Homework 1
Math 270B, Winter 2020, Prof. Roman Vershynin

Problem 1 (Poisson central limit theorem)

Let X_i be i.i.d. random variables each having the Poisson distribution with mean 1, and consider $S_n = X_1 + \cdots + X_n$. Let $x \in \mathbb{R}$. Show that if $k = k(n)$ is such that $(k-n)/\sqrt{n} \to x$ as $n \to \infty$, we have

$$\sqrt{2\pi n}\, \mathbb{P}\left\{S_n = k\right\} \to \exp(-x^2/2).$$

(Hint: first show that S_n has Poisson distribution with mean n. Then use Stirling's formula to analyze the limiting behavior of the probability mass function of S_n.)

Problem 2 (Weak convergence without convergence of densities)

Find an example of random variables X_n with densities f_n so that X_n converge weakly to the uniform distribution on $[0,1]$ but $f_n(x)$ does not converge to 1 for any $x \in [0,1]$.

Problem 3 (Extreme values)

Let X_i be i.i.d. random variables each having exponential distribution with mean 1, and consider $M_n := \max_{i \le n} X_i$. Show that $M_n - \log n$ converges weakly to the standard Gubmel distribution, i.e. the distribution with cumulative distribution function $F(x) = \exp(-e^{-x})$.

Problem 4 (Convergence to a constant)

Let X_n be random variables and c be a constant. Prove that weak convergence of X_n to c is equivalent to convergence of X_n to c in probability.

Problem 5 (Convergence together)

Consider the following statement:

$$\text{if } X_n \to X \text{ weakly and } Y_n \to Y \text{ weakly then } X_n + Y_n \to X + Y \text{ weakly.} \qquad (1)$$

(a) Find an example showing that that implication (1) is false in general.

(b) Prove that if Y is a constant, then implication (1) is true.

(c) Prove that if X_n and Y_n are independent, then implication (1) is true.

Problem 6 (Projection of the sphere is Gaussian)

(a) Prove the following implication: if $X_n \to X$ weakly, $Y_n \geq 0$ and $Y_n \to c$ weakly where c is a constant, then $X_n Y_n \to cX$.

(b) Let Z_n be a random vector uniformly distributed on the unit Euclidean sphere of radius $\sqrt{n}$ in $\mathbb{R}^n$. Prove that the distribution of the first coordinate of Z_n (and actually, of any given coordinate) converges weakly to the standard normal distribution.

(Hint: let X_n be standard normal random vector, and consider $Z_n = X_n \cdot \sqrt{n}/\|X_n\|_2$.)

Problem 8 (Operations on characteristic functions)

Prove that if ϕ is a characteristic function of some random variable, then $\mathrm{Re}\phi$ and $|\phi|^2$ are, too.

Problem 9 (Point masses from characteristic function)

Let X be a random variable with characteristic function ϕ. Prove that for any $a \in \mathbb{R}$, we have

$$\mathbb{P}\{X = a\} = \lim_{T \to \infty} \frac{1}{2T} \int_{T}^{-T} e^{-ita} \phi(t)\, dt.$$

(Hint: imitate the proof of the inversion formula.)

Problem 10 (CLT for a random number of terms)

Let X_i be i.i.d. random variables with mean zero and unit variance. and let $S_n := X_1 + \cdots + X_n$. Let N_n be a sequence of nonnegative integer-valued random variables and a_n be a sequence of nonnegative integers such that $a_n \to \infty$ and $N_n/a_n \to 1$ in probability. Show that

$$S_{N_n}/\sqrt{a_n} \to N(0,1)$$

weakly.

(Hint: use Kolmogorov's maximal inequality to conclude that if $Y_n = S_{N_n}/\sqrt{a_n}$ and $Z_n = S_{a_n}/\sqrt{a_n}$, than $Y_n - Z_n \to 0$ in probability.)

Problem 11 (a non-example for Lindeberg-Feller CLT)

Consider independent random variables X_k such that X_k takes values $\pm k$ with probability $k^{-2}/2$ each and values ± 1 with probability $(1-k^{-2})/2$ each. Show that, although $\mathrm{Var}(S_n)/n \to 2$, $S_n/\sqrt{n}$ does not converge to $N(0,1)$ weakly. Why does this example not contradict Lindeberg-Feller central limit theorem?