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270B - Homework 4

Problem 1. Let (X_n) be an irreducible recurrent Markov chain with doubly-infinite transition matrix P. Let $\psi : \mathbb{N} \to \mathbb{N}$ be a bounded function satisfying

$$\sum_{j=1}^{\infty} P_{ij}\psi(j) = \psi(i) \quad \text{for all } i \in \mathbb{N}.$$

Show that ψ is a constant function.

Proof. First we claim that $\psi(X_n)$ is a martingale. Let $\mathcal{F}_n$ be the filtration generated by $X_1, \ldots, X_n$. We then have by hypothesis

$$\mathbb{E}[\psi(X_{n+1})|\mathcal{F}_n] = \mathbb{E}\left[\sum_j P_{X_{n+1},j}\psi(j) \;\middle|\; \mathcal{F}_n\right] = \sum_j P_{X_n,j}\psi(j) = \psi(X_n).$$

□

Problem 2. Let S and T be stopping times with respect to a filtration $(\mathcal{F}_n)$. Denote by $(\mathcal{F}_T)$ the collection of events F such that $F \cap \{T \leq n\} \in \mathcal{F}_n$ for all n.

(a) Show that $\mathcal{F}_T$ is a σ-algebra.

Proof. That $\emptyset$ and Ω are in $\mathcal{F}_T$ immediately follows from T being a stopping time. If $F \cap \{T \leq n\} \in \mathcal{F}_n$ then

$$F^c \cap \{T \leq n\} = (F \cup \{T > n\})^c \in \mathcal{F}_n,$$

since $\mathcal{F}_n$ is a σ-algebra. □