

270B - Homework 3

Problem 1. Let X_1, X_2, \dots be independent random variables with means μ_i and finite variances σ_i^2 . Consider the sums $S_n = X_1 + \dots + X_n$. Find sequences of real numbers (b_i) and (c_i) such that $S_n^2 + b_n S_n + c_n$ is a martingale with respect to the σ -algebras generated by X_1, \dots, X_n .

Solution. Let's start by centering the sum: define the random variable $M_n = S_n - \sum_{i=1}^n \mu_i$. Since the X_i 's are independent, we have $\text{Var}[M_n] = \sum_{i=1}^n \sigma_i^2$. We claim that

$$V_n = M_n^2 - \sum_{i=1}^n \sigma_i^2 = \left(S_n - \sum_{i=1}^n \mu_i \right)^2 - \sum_{i=1}^n \sigma_i^2$$

is a martingale with respect to the filtration $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$. Let's start the computation.

$$\begin{aligned} \mathbb{E}[V_{n+1} | \mathcal{F}_n] &= \mathbb{E}[S_{n+1}^2] - 2 \left(\sum_{i=1}^{n+1} \mu_i \right) \mathbb{E}[S_{n+1} | \mathcal{F}_n] + \left(\sum_{i=1}^{n+1} \mu_i \right)^2 - \sum_{i=1}^{n+1} \sigma_i^2 \\ &= S_n^2 + 2S_n \mu_{n+1} + \mathbb{E}[X_{n+1}^2] - 2 \left(\sum_{i=1}^{n+1} \mu_i \right) (S_n + \mu_{n+1}) + \left(\sum_{i=1}^{n+1} \mu_i \right)^2 - \sum_{i=1}^{n+1} \sigma_i^2 \\ &= S_n^2 - 2 \left(\sum_{i=1}^n \mu_i \right) S_n + \mathbb{E}[X_{n+1}^2] - 2\mu_{n+1}^2 + \left(\sum_{i=1}^n \mu_i \right)^2 + \mu_{n+1}^2 - \sum_{i=1}^{n+1} \sigma_i^2 \\ &= S_n^2 - 2 \left(\sum_{i=1}^n \mu_i \right) S_n + \left(\sum_{i=1}^n \mu_i \right)^2 - \sum_{i=1}^n \sigma_i^2 \\ &= V_n. \end{aligned}$$

Here we've used the fact that S_n is \mathcal{F}_n -measurable and X_{n+1} is independent of \mathcal{F}_n . The sequences we want are then

$$b_n = -2 \sum_{i=1}^n \mu_i, \quad c_n = \left(\sum_{i=1}^n \mu_i \right)^2 - \sum_{i=1}^n \sigma_i^2.$$

□

Problem 2.

(a) Show that if (X_n) and (Y_n) are martingales with respect to the same filtration, then $X_n \vee Y_n$ is a submartingale.

Proof. We use the trusty identity

$$X_n \vee Y_n = \frac{1}{2}[(X_n + Y_n) + |X_n - Y_n|].$$

Since the sum of martingales is a martingale and conditional Jensen says the absolute value of a martingale is a submartingale, we have

$$\begin{aligned}\mathbb{E}[X_{n+1} \vee Y_{n+1} | \mathcal{F}_n] &= \frac{1}{2}(\mathbb{E}[X_{n+1} + Y_{n+1} | \mathcal{F}_n] + \mathbb{E}[|X_{n+1} - Y_{n+1}| | \mathcal{F}_n]) \\ &\geq \frac{1}{2}[(X_n + Y_n) + |X_n - Y_n|] \\ &= X_n \vee Y_n.\end{aligned}$$

Hence, $X_n \vee Y_n$ is a submartingale. □

(b) Give an example showing that $X_n \vee Y_n$ need not be a martingale.

Proof. □

Problem 3. Give an example of a martingale (X_n) such that $X_n \rightarrow -\infty$ a.s.

Solution. Durrett gives a hint to let $X_n = \xi_1 + \dots + \xi_n$ for some independent centered ξ_i 's. The idea is to concentrate most of the mass of ξ_i around some negative value and put the rest (some summable amount) around some positive value, then apply Borel-Cantelli.

Concretely, let ξ_i be given by

$$\xi_i = \begin{cases} 2^j & \text{with probability } \frac{1}{2^j} \\ -\frac{1}{1-2^{-j}} & \text{with probability } 1 - \frac{1}{2^j} \end{cases}.$$

Clearly ξ_i is centered, so $X_n = \xi_1 + \dots + \xi_n$ is a martingale. Note that

$$\sum_{i=1}^{\infty} \mathbb{P}[\xi_i = 2^j] = \sum_{i=1}^{\infty} \frac{1}{2^j} = 1 < \infty.$$

By Borel-Cantelli, we have that $\xi_i = -\frac{1}{1-2^{-j}}$ eventually with probability 1, so $X_n \rightarrow -\infty$ a.s. □

Problem 4. Let (X_n) be a martingale that is bounded a.s. either above or below by some constant M . Show that $\sup_n \mathbb{E}|X_n| < \infty$.

Proof. If X_n is bounded below, then $X_n + M$ is a nonnegative martingale. By the martingale convergence theorem, $X_n + M$ converges almost surely to some limit Y with $\mathbb{E}|Y| < \infty$. Consequently, X_n also converges a.s. to an integrable function, so $\sup_n \mathbb{E}|X_n| < \infty$. If X_n is bounded above, then $-X_n + M$ is a nonnegative martingale and the same argument works. □

Problem 5. Let Z_1, Z_2, \dots be nonnegative iid random variables with $\mathbb{E}[Z_i] = 1$ and $\mathbb{P}[Z_i = 1] < 1$. Show that as $n \rightarrow \infty$,

$$\prod_{i=1}^n Z_i \rightarrow 0 \quad \text{a.s.}$$

Proof. First note that $M_n = \prod_{i=1}^n Z_i$ is indeed a martingale:

$$\mathbb{E}[M_{n+1}|\mathcal{F}_n] = \mathbb{E}[Z_{n+1}M_n|\mathcal{F}_n] = M_n,$$

where $\mathcal{F}_n = \sigma(Z_1, \dots, Z_n)$. Since M_n is a nonnegative martingale, it converges to some X a.s. with $\mathbb{E}[X] \leq \mathbb{E}[Z_1] = 1$. For any $\epsilon > 0$ we have

$$\mathbb{P}[|M_{n+1} - M_n| > \epsilon] = \mathbb{P}[M_n|Z_{n+1} - 1| > \epsilon].$$

Now if $M_n > \sqrt{\epsilon}$ and $|Z_{n+1} - 1| > \sqrt{\epsilon}$, then clearly $M_n|Z_{n+1} - 1| > \epsilon$. We then have by independence

$$\mathbb{P}[|M_{n+1} - M_n| > \epsilon] \geq \mathbb{P}[M_n > \sqrt{\epsilon}] \cdot \mathbb{P}[|Z_{n+1} - 1| > \sqrt{\epsilon}].$$

Now since $\mathbb{P}[Z_{n+1} = 1] < 1$, we have that for ϵ sufficiently small, $\mathbb{P}[|Z_{n+1} - 1| > \epsilon] \geq \delta > 0$ for some δ . Since M_n converges almost surely, it converges in measure as well, so the left-hand side of the above inequality goes to zero. Since the $\mathbb{P}[|Z_{n+1} - 1| > \sqrt{\epsilon}]$ term is bounded below by a positive constant, we must have that $\mathbb{P}[M_n > \sqrt{\epsilon}] \rightarrow 0$, so $M_n \rightarrow 0$ in probability. Since M_n converges almost surely and the a.s. limit is the same as the probability limit, $M_n \rightarrow 0$ a.s. \square

Problem 6. Let (X_n) be a martingale and let $\Delta_n = X_n - X_{n-1}$ be the martingale differences. Prove that if $X_0 = 0$ and $\sum_{n=1}^{\infty} \Delta_n^2 < \infty$ then X_n converges in L^2 to some random variable X . (I tried doing the problem as stated and it didn't seem to work. I talked with Xiaowen and she said that we need to assume $\sum \mathbb{E}[\Delta_n^2] < \infty$. This is also how it's stated in Durrett.)

Proof. For any m, n we have $X_n - X_m = \sum_{i=m+1}^n \Delta_i$. From this we deduce

$$\mathbb{E}[|X_n - X_m|^2] = \mathbb{E}\left[\left(\sum_{i=m+1}^n \Delta_i\right)^2\right] = \sum_{i=m+1}^n \mathbb{E}[\Delta_i^2] + 2 \sum_{m+1 \leq i < j}^n \mathbb{E}[\Delta_i \Delta_j].$$

Since martingale increments are uncorrelated, $\mathbb{E}[\Delta_i \Delta_j] = \mathbb{E}[\Delta_i] \mathbb{E}[\Delta_j] = 0$. Since $\sum \mathbb{E}[\Delta_i^2] < \infty$, the tail sum goes to zero, so for m, n large enough, the above sum can be made arbitrarily small. (X_n) is then Cauchy in L^2 , so it converges in L^2 by completeness. \square

Problem 7. Construct a branching process (Z_n) as follows. Let X be a random variable with mean μ and variance σ^2 ; it specifies the distribution of the offspring. Set

$$Z_{n+1} = X_1^{(n+1)} + \dots + X_{Z_n}^{(n+1)},$$

to be the size of the population at time $n + 1$, where all $X_i^{(k)}$ are iid random variables distributed identically with X .

- (a) Show that $Y_n = Z_n/\mu^n$ defines a martingale (with respect to the filtration \mathcal{F}_n generated by $X_j^{(k)}$, $1 \leq j \leq Z_n, k \leq n$).

Proof. We compute, critically using the fact that Z_n is \mathcal{F}_n -measurable.

$$\begin{aligned}
\mathbb{E}[Y_{n+1}|\mathcal{F}_n] &= \frac{1}{\mu^{n+1}} \mathbb{E}[Z_{n+1}|\mathcal{F}_n] \\
&= \frac{1}{\mu^{n+1}} \mathbb{E}[X_1^{(n+1)} + \dots + X_{Z_n}^{(n+1)}|\mathcal{F}_n] \\
&= \frac{1}{\mu^{n+1}} \cdot \mu Z_n \\
&= Y_n.
\end{aligned}$$

□

(b) Show that

$$\mathbb{E}[Z_{n+1}^2|\mathcal{F}_n] = \mu^2 Z_n^2 + \sigma^2 Z_n.$$

Proof. We compute.

$$\begin{aligned}
\mathbb{E}[Z_{n+1}^2|\mathcal{F}_n] &= \sum_{i=1}^{Z_n} \mathbb{E}[(X_i^{(n+1)})^2] + 2 \sum_{1 \leq i < j \leq Z_n} \mathbb{E}[X_i^{(n+1)} X_j^{(n+1)}] \\
&= Z_n(\sigma^2 + \mu^2) + Z_n(Z_n - 1)\mu^2 \\
&= \mu^2 Z_n^2 + \sigma^2 Z_n.
\end{aligned}$$

□

(c) Deduce that (Y_n) is bounded in L^2 if and only if $\mu > 1$.

Proof.

□