1. Let X and Y be independent random variables taking values in the positive integers having the same distribution given by

$$\Pr[X = n] = \Pr[Y = n] = 2^{-n}$$

for all $n \in \mathbb{N}$. Find $\Pr[X \text{ divides } Y]$.

$$\Pr[X \mid Y] = \Pr[Y = kX, \text{ some } k]$$

$$= \sum_{x \mid y} f(x,y),$$

where $f(x,y)$ is the joint pmf of X & Y

joint

ie, $f(x,y) = \Pr[X=x, Y=y]$

$$= \underbrace{\Pr[X=x]}_{f_X(x)} \underbrace{\Pr[Y=y]}_{f_Y(y)}$$

marginal pmfs

$$= \sum_{x \mid y} f_X(x) f_Y(y)$$

$$= \sum_{x \mid y} 2^{-x} \cdot 2^{-y}$$

$x \mid y$

$\iff y = kx$

for some k

$$= \sum_{x=1}^{\infty} \underline{\sum_{k=1}^{\infty} 2^{-x} \cdot 2^{-kx}}$$

$$= \sum_{x=1}^{\infty} 2^{-x} \underline{\sum_{k=1}^{\infty} \left(2^{-x}\right)^{k}}$$

geometric

$$= \sum_{x=1}^{\infty} 2^{-x} \frac{2^{-x}}{1-2^{-x}}$$

$$\boxed{= \sum_{x=1}^{\infty} \frac{2^{-2x}}{1-2^{-x}}}$$

2. Let X and Y be independent continuous random variables with densities f_X and f_Y, respectively. Express the density of XY in terms of the densities of X and Y.

· First compute the cdf of XY, then differentiate.

Let $G(t) = \Pr[XY \leq t]$

$= \Pr[XY \leq t, X \leq 0] + \Pr[XY \leq t, X \geq 0]$

$= \Pr[Y \geq t/X, X \leq 0] + \Pr[Y \leq t/X, X \geq 0]$

$= \int_{-\infty}^{0} \int_{t/x}^{\infty} \underline{f_{X,Y}(x,y)} \, dy \, dx$ joint density of X & Y

$+ \int_{0}^{\infty} \int_{-\infty}^{t/x} f_{X,Y}(x,y) \, dy \, dx$

$$= \int_{-\infty}^{0} \int_{t/x}^{\infty} \underbrace{f_X(x) f_Y(y)}_{\text{marginal densities}} \, dy\, dx$$

$$+ \int_{0}^{\infty} \int_{-\infty}^{t/x} f_X(x) f_Y(y) \, dy\, dx$$

$$= \int_{-\infty}^{0} f_X(x) \underline{\int_{t/x}^{\infty} f_Y(y)\, dy} \, dx$$

$$+ \int_{0}^{\infty} f_X(x) \int_{-\infty}^{t/x} f_Y(y)\, dy \, dx$$

differentiate wrt t

$$\frac{d}{dt} \boxed{= \int_{-\infty}^{0} f_X(x) f_Y(t/x) \cdot \frac{-1}{x}\, dx + \int_{0}^{\infty} f_X(x) f(t/x) \frac{1}{x}\, dx}$$