HomeWork 2 Math 271A, Fall 2019.

1. Observe that Jensen's inequality extends to conditional expectations. If $c : \Re \to \Re$ is convex and $c(X) \in \mathcal{L}^1$ then

$$\mathbb{E}[c(X)|\mathcal{G}] \;\; \geq \;\; c(\mathbb{E}[X|\mathcal{G}]), \quad \text{a.s.}$$

Use this to show that:

$$||\mathbb{E}[X|G]||_p \;\; \leq \;\; ||X||_p \quad \text{for p } \geq 1.$$

2. Let $(X_n : n \in N)$ be a sequence of independent random variables, each exponentially distributed:

$$\mathbb{P}[X_n > x] \;\; = \;\; e^{-x}, \quad x \geq 0.$$

   (a) A random variable τ has the lack of memory property if $P(\tau > a + b \mid \tau > a) = P(\tau > b)$. Show that a random variable has the memoryless property if and only if it is exponentially distributed.

   (b) Compute the expectation and variance of X_n. Let $Y = X_n + X_{n+1}$. Find the correlation coefficient between Y and X_n. Find $\mathbb{E}[Y|X_{n+1}]$.

   (c) Show that

$$\mathbb{P}[X_n > \alpha \log(n) \;\text{ for infinitely many }\; n] \;\; = \;\; \begin{cases} 0 & \text{for } \alpha > 1, \\ 1 & \text{else} \end{cases}$$

3. Let B_t be brownian motion.

   (a) Find a matrix A so that in distribution $[B_{t_1}, B_{t_2}, \cdots, B_{t_n}]^T = A[Z_1, Z_2, \cdots, Z_n]^T$ for the $Z_j's$ being iid standard normal random variables.

   (b) Show that $e^{-at}B_{e^{2at}}$ is a Gaussian process (every finite collection of time samples from the process has a multivariate normal distribution), find its covariance function.

4. Consider the random vector $\boldsymbol{x} = [X_1, \cdots, X_n] \in \mathbb{R}^n$

   (a) Show (Chebychev's inequality: for $p > 0$ we have

$$\mathbb{P}[\sum_i |X_i|^p \geq \lambda^p] \leq \lambda^{-p}\mathbb{E}[\sum_i |X_i|^p].$$

   (b) Suppose there exists $k > 0$ so that

$$M = \mathbb{E}[\exp(k(\sum_i |X_i|^p)^{1/p}] < \infty.$$

   Prove that then $\mathbb{P}[\sum_i |X_i|^p \geq \lambda^p] \leq Me^{-k\lambda} \quad \forall \lambda \geq 0.$

5. Let $\Omega = \{1, 2, 3, 4, 5\}$ and $\mathcal{U}$ the collection

$$\mathcal{U} = \{\{1, 2, 3\}, \{3, 4, 5\}\},$$

of subsets of Ω

(a) Find $\mathcal{H}_{\mathcal{U}}$ (the smallest σ-algebra containing $\mathcal{U}$).

(b) Define a random variable by $X(1) = X(2) = 0, X(3) = 10, X(4) = X(5) = 1$, is X measurable wrt $\mathcal{H}_{\mathcal{U}}$.

(b) Define another random variable by $Y(1) = 0, Y(2) = Y(3) = Y(4) = Y(5) = 1$. Find the σ-algebra $\mathcal{H}_Y$ generated by Y.