Liam Hardiman *February 24, 2020*

270B - Homework 3

Problem 1. Let $X_1, X_2, \ldots$ be independent random variables with means μ_i and finite variances σ_i^2. Consider the sums $S_n = X_1 + \cdots + X_n$. Find sequences of real numbers (b_i) and (c_i) such that $S_n^2 + b_n S_n + c_n$ is a martingale with respect to the σ-algebras generated by $X_1, \ldots, X_n$.

Solution. Let's start by centering the sum: define the random variable $M_n = S_n - \sum_{i=1}^n \mu_i$. Since the X_i's are independent, we have $\mathrm{Var}[M_n] = \sum_{i=1}^n \sigma_i^2$. We claim that

$$V_n = M_n^2 - \sum_{i=1}^n \sigma_i^2 = \left(S_n - \sum_{i=1}^n \mu_i\right)^2 - \sum_{i=1}^n \sigma_i^2$$

is a martingale with respect to the filtration $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. Let's start the computation.

$$\begin{aligned}
\mathbb{E}[V_{n+1}|\mathcal{F}_n] &= \mathbb{E}[S_{n+1}^2] - 2\left(\sum_{i=1}^{n+1}\mu_i\right)\mathbb{E}[S_{n+1}|\mathcal{F}_n] + \left(\sum_{i=1}^{n+1}\mu_i\right)^2 - \sum_{i=1}^{n+1}\sigma_i^2 \\
&= S_n^2 + 2S_n\mu_{n+1} + \mathbb{E}[X_{n+1}^2] - 2\left(\sum_{i=1}^{n+1}\mu_i\right)(S_n + \mu_{n+1}) + \left(\sum_{i=1}^{n+1}\mu_i\right)^2 - \sum_{i=1}^{n+1}\sigma_i^2 \\
&= S_n^2 - 2\left(\sum_{i=1}^{n}\mu_i\right)S_n + \mathbb{E}[X_{n+1}^2] - 2\mu_{n+1}^2 + \left(\sum_{i=1}^{n}\mu_i\right)^2 + \mu_{n+1}^2 - \sum_{i=1}^{n+1}\sigma_i^2 \\
&= S_n^2 - 2\left(\sum_{i=1}^{n}\mu_i\right)S_n + \left(\sum_{i=1}^{n}\mu_i\right)^2 - \sum_{i=1}^{n}\sigma_i^2 \\
&= V_n.
\end{aligned}$$

Here we've used the fact that S_n is $\mathcal{F}_n$-measurable and X_{n+1} is independent of $\mathcal{F}_n$. The sequences we want are then

$$b_n = -2\sum_{i=1}^n \mu_i, \qquad c_n = \left(\sum_{i=1}^n \mu_i\right)^2 - \sum_{i=1}^n \sigma_i^2.$$

□

Problem 2.

(a) Show that if (X_n) and (Y_n) are martingales with respect to the same filtration, then $X_n \vee Y_n$ is a submartingale.

Proof. We use the trusty identity

$$X_n \vee Y_n = \frac{1}{2}[(X_n + Y_n) + |X_n - Y_n|].$$

Since the sum of martingales is a martingale and conditional Jensen says the absolute value of a martingale is a submartingale, we have

$$\mathbb{E}[X_{n+1} \vee Y_{n+1} | \mathcal{F}_n] = \frac{1}{2}(\mathbb{E}[X_{n+1} + Y_{n+1} | \mathcal{F}_n] + \mathbb{E}[|X_{n+1} - Y_{n+1}| \mid \mathcal{F}_n])$$
$$\geq \frac{1}{2}[(X_n + Y_n) + |X_n - Y_n|]$$
$$= X_n \vee Y_n.$$

Hence, $X_n \vee Y_n$ is a submartingale. □

(b) Give an example showing that $X_n \vee Y_n$ need not be a martingale.

Proof. □

Problem 3. Give an example of a martingale (X_n) such that $X_n \to -\infty$ a.s.

Solution. Durrett gives a hint to let $X_n = \xi_1 + \cdots + \xi_n$ for some independent centered ξ_i's. The idea is to concentrate most of the mass of ξ_i around some negative value and put the rest (some summable amount) around some positive value, then apply Borel-Cantelli.

Concretely, let ξ_i be given by

$$\xi_i = \begin{cases} 2^j & \text{with probability } \frac{1}{2^j} \\ -\frac{1}{1-2^{-j}} & \text{with probability } 1 - \frac{1}{2^j} \end{cases}.$$

Clearly ξ_i is centered, so $X_n = \xi_1 + \cdots + \xi_n$ is a martingale. Note that

$$\sum_{i=1}^{\infty} \mathbb{P}[\xi_i = 2^j] = \sum_{i=1}^{\infty} \frac{1}{2^j} = 1 < \infty.$$

By Borel-Cantelli, we have that $\xi_i = -\frac{1}{1-2^{-j}}$ eventually with probability 1, so $X_n \to -\infty$ a.s. □

Problem 4. Let (X_n) be a martingale that is bounded a.s. either above or below by some constant M. Show that $\sup_n \mathbb{E}|X_n| < \infty$.

Proof. The main idea is that a nonnegative martingale (X_n) converges a.s. to some X and $\mathbb{E}[X] \leq \mathbb{E}[X_1]$. This is a theorem in Durrett. We establish it by noting that our proof of the Martingale convergence theorem actually shows that we need only assume that $\sup_n \mathbb{E}[X_n^+] < \infty$ in order to guarantee a.s. convergence. Once we have this, note that $-X_n \leq 0$ is a martingale with $\mathbb{E}[(-X_n)^+] = 0$, so by martingale convergence, X_n converges a.s. to some X and $\mathbb{E}[X_1] \geq \mathbb{E}[X]$ by Fatou.
Now if X_n is bounded below, then $X_n + M$ is a nonnegative martingale. By the martingale convergence theorem, $X_n + M$ converges almost surely to some limit Y with $\mathbb{E}|Y| < \infty$. Consequently, X_n also converges a.s. to an integrable function, so $\sup_n \mathbb{E}|X_n| < \infty$. If X_n is bounded above, then $-X_n + M$ is a nonnegative martingale and the same argument works. □

Problem 5. Let $Z_1, Z_2, \ldots$ be nonnegative iid random variables with $\mathbb{E}[Z_i] = 1$ and $\mathbb{P}[Z_i = 1] < 1$. Show that as $n \to \infty$,

$$\prod_{i=1}^{n} Z_i \to 0 \quad \text{a.s.}$$

Proof. First note that $M_n = \prod_{i=1}^{n} Z_i$ is indeed a martingale:

$$\mathbb{E}[M_{n+1}|\mathcal{F}_n] = \mathbb{E}[Z_{n+1}M_n|\mathcal{F}_n] = M_{n+1},$$

where $\mathcal{F}_n = \sigma(Z_1, \ldots, Z_n)$. Since M_n is a nonnegative martingale, it converges to some X a.s. with $\mathbb{E}[X] \leq \mathbb{E}[Z_1] = 1$. For any $\epsilon > 0$ we have

$$\mathbb{P}[|M_{n+1} - M_n| > \epsilon] = \mathbb{P}[M_n|Z_{n+1} - 1| > \epsilon].$$

Now if $M_n > \sqrt{\epsilon}$ and $|Z_{n+1} - 1| > \sqrt{\epsilon}$, then clearly $M_n|Z_{n+1} - 1| > \epsilon$. We then have by independence

$$\mathbb{P}[|M_{n+1} - M_n| > \epsilon] \geq \mathbb{P}[M_n > \sqrt{\epsilon}] \cdot \mathbb{P}[|Z_{n+1} - 1| > \sqrt{\epsilon}].$$

Now since $\mathbb{P}[Z_{n+1} = 1] < 1$, we have that for ϵ sufficiently small, $\mathbb{P}[|Z_{n+1} - 1| > \epsilon] \geq \delta > 0$ for some δ. Since M_n converges almost surely, it converges in measure as well, so the left-hand side of the above inequality goes to zero. Since the $\mathbb{P}[|Z_{n+1} - 1| > \sqrt{\epsilon}]$ term is bounded below by a positive constant, we must have that $\mathbb{P}[M_n > \sqrt{\epsilon}] \to 0$, so $M_n \to 0$ in probability. Since M_n converges almost surely and the a.s. limit is the same as the probability limit, $M_n \to 0$ a.s. □

Problem 6. Let (X_n) be a martingale and let $\Delta_n = X_n - X_{n-1}$ be the martingale differences. Prove that if $X_0 = 0$ and $\sum_{n=1}^{\infty} \Delta_n^2 < \infty$ then X_n converges in L^2 to some random variable X. (I tried doing the problem as stated and it didn't seem to work. I talked with Xiaowen and she said that we need to assume $\sum \mathbb{E}[\Delta_n^2] < \infty$. This is also how it's stated in Durrett.)

Proof. For any m, n we have $X_n - X_m = \sum_{i=m+1}^{n} \Delta_i$. From this we deduce

$$\mathbb{E}[|X_n - X_m|^2] = \mathbb{E}\left[\left(\sum_{i=m+1}^{n} \Delta_i\right)^2\right] = \sum_{i=m+1}^{n} \mathbb{E}[\Delta_i^2] + 2 \sum_{m+1 \leq i < j}^{n} \mathbb{E}[\Delta_i \Delta_j].$$

Since martingale increments are uncorrelated, $\mathbb{E}[\Delta_i \Delta_j] = \mathbb{E}[\Delta_i]\mathbb{E}[\Delta_j] = 0$. Since $\sum \mathbb{E}[\Delta_i^2] < \infty$, the tail sum goes to zero, so for m, n large enough, the above sum can be made arbitrarily small. (X_n) is then Cauchy in L^2, so it converges in L^2 by completeness. □

Problem 7. Construct a branching process (Z_n) as follows. Let X be a random variable with mean μ and variance σ^2; it specifies the distribution of the offspring. Set

$$Z_{n+1} = X_1^{(n+1)} + \cdots + X_{Z_n}^{(n+1)},$$

to be the size of the population at time $n + 1$, where all $X_i^{(k)}$ are iid random variables distributed identically with X.

(a) Show that $Y_n = Z_n/\mu^n$ defines a martingale (with respect to the filtration $\mathcal{F}_n$ generated by $X_j^{(k)}$, $1 \le j \le Z_n$, $k \le n$.)

Proof. We compute, critically using the fact that Z_n is $\mathcal{F}_n$-measurable.

$$\begin{aligned}
\mathbb{E}[Y_{n+1}|\mathcal{F}_n] &= \frac{1}{\mu^{n+1}}\mathbb{E}[Z_{n+1}|\mathcal{F}_n] \\
&= \frac{1}{\mu^{n+1}}\mathbb{E}[X_1^{(n+1)} + \cdots + X_{Z_n}^{(n+1)}|\mathcal{F}_n] \\
&= \frac{1}{\mu^{n+1}} \cdot \mu Z_n \\
&= Y_n.
\end{aligned}$$

□

(b) Show that

$$\mathbb{E}[Z_{n+1}^2|\mathcal{F}_n] = \mu^2 Z_n^2 + \sigma^2 Z_n.$$

Proof. We compute.

$$\begin{aligned}
\mathbb{E}[Z_{n+1}^2|\mathcal{F}_n] &= \sum_{i=1}^{Z_n}\mathbb{E}[(X_i^{(n+1)})^2] + 2\sum_{1\le i<j\le Z_n}\mathbb{E}[X_i^{(n+1)}X_j^{(n+1)}] \\
&= Z_n(\sigma^2+\mu^2) + Z_n(Z_n-1)\mu^2 \\
&= \mu^2 Z_n^2 + \sigma^2 Z_n.
\end{aligned}$$

□

(c) Deduce that (Y_n) is bounded in L^2 if and only if $\mu > 1$.

Proof. We have

$$\mathbb{E}[Y_n^2] = \frac{1}{\mu^{2n}}\mathbb{E}[\mathbb{E}[Z_n^2|\mathcal{F}_{n-1}]] = \frac{1}{\mu^{2n}}(\mu^2\mathbb{E}[Z_{n-1}^2] + \sigma^2\mu^{n-1}) = \mathbb{E}[Y_{n-1}^2] + \frac{\sigma^2}{\mu^{n+1}}.$$

Summing over n and using $Y_0 = 1$ gives

$$\mathbb{E}[Y_n^2] = \begin{cases} 1 + n\sigma^2 & \text{if } \mu = 1 \\ 1 + \frac{\sigma^2}{\mu^2-\mu}(1-\frac{1}{\mu^n}) & \text{otherwise} \end{cases}.$$

If $n \le 1$, this quantity tends toward ∞ and it tends toward $\frac{\sigma^2}{\mu^2-\mu}$ otherwise. □

(d) Show that when $\mu > 1$, the L^2 limit Y of Y_n satisfies

$$\mathrm{Var}[Y] = \frac{\sigma^2}{\mu(\mu-1)}.$$

Proof. Assuming we have convergence in L^2, we have

$$\mathrm{Var}[Y] = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 = \lim_{n\to\infty} \mathbb{E}[Y_n^2] - 1 = \lim_{n\to\infty} \frac{\sigma^2}{\mu(\mu-1)}\left(1 - \frac{1}{\mu^n}\right) = \frac{\sigma^2}{\mu(\mu-1)}.$$

□

Problem 8. Find an example of a martingale (X_n) that converges a.s. to some random variable X, but for which $\limsup_n \mathbb{E}|X_n| = \infty$.

Solution. Define the sequence (a_n) by

$$a_1 = 2, \quad a_n = 4\sum_{i=1}^{n} a_i.$$

Also define the independent random variables Z_n by

$$Z_n = \begin{cases} a_n & \text{with probability } 1/(2n^2) \\ 0 & \text{with probability } 1 - n^{-2} \\ -a_n & \text{with probability } 1/(2n^2) \end{cases}.$$

Finally, define X_n by $X_n = \sum_{i=1}^{n} Z_i$. Since each Z_i is centered, X_n is clearly a martingale with respect to the filtration $\mathcal{F}_n = \sigma(Z_1, \ldots, Z_n)$.

Now $\mathbb{P}[Z_n] \neq 0 = \frac{1}{n^2}$. Since this quantity is summable, we have by Borel-Cantelli that $\mathbb{P}[Z_n \neq 0 \text{ i.o.}] = 0$. Consequently, X_n converges almost surely.

If $Z_n \neq 0$, the minimum possible value for X_n is

$$a_n - \sum_{i=1}^{n-1} a_i = \frac{3}{4}a_n,$$

which corresponds to taking a step of size a_n in one direction and every previous step having size a_i in the opposite direction. By induction we have $a_n = 8 \cdot 5^{n-2}$ for $n \geq 3$. The probability of taking this step is at least $1/(2n^2)$, so by Markov's inequality we have

$$\mathbb{E}|X_n| \geq \frac{3}{4}a_n \mathbb{P}\left[|X_n| \geq \frac{3}{4}a_n\right] \geq \frac{3}{4}\cdot(8\cdot 5^{n-2})\cdot\frac{1}{2n^2} \to \infty.$$

□