$\tau(n) = \#$ divisors of n

a) show $\tau(mn) = \tau(m)\tau(n)$ for $(m,n)=1$

Pf: look at prime factorization.

$$m = p_1^{e_1} \cdots p_k^{e_k}$$

$$\Longrightarrow \tau(m) = (e_1+1)\cdots(e_k+1)$$

b) show $\underbrace{\sum_{d|n} \tau^3(d)}_{f(n)} = \underbrace{\left(\sum_{d|n} \tau(d)\right)^2}_{g(n)}$

Pf: • if f is multiplicative, then so is

$$g(n) = \sum_{d|n} f(d).$$

• so is f^2, f^3, etc.

by these facts,

$\sum_{d|n} \tau^3(d)$ & $\left(\sum_{d|n} \tau(d)\right)^2$ are multiplicative.

suffices to show that

☆ $f(p^k) = g(p^k)$ $\forall$ primes p, $k \geq 0$.

$$f(p^k) = \sum_{d|p^k} \tau^3(d)$$

$$= \sum_{j=0}^{k} \tau^3(p^j)$$

$$= \sum_{j=0}^{k} (j+1)^3$$

$$g(p^k) = \left(\sum_{d|p^k} \tau(d)\right)^2 = \left(\sum_{j=0}^{k} (j+1)\right)^2$$

remains to show that

$$\sum_{j=1}^{k+1} j^3 = \left(\sum_{j=1}^{k+1} j\right)^2$$ induction

Prove that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

~~Pf: Let P(n) be " $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ ".~~

⋮

~~therefore P(n+1) !!~~

~~"Let the pigeons be . . .~~

~~let the pigeonholes be . . .~~

② Show that $g \in \mathbb{Z}/257\mathbb{Z} \sim 0$ is a primitive root if and only if $g^{128} \equiv -1 \pmod{257}$.

$(\mathbb{Z}/p\mathbb{Z})^{\times}$

note: 257 is prime!

$p-1 = 256 = 2^8$

g is a p.r. iff $o(g) = p-1$
$= 256 = 2^8$

$\mathrm{ord}(g) \mid p-1 = 2^8 \Rightarrow \mathrm{ord}(g) = 2^k$ for some k.

$g \neq 0$.

$(g^{128})^2 = g^{256} = g^{p-1} \overset{FLT}{=} 1$

$\Rightarrow g^{128} = \pm 1 \bmod p$ ($x^2 = 1$ has solns ± 1)

if $g^{128} = 1$, then $\mathrm{ord}(g) < 256$
$\Rightarrow g$ not p.r.

conversely, if g is not a p.r.

$\Longrightarrow \text{ord}(g) = 2^k$, $k < 8$ ≥ 0

$$g^{128} = g^{2^7} = g^{2^k \cdot 2^{7-k}} = \left(g^{2^k}\right)^{2^{7-k}}$$

$$= \,)^{2^{7-k}} = ($$

$\boxtimes$

④ Show that for any $k \geq 2$ the number $\sum_{n=1}^{\infty} \frac{1}{k^{n!}}$ is transcendental.

Pf: See lecture notes (10? 11?)

$\sum_{n=1}^{\infty} 1/10^{n!}$ transcendental.

Idea: $\alpha = \sum_{n=1}^{\infty} 1/k^{n!}$

let $r_N = \sum_{n=1}^{N} 1/k^{n!} = \frac{a_N}{k^{N!}}$

$$0 < \alpha - r_N = \sum_{n=N+1}^{\infty} 1/k^{n!} < 2/k^{(N+1)!}$$

$$\frac{1}{k^{(N+1)!}} + \underline{\sum_{n=N+2}^{\infty} 1/k^{n!}}$$

look at base k?

$$\left(\frac{1}{k^{(N+2)!}} + \frac{1}{k^{(N+3)!}} + \cdots \right)$$

$$= \frac{1}{k^{(N+2)!}} \underline{\left(1 + \frac{1}{k^{(N+3)}} + \frac{1}{k^{(N+4)(N+3)}} \right)}$$

$\leq$ geometric

≤ 2

Let $b_N = K^{N!}$

$$0 < \alpha - \frac{a_N}{K^{N!}} < \frac{2}{K^{(N+1)!}}$$

$$0 < \alpha - \frac{a_N}{b_N} < \frac{2}{b_N^{N+1}} = \frac{2}{b_N} \cdot \frac{1}{b_N^N}$$

$$< 1/b_N^N$$

if α were algebraic degree D,

this would have only finitely many solutions for $N \geq D$

but we showed it has solutions $\forall N$

③ Let (x_k, y_k) be the k-th solution to Pell's equation $x^2 - Dy^2 = 1$

Show that the limit $\lim_{k \to \infty} \frac{x_{k+1}}{x_k}$ exists.

Let (x_1, y_1) be the fundamental soln.

then $x_k + y_k\sqrt{D} = (x_1 + \sqrt{D}\, y_1)^k$

$$\Rightarrow x_{k+1} + y_{k+1}\sqrt{D} = (x_k + y_k\sqrt{D})(x_1 + y_1\sqrt{D})$$

$$= (x_1 x_k + y_1 y_k D) + (y_1 x_k + x_1 y_k)\sqrt{D}$$

$$\frac{x_{k+1}}{x_k} = x_1 + y_1 D \frac{y_k}{x_k} \to x_1 + y_1\sqrt{D}$$

$$x_k^2 - y_k^2 D = 1$$

$$\Rightarrow \left(\frac{x_k}{y_k}\right)^2 - D = \frac{1}{y_k^2}$$

as $k \to \infty$, $y_k \to \infty$

$$\Rightarrow \frac{x_k}{y_k} \to \sqrt{D}$$