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270A - Homework 3

Problem 1. Show that if X and Y are independent, integer-valued random variables, then for all $n \in \mathbb{Z}$,

$$\mathbb{P}[X+Y=n] = \sum_{m\in\mathbb{Z}} \mathbb{P}[X=m]\cdot\mathbb{P}[Y=n-m].$$

Proof.

$$\begin{aligned}\mathbb{P}[X+Y=n] &= \mathbb{P}[Y=n-X] \\ &= \sum_{m\in\mathbb{Z}} \mathbb{P}[Y=n-m, X=m] \\ &= \sum_{m\in\mathbb{Z}} \mathbb{P}[Y=n-m]\cdot\mathbb{P}[X=m],\end{aligned}$$

where the last line follows from the independence of X and Y. □

Problem 3. Let $X_1, X_2, \ldots$ be independent random variables that satisfy

$$\frac{\mathrm{Var}[X_i]}{i} \to 0$$

as $i \to \infty$. Let $S_n = X_1 + \cdots + X_n$. Prove that

$$\frac{S_n - \mathbb{E}[S_n]}{n} \to 0 \quad \text{in probability.}$$

Proof. By Chebyshev's inequality, we have that

$$\begin{aligned}\mathbb{P}\left[\left|\frac{S_n-\mathbb{E}[S_n]}{n}\right| > \epsilon\right] &\le \frac{\mathrm{Var}[(S_n-\mathbb{E}[S_n])/n]}{\epsilon^2} \\ &= \frac{1}{n^2\epsilon^2}\sum_{i=1}^n \mathrm{Var}[X_i] \\ &\le \frac{1}{n\epsilon^2}\cdot\max_{i\le n}\mathrm{Var}[X_i] \\ &\to 0.\end{aligned}$$

□

Problem 4.

(a) Show that

$$d(X,Y) = \mathbb{E}\left[\frac{|X-Y|}{1+|X-Y|}\right]$$

defines a metric on the set of random variables.

Proof. It's clear that d is symmetric. Since $\frac{|X-Y|}{1+|X-Y|}$ is nonnegative, its expectation is zero if and only if $|X-Y|$ is zero almost surely, which happens if and only if $X = Y$ almost surely. This just leaves the triangle inequality.

Consider the function $f(t) = \frac{t}{1+t} = 1 - \frac{1}{1+t}$. Since $\frac{1}{1+t}$ is clearly decreasing, we have that f is increasing for $t \geq 0$. For any three random variables X, Y, and Z, we then have

$$\begin{aligned}
d(X,Z) &= f(|X-Z|) \\
&\leq f(|X-Y| + |Y-Z|) \\
&= \frac{|X-Y|}{1+|X-Y|+|Y-Z|} + \frac{|Y-Z|}{1+|X-Y|+|Y-Z|} \\
&\leq \frac{|X-Y|}{1+|X-Y|} + \frac{|Y-Z|}{1+|Y-Z|} \\
&= d(X,Y) + d(Y,Z).
\end{aligned}$$

□

(b) Show that $d(X_n, X) \to 0$ if and only if $X_n \to X$ in probability.

Proof. Suppose that $X_n \to X$ in probability. Fix $\epsilon > 0$ and let $E_n = \{|X_n - X| > \epsilon\}$. For sufficiently large n, we have that $\mathbb{P}[E_n] < \epsilon$. Let's bound the expectation, using the fact that $\frac{t}{1+t} \leq 1$ for all $t \geq 0$.

$$\begin{aligned}
d(X_n, X) &= \int_{E_n} \frac{|X_n - X|}{1+|X_n - X|} d\mathbb{P} + \int_{E_n^c} \frac{|X_n - X|}{1+|X_n - X|} d\mathbb{P} \\
&\leq \mathbb{P}[E_n] \cdot 1 + \mathbb{P}[E_n^c] \cdot \epsilon \\
&\leq 2\epsilon.
\end{aligned}$$

Suppose that X_n doesn't converge to X in probability. Then for some ϵ, there are infinitely many n such that $\mathbb{P}[|X_n - X| > \epsilon] > \epsilon$. Since $t \mapsto \frac{t}{1+t}$ is increasing, for such an n we have (with E_n defined as above)

$$\begin{aligned}
d(X_n, X) &\geq \int_{E_n} \frac{|X_n - X|}{1+|X_n - X|} d\mathbb{P} \\
&\geq \mathbb{P}[E_n] \cdot \frac{\epsilon}{1+\epsilon} \\
&\geq \frac{\epsilon^2}{1+\epsilon},
\end{aligned}$$

so $d(X_n, X)$ doesn't go to zero. □

Problem 5. Let $X_1, X_2, \ldots$ be independent $\mathrm{Ber}(p_n)$ random variables.

(a) Show that $X_n \to 0$ in probability if and only if $p_n \to 0$.

Proof. Fix $\epsilon > 0$. We then have

$$\mathbb{P}[|X_n| > \epsilon] = \mathbb{P}[X_n = 1] = p_n.$$

We then have that $\mathbb{P}[|X_n| > \epsilon] \to 0$ if and only if $p_n \to 0$. $\square$

(b) Show that $X_n \to 0$ a.s. if and only if $\sum p_n < \infty$.

Proof. Fix $\epsilon > 0$ and let $E_n = \{|X_n| > \epsilon\} = \{X_n = 1\}$. We then have

$$\sum \mathbb{P}[E_n] = \sum p_n < \infty.$$

By Borel-Cantelli, we have

$$\mathbb{P}[\limsup E_n] = \mathbb{P}[X_n = 1 \text{ infinitely often}] = 0.$$

Taking complements gives

$$1 = \mathbb{P}[\liminf E_n^c] = \mathbb{P}[X_n = 0 \text{ eventually}],$$

so $X_n \to 0$ almost surely. On the other hand, if $\sum p_n = \infty$, then since the X_j's are independent, Borel-Cantelli says that

$$\mathbb{P}[\limsup E_n] = \mathbb{P}[X_n = 1 \text{ infinitely often}] = 1.$$

Since $X_n = 1$ infinitely often, X_n doesn't converge to zero almost surely. $\square$

Problem 6. Let $X_1, X_2, \ldots$ be a sequence of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ where Ω is a countable set and $\mathcal{F} = 2^\Omega$. Show that $X_n \to X$ in probability implies $X_n \to X$ a.s.

Proof. Suppose X_n didn't converge to X almost surely. Then since Ω is a discrete space, there is some singleton $\{\omega\} \in \mathcal{F}$ with $\mathbb{P}[\{\omega\}] > 0$ (we'll abuse notation and write $\mathbb{P}[\omega] > 0$) and $X_n(\omega)$ doesn't converge to $X(\omega)$. We can then find $\epsilon > 0$ so that for infinitely many n, $|X_n(\omega) - X(\omega)| > \epsilon$. But then X_n can't converge to X in probability since

$$\mathbb{P}[|X_n - X| > \epsilon] \geq \mathbb{P}[\omega] > 0,$$

for infinitely many n. $\square$

Problem 7. Show that for any sequence of random variables $X_1, X_2, \ldots$ there exists a sequence of positive real numbers $c_1, c_2, \ldots$ such that $c_n X_n \to 0$ a.s.

Proof. Since X_n takes values in the real numbers, we have that $\mathbb{P}[|X_n| > t] \to 0$ as $t \to \infty$. There is then some b_n so that $\mathbb{P}[|X_n| > b_n] < 2^{-n}$. Now let $c_n = \frac{1}{n \cdot b_n}$. For any $\epsilon > 0$ we have

$$\sum \mathbb{P}[|c_n X_n| > \epsilon] = \sum \mathbb{P}[|X_n| > \epsilon \cdot n b_n].$$

For n sufficiently large, $\epsilon n \geq 1$ and $\mathbb{P}[|X_n| > \epsilon n b_n] \leq \mathbb{P}[|X_n| > b_n] < 2^{-n}$. The tail of the above series is then summable, and we have

$$\mathbb{P}[\limsup\{|c_n X_n| > \epsilon\}] = 0$$

for all $\epsilon > 0$, so $c_n X_n \to 0$ a.s. □

Problem 8. Let $X_1, X_2, \ldots$ be independent random variables (from $(\Omega, \mathcal{F}, \mathbb{P})$ to $\mathbb{R}$). Show that $\sup_n X_n < \infty$ a.s. if and only if there exists $M \in \mathbb{R}$ such that

$$\sum_n \mathbb{P}[X_n > M] < \infty.$$

Proof. Suppose there exists a real M so that the above sum is finite. Then by Borel-Cantelli we have that $\mathbb{P}[X_n > M \text{ infinitely often}] = 0$, or $\mathbb{P}[X_n \leq M \text{ eventually}] = 1$. For any $\omega \in \Omega$, there is some N_ω so that $X_n(\omega) \leq M$ for all $n > N_\omega$. We then have that

$$X_n(\omega) \leq \max\{X_1(\omega), X_2(\omega), \ldots, X_{N_\omega}(\omega), M\}.$$

Since this holds for a.e. ω, we have that $\sup_n X_n < \infty$ a.s.

Conversely, suppose that for every M we have $\sum \mathbb{P}[X_n > M] = +\infty$. Since the X_n's are independent, Borel-Cantelli tells us that $\mathbb{P}[X_n > M \text{ infinitely often}] = 1$. For almost any $\omega \in \Omega$ and for any real number M, there are infinitely many n such that $X_n(\omega) > M$. Consequently, $\sup_n X_n(\omega)$ is infinite for almost every ω. □

Problem 9. Let $X_0 = 1$ and define X_n inductively by choosing X_{n+1} uniformly at random from the interval $[0, X_n]$. Prove that

$$\frac{\log X_n}{n} \to c$$

a.s. and find the value of c.

Solution. Let $U_1, U_2, \ldots$ be iid Unif$[0,1]$ random variables. We claim that X_n and $U_1 \cdot U_2 \cdots U_n$ have the same distribution. This is clearly true for X_1, so suppose that X_n and $U_1 \cdots U_n$ have the same distribution and consider X_{n+1}. We have that

$$\mathbb{P}[X_{n+1} \leq t] = \frac{t}{X_n}, \quad \text{for all } t \leq X_n.$$

On the other hand, we have

$$\mathbb{P}[U_{n+1} \cdot X_n \leq t] = \mathbb{P}[U_{n+1} \leq t/X_n] = \frac{t}{X_n}.$$

We then have

$$\frac{\log X_n}{n} = \frac{1}{n}\sum_{k=1}^{n}\log U_k.$$

Let's compute an expectation

$$\mathbb{E}[\log U_k] = \int_0^1 \log x \; dx$$

$$= -1.$$

By the strong law of large numbers, we have that

$$\frac{\log X_n}{n} = \frac{1}{n}\sum_{k=1}^{n}\log U_k \to -1 \text{ a.s.}$$

□

Problem 10. Let $X_1, X_2, \ldots$ be independent random variables such that X_n takes value n with probability $1/(2n\log n)$ and value $-n$ with the same probability, and the value 0 with the remaining probability $1 - 1/(n\log n)$. Show that this sequence obeys the weak law, but not the strong law in the sense that

$$\frac{1}{n}\sum_{i=1}^{n} X_i \to 0$$

in probability but not a.s.

Proof. Let $S_n = \sum_{i=2}^{n+1} X_i$. Since the X_i are centered, so is S_n and $\mathbb{E}[S_n/n] = 0$. Let's compute the variance of X_n.

$$\operatorname{Var}[X_n] = \frac{n^2}{2n\log n} + \frac{(-n)^2}{2n\log n}$$

$$= \frac{n}{\log n}.$$

Since the X_n's are independent, the variances add and we have

$$\operatorname{Var}[S_n/n] = \frac{1}{n^2}\sum_{k=2}^{n+1}\frac{k}{\log k}.$$

Now let's fix $\epsilon > 0$ and use Chebyshev

$$\mathbb{P}\left[\left|\frac{S_n}{n}\right| > \epsilon\right] \le \frac{1}{n^2\epsilon^2}\sum_{k=2}^{n+1}\frac{k}{\log k} \tag{1}$$

Now $\frac{k}{\log k}$ is increasing for $k \ge 3$, so we can bound the above probability

$$\mathbb{P}\left[\left|\frac{S_n}{n}\right| > \epsilon\right] \le \frac{1}{n^2\epsilon^2}\cdot\frac{n(n+1)}{\log(n+1)}$$

$$\to 0,$$

so $S_n/n \to 0$ in probability.

Now consider the sum

$$\sum_{n=2}^{\infty} \mathbb{P}[X_n = n] = \sum_{n=2}^{\infty} \frac{1}{2n \log n} = +\infty.$$

Since the X_n's are independent, Borel-Cantelli tells us that $\mathbb{P}[X_n = n \text{ infinitely often}] = 1$. Consequently, $X_n/n = 1$ infinitely often, so the sum $\frac{1}{n} \sum_{k=2}^{n+1} X_n$ cannot converge to zero. $\square$