"We use an idea due to
___ [ ]. ___ ___"

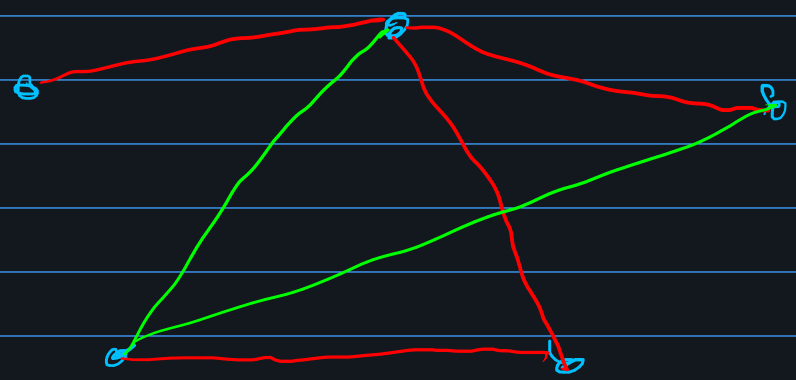
Bibliography | Bibtex

1.
2.
⋮

(a) Show that for every $c \leq 1/1000$ and $p = c/n$, with probability that tends to 1 as n tends to infinity, all the connected components of $G(n, p(n))$ are of size $O(\log n)$. (this is actually true for all $c < 1$, can you prove it?)

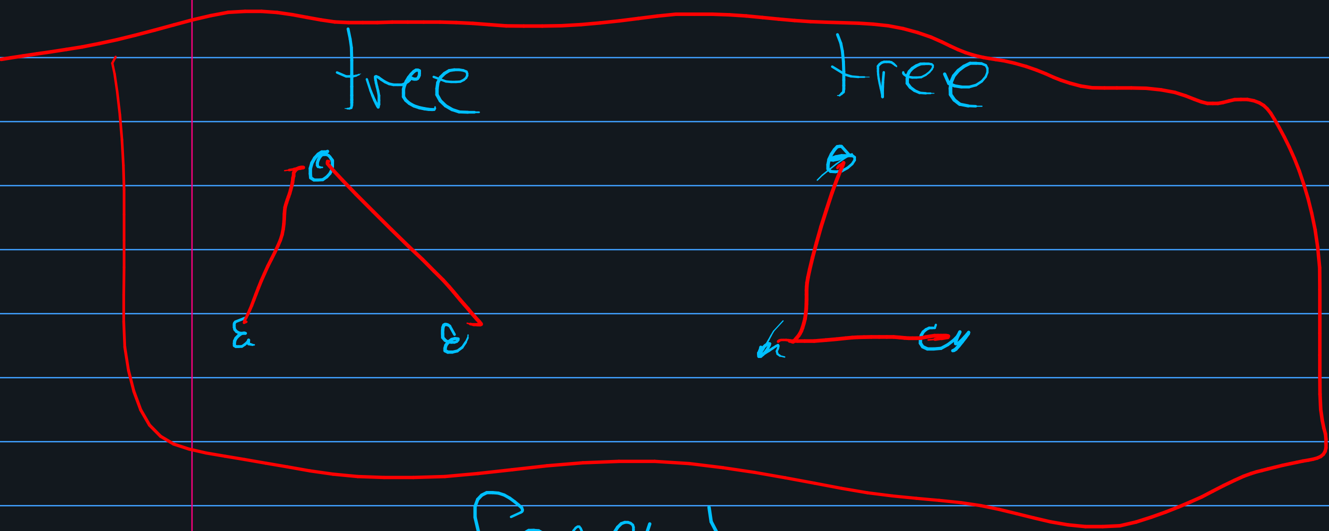
Idea: a set of k vertices forms a connected component iff it contains a spanning tree, & there's no edges into the complement.

Tree: (connected) graph w/ no cycles



edges form a spanning tree.

In a tree, there is exactly one path between any two vertices



Forest

A tree on k vertices has exactly $k-1$ edges

Let S be a set of k vertices.

Pr[S is a connected component] $\sim$

$\leq$ Pr[S contains a spanning tree &
has no edges outside of it]

$\leq$

$\underbrace{\left(\text{\# of spanning trees on } k \text{ vertices}\right)}_{T_k} \cdot$ Pr[you have some specific spanning tree]

$$= p^{k-1}(1-p)^{\underline{k(n-k)}}$$

$\rightarrow$ # of possible edges out of S

Pr[$\exists$ component of size k]

$$\leq T_k \, p^{k-1}(1-p)^{k(n-k)} \binom{n}{k}$$

Set $K = C \log n$ & show that this probability

$\Pr[\exists$ component size bigger than $K] \longrightarrow 0$

$f(n) = o(1) \iff \lim_{n \to \infty} f(n) = 0$

$f(n) = o(g(n)) \iff \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

(b) Show that for every $c \geq 1000$ and $p = c/n$, with probability that tends to 1 as n tends to infinity, there exists a connected component of a linear size. (this is true for all $c > 1$, can you prove something like that?)

$\hookrightarrow \geq kn$ for some k

$k < 1$

- Depth-first search:

Claim: If G is such that every two disjoint sets of size k contain an edge between them in G, then G contains a path of length $\geq n-2k-1$

Now show any two sets of size, say, $(\frac{1}{2}-\varepsilon)n$ in $G_{n,c/n}$ have an edge between them

$$\Rightarrow \exists \text{ path } \geq n-2\left(\tfrac{1}{2}-\varepsilon\right)n-1$$
$$= \Omega(n)$$

Project 2

(b) Show that

$$\mathrm{Var}[X] = \log\log n + O(1)$$

$$X_p = \begin{cases} 1 & \text{if } p \mid x \\ 0 & \text{else} \end{cases}$$

$$p \le M = n^{1/100}$$

$$\mathrm{Var}[X] = \sum_{p \le m} \mathrm{Var}[X_p] + \sum_{p \ne q} \mathrm{Cov}(X_p, X_q)$$

$$X_p X_q = \begin{cases} 1 & \text{if } p \mid x \;\&\; q \mid x \iff pq \mid x \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{Cov}(X_p, X_q) = E[X_p X_q] - E[X_p]E[X_q]$$

X uniform in $[n]$

$$= \Pr[pq \mid X] - \Pr[p \mid X]\Pr[q \mid X]$$

$$\mathbb{E}[X_p] = \Pr[p \mid X] = \frac{\#\text{ multiples of } p \text{ in } [n]}{n}$$

$$= \frac{\lfloor n/p \rfloor}{n}$$

$$= \frac{\lfloor n/pq \rfloor}{n} - \frac{\lfloor n/p \rfloor}{n} \cdot \frac{\lfloor n/q \rfloor}{n}$$

$x - 1 < \lfloor x \rfloor \leq x$ for all x

$$< \frac{n/pq}{n} - \frac{\frac{n}{p} - 1}{n} \cdot \frac{\frac{n}{q} - 1}{n}$$

$$= \frac{1}{pq} - \left(\frac{1}{p} - \frac{1}{n}\right)\left(\frac{1}{q} - \frac{1}{n}\right)$$

$$\cancel{\frac{1}{pq}} - \cancel{\frac{1}{pq}} + \frac{1}{nq} + \frac{1}{np} - \cancel{\frac{1}{n^2}}$$

$$\leq \frac{1}{n}\left(\frac{1}{p} + \frac{1}{q}\right)$$

$$\sum_{p\neq q} \operatorname{Cov}[X_p, X_q] \le \frac{1}{n} \sum_{\substack{p\neq q\\ \le M}} \left(\frac{1}{p}+\frac{1}{q}\right)$$

$$= \frac{2}{n} \sum_{p<q\le M} \left(\frac{1}{p}+\frac{1}{q}\right)$$

$n^{1/100}$

$$< \frac{2M}{n} \underbrace{\sum_{p\le M} \frac{1}{p}}_{\log\log n + O(1)} = o(1)$$

$$\sum_{p\le M} \operatorname{Var}[X_p]$$

$$\operatorname{Var}[X_p] = \mathbb{E}[X_p^2] - \mathbb{E}[X_p]^2$$

$$= \Pr[p \mid x] - \Pr[p \mid x]^2$$

$$= \Pr[p \mid x]\left(1 - \Pr[p \mid x]\right)$$

$x-1 < \lfloor x \rfloor \le x$

$$= \frac{\lfloor n/p \rfloor}{n}\left(1 - \frac{\lfloor n/p \rfloor}{n}\right)$$

$$\le \frac{1}{p}\left(1 - \frac{\frac{n}{p}-1}{n}\right)$$

$$= \frac{1}{p}\left(1 - \frac{1}{p} + \frac{1}{n}\right)$$

$M \le n$

$$= \frac{1}{p}\left(1-\frac{1}{p}\right) + O\left(\frac{1}{n}\right)$$

$$\sum_{p \le M} \mathrm{Var}[X_p] \le \sum_{p \le M} \left[\frac{1}{p}\left(1-\frac{1}{p}\right) + O\left(\frac{1}{n}\right)\right]$$

$1 - \frac{1}{p} \le 1$

$$\le \left(\sum_{p \le M} \frac{1}{p}\right) + O(1)$$

$$= \log\log n + O(1)$$

$$\Longrightarrow \mathrm{Var}[X] = \log\log n + O(1) + o(1)$$

$$= \log\log n + O(1).$$

$f(x) = O(g(x))$ means f doesnt grow more quickly than g

$f(x) = o(g(x))$ means f grows more slowly than g.

e.g. $\frac{1}{n^2} = o\left(\frac{1}{n}\right)$

since $\lim_{n\to\infty} \frac{1/n^2}{1/n} = \lim_{n\to\infty} \frac{1}{n} = 0$

$\Rightarrow \frac{1}{n^2} = o\left(\frac{1}{n}\right)$

$\frac{1}{n} + \frac{1}{n^2} = O\left(\frac{1}{n}\right)$

since $\frac{\frac{1}{n} + \frac{1}{n^2}}{\frac{1}{n}} = 1 + \frac{1}{n} \to 1 < \infty$

Big O notation is explained in the lecture notes that are on Canvas.