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271B - Homework 3

Problem 1. Use Itô's formula to prove that for

$$b_t(k) = \mathbb{E}[B_t^k]$$

with B a standard Brownian motion

$$b_t(k) = \frac{1}{2}k(k-1)\int_0^t b_s(k-2)\ ds.$$

Solution. Consider the function $g(t,x) = x^k$, which is of class C^2 and let $Y_t = g(t, B_t) = B_t^k$. By Itô's formula we have

$$\begin{aligned} dY_t &= \frac{\partial g}{\partial t}(t,B_t)dt + \frac{\partial g}{\partial x}(t,B_t)dB_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,B_t)\cdot(dB_t)^2 \\ &= (kB_t^{k-1})dB_t + \left(\frac{1}{2}k(k-1)B_t^{k-2}\right)dt. \end{aligned}$$

Integrating gives

$$B_t^k = k\int_0^t B_s^{k-1}\ dB_s + \frac{1}{2}k(k-1)\int_0^t B_s^{k-2}\ ds,$$

since $B_0 = 0$ almost surely. Finally, we take the expectation of both sides. The first term will vanish since it's a martingale and we use Fubini to interchange the expectation and the integral of the second term.

$$b_t(k) = \frac{1}{2}k(k-1)\int_0^t b_s(k-2)\ ds.$$

Note that $b_t(1) = \mathbb{E}[B_t] = 0$. By induction, it follows that $b_t(k) = 0$ for all odd k. For $k = 2l$ we have

$$b_t(2l) = \frac{1}{2}(2l)(2l-1)\int_0^t b_s(2l-2)ds = \left[\frac{1}{2}(2l)(2l-1)\right]\left[\frac{1}{2}(2l-2)(2l-3)\right]\int_0^t\int_0^s b_{s'}(2l-4)\ ds'ds.$$

By induction we have

$$b_t(2l) = \frac{(2l)!}{2^l}\int_0^t\int_0^{t_1}\cdots\int_0^{t_{l-1}}\mathbb{E}[B_l^2]\ dt_l\cdots d_{t_1} = \frac{(2l)!t^l}{2^l l!}.$$

□

Problem 2. Find dX and $\langle X\rangle$ when

a) $X(t) = tB_t$,

Solution. Set $g(t,x) = tx$. We then have $X_t = g(t,B_t) = tB_t$. By Itô's formula we have

$$dX_t = B_t dt + t dB_t.$$

We'll prove a general fact about the quadratic variation of an Itô process. Suppose that Y_t is a continuous Itô process given by

$$Y_t = Y_0 + \int_0^t \mu_s\ ds + \int_0^t \sigma_s\ dB_s,$$

where $\int_0^t (\sigma_s^2 + |\mu_s|)ds < \infty$ a.s. for all t. For convenience, let $I_t = \int_0^t \mu_s \, ds$ and $M_t = \int_0^t \sigma_s \, dB_s$. By the bilinearity of $\langle \cdot, \cdot \rangle$ we have

$$\langle Y \rangle_t = \langle Y_0, Y_0 \rangle + \langle I_t, I_t \rangle + \langle M_t, M_t \rangle + 2\langle Y_0, I_t \rangle + 2\langle Y_0, M_t \rangle + 2\langle I_t, M_t \rangle.$$

Now Y_0 and I_t are absolutely continuous, and therefore have finite total variation on any $[0, T]$. Consequently, the terms $\langle Y_0, Y_0 \rangle$, $\langle I_t, I_t \rangle$, and $\langle Y_0, I_t \rangle$ all vanish. The $\langle I_t, M_t \rangle$ and $\langle Y_0, M_t \rangle$ terms also vanish since M_t is a.s. continuous and Y_0 and I_t are of bounded variation. For example, we can choose $\|\Pi\|$ small enough so that $|M_{t_{i+1}} - M_{t_i}| < \epsilon$ for $\Pi = \{0 = t_0 < t_1 < \cdots < t_n = t\}$. We then have

$$\left| \sum_{i=0}^{|\Pi|} (M_{t_{i+1}} - M_{t_i})(I_{t_{i+1}} - I_{t_i}) \right| \leq \epsilon \sum_{i=0}^{|\Pi|} |I_{t_{i+1}} - I_{t_i}|.$$

Since I_t is of bounded variation, this sum can be made arbitrarily small as $\|\Pi\| \to 0$. Consequently, we have $\langle Y \rangle_t = \langle M \rangle_t$, which we showed in class is given by

$$\langle Y \rangle_t = \langle M \rangle_t = \int_0^t \sigma_s^2 \, ds.$$

Finally, we apply this to our problem. Since X_t is given by

$$X_t = \int_0^t B_s \, ds + \int_0^t s \, dB_s,$$

we have

$$\langle X \rangle_t = \int_0^t s^2 \, ds = \frac{1}{3} t^3.$$

□

b) $X(t) = \int_0^t \frac{1-t}{1-s} dB_s$.

Solution. We immediately have

$$dX_t = \frac{1-t}{1-s} dB_s.$$

By our previous discussion, we have

$$\langle X \rangle_t = \int_0^t \left(\frac{1-t}{1-s} \right)^2 ds = t(1-t).$$

□

Problem 3. Show that the Ornstein-Uhlenbeck process

$$X_t = \overline{x} + (x_0 - \overline{x})e^{-at} + \sigma \int_0^t e^{-a(t-s)} dB_s$$

solves

$$dX = a(\overline{x} - X)dt + \sigma dB_t, \ X_0 = x_0.$$

Proof. The trick is to multiply the differential equation through by the integrating factor e^{at}.

$$e^{at}dX_t = a\overline{x}e^{at}dt - ae^{at}X_t dt + \sigma e^{at}dB_t. \tag{1}$$

Now consider the function $g(t,x) = e^{at}x$. By Itô's formula we have

$$d(e^{at}X_t) = ae^{at}X_t dt + e^{at}dX_t.$$

Solving for $e^{at}dX_t$ and substituting into (1) gives

$$d(e^{at}X_t) = a\overline{x}e^{at}dt + \sigma e^{at}dB_t.$$

Integrating from 0 to t gives

$$e^{at}X_t - x_0 = \overline{x}(e^{at} - 1) + \sigma \int_0^t e^{as}dB_s.$$

Multiplying through by e^{-at} and simplifying gives the desired result

$$X_t = \overline{x} + (x_0 - \overline{x})e^{-at} + \sigma \int_0^t e^{-a(t-s)}dB_s.$$

$\square$

Problem 4. Show that the following processes are Martingales.

a) $X_t = \exp(t/2)\cos(B_t)$.

Proof. Consider the function $g(t,x) = e^{t/2}\cos x$. This function is $C^{1,2}$, so by Itô's formula we have

$$d(e^{t/2}\cos B_t) = \frac{1}{2}e^{t/2}\cos(B_t)dt - e^{t/2}\sin(B_t)dB_t - \frac{1}{2}e^{t/2}\cos(B_t)(dB_t)^2.$$

Since $(dB_t)^2 = dt$, the first and third terms on the right-hand side cancel. Integrating from 0 to t gives

$$e^{t/2}\cos(B_t) = 1 - \int_0^t e^{t/2}\sin(B_t)\ dt.$$

Since the Itô integral is a martingale and adding a constant to a martingale yields a martingale, the above quantity is a martingale. $\square$

b) $X_t = (B_t + t)\exp(-B_t - t/2)$.

Proof. Let $g(t,x) = (x+t)\exp(-x - \frac{1}{2}t)$. We then have

$$g_t(t,x) = \exp\left(-x - \frac{1}{2}t\right)\left(1 - \frac{1}{2}x - \frac{1}{2}t\right), \quad g_x(t,x) = \exp\left(-x - \frac{1}{2}t\right)(1 - x - t),$$

$$g_{xx}(t,x) = \exp\left(-x - \frac{1}{2}t\right)(x + t - 2).$$

By Itô's formula we then have

$$d(X_t) = \exp\left(-B_t - \frac{1}{2}t\right)\left(1 - \frac{1}{2}B_t - \frac{1}{2}t\right)dt + \exp\left(-B_t - \frac{1}{2}t\right)(1 - B_t - t)dB_t$$
$$+ \frac{1}{2}\exp\left(-B_t - \frac{1}{2}t\right)(B_t + t - 2)(dB_t)^2.$$

Since $(dB_t)^2 = dt$, the first and last terms on the right-hand side cancel and we have after integrating

$$X_t = (B_t + t)\exp\left(-B_t - \frac{1}{2}t\right) = \int_0^t \exp\left(-B_s - \frac{1}{2}s\right)(1 - B_s - s)\, dB_s.$$

Since the Itô integral is a martingale, X_t is a martingale. □

Problem 5. Consider the vector Itô process $X = [X_1, \ldots, X_m]$:

$$dX_t^{(i)} = \mu_t^{(i)} dt + \sum_{j=1}^{d} \sigma_t^{(i,j)} dB_t^{(j)},$$

with $\mu^{(i)}$, $\sigma^{(i,j)}$ satisfying the standard Itô process conditions. Prove that for f of class $C^{1,2}$:

$$f(t, X_t) - f(0, X_0) = \int_0^t f_t(s, X_s)ds + \int_0^t \nabla_x f(s, X_s) \cdot dX_s + \frac{1}{2}\int_0^t H_x f(s, X_s) : d\langle X\rangle_s,$$

where H_x is the Hessian, : means contraction, and $\langle X\rangle_t^{(i,j)} = \langle X^{(i)}, X^{(j)}\rangle_t$.

Proof. □