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271A- Homework 3

Problem 1

Show that the conditions of Kolmogorov's extension/consistency theorem are satisfied for the finite dimensional distributions associated with the Brownian motion paths.

Proof. Let B_t be a standard Brownian motion and let $t_1, \ldots, t_k \in \mathbb{R}$. The associated finite-dimensional distribution $\nu_{t_1,\ldots,t_k}$ is given by

$$\nu_{t_1,\ldots,t_k}(F_1 \times \cdots \times F_k) = \mathbb{P}[B_{t_1} \in F_1, \ldots, B_{t_k} \in F_k],$$

where the F_j's are arbitrary Borel subsets of $\mathbb{R}$. $\square$

Problem 2

Let B_t be a two-dimensional Brownian motion and

$$D_r = \{x \in \mathbb{R}^2 : |x| < r\}.$$

Compute $\mathbb{P}[B_t \in D_r]$.

Solution. B_t has components $B_t^{(1)}$ and $B_t^{(2)}$, both independent standard one-dimensional Brownian motions. Let's rewrite the desired probability.

$$\mathbb{P}[B_t \in D_r] = \mathbb{P}[(B_t^{(1)})^2 + (B_t^{(2)})^2 < r^2].$$

With the fundamental theorem of calculus, one can show that if a random variable X has density $f_X(s)$, then X^2 has density

$$f_{X^2}(s) = \frac{1}{2\sqrt{s}}[f_X(\sqrt{s}) + f_X(-\sqrt{s})] \cdot \mathbb{1}_{[0,\infty)}(s).$$

We also know that the density of the sum of two random variables is the convolution of their individual densities. Since $B_t^{(i)}$ has density $f(x) = (2\pi t)^{-1/2}e^{-\frac{x^2}{2t}}$, we convolve this with itself to obtain the density of the sum $X := (B_t^{(1)})^2 + (B_t^{(2)})^2$

$$\begin{aligned}
f_X(x) &= \left[(2\pi t)^{-1/2}e^{-\frac{x^2}{2t}}\right] * \left[(2\pi t)^{-1/2}e^{-\frac{x^2}{2t}}\right] \\
&= \frac{\exp(-\frac{x}{2t})}{2\pi t}\int_0^x \frac{dy}{\sqrt{(x-y)y}} \\
&= \frac{\exp(-\frac{x}{2t})}{2t}.
\end{aligned}$$

Thus, we have that the desired probability is given by

$$\mathbb{P}[X < r^2] = \frac{1}{2t} \int_0^{r^2} e^{-\frac{x}{2t}} \, dx$$
$$= 1 - e^{-\frac{r^2}{2t}}.$$

□

Problem 3

Let B_t be a Brownian motion. Show that

(a.) $Y_t = B_T - B_{T-t}$ is a Brownian motion on $[0, T]$.

Proof. We have that $Y_0 = B_T - B_T = 0$. □