HomeWork 1 Math 271B, Winter 2020.

1. The standard Ornstein-Uhlenbeck process X_t is a Gaussian process with mean zero and auto-covariance $C(t,s) = \mathbb{E}[X_t X_s] = \exp(-|t-s|)/2$. Let N_t be the standard Poisson process (intensity $\lambda = 1$) and define the process $Y_t = \zeta(-1)^{N_t}$, where ζ is a random variable independent of the Poisson process that takes values ± 1 with probability $1/2$. Show that X_t and $Z_t = Y_{t/2}/\sqrt{2}$ are both stationary in the strong sense and have the same covariance. Does Y_t satisfy the Kolmogorov continuity condition? Are these processes stochastically continuous?

   Note A process X_t is stochastically continuous if

   $$\lim_{h\to 0} \mathbb{P}\left[|X_{t+h} - X_t| > \delta\right] = 0.,$$

   for all $\delta > 0$ and all t.

2. Let X_n be defined by the stochastic recursion

   $$X_{n+1} = X_n - \Delta t X_n + (B_{(n+1)\Delta t} - B_{n\Delta t}), \quad X_0 = \zeta,$$

   for B_t standard Brownian motion. Identify ζ so that X_n is stationary in the strong sense and give the associated auto-covariance function. What is the continuum limit of this process as $n \to \infty$, $\Delta t \to 0$ so that $n\Delta t = t$?

3. Consider

   $$X_t = \int_0^t (t-s)^{H-1/2} dB_s,$$

   for B_t standard Brownian motion. For which values of H is X_t well defined? Find the distribution of X_t. Compare with the distribution of fractional Brownian motion B_t^H.

4. Prove directly from the definition of Itô integrals the integration by parts relation:

   $$\int_0^t s dB_s = tB_t - \int_0^t B_s ds.$$

5. Prove directly from the definition of the Itô integral that

   $$\int_0^t B_s^2 dB_s = \frac{1}{3}B_t^3 - \int_0^t B_s ds.$$

6. Let

   $$M_t = B_t^3 - 3tB_t,$$

   with B_t standard Brownian motion. Show that M_t is a martingale, first directly and then by using the result of the previous 2 problems.