1. The joint density of X and Y is

$$f(x, y) = c(x^2 - y^2)e^{-x}, \quad 0 \leq x < \infty, \; -x \leq y \leq x.$$

Find the conditional distribution of Y, given $X = x$.

the RV $Y|X=x$

$$\Pr[Y \leq t \mid X = x] = \int_{-\infty}^{t} f_{Y|X}(y|x)\,dy$$

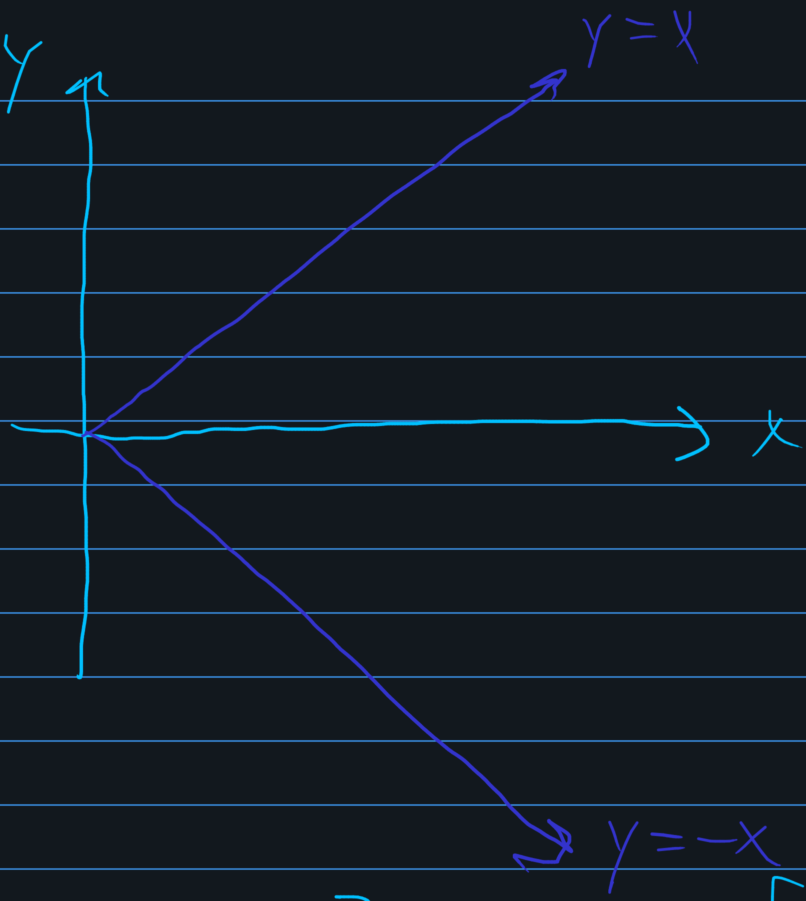
conditional density of Y given $X=x$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

$$\left(\Pr[A|B] = \frac{\Pr[A,B]}{\Pr[B]} \right)$$

Separate the joint into marginals.

$$f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy$$



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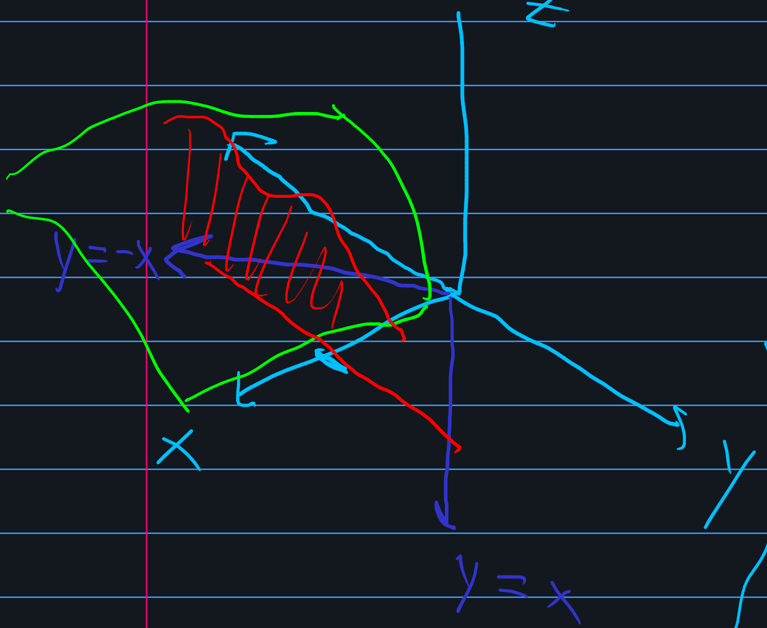
$f_X(x)$

$f_{Y|X}(y|x)$



$0 \leq x < \infty, \quad -x \leq y \leq x$

$$f_X(x) = \int_{-x}^{x} c(x^2 - y^2)e^{-x}\,dy$$

$$= \int_{-x}^{x} \left[cx^2e^{-x} - ce^{-x}y^2\right] dy$$

$$= \left[cx^2 e^{-x} y - \frac{1}{3} c e^{-x} y^3 \right]_{y=-x}^{y=x}$$

$$= \left(cx^3 e^{-x} - \frac{1}{3} c e^{-x} x^3 \right)$$

$$- \left(-cx^3 e^{-x} + \frac{1}{3} c e^{-x} x^3 \right)$$

$$= 2ce^{-x} x^3 - \frac{2}{3} c e^{-x} x^3$$

$$= \frac{4}{3} c e^{-x} x^3 = f_X(x)$$

$$\Rightarrow f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

$$= \frac{c(x^2-y^2)e^{-x}}{\frac{4}{3} c e^{-x} x^3}$$

$$= \frac{3}{4} \frac{x^2-y^2}{x^3}$$

$$\Pr\left[Y \le t \,|\, X=x \right] = \int_{-\infty}^{t} f_{Y|X}(y|x)\,dy = \int_{-\infty}^{t} \frac{3}{4}\left(\frac{x^2-y^2}{x^3} \right) dy$$

2. X and Y have joint density function

$$f(x,y) = \frac{1}{x^2y^2}, \quad x, y \geq 1.$$

(a) Compute the joint density function of $U = XY$, $V = X/Y$.

(b) What are the marginal densities?

$$f_{U,V}(u,v) = f_{X,Y}(x,y)\,|J(x,y)|^{-1}$$

$$J(x,y) = \begin{vmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{vmatrix} = \begin{vmatrix} y & x \\ \frac{1}{y} & -\frac{x}{y^2} \end{vmatrix} = -\frac{x}{y} - \frac{x}{y} = -\frac{2x}{y}$$

$$f_{U,V}(u,v) = f_{X,Y}(x,y)\,|J(x,y)|^{-1} = \frac{1}{x^2y^2}\left|-\frac{2x}{y}\right|^{-1} = \frac{1}{x^2y^2}\,\frac{y}{2x} = \frac{1}{2x^3y}$$

$u = xy$, $v = x/y$

$$xxxy \cdot y \cdot \frac{1}{y} = u^2 v$$

$$= \frac{1}{2u^2v}$$