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271B - Homework 5

Problem 1. Consider

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t, \quad X_0 = 1, \tag{1}$$

with $\mu(x) = x + a$, $\sigma(x) = 4x$. Assuming $X_t > 0$, find dY_t when $Y_t = \sqrt{X_t}$. Can you find Y_t?

Solution. Let $g(t, x) = \sqrt{x}$ so that $Y_t = g(t, X_t)$. By Itô's lemma we have

$$\begin{aligned} dY_t &= \frac{1}{2}X_t^{-1/2}dX_t - \frac{1}{8}X_t^{-3/2}(dX_t)^2 \\ &= \frac{1}{2}X_t^{-1/2}[(X_t + a)dt] - \frac{1}{8}X_t^{-3/2}(16X_t^2 dt) \\ &= \frac{a - 3Y_t^2}{2Y_t}dt + 2Y_t dB_t. \end{aligned}$$

Using dY_t to find Y_t proved difficult. Finding X_t and taking the square root worked however. We use the stochastic version of integrating factors. Define the function

$$F_t = \exp\left(\frac{1}{2}\int_0^t 16\ ds - \int_0^t 4\ dB_s\right) = \exp(8t - 4B_t).$$

From this we get

$$dF_t = 16F_t dt - 4F_t dB_t.$$

We apply the multivariable Itô lemma to obtain (after some tedious algebra)

$$\begin{aligned} d(X_t F_t) &= X_t dF_t + F_t dX_t + d\langle X_t, F_t\rangle \\ &= F_t(a + X_t)dt. \end{aligned}$$

Setting $Z_t = X_t F_t$, we obtain the linear DE

$$\frac{dZ_t}{dt} - Z_t = aF_t.$$

We multiply through by e^{-t} to obtain

$$\frac{d}{dt}(Z_t e^{-t}) = ae^{7t - 4B_t}.$$

Integrating through and substituting X_t back in gives

$$X_t = \frac{1 + \int_0^t \exp(7s - 4B_s)ds}{\exp(7t - 4B_t)}.$$

Taking the square root gives Y_t.

$$Y_t = \left(\frac{1 + \int_0^t \exp(7s - 4B_s)ds}{\exp(7t - 4B_t)}\right)^{1/2}.$$

□

Problem 2. Let X_t be as in (1) but with $\mu(x) = 2x$ and $\sigma(x) = x^a$ and and $Y_t = X_t^b$. Find b so that $\langle Y \rangle_t$ is linear in t.

Solution. By Itô's lemma we have

$$dY_t = bX^{b-1}dX_t + \frac{1}{2}b(b-1)X^{b-2}(dX_t)^2$$

$$= f(X_t)dt + bX_t^{a+b-1}dB_t,$$

for some function f. Consequently we have

$$d\langle Y \rangle_t = (bX_t^{a+b-1})^2 dt.$$

From this we compute the quadratic variation:

$$\langle Y \rangle_t = \int_0^t d\langle Y \rangle_t = b^2 \int_0^t X_s^{2a+2b-2} ds.$$

Setting $b = 1 - a$ makes the exponent in the integrand zero, which makes $\langle Y \rangle_t$ linear in t. $\square$

Problem 3. Let

$$dX_t = \sqrt{1 + X_t}\ dB_t, \quad X_0 = 0.$$

Find $\mathbb{E}[X_t]$ and $\mathbb{E}[X_t^2]$.

Solution. Since $\sqrt{1+x}$ is sublinear and Lipschitz on $[0, \infty)$, the SDE solution existence and uniqueness theorem says that $\sqrt{1 + X_t}$ is in class I^*. The given SDE then describes an Itô process which has only a martingale component. The expectation is then given by

$$\mathbb{E}[X_t] = X_0 = 0.$$

As for the second moment, we use the Itô isometry and the fact that $X_0 = 0$

$$\mathbb{E}[X_t^2] = \mathbb{E}\left[\left(X_0 + \int_0^t \sqrt{1 + X_s} dB_s\right)^2\right]$$

$$= \mathbb{E}\left[\int_0^t (1 + X_s) ds\right]$$

$$= \int_0^t \mathbb{E}[1 + X_s] ds$$

$$= t.$$

$\square$

Problem 4. Let Y be an $\mathcal{F}_T$-measurable random variable such that $\mathbb{E}[|Y|^2] < \infty$ and consider Doob's martingale

$$M_t = \mathbb{E}[Y | \mathcal{F}_t], \quad 0 \le t \le T$$

with respect to the filtration $\{\mathcal{F}_t\}_{0 \le t \le T}$.

(a) Show that $\mathbb{E}[M_t^2] < \infty$ for all $t \in [0, T]$.

Proof. This follows from the conditional form of Jensen's inequality.

$$\mathbb{E}[M_t^2] = \mathbb{E}[\mathbb{E}[Y|\mathcal{F}_t]^2] \le \mathbb{E}[\mathbb{E}[Y^2|\mathcal{F}_t]] = \mathbb{E}[Y^2] < \infty.$$

□

(b) By the martingale representation theorem, there exists a unique process $g(t, \omega)$ in class I* such that

$$M_t = \mathbb{E}[M_0] + \int_0^t g(s, \omega) dB_s, \quad t \in [0, T].$$

Find g in the following cases.

(i) $Y(\omega) = B_T^2(\omega)$.

Solution.

$$\begin{aligned} M_t &= \mathbb{E}[(B_T - B_t + B_t)^2|\mathcal{F}_t] \\ &= \mathbb{E}[(B_T - B_t)^2|\mathcal{F}_t] + 2\mathbb{E}[B_t(B_T - B_t)|\mathcal{F}_t] + \mathbb{E}[B_t^2|\mathcal{F}_t] \\ &= (T - t) + B_t^2 \\ &= T + \int_0^t B_s \, dB_s. \end{aligned}$$

So $g(s, \omega) = B_s(\omega)$. □

(ii) $Y(\omega) = B_T^3(\omega)$.

Solution.

$$\begin{aligned} M_t &= \mathbb{E}[(B_T - B_t + B_t)^3|\mathcal{F}_t] \\ &= \mathbb{E}[(B_T - B_t)^3|\mathcal{F}_t] + 3\mathbb{E}[B_t(B_T - B_t)^2|\mathcal{F}_t] + 3\mathbb{E}[B_t^2(B_T - B_t)|\mathcal{F}_t] + \mathbb{E}[B_t^3|\mathcal{F}_t] \\ &= 3B_t(T - t) + B_t^3. \end{aligned}$$

Let $g(t, X) = 3x(T - t) + x^3$ so that $M_t = g(t, B_t)$. By Itô's lemma

$$dM_t = -3B_t dt + [3(T - t) + 3B_t^2]dB_t + 3B_t dt = 3[(T - t) + B_t^2]dB_t.$$

Consequently, our $g(s, \omega) = 3[(T - t) + B_t^2]$. □

(iii) $Y(\omega) = \exp(\sigma B_T)$, $\sigma \in \mathbb{R}$ is a constant.

Solution. Since $\exp(\sigma B_t - \frac{1}{2}\sigma^2 t)$ is a martingale we have

$$\begin{aligned} M_t &= \mathbb{E}\left[\exp(\sigma B_T - \frac{1}{2}\sigma^2 T)\exp(\frac{1}{2}\sigma^2 T)|\mathcal{F}_t\right] \\ &= \exp(\sigma B_t - \frac{1}{2}\sigma^2 t)\exp(\frac{1}{2}\sigma^2 T). \end{aligned}$$

Applying Itô's lemma to the function $g(t,x) = \exp(\sigma x - \frac{1}{2}\sigma^2 t)\exp(\frac{1}{2}\sigma^2 T)$ shows that M_t solves the SDE

$$dM_t = \sigma M_t \, dB_t.$$

From this we see that

$$g(s,\omega) = \sigma \exp\left(\sigma B_s(\omega) + \frac{1}{2}\sigma^2(T-s)\right).$$

□