Math 175 - Week 2, Discussion 1

1. What's wrong with this proof?

   Claim. *$n(n+1)$ is an odd number for every natural n.*

   Proof. Suppose this is true for n; we prove it for $n+1$. Some quick algebra shows that

   $$(n+1)(n+2) = n(n+1) + 2(n+1).$$

   By the induction hypothesis, $n(n+1)$ is odd. Since $2(n+1)$ is clearly even, we must have that $(n+1)(n+2)$ is odd. $\square$

2. What's wrong with this proof?

   Claim. *If we have n lines in the plane, no two of which are parallel, then they all go through one point.*

   Proof. This is clearly true for one or two lines by definition. Suppose that it is true for any set of n lines and let $S = \{\ell_1, \ell_2, \ell_3, \ell_4, \ldots, \ell_{n+1}\}$ be a set of $n+1$ lines in the plane, no two of which are parallel. Delete the line ℓ_3 to obtain a set S' of n lines, no two of which are parallel. By the induction hypothesis, the lines in S' must all pass through some point P. In particular, ℓ_1 and ℓ_2 pass through P.

   Now put ℓ_3 back and delete ℓ_4 instead to get another set S'' of n lines, no two of which are parallel. Again by the induction hypothesis, they must all pass through some point Q. In particular, ℓ_1 and ℓ_2 pass through Q. But ℓ_1 and ℓ_2 pass through P. Since two lines can pass through at most one point, we must have $P = Q$. But then ℓ_3 goes through P, so all the lines in S go through P. $\square$

3. What is the following sum? Prove your guess by induction.

   $$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(n-1)n}.$$

4. What is the following sum? Give an inductive proof and a combinatorial proof.

   $$0\cdot\binom{n}{0} + 1\cdot\binom{n}{1} + 2\cdot\binom{n}{2} + \cdots + (n-1)\cdot\binom{n}{n-1} + n\cdot\binom{n}{n}$$

5. Prove the following identity. Try to give a combinatorial proof.

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

6. Prove the following identity.

$$\binom{2}{2} + \binom{3}{2} + \cdots + \binom{n}{2} = \binom{n+1}{3}.$$

7. We select 38 even positive integers, all less than 1000. Prove that there will be two of them whose difference is at most 26.

8. Prove that in a group of n people, at least two will have the same number of friends in that group.

9. Let S be a finite set. Prove that $f : S \to S$ is injective (one-to-one) if and only if it is surjective (onto).

10. (If you've seen 120B) Prove that any finite integral domain is a field.

11. (Hard) Every element of the infinite grid $\mathbb{Z} \times \mathbb{Z}$ is colored red, green, or blue. Prove that there is a rectangle whose corners are all the same color.