Math 175 - Graph Theory Exercises

1. (a) Show that every tree with maximum degree k has at least k leaves.

   (b) Which such trees have exactly k leaves?

2. A saturated hydrocarbon is a molecule C_mH_n in which every carbon atom C has four bonds, every hydrogen atom H has one bond, and no sequence of bonds forms a cycle. Show that if C_mH_n is a saturated hydrocarbon, then $n = 2m + 2$.

3. A **center** of a graph $G = (V, E)$ is a vertex u such that $\max\{d(u, v) : v \in V\}$ is as small as possible, where $d(u, v)$ is the length of a shortest path connecting u and v if such a path exists and ∞ otherwise.

   (a) Let T be a tree on at least three vertices and let T' be the tree obtained from T by deleting all of its leaves. Show that T and T' have the same centers.

   (b) Deduce that every tree has either exactly one center or two adjacent centers.

4. Show that at any party with $n \geq 2$ people, there are two people who know the same number of people.

5. Show that in any graph G, the distance function is a **metric**. That is, show that

   (a) $d(u, v) = 0$ if and only if $u = v$.

   (b) $d(u, v) = d(v, u)$.

   (c) $d(u, v) + d(v, w) \geq d(u, w)$.

6. An edge e of a graph G is a **cut edge** if its deletion causes the number of connected components to increase. That is, e is a cut edge of G if $G \setminus e$ has one more connected component than G. Show that e is a cut edge of G if and only if e belongs to no cycle of G.