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271B - Final

Problem 1. Let B be a standard one-dimensional Brownian motion. Consider the SDE

$$dX_t = (t^2\Sigma)dB_t, \quad X_0 = 0 \tag{1}$$

where Σ is an exponentially distributed random variable with parameter λ and independent of the Brownian motion.

(a) Find Y_t so that $M_t = \exp(X_t)Y_t$ is a martingale. Specify the filtration.

Solution. Let $\sigma_t(\omega) = t^2\Sigma(\omega)$ and let $Y_t = \exp(-\frac{1}{2}\int_0^t \sigma_s^2\ ds)$. We claim that $M_t = \exp(X_t)Y_t$ is a martingale. By Itô's lemma we have

$$\begin{aligned} dM_t &= -\frac{1}{2}\sigma_t^2 M_t\ dt + M_t\ dX_t + \frac{1}{2}M_t(dX_t)^2 \\ &= \sigma_t M_t\ dB_t \\ &= t^2\Sigma M_t\ dB_t. \end{aligned}$$

In order for us to conclude that this is a Martingale, have to show that $t^2\Sigma M_t$ is in class I*. To this end, we check the Kazamaki condition (Øksendal, remark after exercise 4.4. I tried using Novikov's condition, which we covered in class, but the expectation wasn't finite): if the following condition holds, then M_t is a martingale.

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T (s^2\Sigma)dB_s\right)\right] < \infty. \tag{2}$$

We've shown on a previous homework assignment that for a deterministic function $f(s)$,

$$\int_0^t f(s)dB_s \sim \mathcal{N}\left(0, \int_0^t f^2(s)ds\right).$$

From this we conclude that

$$\exp\left(\frac{1}{2}\int_0^T s^2\Sigma\ dB_s\right) \sim \exp(\Sigma g),$$

where $g \sim \mathcal{N}(0, T^5/20)$. Since Σ is independent of the Brownian motion, it is also independent of g, hence

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T (s^2\Sigma)dB_s\right)\right] = \mathbb{E}[e^{\Sigma g}] = \mathbb{E}[e^{\Sigma}] \cdot \mathbb{E}[e^g].$$

This quantity is finite if $\lambda > 1$. (I needed $\lambda > 1$ for $\mathbb{E}[e^{\Sigma}] < \infty$. This seems arbitrary, however. Is it still true without this?) Since Kazamaki's condition holds, M_t is indeed a martingale with respect to the filtration generated by the Brownian motion. □

(b) Compute the variance of M_t.

Solution. The variance is given by $\mathrm{Var}[M_t] = \mathbb{E}[M_t^2] - \mathbb{E}[M_t]^2$. Since $X_0 = 0$ a.s. and $Y_0 = 1$ a.s., $M_t = 1$ a.s. To compute $\mathbb{E}[M_t^2]$ we use the Itô isometry:

$$\begin{aligned}\mathbb{E}[M_t^2] &= \mathbb{E}\left[\left(1 + \int_0^t (s^2\Sigma)M_s \, dB_s\right)^2\right] \\ &= 1 + \mathbb{E}\left[\int_0^t (s^2\Sigma)^2 M_s^2 \, ds\right]\end{aligned}$$

This gives

$$\mathrm{Var}[M_t] = \mathbb{E}\left[\int_0^t (s^2\Sigma)^2 M_s^2 \, ds\right].$$

□

(c) Find a bound for $\mathbb{P}[\sup_{0<s<t} |M_t| > \epsilon]$.

Solution. We use Doob's martingale inequality:

$$\begin{aligned}\mathbb{P}\left[\sup_{0\le s\le t} |M_t| \ge \epsilon\right] &\le \frac{1}{\epsilon^2} \cdot \mathbb{E}[|M_t|^2] \\ &= \frac{1}{\epsilon^2}\left(1 + \mathbb{E}\left[\int_0^t (s^2\Sigma)^2 M_s^2 \, ds\right]\right).\end{aligned}$$

□

Problem 2. Consider the Ornstein-Uhlenbeck process

$$dr_t = a(\overline{r} - r_t)dt + \sigma dB_t, \tag{3}$$

where a, $\overline{r}$, σ are constants. This process models an interest rate. The price of a zero-coupon bond at time t when paying 1 at maturity T is

$$P(t,x,T) = \mathbb{E}\left[\exp\left(-\int_t^T r_s \, ds\right) \mid r_t = x\right].$$

(a) Derive the Feynman-Kac formula for the bond price:

$$\begin{cases}\partial_t P + \frac{1}{2}\sigma^2\partial_x^2 P + a(\overline{r} - x)\partial_x P - xP = 0 \\ P(T,x,T) = 1\end{cases}. \tag{4}$$

Solution. We have

$$d\ \exp\left(-\int_t^T r_s \, ds\right) = r_t \cdot \exp\left(-\int_t^T r_s \, ds\right) \ dt. \tag{5}$$

On the other hand, Itô tells us that

$$\begin{aligned}dP &= \partial_t P \ dt + \partial_x P \ dr_t + \frac{1}{2}\partial_x^2 P \cdot (dr_t)^2 \\ &= \partial_t P \ dt + \partial_x P[a(\overline{r} - r_t)dt + \sigma \ dB_t] + \frac{1}{2}\sigma^2\partial_x^2 P \ dt.\end{aligned}$$

We condition (5) on $r_t = x$ and equate this to the above quantity to get

$$\partial_t P\ dt + \partial_x P[a(\overline{r} - r_t)dt + \sigma\ dB_t] + \frac{1}{2}\sigma^2\partial_x^2 P\ dt = r_t P\ dt.$$

Integrating both sides from t to T and using $r_t = x$, we obtain (I wasn't sure how to get rid of the $\sigma\ dB_t$ term)

$$\partial_t P\ dt + a(\overline{r} - x)\partial_x P + \frac{1}{2}\sigma^2\partial_x^2 P - xP = 0.$$

□

(b) Solve this PDE for the price.

Solution. We guess a solution of the form $P(t, x) = A(\tau)\exp[B(\tau)x]$, where $\tau = T - t$. Plugging this into the PDE gives

$$A'(\tau)e^{xB(\tau)} + xA(\tau)B'(\tau)e^{xB(\tau)} = a(\overline{r} - x)A(\tau)B(\tau)e^{xB(\tau)} + \frac{1}{2}\sigma^2 A(\tau)B(\tau)^2 e^{xB(\tau)} - xA(\tau)e^{xB(\tau)}.$$

We separate the A's and B's:

$$\frac{A'(\tau)}{A(\tau)} = a(\overline{r} - x)B(\tau) + \frac{1}{2}\sigma^2 B(\tau)^2 - B'(\tau) - x.$$

Since we're insisting that A does not depend on x, the right-hand side cannot depend on x. Consequently, we have

$$a(\overline{r} - x)B - x = 0 \iff B = -\frac{1}{a}.$$

This gives

$$\frac{A'(\tau)}{A(\tau)} = \frac{\sigma^2}{2a^2} - \overline{r} \implies A(\tau) = Ce^{(\frac{\sigma^2}{2a^2} - \overline{r})\tau} \implies P(t, x) = C\exp\left[\left(\frac{\sigma^2}{2a^2} - \overline{r}\right)\tau - \frac{x}{a}\right],$$

for some constant C. Plugging in the final condition $P(T, x) = 1$ gives $C = e^{x/a}$, and hence

$$P(t, x) = e^{(\frac{\sigma^2}{2a^2} - \overline{r})(T-t)}$$

solves the boundary-value problem. □

Problem 3. Let v be a continuous scalar valued process satisfying

$$0 \le v(t) \le \alpha(t) + \beta \int_0^t v(s)\ ds;\ 0 \le t \le T,$$

with $\beta \ge 0$ and α integrable. Show that

$$v(t) \le \alpha(t) + \beta \int_0^t \alpha(s)e^{\beta(t-s)}ds,\ 0 \le t \le T.$$

Can you relax the assumption about continuity?

Solution. Define the function

$$F(s) = e^{-\beta s} \cdot \beta \int_0^s v(u)\ du. \tag{6}$$

We differentiate:

$$F'(s) = \beta e^{-\beta s} \left(v(s) - \beta \int_0^s v(u)\ du \right) \le \beta \alpha(s) e^{-\beta s}.$$

Integrating from 0 to t gives

$$F(t) \le \beta \int_0^t \alpha(s) e^{-\beta s}\ ds.$$

Now we have from (6) and the above

$$\beta \int_0^t v(s)\ ds = e^{\beta t} F(t) \le \beta \int_0^t \alpha(s) e^{\beta(t-s)} ds.$$

Finally, we know $\beta \int_0^t v(s)\ ds \ge v(t) - \alpha(t)$, so the desired inequality follows. □

Problem 4. Let $B_t = B_t^{(1)} + iB_t^{(2)}$ be a complex Brownian motion.

(a) Let $F(z) = u(z) + iv(z)$ be analytic and define

$$Z_t = F(B_t).$$

Prove that

$$dZ_t = F'(B_t)\ dB_t,$$

where F' is the complex derivative of F.

Proof. We assume the component Brownian motions are independent. By Itô we have

$$dB_t = dB_t^{(1)} + i dB_t^{(2)}.$$

Write Z_t in terms of the component functions of F:

$$Z_t = u(B_t^{(1)}, B_t^{(2)}) + iv(B_t^{(1)}, B_t^{(2)}).$$

By Itô's lemma we have (suppressing the dependence on $B^{(1)}$ and $B^{(2)}$)

$$dZ_t = (u_x + iv_x) dB_t^{(1)} + (u_y + iv_y) dB_t^{(2)} + \frac{1}{2}[(u_{xx} + iv_{xx})dt + (u_{yy} + iv_{yy})dt].$$

Now the components of an analytic function are harmonic, so the bracketed term vanishes. Applying the Cauchy-Riemann equations gives

$$dZ_t = (u_x + iv_x) dB_t = F'(z) dB_t.$$

□

(b) Solve the complex SDE

$$dZ_t = \alpha Z_t dB_t,$$

where α is a constant.

Solution. Pretending that this is a real ODE, we guess that the solution will be exponential. Indeed, by part (a) we have

$$d(e^{\alpha B_t}) = \alpha e^{\alpha B_t} dB_t.$$

Thus, $Z_t = Z_0 + e^{\alpha B_t}$ solves the SDE. $\square$

Problem 5. Consider the SDE

$$dX_t = f(t, X_t)\ dt + c(t)X_t\ dB_t, \quad X_0 = x \tag{7}$$

where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $c : \mathbb{R} \to \mathbb{R}$ are given continuous and deterministic functions.

(a) Define the integrating factor

$$F_t = F_t(\omega) = \exp\left(-\int_0^t c(s)\ dB_s + \frac{1}{2}\int_0^t c^2(s)\ ds\right).$$

Show that (7) can be written

$$d(F_t X_t) = F_t \cdot f(t, X_t)\ dt.$$

Proof. If we define the process Y_t by

$$dY_t = \frac{1}{2}c^2(t)\ dt - c(t)\ dB_t,$$

then by Itô's lemma we have

$$\begin{aligned} dF_t &= F_t\ dY_t + \frac{1}{2}F_t(dY_t)^2 \\ &= F_t c_t^2\ dt - F_t c_t\ dB_t. \end{aligned}$$

Again by Itô we have

$$\begin{aligned} d(F_t X_t) &= F_t\ dX_t + X_t\ dF_t + d\langle F_t, X_t\rangle \\ &= F_t(f(t, X_t)\ dt + c_t X_t\ dB_t) + X_t(F_t c_t^2\ dt - F_t c_t\ dB_t) - c_t^2 F_t X_t\ dt \\ &= F_t \cdot f(t, X_t)\ dt. \end{aligned}$$

$\square$

(b) Now define

$$Y_t(\omega) = F_t(\omega)X_t(\omega)$$

so that

$$X_t = F_t^{-1}Y_t.$$

Deduce that (7) gets the form

$$\frac{dY_t(\omega)}{dt} = F_t(\omega) \cdot f(t, F_t^{-1}(\omega)Y_t(\omega)), \quad Y_0 = x.$$

Proof. By part (a) we have

$$\frac{dY_t(\omega)}{dt} = \frac{d(F_t(\omega)X_t(\omega))}{dt} = F_t(\omega) \cdot f(t, X_t) = F_t(\omega) \cdot f(t, F_t^{-1}(\omega)Y_t(\omega)).$$

Since $F_0 = 1$, we have $Y_0 = x$. □

(c) Apply this method to solve the SDE

$$dX_t = \frac{1}{X_t}\, dt + \alpha X_t\, dB_t, \quad X_0 = x > 0$$

where α is constant.

Solution. As per part (b), define the integrating factor

$$F_t(\omega) = \exp\left(\frac{1}{2}\alpha^2 t - \alpha B_t\right).$$

Letting $Y_t(\omega) = F_t^{-1}(\omega)$, again by part (b) we have

$$\frac{dY_t(\omega)}{dt} = F_t(\omega) \cdot \frac{1}{F_t^{-1}(\omega)Y_t(\omega)} = \frac{F_t(\omega)^2}{Y_t(\omega)}.$$

This is a separable ODE. After separating and integrating we obtain

$$Y_t^2(\omega) - x^2 = 2\int_0^t \exp(\alpha^2 s - 2\alpha B_s)\, ds,$$

which gives

$$X_t(\omega) = \exp\left(\frac{1}{2}\alpha^2 t - \alpha B_t\right)\left[x^2 + 2\int_0^t \exp(\alpha^2 s - 2\alpha B_s)\, ds\right]^{1/2}.$$

□

(d) Apply the method to study the solutions of the SDE

$$dX_t = X_t^{\gamma}\, dt + \alpha X_t\, dB_t; \quad X_0 = x > 0,$$

where α and γ are constants. For what values of γ do we get explosion?

Solution. We use the same integrating factor from part (c). The pathwise ODE we get is

$$Y_t^{-\gamma}(\omega)\, dY_t(\omega) = F_t(\omega)^2\, dt.$$

The left-hand side is integrable if and only if $\gamma < 1$. □

Problem 6. Explain the terms

(a) Martingale

Solution. A martingale is an integrable stochastic process $(M_t)_{t\geq 0}$ adapted to a filtration $(\mathcal{F}_t)_{t\geq 0}$ whose expected value at a future time is its current value: $\mathbb{E}[X_t \mid \mathcal{F}_s] = X_s$ a.s. for $s \leq t$. □

(b) Itô process

Solution. An Itô process, X_t, is a stochastic process that can be written as a sum of stochastic integrals: one with respect to time and the other with respect to Brownian motion, i.e.

$$X_t(\omega) = X_0(\omega) + \int_0^t \mu_t(\omega)\ dt + \int_0^t \sigma_t(\omega)\ dB_t,$$

where μ and σ are integrable (in t) processes for all ω. □

(c) Stopping time

Solution. A random variable $T(\omega)$ is a stopping time with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ if $\{T(\omega) \leq t\} \in \mathcal{F}_t$. The intuition is that one should know whether or not $T \leq t$ based on the "information" $\mathcal{F}_t$. □

(d) Quadratic variation

Solution. The quadratic variation of a process $(X_t)_{t\geq 0}$, in a sense, measures its "roughness." It is given by

$$\langle X \rangle_t = \lim_{\|P\|\to 0} \sum_{k=1}^{n} (X_{t_k} - X_{t_{k-1}})^2,$$

where P ranges over all partitions of $[0, t]$ and the limit is in probability. All finite variation processes have zero quadratic variation, while Brownian motion has infinite total variation and $\langle B \rangle_t = t$. □

(e) Kolmogorov backward equation

Solution. Loosely speaking, for a diffusion process X_t, the Kolmogorov backward equation is a partial differential equation whose solution, $P(x, t)$, is the probability density function:

$$\int_B P(x,t)\ dx = \Pr[X_t \in B \mid X_T = x],$$

where $t \leq T$ and x is fixed. It can be derived from Itô's lemma. □

Questionnaire The top 3 topics I'd like to see are

1. Graph-based models
2. Large deviations
3. Itô calculus for Hilbert space-valued processes

Thank you for asking!

asdf