# Math 130B - Review of 130A

**Do the fourth problem in each section.**

## 1 Combinatorics

1. Give a combinatorial explanation of the identity

$$\binom{n}{r} = \binom{n}{n-r}.$$

2. From ten married couples, we want to select a group of six people that is not allowed to contain a married couple. How many choices are there? How many choices are there if the group must also consist of three men and three women?

3. 1500 teams are competing in a soccer tournament. The organizers let the teams know that every game must have a winner and that the team that loses a game is immediately excluded from the tournament. how many games will be played until the champion is known?

4. Determine the number of solutions to the inequality

$$x_1 + x_2 + \cdots + x_n \leq k,$$

where each $x_i$ is a nonnegative integer.

## 2 Axioms of Probability

1. If a die is rolled four times, what is the probability that 6 comes up at least once?

2. An urn contains $n$ white and $m$ black balls, where $n$ and $m$ are positive numbers.

   (a) If two balls are randomly withdrawn, what is the probability that they are the same color?

   (b) If a ball is randomly withdrawn and then replaced before the second one is drawn, what is the probability that the withdrawn balls are the same color?

   (c) Show that the probability in part (b) is always larger than the one in part (a).

3. Show that if $\Pr[A_i] = 1$ for all $i \geq 1$ then $\Pr[\cap_{i\geq 1} A_i] = 1$.

4. Consider an experiment whose sample space consists of a countably infinite number of points. Show that not all points can be equally likely. Can all points have a positive probability of occurring?

## 3 Conditional Probability and Independence

1. Two fair dice are rolled. What is the conditional probability that at least one lands on 6 given that the dice land on different numbers?

2. Three cards are randomly selected, without replacement, from an ordinary deck of 52 playing cards. Compute the conditional probability that the first card selected is a spade given that the second and third cards are spades.

3. In a class there are four first-year boys, six first-year girls, and six sophomore boys. How many sophomore girls must be present if sex and grade level are to be independent when a student is selected at random?

4. Prove or give a counterexample. If $E_1$ and $E_2$ are independent, then they are conditionally independent given $F$.

## 4 Random Variables

1. If $X$ has distribution function $F$, what is the distribution function of $e^X$?

2. Let $X$ be a binomial random variable with parameters $(n, p)$. What value of $p$ maximizes $\Pr[X = k]$ for $k = 0, 1, \ldots, n$?

3. An urn initially contains one red and one blue ball. At each stage, a ball is randomly chosen and then replaced along with another of the same color. Let $X$ denote the selection number of the first chosen ball that is blue. For instance, if the first selection is red and the second is blue then $X = 2$.

   (a) Find $\Pr[X > i]$, $i \geq 1$.

   (b) Show that with probability 1, a blue ball is eventually chosen. (That is, show that $\Pr[X < \infty] = 1$.)

   (c) Find $\mathbb{E}[X]$.

4. Take a stick of finite length and choose two points on it uniformly at random to cut. What is the probability that you can form a triangle with the three pieces? (*Hint: Set up coordinates and let $X$ and $Y$ be random variables corresponding to the breaking points.*)