Math 180B - Binomial Theorem and Fibonacci Numbers

1. What is the value of the following sum? Try giving both an algebraic proof and a combinatorial proof.
$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n}$$

2. Let $f(x)$ and $g(x)$ be n-times differentiable functions. Show that the n-th derivative of $f(x)g(x)$ is
$$\sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x).$$

3. Show that the Fibonacci sequence F_n satisfies
$$F_{5n+2} > 10^n$$
for all $n \geq 1$.

4. Show that the n-th Fibonacci number can be written
$$F_n = \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \cdots + \binom{n-j}{j-1} + \binom{n-j-1}{j},$$
where j is the largest integer less than or equal to $(n-1)/2$. *Hint: you might want to use the recursive formula for the binomial coefficient,* $\binom{m}{k} = \binom{m-1}{k} + \binom{m-1}{k-1}$.

5. Use Binet's formula to show that for all $n \geq 1$, the Fibonacci sequence F_n satisfies
$$\binom{n}{1}F_1 + \binom{n}{2}F_2 + \binom{n}{3}F_3 + \cdots + \binom{n}{n}F_n = F_{2n}$$
$$-\binom{n}{1}F_1 + \binom{n}{2}F_2 - \binom{n}{3}F_3 + \cdots + (-1)^n\binom{n}{n}F_n = -F_n.$$