_3 = k$ where $x_i \geq 0$ for each i and $x_i \neq x_j$ for all $i \neq j$.

    (b) a_k is the number of solutions to $x_1 + x_2 + x_3 + x_4 = k$ where $x_i \geq 0$ for each i, x_1 is divisible by 4, $x_2 < 5$ and $x_3 \leq x_4$.

    (c) a_k is the number of solutions to $x_1 + \cdots + x_r \leq k$.

19. Find the coefficient of x^{25} in $(1 + x^3 + x^8)^{10}$.

20. Draw n straight lines in the plane so that each pair of lines intersects but no three lines intersect. Find a recurrence relation for a_n, the number of regions the plane is divided into after drawing the n lines. Find an explicit formula for a_n using generating functions.

21. Let G be a d-regular graph on n vertices containing no cycle of length 4.

    (a) Count the number of paths with three vertices in G.

    (b) Show that no two vertices have more than one common neighbor.

(c) Deduce that $d < 2\sqrt{n}$.

22. Let G be a graph with n vertices.

    (a) Show that if G is disconnected, then it can have at most $\binom{n-1}{2}$ edges.

    (b) Show that if the minimum degree of G is at least $n/2$, then it is connected.

    (c) Show that if G has no triangles and the minimum degree is at least $n/4 + 1$, then it is connected.

23. Let $G = (A \cup B, E)$ is a bipartite graph and let k be a positive integer. Suppose that for each $X \subseteq A$ we have $|N(X)| \geq |X| - k$. Prove that there exists a matching which saturates at least $|A| - k$ vertices in A.

24. Let $G = (A \cup B, E)$ be a bipartite graph such that for all $v \in A$, $d(v) = s$ and for all $u \in B$, $d(u) = t$. Show that if $s \geq t$ then G contains a matching saturating A.

25. Let $S = \{1, \ldots, mn\}$ with $m \leq n$. Partition this set into sets $A_1, \ldots, A_m$ each of which has size n. Now take a second partitioning of S into sets $B_1, \ldots, B_m$ each of which as size n. Show that the sets can be renumbered in such a way that for all i, $A_i \cap B_i \neq \emptyset$.

26. Let k be a positive integer and let $G = (A \cup B, E)$ be a bipartite graph with $|A| = n$ and $|B| = kn$. Suppose that for every $X \subseteq A$ we have $|N(X)| \geq k|X|$. Show that one can partition $B = V_1 \cup \ldots \cup V_k$ into disjoint sets V_i such that for each i there exists a perfect matching between A and V_i.