1. Show that the equation $x^2 + y^2 = 9z + 3$ has no integral solution.

Look at divisibility.

RHS is divisible by 3.

$\Rightarrow$ LHS divis by 3. reduce mod 3

$\Rightarrow x^2 + y^2 \equiv 0 \mod 3$

if $x \equiv 0 \Rightarrow x^2 \equiv 0 \mod 3$

$x \equiv 1 \Rightarrow x^2 \equiv 1 \mod 3$

$x \equiv 2 \Rightarrow x^2 \equiv 1 \mod 3$

$\Rightarrow x^2$ & $y^2 \equiv 0 \mod 3$

$\Rightarrow x$ & $y \equiv 0 \mod 3$

$\Rightarrow x^2$ & y^2 divisible by 9

$\Rightarrow x^2 + y^2$ divis by 9

but RHS $= 9z + 3 \equiv 3 \mod 9$.

contradiction $\Rightarrow$ no soln.

3. Here is a descent argument showing the irrationality of $\sqrt{2}$.

- Assume $\sqrt{2}$ is rational and write $\sqrt{2} = 1 + \frac{p}{q}$ where $p < q$.
- Then $2q^2 = q^2 + 2pq + p^2 \implies p^2 = q(q-2p) \implies \frac{p}{q} = \frac{q-2p}{p}$.
- But now $\sqrt{2} = 1 + \frac{q-2p}{p}$ has fractional part with denominator p smaller than q. Repeating the argument, we obtain an argument by descent.

Generalize the argument to prove that $\sqrt{n}$ is irrational whenever $n \in \mathbb{N}$ is not a perfect square. *Hint: write* $\sqrt{n} = m + \frac{p}{q}$ *where* m *is the integer part of* $\sqrt{n}$.

suppose n not a square, but $\sqrt{n} \in \mathbb{Q}$

write $\sqrt{n} = m + \frac{p}{q}$ $\quad p < q$, $p, q > 0$

$m \uparrow$ integer part of $\sqrt{n}$

$\implies q\sqrt{n} = qm + p$

$\implies nq^2 = q^2m^2 + 2mpq + p^2$

$\implies p^2 = (n-m^2)q^2 - 2mpq$

$= q\left[(n-m^2)q - 2mp\right]$

$$\implies \frac{p}{q} = \frac{(n-m^2)q - 2mp}{p}$$

$$\sqrt{n} = m + \frac{p}{q} = m + \frac{(n-m^2)q - 2mp}{p}$$

$p < q \implies$ denom smaller!

by descent (denom. keeps shrinking) we have a contradiction.

$\boxtimes$

4. Show that the equation $x^2 - dy^2 = -1$ has no solutions if $d \equiv 3 \pmod 4$.

mod 4 $\quad x^2 - dy^2 \equiv x^2 - 3y^2$

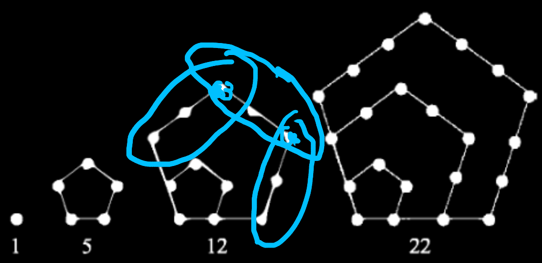
$\underline{3 = -1} \qquad \equiv \underline{x^2 + y^2} \quad$ mod 4

$\Rightarrow x^2 + y^2 \equiv 3 \mod 4$

x^2 & y^2 are either 1 or 0 mod 4.

$\boxtimes$

5. A number n is called a pentagonal number if n pebbles can be arranged in the shape of a filled in pentagon. The first four pentagonal numbers are 1, 5, 12, and 22, as illustrated in Figure 1. You should visualize each pentagon as sitting inside the next larger pentagon. The n-th pentagonal number is formed using an outer pentagon whose sides have n pebbles.



1 2 3

Figure 1: The First Four Pentagonal Numbers

(a) Find a formula for the n-th pentagonal number.

(b) Recall that square-triangular numbers correspond to solutions to the equation $x^2 - 2y^2 = 1$. Find a similar equation for square-pentagonal numbers.

a) $P_n = \sum_{k=1}^{n} (3k-2)$ ✓

$= 3\sum_{k=1}^{n} k - 2n$

$= \frac{3n(n+1)}{2} - 2n = \frac{3n^2 - n}{2}$

$= \frac{n(3n-1)}{2}$ $\boxtimes$

T_n : triangle #s, S_n : square #s
P_n : pent. #s.

Sq-triangular: $T_m = S_n$

$$\frac{m(m+1)}{2} = n^2$$

$\Rightarrow m(m-1) = 2n^2$

↑ complete square ...

$8n^2 = (2m+1)^2 - 1$

$y = 2n,\ x = 2m+1 \Longrightarrow x^2 - 2y^2 = 1$

$P_m = S_n \Longrightarrow \dfrac{m(3m-1)}{2} = n^2$

$m(3m-1) = 2n^2$

$3m^2 - m = 2n^2$

$3\left(m^2 - \frac{m}{3}\right) = 2n^2$

$3\left(m^2 - \frac{m}{3} + \frac{1}{36} - \frac{1}{36}\right) = 2n^2$

$\Rightarrow 3\left(m - \frac{1}{6}\right)^2 - \frac{1}{12} = 2n^2$

$36\left(m - \frac{1}{6}\right)^2 - 1 = 24n^2$

$$(6m-1)^2 - 1 = 24n^2$$

$x = 6m-1$ $\quad$ $y = 2n$

$$\Rightarrow x^2 - 1 = 6y^2$$

$$\Rightarrow \boxed{x^2 - 6y^2 = 1}$$