

271A - Homework 5

Problem 1. Consider the space \mathbb{R}^d and the usual $\|\cdot\|_2$ metric. Show explicitly that a probability measure \mathbb{P} on the measurable space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is uniquely determined by

$$F(x_1, \dots, x_d) = \mathbb{P}[y : y_1 \leq x_1, \dots, y_d \leq x_d].$$

Proof. Our strategy is to use the π - λ theorem. Suppose \mathbb{P}_1 and \mathbb{P}_2 are two probability measures that agree on sets of the form $\{y_1 \leq x_1, \dots, y_d \leq x_d\}$. Define the collection Π by

$$\Pi = \{\{y : y_1 \leq x_1, \dots, y_d \leq x_d\} : x \in \mathbb{R}^d\}.$$

That is, Π consists of all products of rays. As our notation suggests, Π is a π -system since it is clearly nonempty and the intersection of any two products of rays is again a product of rays. We also define the collection Λ to be the sets in $\sigma(\Pi)$, the σ -algebra generated by Π , on which \mathbb{P}_1 and \mathbb{P}_2 agree:

$$\Lambda = \{E \in \sigma(\Pi) : \mathbb{P}_1(E) = \mathbb{P}_2(E)\}.$$

This collection is indeed well-defined since every set in Π is a Borel set, so $\sigma(\Pi) \subseteq \mathcal{B}(\mathbb{R}^d)$ and each $E \in \sigma(\Pi)$ is \mathbb{P}_1 and \mathbb{P}_2 measurable. We again claim that our notation makes sense and that Λ is a λ -system. Let's verify this claim.

- $\mathbb{R}^d \in \Lambda$: We can write \mathbb{R}^d as a union of ray-products, $\mathbb{R}^d = \bigcup_{n=1}^{\infty} (-\infty, n]^d$, so \mathbb{R}^d is indeed in $\sigma(\Pi)$. That $\mathbb{P}_1[\mathbb{R}^d] = \mathbb{P}_2[\mathbb{R}^d]$ follows from the fact that \mathbb{P}_1 and \mathbb{P}_2 are probability measures.
- Closure under complements: Let $E \in \Lambda$. Since \mathbb{P}_1 and \mathbb{P}_2 are probability measures, we can write

$$\mathbb{P}_1[E^c] = 1 - \mathbb{P}_1[E] = 1 - \mathbb{P}_2[E] = \mathbb{P}_2[E^c].$$

- Closure under countable disjoint union: Let $\{E_n\}_{n=1}^{\infty}$ be a countable disjoint family in Λ . By countable additivity of measure, we have

$$\mathbb{P}_1(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mathbb{P}_1[E_n] = \sum_{n=1}^{\infty} \mathbb{P}_2[E_n] = \mathbb{P}_2(\bigcup_{n=1}^{\infty} E_n).$$

Hence, Λ is a λ -system. By the π - λ theorem, $\sigma(\Pi) \subseteq \Lambda$. By construction, we also have $\Lambda \subseteq \sigma(\Pi)$, so $\Lambda = \sigma(\Pi)$. But the products of rays generate all of $\mathcal{B}(\mathbb{R}^d)$, so we must have $\mathbb{P}_1 = \mathbb{P}_2$ on all of $\mathcal{B}(\mathbb{R}^d)$. \square

Problem 2. Show that if a set A of continuous paths on $[0, 1]$ is equicontinuous at each point in $[0, 1]$ then the set is uniformly equicontinuous.

Proof.

\square