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271C - Homework 2

Problem 1. Show that if a, b are deterministic and of class I^* then

(a) if

$$dX_t = a(t)\ dt + b(t)\ dB_t,$$

then $X(t)$ is a Gaussian process with independent increments.

Proof. We have that

$$X_t = X_0 + \int_0^t a_s\ ds + \int_0^t b_s\ dB_s.$$

On a previous homework assignment, we've shown that $\int_0^t b_s\ dB_s \to g$ in L^2, where $g \sim \mathcal{N}(0, \int_0^t b_s^2\ ds)$ (the idea in that proof is to look at the limit definition of the Itô integral). In particular, we have that

$$X_t \sim \mathcal{N}\left(A_t,\ \int_0^t b_s^2\ ds\right),$$

where $A_t = X_0 + \int_0^t a_s\ ds$. Thus, X_t is a Gaussian random variable for each t. To show that the process is Gaussian, fix some integer k and consider times $t_1, \ldots, t_k$ and let $\mathbf{s} = (s_1, \ldots, s_k)$, $\mathbf{X} = (X_{t_1}, \ldots, X_{t_k})$, and $\mathbf{A} = (A_{t_1}, \ldots, A_{t_k})$. Consider the characteristic function of $\mathbf{X}$.

$$\begin{aligned} E[e^{i\mathbf{s}\cdot\mathbf{X}}] &= E\exp\left[i\sum_{j=1}^k s_j\left(A_{t_j} + \int_0^{t_j} b_s\ dB_s\right)\right] \\ &= e^{i\mathbf{s}\cdot\mathbf{A}} \cdot E\exp\left[i\sum_{j=1}^k s_j \int_0^{t_j} b_s\ dB_s\right]. \end{aligned}$$

We need to show that this simplifies to the characteristic function of a multivariate Gaussian. □

(b) If

$$dX_t = a_t X_t\ dt + b_t X_t\ dB_t,$$

then X_t is a log-normal process.

Proof. Dividing the given equation through by X_t gives

$$\frac{dX_t}{X_t} = a_t\ dt + b_t\ dB_t.$$

Now by Itô's lemma we have

$$\begin{aligned} d(\log X_t) &= \frac{dX_t}{X_t} - \frac{d\langle X\rangle_t}{X_t^2} \\ &= (a_t\ dt + b_t\ dB_t) - (b_t^2\ dt) \\ &= (a_t - b_t^2)dt + b_t\ dB_t. \end{aligned}$$

By part (a), $\log X_t$ is then a Gaussian process, so X_t is itself a log-normal process. □

Problem 2. Solve the SDE

$$dX_t = B_t X_t \; dt + B_t X_t \; dB_t, \quad X_0 = 1. \tag{1}$$

Solution. We divide by X_t to obtain

$$\frac{dX_t}{X_t} = B_t \; dt + B_t \; dB_t.$$

By Itô we then have

$$\begin{aligned} d(\log X_t) &= \frac{dX_t}{X_t} - \frac{d\langle X\rangle_t}{X_t^2} \\ &= (B_t \; dt + B_t \; dB_t) - B_t^2 \; dt \\ &= (B_t - B_t^2)dt + B_t \; dB_t. \end{aligned}$$

Using the fact that $X_0 = 1$, we exponentiate and obtain

$$X_t = \exp\left(\int_0^t (B_s - B_s^2)ds + \int_0^t B_s \; dB_s\right).$$

□

Problem 3. Find the stochastic exponential

$$\mathcal{E}(B_t^2 + t).$$

Solution. Let $X_t = B_t^2 + t$. The stochastic exponential of X_t is given by

$$\mathcal{E}(X_t) = \exp(X_t - X_0 - \frac{1}{2}\langle X\rangle_t).$$

We then need to compute $\langle B_t^2 + t\rangle$. To this end, we use Itô to compute $d(X_t^2)$.

$$\begin{aligned} dY_t &= 2X_t \; dX_t + d\langle X\rangle_t \\ &= 2(B_t^2 + t) \cdot d(B_t^2 + t) + d\langle B_t^2 + t\rangle. \end{aligned}$$

From this we deduce

$$d\langle B_t^2 + t\rangle = d[(B_t^2 + t)^2] - 2(B_t^2 + t) \cdot d(B_t^2 + t).$$

Applying Itô again gives

$$d(B_t^2 + t) = 2dt + 2B_t \; dB_t,$$

from which we get

$$d\langle B_t^2 + t\rangle = d[(B_t^2 + t)^2] - 4(B_t^2 + t)dt - 4B_t(B_t^2 + t)dB_t.$$

Putting it all together gives

$$\mathcal{E}(B_t^2 + t) = \exp\left(B_t^2 + t - \frac{1}{2}(B_t^2 + t)^2 + 2\int_0^t (B_s^2 + s)dt + 2\int_0^t B_s(B_s^2 + s)dB_s\right).$$

□

Problem 4. Prove Thomas' Lemma: Let $X, Y, Z \in \mathcal{M}^{c,\ loc}$. Then

$$X_t \circ (Y_t \circ dZ_t) = (X_t Y_t) \circ dZ_t.$$

Proof. For ease of notation, write

$$dX_t = \mu_t^{(X)}\ dt + \sigma_t^{(X)}\ dB_t$$

$$dY_t = \mu_t^{(Y)}\ dt + \sigma_t^{(Y)}\ dB_t$$

$$dZ_t = \mu_t^{(Z)}\ dt + \sigma_t^{(Z)}\ dB_t.$$

(Is assuming $X, Y, Z \in \mathcal{M}^{c,loc}$ enough to let us write this?) Let W_t be defined by

$$W_t = Y_0 + \int_0^t Y_s \circ dZ_s.$$

With this definition, we have $X_t \circ (Y_t \circ dZ_t) = X_t \circ W_t$. Let's convert from Stratonovich to Itô.

$$dW_t = Y_t \circ dZ_t = Y_t\ dZ_t + \frac{1}{2} d\langle Y, Z\rangle_t.$$

From this we deduce

$$X_t \circ dW_t = X_t \circ \left(Y_t\ dZ_t + \frac{1}{2} d\langle Y, Z\rangle_t \right)$$

$$= X_t \circ (Y_t\ dZ_t) + \frac{1}{2} X_t \circ d\langle Y, Z\rangle_t.$$

Now $\langle Y, Z\rangle_t$ has finite total variation, so $X_t \circ d\langle Y, Z\rangle_t = X_t\ d\langle Y, Z\rangle_t$. This gives

$$X_t \circ dW_t = X_t Y_t\ dZ_t + \frac{1}{2} d\left\langle X_t, \int_0^t Y_s\ dZ_s \right\rangle + \frac{1}{2} X_t\ d\langle Y, Z\rangle_t.$$

Now if we could show that

$$d\left\langle X_t, \int_0^t Y_s\ dZ_s \right\rangle = Y_t\ d\langle X, Z\rangle_t,$$

then we'd have

$$X_t \circ dW_t = X_t Y_t\ dZ_t + \frac{1}{2} Y_t\ d\langle X, Z\rangle_t + \frac{1}{2} X_t\ d\langle Y, Z\rangle_t.$$

Then if we could show that

$$\frac{1}{2} Y_t\ d\langle X, Z\rangle_t + \frac{1}{2} X_t\ d\langle Y, Z\rangle_t = \frac{1}{2} d\langle XY, Z\rangle_t,$$

then we'd have

$$X_t \circ dW_t = X_t Y_t\ dZ_t + \frac{1}{2} d\langle XY, Z\rangle_t = X_t Y_t \circ dZ_t.$$

□

Øksendal Problem 4.10 Let $g(x) = |x|$ and define $g_\epsilon(x)$ by

$$g_\epsilon(x) = \begin{cases} |x| & \text{if } |x| \geq \epsilon \\ \frac{1}{2}(\epsilon + x^2/\epsilon) & \text{if } |x| < \epsilon, \end{cases}$$

for $\epsilon > 0$.

(a) Show that

$$g_\epsilon(B_t) = g_\epsilon(B_0) + \int_0^t g_\epsilon'(B_s)\ dB_s + \frac{1}{2\epsilon} \cdot m(E_\epsilon),$$

where $m(\cdot)$ is the Lebesgue measure and E_ϵ is defined by

$$E_\epsilon = \{s \in [0,t] : B_s \in (-\epsilon, \epsilon)\}.$$

Proof. First, we claim that

$$g_\epsilon(B_t) = g_\epsilon(B_0) + \int_0^t g_\epsilon'(B_s)\ dB_s + \frac{1}{2}\int_0^t g_\epsilon''(B_s)\ ds.$$

The desired result easily follows from this since

$$g_\epsilon''(x) = \frac{1}{\epsilon}\mathbb{1}_{\{|x|<\epsilon\}}(x).$$

To prove our claim, first note that g_ϵ is C^1 everywhere and C^2 except for at $\pm\epsilon$. We also have that $|g_\epsilon''(x)| \leq 1/\epsilon$ outside of $x = \pm\epsilon$. Choose $f_k \in C^2$ such that $f_k \to g_\epsilon$ uniformly, $f_k' \to g_\epsilon'$ uniformly, $|f_k''| \leq 1/\epsilon$, and $f_k'' \to g_\epsilon''$ outside of $x = \pm\epsilon$. That such a sequence exists follows from an approximation argument that can be found in Appendix D of Øksendal. Itô's lemma tells us that

$$f_k(B_t) = f_k(B_0) + \int_0^t f_k'(B_s)\ dB_s + \frac{1}{2}\int_0^t f_k''(B_s)\ ds.$$

The sequence f_k was chosen such that taking $k \to \infty$ on both sides of the equation above establishes the claim. □

(b) Prove that

$$\int_0^t g_\epsilon'(B_s) \cdot \mathbb{1}_{B_s \in (-\epsilon,\epsilon)} dB_s = \int_0^t \frac{B_s}{\epsilon} \cdot \mathbb{1}_{B_s \in (-\epsilon,\epsilon)} dB_s \to 0$$

in $L^2(\Pr)$ as $\epsilon \to 0$.

Proof. The equality of the two integrals follows immediately from the definition of g_ϵ. Now by the Itô isometry we have

$$\begin{aligned} E\left[\left(\int_0^t \frac{B_s}{\epsilon} \cdot \mathbb{1}_{B_s\in(-\epsilon,\epsilon)} dB_s\right)^2\right] &= E\left[\int_0^t \frac{B_s^2}{\epsilon^2} \cdot \mathbb{1}_{B_s\in(-\epsilon,\epsilon)} ds\right] \\ &\leq E[m(\{s : B_s \in (-\epsilon,\epsilon)\})]. \end{aligned}$$

By Fubini, this quantity is

$$\int_0^t \Pr[B_s \in (-\epsilon, \epsilon)]\ ds.$$

Since $B_s \sim \mathcal{N}(0, s)$, this quantity goes to zero by the monotone convergence theorem and the result follows. □

(c) By letting $\epsilon \to 0$, prove that

$$|B_t| = |B_0| + \int_0^t \operatorname{sgn}(B_s)\ dB_s + L_t(\omega),$$

where

$$L_t = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \cdot m(E_\epsilon) \quad \text{(limit is in } L^2(\Pr))$$

and

$$\operatorname{sgn}(x) = \begin{cases} -1 & \text{for } x \leq 0 \\ 1 & \text{for } x > 0. \end{cases}$$

Proof. By part (a) we have

$$\begin{aligned} g_\epsilon(B_t) &= g_\epsilon(B_0) + \int_0^t g_\epsilon'(B_s) \cdot \mathbb{1}_{B_s \in (-\epsilon,\epsilon)} dB_s + \int_0^t g_\epsilon'(B_s) \cdot \mathbb{1}_{B_s \notin (-\epsilon,\epsilon)} dB_s + \frac{1}{2\epsilon} \cdot m(E_\epsilon) \\ &= g_\epsilon(B_0) + \int_0^t g_\epsilon'(B_s) \cdot \mathbb{1}_{B_s \in (-\epsilon,\epsilon)} dB_s + \int_0^t \operatorname{sgn}(B_s) \cdot \mathbb{1}_{B_s \notin (-\epsilon,\epsilon)} dB_s + \frac{1}{2\epsilon} \cdot m(E_\epsilon). \end{aligned}$$

By part (b), the second integral goes to zero in $L^2(\Pr)$ as $\epsilon \to 0$ and the claim follows. □

Problem 6. Prove that Tanaka's equation

$$dX_t = \operatorname{sgn}(X_t)\ dB_t, \quad X_0 = 0 \tag{2}$$

has no strong solution.

Proof. This argument comes from Øksendal. Let $\hat{B}_t$ be a Brownian motion generating the filtration $\hat{\mathcal{F}}_t$ and let

$$Y_t = \int_0^t \operatorname{sgn}(\hat{B}_s)\ d\hat{B}_s.$$

By the previous exercise, we have

$$Y_t = |\hat{B}_s| - |\hat{B}_0| - \hat{L}_t(\omega).$$

From this equation, we deduce that Y_t is measurable with respect to the filtration generated by $|\hat{B}_s|$, $s \leq t$, which itself is contained in $\hat{\mathcal{F}}_t$. In particular, the filtration generated by Y_s, $s \leq t$ is strictly contained in $\hat{\mathcal{F}}_t$.

Suppose (2) has strong solution X_t adapted to the filtration $\mathcal{F}_t$ generated by B_s, $s \leq t$. By Theorem 8.4.2 in Øksendal, since $\operatorname{sgn}^2(X_t) = 1$, X_t is a Brownian motion with respect to the underlying probability measure. Suppose X_s, $s \le t$ generates the filtration $\mathcal{G}_t$. Note that by rearranging (2), we have

$$dB_t = \operatorname{sgn}(X_t)\ dX_t.$$

Now we also have that

$$dY_t = \operatorname{sgn}(\hat{B}_s)\ d\hat{B}_s.$$

Combining these and using the argument above, we have that $\mathcal{F}_t$ is strictly contained in $\mathcal{G}_t$. But in order for X_t to be a strong solution, it must be $\mathcal{F}_t$ adapted and $\mathcal{G}_t \subseteq \mathcal{F}_t$ – a contradiction. □

Øksendal Problem 5.11 For fixed $a, b \in \mathbb{R}$, consider the following 1-dimensional equation

$$dY_t = \frac{b - Y_t}{1-t} dt + dB_t; \quad 0 \le t < 1,\ Y_0 = a.$$

Verify that

$$Y_t = a(1-t) + bt + (1-t)\int_0^t \frac{dB_s}{1-s}; \quad 0 \le t < 1$$

solves the equation and prove that $\lim_{t\to 1} Y_t = b$ a.s.

Proof. By Itô we have

$$dY_t = b - a + (1-t)\cdot d\left(\int_0^t \frac{dB_s}{1-s}\right) + \left(\int_0^t \frac{dB_s}{1-s}\right)\cdot d(1-t) + \frac{1}{2} d\left\langle 1-t, \int_0^t \frac{dB_s}{1-s}\right\rangle.$$

Since $1-t$ is absolutely continuous, that quadratic variation term is zero. After some simplification, we obtain the desired

$$dY_t = \frac{b - Y_t}{1-t} dt + dB_t.$$

To show the limit, first note that $M_t = \int_0^t \frac{dB_s}{1-s}$ is a martingale and $(1-t)M_t$ is a submartingale. By Doob's submartingale inequality we have for any $\epsilon > 0$

$$\Pr[\sup_{t\in[1-2^{-n},1+2^{-n}]} (1-t)|M_t| > \epsilon] \le \frac{E[M^2_{1\pm 2^{-n}}]}{\epsilon^2} 2^{-2n}.$$

By the Itô isometry, we then have

$$\Pr[\sup_{t\in[1-2^{-n},1+2^{-n}]} (1-t)|M_t| > \epsilon] \le 2\epsilon^{-2}\cdot 2^{-n}.$$

Setting $\epsilon = 2^{-n/4}$, we obtain a summable sequence, so by Borel-Cantelli, for almost every ω there is some $n(\omega)$ such that $n \ge n(\omega)$ implies that

$$\omega \notin \{\omega : \sup_{t\in[1\pm 2^{-n}]} (1-t)|M_t| > 2^{-n/4}\}.$$

Consequently,

$$\lim_{t\to 1}(1-t)\int_0^t \frac{dB_s}{1-s} = 0,$$

and the desired result follows. □