Homework 1
Math 270A, Fall 2019, Prof. Roman Vershynin

Problem 1

Let Ω be an arbitrary set.

(a). Let $\mathcal{F}$ be the family of all finite subsets of Ω and their complements. Is $\mathcal{F}$ a σ-algebra?

(b). Let $\mathcal{F}$ be the family of all finite or countable subsets of Ω and their complements. Is $\mathcal{F}$ a σ-algebra?

(c). Let $\mathcal{F}$ and $\mathcal{G}$ be two σ-algebras of subsets of Ω. Is $\mathcal{F} \cap \mathcal{G}$ always a σ-algebra?

(d). Let $\mathcal{F}$ and $\mathcal{G}$ be two σ-algebras of subsets of Ω. Is $\mathcal{F} \cup \mathcal{G}$ always a σ-algebra?

Problem 2

A subset $A \subset \mathbb{N}$ is said to have *asymptotic density* if

$$\lim_{n \to \infty} \frac{|A \cap \{1, \ldots, n\}|}{n} \quad \text{exists.}$$

Let $\mathcal{F}$ be the collection of subsets of $\mathbb{N}$ for which the asymptotic density exists. If $\mathcal{F}$ a σ-algebra?

Problem 3

Let X and Y be two random variables on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $E \subset \mathcal{F}$ be an event. Define

$$Z := \begin{cases} X & \text{if } E \text{ occurs} \\ Y & \text{otherwise.} \end{cases}$$

Prove that Z is a random variable.

Problem 4

Let X be a random variable with density (pdf) f. Compute the density of X^2.

(Hint: first compute the distribution function (cdf) of X^2, then differentiate.)

Problem 5

Let X be a nonnegative random variable. Show that

$$\mathbb{E}\,X = \int_0^\infty \mathbb{P}\left\{X > t\right\}\,dt.$$

Problem 6

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a strictly convex function. Let X be a random variable such that $\mathbb{E}\,|X| < \infty$ and $\mathbb{E}\,|\varphi(X)| \leq \infty$. Show that

$$\varphi(\mathbb{E}\,X) = \mathbb{E}(\varphi(X)) \quad \text{implies} \quad X = \mathbb{E}\,X \text{ a.s.}$$

Problem 7

Suppose $0 \leq p_n \leq 1$ and put $\alpha_n := \min(p_n, 1 - p_n)$. Show that, if $\sum_n \alpha_n$ diverges, then no discrete probability space can contain independent events $A_1, A_2, \ldots$ such that $\mathbb{P}(A_n) = p_n$.

Problem 8

Prove that if random variables X and Y are independent, then so are $f(X)$ and $g(X)$, for any Borel measurable functions $f, g : \mathbb{R} \to \mathbb{R}$.

Problem 9

Let $p \geq 3$ be a prime. Let X and Y be independent random variables that are uniformly distributed on $\{0, \ldots, p-1\}$. Define

$$Z_n := (X + nY) \mod p, \quad n = 0, \ldots, p-1.$$

Show that the random variables Z_n are pairwise independent, but not jointly independent.

Problem 10

(a). For any given $\mu \in \mathbb{R}$, $\sigma > 0$, $k > 0$, show that there exists a random variable X with mean μ and variance δ^2 and for which Chebyshev's inequality becomes an identity:

$$\mathbb{P}\left\{|X - \mu| \geq k\sigma\right\} = \frac{1}{k^2}.$$

(b). Show that for any random variable X with mean μ and variance σ^2, one has

$$\mathbb{P}\left\{|X - \mu| \geq k\sigma\right\} = o\Big(\frac{1}{k^2}\Big) \quad \text{as } k \to \infty.$$

Why do parts (a) and (b) not contradict each other?