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271B - Homework 1

Problem 1. The standard Ornstein-Uhlenbeck process X_t is a Gaussian process with mean zero and auto-covariance $C(t,s) = \mathbb{E}[X_t X_s] = \exp(-|t-s|)/2$. Let N_t be the standard Poisson process and define the process $Y_t = \zeta(-1)^{N_t}$, where ζ is a random variable independent of the Poisson process that takes values ± 1 with probability $1/2$.

Show that X_t and $Z_t = Y_{t/2}/\sqrt{2}$ are both stationary in the strong sense and have the same covariance. Does Y_t satisfy the Kolmogorov continuity condition? Are these processes stochastically continuous?

Solution. First we'll show that X_t is stationary. Let $\tau \in \mathbb{R}$. Since X_t is Gaussian with mean zero for all t, so is $X_{t+\tau}$. For any s and t we also have that

$$\mathbb{E}[X_{s+\tau}X_{t+\tau}] = \frac{1}{2}e^{-|(s+\tau)-(t+\tau)|} = \frac{1}{2}e^{-|s-t|} = \mathbb{E}[X_s X_t].$$

Since a Gaussian process is determined by its mean and covariance, we have that X_t and $X_{t+\tau}$ are equal in distribution, so the process is stationary.

Now for Z_t. We claim that Z_t has the Markov property, i.e. for any $t_1 < t_2 < \ldots < t_n$ and $\alpha_i = \pm 1/\sqrt{2}$

$$\begin{aligned}&\mathbb{P}[Z_{t_1} = \alpha_1,\ Z_{t_2} = \alpha_2, \ldots, Z_{t_n} = \alpha_n] \\ &\qquad = \mathbb{P}[Z_{t_1} = \alpha_1]\mathbb{P}[Z_{t_2} = \alpha_2 \mid Z_{t_1} = \alpha_1] \cdots \mathbb{P}[Z_{t_n} = \alpha_n \mid Z_{t_{n-1}} = \alpha_{n-1}] \quad (1)\end{aligned}$$

Informally, the value of Z_{t_j} given $Z_{t_1}, \ldots, Z_{t_{j-1}}$ depends only on the number of sign flips of Z over the interval $(t_{j-1}, t_j]$. This only depends on the parity of $N_{t_j} - N_{t_{j-1}}$. Let's look at the terms on the right-hand side of (1).

$$\begin{aligned}\mathbb{P}[Z_{t_j} = \alpha_j \mid Z_{t_{j-1}} = \alpha_{j-1}] &= \begin{cases}\mathbb{P}[N_{t_j - t_{j-1}} \text{ is even}], & \text{if } \alpha_j = \alpha_{j-1} \\ \mathbb{P}[N_{t_j - t_{j-1}} \text{ is odd}], & \text{if } \alpha_j = -\alpha_{j-1}\end{cases} \\ &= \mathbb{P}[Z_{t_j + m} = \alpha_j \mid Z_{t_{j-1}+m} = \alpha_{j-1}]\end{aligned} \quad (2)$$

The last equality follows from the stationarity of Poisson increments. Equations (1) and (2) imply that Z is indeed stationary.

Let's compute the covariance of Z_t. Since ζ is independent of N_t we have

$$\mathbb{E}[Z_t] = \frac{1}{\sqrt{2}}\mathbb{E}[\zeta] \cdot \mathbb{E}[Y_{t/2}] = 0.$$

Consequently, for any s and t, the covariance is given by

$$\mathbb{E}[Z_s Z_t] = \mathbb{E}[Z_0 Z_{|t-s|}] = \frac{1}{2}\mathbb{E}[\zeta^2]\mathbb{E}\left[(-1)^{N_{|t-s|/2}}\right] = \frac{1}{2}\left(\mathbb{P}[N_{|t-s|/2} \text{ is even}] - \mathbb{P}[N_{|t-s|/2} \text{ is odd}]\right)$$

$$= \frac{1}{2}(2\mathbb{P}[N_{|t-s|/2} \text{ is even}] - 1). \quad (3)$$

As for that probability, we have

$$\mathbb{P}[N_{|t-s|/2} \text{ is even}] = \sum_{n=0}^{\infty} \mathbb{P}[N_{|t-s|/2} = 2n] = \sum_{n=0}^{\infty} \frac{(|t-s|/2)^{2n} e^{-|t-s|}}{(2n)!} = e^{-|t-s|/2}\cosh(|t-s|/2).$$

Substituting this expression into (3) gives

$$\mathbb{E}[Z_s Z_t] = \frac{1}{2}e^{-|t-s|/2} = \mathbb{E}[X_s X_t],$$

as desired.

Let's check to see if Y_t satisfies the Kolmogorov continuity condition. For any s and t, the quantity $|(-1)^{N_t} - (-1)^{N_s}|$ will be zero if N_t and N_s have the same parity and 2 if they have opposite parity. By the stationarity of Poisson increments, we have that

$$\left|(-1)^{N_t} - (-1)^{N_s}\right| = \begin{cases} 0, & N_{|t-s|} \text{ is even} \\ 2, & N_{|t-s|} \text{ is odd} \end{cases}.$$

Let $\alpha > 0$. By the above reasoning, we have that

$$\mathbb{E}[|Y_t - Y_s|^\alpha] = 2^\alpha \mathbb{P}[N_{|t-s|} \text{ is odd}] = 2^\alpha e^{-|t-s|}\sinh|t-s| = 2^{\alpha-1}\left(1 - e^{-2|t-s|}\right). \quad (4)$$

We claim that there are no positive K or β such that

$$\mathbb{E}[|Y_t - Y_s|^\alpha] \le K|t-s|^{1+\beta}$$

for all s, t. The right-hand side of (4) is $\Theta(|t-s|)$ as $|t-s| \to 0$, while $K|t-s|^{1+\beta}$ is $o(|t-s|)$ as $|t-s| \to 0$. We conclude that Y_t does *not* satisfy the Kolmogorov continuity condition.

Let's check for stochastic continuity. By Markov we have

$$\mathbb{P}[|X_{t+h} - X_t| > \delta] \le \frac{1}{\delta^2}\mathbb{E}[(X_{t+h} - X_t)^2]$$

$$= \frac{1}{\delta^2}\left(1 - e^{-|h|}\right),$$

which goes to zero as $h \to 0$ for any $\delta > 0$, so X is stochastically continuous. Now for Y. The quantity $|Y_{t+h} - Y_t|$ is zero when N_{t+h} and N_t have the same parity and is 2 when they have opposite parity. For $\delta < 2$ we have

$$\mathbb{P}[|Y_{t+h} - Y_t| > \delta] = \mathbb{P}[N_{|h|} \text{ is odd}]$$

$$= e^{-|h|}\sinh|h|,$$

which goes to zero as $h \to 0$, so Y is stochastically continuous. The same argument shows that Z is stochastically continuous as well. □

Problem 2. Let X_n be defined by the stochastic recursion

$$X_{n+1} = X_n - \Delta t X_n + (B_{(n+1)\Delta t} - B_{n\Delta t}),\ X_0 = \zeta, \tag{5}$$

for B_t standard Brownian motion. Find ζ so that X_n is stationary in the strong sense and give the associated auto-covariance function. What is the continuum limit of this process as $n \to \infty$, $\Delta t \to 0$ so that $n\Delta t = t$.

Solution. By induction we have that

$$X_{n+1} = (1-\Delta t)^{n+1}\zeta + \sum_{k=0}^{n}(1-\Delta t)^{n-k}(B_{(k+1)\Delta t} - B_{k\Delta t}). \tag{6}$$

By the above expansion, we can see that for $0 < \Delta t < 1$, ζ contributes less to X_{n+1}. The sum term is a sum of independent Gaussians, and hence Gaussian. We conclude that for n large, X_n approaches a Gaussian. In order for the process to be stationary, ζ must also be Gaussian.

Since ζ is Gaussian, it is determined by its mean and variance. Taking the expectation on both sides of the recursive formula (5) gives

$$\mathbb{E}[X_{n+1}] = (1-\Delta t)\mathbb{E}[X_n].$$

By stationarity, $\mathbb{E}[X_{n+1}] = \mathbb{E}[X_n]$. The above equation then forces $\mathbb{E}[X_n] = 0$ for all n, so $\mathbb{E}[\zeta] = 0$ as well. Taking the variance of both sides of the recursive formula and using stationarity gives

$$\mathrm{Var}[\zeta] = \mathrm{Var}[X_n] = (1-\Delta t)^2\mathrm{Var}[\zeta] + \Delta t.$$

Solving for $\mathrm{Var}[\zeta]$ gives $\mathrm{Var}[\zeta] = \frac{1}{2-\Delta t}$.

Now let's show that the choice $\zeta \sim \mathcal{N}(0, \frac{1}{2-\Delta t})$ makes X_n stationary. It's clear that this choice of ζ makes X_n a Gaussian process with zero mean for all n, so to check stationarity, it suffices to show that $\mathrm{Cov}(X_n X_{n+1})$ is independent of n. The same calculation that we used to find $\mathrm{Var}[\zeta]$ shows that $\mathrm{Var}[X_n] = \frac{1}{2-\Delta t}$. Now we compute the covariance.

$$\mathrm{Cov}(X_n, X_{n+1}) = (1-\Delta t)\mathrm{Var}[X_n] + \mathrm{Cov}(X_n, B_{(n+1)\Delta t} - B_{n\Delta t}) = \frac{1-\Delta t}{2-\Delta t}.$$

Here we've used the fact that disjoint increments of Brownian motion are independent. Since the covariance is independent of n, we conclude that this choice of ζ does indeed make the process stationary. By induction, the auto-covariance is given by

$$\mathrm{Cov}(X_n, X_{n+m}) = \frac{(1-\Delta t)^m}{2-\Delta t}.$$

Now for the continuous time limit. By stationarity, $X_n \sim \mathcal{N}(0, \frac{1}{2-\Delta t})$ for any $\Delta t > 0$. As $\Delta t \to 0$ we then have $X_t \sim \mathcal{N}(0, \frac{1}{2})$, where the limit is in L^2. As for the covariance, set $Y_t = X_{t/\Delta t}$. We then have

$$\begin{aligned}
\operatorname{Cov}(Y_s, Y_t) &= \operatorname{Cov}(X_{s/\Delta t}, X_{t/\Delta t}) \\
&= \frac{1}{2-\Delta t}(1-\Delta t)^{|s-t|/\Delta t} \\
&\to \frac{1}{2}e^{-|s-t|}, \text{ as } \Delta t \to 0.
\end{aligned}$$

□

Problem 3. Consider
$$X_t = \int_0^t (t-s)^{H-1/2} dB_s,$$
for B_t standard Brownian motion. For which values of H is X_t well defined? Find the distribution of X_t. Compare with the distribution of fractional Brownian motion B_t^H.

Solution. The family $(t-s)^{H-1/2}$ is deterministic and càdlàg when $H \geq 1/2$. We also have
$$\mathbb{E}\left[\int_0^t (t-s)^{2H-1}\ ds\right] = \int_0^t (t-s)^{2H-1}\ ds < \infty \iff H > 0.$$
Consequently, X_t is well defined for $H \geq 1/2$. The family $f_s^{(t)} = (t-s)^{H-1/2}$ is uniformly $\mathbb{P}$-integrable, so we have
$$X_t = \int_0^t (t-s)^{H-1/2}\ dB_s = \lim_{\Delta t \to 0} \sum_{k=1}^{t/\Delta t} f_{t_i}^{(t)} \Delta B_{t_{i+1}},$$
where $t_i = i\Delta t$, and $\Delta B_{t_{i+1}} = B_{t_{i+1}} - B_{t_i}$. Now for any fixed t, the values $f_{t_i}^{(t)}$ are deterministic constants. The increments $\Delta B_{t_{i+1}}$ are independent Gaussians, so the above sum is a limit (in some sense) of Gaussians. We'll show that the above sum weakly converges to a Gaussian by showing that its mean and variance converge.

The increment $\Delta B_{t_{i+1}}$ has mean zero, so for any $\Delta t > 0$, the above sum has mean zero. As the increments are independent and have variance Δt, the variance of the sum is given by
$$\sigma_{\Delta t}^2 = \sum_{k=1}^{t/\Delta t} \left(f_{t_i}^{(t)}\right)^2 \Delta t,$$
which we recognize as a Riemann sum that converges to $\int_0^t (f^{(t)})^2\ ds$. We conclude that
$$X_t \sim \mathcal{N}\left(0, \int_0^t (t-s)^{2H-1}\ ds\right) = \mathcal{N}\left(0, \frac{t^{2H}}{2H}\right).$$
This almost has the same distribution as fractional Brownian motion, which satisfies
$$B_t^H \sim \mathcal{N}(0, t^{2H}).$$

□

Problem 4. Prove directly from the definition of Itô integrals the integration by parts relation:

$$\int_0^t s\, dB_s = tB_t - \int_0^t B_s\, ds.$$

Proof. Since the function $(s, \omega) \mapsto s$ is deterministic, it is adapted and uniformly integrable, so we have by summation by parts

$$\begin{aligned}
\int_0^t s\, dB_s &= \lim_{\Delta t \to 0} \sum_{k=1}^{t/\Delta t} t_i (B_{t_{i+1}} - B_{t_i}) \\
&= \lim_{\Delta t \to 0} \left[t_{t/\Delta t}(B_{t_{t/\Delta t}} - B_0) - \sum_{k=1}^{t/\Delta t} B_{t_i}(t_{i+1} - t_i) \right] \\
&= \lim_{\Delta t \to 0} \left[tB_t - \sum_{k=1}^{t/\Delta t} B_{t_i} \Delta t \right] \\
&= tB_t - \int_0^t B_s\, ds.
\end{aligned}$$

□

Problem 5. Prove directly form the definition of the Itô integral that

$$\int_0^t B_s^2\, dB_s = \frac{1}{3} B_t^3 - \int_0^t B_s\, ds.$$

Proof. The family of functions $\{B_s^2\}_{s \le t}$ is uniformly $\mathbb{P}$-integrable since $\mathbb{E}[B_s^2] = s$. We then have

$$\begin{aligned}
\int_0^t B_s^2\, dB_s &= \lim_{\Delta t \to 0} \sum_{k=0}^{t/\Delta t} B_{t_i}^2 (B_{t_{i+1}} - B_{t_i}) \\
&= \lim_{\Delta t \to 0} \sum_{k=0}^{t/\Delta t} \left[-\frac{1}{3}(B_{t_{i+1}} - B_{t_i})^3 + \frac{1}{3}(B_{t_{i+1}}^3 - B_{t_i}^3) - B_{t_i}(B_{t_{i+1}} - B_{t_i})^2 \right].
\end{aligned}$$

Now the first term approaches the cubic variation of Brownian motion, which is zero. The second term telescopes, leaving us with $\frac{1}{3}B_t$. The squared part of the last term approaches the quadratic variation of Brownian motion, which is t, so the last term approaches $\int_0^t B_s\, ds$. We then have

$$\int_0^t B_s^2\, dB_s = \frac{1}{3} B_t^3 - \int_0^t B_s\, ds,$$

as desired. □

Problem 6. Let

$$M_t = B_t^3 - 3tB_t,$$

with B_t standard Brownian motion. Show that M_t is a martingale, first directly and then by using the result of the previous 2 problems.

Proof. Let $\mathcal{F}_t$ be the filtration generated by M_t. We then have for $s < t$

$$\begin{aligned}
\mathbb{E}[M_t \mid \mathcal{F}_s] &= \mathbb{E}[B_t^3 - 3tB_t \mid \mathcal{F}_s] \\
&= \mathbb{E}[(B_t - B_s + B_s)^3 - 3tB_t + 3sB_s - 3sB_s \mid \mathcal{F}_s] \\
&= \mathbb{E}[(B_t - B_s)^3] + 3\mathbb{E}[(B_t - B_s)^2 B_s] + 3\mathbb{E}[(B_t - B_s)B_s^2] + B_s^3 - 3(t-s)\mathbb{E}[B_t - B_s] - 3sB_s \\
&= B_s^3 - 3sB_s \\
&= M_s.
\end{aligned}$$

Here we've used that the increments $(B_t - B_s)$ are Gaussians independent of $\mathcal{F}_s$ whose odd moments are zero.

By the previous two exercises we have

$$B_t^3 - 3tB_t = 3\int_0^t (B_s^2 - s)\ dB_s.$$

Since the Itô integral is a martingale, M_t is a martingale. $\square$