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271C - Homework 4

Karatzas and Shreve Problem 2.8.6

Derive the transition density for Brownian motion absorbed at the origin, $\{B_{t\wedge T_0}, \mathcal{F}_t; 0 \le t < \infty\}$, by verifying that

$$P^x[B_t \in dy, T_0 > t] = p_-(t;x,y)dy$$
$$:= \big[p(t;x,y) - p(t;x,-y)\big]dy; \quad t>0,\ x,y,>0.$$

Here, p is the Gaussian kernel

$$p(t;x,y) = \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t}, \quad t>0,\ x,y \in \mathbb{R}.$$

Proof. Fix $x > 0$. By symmetry, we can start our Brownian motion at $-x$ and kill it when it reaches the origin. Let T_0 denote the stopping time $T_0 = \inf\{t \ge 0 : B_t = 0,\ B_0 = -x\}$. Let's compute the distribution function. For $y \le 0$ we have

$$P^x[B_{t\wedge T_0} \le y] = P^x\left[B_t \le y,\ \max_{0\le s\le t} B_s < 0\right]$$
$$= P^x[B_t \le y] - P^x\left[B_t \le y,\ \max_{0\le s\le t} B_s \ge 0\right]$$
$$= P^x[B_t \le y] - P^x[B_t \le y,\ T_0 \le t].$$

By the reflection principle,

$$P^x[B_t \le y,\ T_0 \le t] = P^x[B_t \ge -y,\ T_0 \le t] = \Pr[B_t \ge -y].$$

We then have

$$P^x[B_{t\wedge T_0} \le y] = P^x[B_t \le y] + P^x[B_t \ge -y].$$

Differentiating with respect to t gives the desired density. □

Karatzas and Shreve Problem 2.8.7

Show that under P^0, reflected Brownian motion, $|B| = \{|B_t|, \mathcal{F}_t; 0 \le t < \infty\}$ is a Markov process with transition density

$$P^0\big[|B_{t+s}| \in dy \mid |B_t| = x\big] = p_+(s;x,y)dy$$
$$:= [p(s;x,y) + p(s;x,-y)]dy; \quad s>0, t\ge 0,\ x,y \ge 0.$$

Proof. To show that $|B|$ satisfies the Markov property, we'll show that

$$P^x\big[|B_s| \in A\big] = P^0\big[|B|_{t+s} \in A \mid |B_t| = x\big]. \tag{1}$$

Let's split up the condition

$$P^0\big[|B_{t+s}| \in A \mid |B_t| = x\big] = P^0\big[|B_{t+s}| \in A \mid B_t = x\big]P^0\big[B_t = x \mid |B_t| = x\big]$$

$$+ P^0\big[|B_{t+s}| \in A \mid B_t = -x\big]P^0\big[B_t = -x \mid |B_t| = x\big].$$

Since the distribution of B_t is symmetric, each of those second factors is $1/2$.

$$\begin{aligned} P^0\big[|B_{t+s}| \in A \mid |B_t| = x\big] &= \frac{1}{2}P^0\big[|B_{t+s}| \in A \mid B_t = x\big] + \frac{1}{2}P^0\big[|B_{t+s}| \in A \mid B_t = -x\big] \\ &= \frac{1}{2}P^x\big[|B_s| \in A\big] + \frac{1}{2}P^{-x}\big[|B_s| \in A\big] \\ &= P^x\big[|B_s| \in A\big]. \end{aligned}$$

Thus, the process satisfies the Markov property. As for the transition density, we start by computing its distribution function.

$$\begin{aligned} P^0\big[|B_{t+s}| \le y \mid |B_t| = x\big] &= P^0\big[|B_{t+s}| \le y \mid B_t = x\big]P^0\big[B_t = x \mid |B_t| = x\big] \\ &\quad + P^0\big[|B_{t+s}| \le y \mid B_t = -x\big]P^0\big[B_t = -x \mid |B_t| = x\big] \\ &= \frac{1}{2}P^0\big[|B_{t+s}| \le y \mid B_t = x\big] + \frac{1}{2}P^0\big[|B_{t+s}| \le y \mid B_t = -x\big] \\ &= \frac{1}{2}P^x\big[|B_s| \le y\big] + \frac{1}{2}P^{-x}\big[|B_s| \le y\big] \\ &= P^x\big[|B_s| \le y\big] \\ &= \int_{-y}^{y} p(s; x, t)\ dt. \end{aligned}$$

Differentiating with respect to y establishes the desired result. □

Karatzas and Shreve Problem 2.8.8

Define $Y_t = M_t - B_t$, $0 \le t < \infty$, where $M_t = \max_{0 \le s \le t} B_s$ is the running maximum. Show that under P^0, the process $Y = \{Y_t, \mathcal{F}_t, 0 \le t < \infty\}$ is Markov and has transition density

$$P^0[Y_{t+s} \in dy \mid Y_t = x] = p_+(s; x, y)dy; \quad s > 0,\ t \ge 0,\ x, y \ge 0.$$

Conclude that under P^0 the processes $|B|$ and Y have the same finite dimensional distributions.

Proof. (This argument comes from page 55 in a book by Mörters and Peres) Fix $s \geq 0$ and let $\tilde{B}_t$ be our original Brownian motion B_t "started anew" at $t = s$:

$$\tilde{B}_t = B_{t+s} - B_s.$$

We showed ages ago that $\tilde{B}_t$ is also a Brownian motion. Similarly, define $\tilde{M}_t = \max_{s \leq t} \tilde{B}_s$. If we can show that, conditional on $\mathcal{F}_s$, the process Y_{s+t} has the same distribution as $|Y(s) + \tilde{B}_t| = |B_{s+t}|$, then since the reflected Brownian motion is Markovian, we will have shown that Y is also Markovian with the same transition density.

$M(s+t)$ is the larger between the running maximum of B before s or between s and $s+t$. We can write the latter as $B_s + \tilde{M}_t$, so we have

$$M_{s+t} = M_s \vee (B_s + \tilde{M}_t).$$

From this we deduce

$$Y_{s+t} = M_{s+t} - B_{s+t} = (M_s \vee (B_s + \tilde{M}_t)) - (B_s + \tilde{B}_t) = (Y_s \vee \tilde{M}_t) - \tilde{B}_t.$$

From here, it suffices to show that $(y \vee \tilde{M}_t) - \tilde{B}_t$ has the same distribution as $|y + \tilde{B}_t|$ for any $y \geq 0$. For any $a \geq 0$ we can write

$$\Pr[(y \vee \tilde{M}_t) - \tilde{B}_t > a] = \Pr[y - \tilde{B}_t > a] + \Pr[y - \tilde{B}_t \leq a, \ \tilde{M}_t - \tilde{B}_t > a]$$

$$= p_1 + p_2.$$

Since the distribution of $\tilde{B}_t$ is symmetric, we have $p_1 = \Pr[y + \tilde{B}_t > a]$. The plan is to use the reflection principle on p_2. To this end, define the time reversed Brownian motion W by

$$W_u = \tilde{B}_{t-u} - \tilde{B}_t, \quad 0 \leq u \leq t.$$

Similarly, let $(M_W)_t = \max_{0 \leq u \leq t} W_t$ be its running maximum. Clearly, we have that $(M_W)_t = \tilde{M}_t - \tilde{B}_t$. Since $W_t = -\tilde{B}_t$, we have

$$p_2 = \Pr[y + W_t \leq a, \ (M_W)_t > a].$$

By the reflection principle, we have

$$p_t = \Pr[y + \tilde{B}_t \leq -a].$$

Adding p_1 and p_2 gives the desired

$$\Pr[y \vee \tilde{M}_t - \tilde{B}_t > a] = \Pr[|y + \tilde{B}_t| > a].$$

□

Øksendal Problem 8.15

Let $f \in C_0^2(\mathbb{R}^n)$ and $\alpha(x) = (\alpha_1(x), \ldots, \alpha_n(x))$ with $\alpha_i \in C_0^2(\mathbb{R}^n)$ be given functions and consider the PDE

$$\begin{cases} \partial_t u = \left(\sum_{i=1}^n \alpha_i(x)\partial_{x_i} + \frac{1}{2}\sum_{i=1}^n \partial_{x_i}^2\right) u; \quad t > 0,\ x \in \mathbb{R}^n \\ u(0,x) = f(x); \quad x \in \mathbb{R}^n. \end{cases}$$

(a) Use the Girsanov theorem to show that the unique bounded solution $u(t,x)$ of this equation can be expressed by

$$u(t,x) = E^x\left[\exp\left(\int_0^t \alpha(B_s)\cdot dB_s - \frac{1}{2}\int_0^t |\alpha(B_s)|^2 ds\right) f(B_t)\right],$$

where E^x is the expectation w.r.t. P^x.

Proof. We follow the lead of example 8.6.9. The PDO in the above PDE is the generator of the diffusion whose SDE is given by

$$dX_t = \alpha(X_t)\ dt + dB_t.$$

By theorem 8.1.1 (Kolmogorov's backward equation), the solution to this boundary value problem is unique and given by

$$u(t,x) = E^x[f(X_t)].$$

Since $f \in C_0^2$, the solution is bounded. Define the measure Q by $dQ(\omega) = M_T(\omega)dP(\omega)$, where M_t is given by

$$M_t = \exp\left(\int_0^t \alpha(B_s)\cdot dB_s - \frac{1}{2}\int_0^t |\alpha(B_s)|^2 ds\right).$$

By Girsanov's theorem, $\tilde{B}_t$ given by

$$\tilde{B}_t = B_t - \int_0^t \alpha(B_s)\ ds$$

is a Brownian motion w.r.t. Q for $t \leq T$. As per example 8.6.9, B_t satisfies the SDE

$$dB_t = \alpha(B_t)\ dt + d\tilde{B}_t.$$

The solution is then given by

$$E_Q^x[f(B_t)] = E^x\left[\exp\left(\int_0^t \alpha(B_s)\cdot dB_s - \frac{1}{2}\int_0^t |\alpha(B_s)|^2 ds\right) f(B_t)\right].$$

$\square$

(b) Now assume that α is a gradient, i.e., that there exists $\gamma \in C^1(\mathbb{R}^n)$ such that

$$\nabla\gamma = \alpha.$$

Assume for simplicity that $\gamma \in C_0^2(\mathbb{R}^n)$. Use Itô's formula to prove that

$$u(t,x) = \exp[-\gamma(x)]E^x\left[\exp\left(-\frac{1}{2}\int_0^t \left(\nabla\gamma^2(B_s) + \Delta\gamma(B_s)\right) ds\right)\exp[\gamma(B_t)]f(B_t)\right].$$

Proof. We substitute $\alpha = \nabla\gamma$ into what we got in part (a).

$$\begin{aligned}\exp\left[\int_0^t \alpha(B_s)\cdot dB_s - \frac{1}{2}\int_0^t |\alpha(B_s)|^2 ds\right] &= \exp\left[\int_0^t \nabla\gamma(B_s)\ dB_s - \frac{1}{2}\int_0^t |\nabla\gamma(B_s)|^2\ ds\right] \\ &= \exp\left[\gamma(B_t) - \gamma(B_0) - \frac{1}{2}\int_0^t \Delta\gamma(B_s)\ ds - \frac{1}{2}\int_0^t |\nabla\gamma(B_s)|^2\ ds\right].\end{aligned}$$

Assuming $B_0 = x$, we get what we want. □

(c) Put $v(t,x) = \exp[\gamma(x)]u(t,x)$. Use the Feynman-Kac formula to show that $v(t,x)$ satisfies the PDE

$$\begin{cases}\partial_t v = -\frac{1}{2}\left(|\nabla\gamma|^2 + \Delta\gamma\right)\cdot v + \frac{1}{2}\Delta v; & t > 0,\ x \in \mathbb{R}^n \\ v(0,x) = \exp[\gamma(x)]f(x); & x \in \mathbb{R}^n.\end{cases}$$

Proof. This follows directly from Feynman-Kac. We have that

$$v(t,x) = E^x\left[\exp\left(-\int_0^t \left(\nabla\gamma^2(B_s) + \Delta\gamma(B_s)\right) ds\right) e^{\gamma(B_t)} f(B_t)\right].$$

If we call the integrand in the exponent q, then Feynman-Kac says that

$$\begin{cases}\partial_t v = Av - qv; & t > 0,\ x \in \mathbb{R}^n \\ v(0,x) = e^{\gamma(x)} f(x); & x \in \mathbb{R}^n,\end{cases}$$

where A is the generator of B_t. The generator of Brownian motion is the Laplacian, so $A = \Delta$ and we're done. □

Øksendal 8.16

Let B_t denote Brownian motion in $\mathbb{R}^n$ and consider the diffusion X_t in $\mathbb{R}^n$ defined by

$$dX_t = \nabla h(X_t)dt + dB_t; \quad X_0 = x \in \mathbb{R}^n,$$

where $h \in C_0^1(\mathbb{R}^n)$.

(a) Prove that for $f \in C_0(\mathbb{R}^n)$ we have

$$E^x[f(X_t)] = E^x\left[\exp\left(-\int_0^t V(B_s)\ ds\right)\cdot \exp[h(B_t) - h(x)]\cdot f(B_t)\right], \tag{2}$$

where

$$V(x) = \frac{1}{2}|\nabla h(x)|^2 + \frac{1}{2}\Delta h(x).$$

Proof. As per Øksendal's hint, we use Girsanov on the LHS of (2) to write it in terms of B_t. We again follow the example 8.6.9. Define M_t by

$$M_t = \exp\left[\int_0^t \nabla h(B_s)\cdot B_s - \frac{1}{2}\int_0^t |\nabla h(B_s)|^2\ ds\right].$$

Since $h \in C_0^1$, it satisfies the Novikov condition and M_t is a Martingale. Define the measure Q by $dQ = M_t\ dP$ for some $T < \infty$. Then $\tilde{B}_t$ given by

$$\tilde{B}_t = -\int_0^t \nabla(B_s)\ ds + B_t$$

is a Brownian motion w.r.t. Q and

$$dB_t = \nabla h(B_s)dt + d\tilde{B}_t.$$

Now we can rewrite that expectation.

$$\begin{aligned}
E_P^x[f(X_t)] &= E_Q^x[M_t^{-1} f(X_t)] \\
&= E_Q^x\left[\exp\left(\int_0^t \nabla h(X_s)\cdot dB_s - \frac{1}{2}\int_0^t |\nabla h(X_s)|^2 ds\right) f(X_t)\right] \\
&= E_P^x\left[\exp\left(\int_0^t \nabla h(B_s)\cdot dB_s - \frac{1}{2}\int_0^t |\nabla h(B_s)|^2 ds\right) f(X_t)\right]
\end{aligned}$$

Now by Itô we have

$$h(B_t) - h(B_0) = \int_0^t \nabla h(B_s)\cdot dB_s + \frac{1}{2}\int_0^t \Delta h(B_s)\ ds.$$

Solving for that first integral and plugging it into the last line of the previous computation establishes the desired result. $\square$

(b) Use Feynman-Kac to restate (2) as follows (assuming $V \geq 0$):

$$T_t^X(f, x) = \exp[-h(x)] \cdot T_t^Y(f \cdot \exp h, x),$$

where T_t^X and T_t^Y denote the transition operators of the process X and Y, respectively, i.e.

$$T_t^X(f, x) = E^x[f(X_t)].$$

Proof. There's a remark in Øksendal that says that if Y is the process obtained from killing B_t at the rate V, then Y has transition operator

$$T_t^Y(f, x) = E^x\left[\exp\left(-\int_0^t V(B_s)\ ds\right) f(B_t)\right]$$

If we just pull the $e^{-h(x)}$ out of the expectation in (2), then we can write it as

$$T_t^X(f, x) = e^{-h(x)} T_t^Y(f \cdot \exp h, x).$$

$\square$