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270B - Homework 4

Problem 1. Let (X_n) be an irreducible recurrent Markov chain with doubly-infinite transition matrix P. Let $\psi : \mathbb{N} \to \mathbb{N}$ be a bounded function satisfying

$$\sum_{j=1}^{\infty} P_{ij}\psi(j) = \psi(i) \quad \text{for all } i \in \mathbb{N}.$$

Show that ψ is a constant function.

Proof. First we claim that $\psi(X_n)$ is a martingale. Let $\mathcal{F}_n$ be the filtration generated by $X_1, \ldots, X_n$. We then have by hypothesis

$$\mathbb{E}[\psi(X_{n+1})|\mathcal{F}_n] = \sum_j P_{X_n,j}\psi(j) = \psi(X_n),$$

so $\psi(X_n)$ is a martingale with respect to this filtration. We showed on a previous homework assignment that bounded martingales converge almost surely, so $\psi(X_n) \to s$ almost surely for some $s \in \mathbb{N}$.

Suppose ψ is non-constant, so $\psi(i) = u$ and $\psi(j) = v$ for some $i \neq j$, $u \neq v$. Intuitively, since the Markov chain is irreducible and recurrent, it should return to states i and j infinitely often with probability 1. But since $\psi(i) \neq \psi(j)$, this means that $\psi(X_n)$ cannot converge almost surely. □

Problem 2. Let S and T be stopping times with respect to a filtration $(\mathcal{F}_n)$. Denote by $(\mathcal{F}_T)$ the collection of events F such that $F \cap \{T \leq n\} \in \mathcal{F}_n$ for all n.

(a) Show that $\mathcal{F}_T$ is a σ-algebra.

Proof. That $\emptyset$ and Ω are in $\mathcal{F}_T$ immediately follows from T being a stopping time. To show closure under complementation, write $F \in \mathcal{F}_T$ like so

$$F = (F \cap \{T \leq n\}) \cup (F \cap \{T > n\}).$$

This gives

$$F^c \cap \{T \leq n\} = (F \cap \{T \leq n\})^c \cap (F \cap \{T > n\})^c \cap \{T \leq n\}.$$

Since $F \cap \{T \leq n\}$ is in $\mathcal{F}_n$, the above set is also in $\mathcal{F}_n$.

As for countable unions, let $F_1, F_2, \ldots \in \mathcal{F}_T$. Then

$$\left(\bigcup_{k=1}^{\infty} F_k\right) \cap \{T \leq n\} = \bigcup_{k=1}^{\infty} (F_k \cap \{T \leq n\}).$$

Since $F_k \cap \{T \leq n\} \in \mathcal{F}_n$ for all n, the above set is in $\mathcal{F}_n$. □

(b) Show that T is measurable with respect to $\mathcal{F}_T$.

Proof. T is measurable with respect to $\mathcal{F}_T$ if and only if $\{T \leq n\} \in \mathcal{F}_T$ for all n. This follows immediately from the fact that T is a stopping time with respect to the filtration $\mathcal{F}_n$. $\square$

(c) If $E \in \mathcal{F}_S$, show that $E \cap \{S \leq T\} \in \mathcal{F}_T$.

Proof. The idea is to write $\{S \leq T\}$ as $\cup_{k=1}^{\infty}\{T = k\} \cap \{S \leq k\}$. For any $E \in \mathcal{F}_S$ we then have

$$\begin{aligned}(E \cap \{S \leq T\}) \cap \{T \leq n\} &= \bigcup_{k=1}^{\infty}(E \cap \{S \leq k\} \cap \{T = k\} \cap \{T \leq n\}) \\ &= \bigcup_{k=1}^{n}(E \cap \{S \leq k\} \cap \{T = k\}).\end{aligned}$$

Since $E \in \mathcal{F}_S$, $E \cap \{S \leq k\} \in \mathcal{F}_k$ for all k. Since T is a stopping time with respect to $\mathcal{F}_n$, $\{T = k\} \in \mathcal{F}_k$ for all k. Consequently, the above union is in $\mathcal{F}_n$, so $E \cap \{S \leq T\} \in \mathcal{F}_T$. $\square$

(d) Show that if $S \leq T$ a.s. then $\mathcal{F}_S \subset \mathcal{F}_T$.

Proof. If $S \leq T$ a.s., then $\{S \leq T\}$ is a set with probability 1. By part (c), if $E \in \mathcal{F}_S$, then all but a measure zero subset of E is in $\mathcal{F}_T$. $\square$

Problem 3. Let (X_n) be a uniformly bounded [integrable, right?] martingale with respect to the filtration $(\mathcal{F}_n)$. Let S and T be two stopping times satisfying $S \leq T$ a.s. Prove that

$$X_T = \mathbb{E}[X|\mathcal{F}_T] \quad \text{and} \quad X_S = \mathbb{E}[X_T|\mathcal{F}_S]$$

where X is the almost sure limit of X_n.

Proof. Since X_n is uniformly integrable, we can reconstruct X_n from its a.s. (and L^1) limit:

$$X_n = \mathbb{E}[X|\mathcal{F}_n].$$

Now for any $F \in \mathcal{F}_T$ we have $F \cap \{T = n\} \in \mathcal{F}_n$ for all n.

$$\int_F X_T \; d\mathbb{P} = \sum_{n=0}^{\infty} \int_{F \cap \{T=n\}} X_T \; d\mathbb{P} = \sum_{n=0}^{\infty} \int_{F \cap \{T=n\}} X_n \; d\mathbb{P}.$$

By the definition of conditional expectation, the last integral above is equal to

$$\sum_{n=1}^{\infty} \int_{F \cap \{T=n\}} X \; d\mathbb{P} = \int_F X \; d\mathbb{P}.$$

Again, by the definition of conditional expectation, we have $X_T = \mathbb{E}[X|\mathcal{F}_T]$.

By problem 2(d), we have that since $S \leq T$ a.s., $\mathcal{F}_S \subset \mathcal{F}_T$. Consequently, we have by the law of total expectation

$$\mathbb{E}[X_T|\mathcal{F}_S] = \mathbb{E}[\mathbb{E}[X|\mathcal{F}_T] \mid \mathcal{F}_S] = \mathbb{E}[X|\mathcal{F}_S] = X_S.$$

□

Problem 4. A die is rolled repeatedly. Which of the following are Markov chains? For those that are, compute the transition matrix.

(a) The largest number X_n shown up to the n-th roll.

Solution. Intuitively, this should be a Markov chain: to check if the current roll is the largest thus far, we need only compare it to the largest roll seen before. More concretely, consider $i_1 \leq i_2 \leq \cdots \leq i_n$. Then

$$\mathbb{P}[X_{n+1} = i_{n+1} \mid X_n = i_n, \ldots, X_1 = i_1] = \begin{cases} \frac{1}{6} & \text{if } i_{n+1} \geq i_n \\ 0 & \text{otherwise} \end{cases} = \mathbb{P}[X_{n+1} = i_{n+1} \mid X_n = i_n].$$

□

(b) The number N_n of sixes in n rolls.

Solution. Again, intuition tells us that this should be a Markov chain: the number of sixes on the $n+1$-st roll will either be the same as or one greater than the number of sixes on the n-th roll. □