Today's
Problems on
Canvas under
Files > Discussion
Documents

- Make an account
  on Overleaf.com
- or download something
  like Miktex

3. Prove that the number of positive irreducible fractions $\leq 1$ with denominator $\leq n$ is $\phi(1)+\phi(2)+\cdots+\phi(n)$.

3,

Induction

$n=1$ $\{\frac{1}{1}\}$ is the set of all such fractions

& $|\{\frac{1}{1}\}| = 1 = \varphi(1)$

- say it holds up to $n$,

# such fractions denom $\leq n+1$

$=$ # such fractions denom $\leq n$

$+$ # " denom $= n+1$

$\overset{\text{induct}}{=} \varphi(1)+\varphi(2)+\cdots+\varphi(n)$

$+\varphi(n+1)$ $\blacksquare$

5. Define the function $\mu : \mathbb{N} \to \{-1, 0, 1\}$ by

$$\mu(n) = \begin{cases} 1, & \text{if } n \text{ is square-free and has an even number of prime divisors} \\ -1, & \text{if } n \text{ is square-free and has an odd number of prime divisors} \\ 0, & \text{if } n \text{ has a squared prime factor} \end{cases}.$$

(a) Prove that $\mu(n)$ is multiplicative and

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n > 1 \end{cases}.$$

let $\omega(n) = \#$ prime divisors of $n$

$$\mu(n) = \begin{cases} (-1)^{\omega(n)}, & n \text{ square-free} \\ 0, & n \text{ divisible by a square} \end{cases}$$

suppose $(m,n) = 1$

$\Rightarrow \{\text{primes } p \mid m\} \cap \{\text{primes } p \mid n\} = \emptyset$

$\{\text{primes } p \mid mn\}$

$= \{\text{primes } p \mid m\}$

$\sqcup \{\text{primes } p \mid n\}$

- if $m$ or $n$ divisible by a square, wlog, $m$ is. Then so is $mn$

$\Rightarrow 0 = \mu(mn) = \mu(m)\mu(n)$

$= 0 \; \mu(n)$

$= 0$

if $m$ & $n$ sq-free,

then $\omega(mn) = \omega(m) + \omega(n)$

$$\Rightarrow \mu(mn) = (-1)^{\omega(mn)}$$

$$= (-1)^{\omega(m)} (-1)^{\omega(n)}$$

$$= \mu(m)\mu(n).$$

---

on your HW, you prove $f$ multiplicative

$\Longrightarrow \sum_{d \mid n} f(d)$ multi-plicative

So $\sum_{d|n} \mu(d)$ is multiplicative

Case 1: $n=1$

$$\sum_{d|1} \mu(d) = \mu(1)$$

$$= 1$$

Case 2: $n = p^k$ $p$ prime

The divisors of $p^k$ are

$p^0, p^1, \ldots, p^k$

So $\sum_{d \mid p^k} \mu(d)$

$$= \sum_{j=0}^{k} \mu(p^j) = \mu(p^0) + \mu(p^1)$$

$$= 1 - 1 = 0$$

general case,

$$n = p_1^{e_1} \cdots p_k^{e_k}$$

let $F(n) = \sum_{d \mid n} \mu(d)$

$$F(n) = F(p_1^{e_1} \cdots p_k^{e_k})$$

$$= F(p_1^{e_1}) \cdots F(p_k^{e_k}) \quad \boxtimes$$

$$= 0 \cdot 0 \cdots 0 = 0$$

Mobius inversion

b) (b) Prove that if $F(n) = \sum_{d|n} f(d)$ for every positive integer $n$, then $f(n) = \sum_{d|n} \mu(d)F(n/d)$.

Pf: $\sum_{d|n} \mu(d) F(n/d)$

$= \sum_{d|n} \mu(d) \sum_{k|n/d} f(k)$

$d|n$ & $k|\frac{n}{d}$ $\Leftrightarrow dk|n$

$= \sum_{\substack{(d,n) \\ : dk|n}} \mu(d) f(k)$

is symmetric in $d$ & $k$, so we can switch them

$= \sum_{k|n} f(k) \sum_{d|\frac{n}{k}} \mu(d)$ → part a only term that survives is $k=n$

$= f(n)$

c) (c) Prove the converse to (b): if $f(n) = \sum_{d|n} \mu(d)F(n/d)$, then $F(n) = \sum_{d|n} f(d)$.

Pf: $\sum_{d|n} f(d) = \sum_{d|n} \sum_{k|d} \mu(k) F\left(\frac{d}{k}\right)$

$= \sum_{d|n} \sum_{k|d} \mu\left(\frac{d}{k}\right) F(k) \quad (*)$

group by $F(k)$, get

$\mu(d/k) F(k) \ \forall\ k|d|n$

$\iff \frac{d}{k} \Big| \frac{n}{k}$

check!

$\checkmark$ $\mu(d/k)$

One term for every $\frac{d}{k} \mid \frac{n}{k}$. Let $r = d/k$

$$(*) = \sum_{k \mid n} \underbrace{\sum_{r \mid n/k} \mu(r)}_{} F(k)$$

Part a

only term that survives is

$k = n$

$= F(n)$

$\boxtimes$