Liam Hardiman *March 2, 2020*

271B - Homework 4

Problem 1. Consider the process

$$X_t = B_t^{(1)} B_t^{(2)} + e^t - t_0 + (B_t^{(3)})^2,$$

where $\mathbb{E}[B_t^{(i)} B_t^{(j)}] = \rho_{i,j} t$. Find the stochastic differential equation satisfied by this process.

Solution. Let $g(t, x_1, x_2, x_3) = x_1 x_2 + e^t - t_0 + x_3^2$. We apply Itô's lemma to obtain

$$\begin{aligned} dX_t &= e^t dt + B_t^{(2)} dB_t^{(1)} + B_t^{(1)} dB_t^{(2)} + 2B_t^{(3)} dB_t^{(3)} + (dB_t^{(3)})^2 + \rho_{1,2} dt \\ &= (e^t + 1 + \rho_{1,2}) dt + B_t^{(2)} dB_t^{(1)} + B_t^{(1)} dB_t^{(2)} + 2B_t^{(3)} dB_t^{(3)} \end{aligned}$$

□

Problem 2. Consider

$$Z_t = \mu_t dt + \theta_t \cdot dB_t, \ Z_0 = z_0,$$

for B a standard n-dimensional Brownian motion and θ a bounded deterministic n-dimensional vector process and μ a bounded deterministic process, all progressively measurable.

(a) Use the Kolmogorov forward equation to derive the density of Z_T.

Solution. Let $p_t(x)$ be the density of Z_t. The Kolmogorov forward equation says that

$$\begin{cases} \partial_t p_t(x) = \left(-\mu_t \partial_x + \frac{1}{2}|\theta_t|^2 \partial_x^2\right) p_t(x) \\ p_0(x) = \delta(x - z_0) \end{cases}.$$

If $\widehat{p}_t(\omega)$ is the Fourier transform of $p_t(x)$, then after integrating by parts twice we arrive at

$$\begin{cases} \partial_t \widehat{p}_t(\omega) = \left(i\omega\mu_t - \frac{1}{2}\omega^2 |\theta_t|^2\right) \widehat{p}_t(\omega) \\ \widehat{p}_0(\omega) = e^{iz_0\omega} \end{cases}.$$

The first equation is a separable ODE, so we obtain

$$\widehat{p}_t(\omega) = e^{iz_0\omega} \exp\left[\int_0^t \left(i\omega\mu_s - \frac{1}{2}\omega^2 |\theta_s|^2\right) ds\right].$$

Taking the inverse Fourier transform gives $p_t(x)$. □

Problem 3. Show that

$$\int_0^t B_s \circ dB_s = \frac{1}{2} B_t^2.$$

Proof. For all i let $t_i^* = \frac{1}{2}(t_i + t_{i-1})$. We have

$$\sum_{i=i}^{t/\Delta t} B_{t_i^*}(B_{t_i} - B_{t_{i-1}}) = \sum_{i=i}^{t/\Delta t} (B_{t_{i-1}} + (B_{t_i^*} - B_{t_{i-1}}))(B_{t_i} - B_{t_{i-1}})$$

$$= \sum_{i=1}^{t/\Delta t} B_{t_{i-1}}(B_{t_i} - B_{t_{i-1}}) + \sum_{i=1}^{t/\Delta t} (B_{t_i^*} - B_{t_{i-1}})[(B_{t_i^*} - B_{t_{i-1}}) + (B_{t_i} - B_{t_i^*})].$$

The first sum converges to the Itô integral $\int_0^t B_s dB_s = \frac{1}{2}B_t^2 - \frac{1}{2}t^2$, while the second sum converges to $\frac{1}{2}t^2$. In all, we have that $\int_0^t B_s \circ dB_s = \frac{1}{2}B_t^2$. ☐

Problem 4. Show that the following quantities define metrics on the appropriate spaces.

$$[f] = \sum_{n=1}^{\infty} 2^{-n} \left(1 \wedge \sqrt{\mathbb{E} \int_0^n f_s^2 \; ds} \right)$$

$$\|X\| = \sum_{n=1}^{\infty} 2^{-n} \left(1 \wedge \sqrt{\mathbb{E} X_n^2} \right).$$