Homework 3
Math 270B, Winter 2020, Prof. Roman Vershynin

Problem 1 (Quadratic martingales)

Let $X_1, X_2, \ldots$ be independent random variables with means μ_i and finite variances $\sigma_i = \mathbb{E}\, X_i^2$. Consider the sums $S_n := X_1 + \cdots + X_n$. Find sequences of real numbers (b_i) and (c_i) such that $S_n^2 + b_n S_n + c_n$ is a martingale (with respect to the σ-algebras generated by $X_1, \ldots, X_n$).

Problem 2 (Maximum of two martingales)

(a) Show that if (X_n) and (Y_n) are martingales with respect to the same filtration, then $X_n \vee Y_n$ is a aubmartingale.

(b) Give an example showing that $X_n \vee Y_n$ needs not be a martingale.

Problem 3 (An unbalanced martingale)

Give an example of a martingale (X_n) such that $X_n \to -\infty$ a.s.

Problem 4 (One-sided bounded martingales)

Let (X_n) be a martingale that is bounded a.s. either above or below by some constant M. Show that $\sup_n \mathbb{E}\,|X_n| < \infty$.

Problem 5 (A product martingale vanishes)

Let $Z_1, Z_2, \ldots$ be nonnegative i.i.d. random variables with $\mathbb{E}\, Z_i = 1$ and $\mathbb{P}\,\{Z_i = 1\} < 1$. Show that, as $n \to \infty$,

$$\prod_{i=1}^{n} Z_i \to 0 \quad \text{a.s.}$$

Problem 6 (Square integrable martingales)

Let (X_n) be a martingale and let $\Delta_n := X_n - X_{n-1}$ be the martingale differences. Prove that if $X_0 = 0$ and $\sum_{n=1}^{\infty} \Delta_n^2 < \infty$ then X_n converges in L^2 to some random variable X.

Problem 7 (Branching process)

Construct a branching process (Z_n) in the usual way. Namely, let X be a random variable with mean μ and variance σ^2; it specifies the distribution of the offspring. Set

$$Z_{n+1} := X_1^{(n+1)} + \cdots + X_{Z_n}^{(n+1)}$$

to be the size of the population at time $n+1$, where all $X_i^{(k)}$ are i.i.d. random variables distributed identically with X.

(a) Show that $X_n := Z_n/\mu^n$ defines a martingale.

(b) Show that

$$\mathbb{E}[Z_{n+1}^2|\mathcal{F}_n] = \mu^2 Z_n^2 + \sigma^2 Z_n.$$

(c) Deduce that (X_n) is bounded in L^2 if an only $\mu > 1$.

(d) Show that when $\mu > 1$, the L^2-limit X of X_n (assume it exists) satisfies

$$\mathrm{Var}(X) = \frac{\sigma^2}{\mu(\mu-1)}.$$

Problem 8 (Unbounded martingale that converges a.s.)

Find an example of a martingale (X_n) that converges a.s. to some random variable X, but for which $\limsup_n \mathbb{E}\,|X_n| = \infty$.

(Hint: define the sequence $a_1 := 2$, $a_n := 4\sum_{i=1}^{n-1} a_i$. Consider independent random variables Z_n that take value $\pm a_n$ with probability $(2n)^{-2}$ and 0 with probability $1-n^{-2}$. Define $X_n := \sum_{i=1}^{n} Z_i$.)