3. 3. If X has distribution function F, what is the distribution function of e^X?

By definition, $\mathbb{P}[X \leq t] = F(t)$ $\forall t \in \mathbb{R}$.

We want $\mathbb{P}[e^X \leq t]$.

$\mathbb{P}[e^X \leq t] = \mathbb{P}[X \leq \log t] = F(\log t)$ $\boxtimes$

⚠ any time I write log & I don't specify otherwise, it's $\log_e$

4. 4. Let X be a binomial random variable with parameters (n, p). What value of p maximizes $\Pr[X = k]$ for $k = 0, 1, \ldots, n$?

Recall that $\mathbb{P}[X=k] = \binom{n}{k} p^k (1-p)^{n-k} =: f(p)$ for $X \sim \text{Bin}(n,k)$.

we want to find p that maximizes this.

Since $\log(\cdot)$ is monotone increasing, maximizing $\log(f(p))$ will maximize $f(p)$ too.

Use calculus to maximize:

$$\frac{d}{dp} \log(f(p)) = \frac{d}{dp}\left[\log\binom{n}{k} + k \log p + (n-k)\log(1-p)\right]$$

$$= \frac{k}{p} - \frac{n-k}{1-p}$$

Set derivative $= 0$

$$\Rightarrow \frac{k}{p} - \frac{n-k}{1-p} = 0 \iff p = k/n.$$

- So $p = k/n$ is a critical point.
- $p \in [0,1]$, so have to check endpoints too

$f(0) = \binom{n}{k} 0^k (1-0)^{n-k} = 0$

$f(1) = \binom{n}{k} 1^k (1-1)^{n-k} = 0$

$f(\frac{k}{n}) = \binom{n}{k} (\frac{k}{n})^k (\frac{n-k}{n})^{n-k} \geq 0$,

so $p = k/n$ indeed maximizes $f(p)$

$\boxtimes$

2. 2. Consider an experiment whose sample space consists of a countably infinite number of points. Show that not all points can be equally likely. Can all points have a positive probability of occurring?

Call the sample space Ω.

We need $\sum_{\omega \in \Omega} \mathbb{P}[\omega] = 1$ by the axioms of probability.

- Suppose every $\omega \in \Omega$ is equally likely. Then $\mathbb{P}[\omega] = p$ for some $p \in [0,1]$.

But then

$$\mathbb{P}[\Omega] = \sum_{\omega\in\Omega} \mathbb{P}[\omega] = \sum_{\omega\in\Omega} p = \begin{cases} 0 & \text{if } p=0 \\ +\infty & \text{if } p>0 \end{cases}$$

since Ω is infinite. $\neq 1$. $\boxtimes$

Consequence: You can't randomly select an integer from all of $\mathbb{Z}$ uniformly at random!

The probabilities can be positive, just not all equal.

Since Ω is countably infinite, we can enumerate it: $\Omega = \{\omega_1, \omega_2, \omega_3, \ldots\}$

Let $\mathbb{P}[\omega_n] = 1/2^n$. Then

$$\mathbb{P}[\Omega] = \sum_{\omega\in\Omega} \mathbb{P}[\omega] = \sum_{n=1}^{\infty} 1/2^n = 1.$$

$\boxtimes$

1. A man has five coins, two of which are double-headed, one is double-tailed, and two are normal. He shuts his eyes, picks a coin at random, and tosses it.

1.a) (a) What is the probability that the lower face of the coin is a head?

label the coins C_1 = double heads, C_2 = double tails, C_3 = fair,

- $\{\text{lower face is heads}\} = \{\text{Picked } C_1, \text{ lower is } H\} \sqcup \{\text{Picked } C_3, \text{ lower is } H\}$

- $\mathbb{P}[\text{Picked } C_1, \text{ lower is } H] = \underbrace{\tfrac{1}{3}}_{\mathbb{P}[\text{picked } C_1]} \cdot \underbrace{1}_{\mathbb{P}[\text{lower of } C_1 \text{ is } H]}$

- $\mathbb{P}[\text{Picked } C_3, \text{ lower is } H] = \tfrac{1}{3} \cdot \tfrac{1}{2}$

$\Longrightarrow \mathbb{P}[\text{lower is } H] = \tfrac{1}{3} + \tfrac{1}{6} = 1/2$

(b) He opens his eyes and see that the coin is showing heads. What is the probability that the lower face is heads as well?

if both sides are H, they must've picked coin 1, so we're interested in

$$\mathbb{P}[C_1 \mid \text{showing heads}] = \mathbb{P}[C_1, \text{ showing heads}] \big/ \mathbb{P}[\text{showing heads}]$$

$\mathbb{P}[\text{showing heads}] = \mathbb{P}[\text{lower is heads}]$, so

$$\mathbb{P}[C_1 \mid \text{showing heads}] = \frac{\frac{1}{3} \cdot 1}{1/2} = 2/3$$

(c) He shuts his eyes again, and tosses the coin again. What is the probability that the lower face is head?

$$\mathbb{P}[\text{second toss lower is H} \mid \text{first toss shows H}]$$

$$= \frac{\mathbb{P}[\text{first toss shows H, second lower H}]}{\mathbb{P}[\text{first toss shows H}]}$$

$$= \frac{\left(\mathbb{P}[C_1, \text{first shows H, second lower H}] + \mathbb{P}[C_3, \text{first shows H, second lower H}]\right)}{\mathbb{P}[\text{first toss shows H}]}$$

$$= \frac{\frac{1}{3} \cdot 1 \cdot 1 + \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{2}}{1/2} = 5/6$$

(d) He opens his eyes and sees that the coin is showing heads. What is the probability that the lower face is a head?

$$\mathbb{P}[C_1 \mid \text{first shows } H, \text{ second shows } H]$$

$$= \frac{\mathbb{P}[C_1, \text{first shows } H, \text{second shows } H]}{\mathbb{P}[\text{first } H, \text{ second } H]}$$

$$= \frac{\frac{1}{3}\cdot 1 \cdot 1}{\mathbb{P}[C_1, H, H] + \mathbb{P}[C_3, H, H]} = \frac{1/3}{\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{2}\cdot\frac{1}{2}}$$

$$= \frac{1/3}{5/12} = 4/5$$

(e) He discards this coin, picks another one at random (but not the same coin), and tosses it. What is the probability that it shows heads?

$$\mathbb{P}[\text{new coin heads} \mid \text{first coin } H, H]$$

$$= \frac{\mathbb{P}[\text{new coin } H, \text{ first coin } H, H]}{\mathbb{P}[\text{first coin } HH]} \quad (*)$$

Possible paths:

$C_1 \to H \to H \to C_3 \to H : \frac{1}{3}\cdot 1 \cdot 1 \cdot \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{12}$

$C_3 \to H \to H \to C_1 \to H : \frac{1}{3}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot 1 = \frac{1}{24}$

So $(*) = \dfrac{\frac{1}{12} + \frac{1}{24}}{5/12} = 3/10$

from before