Double Counting and Inclusion-Exclusion

1. Use the following steps to show that the sequence

$$\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n$$

has a limit.

   (a) Sketch a graph for $f(x) = \frac{1}{x}$ and interpret γ_n as an area to show that $\gamma_n > 0$ for all n.

   (b) Interpret

$$\gamma_n - \gamma_{n+1} = [\log(n+1) - \log n] - \frac{1}{n+1}$$

   as a difference of areas to show that γ_n is a decreasing sequence.

   (c) Deduce that $\lim_{n\to\infty} \gamma_n$ converges.

2. Use your proof in exercise 1 to finish the proof from lecture about the divisor counting function: if we let $t(n)$ be the number of divisors of n and define $\tau(n) = \frac{1}{n}\sum_{k=1}^{n} t(k)$, then

$$|\tau(n) - \log n| \leq 1.$$

3. Prove that $(x+y)^p \equiv 0 \pmod{p}$, where p is a prime.

4. Let n and k be positive integers and let S be a set of n points in the plane such that no three points of S are colinear and for every point P of S there are at least k points of S equidistant from P. Use these steps to prove that

$$k < \frac{1}{2} + \sqrt{2n}.$$

   (a) Fix a point P in S. At *least* how many isosceles triangles in S have P as their apex?

   (b) Fix two points P and Q in S. At *most* how many isosceles triangles in S contain points P and Q? *Hint: Consider the perpendicular bisector of A and B.*

   (c) Use your answers to (a) and (b) to do a double-counting argument.

5. How many positive integers below 100 are divisible by 2, 3, or 5?

6. (a) A class has 20 students. How many ways are there to form a study group, assuming a study group must have at least two members?

   (b) How many ways are there to form a study group that contains at least one of Alice, Bob, and Carol?

7. At a 2020 Tokyo Olympics press conference, there are 200 journalists. 175 of them speak Japanese, 150 of them speak German, 180 of them speak English, and 160 of them speak Mandarin. What is the minimum number of journalists that can speak all four languages?

8. There are 50 students in a class who are given a test with three questions on it. All of the students answer at least one question. If 12 students did not answer question 1, 14 didn't answer question 2, 10 didn't answer question 3, and 25 answered all three questions, then how many students answered exactly 1 question?