1. A mouse is placed in a maze with two rooms pictured in Figure 1. Starting from room 1, what is the expected number of steps the mouse takes before it reaches the exit?

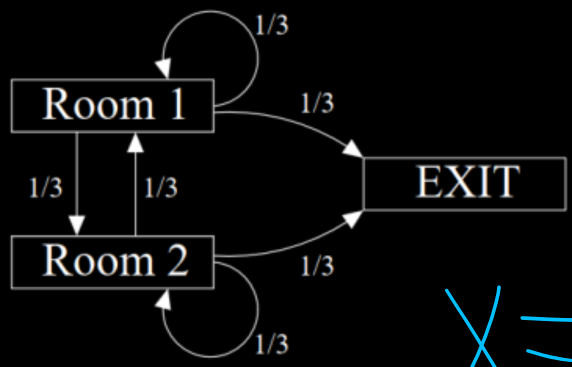


Figure 1: Maze

$X = \#$ steps to finish

Trick: Condition on the first step.

want

Let $S_1 = E[X \mid \text{mouse starts in room 1}]$

$S_2 = E[X \mid \text{mouse starts in 2}]$

$$S_1 = \frac{1}{3} \cdot 1 + \frac{1}{3}(S_1 + 1) + \frac{1}{3}(S_2 + 1)$$

$$S_2 = \frac{1}{3} \cdot 1 + \frac{1}{3}(S_2 + 1) + \frac{1}{3}(S_1 + 1)$$

$\Rightarrow S_1 = S_2$. replace all S_2's w/ S_1

$$S_1 = \frac{1}{3} + \frac{1}{3}(S_1 + 1) + \frac{1}{3}(S_1 + 1)$$

$$3S_1 = 1 + 2S_1 + 2 \Rightarrow \boxed{S_1 = 3}$$

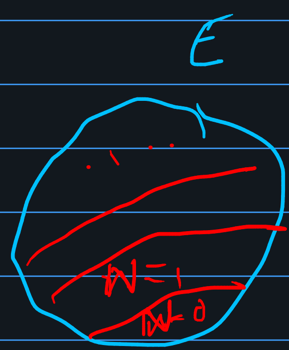
2. On the day before an exam, Math 130B students go to Liam's office hours to ask questions. Each question asked will appear on the exam with probability p. The number of questions asked is a Poisson distributed random variable with mean λ. What is the probability that Liam does not have to answer an exam question?

Looks like a binomial.

- Let E be the event "Liam doesn't answer an exam question"

we want $\Pr[E]$

- Let N = # questions asked.
$\sim \text{Pois}(\lambda)$



$$\Pr[E] = \sum_{n=0}^{\infty} \Pr[E \,\&\, N=n]$$

$$= \sum_{n=0}^{\infty} \Pr[E \mid N=n] \cdot \Pr[N=n]$$

None of the n questions on exam

$$= \sum_{n=0}^{\infty} (1-p)^n \frac{\lambda^n}{n!} e^{-\lambda}$$

$$= e^{-\lambda} \sum_{n=0}^{\infty} \frac{[\lambda(1-p)]^n}{n!}$$

$$= e^{-\lambda} \cdot e^{\lambda(1-p)} = \boxed{e^{-\lambda p}}$$

☑

3. If X and Y are independent continuous random variables, show that

$$\Pr[Y < X] = \int_{\mathbb{R}} F_Y(x) f_X(x)\, dx,$$

where $F_Y(\cdot)$ is the cdf of Y and $f_X(\cdot)$ is the pdf of X. Use this to compute $\Pr[Y < X]$ where $X \sim Exp(\mu)$ and $Y \sim Exp(\lambda)$ are independent.

usual way: find joint $\to$ take an integral

Pf: $$\Pr[Y \le X] = \iint_{y \le x} f_{X,Y}(x,y)\, dy\, dx$$

$$= \iint_{y \le x} f(y|x) f_X(x)\, dy\, dx$$

$$= \int_{\mathbb{R}} \int_{-\infty}^{x} f(y|x) f_X(x)\, dy\, dx$$

$$= \int_{\mathbb{R}} \Pr[Y \le X \mid X = x] f_X(x)\, dx$$

$X \perp\!\!\!\perp Y$

$$= \int_{\mathbb{R}} \Pr[Y \le x \mid X = x] f_X(x)\, dx$$

$$= \int_{\mathbb{R}} \Pr[Y \le x] f_X(x)\, dx$$

$$= \int_{\mathbb{R}} F_Y(x) f_X(x)\, dx$$

$\boxtimes$

$X \sim Exp(\mu)$, $Y \sim Exp(\lambda)$

$$\Rightarrow Pr[Y \leq X] = \int_{\mathbb{R}} F_Y(x) f_X(x)\, dx$$

$$= \int_0^\infty \left(1 - e^{-\lambda x}\right) \mu e^{-\mu x}\, dx$$

$$= \int_0^\infty \mu e^{-\mu x} - \mu e^{-(\lambda+\mu)x}\, dx$$

$$= -e^{-\mu x}\Big|_0^\infty + \frac{\mu}{\mu+\lambda} e^{-(\lambda+\mu)x}\Big|_0^\infty$$

$$= 1 - \frac{\mu}{\mu+\lambda} = \boxed{\frac{\lambda}{\mu+\lambda}}$$