Math 175 - Homework 3

1. There are twenty-five beads in a bag, each colored red, blue or green. Prove that there are at least nine beads of the same color.

2. Let $A = \{a_1, a_2, a_3, a_4, a_5\}$ be a set of five positive integers. Show that for any permutation $a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4}, a_{i_5}$ of A, the product

$$(a_{i_1} - a_1)(a_{i_2} - a_2)(a_{i_3} - a_3)(a_{i_4} - a_4)(a_{i_5} - a_5).$$

   is always even.

3. Let K_n be the complete graph on n vertices, i.e. the graph with n vertices where every pair of vertices is connected by an edge.
   (a) Suppose we color each edge of K_6 either red or blue. Prove that there is at least one monochromatic triangle. A triangle is monochromatic if all of its edges have the same color.
   (b) Is part (a) still true for K_5?
   (c) Prove that in any coloring of K_6 we actually have at least *two* monochromatic triangles. *Hint: Count the complement.*

4. (a) Is it possible to cover a chessboard (an 8 by 8 grid where squares alternate between two colors) with dominoes that cover exactly two squares?
   (b) What if we delete two white squares?

5. (a) Let A be a set of m positive integers where $m \geq 1$. Show that there exists a nonempty subset B of A such that the sum $\sum_{x \in B} x$ is divisible by m.
   (b) Let $X \subseteq \{1, 2, \ldots, 99\}$ and $|X| = 10$. Show that it is possible to select two disjoint nonempty proper subsets Y, Z of X such that $\sum_{y \in Y} y = \sum_{z \in Z} z$.