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271A- Homework 2

1. Use Jensen's inequality to show that for $p \geq 1$.

$$\|\mathbb{E}[X|\mathcal{G}]\|_{L^p} \leq \|X\|_{L^p}.$$

Proof. Let's look at the p-norm to the p-th power.

$$\begin{aligned}
\|\mathbb{E}[X|\mathcal{G}]\|_{L^p}^p &= \int |\mathbb{E}[X|\mathcal{G}]|^p \, d\mathbb{P} \\
&\leq \int \mathbb{E}[|X|^p|\mathcal{G}] \, d\mathbb{P} \quad \text{(by Jensen's inequality)} \\
&= \int |X|^p \, d\mathbb{P} \quad \text{(by definition of conditional expectation)} \\
&= \|X\|_{L^p}^p.
\end{aligned}$$

Taking the p-th root of both sides establishes the claim. □

2. Let $(X_n : n \in \mathbb{N})$ be a sequence of independent random variables, each exponentially distributed:

$$\mathbb{P}[X_n > x] = e^{-x}, \quad x \geq 0.$$

(a) A random variable τ has the lack of memory property if

$$\mathbb{P}[\tau > a + b \mid \tau > a] = \mathbb{P}[\tau > b].$$

Show that a random variable has the memoryless property if and only if it is exponentially distributed.

Proof. Suppose τ is exponentially distributed, i.e.

$$\mathbb{P}[\tau > x] = \begin{cases} e^{-\lambda x}, & \text{if } x \geq 0, \\ 1, & \text{if } x < 0. \end{cases}$$

By the definition of conditional probability we have

$$\mathbb{P}[\tau > a + b \mid \tau > a] = \frac{\mathbb{P}[(\tau > a + b) \wedge (\tau > a)]}{\mathbb{P}[\tau > a]}.$$

□