Liam Hardiman *January 29, 2020 (Late)*

270B - Homework 1

Problem 1. Let X_i be i.i.d. random variables each having the Poisson distribution with mean 1, and consider $S_n = X_1 + \cdots + X_n$. Let $x \in \mathbb{R}$. Show that if $k = k(n)$ is such that $(k-n)/\sqrt{n} \to x$ as $n \to \infty$, we have

$$\sqrt{2\pi n}\mathbb{P}[S_n = k] \to \exp(-x^2/2).$$

Proof. First we claim that S_n has Poisson distribution with mean n. To see this, observe that by independence we have

$$\varphi_{S_n}(t) = \mathbb{E}\left[e^{it(X_1+\cdots+X_n)}\right] = \mathbb{E}\left[e^{itX_1}\right]\cdots\mathbb{E}\left[e^{itX_n}\right] = \varphi_1(t)\cdots\varphi_n(t), \tag{1}$$

where φ_j is the characteristic function of X_j. Now if the random variable X has Poisson distribution with intensity λ, its characteristic function is given by

$$\mathbb{E}\left[e^{itX}\right] = \sum_{k=0}^{\infty} e^{itk} \cdot \frac{\lambda^k e^{-\lambda}}{k!} = \exp(\lambda(e^{it}-1)).$$

Using this, we see that

$$\varphi_{S_n}(t) = \exp(e^{it}-1)^n = \exp(n(e^{it}-1)),$$

which is the characteristic function of the Poisson with intensity λ. Since a distribution is determined by its characteristic function, we conclude that S_n has Poisson distribution with intensity λ. $\square$