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271B - Homework 4

Problem 1. Consider the process

$$X_t = B_t^{(1)} B_t^{(2)} + e^t - t_0 + (B_t^{(3)})^2,$$

where $\mathbb{E}[B_t^{(i)} B_t^{(j)}] = \rho_{i,j} t$. Find the stochastic differential equation satisfied by this process.

Solution. Let $g(t, x_1, x_2, x_3) = x_1 x_2 + e^t - t_0 + x_3^2$. We apply Itô's lemma to obtain

$$\begin{aligned} dX_t &= e^t dt + B_t^{(2)} dB_t^{(1)} + B_t^{(1)} dB_t^{(2)} + 2B_t^{(3)} dB_t^{(3)} + (dB_t^{(3)})^2 + \rho_{1,2} dt \\ &= (e^t + 1 + \rho_{1,2}) dt + B_t^{(2)} dB_t^{(1)} + B_t^{(1)} dB_t^{(2)} + 2B_t^{(3)} dB_t^{(3)} \end{aligned}$$

□

Problem 2. Consider

$$Z_t = \mu_t dt + \theta_t \cdot dB_t, \; Z_0 = z_0,$$

for B a standard n-dimensional Brownian motion and θ a bounded deterministic n-dimensional vector process and μ a bounded deterministic process, all progressively measurable.

(a) Use the Kolmogorov forward equation to derive the density of Z_T.

Solution. Let $p_t(x)$ be the density of Z_t. The Kolmogorov forward equation says that

$$\begin{cases} \partial_t p_t(x) = \left(-\mu_t \partial_x + \frac{1}{2}|\theta_t|^2 \partial_x^2\right) p_t(x) \\ p_0(x) = \delta(x - z_0) \end{cases}.$$

If $\widehat{p}_t(\omega)$ is the Fourier transform of $p_t(x)$, then after integrating by parts twice we arrive at

$$\begin{cases} \partial_t \widehat{p}_t(\omega) = \left(i\omega\mu_t - \frac{1}{2}\omega^2 |\theta_t|^2\right) \widehat{p}_t(\omega) \\ \widehat{p}_0(\omega) = e^{iz_0\omega} \end{cases}.$$

The first equation is a separable ODE, so we obtain

$$\widehat{p}_t(\omega) = e^{iz_0\omega} \exp\left[\int_0^t \left(i\omega\mu_s - \frac{1}{2}\omega^2 |\theta_s|^2\right) ds\right].$$

Taking the inverse Fourier transform gives $p_t(x)$. □

Problem 3. Show that

$$\int_0^t B_s \circ dB_s = \frac{1}{2} B_t^2.$$

Proof. For all i let $t_i^* = \frac{1}{2}(t_i + t_{i-1})$. We have

$$\sum_{i=i}^{t/\Delta t} B_{t_i^*}(B_{t_i} - B_{t_{i-1}}) = \sum_{i=i}^{t/\Delta t} (B_{t_{i-1}} + (B_{t_i^*} - B_{t_{i-1}}))(B_{t_i} - B_{t_{i-1}})$$

$$= \sum_{i=1}^{t/\Delta t} B_{t_{i-1}}(B_{t_i} - B_{t_{i-1}}) + \sum_{i=1}^{t/\Delta t} (B_{t_i^*} - B_{t_{i-1}})[(B_{t_i^*} - B_{t_{i-1}}) + (B_{t_i} - B_{t_i^*})].$$

The first sum converges to the Itô integral $\int_0^t B_s dB_s = \frac{1}{2}B_t^2 - \frac{1}{2}t^2$, while the second sum converges to $\frac{1}{2}t^2$. In all, we have that $\int_0^t B_s \circ dB_s = \frac{1}{2}B_t^2$. □

Problem 4. Show that the following quantities define metrics on the appropriate spaces.

$$[f] = \sum_{n=1}^{\infty} 2^{-n} \left(1 \wedge \sqrt{\mathbb{E} \int_0^n f_s^2 \, ds} \right)$$

$$\|X\| = \sum_{n=1}^{\infty} 2^{-n} \left(1 \wedge \sqrt{\mathbb{E} X_n^2} \right).$$

Proof. Symmetry and "zero if and only if zero" are obvious, so we check the triangle inequality. This essentially follows from the triangle inequality of the $L^2(\mathbb{P} \otimes \text{Leb})$ norm: we recognize $(\mathbb{E} \int_0^n f_s^2 \, ds)^{1/2}$ as the L^2 norm of $f \cdot \mathbb{1}_{\Omega \times [0,n]}$, which we write as $\|f\|_{2(n)}$. We then have

$$1 \wedge \|f + g\|_{2(n)} \leq 1 \wedge (\|f\|_{2(n)} + \|g\|_{2(n)}).$$

The desired conclusion follows from considering the two cases of $\|f\|_{2(n)} + \|g\|_{2(n)} \leq 1$ and $\|f\|_{2(n)} + \|g\|_{2(n)} \geq 1$. The exact same argument works for $\|X\|$. □

Problem 5. Consider

$$Z_t = \exp\left(\int_0^t \theta_s \cdot dB_s - \frac{1}{2} \int_0^t |\theta_s|^2 ds \right),$$

for B an n-dimensional standard Brownian motion and θ a bounded n-dimensional progressively measurable vector process.

(a) Show that Z is a martingale.

Proof. Let Y_t be the expression inside the exponential defining Z_t:

$$Y_t = \int_0^t \theta_s \cdot dB_s - \frac{1}{2} \int_0^t |\theta_s|^2 ds.$$

This gives

$$dY_t = -\frac{1}{2}|\theta_t|^2 dt + \theta_t \cdot dB_t, \quad (dY_t)^2 = |\theta_t|^2.$$

By Itô's lemma we have

$$dZ_t = Z_t dY_t + \frac{1}{2} Z_t (dY_t)^2$$

$$= Z_t \theta_t \cdot dB_t.$$

The restrictions on θ guarantee that $Z_t\theta_t$ is of class II*, so Z_t is an Itô process with only a martingale component, and is hence a martingale. $\square$

(b) Assume that θ is independent of B. Derive an expression for the variance of Z_t.

Solution. The variance of Z_t is given by

$$\mathrm{Var}[Z_t] = \mathbb{E}[Z_t^2] - \mathbb{E}[Z_t]^2.$$

Since Z_t is a martingale, $\mathbb{E}[Z_t] = \mathbb{E}[Z_0] = 1$. By the Itô isometry, martingale property, and independence, we have

$$\begin{aligned}
\mathbb{E}[Z_t^2] &= \mathbb{E}\left(\int_0^t Z_s\theta_s \cdot dB_s + 1\right)^2 \\
&= \mathbb{E}\left(\int_0^t Z_s\theta_s \cdot dB_s\right)^2 + 2\mathbb{E}\int_0^t Z_s\theta_s \cdot dB_s + 1 \\
&= \mathbb{E}\left[\int_0^t Z_s^2|\theta_s|^2 ds\right] + 2\sum_{i<j}\mathbb{E}\left(\int_0^t Z_s\theta_s^{(i)} dB_s^{(i)}\right)\mathbb{E}\left(\int_0^t Z_s\theta_s^{(j)} dB_s^{(j)}\right) + 1 \\
&= \mathbb{E}\left[\int_0^t Z_s^2|\theta_s|^2 ds\right] + 1.
\end{aligned}$$

Thus, we have

$$\mathrm{Var}[Z_t] = \mathbb{E}\left[\int_0^t Z_s^2|\theta_s|^2 ds\right].$$

$\square$

Problem 6. Consider the scalar process

$$dZ_t = \mu_t dt + dB_t,$$

for μ_t bounded and progressively measurable. Find X_t so that Z_tX_t is a martingale.

Solution. Let X_t be defined by

$$X_t = \exp\left(-\int_0^t \mu_t\ dB_t - \frac{1}{2}\int_0^t \mu_t^2\ dt\right).$$

By Itô's lemma and the same reasoning used in the previous problem, we have

$$dX_t = -X_t\mu_t\ dB_t.$$

We also have

$$\begin{aligned}
d(Z_tX_t) &= Z_t dX_t + X_t dZ_t + d\langle Z_t, X_t\rangle \\
&= Z_t(-X_t\mu_t dB_t) + X_t(\mu_t dt + dB_t) - X_t\mu_t dt \\
&= X_t(1 - Z_t\mu_t)dB_t.
\end{aligned}$$

This is an Itô process with only a martingale component, so Z_tX_t is a martingale. $\square$