Math 175 - Homework 7

1. Let G be bipartite graph with parts X and Y.

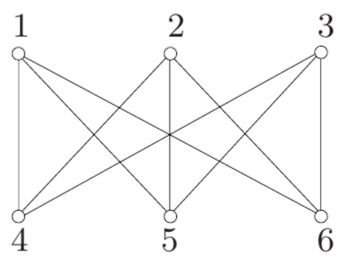
   (a) Show that $\sum_{v \in X} d(v) = \sum_{v \in Y} d(v)$.

   (b) Deduce that if G is k-regular (every vertex has degree k) with $k \geq 1$, then $|X| = |Y|$.

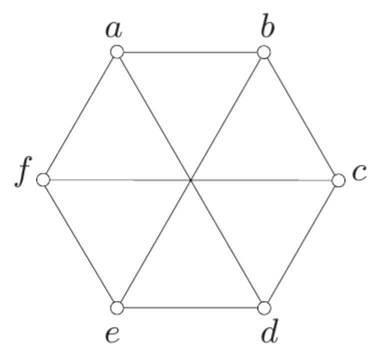
2. Let G be the graph whose vertex set is the set of k-tuples with coordinates in $\{0, 1\}$, with x adjacent to y when x and y differ in exactly one position. Determine whether G is bipartite.

3. An *isomorphism* between simple graphs G and H is a bijection $f : V(G) \to V(H)$ that preserves adjacency (that is, the vertices u and v are adjacent in G if and only if their images $f(u)$ and $f(v)$ are adjacent in H).

   (a) Are the graphs G and H shown below isomorphic? If so, give an isomorphism. If not, explain why not.



G



H

   (b) If two graphs have the same number of vertices and edges, are they necessarily isomorphic? If so, prove it. If not, give a counterexample.

   (c) Show that any isomorphism between two graphs maps each vertex to a vertex of the same degree.

   (d) Let G be a connected graph. Show that every graph which is isomorphic to G is connected.

4. A *triangle-free* graph is one which contains no triangles. Let G be a simple n-vertex, m-edge, triangle-free graph.

   (a) Show that $d(x) + d(y) \leq n$ for all adjacent x and y.

   (b) Deduce that $\sum_{v \in V} d(v)^2 \leq mn$.

   (c) Apply the Cauchy-Schwarz inequality to deduce that $m \leq n^2/4$. Recall the Cauchy-Schwarz inequality says that
   $$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \sum_{i=1}^{n} a_i^2 \cdot \sum_{i=1}^{n} b_i^2.$$

   (d) For each positive integer n, show that this bound can be realized. That is, find a simple triangle-free graph G with $m = \lfloor n^2/4 \rfloor$.