3. Recall that the Farey sequence of order n, $\mathbf{F}_n$, is the sequence of reduced fractions $0 \leq \frac{a}{b} \leq 1$ arranged in increasing order. Prove that the number of terms in $\mathbf{F}_n$ is $1 + \sum_{j=1}^{n} \phi(j)$ and that their sum is exactly half this value. Using this, come up with a recursive formula for $|\mathbf{F}_n|$.

$1 \leq b \leq n$

$$|F_n| = 1 + \sum_{j=1}^{n} \phi(j)$$

$$|F_{n+1}| = 1 + \sum_{j=1}^{n+1} \phi(j)$$

$$= 1 + \sum_{j=1}^{n} \phi(j) + \phi(n+1)$$

$$= |F_n| + \phi(n+1)$$

1. Show that if d is divisible by a prime congruent to 3 mod 4 then $x^2 - dy^2 = -1$ has no solutions in integers.

say $p \mid d$, $p \equiv 3 \mod 4$.

Suppose $x^2 - dy^2 = -1$ has solution (x_0, y_0)

look mod $p \Rightarrow x_0^2 \equiv -1 \mod p$

$\Rightarrow \left(\frac{-1}{p}\right)$ contradiction since $\left(\frac{-1}{p}\right) = 1$ iff $p \equiv 1 \mod 4$ or $p = 2$.

real

2. Prove that there exist irrational numbers α and β such that α^β is rational.

we know $\sqrt{2} \notin \mathbb{Q}$

- Consider $\sqrt{2}^{\sqrt{2}}$. if it's rational, done!
- if $\sqrt{2}^{\sqrt{2}} \notin \mathbb{Q}$

consider $\underbrace{\left(\sqrt{2}^{\sqrt{2}}\right)}_{\notin \mathbb{Q}}^{\overbrace{\sqrt{2}}^{\notin \mathbb{Q}}} = \sqrt{2}^2 = 2 \in \mathbb{Q}$.

α be alg deg n / $\mathbb{Q}$

$\sqrt{2}^{2/e}$ if this $\in \mathbb{Q}$, done.

otherwise, consider

$$\left(\sqrt{2}^{2/e}\right)^e = \sqrt{2}^2 = 2.$$

3. Recall that the Farey sequence of order n, $\mathbf{F}_n$, is the sequence of reduced fractions $0 \leq \frac{a}{b} \leq 1$ arranged in increasing order. Prove that the number of terms in $\mathbf{F}_n$ is $1 + \sum_{j=1}^{n} \phi(j)$ and that their sum is exactly half this value. Using this, come up with a recursive formula for $|\mathbf{F}_n|$.

$b \leq n$

a) count the # fractions w/ denominator j.

$1 \leq j \leq n$

- Let $a/j \in F_n$. Since a/j is reduced.

$\Rightarrow (a, j) = 1$

$a \leq j$

- if $(a,j) = 1$, $a/j \in F_n$

$\Rightarrow \varphi(j)$ fractions in F_n w/ denom j

but $0 \in F_n$ too, so $|F_n|$

$$= 1 + \sum_{j=1}^{n} \varphi(j)$$

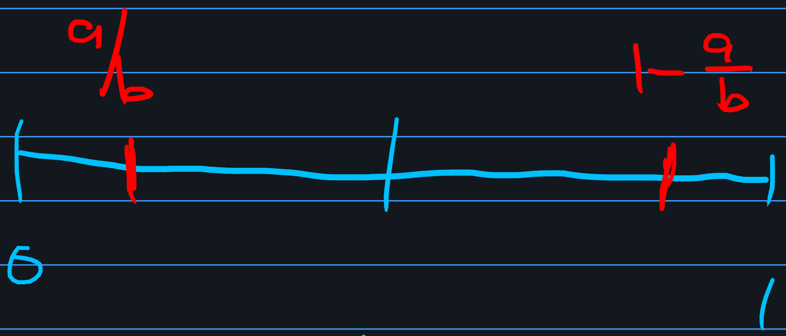
$\boxtimes$

. wts $\displaystyle\sum_{\frac{a}{b}\in F_n} \frac{a}{b} = \frac{1}{2}\left(1+\sum_{j=1}^{n}\varphi(j)\right)$

$= \frac{1}{2}|F_n|$

▷ Suppose $\frac{a}{b} \in F_n$

consider $1-\frac{a}{b} = \frac{b-a}{b}$



$\frac{b-a}{b} \in F_n$ too. need $(b-a, b) = 1$

$\gcd(b-a, b) = \gcd(a, b) = 1$ ✓

note that $\frac{a}{b} + \left(1-\frac{a}{b}\right) = 1$

idea: group the pairs $\frac{a}{b}$ & $1-\frac{a}{b}$.

if $\frac{a}{b} = 1-\frac{a}{b} = \frac{b-a}{b}$

$\Rightarrow a = b-a \Rightarrow 2a = b$

$\gcd(a,b) = 1 \Rightarrow b = 2, a = 1$

$\Rightarrow \frac{a}{b} = 1/2$

term corresponding to $\frac{1}{2}$

$1 + \sum_{j=1}^{n} \varphi(j)$ terms in total.

$\frac{1}{2} \sum_{j=1}^{n} \varphi(j)$ pairs that sum to 1.

$\Rightarrow$ total sum is

$$\underline{\frac{1}{2} \sum_{j=1}^{n} \varphi(j)} + \frac{1}{2}$$

$$= \frac{1}{2} \left(\sum_{j=1}^{n} \varphi(j) + 1 \right) \quad \boxtimes$$

$$= \frac{1}{2} |F_n|$$

4. idea: if $\frac{a}{b} < \frac{c}{d}$ are consecutive

terms in F_n $\Rightarrow$ $\underline{bc - ad} = 1$

$$\sum_{k=1}^{n-1} \frac{1}{b_k b_{k+1}} .$$

$$\frac{a_{k+1}}{b_{k+1}} - \frac{a_k}{b_k} = \frac{a_{k+1} b_k - a_k b_{k+1}}{b_{k+1} b_k} = \frac{1}{b_{k+1} b_k}$$

$$\|$$

$$\sum_{k=1}^{n-1} \left(\frac{a_{k+1}}{b_{k+1}} - \frac{a_k}{b_k} \right)$$

$$\|$$

$$1 - 0 = 1$$