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271B - Homework 1

Problem 1. The standard Ornstein-Uhlenbeck process X_t is a Gaussian process with mean zero and auto-covariance $C(t,s) = \mathbb{E}[X_t X_s] = \exp(-|t-s|)/2$. Let N_t be the standard Poisson process and define the process $Y_t = \zeta(-1)^{N_t}$, where ζ is a random variable independent of the Poisson process that takes values ± 1 with probability $1/2$.

Show that X_t and $Z_t = Y_{t/2}/\sqrt{2}$ are both stationary in the strong sense and have the same covariance. Does Y_t satisfy the Kolmogorov continuity condition? Are these processes stochastically continuous?

Solution. First we'll show that X_t is stationary. Let $\tau \in \mathbb{R}$. Since X_t is Gaussian with mean zero for all t, so is $X_{t+\tau}$. For any s and t we also have that

$$\mathbb{E}[X_{s+\tau}X_{t+\tau}] = \frac{1}{2}e^{-|(s+\tau)-(t+\tau)|} = \frac{1}{2}e^{-|s-t|} = \mathbb{E}[X_s X_t].$$

Since a Gaussian process is determined by its mean and covariance, we have that X_t and $X_{t+\tau}$ are equal in distribution, so the process is stationary.

Now for Z_t. We claim that Z_t has the Markov property, i.e. for any $t_1 < t_2 < \ldots < t_n$ and $\alpha_i = \pm 1/\sqrt{2}$

$$\begin{aligned}\mathbb{P}[Z_{t_1} = \alpha_1,\ Z_{t_2} = \alpha_2, \ldots, Z_{t_n} = \alpha_n]& \\ = \mathbb{P}[Z_{t_1} = \alpha_1]\mathbb{P}[Z_{t_2} = \alpha_2 \mid Z_{t_1} = \alpha_1] \cdots \mathbb{P}[Z_{t_n} = \alpha_n \mid Z_{t_{n-1}} = \alpha_{n-1}]&\end{aligned} \tag{1}$$

Informally, the value of Z_{t_j} given $Z_{t_1}, \ldots, Z_{t_{j-1}}$ depends only on the number of sign flips of Z over the interval $(t_{j-1}, t_j]$. This only depends on the parity of $N_{t_j} - N_{t_{j-1}}$. Let's look at the terms on the right-hand side of (1).

$$\begin{aligned}\mathbb{P}[Z_{t_j} = \alpha_j \mid Z_{t_{j-1}} = \alpha_{j-1}] &= \begin{cases} \mathbb{P}[N_{t_j - t_{j-1}} \text{ is even}], & \text{if } \alpha_j = \alpha_{j-1} \\ \mathbb{P}[N_{t_j - t_{j-1}} \text{ is odd}], & \text{if } \alpha_j = -\alpha_{j-1} \end{cases} \\ &= \mathbb{P}[Z_{t_j+m} = \alpha_j \mid Z_{t_{j-1}+m} = \alpha_{j-1}]\end{aligned} \tag{2}$$

The last equality follows from the stationarity of Poisson increments. Equations (1) and (2) imply that Z is indeed stationary.

Let's compute the covariance of Z_t. Since ζ is independent of N_t we have

$$\mathbb{E}[Z_t] = \frac{1}{\sqrt{2}}\mathbb{E}[\zeta] \cdot \mathbb{E}[Y_{t/2}] = 0.$$

Consequently, for any s and t, the covariance is given by

$$\mathbb{E}[Z_s Z_t] = \mathbb{E}[Z_0 Z_{|t-s|}] = \frac{1}{2}\mathbb{E}[\zeta^2]\mathbb{E}\left[(-1)^{N_{|t-s|/2}}\right] = \frac{1}{2}\left(\mathbb{P}[N_{|t-s|/2} \text{ is even}] - \mathbb{P}[N_{|t-s|/2} \text{ is odd}]\right)$$

$$= \frac{1}{2}(2\mathbb{P}[N_{|t-s|/2} \text{ is even}] - 1). \quad (3)$$

As for that probability, we have

$$\mathbb{P}[N_{|t-s|/2} \text{ is even}] = \sum_{n=0}^{\infty} \mathbb{P}[N_{|t-s|/2} = 2n] = \sum_{n=0}^{\infty} \frac{(|t-s|/2)^{2n} e^{-|t-s|}}{(2n)!} = e^{-|t-s|/2}\cosh(|t-s|/2).$$

Substituting this expression into (3) gives

$$\mathbb{E}[Z_s Z_t] = \frac{1}{2}e^{-|t-s|/2} = \mathbb{E}[X_s X_t],$$

as desired.

Let's check to see if Y_t satisfies the Kolmogorov continuity condition. For any s and t, the quantity $|(-1)^{N_t} - (-1)^{N_s}|$ will be zero if N_t and N_s have the same parity and 2 if they have opposite parity. By the stationarity of Poisson increments, we have that

$$\left|(-1)^{N_t} - (-1)^{N_s}\right| = \begin{cases} 0, & N_{|t-s|} \text{ is even} \\ 2, & N_{|t-s|} \text{ is odd} \end{cases}.$$

Let $\alpha > 0$. By the above reasoning, we have that

$$\mathbb{E}[|Y_t - Y_s|^\alpha] = 2^\alpha \mathbb{P}[N_{|t-s|} \text{ is odd}] = 2^\alpha e^{-|t-s|}\sinh|t-s| = 2^{\alpha-1}\left(1 - e^{-2|t-s|}\right). \quad (4)$$

We claim that there are no positive K or β such that

$$\mathbb{E}[|Y_t - Y_s|^\alpha] \le K|t-s|^{1+\beta}$$

for all s, t. The right-hand side of (4) is $\Theta(|t-s|)$ as $|t-s| \to 0$, while $K|t-s|^{1+\beta}$ is $o(|t-s|)$ as $|t-s| \to 0$. We conclude that Y_t does *not* satisfy the Kolmogorov continuity condition.

Let's check for stochastic continuity. By Markov we have

$$\mathbb{P}[|X_{t+h} - X_t| > \delta] \le \frac{1}{\delta^2}\mathbb{E}[(X_{t+h} - X_t)^2]$$

$$= \frac{1}{\delta^2}\left(1 - e^{-|h|}\right),$$

which goes to zero as $h \to 0$ for any $\delta > 0$, so X is stochastically continuous. Now for Y. The quantity $|Y_{t+h} - Y_t|$ is zero when N_{t+h} and N_t have the same parity and is 2 when they have opposite parity. For $\delta < 2$ we have

$$\mathbb{P}[|Y_{t+h} - Y_t| > \delta] = \mathbb{P}[N_{|h|} \text{ is odd}]$$

$$= e^{-|h|}\sinh|h|,$$

which goes to zero as $h \to 0$, so Y is stochastically continuous. The same argument shows that Z is stochastically continuous as well. $\square$

Problem 2. Let X_n be defined by the stochastic recursion

$$X_{n+1} = X_n - \Delta t X_n + (B_{(n+1)\Delta t} - B_{n\Delta t}), \ X_0 = \zeta, \tag{5}$$

for B_t standard Brownian motion. Find ζ so that X_n is stationary in the strong sense and give the associated auto-covariance function. What is the continuum limit of this process as $n \to \infty$, $\Delta t \to 0$ so that $n\Delta t = t$.

Solution. By induction we have that

$$X_{n+1} = (1-\Delta t)^{n+1}\zeta + \sum_{k=0}^{n}(1-\Delta t)^{n-k}(B_{(k+1)\Delta t} - B_{k\Delta t}). \tag{6}$$

By the above expansion, we can see that for $0 < \Delta t < 1$, ζ contributes less to X_{n+1}. The sum term is a sum of independent Gaussians, and hence Gaussian. We conclude that for n large, X_n approaches a Gaussian. In order for the process to be stationary, ζ must also be Gaussian.

Since ζ is Gaussian, it is determined by its mean and variance. Taking the expectation on both sides of the recursive formula (5) gives

$$\mathbb{E}[X_{n+1}] = (1-\Delta t)\mathbb{E}[X_n].$$

By stationarity, $\mathbb{E}[X_{n+1}] = \mathbb{E}[X_n]$. The above equation then forces $\mathbb{E}[X_n] = 0$ for all n, so $\mathbb{E}[\zeta] = 0$ as well. Taking the variance of both sides of the inductive formula (6) and using stationarity gives

$$\begin{aligned}\mathrm{Var}[\zeta] = \mathrm{Var}[X_{n+1}] &= (1-\Delta t)^{2(n+1)}\mathrm{Var}[\zeta] + \Delta t\sum_{k=0}^{n}(1-\Delta t)^{2(n-k)} \\ &= (1-\Delta t)^{2(n+1)}\mathrm{Var}[\zeta] + \Delta t(1-\Delta t)^{2n}\cdot\frac{1-(1-\Delta t)^{-2(n+1)}}{1-(1-\Delta t)^{-2}}.\end{aligned}$$

Solving for $\mathrm{Var}[\zeta]$ gives $\mathrm{Var}[\zeta] = \frac{1}{2-\Delta t}$.

Now let's show that the choice $\zeta \sim \mathcal{N}(0, \frac{1}{2-\Delta t})$ makes X_n stationary. It's clear that this choice of ζ makes X_n a Gaussian process with zero mean for all n, so to check stationarity, it suffices to show that $\mathrm{Cov}(X_n X_{n+1})$ is independent of n. The same calculation that we used to find $\mathrm{Var}[\zeta]$ shows that $\mathrm{Var}[X_n] = \frac{1}{2-\Delta t}$. Now we compute the covariance.

$$\mathrm{Cov}(X_n, X_{n+1}) = (1-\Delta t)\mathrm{Var}[X_n] + \mathrm{Cov}(X_n, B_{(n+1)\Delta t} - B_{n\Delta t}) = \frac{1-\Delta t}{2-\Delta t}.$$

Here we've used the fact that disjoint increments of Brownian motion are independent. Since the covariance is independent of n, we conclude that this choice of ζ does indeed make the process stationary. By induction, the auto-covariance is given by

$$\mathrm{Cov}(X_n, X_{n+m}) = \frac{(1-\Delta t)^m}{2-\Delta t}.$$

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