Plan for final:

June 8

Cumulative Exam

5 questions, choose 4.

- each question will have ≥ 2 parts

1. A biased coin lands heads with probability 1/10. The coin is flipped 200 times. Use Markov's inequality to give an upper bound on the probability that the coin shows heads at least 120 times. Improve this bound using Chebyshev's inequality. Improve it even further with Chernoff's inequality.

Concentration inequality

Markov's ineq: if X is a rv, s.t. $X \geq 0$, then

$$\Pr[X \geq a] \leq \frac{\mathbb{E}[X]}{a} \quad \forall a > 0$$

Let $X = \#H$, want

$$\Pr[X \geq 120] \leq \frac{\mathbb{E}[X]}{120} = \frac{20}{120} = 1/6$$

$$\| \sum_{k=120}^{200} \binom{200}{k} \left(\frac{1}{10}\right)^k \left(\frac{9}{10}\right)^{200-k}$$

$X \sim Bin(200, 1/10)$

Chernoff's inequality:

Say $X = \sum_{i=1}^{n} X_i$, where

$X_i \sim \text{Bern}(p_i)$ independent.

let $\mu = \mathbb{E}[X] = \sum_{i=1}^{n} p_i$

i) $\Pr[X \geq (1+\delta)\mu] \leq \exp\left(-\frac{\delta^2}{2+\delta}\mu\right)$

$\forall \delta > 0$

Corr: $\Pr[|X-\mu| \geq \delta\mu] \leq 2e^{\frac{-\delta^2\mu}{3}}$

$\forall\, 0 < \delta < 1$

Using Chernoff: $X = \#$ H.

$\mu = 200 \cdot 1/10 = 20$

$\Pr[X \geq 120] = \Pr[X \geq 6 \cdot \mu] = \Pr[X \geq (1+5)\mu]$

$\leq \exp\left(\frac{-5^2}{2+5} 20\right) = \exp\left(-\frac{25 \cdot 20}{7}\right)$

$\approx 9.53 \times 10^{-32}$

2. The average height of an anteater is 10 inches.

(a) Give an upper bound on the probability that a certain ~~raccoon~~ is at least 15 inches tall.

(b) The standard deviation for this height distribution is 2 inches. Find a lower bound on the probability that a certain ~~raccoon~~ is between 5 and 15 inches tall.

$Var = 4$

(c) Now assume that this distribution is normal. Use a normal table to repeat the calculation from part (b). How close was your bound to the true probability?

a) Let X = height of anteater

Markov $\Rightarrow \Pr[X \geq 15] \leq \frac{10}{15} = 2/3$

b) Chebyshev's inequality: if X is an rv then $\forall t > 0$,

$$\Pr[|X - E[X]| > t] \leq \frac{Var[X]}{t^2}$$

$\mu = 10, \sigma^2 = 4$

Cheb: $\Pr[|X - 10| > 5] = \Pr[X > 15 \text{ or } X < 5]$

$$\leq \frac{Var[X]}{5^2} = \frac{4}{25}$$

$$\Rightarrow \Pr[|X - 10| \leq 5] = 1 - \Pr[|X - 10| > 5]$$

$$\geq 1 - \frac{4}{25} = \underline{21/25}$$