Math 130B - Graphs and Expectation

1. Let $G = (V, E)$ be a (simple) graph with finite vertex set. Show that $\sum_{v \in V} d(v) = 2|E|$, where $d(v)$ is the *degree* of the vertex v (the number of edges incident to v).

2. Recall that the random graph $G \sim \mathcal{G}(n, p)$ has n vertices and each pair of vertices has an edge between them with probability p, independent of one another. Show that if $p = o(1/n)$ then $G \sim \mathcal{G}(n, p)$ has no cycles with probability $1 - o(1)$. *Hint: Fix some $t \geq 3$ and let X_t be the number of cycles of length t. Calculate $E[X_t]$. You might want to use the fact that $\binom{n}{k} \leq \frac{n^k}{k!}$.*

3. Let $G = V(, E)$ be a simple graph. Recall that an independent set in G is a set of vertices, no two of which have an edge between them. The *independence number* $\alpha(G)$ is the size of the largest independent set in G. Here we'll prove that

$$\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{d(v) + 1}. \tag{1}$$

   (a) Choose an ordering $v_1, \ldots, v_n$ of $V(G)$ uniformly at random. Let S be the set of all vetices that appear before all of their neighbors in the ordering. Show that S is an independent set.

   (b) For each vertex v, let X_v be the indicator random variable which has value 1 if $v \in S$ and 0 otherwise. Compute $E[X_v]$. Use this to compute $E[|S|]$ and deduce (1).