Math 180B - More on Pell's Equation

1. (a) For each $2 \leq D \leq 15$ (pick a few) that is not a perfect square, determine whether or not the equation $x^2 - Dy^2 = -1$ has a solution in positive integers. Can you determine a pattern that lets you predict for which D's it has a solution?

   (b) If (x_0, y_0) is a solution to $x^2 - Dy^2 = -1$ in positive itnegers, show that $(x_0^2 + Dy_0^2, 2x_0y_0)$ is a solution to Pell's equation $x^2 - Dy^2 = 1$.

   (c) Find a solution to $x^2 - 41y^2 = -1$ by plugging in values for y until you get a perfect square. Use your answer to (b) to find a solution to Pell's equation $x^2 - 41y^2 = 1$ in positive integers.

   (d) If (x_0, y_0) is a solution to the equation $x^2 - Dy^2 = M$, and if (x_1, y_1) is a solution to Pell's equation $x^2 - Dy^2 = 1$, show that $(x_0x_1 + Dy_0y_1, x_0y_1 + y_0x_1)$ is also a solution to the equation $x^2 - Dy^2 = M$. Use this to find five different solutions in positive integers to the equation $x^2 - 2y^2 = 7$.

2. If x_1, y_1 is the fundamental solution of $x^2 - Dy^2 = 1$ and

$$x_n + y_n\sqrt{D} = (x_1 + y_1\sqrt{D})^n$$

prove that the pair of integers x_n, y_n can be calculated from the formulas

$$x_n = \frac{1}{2}\left[\left(x_1 + y_1\sqrt{D}\right)^n + \left(x_1 - y_1\sqrt{D}\right)^n\right]$$

$$y_n = \frac{1}{2\sqrt{D}}\left[\left(x_1 + y_1\sqrt{D}\right)^n - \left(x_1 - y_1\sqrt{D}\right)^n\right].$$

3. Show that the equation $y^2 = x^3 + xz^4$ has no solutions in nonzero integers x, y, z. *Hint: First show that it can be reduced to a solution satisfying $(x, z) = 1$. Then use the fact that $x^3 + xz^4 = x(x^2 + x^4)$ is a perfect square to show that there are no solutions other than $x = y = 0$.*