1. Let X and Y be independent variables having the exponential distribution with parameters λ and μ, respectively. Find the density function of $X+Y$.

$X \sim \text{Exp}(\lambda)$, $Y \sim \text{Exp}(\mu)$

Let $f_{X+Y}(t)$ be the density of $X+Y$

Since X & Y are independent,

1 $f_{X+Y}(t) = \int_{\mathbb{R}} f_X(t-y) f_Y(y)\,dy$ (*)

2 $F_{X+Y}(t) = \Pr[X+Y \leq t]$

3 $= \Pr[X \leq t - Y]$

4 $= \int_{\mathbb{R}} F_X(t-y) f_Y(y)\,dy$

differentiate wrt t

$\Rightarrow f_{X+Y}(t) = \int_{\mathbb{R}} f_X(t-y) f_Y(y)\,dy$

$$f_X(x) = \lambda e^{-\lambda x} \chi_{\{x \geq 0\}}(x)$$

$$f_Y(y) = \mu e^{-\mu y} \chi_{\{y \geq 0\}}(y)$$

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{else} \end{cases}$$

$\uparrow$ chi

$$f_{X+Y}(t) = \int_{\mathbb{R}} f_X(t-y) f_Y(y)\, dy$$

$$= \int_{\mathbb{R}} \left(\lambda e^{-\lambda(t-y)} \chi_{\{t-y \geq 0\}}(y)\right) \cdot \left(\mu e^{-\mu y} \chi_{\{y \geq 0\}}(y)\right) dy$$

$\exp(t) = e^t$

$$= \lambda\mu e^{-\lambda t} \int_{\mathbb{R}} \exp\left((\lambda-\mu)y\right) \underbrace{\chi_{\{y \geq 0\}}(y)\, \chi_{\{t-y \geq 0\}}(y)}_{= 1 \text{ iff } y \geq 0 \text{ AND } t-y \geq 0} dy$$

$= 1$ iff $y \geq 0$ AND $t - y \geq 0$

$t \geq y$

$$= \lambda\mu e^{-\lambda t}\int_0^t \exp((\lambda-\mu)y)\,dy$$

$$= \lambda\mu e^{-\lambda t}\left.\frac{\exp((\lambda-\mu)y)}{\lambda-\mu}\right|_0^t$$

$$\chi_E \quad \mathbb{1}_E = \begin{cases} 1 & \text{if } x\in E \\ 0 & \text{otherwise} \end{cases}$$

"characteristic fn" "indicator fn"

$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

2. Let $\phi(x)$ be the standard normal density and let $\Phi(x)$ be the standard normal cdf. Show that $\phi'(x) + x\phi(x) = 0$. Deduce that

$\phi' = -x\phi$

$$\frac{1}{x} - \frac{1}{x^3} < \frac{1-\Phi(x)}{\phi(x)} < \frac{1}{x} - \frac{1}{x^3} + \frac{3}{x^5}, \qquad x > 0.$$

This inequality is useful since it helps estimate $\Phi(x)$, which doesn't have a closed-form expression.

$$\Phi(x) = \Pr[N \in x] = \int_{-\infty}^{x} \phi(t)\,dt$$

$$\Rightarrow 1 - \Phi(x) = \int_x^{\infty} \phi(t)\,dt$$

$$= \int_x^{\infty} -\frac{\phi'(t)}{t}\,dt$$

$u = -\frac{1}{t}$ $\quad\quad$ $v = \phi(t)$

$du = \frac{1}{t^2}\,dt$ $\quad\quad$ $dv = \phi'(t)\,dt$

$$= -\frac{\phi(t)}{t}\Big|_x^\infty - \int_x^\infty \frac{\phi(t)}{t^2}\,dt$$

$$= \frac{\phi(x)}{x} - \int_x^\infty \frac{\phi(t)}{t^2}\,dt$$

$$= \frac{\phi(x)}{x} + \int_x^\infty \frac{\phi'(t)}{t^3}\,dt$$

Parts again . . .