39.11. Prove that the Fibonacci sequence modulo m eventually repeats ~~with two consecutive 1's.~~ [*Hint.* The Fibonacci recursion can also be used backwards. Thus if you know the values of F_n and F_{n+1}, then you can recover the value of F_{n-1} using the formula $F_{n-1} = F_{n+1} - F_n$.]

$$(F_n \bmod m,\ F_{n+1} \bmod m) \in (\mathbb{Z}/m\mathbb{Z})^2$$

infinite seq. — finite

Pigeonhole $\Rightarrow$ some (r,s) will have to repeat

ie, $(F_n, F_{n+1}) = (a, b)$

$\parallel$

$(F_{n'}, F_{n'+1})$ //

$\exists\, r, s$ s.t. $F_r \equiv F_{r+s}$, $F_{r+1} \equiv F_{r+1+s}$

ie, $(F_r, F_{r+1}) = (F_{r+s}, F_{r+1+s}) = (a, b)$

$$F_{r+1} = F_r + F_{r-1} \Rightarrow \underline{F_{r-1}} = F_{r+1} - F_r$$

$$\equiv F_{r+1+s} - F_{r+s}$$

$$\equiv \underline{F_{r-1+s}}$$

$F_{r-1}, \underline{F_r}, \underline{F_{r+1}}, \ldots \underline{F_{r+s}}, \underline{F_{r+s}}, \underline{F_{r+1+s}}$

s

iterate this $\Rightarrow F_1 \equiv F_{s+1}, \; F_2 \equiv F_{s+2}$

$$\begin{array}{ccc} F_1 \equiv F_{s+1} & , & F_2 \equiv F_{s+2} \\ \| & & \| \\ 1 & & 1 \end{array}$$

pigeonhole $\rightarrow$ iterate backwards

if $\lim_{n \to \infty} \left| \frac{f(n)}{g(n)} \right| < \infty$

$$\Rightarrow f(n) = O(g(n))$$

$$f(n) = O(g(n)) \not\Rightarrow \lim_{n \to \infty} \left| \frac{f(n)}{g(n)} \right| < \infty$$

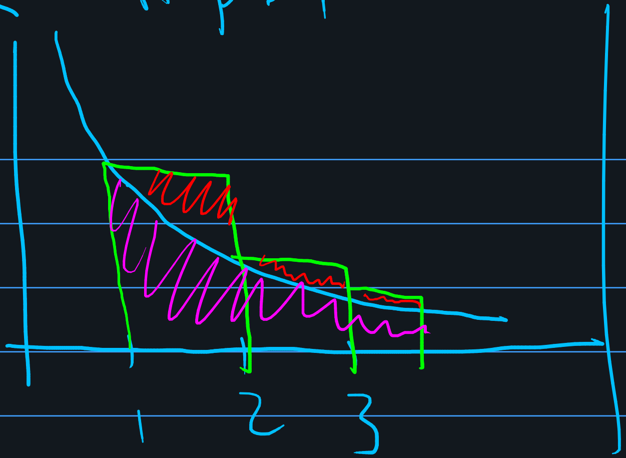
$$(x+2)\cos(x) = O(x)$$

$$f(n) = O(g(n)) \Leftrightarrow \limsup_{x \to \infty} \left| \frac{f(x)}{g(x)} \right| < \infty$$

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} = O(1)$$

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n} = \log(n) + O(1)$$

$\sum 1/n^p < \infty$ iff $p > 1$



Since $\sum_{n=1}^{\infty} 1/n^2 < \infty$

$$\cos(n) = O(1)$$

since $|\cos(n)| \le 1$

$$\sum \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$y^2 = x^3 - 1 \qquad (1,0)$$

$$\Rightarrow x^3 = y^2 + 1 = (y+i)(y-i)$$

if $\gcd(y+i, y-i) = 1 \Rightarrow y \pm i$ are cubes

$\uparrow$ since $\mathbb{Z}[i]$ is a UFD

say $(y+i) = (m+ni)^3$

$\gamma \mid y+i \Rightarrow N(\gamma) \mid N(y+i) = y^2+1 = x^3$

Let γ be a common divisor of $y \pm i$

$\Rightarrow \gamma \mid (y+i) - (y-i) = 2i \Rightarrow N(\gamma) \mid 4$