Math 180B - More on Gaussian Integers

1. Find a greatest common divisor for each of the following pairs of Gaussian integers.

   (a) $\alpha = 8 + 38i$ and $\beta = 9 + 59i$

   (b) $\alpha = -9 + 19i$ and $\beta = -19 + 4i$

2. Let R be the following set of complex numbers:

$$R = \{a + bi\sqrt{5} : a, b \in \mathbb{Z}\}.$$

   (a) Verify that R is a ring.

   (b) Show that the only units in R are 1 and -1.

   (c) Recall that an element α of R is irreducible if and only if its only divisors in R are units and unit multiples of α. Prove that 2 is an irreducible element of R.

   (d) Define the norm of an element $\alpha = a + bi\sqrt{5}$ to be $N(\alpha) = \alpha\bar{\alpha} = a^2 + 5b^2$. Let $\alpha = 11 + 2i\sqrt{5}$ and $\beta = 1 + i\sqrt{5}$. Show that it is not possible to find elements γ and ρ in R satisfying

$$\alpha = \beta\gamma + \rho \quad \text{and} \quad N(\rho) < N(\beta).$$

   Thus, R does not have the division with remainder property.

   (e) Show that 2 does not divide either factor in the product.

$$(1 + i\sqrt{5})(1 - i\sqrt{5}) = 6.$$

   Conclude that 2 is not prime in R.

   (f) Conclude that R does not have the unique factorization property