Math 130B - Big O Notation

The following definition gives us a rigorous way of saying one function is “larger” than another.

Definition 1. Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions.

(a) We write $f(x) = O(g(x))$ if there exist positive constants c_1 and c_2 such that $|f(x)| \leq c_1|g(x)|$ for all $x \geq c_2$. That is, f is *eventually* at most a constant multiple of g.

(b) $f(x) = \Omega(g(x))$ if there exist positive constants c_1 and c_2 such that $|f(x)| \geq c_1|g(x)|$ for all $x \geq c_2$. That is, f is eventually at least a constant multiple of g.

(c) $f(x) = \Theta(g(x))$ if $f(x) = O(g(x))$ and $f(x) = \Omega(g(x))$.

Exercise 1. (a) Let $f(x) = ax + b$ and $g(x) = cx + d$ where $a, c \neq 0$. Show that $f(x) = O(g(x))$.

(b) Let $f(x) = ax^2 + bx + c$, $a > 0$. Show that $f(x) = \Omega(x^2)$.

(c) Show that $\sin x = \Theta(1)$.

These definitions can be a bit cumbersome to work with sometimes. The following is sometimes easier to check. Try proving it!

Theorem 1. (a) If $\lim_{x\to\infty} \frac{|f(x)|}{|g(x)|} < \infty$ then $f(x) = O(g(x))$.

(b) If $\lim_{x\to\infty} \frac{|f(x)|}{|g(x)|} > 0$ then $f(x) = \Omega(g(x))$.

Exercise 2. Prove the following.

(a) $x^2 + \sqrt{x} = O(x^2)$.

(b) If $\lim_{x\to\infty} f(x) = \infty$, then $f(x) + \sin x = \Theta(f(x))$.

(c) $k^2 2^k = O(e^{2k})$.

(d) $N^{10} 2^N = O(e^N)$.

We also have notation to express the idea of one function being *strictly* less or greater than another.

Definition 2. Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions.

(a) We write $f(x) = o(g(x))$ if for all $c_1 > 0$ there exists a $c_2 > 0$ so that $|f(x)| \leq c_1|g(x)|$ for all $x \geq c_2$. That is, f is eventually smaller than *any* constant multiple of g.

(b) We write $f(x) = \omega(g(x))$ if for all $c_1 > 0$ there exists a $c_2 > 0$ so that $|f(x)| \geq c_1|g(x)|$ for all $x \geq c_2$. That is, g is eventually greater than any multiple of g.

Just like with O and Ω, we can take limits to show $f(x) = o(g(x))$ or $\omega(g(x))$.

Theorem 2. (a) $f(x) = o(g(x))$ if and only if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.

(b) $f(x) = \omega(g(x))$ if and only if $\lim_{x \to \infty} \frac{|f(x)|}{|g(x)|} = \infty$.

Exercise 3. Prove the following.

(a) We often say that a sequence of events E_n happens "with high probability" if $\Pr[E_n] = 1 - o(1)$. Why does this make sense?

(b) If $f(x) = o(g(x))$ then $f(x) = O(g(x))$. If $f(x) = \omega(g(x))$ then $f(x) = \Omega(g(x))$. Give examples to show that the converses to these statements are false.

(c) $k^{300} = o(2^k)$.

(d) $k^{0.001} = \omega((\log k)^{375})$.