Sample Final Exam – Math 130B

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Instructions:

- You must show your work and clearly explain your line of reasoning.
- You need to solve FOUR of the five problems below. The maximum score is 100
- The exam is 2 hours, but you will automatically get extra half an hour for scanning and uploading your solution (and making sure that the file is readable. If not, then please rewrite).

GOOD LUCK!

1. Let X be uniformly distributed on the interval $[0, \frac{\pi}{2}]$ and let Y be exponentially distributed with mean $1/3$. Assume X and Y are independent.

   (a) Compute the joint density function of U and V, where U and V are given by

   $$U = e^Y \cos(X), \quad V = e^Y \sin(X).$$

   (b) Compute $E[U+V]$ and $\mathrm{Var}[U+V]$.

2. A factory produces bolts of a certain width. The bolts don't come out perfectly each time – their widths deviate from the specified width by a random amount X, measured in micrometers. Suppose X has density function $f(x) = \frac{C}{1+(x/10)^2}$, where $-20 \leq x \leq 20$.

   (a) Find the value of C such that f is indeed a probability density function.

   (b) Calculate the mean and variance of X.

   (c) Suppose the factory produces 10,000 bolts a day. Assuming that each bolt is produced independently, estimate the number of bolts whose widths deviate from specification by more than 10 micrometers.

3. You and one million other people have bought tickets for this week's lottery. Each person has a one in one million chance of selecting winning numbers. If there is more than one person with winning numbers, one winner is randomly chosen from them to win the prize.
   Suppose your ticket has winning numbers. Let X be the number of other matching tickets belonging to the other one million players. Since there are one million other tickets, each of which has a one in one million chance of winning, we can assume that X is approximately a Poisson random variable with parameter 1.

   (a) What is the probability that you have the only winning ticket?

   (b) The prize winner is chosen at random from all the winning ticket holders. What is the probability that you win, given that there are x other winners?

   (c) What is the probability that you win, given that you have a matching ticket? *Hint: conditional expectation.*

4. Suppose that X is uniformly distributed on the set of n even numbers $\{2, 4, 6, 8, \ldots, 2n\}$.

   (a) Calculate $M_X(t)$.

   (b) Calculate $\mathbb{E}[X^3]$.

   (c) Let $X_1, \ldots, X_k$ be independent copies of X and let $S_k = X_1 + \ldots + X_k$. Calculate $M_{S_k}(t)$.

5. (a) Let X be a random variable. Use Chebyshev's inequality to show that

   $$\Pr[X = 0] \leq \frac{\mathrm{Var}[X]}{E[X]^2}.$$

   Deduce that if X_n is a sequence of random variables and $\mathrm{Var}[X_n] = o(E[X_n]^2)$, then $X_n = 0$ with high probability.

   (b) Show that if $p \geq Cn^{-2/3}$ for some large constant C, then the random graph $G(n, p)$ contains a clique of size 4 with high probability.