*Liam Hardiman* *November 25, 2019*

# 270A - Homework 3

**Problem 1.** Show that if $X$ and $Y$ are independent, integer-valued random variables, then for all $n \in \mathbb{Z}$,

$$\mathbb{P}[X + Y = n] = \sum_{m \in \mathbb{Z}} \mathbb{P}[X = m] \cdot \mathbb{P}[Y = n - m].$$

*Proof.*

$$\begin{aligned}
\mathbb{P}[X + Y = n] &= \mathbb{P}[Y = n - X] \\
&= \sum_{m \in \mathbb{Z}} \mathbb{P}[Y = n - m, X = m] \\
&= \sum_{m \in \mathbb{Z}} \mathbb{P}[Y = n - m] \cdot \mathbb{P}[X = m],
\end{aligned}$$

where the last line follows from the independence of $X$ and $Y$. $\square$

**Problem 3.** Let $X_1, X_2, \ldots$ be independent random variables that satisfy

$$\frac{\mathrm{Var}[X_i]}{i} \to 0$$

as $i \to \infty$. Let $S_n = X_1 + \cdots + X_n$. Prove that

$$\frac{S_n - \mathbb{E}[S_n]}{n} \to 0 \quad \text{in probability.}$$

*Proof.* By Chebyshev's inequality, we have that

$$\begin{aligned}
\mathbb{P}\left[\left|\frac{S_n - \mathbb{E}[S_n]}{n}\right| > \epsilon\right] &\leq \frac{\mathrm{Var}[(S_n - \mathbb{E}[S_n])/n]}{\epsilon^2} \\
&= \frac{1}{n^2\epsilon^2} \sum_{i=1}^{n} \mathrm{Var}[X_i] \\
&\leq \frac{1}{n\epsilon^2} \cdot \max_{i \leq n} \mathrm{Var}[X_i] \\
&\to 0.
\end{aligned}$$

$\square$

**Problem 4.**

(a) Show that

$$d(X, Y) = \mathbb{E}\left[\frac{|X - Y|}{1 + |X - Y|}\right]$$

defines a metric on the set of random variables.

*Proof.* It's clear that $d$ is symmetric. Since $\frac{|X-Y|}{1+|X-Y|}$ is nonnegative, its expectation is zero if and only if $|X-Y|$ is zero almost surely, which happens if and only if $X = Y$ almost surely. This just leaves the triangle inequality.

Consider the function $f(t) = \frac{t}{1+t} = 1 - \frac{1}{1+t}$. Since $\frac{1}{1+t}$ is clearly decreasing, we have that $f$ is increasing for $t \geq 0$. For any three random variables $X$, $Y$, and $Z$, we then have

$$\begin{aligned}
d(X, Z) &= f(|X-Z|) \\
&\leq f(|X-Y| + |Y-Z|) \\
&= \frac{|X-Y|}{1+|X-Y|+|Y-Z|} + \frac{|Y-Z|}{1+|X-Y|+|Y-Z|} \\
&\leq \frac{|X-Y|}{1+|X-Y|} + \frac{|Y-Z|}{1+|Y-Z|} \\
&= d(X, Y) + d(Y, Z).
\end{aligned}$$

□

(b) Show that $d(X_n, X) \to 0$ if and only if $X_n \to X$ in probability.

*Proof.* Suppose that $X_n \to X$ in probability. Fix $\epsilon > 0$ and let $E_n = \{|X_n - X| > \epsilon\}$. For sufficiently large $n$, we have that $\mathbb{P}[E_n] < \epsilon$. Let's bound the expectation, using the fact that $\frac{t}{1+t} \leq 1$ for all $t \geq 0$.

$$\begin{aligned}
d(X_n, X) &= \int_{E_n} \frac{|X_n - X|}{1+|X_n - X|} d\mathbb{P} + \int_{E_n^c} \frac{|X_n - X|}{1+|X_n - X|} d\mathbb{P} \\
&\leq \mathbb{P}[E_n] \cdot 1 + \mathbb{P}[E_n^c] \cdot \epsilon \\
&\leq 2\epsilon.
\end{aligned}$$

Suppose that $X_n$ doesn't converge to $X$ in probability. Then for some $\epsilon$, there are infinitely many $n$ such that $\mathbb{P}[|X_n - X| > \epsilon] > \epsilon$. Since $t \mapsto \frac{t}{1+t}$ is increasing, for such an $n$ we have (with $E_n$ defined as above)

$$\begin{aligned}
d(X_n, X) &\geq \int_{E_n} \frac{|X_n - X|}{1+|X_n - X|} d\mathbb{P} \\
&\geq \mathbb{P}[E_n] \cdot \frac{\epsilon}{1+\epsilon} \\
&\geq \frac{\epsilon^2}{1+\epsilon},
\end{aligned}$$

so $d(X_n, X)$ doesn't go to zero. □

**Problem 5.** Let $X_1, X_2, \dots$ be independent $\mathrm{Ber}(p_n)$ random variables.

(a) Show that $X_n \to 0$ in probability if and only if $p_n \to 0$.

*Proof.* Fix $\epsilon > 0$. We then have

$$\mathbb{P}[|X_n| > \epsilon] = \mathbb{P}[X_n = 1] = p_n.$$

We then have that $\mathbb{P}[|X_n| > \epsilon] \to 0$ if and only if $p_n \to 0$. $\square$

(b) Show that $X_n \to 0$ a.s. if and only if $\sum p_n < \infty$.

*Proof.* Fix $\epsilon > 0$ and let $E_n = \{|X_n| > \epsilon\} = \{X_n = 1\}$. We then have

$$\sum \mathbb{P}[E_n] = \sum p_n < \infty.$$

By Borel-Cantelli, we have

$$\mathbb{P}[\limsup E_n] = \mathbb{P}[X_n = 1 \text{ infinitely often}] = 0.$$

Taking complements gives

$$1 = \mathbb{P}[\liminf E_n^c] = \mathbb{P}[X_n = 0 \text{ eventually}],$$

so $X_n \to 0$ almost surely. On the other hand, if $\sum p_n = \infty$, then since the $X_j$'s are independent, Borel-Cantelli says that

$$\mathbb{P}[\limsup E_n] = \mathbb{P}[X_n = 1 \text{ infinitely often}] = 1.$$

Since $X_n = 1$ infinitely often, $X_n$ doesn't converge to zero almost surely. $\square$

**Problem 6.** Let $X_1, X_2, \dots$ be a sequence of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ where $\Omega$ is a countable set and $\mathcal{F} = 2^\Omega$. Show that $X_n \to X$ in probability implies $X_n \to X$ a.s.

*Proof.* Suppose $X_n$ didn't converge to $X$ almost surely. Then since $\Omega$ is a discrete space, there is some singleton $\{\omega\} \in \mathcal{F}$ with $\mathbb{P}[\{\omega\}] > 0$ (we'll abuse notation and write $\mathbb{P}[\omega] > 0$) and $X_n(\omega)$ doesn't converge to $X(\omega)$. We can then find $\epsilon > 0$ so that for infinitely many $n$, $|X_n(\omega) - X(\omega)| > \epsilon$. But then $X_n$ can't converge to $X$ in probability since

$$\mathbb{P}[|X_n - X| > \epsilon] \geq \mathbb{P}[\omega] > 0,$$

for infinitely many $n$. $\square$

**Problem 7.** Show that for any sequence of random variables $X_1, X_2, \dots$ there exists a sequence of positive real numbers $c_1, c_2, \dots$ such that $c_n X_n \to 0$ a.s.

*Proof.* Since $X_n$ takes values in the real numbers, we have that $\mathbb{P}[|X_n| > t] \to 0$ as $t \to \infty$. There is then some $b_n$ so that $\mathbb{P}[|X_n| > b_n] < 2^{-n}$. Now let $c_n = \frac{1}{n \cdot b_n}$. For any $\epsilon > 0$ we have

$$\sum \mathbb{P}[|c_n X_n| > \epsilon] = \sum \mathbb{P}[|X_n| > \epsilon \cdot n b_n].$$

For $n$ sufficiently large, $\epsilon n \geq 1$ and $\mathbb{P}[|X_n| > \epsilon n b_n] \leq \mathbb{P}[|X_n| > b_n] < 2^{-n}$. The tail of the above series is then summable, and we have

$$\mathbb{P}[\limsup\{|c_n X_n| > \epsilon\}] = 0$$

for all $\epsilon > 0$, so $c_n X_n \to 0$ a.s. $\square$

**Problem 8.** Let $X_1, X_2, \ldots$ be independent random variables (from $(\Omega, \mathcal{F}, \mathbb{P})$ to $\mathbb{R}$). Show that $\sup_n X_n < \infty$ a.s. if and only if there exists $M \in \mathbb{R}$ such that

$$\sum_n \mathbb{P}[X_n > M] < \infty.$$

*Proof.* Suppose there exists a real $M$ so that the above sum is finite. Then by Borel-Cantelli we have that $\mathbb{P}[X_n > M \text{ infinitely often}] = 0$, or $\mathbb{P}[X_n \leq M \text{ eventually}] = 1$. For any $\omega \in \Omega$, there is some $N_\omega$ so that $X_n(\omega) \leq M$ for all $n > N_\omega$. We then have that

$$X_n(\omega) \leq \max\{X_1(\omega), X_2(\omega), \ldots, X_{N_\omega}(\omega), M\}.$$

Since this holds for a.e. $\omega$, we have that $\sup_n X_n < \infty$ a.s.

Conversely, suppose that for every $M$ we have $\sum \mathbb{P}[X_n > M] = +\infty$. Since the $X_n$'s are independent, Borel-Cantelli tells us that $\mathbb{P}[X_n > M \text{ infinitely often}] = 1$. For almost any $\omega \in \Omega$ and for any real number $M$, there are infinitely many $n$ such that $X_n(\omega) > M$. Consequently, $\sup_n X_n(\omega)$ is infinite for almost every $\omega$. $\square$

**Problem 9.** Let $X_0 = 1$ and define $X_n$ inductively by choosing $X_{n+1}$ uniformly at random from the interval $[0, X_n]$. Prove that

$$\frac{\log X_n}{n} \to c$$

a.s. and find the value of $c$.

*Solution.* Let $U_1, U_2, \ldots$ be iid Unif$[0,1]$ random variables. We claim that $X_n$ and $U_1 \cdot U_2 \cdots U_n$ have the same distribution. This is clearly true for $X_1$, so suppose that $X_n$ and $U_1 \cdots U_n$ have the same distribution and consider $X_{n+1}$. $\square$