Class - Homework N

Problem 1

Prove the following theorem.

Theorem 1 (Markov's Inequality)**.** If $X : \Omega \to \mathbb{R}$ is a random variable and $E[|X|] < \infty$, then for all $a \geq 0$

$$\Pr[|X| \geq a] \leq \frac{E[|X|]}{a}. \tag{1}$$

Solution. Define the random variable Y by

$$Y(\omega) = \begin{cases} 0, & \text{if } |X(\omega)| < a \\ a, & \text{if } |X(\omega)| \geq a \end{cases}.$$

Note that $Y(\omega) \leq |X(\omega)|$ for all $\omega \in \Omega$. This gives

$$\begin{aligned} E[|X|] &\geq E[Y] \\ &= a \cdot \Pr[|X| \geq a]. \end{aligned}$$

Dividing through by a establishes (1). $\square$

Problem 2

Prove that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}. \tag{2}$$

Solution. You could prove this by induction, but that's pretty boring. Check out this argument allegedly due to Gauss. Call the left-hand side of (2) S then write it forward and backwards.

$$S = 1 + 2 + 3 + \cdots + n$$

$$S = n + (n-1) + (n-2) + \cdots + 1$$

The first terms in both rows sum to $n+1$, the second terms in both rows also sum to $n+1$, and so on. There are n terms in each row, so adding these rows together gives $n(n+1)$ on the right-hand side and $2S$ on the left. Dividing by 2 establishes (2). $\square$