HomeWork 4 Math 271B, Winter 2020.

1. Consider

$$\beta_t^{(1)}\beta_t^{(2)} + \exp(t) - t_0 + (\beta_t^{(3)})^2.$$

Find the stochastic differential equation satisfied by this process.

2. Consider

$$dZ_t = \mu_t dt + \sum_{i=1}^{n} \theta_t^{(i)} d\beta_t^{(i)}, \quad Z_0 = z_0,$$

for β n-dimensional Brownian motion and θ a bounded deterministic n-dimensional vector process and μ a bounded deterministic process (all progressively measurable).

a) Use Kolmogorov forward equation to derive the density of Z_T.

b) Use Kolmogorov backward equation to compute $\mathbb{E}[Z_T \mathcal{I}_{Z_T > K}]$ for K a deterministic constant.

3. Show that

$$\int_0^t \beta_s \circ d\beta_s = \beta_t^2/2,$$

when $\circ$ means the Stratonovich interpretation of the stochastic integral by going back to the definition of the stochastic integral.

4. Recall from class

$$[f] = \sum_{n=1}^{\infty} 2^{-n} \left(1 \wedge \sqrt{\mathbb{E}[\int_0^n f_s^2 ds]}\right)$$

$$\|X\| = \sum_{n=1}^{\infty} 2^{-n} \left(1 \wedge \sqrt{\mathbb{E}[X_n^2 ds]}\right).$$

Show that these are metrics on appropriate spaces.

5. Consider

$$Z_t = \exp(\int_0^t \theta_s \cdot \beta_s - \int_0^t |\theta|_s^2 ds/2),$$

for β n-dimensional Brownian motion and θ a bounded n-dimensional progressively measurable vector process.

a) Show that Z is a martingale.

b) Assume that θ is independent of β. Derive an expression for the variance if Z_T.

6. Consider (scalar processes)

$$dZ_t = \mu_t dt + d\beta_t,$$

for μ_t bounded and progressively measurable. Find X_t so that $Z_t X_t$ is a martingale. (Hint: look at processes of the form in the previous problem).