Homework 3
Math 270A, Fall 2019, Prof. Roman Vershynin

Problem 1 (discrete convolution)

Show that if X and Y are independent, integer-valued random variables, then

$$\mathbb{P}\{X+Y=n\} = \sum_{m\in\mathbb{Z}} \mathbb{P}\{X=m\}\,\mathbb{P}\{Y=n-m\} \quad \text{for all } n\in\mathbb{Z}.$$

Problem 2 (A direct construction of independence)

Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\Omega = (0,1)$, $\mathcal{F}$ is the Borel σ-algebra, and $\mathbb{P}$ is the Lebesgue measure. Define a sequence of random variables $Y_1, Y_2, \ldots$ by

$$Y_n(\omega) := \begin{cases} 1 & \text{if } \lceil 2^n\omega \rceil \text{ is even,} \\ 0 & \text{if } \lceil 2^n\omega \rceil \text{ is odd.} \end{cases}$$

Show that $Y_1, Y_2, \ldots$ are independent $\mathrm{Ber}(1/2)$ random variables.

Problem 3 (WLLN for non-identically distributed r.v.'s)

Let $X_1, X_2, \ldots$ be independent random variables that satisfy

$$\frac{\mathrm{Var}(X_i)}{i} \to 0 \quad \text{as } i \to \infty.$$

Let $S_n := X_1 + \cdots + X_n$. Prove that

$$\frac{S_n - \mathbb{E}[S_n]}{n} \to 0 \quad \text{in probability.}$$

Problem 4 (Metric for convergence in probability)

(a). Show that

$$d(X,Y) := \mathbb{E}\left[\frac{|X-Y|}{1+|X-Y|}\right]$$

defines a metric on the set of random variables (more formally, on the set of equivalence classes defined by the equivalence relation $X=Y$ a.s.)

(b). Prove that $d(X_n, X) \to 0$ if and only if $X_n \to X$ in probability.

Problem 5 (Convergence in probability and a.s.)

Let $X_1, X_2, \ldots$ be independent $\mathrm{Ber}(p_n)$ random variables.

(a). Show that $X_n \to 0$ in probability if and only if $p_n \to 0$.

(b). Show that $X_n \to 0$ a.s. if and only if $\sum_n p_n < \infty$.

Problem 6 (Convergence in probability and a.s. on discrete spaces)

Let $X_1, X_2, \ldots$ be a sequence of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ where Ω is a countable set and $\mathcal{F} = 2^\Omega$ (the power set). Show that $X_n \to X$ in probability implies $X_n \to X$ a.s.

Problem 7 (Suppression)

Show that for any sequence of random variables $X_1, X_2, \ldots$. there exists a sequence of positive real numbers $c_1, c_2, \ldots$ such that Show that $c_n X_n \to 0$ a.s.

Problem 8 (Records)

Let $X_1, X_2, \ldots$ be independent random variables. Show that $\sup_n X_n < \infty$ if and only if there exists $M \in \mathbb{R}$ such that

$$\sum_n \mathbb{P}\{X_n > M\} < \infty.$$

Problem 9 (Keep breaking the stick)

Let $X_0 = 1$ and define X_n inductively by choosing X_{n+1} uniformly at random from the interval $[0, X_n]$. Prove that

$$\frac{\ln X_n}{n} \to c \quad \text{a.s.}$$

and find the value of the constant c.

Problem 10 (Weak vs. strong LLN)

Let $X_2, X_3, \ldots$ be independent random variables such that X_n takes value n with probability $1/(2n \ln n)$, value $-n$ with the same probability, and value 0 with the remaining probability $1 - 1/(n \ln n)$. Show that this sequence obeys the weak law but not the strong law, in the sense that

$$\frac{1}{n} \sum_{i=1}^{n} X_i \to 0$$

in probability but not a.s.