1. $X \sim \text{Unif}[0, \pi/2]$, $Y \sim \text{Exp}(3)$.

$X \perp\!\!\!\perp Y$

a) $U = e^{Y}\cos X$, $V = e^{Y}\sin X$

find joint density $f_{U,V}$.

Soln: $f_{U,V}(u,v) = f_{X,Y}(x,y)\,|J(x,y)|^{-1}$

$$J = \begin{vmatrix} \partial_x U & \partial_y U \\ \partial_x V & \partial_y V \end{vmatrix}$$

$$= \begin{vmatrix} -e^{y}\sin x & e^{y}\cos x \\ e^{y}\cos x & e^{y}\sin x \end{vmatrix} = \begin{matrix} -e^{2y}\sin^2 x \\ -e^{2y}\cos^2 x \end{matrix} = -e^{2y}$$

Since $X \perp\!\!\!\perp Y$, $f_{X,Y}(x,y) = f_X(x) f_Y(y)$

$$= 3e^{-3y} \qquad \begin{matrix} 0 \le x \le \pi/2 \\ 0 \le y < \infty \end{matrix}$$

$$\Rightarrow f_{X,Y}(x,y) = 3e^{-3y}\left|e^{-2y}\right|^{-1}$$

$$= 3e^{-y} \qquad 0 \le x \le \pi/2$$

$$y \geq 0$$

$$u = e^{y}\cos x \quad , v = e^{y}\sin x$$

- $u^2+v^2 = e^{2y} \Rightarrow y = \frac{1}{2}\log(u^2+v^2)$
- $v/u = \tan x \Rightarrow x = \arctan(v/u)$
- $0 \le x \le \pi/2 \,,\; y \geq 0 \Rightarrow 0 \le u < \infty \,, 0 \le v < \infty$

$$\Rightarrow f_{U,V}(u,v) = 3e^{-1/2\log(u^3+v^3)} \qquad u,v \geq 0$$

$\boxed{//}$

b) find $E[U+V]$ & $Var[U+V]$

Soln:

$$E[U+V] = E[U] + E[V]$$
$$= E[e^Y \cos X] + E[e^Y \sin X]$$

$$(\text{since } X \perp\!\!\!\perp Y) = E[e^Y]E[\cos X] + E[e^Y]E[\sin X]$$

$$\cdot\ E[\cos X] = \frac{2}{\pi}\int_0^{\pi/2} \cos x\, dx = \frac{2}{\pi} \sin x \Big|_0^{\pi/2}$$
$$= 2/\pi$$

$$\cdot\ E[\sin X] = \frac{2}{\pi}\int_0^{\pi/2} \sin x\, dx = \frac{2}{\pi}(-\cos x)\Big|_0^{\pi/2}$$
$$= 2/\pi$$

$$\cdot\ E[e^Y] = 3\int_0^{\infty} e^y \cdot e^{-3y}\, dy = 3\int_0^{\infty} e^{-2y}\, dy$$
$$= 3/2$$

$$\Rightarrow E[U+V] = 6/\pi$$

$$\mathrm{Var}[U+V] = \mathrm{Var}[U] + \mathrm{Var}[V] + 2\,\mathrm{Cov}(U,V) \quad (*)$$

- $\mathrm{Var}[U] = E[U^2] - E[U]^2$ know this already

- $E[U^2] = E[e^{2Y}\cos^2 X] = E[e^{2Y}]E[\cos^2 X] \quad (X \perp\!\!\!\perp Y)$

- $E[e^{2Y}] = 3\int_0^\infty e^{2y}e^{-3y}\,dy = 3\int_0^\infty e^{-y}\,dy = 3$

- $E[\cos^2 X] = \frac{2}{\pi}\int_0^{\pi/2}\cos^2 x\,dx = \frac{1}{\pi}\int_0^{\pi/2}(1+\cos(2x))\,dx$

$$= \frac{1}{\pi}\left[x + \frac{1}{2}\sin(2x)\right]_0^{\pi/2} = 1/2$$

$\Rightarrow E[U^2] = 3\cdot 1/2 = 3/2$

$\Rightarrow \mathrm{Var}[U] = \frac{3}{2} - (3/\pi)^2$

$\cdot \mathrm{Var}[V] = E[V^2] - E[V]^2$

$\cdot E[V^2] = E[e^{2Y}\sin^2 X] = E[e^{2Y}]E[\sin^2 X]$

$\cdot E[\sin^2 X] = \frac{2}{\pi}\int_0^{\pi/2} \sin^2 x\,dx = \frac{1}{\pi}\int_0^{\pi/2}(1-\cos(2x))\,dx$

$= \frac{1}{\pi}\left(x - \frac{1}{2}\cos(2x)\right)_0^{\pi/2} = 1/2$

$\Rightarrow E[V^2] = 3/2$

$\Rightarrow \mathrm{Var}[V] = 3/2 - (3/\pi)^2$

$\cdot \mathrm{Cov}(U,V) = E[UV] - \underbrace{E[U]}_{3/\pi}\underbrace{E[V]}_{3/\pi}$

$\cdot E[UV] = E[e^{2Y}\sin X\cos X] = E[e^{2Y}]E[\sin X\cos X]$

$E[\sin X\cos X] = \frac{2}{\pi}\int_0^{\pi/2} \sin x\cos x\,dx \quad u = \sin x \Rightarrow du = \cos x\,dx$

$= \frac{2}{\pi}\int_0^1 u\,du = 1/\pi$

$\Rightarrow \mathrm{Cov}(U,V) = 3/\pi - (3/\pi)^2$

Thus,

$$Var[U+V] = Var[U] + Var[V] + 2Cov(U,V)$$

$$= \left[\frac{3}{2} - \left(\frac{3}{\pi}\right)^2\right] + \left[\frac{3}{2} - \left(\frac{3}{\pi}\right)^2\right] + 2\left[\frac{3}{\pi} - \left(\frac{3}{\pi}\right)^2\right]$$

$$= 3 + 6/\pi - 36/\pi^2$$

$\boxed{/\!/}$

2. X RV w/ density $f(x) = \frac{C}{1+(x/10)^2}$, $-20 \leq x \leq 20$

a) find C s.t. f is a density.

Soln: $\int_{-20}^{20} f(x)dx = 1$

$\parallel$

$C\int_{-20}^{20} \frac{1}{1+(x/10)^2} dx = 10C \arctan\left(\frac{x}{10}\right)\Big|_{-20}^{20}$

$= 10C\left[\arctan 2 - \arctan(-2)\right]$

$= 20C \arctan(2) \Rightarrow C = 1/20\arctan(2)$ $\boxtimes$

b) $E[X] = \int_{-20}^{20} x f(x) dx = C\int_{-20}^{20} \frac{x}{1+(x/10)^2} dx = 0$

(integral of odd function over symmetric interval)

$$E[X^2] = \int_{-20}^{20} x^2 f(x)\,dx = C\int_{-20}^{20} \frac{x^2}{1+(x/10)^2}$$

$$= 100C\int_{-20}^{20} \frac{(x/10)^2}{1+(x/10)^2}\,dx$$

$$= 100C\int_{-20}^{20}\left(1 - \frac{1}{1+(x/10)^2}\right)dx =$$

$$= 100C\left[40 - \int_{-20}^{20} \frac{1}{1+(x/10)^2}\,dx\right]$$

$$= 100C\left[40 - 1/C\right] = 4000C - 100$$

$$\Rightarrow \mathrm{Var}[X] = 4000C - 100$$

c) Produce 10000 bolts. Estimate the probability that more than 2000 bolts differ by ≥ 10.

Idea: DeMoivre-Laplace Limit Thm

Let $p = \Pr[\text{bolt differs by} \geq 10]$

$$= \Pr[|X| \geq 10] = 1 - \int_{-10}^{10} f(x)\,dx$$

$$= 1 - 5\pi C$$

If $S_n = \#$ bolts in n that differ by ≥ 10, then

$$\Pr[2000 \leq S_n] = \Pr\left[\frac{2000 - np}{\sqrt{np(1-p)}} \leq \frac{S_n - np}{\sqrt{np(1-p)}}\right]$$

$$\approx \Pr\left[\frac{2000-np}{\sqrt{np(1-p)}} \leq g\right]$$

where $g \sim N(0,1)$

Set $n = 10{,}000 \Rightarrow$ $$= 1 - \Phi\left(\frac{2000 - np}{\sqrt{np(1-p)}}\right)$$

$\boxtimes$

3. • You + 1M others. Each has $1/1M$ chance of winning.
• prize goes to a random winner.

• You win. $X = \#$ other winners $\sim Pois(1)$

a) $Pr[\text{you're the only winner}]$

$= Pr[X=0] = 1^0 \cdot e^{-1}/0! = 1/e$

b) $Pr[\text{you get the prize} \mid x \text{ other winners}]$

$= 1/(x+1)$

c) $Pr[\text{you get the prize}] =$

$= \sum_{x=0}^{\infty} Pr[\text{get prize} \mid X=x] Pr[X=x]$

$= \sum_{x=0}^{\infty} \frac{1}{x+1} \cdot \frac{1^x e^{-1}}{x!} = \frac{1}{e} \sum_{x=0}^{\infty} \frac{1}{(x+1)!} = \frac{1}{e}(e-1)$

$= 1 - 1/e$ $\square$

4. $X \sim \text{Unif}\{2, 4, \ldots, 2n\}$

a) find $M_X(t)$

Soln: $M_X(t) = \mathbb{E}[e^{tX}]$

$$= \sum_{k=1}^{n} \frac{1}{n} \cdot e^{2kt} = \frac{e^{2t}}{n} \sum_{k=1}^{n} e^{2t(k-1)}$$

$$= \frac{e^{2t}}{n} \frac{e^{2tn} - 1}{e^{2t} - 1} = \frac{e^{2t(n+1)} - e^{2t}}{n(e^{2t} - 1)}$$

b) $\mathbb{E}[X^3] = \frac{d^3}{dt^3} M_X(t) \Big|_{t=0}$

c) $X_i \sim X$ iid. $S_k = X_1 + \cdots + X_k$.

$M_{S_k}(t) \underset{\uparrow}{=} M_X(t)^k$

independence

5. a) Cheb says $\Pr[|X-\mu| \geq t] \leq \text{Var}[X]/t^2$

if $X=0 \Rightarrow |X-\mu| = \mu$

$\Rightarrow \Pr[X=0] \leq \Pr[|X-\mu| \geq \mu] \leq \text{Var}[X]/\mu^2$

if $\text{Var}[X_n] = o(E[X]^2) \Rightarrow \text{Var}[X_n]/E[X_n]^2 \to 0$

$\Rightarrow \Pr[X_n=0] \leq \frac{\text{Var}[X_n]}{E[X_n]^2} \to 0.$

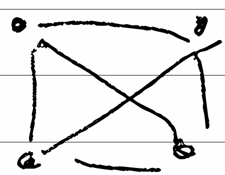
$p = \omega(n^{-2/3})$

b) let $X = \#$ 4 cliques in $G(n,p)$

Idea: use (a) & show $\Pr[X=0] \le \frac{\mathrm{Var}[X]}{E[X]^2} \to 0$

write $X = \sum_C X_C$, where C ranges over all sets of 4 vertices.

$\Rightarrow E[X] = \sum_C E[X_C] = \sum_C \Pr[C \text{ is a 4 clique}]$



4 clique has 6 edges

$\Rightarrow \Pr[C \text{ is a 4 clique}] = p^6$

$\Rightarrow E[X] = \binom{n}{4} p^6 = \Theta(n^4 p^6)$

$$\mathrm{Var}[X] = \sum_{C} \mathrm{Var}[X_C] + 2\sum_{C \neq C'} \mathrm{Cov}(X_C, X_{C'})$$

- $$\mathrm{Var}[X_C] = E[X_C^2] - E[X_C]^2$$

$$= E[X_C] - E[X_C]^2 \quad (\text{since } X_C \in \{0,1\})$$

$$= p^6 - p^{12}$$

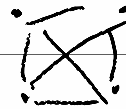
$$\Rightarrow \sum_{C} \mathrm{Var}(X_C) = \binom{n}{4}(p^6 - p^{12}) = \Theta(n^4 p^6)$$

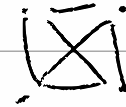
(since $p < 1$)

$$\mathrm{Cov}(X_C, X_{C'}) = E[X_C X_{C'}] - E[X_C]E[X_{C'}]$$
$$\leq E[X_C X_{C'}]$$
$$= \Pr[C \,\&\, C' \text{ are 4 cliques}]$$

depends on $|C \cap C'|$

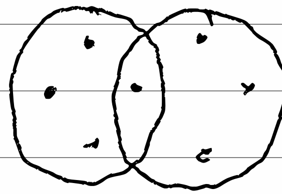
$|C \cap C'| = 0$





independent

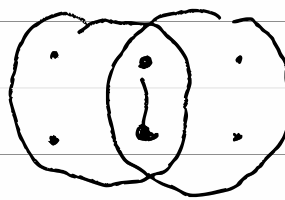
$|C \cap C'| = 1$



no edges in common

$\Rightarrow$ independent

$|C \cap C'| = 2$

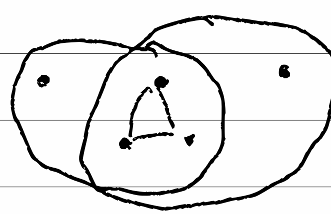


common edges

$\Rightarrow$ not independent

$\binom{n}{6}\binom{6}{2}$ ways for this to happen

$|C \cap C'| = 3$



common edges

$\Rightarrow$ not independent

$\binom{n}{5}\binom{5}{3}$ ways for this to happen.

If $|C \cap C'| = 2$, need $6+6-\underset{\text{overlap}}{1}$ edges

$\Rightarrow \Pr[C \,\&\, C' \text{ are 4 cliques}] = p^{11}$

if $|C \cap C'| = 3$ need $6+6-3 = 9$ edges

$\Rightarrow \Pr[C \,\&\, C' \text{ are 4 cliques}] = p^{9}$

$$\Rightarrow \frac{\mathrm{Var}[X]}{\mathbb{E}[X]^2} = \frac{\left[\sum \mathrm{Var}[X_C] + 2\sum_{|C\cap C'|=2} \mathrm{Cov}(X_C, X_{C'}) + 2\sum_{|C\cap C'|=3} \mathrm{Cov}(X_C, X_{C'})\right]}{\mathbb{E}[X]^2}$$

$$\leq \frac{\Theta(n^4p^6) + \Theta(n^6p^{11}) + \Theta(n^5p^9)}{\Theta(n^4p^6)^2}$$

$$= O(1/n^4p^6) + O(1/n^2p) + O(1/n^3p^3)$$

Since $p = \omega(n^{-2/3})$, these denominators go to $\infty \Rightarrow \Pr[X=0]$

$= \Pr[\text{there are } \underline{\text{no}} \text{ 4 cliques}]$

$\longrightarrow 0$

$\Rightarrow \Pr[\text{there is a 4-clique}] \longrightarrow 1$

$\boxtimes$