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Induction

Prove that

$$1+2+\cdots+n=\frac{n(n+1)}{2}.$$

Proof: Induction.

Base case: $n=1$

$$1 \stackrel{?}{=} \frac{1(1+1)}{2} = \frac{2}{2} = 1 \checkmark$$

Inductive step: suppose the proposition is true for all $k \le n$

ie $1+2+\cdots+k = \frac{k(k+1)}{2}$

$\forall k \le n$

Show it's true for $n+1$.

$$\underbrace{1+2+\cdots+n}_{\text{inductive hypothesis}}+(n+1)$$

$$= \frac{n(n+1)}{2}+(n+1)$$

$$=\frac{n^2+n+2n+2}{2}$$

$$=\frac{n^2+3n+2}{2}=\frac{(n+1)(n+2)}{2}$$

The claim follows by induction $\square$

Erdős (19__):

$$\prod_{p \le n} p \le 4^{n-1}$$

$\underbrace{p \le n}$ primes $\le n$.

Pf: Base case

$n = 1$ ⚠ 1 is not prime

$$\prod_{p \le 1} p = \text{empty product} = 1$$

$$\overset{?}{\le} 4^{1-1} = 4^0 = 1 \checkmark$$

Inductive step: suppose

$$\prod_{p \le k} p \le 4^{k-1} \quad \forall k \le n$$

strong induction

$n \geq 2$

$$\prod_{p \le n+1} p$$

• if $n+1$ is even, it's not prime

$$= \prod_{p \le n} p$$

$$\text{(ind. hyp)} \le 4^{n-1} < 4^n \checkmark$$

• Assume then $n = 2k$

$$\prod_{p \le n+1} p = \Big(\prod_{p \le k+1} p \Big) \Big(\prod_{k+2 \le p \le 2k+1} p \Big) \quad (*)$$

Consider the binomial coefficient

$$\binom{2k+1}{k} = \frac{(2k+1)!}{k!\,(k+1)!}$$

$$= \frac{(2k+1)(2k)\cdots(k+2)}{k!}$$

is divisible by all primes in $[k+2, 2k+1]$.

Since $k!$ divisible only by primes $\leq k$.

$$(*) \le \left(\prod_{p \le k+1} p\right)\binom{2k+1}{k}$$

induct:

$$\le 4^k \binom{2k+1}{k}$$

Claim: $\binom{2k+1}{k} \le 4^k$

$$2 \cdot 4^k = 2^{2k+1} = (1+1)^{2k+1}$$

$$= \sum_{j=0}^{2k+1} \binom{2k+1}{j}$$

$$= \binom{2k+1}{k} + \binom{2k+1}{k+1} + \text{stuff}$$

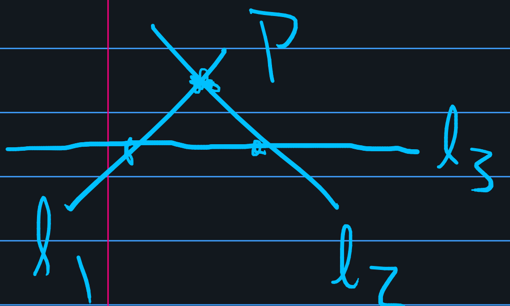
$$= 2\binom{2k+1}{k} + \text{stuff}$$

5. What's wrong with this proof?

Claim. *If we have n lines in the plane, no two of which are parallel, then they all go through one point.*

Proof. This is clearly true for one or two lines by definition. Suppose that it is true for any set of n lines and let $S = \{\ell_1, \ell_2, \ell_3, \ell_4, \ldots, \ell_{n+1}\}$ be a set of $n+1$ lines in the plane, no two of which are parallel. Delete the line ℓ_3 to obtain a set S' of n lines, no two of which are parallel. By the induction hypothesis, the lines in S' must all pass through some point P. In particular, ℓ_1 and ℓ_2 pass through P.

Now put ℓ_3 back and delete ℓ_4 instead to get another set S'' of n lines, no two of which are parallel. Again by the induction hypothesis, they must all pass through some point Q. In particular, ℓ_1 and ℓ_2 pass through Q. But ℓ_1 and ℓ_2 pass through P. Since two lines can pass through at most one point, we must have $P = Q$. But then ℓ_3 goes through P, so all the lines in S go through P. $\square$



no ℓ_4

Inductive step is assuming a base case of 3.

but their base case is 2.