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270A - Homework 1

1. (a) Let $\mathcal{F}$ be the family of all finite subsets of Ω and their complements. Is $\mathcal{F}$ a σ-algebra?

   Solution. If Ω is finite then $\mathcal{F}$ is simply the power set of Ω, which is definitely a σ-algebra. However, if $\mathcal{F}$ is infinite, then $\mathcal{F}$ is never a σ-algebra. To see this, let (x_n) be a countable sequence of distinct elements in Ω and consider the set of even-indexed terms

   $$F = \{x_n : n = 2k,\ k \in \mathbb{N}\}.$$

   This set is a countable union of singletons and all singletons belong to $\mathcal{F}$. F is clearly infinite, but so is its complement, which contains the (infinite) set of odd-indexed terms. We conclude that F is neither finite nor co-finite, so $\mathcal{F}$ is not closed under countable unions when Ω is an infinite set. □

   (b) Let $\mathcal{F}$ be the family of all countable subsets of Ω and their complements. Is $\mathcal{F}$ a σ-algebra?

   Solution. $\mathcal{F}$ is indeed a σ-algebra. The empty set is clearly countable, and $\Omega^C = \emptyset$. Let F_n be a countable collection of sets in $\mathcal{F}$ and consider their union, $F = \cup_{n=1}^{\infty} F_n$. If each F_n is countable, then F is just a countable union of countable sets: countable. If one of the F_n's, say F_k, were co-countable, then $F^C \subseteq F_k^C$, which is countable, so F is co-countable. Since $\mathcal{F}$ contains the empty set and Ω and is closed under countable unions and complements, it is a σ-algebra. □

   (c) Let $\mathcal{F}$ and $\mathcal{G}$ be two σ-algebras of subsets of Ω. Is $\mathcal{F} \cap \mathcal{G}$ always a σ-algebra?

   Solution. $\mathcal{F}$ is a σ-algebra. Since $\mathcal{F}$ and $\mathcal{G}$ both contain $\emptyset$ and Ω, so does their intersection. Let E_n be a countable collection of sets in $\mathcal{F} \cap \mathcal{G}$. Since $\mathcal{F}$ and $\mathcal{G}$ are both σ-algebras, the union $E = \cup_{n=1}^{\infty} E_n$ is in both $\mathcal{F}$ and $\mathcal{G}$ and each E_n^C is in both $\mathcal{F}$ and $\mathcal{G}$ as well. □

   (d) Let $\mathcal{F}$ and $\mathcal{G}$ be two σ-algebras of subsets of Ω. Is $\mathcal{F} \cup \mathcal{G}$ always a σ-algebra?

   Solution. The union need not be a σ-algebra. Let $\Omega = \{1, 2, 3, 4\}$, $\mathcal{F} = \{\emptyset, \Omega, \{1\}, \{2, 3, 4\}\}$, and $\mathcal{G} = \{\emptyset, \Omega, \{2\}, \{1, 3, 4\}\}$. $\mathcal{F}$ and $\mathcal{G}$ are σ-algebras, but the set $\{1\} \cup \{2\} = \{1, 2\}$ is not in their union. □

2. A subset $A \subset \mathbb{N}$ is said to have asymptotic density if

   $$\lim_{n\to\infty} \frac{|A \cap \{1, \ldots, n\}|}{n}$$

   exists. Let $\mathcal{F}$ be the collection of subsets of $\mathbb{N}$ for which the asymptotic density exists. Is $\mathcal{F}$ a σ-algebra?

Solution. $\mathcal{F}$ is not a σ-algebra. First let's construct a set not in $\mathcal{F}$. The idea is to build a set that has long gaps followed by even longer "runs". Let $F_0 = \{1\}$ and $F_i = \{2^i, \dots, 2^{i+1} - 1\}$. Define the set A by $A = \cup_{j=0}^{\infty} F_{2j}$. A consists of a run of length 2^{2j} followed by a gap of length 2^{2j+1} for each $j = 0, 1, \dots$. Our set A does not have asymptotic density since

$$\begin{aligned}\frac{|A \cap [2^{2k}]|}{2^{2k}} &= \frac{\sum_{j=0}^{k-1} 2^{2j} + 1}{2^{2k}} \\ &= \frac{1}{3}\end{aligned}$$

while on the other hand,

$$\begin{aligned}\frac{|A \cap [2^{2k+1}]|}{2^{2k+1}} &= \frac{\sum_{j=0}^{k} 2^{2j}}{2^{2k+1}} \\ &= \frac{1}{3}\left(2 - \frac{1}{2^{2k+1}}\right) \\ &\to \frac{2}{3}.\end{aligned}$$

Hence, A is not in $\mathcal{F}$. Since A is a countable union of singletons, which clearly have asymptotic density zero, we conclude that $\mathcal{F}$ is not a σ-algebra.

□

3. Let X and Y be two random variables on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $E \in \mathcal{F}$ be an event. Define

$$Z = \begin{cases} X & \text{if } E \text{ occurs} \\ Y & \text{otherwise.} \end{cases}$$

Prove that Z is a random variable.

Proof. We can write $Z = X \cdot \mathbb{1}_E + Y \cdot \mathbb{1}_{E^c}$. Since E is an event, the indicator functions $\mathbb{1}_E$ and $\mathbb{1}_{E^c}$ are measurable. Since X and Y are measurable and products and sums of measurable functions are measurable, we have that Z is measurable, and hence a random variable. □

4. Let X be a random variable with density f. Compute the density of X^2.

Solution. First let's compute the distribution of X^2. Let $t \geq 0$.

$$\begin{aligned}\mathbb{P}[X^2 < t] &= \mathbb{P}[-\sqrt{t} < X < \sqrt{t}] \\ &= \int_{-\sqrt{t}}^{\sqrt{t}} f(s)\ ds.\end{aligned}$$

By the Lebesgue differentiation theorem, the above integral is an almost everywhere differentiable function of t and we can apply the fundamental theorem of calculus. If we let g be the density of X^2 then

$$
\begin{aligned}
g(t) &= \frac{d}{dt} \int_{-\sqrt{t}}^{\sqrt{t}} f(s)\ ds \\
&= \frac{1}{2\sqrt{t}} [f(\sqrt{t}) + f(-\sqrt{t})],
\end{aligned}
$$

for $t \geq 0$. Since X^2 is clearly nonnegative, we then have

$$
g(t) = \begin{cases} \frac{1}{2\sqrt{t}} [f(\sqrt{t}) + f(-\sqrt{t})], & t > 0 \\ 0, & t \leq 0. \end{cases}
$$

$\square$

5. Let X be a nonnegative random variable. Show that

$$
\mathbb{E}[X] = \int_0^\infty \mathbb{P}[X > t]\ dt.
$$

Proof. Since X is nonnegative (this is important – the Lebesgue integral is orientation-independent, unlike the Riemann integral!),

$$
X = \int_0^X dt.
$$

We can then take the expectation of both sides and apply Fubini's theorem.

$$
\begin{aligned}
\mathbb{E}[X] &= \int_\Omega \int_0^X dt\ d\mathbb{P} \\
&= \int_\Omega \int_0^\infty \mathbb{1}_{X>t}(t)\ dt\ d\mathbb{P} \\
&= \int_0^\infty \int_\Omega \mathbb{1}_{X>t}(x)\ d\mathbb{P}\ dt \\
&= \int_0^\infty \mathbb{P}[X > t]\ dt.
\end{aligned}
$$

$\square$

6. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a strictly convex function. Let X be a random variable such that $\mathbb{E}[|X|] < \infty$ and $\mathbb{E}[|\varphi(X)|] \leq \infty$. Show that

$$
\varphi(\mathbb{E}[X]) = \mathbb{E}[\varphi(X)] \implies X = \mathbb{E}[X] \text{ a.s.}
$$

Proof. Since φ is strictly convex, for every $t \in \mathbb{R}$ there exists an affine linear function $F_t(x)$ such that $F_t(t) = \varphi(t)$ and $F_t(x) < \varphi(x)$ for all $x \neq t$. We can set $t = \mathbb{E}[X]$ compose with X to obtain $F_t(X) \leq \varphi(X)$, with equality if and only if $X = \mathbb{E}[X]$. Note that since F_t is affine linear we have that $\mathbb{E}[F_t(X)] = F_t(\mathbb{E}[X])$.

Suppose that $\varphi(\mathbb{E}[X]) = \mathbb{E}[\varphi(X)]$. When $t = \mathbb{E}[X]$, F_t and φ agree at $\mathbb{E}[X]$, so $\varphi(\mathbb{E}[X]) = F_t(\mathbb{E}[X])$, yielding

$$\begin{aligned}
\mathbb{E}[\varphi(X)] &= \varphi(\mathbb{E}[X]) \\
&= F_t(\mathbb{E}[X]) \\
&= \mathbb{E}[F_t(X)].
\end{aligned}$$

By the linearity of expectation we then have $\mathbb{E}[\varphi(X) - F_t(X)] = 0$. By convexity, $\varphi(X) - F_t(X) \geq 0$, so since this expectation is zero, we must have that $\varphi(X) = F_t(X)$ almost surely. By strict convexity, this implies that $X = t = \mathbb{E}[X]$ almost surely. □

7. Suppose $0 \leq p_n \leq 1$ and put $\alpha_n = \min(p_n, 1 - p_n)$. Show that if $\sum_n \alpha_n$ diverges, then no discrete probability space can contain independent events $A_1, A_2, \ldots$ such that $\mathbb{P}[A_n] = p_n$.