Problem Set 5 Math 271C, Spring 2020.

1. a) Let l_T^x be local time (at origin) at time T for $Y_t = \sigma\beta_t + x$ with β standard Brownian motion. Set up problems for determining $\mathbb{E}[l_T^x]$ and $\mathbb{E}[(l_T^x)^2]$ (the pdes).

   b) Solve the problem for $\mathbb{E}[l_T^x]$.

2. a) Show that the diffusion matrix a with

$$a_{i,j}(\boldsymbol{x}) = \int_{-\infty}^{\infty} \mathbb{E}[F_i(\boldsymbol{x}, Y(0))F_j(\boldsymbol{x}, Y(t))]dt$$

   is non-negative definite (for Y and ergodic Markov process with notation as in (last) class and notes on Canvas).

   b) Consider the reduced SIR model:

$$\frac{d}{dt}I_t = \kappa I_t(1 - I_t) - \lambda I_i,$$

   with the model for the contact rate and the death/recovery rate:

$$\begin{aligned} \kappa &= \kappa_t = \bar{\kappa} + F(Y_{t/\varepsilon}), \\ \lambda &= \bar{\lambda} + \varepsilon\tilde{\lambda}, \end{aligned}$$

   with $\varepsilon \ll 1$ and with $|F| < \bar{\kappa}$ and F centered w.r.t. the invariant distribution for the ergodic diffusion Y and where the contact rate fluctuations are relatively rapid (happens on a fast, say daily, scale relative to the recovery/death time scale $1/\bar{\lambda}$). Introduce the time and amplitude rescaling $t' = \varepsilon^a t, I_0 = \varepsilon^b \bar{I}_0$ and identify values for $a, b, \bar{\lambda}$ so that the problem is in diffusion limit form.

   c) Identify the (weak) limit Itô diffusion and write it both in Itô and Stratonovich form.

3. Problem 11.1 page 254 Øksendal.

4. **Variance reduction**:

   Consider the sde

$$dX_t = \mu(X_t)dt + \sigma(X_t)d\beta_t,$$

   and $u(t, x) = \mathbb{E}^x[f(X_t)]$ with μ, σ, f smooth and bounded, and x, β scalar valued. Assume that we can solve the SDE, but solving the pde problem for u is expensive (computing u may in particular be expensive in cases where the dimension of x is large). Define the Monte-Carlo estimate for u:

$$u(T, x) \approx \hat{u}^N = \frac{1}{N}\sum_{i=1}^{N} f(X_T^{(i)}),$$

with $X_t^{(i)}$ independent realizations for X_t (assume the sde can be solved weakly "exactly").

a) Show that $\hat{u}^N$ is an unbiased estimate and that

$$\mathrm{Var}(\hat{u}^N) = \frac{1}{N}\mathbb{E}^x\left[\int_0^T \sigma^2(X_s)u_x^2(T-s,X_s)ds\right].$$

b) Consider

$$dY_t = (\mu(Y_t) + v(t,Y_t))dt + \sigma(Y_t)d\beta_t, \tag{1}$$

and assume uniform ellipticity: $\sigma(x) > \Delta > 0$. Denote the stochastic exponentials:

$$M_t^{\pm} = \mathcal{E}\left(\pm\int_0^t v(s,Y_s)\sigma(Y_s)d\beta_s\right),$$

(assume that v satisfies appropriate integrability conditions) and show that (β Brownian Motion under $\mathbb{E}$):

$$u(t,x) = \mathbb{E}^x[f(X_t)] = \mathbb{E}^x[f(Y_t)M_T^-].$$

c) Introduce the estimator (given N independent realizations of the paths Y)

$$u(T,x) \approx \check{u}^N = \frac{1}{N}\sum_{i=1}^N f(Y_T^{(i)})M_T^{-,(i)}.$$

d) Assume that $f > \Delta > 0$ and show that v satisfying

$$\partial_x u(t,x)\sigma(x) = v(t,x)u(t,x)/\sigma(x), \tag{2}$$

is well defined. Show that with this choice of v in Eq. (1): $\check{u}^1 = u(T,x)$.

Hint: derive an expression for the variance of $f(Y_T)M_T^-$ using Itô's rule and isometry, you may alternatively be able to show $\check{u}^1 = u(T,x)$ directly, again using Itô calculus in view of the form of v.

Note that to find this "optimal control" or "importance sampler" v we need the function u which is the unknown. In practice we may use an estimate for u and an iteration in the Monte-Carlo estimation to improve this a priori estimate. (This process may in fact involve considering an augmented Markov process that allows for joint estimation of $u, \partial_x u$).

d) Consider now v in Eq. (1) as a control. Set up an optimal control problem such that the optimal control is given by Eq. (2).

Hint: consider

$$V(t,x) = \min_v \mathbb{E}^{t,x}[\int_t^T \sigma^2(Y_s)v_s^2/2ds - \log(f(Y_T))].$$

5. a) By considering the null-space of the generator for Brownian motion show that this process cannot have an invariant distribution.

   b) For the n-dimensional Ornstein Uhlenbeck process show that the null-space is spanned by the constant functions and the null-space of the adjoint by the invariant distribution.

   c) Set up a (pde) problem that (when solved) gives the probability that an OU process Y_t starting at $y, (|y| < R)$ satisfies $\max_{0<t<T} |Y_t| < R$.

6. Problem 8.17 page 102 Karatzas & Schreve.

7. Problem 8.18 page 102 Karatzas & Schreve.

8. Consider Brownian Motion starting at $x \in (0,1)$ and reflected at $0, 1$.

   a) Use the associated spectral representation (as in class) to show that the 1d Laplacian satisfies the Fredhom Alternative with respect to square integrable functions on $[0,1]$ satisfying homogeneous Neumann boundary conditions.

   b) **The Resolvent & Shifting spectrum:**

   Show that $\Delta/2 - \lambda I$ (I the identity operator) is invertible w.r.t. functions square integrable on $[0,1]$ satisfying homogeneous Neumann boundary conditions, but not necessarily a solvability condition as in a).

   Show that for such a function f

$$\left((\Delta/2 - \lambda I)^{-1} f\right) = (-R_\lambda f)(x) = -\int_0^\infty \mathbb{E}^{0,x}[f(X_t)e^{-\lambda t}dt]$$

   for appropriate X_t and comment on how we can interpret this in terms of a killed diffusion (which?) (here R_λ called resolvent). Express also $(R_\lambda f)(x)$ in terms of the spectral decomposition in a).