**HomeWork 3 Math 271B, Winter 2020**.

1. Use Itô's formula to prove that for

$$b_t(k) = \mathbb{E}[\beta_t^k]$$

with $\beta$ standard Brownian motion

$$\frac{db_t(k)}{dt} = \frac{1}{2}k(k-1)\int_0^t b_s(k-2)ds, k \geq 2,$$

and use this to find $b_t(k)$ for general k.

2. Find $dX$ and $\langle X \rangle$ when

a) $X(t) = t\beta_t$, b) $X(t) = \int_0^t \frac{1-t}{1-s} d\beta_s$, with again $\beta_t$ standard Brownian motion.

3. Show that the Ornstein-Uhlenbeck process

$$X_t = \bar{x} + (x_0 - \bar{x})e^{-at} + \sigma \int_0^t e^{-a(t-s)} d\beta_s,$$

solves

$$dX = a(\bar{x} - X)dt + \sigma d\beta, \quad X_0 = x_0.$$

4. Show that

a) $\exp(t/2)\cos(\beta_t)$, b) $(\beta_t + t)\exp(-\beta_t - t/2)$

are martingales for $\beta_t$ standard Brownian motion.

5. Consider the vector Itô process: $\boldsymbol{X} = [X_1, \cdots, X_m]$:

$$dX_t^{(i)} = \mu_i dt + \sum_{j=1}^{d} \sigma^{(i,j)} d\beta^{(j)},$$

with $\mu^{(i)}, \sigma(i,j)$ satisfying the standard Itô process conditions mentioned in class, and the $\beta^{(j)}$s independent standard Brownian motions.

Prove that for $f$ of class $C^{1,2}$:

$$f(t, \boldsymbol{X}_t) - f(0, \boldsymbol{X}_0) = \int_0^t f_t(s, \boldsymbol{X}_s)ds + \int_0^t \nabla_x f(s, \boldsymbol{X}_s) \cdot d\boldsymbol{X}_s + \frac{1}{2}\int_0^t \mathbf{H}_x f(s, \boldsymbol{X}_s) : d\langle \boldsymbol{X} \rangle_s,$$

where $\mathbf{H}_x$ is the Hessian and : means contraction (or matrix inner product) and $\langle \boldsymbol{X} \rangle^{(i,j)} = \langle X^{(i)}, X^{(j)} \rangle$.