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# 271A - Homework 5

**Problem 1.** Consider the space $\mathbb{R}^d$ and the usual $\|\cdot\|_2$ metric. Show explicitly that a probability measure $\mathbb{P}$ on the measurable space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is uniquely determined by

$$F(x_1, \ldots, x_d) = \mathbb{P}[y : y_1 \leq x_1, \ldots, y_d \leq x_d].$$

*Proof.* Our strategy is to use the $\pi$-$\lambda$ theorem. Suppose $\mathbb{P}_1$ and $\mathbb{P}_2$ are two probability measures that agree on sets of the form $\{y_1 \leq x_1, \ldots, y_d \leq x_d)$. Define the collection $\Pi$ by

$$\Pi = \big\{\{y : y_1 \leq x_1, \ldots, y_d \leq x_d\} : x \in \mathbb{R}^d\big\}.$$

That is, $\Pi$ consists of all products of rays. As our notation suggests, $\Pi$ is a $\pi$-system since it is clearly nonempty and the intersection of any two products of rays is again a product of rays. We also define the collection $\Lambda$ to be the sets in $\sigma(\Pi)$, the $\sigma$-algebra generated by $\Pi$, on which $\mathbb{P}_1$ and $\mathbb{P}_2$ agree:

$$\Lambda = \{E \in \sigma(\Pi) : \mathbb{P}_1(E) = \mathbb{P}_2(E)\}.$$

This collection is indeed well-defined since every set in $\Pi$ is a Borel set, so $\sigma(\Pi) \subseteq \mathcal{B}(\mathbb{R}^d)$ and each $E \in \sigma(\Pi)$ is $\mathbb{P}_1$ and $\mathbb{P}_2$ measurable. We again claim that our notation makes sense and that $\Lambda$ is a $\lambda$-system. Let's verify this claim.

- $\mathbb{R}^d \in \Lambda$: We can write $\mathbb{R}^d$ as a union of ray-products, $\mathbb{R}^d = \bigcup_{n=1}^{\infty}(-\infty, n]^d$, so $\mathbb{R}^d$ is indeed in $\sigma(\Pi)$. That $\mathbb{P}_1[\mathbb{R}^d] = \mathbb{P}_2[\mathbb{R}^d]$ follows from the fact that $\mathbb{P}_1$ and $\mathbb{P}_2$ are probability measures.
- Closure under complements:

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