Math 130B - Homework 3

1. Let Z be a standard normal random variable and let g be a differentiable function with derivative g'.

   (a) Show that $E[g'(Z)] = E[Zg(Z)]$. What assumptions do you need in order for this to be true?

   (b) Show that $E[Z^{n+1}] = nE[Z^{n-1}]$.

   (c) Find $E[Z^4]$.

   (d) Find $E[Z^{2n}]$ and $E[Z^{2n+1}]$.

   (e) Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Show that $E[(X-\mu)g(X)] = \sigma^2 E[g'(X)]$ when both sides exist.

2. Let $\phi(x)$ be the standard normal density and let $\Phi(x)$ be the standard normal cdf. Show that $\phi'(x) + x\phi(x) = 0$. Deduce that

$$\frac{1}{x} - \frac{1}{x^3} < \frac{1-\Phi(x)}{\phi(x)} < \frac{1}{x} - \frac{1}{x^3} + \frac{3}{x^5}, \quad x > 0.$$

   This inequality is useful since it helps estimate $\Phi(x)$, which doesn't have a closed-form expression.

3. Four witnesses, Alice, Bob, Carol, and Dave, at a trial each tell the truth with probability $\frac{1}{3}$ independent of each other. In their testimonies, Alice claimed that Bob denied that Carol declared that Dave lied. What is the conditional probability that Dave told the truth?

4. Consider a stick of unit length. Break the stick at two points chosen uniformly at random. What is the probability that you can form a triangle from the three pieces that remain.

5. A roulette wheel has 38 slots: 18 red, 18 black, and 2 green. A ball is tossed into the wheel and eventually settles in a slot at random. You play many games, betting \$1 on red each time. If the ball lands in a red slot you win \$1, otherwise you lose the dollar you bet. After n games, what is the probability that you have more money than you started with? Give (approximate) numerical answers for $n = 100$ and $n = 1000$. How would the answer change if there were no green slots?

6. How many times do you need to toss a fair coin in order to get 100 heads with probability at least 0.9?