work on Canvas problems

until ~3:20 ish

$R \subseteq \mathbb{C}$ is a ring if

1) $r+s \in R \quad \forall r,s \in R$,
2) $rs \in R \quad \forall r,s \in R$
3) $0,1 \in R$
4) if $r \in R \implies -r \in R$
5) $r(s+t) = rs + rt \quad \forall r,s,t \in R$

$r \in R$ is a unit if $rs = 1$ for some $s \in R$.

$a+bi \mid c+di$

$$\Rightarrow \quad \mathbb{Z}[i] \ni \frac{c+di}{a+bi} = \frac{(c+di)(a-bi)}{a^2+b^2}$$

$$= \underbrace{\frac{ac+bd}{a^2+b^2}}_{\in \mathbb{Z}} + \underbrace{\frac{ad-bc}{a^2+b^2}}_{\in \mathbb{Z}} \cdot i$$

$a+bi \mid c+di \Longrightarrow a^2+b^2 \mid c^2+d^2$

Pf: $\Downarrow$ $(a+bi)\gamma = c+di$, some $\gamma \in \mathbb{Z}[i]$

$$\Rightarrow \quad N((a+bi)\gamma) = N(c+di)$$

$$N((a+bi)\gamma) = N(a+bi)N(\gamma) = (a^2+b^2)\underbrace{N(\gamma)}_{\in \mathbb{Z}} = c^2+d^2$$

$\Longrightarrow a^2+b^2 \mid c^2+d^2$

$\boxtimes$

2. Verify that each of the following subsets R_1, R_2, R_3, and R_4 of the complex numbers is a ring. In each case, describe the ring's units.

(a) $R_1 = \{a + bi\sqrt{2} : a, b \in \mathbb{Z}\}$.

(b) Let $\omega = -\frac{1}{2} + \frac{1}{2}i\sqrt{3}$. $R_2 = \{a + b\omega : a, b \in \mathbb{Z}\}$. $\rightarrow 1 + \omega + \omega^2 = 0$

$\Rightarrow \omega^2 = -1 - \omega$

(c) Let p be a fixed (integer) prime. $R_3 = \{a/d : a, d \in \mathbb{Z},\ p \nmid d\}$.

(d) $R_4 = \{a + b\sqrt{3} : a, b \in \mathbb{Z}\}$.

just have to show closure + inverses + identities

↑ +

- $(a + bi\sqrt{2}) + (c + di\sqrt{2})$

$$= (a+c) + (b+d)i\sqrt{2}$$

- $(a + bi\sqrt{2})(c + di\sqrt{2}) = (ac - 2bd) + (ad + bc)i\sqrt{2}$

$$\frac{a}{d} + \frac{a'}{d'} = \frac{ad' + a'd}{dd'}$$

$$p \nmid d,\ p \nmid d' \Rightarrow p \nmid dd'$$

$$\frac{aa'}{dd'}$$

$R_1 = \{a + bi\sqrt{2} \;;\; a, b \in \mathbb{Z}\}$

Find the units.

$a + bi\sqrt{2}$ is a unit if $\exists\, c + di\sqrt{2}$ s.t.

$$(a + bi\sqrt{2})(c + di\sqrt{2}) = 1$$

Consider $N : R_1 \to \mathbb{Z}^+$,

$$a + bi\sqrt{2} \mapsto (a + bi\sqrt{2})(a - bi\sqrt{2})$$
$$= a^2 + 2b^2$$

It's a norm: $N(\alpha\beta) = N(\alpha)N(\beta)$

α is a unit iff $N(\alpha) = 1$

$\Rightarrow a + bi\sqrt{2}$ unit iff

$$N(a + bi\sqrt{2}) = a^2 + 2b^2 = 1$$

if $b \geq 1$, LHS $> 1 \Rightarrow b = 0$

$\Rightarrow a^2 = 1 \Rightarrow a = \pm 1$

$\Rightarrow$ units are ± 1

$R_2 = \{a+b\omega : a,b \in \mathbb{Z},\ 1+\omega+\omega^2=0\}$

$\omega = -1/2 + \sqrt{3}/2\, i$

$\overline{\omega} = -1/2 - \sqrt{3}/2\, i$

$a+b\omega$ is a unit iff

$$1 = N(a+b\omega) = (a+b\omega)\overline{(a+b\omega)}$$

$$= (a+b\omega)(a+b\overline{\omega})$$

$$= a^2 + ab(\omega+\overline{\omega}) + b^2\omega\overline{\omega}$$

$$= a^2 - ab + b^2$$

find a,b s.t. $a^2 - ab + b^2 = 1$

$$4a^2 - 4ab + 4b^2 = 4$$

$$(2a-b)^2 + 3b^2 = 4$$

need $-1 \leq b \leq 1$

$b = 1 \quad (2a-1)^2 + 3 = 4$

$$(2a-1)^2 = 1$$

$2a-1 = 1 \qquad 2a-1 = -1$

$2a = 2 \qquad 2a = 0$

$a = 1 \qquad a = 0$

$$b=0 \Rightarrow (2a)^2+0=4$$

$$\Rightarrow 4a^2=4 \Rightarrow a=\pm 1$$

$$b=-1 \Rightarrow (2a+1)^2+3=4$$

$$(2a+1)^2=1$$

$$2a+1=1 \qquad 2a+1=-1$$

$$a=0 \qquad a=-1$$

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units are $\{1, -1, \omega, 1+\omega, -\omega, -1-\omega\}$

$$1+\omega+\omega^2=0$$

$$\omega^2=-1-\omega$$

$$=\{1, -1, \omega, -\omega^2, -\omega, \omega^2\}$$