

## 270B - Homework 2

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### Problem 1.

- (a) Prove that if the probability density functions of  $X_n$  converge pointwise to the probability density function of  $X$ , then  $X_n$  converges to  $X$  weakly.

*Proof.* Let  $f_n$  and  $f$  be the density functions of  $X_n$  and  $X$ , respectively, and let  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  be bounded and continuous. We then have

$$\begin{aligned} |\mathbb{E}_n[\varphi] - \mathbb{E}[\varphi]| &= \left| \int_{\mathbb{R}} f_n(x) \varphi(x) dx - \int_{\mathbb{R}} f(x) \varphi(x) dx \right| \\ &\leq \|\varphi\|_{\infty} \int_{\mathbb{R}} |f_n(x) - f(x)| dx. \end{aligned} \tag{1}$$

Now we claim that  $\|f_n - f\|_{L^1} \rightarrow 0$  (argument taken from 210 notes). Since  $\int f_n = \int f = 1$  we have by Fatou

$$2 \int_{\mathbb{R}} f dx = \int_{\mathbb{R}} \liminf_{n \rightarrow \infty} (f + f_n - |f_n - f|) dx \leq 2 \int_{\mathbb{R}} f dx - \limsup_{n \rightarrow \infty} \int_{\mathbb{R}} |f_n - f| dx.$$

In particular, we have  $\limsup_{n \rightarrow \infty} \int |f_n - f| = 0$ , so  $\|f_n - f\|_{L^1} \rightarrow 0$ . Consequently, the right-hand side of (1) goes to zero as  $n \rightarrow \infty$ . We then have that  $\mathbb{E}_n[\varphi] \rightarrow \mathbb{E}[\varphi]$  for all  $\varphi$  bounded continuous, so  $X_n \rightarrow X$  weakly.  $\square$

- (b) Prove that if the probability mass functions of  $X_n$  converge to the probability mass function of  $X$  pointwise then  $X_n$  converges to  $X$  weakly.

*Proof.* We essentially mirror our proof of part (a) but with the counting measure. Let  $f_n$  and  $f$  be the mass functions of  $X_n$  and  $X$  respectively and suppose they take values in the discrete set  $S \subset \mathbb{R}$ . Suppose  $\varphi$  is a bounded function on  $S$  (note that any function from a discrete space is continuous). We then have

$$\begin{aligned} |\mathbb{E}_n[\varphi] - \mathbb{E}[\varphi]| &= \left| \sum_{x \in S} f_n(x) \varphi(x) - \sum_{x \in S} f(x) \varphi(x) \right| \\ &\leq \|\varphi\|_{\infty} \sum_{x \in S} |f_n(x) - f(x)|. \end{aligned} \tag{2}$$

Fatou still works with the counting measure, so we have

$$2 \sum_{x \in S} f(x) = \sum_{x \in S} \liminf_{n \rightarrow \infty} (f(x) + f_n(x) - |f_n(x) - f(x)|) \leq 2 \sum_{x \in S} f(x) - \limsup_{n \rightarrow \infty} \sum_{x \in S} |f_n(x) - f(x)|.$$

We conclude that  $\sum_{x \in S} |f_n(x) - f(x)| \rightarrow 0$  as  $n \rightarrow \infty$ . The right-hand side of (2) vanishes as  $n \rightarrow \infty$ , so  $X_n \rightarrow X$  weakly.  $\square$