Say X & Y are RVs w/

joint density $f_{XY}(x,y)$

Consider $U = g_1(X,Y)$

$V = g_2(X,Y)$

density $f_{U,V}(u,v) = ?$

1. If X and Y are independent and identically distributed uniform random variables on $(0,1)$, compute the joint density of

(a) $U = X + Y$, $V = X/Y$,

(b) $U = X$, $V = X/Y$,

(c) $U = X + Y$, $V = X/(X + Y)$.

$$f_{U,V}(u,v) = f_{X,Y}(x,y)\,\left|J(x,y)\right|^{-1}$$

where $J(x,y) = \begin{vmatrix} U_x & U_y \\ V_x & V_y \end{vmatrix}$

$$U_x = \partial/\partial x\; U = \partial_x U$$

$$U_y = \partial/\partial y\; U = \partial_y U$$

* need $f_{XY}(x,y)$

here, $f_{XY}(x,y) = f_X(x) f_Y(y)$

since $X \perp\!\!\!\perp Y$

independent

$X, Y \sim \mathrm{Unif}([0,1])$

$$\Rightarrow f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{else} \end{cases}$$

$$f_Y(y) = \begin{cases} 1 & 0 \le y \le 1 \\ 0 & \text{else} \end{cases}$$

$$\Rightarrow f_{XY}(x,y) = \begin{cases} 1 & \text{if } 0 \le x, y \le 1 \\ 0 & \text{else} \end{cases}$$

$U = X + Y, \quad V = X/Y = \frac{1}{Y} X$

$$J(x,y) = \begin{vmatrix} 1 & 1 \\ \frac{1}{y} & -\frac{x}{y^2} \end{vmatrix} = -\frac{x}{y^2} - \frac{1}{y}$$

$$= -\frac{x+y}{y^2}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$f_{U,V}(u,v) = f_{X,Y}(x,y)\,\left|J(x,y)\right|^{-1}$$

$$= \left|-\frac{x+y}{y^2}\right|^{-1} = \frac{y^2}{x+y}$$

- get x, y in terms of u, v

$u = x+y$, $\quad v = x/y$

$$\frac{u}{v+1} = \frac{x+y}{\frac{x}{y}+1} = \frac{x+y}{\frac{x+y}{y}} = y$$

$$\Rightarrow f_{U,V}(u,v) = \frac{\left(u/(v+1)\right)^2}{u} = \frac{u}{(v+1)^2}$$

$$u = x, \quad v = x/y \Rightarrow y = \frac{x}{v} = \frac{u}{v}$$

$$f_{u,v}(u,v) = f_{x,y}(x,y)\,\left|J(x,y)\right|^{-1}$$

$$J(x,y) = \begin{vmatrix} 1 & 0 \\ \frac{1}{y} & -\frac{x}{y^2} \end{vmatrix} = \frac{-x}{y^2}$$

$$= \left|-\frac{x}{y^2}\right|^{-1} = \frac{y^2}{x} = \frac{(u/v)^2}{u}$$

$$= \frac{u}{v^2}$$

2. If X_1 and X_2 are independent exponential random variables, each having parameter λ, find the joint density function of $Y_1 = X_1 + X_2$ and $Y_2 = e^{X_1}$.

Same idea as previous problem.

$$f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}(x_1,x_2)\,\left|J(x_1,x_2)\right|^{-1}$$

- by independence, $f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1)\, f_{X_2}(x_2)$

$$f_{X_1}(x_1) = \begin{cases} \lambda e^{-\lambda x_1} & x_1 \geq 0 \\ 0 & \text{else} \end{cases}$$

$$f_{X_2}(x_2) = \begin{cases} \lambda e^{-\lambda x_2} & x_2 \geq 0 \\ 0 & \text{else} \end{cases}$$

$$\Rightarrow f_{X_1,X_2}(x_1,x_2) = \begin{cases} \lambda^2 \exp(-\lambda(x_1+x_2)) & \text{if } 0 \leq x_1, x_2 \\ 0 & \text{else} \end{cases}$$