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271B - Homework 2

Problem 1. Let S, T, and T_n, $n = 1, 2, \dots$ be stopping times (with respect to some filtration $\{\mathcal{F}_t\}_{t \geq 0}$). Show that $T \vee S$, $T \wedge S$, $T + S$, $\sup_n T_n$ are also stopping times.

Proof. The pointwise minimum, maximum, sum, and supremum of measurable functions are measurable. For the minimum and maximum we have

$$\{(T \wedge S) \leq t\} = \{T \leq t\} \cup \{S \leq t\}$$

$$\{(T \vee S) \leq t\} = \{T \leq t\} \cap \{S \leq t\}.$$

Unions and intersections of measurable sets are measurable, so both of these sets live in $\mathcal{F}_t$. Thus, $T \wedge S$ and $T \vee S$ are stopping times. For the sum, we can write the set $\{T + S \leq t\}$ as a countable union:

$$\{T + S \leq t\} = \bigcup_{\alpha, \beta \in \mathbb{Q},\ \alpha + \beta \leq t} \{T \leq \alpha\} \cap \{S \leq \beta\}.$$

As $\mathcal{F}_t$-measurability is closed under countable union and intersection, the sum is a stopping time. Finally, we have

$$\{\sup_n T_n \leq t\} = \bigcap_{n=1}^{\infty} \{T_n \leq t\},$$

which is measurable, so the supremum is also a stopping time. □

Problem 2. Let X_t be an adapted and continuous stochastic process, and define

$$T_\Gamma = \inf\{t \geq 0 : X_t \in \Gamma\}$$

for Γ a closed set. Show that T_Γ is a stopping time.

Proof. asdf □

Problem 3. Show that if X_t is a martingale with respect to some filtration (say $\mathcal{F}_t$) then it is also a martingale with respect to the filtration generated by itself.

Proof. Let $\mathcal{G}_t = \sigma(X_s : s \leq t)$ be the filtration X generates. We then have $\mathcal{G}_t \subseteq \mathcal{F}_t$ for all t since $\mathcal{G}_t$ is the smallest σ-algebra with respect to which X_t is measurable. By the law of total expectation and the martingale property of X_t with respect to $\mathcal{F}_t$ we have for any $s \leq t$

$$\mathbb{E}[X_t \mid \mathcal{G}_s] = \mathbb{E}[\mathbb{E}[X_t \mid \mathcal{F}_s] \mid \mathcal{G}_s] = \mathbb{E}[X_s \mid \mathcal{G}_s] = X_s.$$

Thus, X_t is a martingale with respect to $\{\mathcal{G}_t\}$. □

Problem 4. Let a, b be deterministic and f, g of class I. Show that if

$$a + \int_0^T f_s \, dB_s = b + \int_0^T g_s \, dB_s \tag{1}$$

then $a = b$ and $f = g$ a.a. for $(t, \omega) \in (0, T) \times \Omega$.

Proof. Since f and g are of class I, $\int_0^t f_s\, dB_s$ and $\int_0^t g_s\, dB_s$ are martingales and $\int_0^0 f_s\, dB_s = 0$ a.s. (the same holds for g). Taking the expectation of both sides of the given relation shows that $a = b$ a.s. and

$$\int_0^T (f_s - g_s)\, dB_s = 0.$$

By the Itô isometry we have

$$0 = \mathbb{E}\left[\left(\int_0^T (f_s - g_s)\, dB\right)^2\right] = \mathbb{E}\left[\int_0^T (f_s - g_s)^2\, ds\right].$$

We conclude that $f_t(\omega) = g_t(\omega)$ for almost all $(t, \omega) \in (0, T) \times \Omega$. □

Problem 5. Assume that X_t is of class I and continuous in mean square on $[0, T]$, that is for $t \in [0, T]$

$$\mathbb{E}[X_t^2] < \infty, \quad \lim_{s \to t} \mathbb{E}[(X_t - X_s)^2] = 0.$$

Define

$$\phi_t^{(n)} = \sum_j X_{t_{j-1}^{(n)}} \chi_{[t_{j-1}^{(n)}, t_j^{(n)})}(t), \; t_j^{(n)} = j2^{-n}.$$

Show that for $0 \leq t \leq T$

$$\int_0^t X_s\, dB_s = \lim_{n \to \infty} \int_0^t \phi_s^{(n)}\, dB_s,$$

where the limit is n $L^2(\mathbb{P})$.

Proof. For any n we have by the Itô isometry

$$\mathbb{E}\left[\left(\int_0^t (X_s - \phi_s^{(n)})\, dB_s\right)^2\right] = \mathbb{E}\left[\int_0^t (X_s - \phi_s^{(n)})^2\, ds\right] = \mathbb{E}\left[\sum_j \int_{t_{j-1}^{(n)}}^{t_j^{(n)}} (X_s - X_{t_{j-1}^{(n)}})^2\, ds\right].$$

Now we claim that continuity in mean square on the compact set $[0, T]$ implies uniform continuity in mean square. Assuming this claim, we can choose n large enough so that $\mathbb{E}[(X_s - X_{t_{j-1}^{(n)}})^2]$ is smaller than say ϵ for all j. For n at least this large we have

$$\mathbb{E}\left[\left(\int_0^t (X_s - \phi_s^{(n)})\, dB_s\right)^2\right] \leq \sum_j (t_j^{(n)} - t_{j-1}^{(n)})\epsilon = \epsilon T.$$

Since the L^2 distance between $\int_0^t \varphi_s^{(n)}\, dB_s$ and $\int_0^t X_s\, dB_s$ can be made arbitrarily small, we conclude that $\int_0^t \phi_s^{(n)}\, dB_s \to \int_0^t X_s\, dB_s$ in L^2. □