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270B - Homework 4

Problem 1. Let (X_n) be an irreducible recurrent Markov chain with doubly-infinite transition matrix P. Let $\psi : \mathbb{N} \to \mathbb{N}$ be a bounded function satisfying

$$\sum_{j=1}^{\infty} P_{ij}\psi(j) = \psi(i) \quad \text{for all } i \in \mathbb{N}.$$

Show that ψ is a constant function.

Proof. First we claim that $\psi(X_n)$ is a martingale. Let $\mathcal{F}_n$ be the filtration generated by $X_1, \ldots, X_n$. We then have by hypothesis

$$\mathbb{E}[\psi(X_{n+1})|\mathcal{F}_n] = \sum_j P_{X_n,j}\psi(j) = \psi(X_n),$$

so $\psi(X_n)$ is a martingale with respect to this filtration. We showed on a previous homework assignment that bounded martingales converge almost surely, so $\psi(X_n) \to s$ almost surely for some $s \in \mathbb{N}$.

Suppose ψ is non-constant, so $\psi(i) = u$ and $\psi(j) = v$ for some $i \neq j$, $u \neq v$. Intuitively, since the Markov chain is irreducible and recurrent, it should return to states i and j infinitely often with probability 1. But since $\psi(i) \neq \psi(j)$, this means that $\psi(X_n)$ cannot converge almost surely. □

Problem 2. Let S and T be stopping times with respect to a filtration $(\mathcal{F}_n)$. Denote by $(\mathcal{F}_T)$ the collection of events F such that $F \cap \{T \leq n\} \in \mathcal{F}_n$ for all n.

(a) Show that $\mathcal{F}_T$ is a σ-algebra.

Proof. That $\emptyset$ and Ω are in $\mathcal{F}_T$ immediately follows from T being a stopping time. To show closure under complementation, write $F \in \mathcal{F}_T$ like so

$$F = (F \cap \{T \leq n\}) \cup (F \cap \{T > n\}).$$

This gives

$$F^c \cap \{T \leq n\} = (F \cap \{T \leq n\})^c \cap (F \cap \{T > n\})^c \cap \{T \leq n\}.$$

Since $F \cap \{T \leq n\}$ is in $\mathcal{F}_n$, the above set is also in $\mathcal{F}_n$.

As for countable unions, let $F_1, F_2, \ldots \in \mathcal{F}_T$. Then

$$\left(\bigcup_{k=1}^{\infty} F_k\right) \cap \{T \leq n\} = \bigcup_{k=1}^{\infty} (F_k \cap \{T \leq n\}).$$

Since $F_k \cap \{T \leq n\} \in \mathcal{F}_n$ for all n, the above set is in $\mathcal{F}_n$. □

(b) Show that T is measurable with respect to $\mathcal{F}_T$.

Proof. T is measurable with respect to $\mathcal{F}_T$ if and only if $\{T \leq n\} \in \mathcal{F}_T$ for all n. This follows immediately from the fact that T is a stopping time with respect to the filtration $\mathcal{F}_n$. $\square$

(c) If $E \in \mathcal{F}_S$, show that $E \cap \{S \leq T\} \in \mathcal{F}_T$.

Proof. The idea is to write $\{S \leq T\}$ as $\cup_{k=1}^{\infty}\{T = k\} \cap \{S \leq k\}$. For any $E \in \mathcal{F}_S$ we then have

$$
\begin{aligned}
(E \cap \{S \leq T\}) \cap \{T \leq n\} &= \bigcup_{k=1}^{\infty} (E \cap \{S \leq k\} \cap \{T = k\} \cap \{T \leq n\}) \\
&= \bigcup_{k=1}^{n} (E \cap \{S \leq k\} \cap \{T = k\}).
\end{aligned}
$$

Since $E \in \mathcal{F}_S$, $E \cap \{S \leq k\} \in \mathcal{F}_k$ for all k. Since T is a stopping time with respect to $\mathcal{F}_n$, $\{T = k\} \in \mathcal{F}_k$ for all k. Consequently, the above union is in $\mathcal{F}_n$, so $E \cap \{S \leq T\} \in \mathcal{F}_T$. $\square$

(d) Show that if $S \leq T$ a.s. then $\mathcal{F}_S \subset \mathcal{F}_T$.

Proof. Suppose $F \in \mathcal{F}_S$. We then have

$$F \cap \{T \leq n\} = (F \cap \{S \leq n\}) \cap \{T \leq n\} \in \mathcal{F}_n,$$

so $F \in \mathcal{F}_T$. $\square$

Problem 3. Let (X_n) be a uniformly bounded [integrable, right?] martingale with respect to the filtration $(\mathcal{F}_n)$. Let S and T be two stopping times satisfying $S \leq T$ a.s. Prove that

$$X_T = \mathbb{E}[X|\mathcal{F}_T] \quad \text{and} \quad X_S = \mathbb{E}[X_T|\mathcal{F}_S]$$

where X is the almost sure limit of X_n.

Proof. Since X_n is uniformly integrable, we can reconstruct X_n from its a.s. (and L^1) limit:

$$X_n = \mathbb{E}[X|\mathcal{F}_n].$$

Now for any $F \in \mathcal{F}_T$ we have $F \cap \{T = n\} \in \mathcal{F}_n$ for all n.

$$\int_F X_T \, d\mathbb{P} = \sum_{n=0}^{\infty} \int_{F \cap \{T=n\}} X_T \, d\mathbb{P} = \sum_{n=0}^{\infty} \int_{F \cap \{T=n\}} X_n \, d\mathbb{P}.$$

By the definition of conditional expectation, the last integral above is equal to

$$\sum_{n=1}^{\infty} \int_{F \cap \{T=n\}} X \, d\mathbb{P} = \int_F X \, d\mathbb{P}.$$

Again, by the definition of conditional expectation, we have $X_T = \mathbb{E}[X|\mathcal{F}_T]$.

By problem 2(d), we have that since $S \le T$ a.s., $\mathcal{F}_S \subset \mathcal{F}_T$. Consequently, we have by the law of total expectation

$$\mathbb{E}[X_T|\mathcal{F}_S] = \mathbb{E}[\mathbb{E}[X|\mathcal{F}_T] \mid \mathcal{F}_S] = \mathbb{E}[X|\mathcal{F}_S] = X_S.$$

□

Problem 4. A die is rolled repeatedly. Which of the following are Markov chains? For those that are, compute the transition matrix.

(a) The largest number X_n shown up to the n-th roll.

Solution. Intuitively, this should be a Markov chain: to check if the current roll is the largest thus far, we need only compare it to the largest roll seen before. More concretely, consider $1 \le i_1, i_2, \ldots, i_n \le 6$. Then

$$\mathbb{P}[X_{n+1} = i_{n+1} \mid X_n = i_n, \ldots, X_1 = i_1] = \begin{cases} i_n/6 & \text{if } i_{n+1} = i_n \\ 1/6 & \text{if } i_{n+1} > i_n \\ 0 & \text{otherwise} \end{cases} = \mathbb{P}[X_{n+1} = i_{n+1} \mid X_n = i_n].$$

The state space of this Markov chain is $\{1, 2, 3, 4, 5, 6\}$ and the transition matrix $P_{i,j}$ is given by

$$P_{i,j} = \begin{cases} i/6 & \text{if } j = i \\ 1/6 & \text{if } j > i \\ 0 & \text{otherwise} \end{cases}$$

□

(b) The number N_n of sixes in n rolls.

Solution. N_n is a Markov chain: the number of sixes on the $n+1$-st roll will either be the same as or one greater than the number of sixes on the n-th roll. N_{n+1} will only depend on N_n since the dice rolls are independent. The state space is the set of nonnegative integers and the transition matrix is given by

$$P_{i,j} = \begin{cases} 5/6 & \text{if } j = i \\ 1/6 & \text{if } j = i + 1 \\ 0 & \text{otherwise} \end{cases}.$$

□

(c) At time r, the time C_r since the most recent six.

Solution. If E_n is the even that the n-th roll is a six, then we have $C_{n+1} = C_n + \mathbb{1}_{E_{n+1}}$. Since the $(n+1)$-st roll is independent of the previous states of the process, we see that C_n is a Markov chain. The state space is the set of nonnegative integers and its transition matrix is given by

$$P_{i,j} = \begin{cases} 1/6 & \text{if } j = i+1 \\ 5/6 & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}.$$

□

(d) At time r, the time B_r until the next six.

Solution. If $B_r > 1$, then $B_{r+1} = B_r - 1$ almost surely - if the next six will come in $B_r > 1$ rolls, then on the next roll it will come in $B_r - 1$ rolls. Now if $B_r = 1$, then the process "restarts" and the time until the next six will follow a geometric distribution with parameter 1/6, i.e. $\mathbb{P}[B_{r+1} = j] = \frac{1}{6}(\frac{5}{6})^{j-1}$. In either case, the distribution of B_{r+1} depends only on the value of B_r and we have a Markov chain. The transition matrix is given by

$$P_{i,j} = \begin{cases} 1 & \text{if } i > 1,\ j = i-1 \\ \frac{1}{6}(\frac{5}{6})^{j-1} & \text{if } i = 1,\ j \geq 1 \\ 0 & \text{otherwise} \end{cases}.$$

□

Problem 5. Let S_n be a simple random walk starting at $S_0 = 0$. Show that $X_n = |S_n|$ is a Markov chain.

Proof. If $|S_n| = i_n$, then $S_n = i_n$ or $S_n = -i_n$. We then have for $i_n > 0$

$$\begin{aligned} \mathbb{P}[|S_{n+1}| = i_{n+1} \mid |S_n| = i_n, \ldots, |S_1| = i_1] &= \mathbb{P}[S_{n+1} = i_{n+1} \mid S_n = i_n, \ldots, |S_1| = i_1] \\ &\quad + \mathbb{P}[S_{n+1} = -i_{n+1} \mid S_n = -i_n, \ldots, |S_1| = i_1] \end{aligned}$$

□

Problem 6. Let (X_n) be a Markov chain, and let T be a stopping time with respect to the filtration $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. Show that

$$\mathbb{P}[X_{T+1} = j \mid X_k = x_k \text{ for } 0 \leq k < T,\ X_T = i] = \mathbb{P}[X_{T+1} = j \mid X_T = i]$$

for all $m \geq 0$ i, j and x_k.

Proof. For ease of notation, let $E_m = \{X_k = x_k \text{ for } 0 \le k \le m\}$. We then have

$$\mathbb{P}[X_{T+1} = j \mid E_T] = \sum_{n=1}^{\infty} \mathbb{P}[X_{T+1} = j \mid E_T,\ T = n] \cdot \mathbb{P}[T = n \mid E_T]. \tag{1}$$

By the definition of conditional probability we have

$$\mathbb{P}[X_{T+1} = j \mid E_T,\ T = n] = \frac{\mathbb{P}[X_{T+1} = j,\ E_T,\ T = n]}{\mathbb{P}[E_T,\ T = n]} = \frac{\mathbb{P}[X_{n+1} = j,\ E_n,\ T = n]}{\mathbb{P}[E_n,\ T = n]}.$$

Now since T is a stopping time, it is $\mathcal{F}_n$ measurable. I want to say that $E_n \cap \{T = n\} = E_n$. This gives, by the Markov property

$$\mathbb{P}[X_{T+1} = j \mid E_T,\ T = n] = \frac{\mathbb{P}[X_{n+1} = j, E_n]}{\mathbb{P}[E_n]} = \mathbb{P}[X_{n+1} = j \mid E_n] = \mathbb{P}[X_{n+1} = j \mid X_n = i_n].$$

Combining this with (1), and using the presumed stationarity of the chain, we have

$$\mathbb{P}[X_{T+1} = j \mid E_T] = \mathbb{P}[X_{n+1} = j \mid X_n = i_n] = \mathbb{P}[X_{T+1} = j \mid X_T = i_T].$$

□

Problem 7. Find an example of two Markov chains (X_n) and (Y_n) such that $X_n + Y_n$ is not a Markov chain.

Solution. Let X_n be a simple symmetric random walk on $\mathbb{Z}$ that starts at the origin and let Y_n be given by

$$Y_n = \begin{cases} 1 & \text{if } X_1 = 1 \\ 0 & \text{if } X_1 = -1 \end{cases}.$$

We've established that X_n is a Markov chain. Since Y_n is constant, it is also a Markov chain. However, we claim that $Z_n = X_n + Y_n$ is not a Markov chain. Intuitively, the steps Z_n can take to get to Z_{n+1} depend on the first step, Z_0. Concretely, consider $\mathbb{P}[Z_5 = 3 \mid Z_4 = 2]$. Z_4 could've reached 2 in several different ways, e.g.: $-1, +1, +1, +1$ or $+2, 0, 0, 0$. If $Z_1 = -1$, then the probability that $Z_5 = 3$ is $1/2$. On the other hand, if $Z_1 = 2$, then the probability that $Z_5 = 3$ is 0. □

Problem 8. A particle performs a random walk on the vertices of a three-dimensional cube. At each step it remains where it is with probability $1/4$ or moves to one of its neighboring vertices each having probability $1/4$. Compute the mean number of steps until the particle returns to the vertex from which the walk started.

Solution. Suppose that the particle starts at vertex u. Let μ_u be the expected return time. At each step the particle can stay put with positive probability, so the Markov chain is aperiodic. Furthermore, the cube is connected and finite, so the Markov chain is also irreducible. This chain then admits a stationary distribution π. We also know that $\mu_u = \frac{1}{\pi_u}$. The uniform measure $\pi_u = \frac{1}{8}$ satisfies the detailed balance equations: for u and v adjacent (note that u is adjacent to itself here)

$$\pi_u P_{u,v} = \pi_v P_{v,u} \iff \frac{1}{8} \cdot \frac{1}{4} = \frac{1}{8} \cdot \frac{1}{4}.$$

We conclude that the mean number of steps until the particle returns to its starting location is 8. □

Problem 9. Prove that the symmetric random walk on $\mathbb{Z}^2$ is recurrent while the symmetric random walk on $\mathbb{Z}^3$ is transient.

Proof. $\square$