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271B - Final

Problem 1. Let B be a standard one-dimensional Brownian motion. Consider the SDE

$$dX_t = (t^2\Sigma)dB_t, \quad X_0 = 0 \tag{1}$$

where Σ is an exponentially distributed random variable with parameter λ and independent of the Brownian motion.

(a) Find Y_t so that $M_t = \exp(X_t)Y_t$ is a martingale. Specify the filtration.

Solution. Let $\sigma_t(\omega) = t^2\Sigma(\omega)$ and let $Y_t = \exp(-\frac{1}{2}\int_0^t \sigma_s^2\ ds)$. We claim that $M_t = \exp(X_t)Y_t$ is a martingale. By Itô's lemma we have

$$\begin{aligned} dM_t &= -\frac{1}{2}\sigma_t^2 M_t\ dt + M_t\ dX_t + \frac{1}{2}M_t(dX_t)^2 \\ &= \sigma_t M_t\ dB_t \\ &= t^2\Sigma M_t\ dB_t. \end{aligned}$$

In order for us to conclude that this is a Martingale, have to show that $t^2\Sigma M_t$ is in class I*. To this end, we check the Kazamaki condition (Øksendal, remark after exercise 4.4. I tried using Novikov's condition, which we covered in class, but the expectation wasn't finite): if the following condition holds, then M_t is a martingale.

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T (s^2\Sigma)dB_s\right)\right] < \infty. \tag{2}$$

We've shown on a previous homework assignment that for a deterministic function $f(s)$,

$$\int_0^t f(s)dB_s \sim \mathcal{N}\left(0, \int_0^t f^2(s)ds\right).$$

From this we conclude that

$$\exp\left(\frac{1}{2}\int_0^T s^2\Sigma\ dB_s\right) \sim \exp(\Sigma g),$$

where $g \sim \mathcal{N}(0, T^5/20)$. Since Σ is independent of the Brownian motion, it is also independent of g, hence

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T (s^2\Sigma)dB_s\right)\right] = \mathbb{E}[e^{\Sigma g}] = \mathbb{E}[e^{\Sigma}] \cdot \mathbb{E}[e^g].$$

This quantity is finite if $\lambda > 1$. (I needed $\lambda > 1$ for $\mathbb{E}[e^{\Sigma}] < \infty$. This seems arbitrary, however. Is it still true without this?) Since Kazamaki's condition holds, M_t is indeed a martingale with respect to the filtration generated by the Brownian motion. □

(b) Compute the variance of M_t.

Solution. The variance is given by $\mathrm{Var}[M_t] = \mathbb{E}[M_t^2] - \mathbb{E}[M_t]^2$. Since $X_0 = 0$ a.s. and $Y_0 = 1$ a.s., $M_t = 1$ a.s. To compute $\mathbb{E}[M_t^2]$, we use the Itô isometry.

$$\begin{aligned}\mathbb{E}[M_t^2] &= \mathbb{E}\left[\left(1 + \int_0^t (s^2\Sigma) M_s \, dB_s\right)^2\right] \\ &= 1 + \mathbb{E}\left[\int_0^t (s^2\Sigma)^2 M_s^2 \, ds\right]\end{aligned}$$

□

Problem 2. Consider the Ornstein-Uhlenbeck process

$$dr_t = a(\overline{r} - r_t)dt + \sigma dB_t, \tag{3}$$

where a, $\overline{r}$, σ are constants. This process models an interest rate. The price of a zero-coupon bond at time t when paying 1 at maturity T is

$$P(t,x,T) = \mathbb{E}\left[\exp\left(-\int_t^T r_s \, ds\right) \mid r_t = x\right].$$

(a) Derive the Feynman-Kac formula for the bond price:

$$\begin{cases}\partial_t P + \frac{1}{2}\sigma^2 \partial_x^2 P + a(\overline{r} - x)\partial_x P - xP = 0 \\ P(T,x,T) = 1\end{cases}. \tag{4}$$

Solution. I think the idea here is to use Itô and the Kolmogorov backward equation. □

Problem 3. Let v be a continuous scalar valued process satisfying

$$0 \leq v(t) \leq \alpha(t) + \beta \int_0^t v(s) \, ds; \ 0 \leq t \leq T,$$

with $\beta \geq 0$ and α integrable. Show that

$$v(t) \leq \alpha(t) + \beta \int_0^t \alpha(s) e^{\beta(t-s)} ds, \ 0 \leq t \leq T.$$

Can you relax the assumption about continuity?

Solution. Define the function

$$F(s) = e^{-\beta s} \cdot \beta \int_0^s v(u) \, du. \tag{5}$$

We differentiate:

$$F'(s) = \beta e^{-\beta s}\left(v(s) - \beta \int_0^s v(u) \, du\right) \leq \beta \alpha(s) e^{-\beta s}.$$

Integrating from 0 to t gives

$$F(t) \leq \beta \int_0^t \alpha(s) e^{-\beta s} \, ds.$$

Now we have from (5) and the above

$$\beta \int_0^t v(s) \, ds = e^{\beta t} F(t) \leq \beta \int_0^t \alpha(s) e^{\beta(t-s)} ds.$$

Finally, we know $\beta \int_0^t v(s) \, ds \geq v(t) - \alpha(t)$, so the desired inequality follows. □

Problem 4. Let $B_t = B_t^{(1)} + iB_t^{(2)}$ be a complex Brownian motion.

(a) Let $F(z) = u(z) + iv(z)$ be analytic and define

$$Z_t = F(B_t).$$

Prove that

$$dZ_t = F'(B_t)\ dB_t,$$

where F' is the complex derivative of F.

Proof. We assume the component Brownian motions are independent. By Itô we have

$$dB_t = dB_t^{(1)} + idB_t^{(2)}.$$

Write Z_t in terms of the component functions of F:

$$Z_t = u(B_t^{(1)}, B_t^{(2)}) + iv(B_t^{(1)}, B_t^{(2)}).$$

By Itô's lemma we have (suppressing the dependence on $B^{(1)}$ and $B^{(2)}$)

$$dZ_t = (u_x + iv_x)dB_t^{(1)} + (u_y + iv_y)dB_t^{(2)} + \frac{1}{2}[(u_{xx} + iv_{xx})dt + (u_{yy} + iv_{yy})dt].$$

Now the components of an analytic function are harmonic, so the bracketed term vanishes. Applying the Cauchy-Riemann equations gives

$$dZ_t = (u_x + iv_x)dB_t = F'(z)dB_t.$$

$\square$

(b) Solve the complex SDE

$$dZ_t = \alpha Z_t dB_t,$$

where α is a constant.

Solution. Pretending that this is a real ODE, we guess that the solution will be exponential. Indeed, by part (a) we have

$$d(e^{\alpha B_t}) = \alpha e^{\alpha B_t} dB_t.$$

Thus, $Z_t = Z_0 + e^{\alpha B_t}$ solves the SDE. $\square$