270B - Homework 3

Problem 1. Let $X_1, X_2, \ldots$ be independent random variables with means μ_i and finite variances σ_i^2. Consider the sums $S_n = X_1 + \cdots + X_n$. Find sequences of real numbers (b_i) and (c_i) such that $S_n^2 + b_n S_n + c_n$ is a martingale with respect to the σ-algebras generated by $X_1, \ldots, X_n$.

Solution. Let's start by centering the sum: define the random variable $M_n = S_n - \sum_{i=1}^n \mu_i$. Since the X_i's are independent, we have $\mathrm{Var}[M_n] = \sum_{i=1}^n \sigma_i^2$. We claim that

$$V_n = M_n^2 - \sum_{i=1}^n \sigma_i^2 = \left(S_n - \sum_{i=1}^n \mu_i\right)^2 - \sum_{i=1}^n \sigma_i^2$$

is a martingale with respect to the filtration $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. Let's start the computation.

$$\begin{aligned}
\mathbb{E}[V_{n+1}|\mathcal{F}_n] &= \mathbb{E}[S_{n+1}^2] - 2\left(\sum_{i=1}^{n+1}\mu_i\right)\mathbb{E}[S_{n+1}|\mathcal{F}_n] + \left(\sum_{i=1}^{n+1}\mu_i\right)^2 - \sum_{i=1}^{n+1}\sigma_i^2 \\
&= S_n^2 + 2S_n\mu_{n+1} + \mathbb{E}[X_{n+1}^2] - 2\left(\sum_{i=1}^{n+1}\mu_i\right)(S_n + \mu_{n+1}) + \left(\sum_{i=1}^{n+1}\mu_i\right)^2 - \sum_{i=1}^{n+1}\sigma_i^2 \\
&= S_n^2 - 2\left(\sum_{i=1}^{n}\mu_i\right)S_n + \mathbb{E}[X_{n+1}^2] - 2\mu_{n+1}^2 + \left(\sum_{i=1}^{n}\mu_i\right)^2 + \mu_{n+1}^2 - \sum_{i=1}^{n+1}\sigma_i^2 \\
&= S_n^2 - 2\left(\sum_{i=1}^{n}\mu_i\right)S_n + \left(\sum_{i=1}^{n}\mu_i\right)^2 - \sum_{i=1}^{n}\sigma_i^2 \\
&= V_n.
\end{aligned}$$

Here we've used the fact that S_n is $\mathcal{F}_n$-measurable and X_{n+1} is independent of $\mathcal{F}_n$. The sequences we want are then

$$b_n = -2\sum_{i=1}^n \mu_i, \qquad c_n = \left(\sum_{i=1}^n \mu_i\right)^2 - \sum_{i=1}^n \sigma_i^2.$$

□

Problem 2.

(a) Show that if (X_n) and (Y_n) are martingales with respect to the same filtration, then $X_n \vee Y_n$ is a submartingale.

Proof. We use the trusty identity

$$X_n \vee Y_n = \frac{1}{2}[(X_n + Y_n) + |X_n - Y_n|].$$

Since the sum of martingales is a martingale and conditional Jensen says the absolute value of a martingale is a submartingale, we have

$$\begin{aligned}\mathbb{E}[X_{n+1} \vee Y_{n+1} | \mathcal{F}_n] &= \frac{1}{2}(\mathbb{E}[X_{n+1} + Y_{n+1} | \mathcal{F}_n] + \mathbb{E}[|X_{n+1} - Y_{n+1}| \mid \mathcal{F}_n]) \\ &\geq \frac{1}{2}[(X_n + Y_n) + |X_n - Y_n|] \\ &= X_n \vee Y_n.\end{aligned}$$

Hence, $X_n \vee Y_n$ is a submartingale. □

(b) Give an example showing that $X_n \vee Y_n$ need not be a martingale.

Proof. □

Problem 3. Give an example of a martingale (X_n) such that $X_n \to -\infty$ a.s.

Solution. Durrett gives a hint to let $X_n = \xi_1 + \cdots + \xi_n$ for some independent centered ξ_i's. The idea is to concentrate most of the mass of ξ_i around some negative value and put the rest (some summable amount) around some positive value, then apply Borel-Cantelli.

Concretely, let ξ_i be given by

$$\xi_i = \begin{cases} 2^j & \text{with probability } \frac{1}{2^j} \\ -\frac{1}{1-2^{-j}} & \text{with probability } 1 - \frac{1}{2^j} \end{cases}.$$

Clearly ξ_i is centered, so $X_n = \xi_1 + \cdots + \xi_n$ is a martingale. Note that

$$\sum_{i=1}^{\infty} \mathbb{P}[\xi_i = 2^j] = \sum_{i=1}^{\infty} \frac{1}{2^j} = 1 < \infty.$$

By Borel-Cantelli, we have that $\xi_i = -\frac{1}{1-2^{-j}}$ eventually with probability 1, so $X_n \to -\infty$ a.s. □

Problem 4. Let (X_n) be a martingale that is bounded a.s. either above or below by some constant M. Show that $\sup_n \mathbb{E}|X_n| < \infty$.

Proof. If X_n is bounded below, then $X_n + M$ is a nonnegative martingale. By the martingale convergence theorem, $X_n + M$ converges almost surely to some limit Y with $\mathbb{E}|Y| < \infty$. Consequently, X_n also converges a.s. to an integrable function, so $\sup_n \mathbb{E}|X_n| < \infty$. If X_n is bounded above, then $-X_n + M$ is a nonnegative martingale and the same argument works. □

Problem 5. Let $Z_1, Z_2, \ldots$ be nonnegative iid random variables with $\mathbb{E}[Z_i] = 1$ and $\mathbb{P}[Z_i = 1] < 1$. Show that as $n \to \infty$,

$$\prod_{i=1}^{n} Z_i \to 0 \quad \text{a.s.}$$

Proof. □