Liam Hardiman *February 10, 2020*

270B - Homework 2

Problem 1.

(a) Prove that if the probability density functions of X_n converge pointwise to the probability density function of X, then X_n converges to X weakly.

Proof. Let f_n and f be the density functions of X_n and X, respectively, and let $\varphi : \mathbb{R} \to \mathbb{R}$ be bounded and continuous. We then have

$$\begin{aligned} |\mathbb{E}_n[\varphi] - \mathbb{E}[\varphi]| &= \left| \int_{\mathbb{R}} f_n(x)\varphi(x)\ dx - \int_{\mathbb{R}} f(x)\varphi(x)\ dx \right| \\ &\leq \|\varphi\|_\infty \int_{\mathbb{R}} |f_n(x) - f(x)|\ dx. \end{aligned} \tag{1}$$

Now we claim that $\|f_n - f\|_{L^1} \to 0$ (argument taken from 210 notes). Since $\int f_n = \int f = 1$ we have by Fatou

$$2\int_{\mathbb{R}} f\ dx = \int_{\mathbb{R}} \liminf_{n\to\infty} (f + f_n - |f_n - f|)\ dx \leq 2\int_{\mathbb{R}} f\ dx - \limsup_{n\to\infty} |f_n - f|\ dx.$$

In particular, we have $\limsup_{n\to\infty} \int |f_n - f| = 0$, so $\|f_n - f\|_{L^1} \to 0$. Consequently, the right-hand side of (1) goes to zero as $n \to \infty$. We then have that $\mathbb{E}_n[\varphi] \to \mathbb{E}[\varphi]$ for all φ bounded continuous, so $X_n \to X$ weakly. $\square$

(b) Prove that if the probability mass functions of X_n converge to the probability mass function of X pointwise then X_n converges to X weakly.

Proof. We essentially mirror our proof of part (a) but with the counting measure. Let f_n and f be the mass functions of X_n and X respectively and suppose they take values in the discrete set $S \subset \mathbb{R}$. Suppose φ is a bounded function on S (note that any function from a discrete space is continuous). We then have

$$\begin{aligned} |\mathbb{E}_n[\varphi] - \mathbb{E}[\varphi]| &= \left| \sum_{x\in S} f_n(x)\varphi(x) - \sum_{x\in S} f(x)\varphi(x) \right| \\ &\leq \|\varphi\|_\infty \sum_{x\in S} |f_n(x) - f(x)|. \end{aligned} \tag{2}$$

Fatou still works with the counting measure, so we have

$$2\sum_{x\in S} f(x) = \sum_{x\in S} \liminf_{n\to\infty} (f(x) + f_n(x) - |f_n(x) - f(x)|) \leq 2\sum_{x\in S} f(x) - \limsup_{n\to\infty} \sum_{x\in S} |f_n(x) - f(x)|.$$

We conclude that $\sum_{x\in S} |f_n(x) - f(x)| \to 0$ as $n \to \infty$. The right-hand side of (2) vanishes as $n \to \infty$, so $X_n \to X$ weakly. $\square$