1. Find a greatest common divisor for each of the following pairs of Gaussian integers.

(a) $\alpha = 8 + 38i$ and $\beta = 9 + 59i$

(b) $\alpha = -9 + 19i$ and $\beta = -19 + 4i$

idea: Euclidean algorithm

a) Start by dividing the element w/ larger norm by the one w/ smaller norm

$$\frac{\beta}{\alpha} = \frac{9+59i}{8+38i} = \frac{89}{58} + \frac{5}{58}i$$

$$\frac{(9+59i)(8-38i)}{(8+38i)(8-38i)}$$

round to nearest Gaussian integer

β/α rounds to $2 + 0i$

$$\Rightarrow 9+59i = 2(8+38i) + (-7-17i)$$

$$\frac{8+38i}{-7-17i} = -\frac{27}{13} - \frac{5}{13}i$$

rounds to $-2 + 0i$

$$8+38i = -2(-7-17i) + (-6+4i)$$

⋮

2. Let R be the following set of complex numbers:

$$R = \{a + bi\sqrt{5} : a, b \in \mathbb{Z}\}.$$

(a) Verify that R is a ring.

easy check.

(b) Show that the only units in R are 1 and -1.

b) consider the norm $N(a+bi\sqrt{5}) = (a+bi\sqrt{5})(a-bi\sqrt{5})$

$= a^2 + 5b^2$

$\in \mathbb{Z}^+$.

if α is a unit in R, $\alpha\beta = 1$ for some $\beta \in R$

$N(\alpha\beta) = N(1)$ since $N(\alpha), N(\beta) \in \mathbb{Z}^+$.

$N(\alpha)N(\beta) = 1 \implies N(\alpha) = N(\beta) = \pm 1$

$= 1$

Show the converse; if $N(\alpha) = 1$, then α is a unit.

α unit $\iff$ $N(\alpha) = 1 \iff a^2 + \underbrace{5b^2}_{\geq 0} = 1$

Needs $b = 0 \implies a^2 = 1 \implies a = \pm 1$

$\implies \alpha = \pm 1 \implies$ units are ± 1

$\{a + b\omega : \omega^2 + \omega + 1 = 0\}$

$R = \mathbb{Z}[\sqrt{-5}]$

(c) Recall that an element α of R is irreducible if and only if its only divisors in R are units and unit multiples of α. Prove that 2 is an irreducible element of R.

Suppose $2 = \alpha\beta$

$\Rightarrow N(2) = N(\alpha)N(\beta)$

$N(2) = 2 \cdot 2 = 4$

$\Rightarrow (N(\alpha), N(\beta)) \in \{(1,4), (4,1), (2,2)\}$

if $N(\alpha) = 1 \Rightarrow \alpha$ is a unit

$\beta = a + b\sqrt{5}$

if $N(\beta) = 4 \Rightarrow a^2 + 5b^2 = 4$

$\Rightarrow a = \pm 2$

$\Rightarrow \beta = \pm 2$

only other possibility is $N(\alpha) = N(\beta) = 2$

if $\alpha = a + bi\sqrt{5} \Rightarrow N(\alpha) = a^2 + 5b^2 = 2$

b needs to be 0

$\Rightarrow a^2 = 2$ no solns

there are no elements w/ norm 2

$\Longrightarrow$ if $2 = \alpha\beta \Rightarrow \alpha$ or β is a unit $\Rightarrow$ 2 irreducible.

$\blacksquare$

(e) 2 is not prime in R.

π is ....

- irreducible: if $\pi = \alpha\beta \Rightarrow \alpha$ or β is a unit
- prime: if $\pi \mid \alpha\beta \Rightarrow \pi \mid \alpha$ or $\pi \mid \beta$