Math 180B - Induction

1. Prove the following equations by induction. In each case, n is a positive integer.

   (a) $1 + 4 + 7 + \cdots + (3n-2) = \frac{n(3n-1)}{2}$.

   (b) $1 + x + x^2 + \cdots + x^n = \frac{1-x^{n+1}}{1-x}$. What happens when $x = 1$?

   (c) $\lim_{x\to\infty} \frac{x^n}{e^x} = 0$.

   (d) $n! = \int_0^\infty x^n e^{-x}\, dx$.

2. Prove the following inequalities by induction. n is a positive integer.

   (a) $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n} \geq 1 + \frac{n}{2}$.

   (b) $\binom{2n}{n} < 4^n$.

   (c) $n! \leq n^n$.

3. Prove that the sum of the angles of a convex n-gon ($n \geq 3$) is $180(n-2)$ degrees or $\pi(n-2)$ radians.

4. Prove the binomial theorem
$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

5. What's wrong with this proof?

   Claim. *If we have n lines in the plane, no two of which are parallel, then they all go through one point.*

   Proof. This is clearly true for one or two lines by definition. Suppose that it is true for any set of n lines and let $S = \{\ell_1, \ell_2, \ell_3, \ell_4, \ldots, \ell_{n+1}\}$ be a set of $n+1$ lines in the plane, no two of which are parallel. Delete the line ℓ_3 to obtain a set S' of n lines, no two of which are parallel. By the induction hypothesis, the lines in S' must all pass through some point P. In particular, ℓ_1 and ℓ_2 pass through P.

   Now put ℓ_3 back and delete ℓ_4 instead to get another set S'' of n lines, no two of which are parallel. Again by the induction hypothesis, they must all pass through some point Q. In particular, ℓ_1 and ℓ_2 pass through Q. But ℓ_1 and ℓ_2 pass through P. Since two lines can pass through at most one point, we must have $P = Q$. But then ℓ_3 goes through P, so all the lines in S go through P. $\square$