1. Find the continued fraction expansion for the following numbers.

(a) 61/14 $\quad \models p(x) = 14x - 61$

Recall: the continued fraction expansion of $\alpha \in \mathbb{R}$ is periodic iff α is a quadratic irrationality.

i.e., α is a zero of a polynomial $p(x) \in \mathbb{Z}[x]$, $\deg(p) \leq 2$.

$$\frac{61}{14} = 4 + \left(\frac{61}{14} - 4\right) = 4 + \frac{5}{14}$$

$$= 4 + \frac{1}{14/5}$$

$$\frac{14}{5} = 2 + \frac{4}{5} = 2 + 1/(5/4)$$

$$\frac{5}{4} = 1 + \frac{1}{4} \quad \text{done}$$

$$61/14 = 4 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{4}}}$$

terminates iff $\alpha \in \mathbb{Q}$

c) $\sqrt{13}$. $\quad 3 < \sqrt{13} < 4$

$$\Rightarrow \sqrt{13} = 3 + (\sqrt{13} - 3) = 3 + \frac{1}{\frac{1}{\sqrt{13}-3}}$$

$$\frac{1}{\sqrt{13}-3} = \frac{\sqrt{13}+3}{4} = 1 + \frac{\sqrt{13}-1}{4} = 1 + \frac{1}{\frac{4}{\sqrt{13}-1}}$$

$$\frac{4}{\sqrt{13}-1} = \frac{(\sqrt{13}+1)4}{12} = \frac{\sqrt{13}+1}{3} = 1 + \frac{\sqrt{13}-2}{3}$$

$$= 1 + \frac{1}{\frac{3}{\sqrt{13}-2}}$$

$$\frac{3}{\sqrt{13}-2} = \frac{3(\sqrt{13}+2)}{9} = \frac{\sqrt{13}+2}{3} = 1 + \frac{\sqrt{13}-1}{3} = 1 + \frac{1}{\frac{3}{\sqrt{13}-1}}$$

$$\frac{3}{\sqrt{13}-1} = \frac{3(\sqrt{13}+1)}{12} = \frac{\sqrt{13}+1}{4} = 1 + \frac{\sqrt{13}-3}{4}$$

$$= 1 + \frac{1}{\frac{4}{\sqrt{13}-3}}$$

$$\frac{4}{\sqrt{13}-3} = \frac{4(\sqrt{13}+3)}{4} = \sqrt{13}+3 = 6 + (\sqrt{13}-3)$$

$$= 6 + \frac{1}{\frac{1}{\sqrt{13}-3}}$$

$$\sqrt{13} = 3 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \cdots}}}}}$$

$$= [3; \overline{1, 1, 1, 1, 6}]$$

You can do this for <u>any</u> $\alpha \in \mathbb{R}$

only repeats for quadratic irrationalities.

2. Let $c_1, \ldots, c_n$ be integers such that the continued fraction $[c_1; \ldots, c_n]$ exists. Show that we can describe the continued fraction in terms of matrix multiplication $\in \mathbb{Q}$

$$\begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_n & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \implies [c_1; \ldots, c_n] = \frac{A}{C}.$$

" $c_1 + \cfrac{1}{c_2 + \cfrac{1}{c_3 + \cfrac{1}{\ddots}}}$

Pf: Induct on n.

Base case: $n = 1$ $[c_1]$

✓ $\begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix} \rightsquigarrow \frac{c_1}{1} = c_1 = [c_1]$

suppose it holds for sequences of length $n-1$. Then

$$[c_2; c_3, \dots, c_n] = \frac{A}{C}$$

where $\begin{pmatrix} c_2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c_3 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_n & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$

$$[c_1; c_2, \dots c_n] = c_1 + \frac{1}{[c_2; c_3, \dots, c_n]}$$

$$= c_1 + \frac{C}{A} = \underline{\frac{Ac_1 + C}{A}}$$

Also, we have

$$\begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix}\underline{\begin{pmatrix} c_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_n & 1 \\ 1 & 0 \end{pmatrix}}$$

$$= \begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} Ac_1 + C & Bc_1 + D \\ A & B \end{pmatrix}$$

$$\rightsquigarrow \underline{\frac{Ac_1 + C}{A}}$$

$\boxtimes$