HomeWork 1 Math 271A, Fall 2019.

1. (a) Let $\Delta_1, \Delta_2, \cdots$ be independent random variables with mean 0 and variance 1. Let $X_1 = \Delta_1$ and for $n = 1, 2, \cdots$ let $X_{n+1} = X_n + \Delta_{n+1} f_n(X_1, \cdots, X_n)$ for f_n given bounded deterministic functions. Show that $\{X_n\}$ is a martingale (specify the filtration). The martingales of gambling have this form.

   (b) Let $Y_1, Y_2, \cdots$ be independent random variables with mean 0 and variance σ^2. Let $X_n = (\sum_{k=1}^n Y_k)^2 - n\sigma^2$ and show that $\{X_n\}$ is a martingale (give the filtration).

2. (a) Show that if $X_n \to X$ in $L^p, p \geq 1$, then

$$X_n \quad \to \quad X \quad \text{in probability}$$

   (b) Construct an example with a sequence X_n of random variables that converges in L^p, but not almost surely.

3. A stochastic process $X(t)$ is a martingale relative to the filtration $\mathcal{F}_t$ if

$$X(t) \in L^1(\mathbb{P}) \quad \forall t \geq 0\,,$$
$$\mathbb{E}[X(t)|\mathcal{F}_s] = X(s)\,, \quad \text{for}\, t \geq s\,.$$

   Prove that

$$B^2(t) - t\,,$$

   is a martingale.

4. Let $\mathbf{W}_t$ be a standard n-dimensional Brownian motion and fix $t_0 \geq 0$ (a standard n-dimensional Brownian is a vector process where each component is a standard Brownian motion and the components are independent of each other). Prove that

$$\tilde{\mathbf{W}}(t) = \mathbf{U}\left\{\mathbf{W}(t_0 + t) - \mathbf{W}(t_0)\right\}\;; \quad t \geq 0\,,$$

   is a standard n-dimensional Brownian motion for $\mathbf{U}$ an orthogonal matrix. (that is, show that each process component has independent Gaussian increments and that they are independent of 'each other using that uncorrelated Gaussians are independent and linear combinations of jointly Gaussians are Gaussian).