3. Find all positive solutions to $x^2 - 2y^2 = 1$ for which $y < 250$.

Let (x_0, y_0) be the solution w/ smallest y-coord (fundamental soln)

Then $\Rightarrow$ all solns have the form

$$(x_0 + y_0\sqrt{d})^k = x_k + y_k\sqrt{d}$$

$x=2$: $4 - 2y^2 = 1 \Rightarrow 2y^2 = 3$ no solns

$x=3$: $9 - 2y^2 = 1 \Rightarrow 2y^2 = 8 \Rightarrow y = 2$

$\Rightarrow (3,2)$ smallest solution.

k-th smallest soln, (x_k, y_k) satisfies

$$x_k + y_k\sqrt{2} = (3 + 2\sqrt{2})^k$$

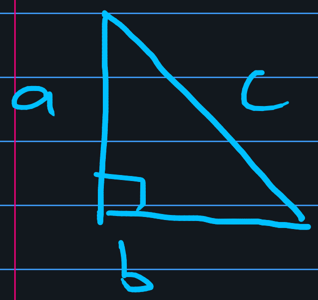
$$x_2 + y_2\sqrt{2} = (3 + 2\sqrt{2})^2 = 9 + 2\cdot 3\cdot 2\sqrt{2} + 8$$

$$= 17 + 12\sqrt{2}$$

$$x_3 + y_3\sqrt{2} = (3 + 2\sqrt{2})^3 = (17 + 12\sqrt{2})(3 + 2\sqrt{2})$$

...

5. Consider a right triangle, the lengths of whose sides are integers. Prove that the area cannot be a perfect square.



$a^2 + b^2 = c^2$

Pythagorean triples.

<u>wlog</u>, assume $\gcd(a,b,c) = 1$

$\Rightarrow \exists s, t$ not both odd, $(s,t) = 1$ (even)

s.t. $a = 2st$, $b = s^2 - t^2$, $c = s^2 + t^2$

$A = \frac{1}{2} ba = st(s^2 - t^2) = st(s-t)(s+t)$

- Suppose A is a perfect square.
- Note that $s, t, s \pm t$ are mutually coprime

$\Rightarrow s, t, s \pm t$ are perfect squares. (think prime factorization)

Write $s + t = p^2$, $s - t = q^2$ some p, q

p & q must be odd since s & t have opposite parity.

claim: $p-q, p+q$ both even, one is divisible by 4.

$\Rightarrow$ p & q odd $\Rightarrow$ $p \pm q$ even ✓

consider cases $p = 1, 3 \mod 4$
$q = 1, 3 \mod 4$

$\Leftarrow$

define $u = \frac{p-q}{2}$, $v = \frac{p+q}{2}$ $\in \mathbb{Z}$

one of them is even, other is odd

note that $u^2 + v^2 = s$ (check)

& s is a perfect square.

$\Rightarrow$ (u, v, s) is an integer right triangle

whose area $= \frac{1}{2} uv = \frac{1}{2} \frac{p^2 - q^2}{4} = t/4 \in \mathbb{Z}$

since one of u, v is even

$t/4$ is a perfect square since t is a square

that is smaller than $st(s^2 - t^2) = A$

get contradiction by descent. $\blacksquare$