1. A random variable X has density

$$f(x) = \begin{cases} 3cx^3 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$2x(x+c)$

$c = \frac{4}{3} \Rightarrow f(x) = 4x^3$

(i) (10 points) Determine c (that is, for which value of c the function f is indeed a density function?).

(ii) (15 points) Compute the density function of X^2.

(iii) (15 points) Compute the density function of $1/X$

(iv) (15 points) Compute $Var(1/X)$.

(v) (15 points) A trial has a success probability $q = \Pr[1/4 \le X \le 1]$. Suppose that independent such trials are being performed and let Y be the first time that a success occurs. Calculate $\mathbb{E}[Y^2]$.

$$Var[Y] = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2$$

$$Var\left[\frac{1}{X}\right] = \mathbb{E}\left[\frac{1}{X^2}\right] - \mathbb{E}\left[\frac{1}{X}\right]^2$$

$$\mathbb{E}\left[\frac{1}{X^2}\right] = \int_0^1 \frac{1}{x^2} f(x)\,dx$$

$$= \int_0^1 \frac{1}{x^2}(4x^3)\,dx = 4\int_0^1 x\,dx$$

if you used $f(x) = x^2(x+c)$

$$\mathbb{E}\left[\frac{1}{x^2}\right] = \int_0^1 \frac{1}{x^2} x^2(x+c)\,dx$$

$$= \int_0^1 (x+c)\,dx$$

iii) density of $1/X$

· compute CDF of $1/X$, then differentiate.

$\hookrightarrow G(t)$ $t \in [1, \infty)$

$$G(t) = \Pr\left[\frac{1}{X} \leq t\right] = \Pr\left[\frac{1}{t} \leq X\right]$$

$$= 1 - \Pr\left[X < \frac{1}{t}\right] = 1 - F(1/t)$$

where F is the distribution of X.

So the density, g, of $1/X$ is given by

$$g(x) = G'(x) = \left[1 - F(1/t)\right]'$$

$$= -F'(1/t)\left(-\frac{1}{t^2}\right) = \frac{F'(1/t)}{t^2}$$

$$= \frac{f(1/t)}{t^2}$$ valid for $1 \leq t < \infty$

$Y \sim \mathrm{Geo}(p)$

$\mathbb{E}[Y^2]$

$\mathrm{Var}[Y] = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2$

$\Rightarrow \mathbb{E}(Y^2) = \mathrm{Var}[Y] + \mathbb{E}[Y]^2$

$= \frac{1-p}{p^2} + \frac{1}{p^2} = \frac{2-p}{p^2}$

2. Every day Jo practices her tennis serve by continually serving until she has had a total of 50 successful serves. If each of her serves is, independently of previous ones, successful with probability 0.4, approximately what is the probability that she will need more than 100 serves to accomplish her goal? Use the normal approximation.

exact answer:

$\Pr[\text{needs} > 100 \text{ serves}]$

$= \Pr[< 50 \text{ successes in } 100 \text{ trials}]$

Think binomial w/ 100 trials, < 50 successes

$= \sum_{k=0}^{49} \Pr[k \text{ successes}]$

$= \sum_{k=0}^{49} \binom{100}{k} (0.4)^k (0.6)^{100-k}$

De-Moivre Laplace: (sec 5.4)

Let S_n = # successes in n trials

w/ success prob p.

$$\Pr\left[a \leq \frac{S_n - np}{(np(1-p))^{1/2}} \leq b\right]$$

$$\xrightarrow{n\to\infty} \Phi(b) - \Phi(a)$$

where $\Phi(t) = \Pr[N \leq t]$ for $N \sim \mathcal{N}(0,1)$

Applied here...

Let X = number of successful serves.

$np = 100 \cdot .4 = 40$

$np(1-p) = 100 \cdot .4 \cdot .6 = 24$

$\sqrt{np(1-p)} = \sqrt{24} = 2\sqrt{6}$

want $\Pr[0 \le X \le 49]$

$$= \Pr[0 - np \le X - np \le 49 - np]$$

$$= \Pr\left[\frac{0-np}{(np(1-p))^{1/2}} \le \frac{X-np}{(np(1-p))^{1/2}} \le \frac{49-np}{(np(1-p))^{1/2}}\right]$$

$$= \Pr\left[\frac{-40}{2\sqrt{6}} \le \downarrow \le \frac{49-40}{2\sqrt{6}}\right]$$

$$= \Pr\left[\frac{-20}{\sqrt{6}} \le \downarrow \le 9/2\sqrt{6}\right]$$

$$\approx \Phi\left(9/2\sqrt{6}\right) - \Phi\left(\frac{-20}{\sqrt{6}}\right)$$

$$\Pr[X = 50]$$