HomeWork 5 Math 271A, Fall 2019.

1. Consider the space $\mathbb{R}^d$ and the usual $\|\cdot\|_2$ metric. Show explicitly that a probability measure $\mathbb{P}$ on the measurable space $\big(\mathbb{R}^d, \beta(\mathbb{R}^d)\big)$ is uniquely determined by
$$F(x_1, \cdots, x_d) = \mathbb{P}(y : y_1 \leq x_1, \cdots, y_d \leq x_d).$$

2. Show that if a set A of continuous paths on $[0,1]$ is equicontinuous at each point in $[0,1]$ then the set is uniformly equicontinuous.

3. Let $\zeta_i, i = 1, 2, 3, \cdots$ and consider the random walk $S_n = \sum_{i=1}^n \zeta_i$. By interpolation of this process (as in class $\mapsto Y_t$) and proper rescaling and normalization construct a family of processes that converges in distribution to standard Brownian motion.

4. Suppose $\{X_n\}_{n=1}^\infty$ is a sequence of random variables taking values in a metric space (S_1, ρ_1) and converging in distribution to X. Suppose (S_2, ρ_2) is another metric space, and $\phi : S_1 \to S_2$ is continuous. Show that $Y_n \equiv \phi(X_n)$ converges in distribution to $Y \equiv \phi(X)$.

5. Consider the space $C[0,1]$ of continuous function on $[0,1]$ with the supremum metric $\rho(\omega) = \max_{0<t<1} |\omega(t)|$ and associated norm. Show that this metric space is separable and complete. Show that a probability measure on $(C[0,1], \beta(C[0,1])$ is tight.

6. Let $X_t, 0 < t < 2^N$ be a stochastic process. Define the Haar detail coefficients by
$$d_n(j) = \frac{1}{\sqrt{2^n}} \int_{-\infty}^{\infty} \psi(t/2^n - j) X(t) dt, \quad n = 1, 2, \cdots, N, \quad j = 1, 2, \cdots, 2^{N-n},$$
with the mother wavelet defined by
$$\psi(t) \;=\; \begin{cases} -1 & \text{if} \quad -1 \leq t < -1/2 \\ 1 & \text{if} \quad -1/2 \leq t < 0 \\ 0 & \text{otherwise} \end{cases} .$$
The difference coefficients correspond to probing the process at different scales and locations, with j representing location and n scale.

   The scale spectrum of X relative to the Haar wavelet basis is the sequence S_j defined by
$$S_n = \frac{1}{2^{N-j}} \sum_{k=1}^{2^{N-n}} (d_n(j))^2 \;, \quad n = 1, 2, ..., N.$$
Assume that X is centered, continuous, Gaussian process, starting at the origin, with homogeneous increments and covariance function (for the parameter $H \in (0,1)$:
$$\mathbb{E}[X_t X_s] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$
$\to$ compute $\mathbb{E}[S_j]$.