$X \qquad E[X]$

$$\Pr[|X - E[X]| > t] \le f(t)$$

concentration inequalities

- Markov's ineq | $\Pr[X > t] \le E[X]/t$
- ☆ Chebyshev's "
- Chernoff's "

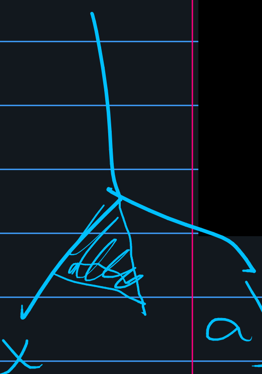
2. The joint density of X, Y is given by

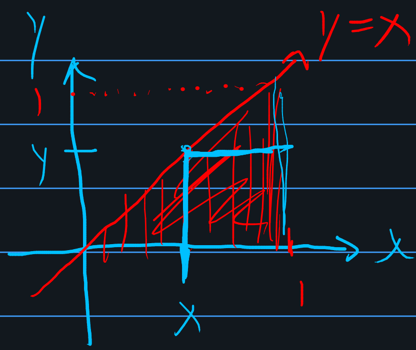
$$f(x,y) = \begin{cases} 3x & \text{if } 0 \le y \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the marginal densities f_X and f_Y.

(b) Compute $COV(X,Y)$.

(c) Compute $Var(X+Y)$.





a) let $f_X(x)$ be the density of X

$f_Y(y)$ be " " of Y

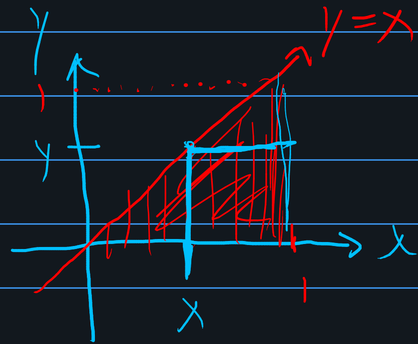
$$f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy$$

$$= \int_0^x 3x\,dy = 3xy\Big|_0^x = 3x^2$$

$$f_Y(y) = \int_y^1 3x\,dx = \frac{3}{2}x^2\Big|_y^1 = \frac{3}{2}(1-y^2)$$

b) $\mathrm{Cov}(X,Y) = \underline{E[XY]} - \underline{E[X]}\,\underline{E[Y]}$

$$\underline{E[XY]} = \int_{0\le y\le x\le 1} xy\, f(x,y)\, dy\, dx$$



$$= \int_{0\le y\le x\le 1} 3x^2 y\, dy\, dx$$

$$= \int_0^1 \int_0^x 3x^2 y\, dy\, dx = \int_0^1 \frac{3}{2} x^2 y^2 \Big|_0^x dx$$

$$= \int_0^1 \frac{3}{2} x^4\, dx = \frac{3}{10} x^5 \Big|_0^1 = \underline{3/10}$$

$$\underline{E[X]} = \int_0^1 x f_X(x)\, dx = \int_0^1 3x^3\, dx$$

$$= \underline{3/4}$$

$$\underline{E[Y]} = \int_0^1 y f_Y(y)\, dy = \int_0^1 \frac{3}{2} y - \frac{3}{2} y^3\, dy$$

$$= \frac{3}{4} - \frac{3}{8} = \underline{3/8}$$

$$\Rightarrow \mathrm{Cov}(X,Y) = \frac{3}{10} - \frac{3}{4}\cdot\frac{3}{8} = \frac{3}{2}\left(\frac{1}{5} - \frac{3}{16}\right)$$

c) $\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\mathrm{Cov}(X,Y)$

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$\mathbb{E}[X^2] = \int_0^1 x^2 f_X(x)\,dx = 3\int_0^1 x^4\,dx$$

$$= 3/5$$

$$\Rightarrow \mathrm{Var}[X] = \frac{3}{5} - \left(3/4\right)^2$$

$$= 3\left(\frac{1}{5} - \frac{3}{16}\right)$$

same idea for $\mathrm{Var}[Y]$.

3. Show that there exists some constant $c > 0$ such that for all n sufficiently large, there exists a subset $S \subseteq \{1, \ldots, n\}$ of size at least $|S| \geq cn^{1/3}$ such that S does not contain non-trivial solutions to $a + b = c + d$ (where $a, b, c, d \in S$, $a \neq b$, $c \neq d$, and $\{a, b\} \neq \{c, d\}$). That is, show that no two pairs of elements of S have the same sum.

$S \neq \{1, 2, 3, 4\}$ $\quad N = 1$

$\{(1,4), (2,3)\}$

5 5

Approach: consider a random subset S of $[n]$, where each element of $[n]$ has probability p of appearing in S. (we'll choose p later).

- Let $N = \#$ bad pairs of pairs in X ie,

$(a,b), (c,d) : a+b = c+d$ $\quad$ a,b,c,d distinct

- upper bound on N:

$$N \leq \sum_{k=1}^{2n} (\text{\# bad pairs of pairs summing to } k)$$

$$\sum_{k=1}^{n} k^p \leq cn^{p+1}$$

$$< \sum_{k=1}^{2n} \binom{k/2}{2} \leq \sum_{k=1}^{2n} ck^2 \leq cn^3$$

$\hookrightarrow \frac{(k/2)(k/2 - 1)}{2}$

different c's

1, 2, 3, 4 $\qquad$ 1, 2, 3, 5, 6

$$\Pr[\text{bad pair } \{(a,b),(c,d)\} \text{ is in } S]$$

$$= \Pr[a,b,c,d \text{ in } S] = p^4$$

$$\Rightarrow E[N] \leq c n^3 p^4$$