Liam Hardiman *February 20, 2020*

271B - Homework 2

Problem 1. Let S, T, and T_n, $n = 1, 2, \dots$ be stopping times (with respect to some filtration $\{\mathcal{F}_t\}_{t\geq 0}$). Show that $T \vee S$, $T \wedge S$, $T + S$, $\sup_n T_n$ are also stopping times.

Proof. The pointwise minimum, maximum, sum, and supremum of measurable functions are measurable. For the minimum and maximum we have

$$\{(T \wedge S) \leq t\} = \{T \leq t\} \cup \{S \leq t\}$$

$$\{(T \vee S) \leq t\} = \{T \leq t\} \cap \{S \leq t\}.$$

Unions and intersections of measurable sets are measurable, so both of these sets live in $\mathcal{F}_t$. Thus, $T \wedge S$ and $T \vee S$ are stopping times. For the sum, we can write the set $\{T + S \leq t\}$ as a countable union:

$$\{T + S \leq t\} = \bigcup_{\alpha, \beta \in \mathbb{Q},\ \alpha+\beta \leq t} \{T \leq \alpha\} \cap \{S \leq \beta\}.$$

As $\mathcal{F}_t$-measurability is closed under countable union and intersection, the sum is a stopping time. Finally, we have

$$\{\sup_n T_n \leq t\} = \bigcap_{n=1}^{\infty} \{T_n \leq t\},$$

which is measurable, so the supremum is also a stopping time. □

Problem 2. Let X_t be an adapted and continuous stochastic process, and define

$$T_\Gamma = \inf\{t \geq 0 : X_t \in \Gamma\}$$

for Γ a closed set. Show that T_Γ is a stopping time.

Proof. As Γ is closed, for every x there is a well-defined "distance to Γ" function

$$d(x, \Gamma) = \inf_{y \in \Gamma} |x - y|.$$

In fact, this function is continuous. Since X_t has continuous paths and is $\mathcal{F}_t$ measurable, the composition $Y_t = d(X_t, \Gamma)$ is $\mathcal{F}_t$ measurable.

Since Γ is closed, $X_t \in \Gamma$ if and only if $Y_t = d(X_t, \Gamma) = 0$. From this it follows that $T_\Gamma > t$ if and only if $Y_s > 0$ for all $s \leq t$. Intuitively, if $T_\Gamma > t$, then X_t arrives in Γ at some time strictly later than t. In order for this to happen, X_t must be outside of Γ at all times $s \leq t$, in which case $Y_s = d(X_s, \Gamma) > 0$. This set is ostensibly an uncountable intersection, but we can write it as a union of countable intersections by approximating by rational points.

$$\{T_\Gamma > t\} = \bigcap_{s \leq t} \{Y_s > 0\} = \bigcup_{n \geq 1} \bigcap_{q \in \mathbb{Q} \cap [0,t]} \{Y_q > 1/n\} \in \mathcal{F}_t.$$

Hence, T_Γ is a stopping time. □

Problem 3. Show that if X_t is a martingale with respect to some filtration (say $\mathcal{F}_t$) then it is also a martingale with respect to the filtration generated by itself.

Proof. Let $\mathcal{G}_t = \sigma(X_s : s \le t)$ be the filtration X generates. We then have $\mathcal{G}_t \subseteq \mathcal{F}_t$ for all t since $\mathcal{G}_t$ is the smallest σ-algebra with respect to which X_t is measurable. By the law of total expectation and the martingale property of X_t with respect to $\mathcal{F}_t$ we have for any $s \le t$

$$\mathbb{E}[X_t \mid \mathcal{G}_s] = \mathbb{E}[\mathbb{E}[X_t \mid \mathcal{F}_s] \mid \mathcal{G}_s] = \mathbb{E}[X_s \mid \mathcal{G}_s] = X_s.$$

Thus, X_t is a martingale with respect to $\{\mathcal{G}_t\}$. $\square$

Problem 4. Let a, b be deterministic and f, g of class I. Show that if

$$a + \int_0^T f_s \, dB_s = b + \int_0^T g_s \, dB_s \tag{1}$$

then $a = b$ and $f = g$ a.a. for $(t, \omega) \in (0, T) \times \Omega$.

Proof. Since f and g are of class I, $\int_0^t f_s \, dB_s$ and $\int_0^t g_s \, dB_s$ are martingales and $\int_0^0 f_s \, dB_s = 0$ a.s. (the same holds for g). Taking the expectation of both sides of the given relation shows that $a = b$ a.s. and

$$\int_0^T (f_s - g_s) \, dB_s = 0.$$

By the Itô isometry we have

$$0 = \mathbb{E}\left[\left(\int_0^T (f_s - g_s) \, dB\right)^2\right] = \mathbb{E}\left[\int_0^T (f_s - g_s)^2 \, ds\right].$$

We conclude that $f_t(\omega) = g_t(\omega)$ for almost all $(t, \omega) \in (0, T) \times \Omega$. $\square$

Problem 5. Assume that X_t is of class I and continuous in mean square on $[0, T]$, that is for $t \in [0, T]$

$$\mathbb{E}[X_t^2] < \infty, \quad \lim_{s \to t} \mathbb{E}[(X_t - X_s)^2] = 0.$$

Define

$$\phi_t^{(n)} = \sum_j X_{t_{j-1}^{(n)}} \chi_{[t_{j-1}^{(n)}, t_j^{(n)})}(t), \; t_j^{(n)} = j 2^{-n}.$$

Show that for $0 \le t \le T$

$$\int_0^t X_s \, dB_s = \lim_{n \to \infty} \int_0^t \phi_s^{(n)} \, dB_s,$$

where the limit is in $L^2(\mathbb{P})$.

Proof. For any n we have by the Itô isometry

$$\mathbb{E}\left[\left(\int_0^t (X_s - \phi_s^{(n)}) \, dB_s\right)^2\right] = \mathbb{E}\left[\int_0^t (X_s - \phi_s^{(n)})^2 \, ds\right] = \mathbb{E}\left[\sum_j \int_{t_{j-1}^{(n)}}^{t_j^{(n)}} (X_s - X_{t_{j-1}^{(n)}})^2 \, ds\right].$$

Now we claim that continuity in mean square on the compact set $[0,T]$ implies uniform continuity in mean square. Assuming this claim, we can choose n large enough so that $\mathbb{E}[(X_s - X_{t_{j-1}^{(n)}})^2]$ is smaller than say ϵ for all j. For n at least this large we have

$$\mathbb{E}\left[\left(\int_0^t (X_s - \phi_s^{(n)})\ dB_s\right)^2\right] \leq \sum_j (t_j^{(n)} - t_{j-1}^{(n)})\epsilon = \epsilon T.$$

Since the L^2 distance between $\int_0^t \varphi_s^{(n)}\ dB_s$ and $\int_0^t X_s\ dB_s$ can be made arbitrarily small, we conclude that $\int_0^t \phi_s^{(n)}\ dB_s \to \int_0^t X_s\ dB_s$ in L^2.

Now we show uniform mean square continuity. Suppose for the sake of contradiction that for some ϵ there is no δ such that $|s-t| < \delta$ implies that $\|X_s - X_s\|_{L^2} < \epsilon$. Then we can find a sequence s_n, t_n so that $|s_n - t_n| < 1/n$ but $\|X_s - X_t\|_{L^2} > \epsilon$. By the compactness of $[0,T]$, we can assume that $s_n \to s^* \in [0,T]$. We then have $\|X_{s^*} - X_{t_n}\|_{L^2} > \epsilon$, but this contradicts the mean square continuity of X at s^*. $\square$

Problem 6. Let X_t be a deterministic continuous function and

$$Y_t = \int_0^t X_s\ dB_s.$$

Deduce the law of the process Y.

Solution. We assume $t \in [0,T]$ for some $T < \infty$. Since X_t is continuous, it is bounded and $\int_0^t X_s\ ds < \infty$ for all t and ω. In particular, the family $\{X_t\}_{t\in[0,T]}$ is uniformly integrable in ω, so we have

$$\int_0^t X_s\ dB_s = \lim_{n\to\infty} \sum_{j=1}^{t/\Delta t} X_{t_{j-1}} \Delta B_{t_j},$$

where $\Delta B_{t_j} = B_{t_j} - B_{t_{j-1}}$ and the limit is in L^2. (Alternatively, X satisfies the hypotheses of problem 5, so we could have used the result from that problem to get this limit.) Since the Brownian increments on the right-hand side are disjoint, they are independent normal random variables, so the whole sum on the right is a normal random variable with distribution

$$\mathcal{N}\left(0, \sum_{j=1}^{t/\Delta t} X_{t_{j-1}}^2 \Delta t\right).$$

Since X is continuous and deterministic, we recognize the above sum as a Riemann sum as $\Delta t \to 0$. Since the L^2 limit of normal random variables is normal when the mean and variance converge as sequences of real numbers, we have

$$Y_t = \int_0^t X_s\ dB_s \sim \mathcal{N}\left(0, \int_0^t X_s^2\ ds\right).$$

Now the process Y is Gaussian if and only if for every $t_1 < \cdots < t_k \in T$, any linear combination of the Y_{t_j}'s has univariate normal distribution. Since $\int_a^b X_s\ dB_s = \int_a^c X_s\ dB_s + \int_c^b X_s\ dB_s$ for any $a < c < b$, we have

$$c_1 Y_{t_1} + c_2 Y_{t_2} + \cdots + c_k Y_{t_k} = (c_1 + \cdots + c_k) Y_{t_1} + (c_2 + \cdots + c_k)(Y_{t_2} - Y_{t_1}) + \cdots + c_k (Y_{t_k} - Y_{t_{k-1}}).$$

The variables $Y_{t_j} - Y_{t_{j-1}}$ are themselves Itô integrals over disjoint intervals, so they are independent normal random variables. We conclude that the above linear combination is normally distributed, so the process Y is Gaussian.

A Gaussian process, and therefore its law, is determined by its mean and covariance. Since $Y_t \sim \mathcal{N}(0, \int_0^t X_s^2\, ds)$, the process Y has zero mean. As for the covariance, we have for any s, t

$$\begin{aligned}
\operatorname{Cov}(Y_s, Y_t) &= \mathbb{E}\left[\int_0^s X_u\, dB_u \cdot \int_0^t X_u\, dB_u\right] \\
&= \mathbb{E}\left[\left(\int_0^{s\wedge t} X_u\, dB_u\right)^2\right] + \mathbb{E}\left[\int_0^{s\wedge t} X_u\, dB_u \cdot \int_{s\wedge t}^{s\vee t} X_u\, dB_u\right] \\
&= \mathbb{E}\left[\int_0^{s\wedge t} X_u^2\, du\right] = \int_0^{s\wedge t} X_u^2\, du.
\end{aligned}$$

The last line follows from the Itô isometry and the independence of Itô integrals over disjoint intervals.

□