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270A - Homework 1

1. (a) Let $\mathcal{F}$ be the family of all finite subsets of Ω and their complements. Is $\mathcal{F}$ a σ-algebra?

   Solution. If Ω is finite then $\mathcal{F}$ is simply the power set of Ω, which is definitely a σ-algebra. However, if $\mathcal{F}$ is infinite, then $\mathcal{F}$ is never a σ-algebra. To see this, let (x_n) be a countable sequence of distinct elements in Ω and consider the set of even-indexed terms

   $$F = \{x_n : n = 2k,\ k \in \mathbb{N}\}.$$

   This set is a countable union of singletons and all singletons belong to $\mathcal{F}$. F is clearly infinite, but so is its complement, which contains the (infinite) set of odd-indexed terms. We conclude that F is neither finite nor co-finite, so $\mathcal{F}$ is not closed under countable unions when Ω is an infinite set. $\square$

   (b) Let $\mathcal{F}$ be the family of all countable subsets of Ω and their complements. Is $\mathcal{F}$ a σ-algebra?

   Solution. $\mathcal{F}$ is indeed a σ-algebra. The empty set is clearly countable, and $\Omega^C = \emptyset$. Let F_n be a countable collection of sets in $\mathcal{F}$ and consider their union, $F = \cup_{n=1}^{\infty} F_n$. If each F_n is countable, then F is just a countable union of countable sets: countable. If one of the F_n's, say F_k, were co-countable, then $F^C \subseteq F_k^C$, which is countable, so F is co-countable. Since $\mathcal{F}$ contains the empty set and Ω and is closed under countable unions and complements, it is a σ-algebra. $\square$

   (c) Let $\mathcal{F}$ and $\mathcal{G}$ be two σ-algebras of subsets of Ω. Is $\mathcal{F} \cap \mathcal{G}$ always a σ-algebra?

   Solution. $\mathcal{F}$ is a σ-algebra. Since $\mathcal{F}$ and $\mathcal{G}$ both contain $\emptyset$ and Ω, so does their intersection. Let E_n be a countable collection of sets in $\mathcal{F} \cap \mathcal{G}$. Since $\mathcal{F}$ and $\mathcal{G}$ are both σ-algebras, the union $E = \cup_{n=1}^{\infty} E_n$ is in both $\mathcal{F}$ and $\mathcal{G}$ and each E_n^C is in both $\mathcal{F}$ and $\mathcal{G}$ as well. $\square$

   (d) Let $\mathcal{F}$ and $\mathcal{G}$ be two σ-algebras of subsets of Ω. Is $\mathcal{F} \cup \mathcal{G}$ always a σ-algebra?

   Solution. The union need not be a σ-algebra. Let $\Omega = \{1, 2, 3, 4\}$, $\mathcal{F} = \{\emptyset, \Omega, \{1\}, \{2, 3, 4\}\}$, and $\mathcal{G} = \{\emptyset, \Omega, \{2\}, \{1, 3, 4\}\}$. $\mathcal{F}$ and $\mathcal{G}$ are σ-algebras, but the set $\{1\} \cup \{2\} = \{1, 2\}$ is not in their union. $\square$

2. A subset $A \subset \mathbb{N}$ is said to have asymptotic density if

   $$\lim_{n\to\infty} \frac{|A \cap \{1, \dots, n\}|}{n}$$

   exists. Let $\mathcal{F}$ be the collection of subsets of $\mathbb{N}$ for which the asymptotic density exists. Is $\mathcal{F}$ a σ-algebra?

Solution. $\mathcal{F}$ is not a σ-algebra. First let's construct a set not in $\mathcal{F}$. The idea is to build a set that has long gaps followed by even longer "runs". Let $F_0 = \{1\}$ and $F_i = \{2^i, \dots, 2^{i+1} - 1\}$. Define the set A by $A = \cup_{j=0}^{\infty} F_{2j}$. A consists of a run of length 2^{2j} followed by a gap of length 2^{2j+1} for each $j = 0, 1, \dots$. Our set A does not have asymptotic density since

$$\begin{aligned} \frac{|A \cap [2^{2k}]|}{2^{2k}} &= \frac{\sum_{j=0}^{k-1} 2^{2j} + 1}{2^{2k}} \\ &= \frac{1}{3} \end{aligned}$$

while on the other hand,

$$\begin{aligned} \frac{|A \cap [2^{2k+1}]|}{2^{2k+1}} &= \frac{\sum_{j=0}^{k} 2^{2j}}{2^{2k+1}} \\ &= \frac{1}{3}\left(2 - \frac{1}{2^{2k+1}}\right) \\ &\to \frac{2}{3}. \end{aligned}$$

Hence, A is not in $\mathcal{F}$. Since A is a countable union of singletons, which clearly have asymptotic density zero, we conclude that $\mathcal{F}$ is not a σ-algebra.

□

3. Let X and Y be two random variables on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $E \in \mathcal{F}$ be an event. Define

$$Z = \begin{cases} X & \text{if } E \text{ occurs} \\ Y & \text{otherwise.} \end{cases}$$

Prove that Z is a random variable.

Proof. We can write $Z = X \cdot \mathbb{1}_E + Y \cdot \mathbb{1}_{E^c}$. Since E is an event, the indicator functions $\mathbb{1}_E$ and $\mathbb{1}_{E^c}$ are measurable. Since X and Y are measurable and products and sums of measurable functions are measurable, we have that Z is measurable, and hence a random variable. □

4. Let X be a random variable with density f. Compute the density of X^2.

Solution. First let's compute the distribution of X^2. Let $t \geq 0$.

$$\begin{aligned} \mathbb{P}[X^2 < t] &= \mathbb{P}[-\sqrt{t} < X < \sqrt{t}] \\ &= \int_{-\sqrt{t}}^{\sqrt{t}} f(s)\ ds. \end{aligned}$$

By the Lebesgue differentiation theorem, the above integral is an almost everywhere differentiable function of t and we can apply the fundamental theorem of calculus. If we let g be the density of X^2 then

$$\begin{aligned} g(t) &= \frac{d}{dt}\int_{-\sqrt{t}}^{\sqrt{t}} f(s)\ ds \\ &= \frac{1}{2\sqrt{t}}[f(\sqrt{t}) + f(-\sqrt{t})], \end{aligned}$$

for $t \geq 0$. Since X^2 is clearly nonnegative, we then have

$$g(t) = \begin{cases} \frac{1}{2\sqrt{t}}[f(\sqrt{t}) + f(-\sqrt{t})], & t > 0 \\ 0, & t \leq 0. \end{cases}$$

□

5. Let X be a nonnegative random variable. Show that

$$\mathbb{E}[X] = \int_0^\infty \mathbb{P}[X > t]\ dt.$$

Proof. Since X is nonnegative (this is important – the Lebesgue integral is orientation-independent, unlike the Riemann integral!),

$$X = \int_0^X dt.$$

We can then take the expectation of both sides and apply Fubini's theorem.

$$\begin{aligned} \mathbb{E}[X] &= \int_\Omega \int_0^X dt\ d\mathbb{P} \\ &= \int_\Omega \int_0^\infty \mathbb{1}_{X>t}(t)\ dt\ d\mathbb{P} \\ &= \int_0^\infty \int_\Omega \mathbb{1}_{X>t}(x)\ d\mathbb{P}\ dt \\ &= \int_0^\infty \mathbb{P}[X > t]\ dt. \end{aligned}$$

□

6. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a strictly convex function. Let X be a random variable such that $\mathbb{E}[|X|] < \infty$ and $\mathbb{E}[|\varphi(X)|] \leq \infty$. Show that

$$\varphi(\mathbb{E}[X]) = \mathbb{E}[\varphi(X)] \implies X = \mathbb{E}[X] \text{ a.s.}$$

Proof. Since φ is strictly convex, for every $t \in \mathbb{R}$ there exists an affine linear function $F_t(x)$ such that $F_t(t) = \varphi(t)$ and $F_t(x) < \varphi(x)$ for all $x \neq t$. We can set $t = \mathbb{E}[X]$ compose with X to obtain $F_t(X) \leq \varphi(X)$, with equality if and only if $X = \mathbb{E}[X]$. Note that since F_t is affine linear we have that $\mathbb{E}[F_t(X)] = F_t(\mathbb{E}[X])$.

Suppose that $\varphi(\mathbb{E}[X]) = \mathbb{E}[\varphi(X)]$. When $t = \mathbb{E}[X]$, F_t and φ agree at $\mathbb{E}[X]$, so $\varphi(\mathbb{E}[X]) = F_t(\mathbb{E}[X])$, yielding

$$\begin{aligned}\mathbb{E}[\varphi(X)] &= \varphi(\mathbb{E}[X]) \\ &= F_t(\mathbb{E}[X]) \\ &= \mathbb{E}[F_t(X)].\end{aligned}$$

By the linearity of expectation we then have $\mathbb{E}[\varphi(X) - F_t(X)] = 0$. By convexity, $\varphi(X) - F_t(X) \geq 0$, so since this expectation is zero, we must have that $\varphi(X) = F_t(X)$ almost surely. By strict convexity, this implies that $X = t = \mathbb{E}[X]$ almost surely. □

7. Suppose $0 \leq p_n \leq 1$ and put $\alpha_n = \min(p_n, 1 - p_n)$. Show that if $\sum_n \alpha_n$ diverges, then no discrete probability space can contain independent events $A_1, A_2, \ldots$ such that $\mathbb{P}[A_n] = p_n$.

Proof. First let's strengthen the Borel-Cantelli lemma by proving a partial converse. Consider the tail $\cup_{n=M}^{\infty} A_n$ for M large. We compute the probability of the tail's complement using the fact that $1 - x \leq e^{-x}$.

$$\begin{aligned}\mathbb{P}[\cap_{n=M}^{N} A_n^c] &= \prod_{n=M}^{N} (1 - \mathbb{P}[A_n]) \\ &\leq \exp\left(-\sum_{n=M}^{N} \mathbb{P}[A_n]\right) \\ &\to 0 \quad \text{as } N \to \infty.\end{aligned}$$

Since the complement of the tail goes to zero in probability, we have that $\mathbb{P}[\cup_{n=M}^{\infty} A_n] = 1$ for all M. Since $\limsup A_n = \cap_{M=1}^{\infty} \cup_{n=M}^{\infty} A_n$, continuity of measure tells us that $\mathbb{P}[\limsup A_n] = 1$.

Suppose that these events live in a discrete probability space Ω (equipped with the power set σ-algebra). Since Ω is discrete, there must be some $\omega \in \Omega$ with $\mathbb{P}[\{\omega\}] > 0$. Define the sequence of events E_n by

$$E_n = \begin{cases} A_n^c, & \text{if } \omega \in A_n \\ A_n, & \text{if } \omega \notin A_n. \end{cases}$$

In particular, each E_n misses ω and the E_n's are independent. We also have that

$$\sum \mathbb{P}[E_n] \geq \sum \alpha_n = \infty.$$

By our strengthened Borel-Cantelli lemma, $\mathbb{P}[\limsup E_n] = 1$. By discreteness, we must then have that $\Omega = \limsup E_n$. But ω, which has positive probability, isn't in $\limsup E_n$ – a contradiction. We conclude that Ω is not discrete. $\square$

8. Prove that if random variables X and Y are independent, then so are $f(X)$ and $g(Y)$, for any Borel measurable functions $f, g : \mathbb{R} \to \mathbb{R}$.

Proof. We need to show that for any Borel sets $B_1, B_2 \in \mathcal{B}(\mathbb{R})$, the events $(f \circ X)^{-1}[B_1]$ and $(g \circ Y)^{-1}[B_2]$ are independent. We can rewrite these preimages as

$$(f \circ X)^{-1}[B_1] = X^{-1}[f^{-1}[B_1]], \quad (g \circ Y)^{-1}[B_2] = Y^{-1}[g^{-1}[B_2]].$$

Since f and g are measurable, the preimages $f^{-1}[B_1]$ and $g^{-1}[B_2]$ are Borel sets. Similarly, since X and Y are random variables, the preimages $X^{-1}[f^{-1}[B_1]]$ and $Y^{-1}[g^{-1}[B_2]]$ are events. Since X and Y are independent, any preimage under X is independent of any preimage under Y. $\square$

9. Let $p \geq 3$ be prime. Let X and Y be independent random variables that are uniformly distributed on $\{0, \ldots, p-1\}$. Define

$$Z_n = (X + nY) \pmod{p}, \quad n = 0, \ldots, p-1.$$

Show that the random variables Z_n are pairwise independent, but not jointly independent.

Proof. First, we claim it suffices to prove that for all s, t in $S = \{0, \ldots, p-1\}$ we have

$$\mathbb{P}[Z_{n_1} = s,\ Z_{n_2} = t] = \mathbb{P}[Z_{n_1} = s] \cdot \mathbb{P}[Z_{n_2} = t]. \tag{1}$$

Define $\mathcal{A}_1$ and $\mathcal{A}_2$ as follows.

$$\mathcal{A}_i = \{\Omega\} \cup \big\{\{Z_{n_i} = s\} : s \in S\big\}.$$

$\mathcal{A}_i$ is a π-system that contains Ω. Since the singletons generate the power set σ-algebra, we have that $\sigma(\mathcal{A}_i) = \sigma(Z_{n_i})$. Since (1) says that $\mathcal{A}_1$ and $\mathcal{A}_2$ are independent, we have that Z_{n_1} and Z_{n_2} are independent by the $\pi - \lambda$ theorem.

Now let's actually verify (1). Let's start with the right-hand side. Since X and Y are independent, we have

$$\begin{aligned}
\mathbb{P}[Z_{n_1} = s] &= \mathbb{P}[X = s - n_1 Y] \\
&= \sum_{y=0}^{p-1} \mathbb{P}[X = s - n_1 y,\ Y = y] \\
&= \frac{1}{p} \sum_{y=0}^{p-1} \mathbb{P}[X = s - n_1 y] \\
&= \frac{1}{p}.
\end{aligned}$$

The right-hand side of (1) is then $\frac{1}{p^2}$. Now for the joint. If we assume that $n_1 \neq n_2$ then the system

$$\begin{array}{ccccc}
X & + & n_1 Y & = & s \\
X & + & n_2 Y & = & t
\end{array}$$

has a unique solution (α, β) for X, Y since $\mathbb{Z}/p\mathbb{Z}$ is a field. Sine X and Y are independent, the joint probability becomes

$$\begin{aligned}
\mathbb{P}[Z_{n_1} = s,\ Z_{n_2} = t] &= \mathbb{P}[X = \alpha,\ Y = \beta] \\
&= \mathbb{P}[X = \alpha] \cdot \mathbb{P}[Y = \beta] \\
&= \frac{1}{p^2}.
\end{aligned}$$

We have then verified (1), so the Z_{n_i}'s are pairwise independent.

To see that the Z_{n_i}'s are not jointly independent, consider three variables Z_{n_1}, Z_{n_2}, Z_{n_3}, with distinct n_i. Such a trio exists since we've assumed $p \geq 3$. The joint event $\mathbb{P}[Z_{n_1} = r,\ Z_{n_2} = s,\ Z_{n_3} = t]$ represents the overdetermined system

$$\begin{array}{ccccc}
X & + & n_1 Y & = & r \\
X & + & n_2 Y & = & s \\
X & + & n_3 Y & = & t
\end{array}.$$

For any $[r\ s\ t]^T$ not in the span of $[1\ 1\ 1]^T$ and $[n_1\ n_2\ n_3]^T$, this system will have no solutions and $\mathbb{P}[Z_{n_1} = r,\ Z_{n_2} = s,\ Z_{n_3} = t] = 0$. However, $\mathbb{P}[Z_{n_1} = r] \cdot \mathbb{P}[Z_{n_2} = s] \cdot \mathbb{P}[Z_{n_3} = t] = \frac{1}{p^3}$. We conclude that the Z_{n_i}'s are not jointly independent. $\square$

10. (a) For any given $\mu \in \mathbb{R}$, $\sigma > 0$, $k \geq 1$, show that there exists a random variable X with mean μ and variance σ^2 for which Chebyshev's inequality becomes an equality:

$$\mathbb{P}[|X - \mu| \geq k\sigma] = \frac{1}{k^2}.$$

Proof. Let's construct a random variable X that takes values in $\{-k\sigma, 0, k\sigma\}$. To make things simple, let's construct X to have zero mean (we'll shift it to μ later). In order for this to work, we need $\mathbb{P}[X = -k\sigma] = \mathbb{P}[X = k\sigma] = \beta$ and $\mathbb{P}[X = 0] = \alpha$ for some nonnegative α, β with $\alpha + 2\beta = 1$. The variance of X will then be given by

$$\mathrm{Var}[X] = \mathbb{E}[X^2] = 2\beta k^2\sigma^2.$$

In order for $\mathrm{Var}[X] = \sigma^2$ to hold, we need $\beta = \frac{1}{2k^2}$. Consider then the variable X with

$$X = \begin{cases} -k\sigma, & \text{with probability } \frac{1}{2k^2} \\ 0, & \text{with probability } 1 - \frac{1}{k^2} \\ k\sigma, & \text{with probability } \frac{1}{2k^2} \end{cases}.$$

We then have

$$\begin{aligned} \mathbb{P}[|X| \geq k\sigma] &= \mathbb{P}[X = \pm k\sigma] \\ &= \frac{1}{k^2}. \end{aligned}$$

By linearity of expectation, the variable $X + \mu$ will have mean μ, variance σ^2 and the above equality will still hold. □

(b) Show that for any random variable X with mean μ and variance σ^2, one has

$$\mathbb{P}[|X - \mu| \geq k\sigma] = o\left(\frac{1}{k^2}\right) \quad \text{as } k \to \infty.$$

Proof. By Markov's inequality we have

$$\begin{aligned} k^2 \cdot \mathbb{P}[|X - \mu| \geq k\sigma] &= k^2 \cdot \mathbb{P}[(X - \mu)^2 \geq k^2\sigma^2] \\ &\leq \frac{1}{\sigma^2} \int_{|X-\mu| \geq k\sigma} (X - \mu)^2 \, d\mathbb{P}. \end{aligned} \tag{2}$$

Since X has finite variance, the function $(X - \mu)^2$ is integrable. Consequently, the set function $E \mapsto \int_E (X - \mu)^2 \, d\mathbb{P}$ is a measure absolutely continuous with respect to $\mathbb{P}$. This means that for any ϵ, there is a δ so that $\mathbb{P}[E] < \delta$ implies that $\int_E (X - \mu)^2 \, d\mathbb{P} < \epsilon$.

Fix $\epsilon > 0$. If we can show that for k sufficiently large, $\mathbb{P}[|X - \mu| \geq k\sigma] < \delta$, then the last line in (2) will tend to zero as $k \to \infty$ by the above discussion. Consider the inequality

$$\frac{|X - \mu|}{\sigma} - 1 \leq \sum_{k=1}^{\infty} \mathbb{1}_{|X-\mu| \geq k\sigma} \leq \frac{|X - \mu|}{\sigma}.$$

Integrating through this inequality with respect to $\mathbb{P}$ and using the fact that $\mathbb{P}$ is a finite measure shows that $|X - \mu|/\sigma$ is integrable if and only if $\sum_{k=1}^{\infty} \mathbb{P}[|X - \mu| \geq k\sigma] < \infty$. Since

$|X-\mu|/\sigma$ is indeed integrable, we have that $\mathbb{P}[|X-\mu| \geq k\sigma] \to 0$ as $k \to \infty$. We can then pick k sufficiently large so that $\int_{|X-\mu|\geq k\sigma} (X-\mu)^2 \, d\mathbb{P} < \epsilon$ so that

$$\begin{aligned} k^2 \cdot \mathbb{P}[|X-\mu| \geq k\sigma] &\leq \frac{1}{\sigma^2} \int_{|X-\mu|\geq k\sigma} (X-\mu)^2 \, d\mathbb{P} \\ &\leq \frac{\epsilon}{\sigma^2}. \end{aligned}$$

We conclude that $\mathbb{P}[|X-\mu| \geq k\sigma]$ is $o(1/k^2)$ as $k \to \infty$. $\square$