MATH 271B 2020 FINAL

Name:____________________________

Question	Score	Maximum
1		10
2		20
3		20
4		20
5		20
6		10
Total		100

1. Consider for β standard one-dimensional Brownian motion:

$$dX_t = (t^2\Sigma)d\beta_t, \quad X_0 = 0,$$

where Σ is an exponentially distributed random variable with parameter λ and independent of the Brownian motion.

a) Find Y_t so that $M_t = \exp(X_t)Y_t$ is a martingale, specify with respect to which filtration.

b) Compute the variance of M_t.

c) Find a bound for $\mathbb{P}[sup_{0<s<t}|M_t| > \varepsilon]$.

2. Consider the Ornstein-Uhlenbeck process

$$dr_t = a(\bar{r} - r_t)dt + \sigma d\beta_t,$$

with $a, \bar{r}, \sigma$ constants as a model for for the interest rate, (note that the rate can go negative). The price of a zero-coupon bond at time t when paying 1 at maturity T is

$$P(t, x, T) = \mathbb{E}[\exp(-\int_t^T r_s ds) \mid r_t = x].$$

a) Derive the Feynman-Kac formula for the bond price:

$$\frac{\partial P}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 P}{\partial x^2} + a(\bar{r} - x)\frac{\partial P}{\partial x} - xP = 0,$$
$$P(T, x, T) = 1.$$

b) Solve this partial differential equation for the price (Hint: try a solution of the form $A(\tau)\exp(B(\tau)x)$ with $\tau = T - t$ being time to maturity).

3. Let v be a continuous scalar valued process satisfying

$$0 \leq v(t) \leq \alpha(t) + \beta \int_0^t v(s)ds; 0 \leq t \leq T,$$

with $\beta \geq 0$ and α integrable. Show that

$$v(t) \leq \alpha(t) + \beta \int_0^t \alpha(s)e^{\beta(t-s)}ds, 0 \leq t \leq T.$$

Can you relax the assumption about continuity?

4. Do problem 5.14 page 78 Øksendal.

5. Do problem 5.16 page 79 Øksendal.

6. Explain the terms:
   a) Martingale.
   b) Itô process.
   c) Stopping time.
   d) Quadratic variation.
   e) Kolmogorov Backward Equation.

7. Please order the topics below on a scale 1-10 for topics you would like to see covered in spring:

   - Optimal Control.
   - Diffusion limit, deriving SDE models from problems with rapidly fluctuating parameters.
   - Itô-Calculus for Hibert Space Valued Processes
   - Numerics for SDEs.
   - Waves in random media and localization.
   - Aspects of stochastic gradient descent in Machine Learning.
   - Uncertainty Quantification and Kriging.
   - Large Deviations.
   - Graph based models.
   - Mathematical Finance.
   - Other topic(s) you would like to see covered: __________.