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270B - Homework 2

Problem 1.

(a) Prove that if the probability density functions of X_n converge pointwise to the probability density function of X, then X_n converges to X weakly.

Proof. Let f_n and f be the density functions of X_n and X, respectively, and let $\varphi : \mathbb{R} \to \mathbb{R}$ be bounded and continuous. We then have

$$\begin{aligned} |\mathbb{E}_n[\varphi] - \mathbb{E}[\varphi]| &= \left| \int_{\mathbb{R}} f_n(x)\varphi(x)\ dx - \int_{\mathbb{R}} f(x)\varphi(x)\ dx \right| \\ &\leq \|\varphi\|_\infty \int_{\mathbb{R}} |f_n(x) - f(x)|\ dx. \end{aligned} \tag{1}$$

Now we claim that $\|f_n - f\|_{L^1} \to 0$ (argument taken from 210 notes). Since $\int f_n = \int f = 1$ we have by Fatou

$$2\int_{\mathbb{R}} f\ dx = \int_{\mathbb{R}} \liminf_{n\to\infty}(f + f_n - |f_n - f|)\ dx \leq 2\int_{\mathbb{R}} f\ dx - \limsup_{n\to\infty} |f_n - f|\ dx.$$

In particular, we have $\limsup_{n\to\infty} \int |f_n - f| = 0$, so $\|f_n - f\|_{L^1} \to 0$. Consequently, the right-hand side of (1) goes to zero as $n \to \infty$. We then have that $\mathbb{E}_n[\varphi] \to \mathbb{E}[\varphi]$ for all φ bounded continuous, so $X_n \to X$ weakly. □

(b) Prove that if the probability mass functions of X_n converge to the probability mass function of X pointwise then X_n converges to X weakly.

Proof. We essentially mirror our proof of part (a) but with the counting measure. Let f_n and f be the mass functions of X_n and X respectively and suppose they take values in the discrete set $S \subset \mathbb{R}$. Suppose φ is a bounded function on S (note that any function from a discrete space is continuous). We then have

$$\begin{aligned} |\mathbb{E}_n[\varphi] - \mathbb{E}[\varphi]| &= \left| \sum_{x\in S} f_n(x)\varphi(x) - \sum_{x\in S} f(x)\varphi(x) \right| \\ &\leq \|\varphi\|_\infty \sum_{x\in S} |f_n(x) - f(x)|. \end{aligned} \tag{2}$$

Fatou still works with the counting measure, so we have

$$2\sum_{x\in S} f(x) = \sum_{x\in S} \liminf_{n\to\infty}(f(x) + f_n(x) - |f_n(x) - f(x)|) \leq 2\sum_{x\in S} f(x) - \limsup_{n\to\infty} \sum_{x\in S} |f_n(x) - f(x)|.$$

We conclude that $\sum_{x\in S} |f_n(x) - f(x)| \to 0$ as $n \to \infty$. The right-hand side of (2) vanishes as $n \to \infty$, so $X_n \to X$ weakly. □

(c) In general, weak convergence does not imply pointwise convergence of probability density functions. Show this by example.

Solution. Let $f_{n,k}(x)$ be a typewriter sequence weighted to have integral 1, that is

$$f_{n,k}(x) = \frac{1}{1-2^{-n}} \mathbb{1}_{[0,1]\setminus[k\cdot 2^{-n},\ (k+1)\cdot 2^{-n}]}(x), \quad n = 1, 2, \ldots,\ k = 0, 1, \ldots, 2^n - 1.$$

Intuitively, $f_{n,k}$ is a flat line of height $\frac{1}{1-2^{-n}}$ over $[0,1]$ except for a gap at $[k \cdot 2^{-n}, (k+1) \cdot 2^{-n}]$, where it is zero. Since this gap slides along the unit interval indefinitely, $f_{n,k}$ does not converge pointwise anywhere. Now for any φ bounded and continuous we have

$$|\mathbb{E}_{n,k}[\varphi] - \mathbb{E}[\varphi]| = \int_0^1 \varphi(x)(f_{n,k}(x) - 1)dx \leq \|\varphi\|_\infty \cdot 2^{-n} \to 0,$$

so X_n converges weakly to the uniform distribution on $[0,1]$ even though the densities don't converge pointwise at all. □

Problem 2. Consider normal random variables $X_n \sim \mathcal{N}(\mu_n, \sigma_n^2)$. Assume X_n converges weakly to some random variable X. Prove that $X \sim \mathcal{N}(\mu, \sigma^2)$ where $\mu = \lim_{n\to\infty} \mu_n$ and $\sigma^2 = \lim_{n\to\infty} \sigma_n^2$ and both limits exits.

Proof. Since $x \mapsto e^{itx}$ is bounded and continuous, the weak convergence of X_n to X implies that the characteristic functions ϕ_n of X_n converge pointwise to the characteristic function ϕ of X. After completing the square in the exponent, ϕ_n is given by

$$\begin{aligned}\phi_n(t) &= \frac{1}{\sqrt{2\pi\sigma_n^2}} \int_{\mathbb{R}} e^{-\frac{1}{2}\left(\frac{x-\mu_n}{\sigma_n}\right)^2} \cdot e^{itx}\, dx \\ &= e^{i\mu_n t - \sigma_n^2 t^2/2}.\end{aligned}$$

As n goes to infinity we have

$$\lim_{n\to\infty} \phi_n(t) = \lim_{n\to\infty} e^{i\mu_n t - \sigma_n^2 t^2/2} = e^{i\mu t - \sigma^2 t^2/2},$$

which is the characteristic function for the random variable $Y \sim \mathcal{N}(\mu, \sigma^2)$. Since the characteristic function determines the distribution of the random variable, we conclude that $X \sim \mathcal{N}(\mu, \sigma^2)$. □

Problem 3. Let $X_1, X_2, \ldots$ be independent Rademacher random variables. Let $S_n = X_1 + \cdots + X_n$.

(a) Prove that the sequence $S_n/\sqrt{n}$ is unbounded almost surely.

Proof. □