Homework 4
Math 270A, Fall 2019, Prof. Roman Vershynin

Problem 1 (Generalization of Borel-Cantelli)

Let $E_1, E_2, \dots$ be events on the same probability space. Assume that

$$\mathbb{P}(E_n) \to 0 \quad \text{and} \quad \sum_n \mathbb{P}(E_n \cap E_{n-1}^c) < \infty.$$

Show that

$$\mathbb{P}(E_n \text{ occur i.o.}) = 0.$$

Problem 2 (Extreme values)

Let $X_1, X_2, \dots$ be i.i.d. random variables with the standard exponential distribution, i.e.

$$\mathbb{P}\left\{X_i > x\right\} = e^{-x}, \quad x \geq 0.$$

(a) Show that

$$\limsup_n \frac{X_n}{\log n} = 1 \text{ a.s.}$$

(b) Let $M_n := \max_{1 \leq k \leq n} X_k$. Show that

$$\limsup_n \frac{M_n}{\log n} = 1 \text{ a.s.}$$

Problem 3 (Generalization of Kolmogorov three-series theorem)

Let

$$\psi(x) := \begin{cases} x^2 & \text{when } |x| \leq 1 \\ |x| & \text{when } |x| \geq 1. \end{cases}$$

Let $X_1, X_2, \dots$ be independent mean zero random variables. Show that if $\sum_n \mathbb{E}\, \psi(X_n) < \infty$, then $\sum_n X_n$ converges a.s.

Problem 4 (Failure of the strong law of large numbers)

Construct a sequence of independent mean zero random variables $X_1, X_2, \dots$ such that

$$\frac{1}{n} \sum_{k=1}^n X_k \to \infty \text{ a.s.}$$

Why does not this example contradict the strong law of large numbers?

Problem 5 (Recurrent events)

Suppose disasters occur at random times X_i apart from each other. Precisely, k-th disaster occur at time $T_k := X_1 + \cdots + X_k$ where X_i are i.i.d. random variables taking positive values and with finite mean μ. Let

$$N(t) := \max\{n : \ T_n \le t\}$$

be the number of disasters that have occurred by time t. Prove that

$$N(t) \to \infty \quad \text{and} \quad \frac{N(t)}{t} \to \frac{1}{\mu}$$

almost surely as $t \to \infty$.

(Hint: check that $N(t) < n$ iff $T_n > t$, and $T_{N(t)} \le t < T_{N(t)+1}$. Use the strong law of large numbers for T_n/n.)

Problem 6 (Convergence of series)

Let $X_1, X_2, \ldots$ be independent random variables. Show that $\sum_n X_n$ converges in probability if and only if $\sum_n X_n$ converges almost surely.

(Hint: to prove that the sequence of partial sums S_n is Cauchy, argue as in the proof of Kolmogorov's two-series theorem but use Etemadi's maximal inequality.)

Problem 7 (Divergence of series with positive i.i.d. terms)

Let $X_1, X_2, \ldots$ be i.i.d. random variables taking non-negative values, and such that $\mathbb{P}\{X_i > 0\} > 0$. Prove that

$$\sum_n X_n = \infty \text{ a.s.}$$

Problem 8 (Diophantine approximation)

Call number $x \in [0,1]$ *badly approximable* (by rationals) if there exists $c = c(x) > 0$ and $\varepsilon = \varepsilon(x) > 0$ such that for any $p, q \in \mathbb{N}$ we have

$$\left| x - \frac{p}{q} \right| > \frac{c}{q^{2+\varepsilon}}.$$

Prove that almost all numbers in $[0,1]$ are badly approximable (i.e. all except a set of Lebesgue measure zero).

(Hint: fix c, ε. For each q, consider the set E_q of numbers x that satisfy the reverse inequality. Use Borel-Cantelli lemma for these sets.)

Problem 9 (Random harmonic series)

Let $X_1, X_2, \ldots$ be i.i.d. random variables with finite mean μ. Prove that

$$\frac{1}{\ln n} \sum_{k=1}^{n} \frac{X_k}{k} \to \mu \text{ a.s.}$$

(Hint: work along the subsequence 2^{2^n}*.)*