2. Let $f(x)$ and $g(x)$ be n-times differentiable functions. Show that the n-th derivative of $f(x)g(x)$ is

$$\sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x).$$

Problems are on Canvas

7_1

work until ~3:20

1. What is the value of the following sum? Try giving both an algebraic proof and a combinatorial proof.

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n}$$

$$\underline{2^n} = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot 1^k \cdot 1^{n-k}$$

$$= \underline{\sum_{k=0}^{n} \binom{n}{k}}$$

Combinatorial (counting):

consider a set of size n, S. $|S| = n$

S has 2^n subsets.

(each element is or isn't in the subset)

$$\#\text{subsets} = \sum_{k=0}^{n} (\#\text{subsets of size } k)$$

$$= \sum_{k=0}^{n} \binom{n}{k}$$

2. Let $f(x)$ and $g(x)$ be n-times differentiable functions. Show that the n-th derivative of $f(x)g(x)$ is

$$\sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x). = [f(x)g(x)]^{(n)}$$

base $n=0$: $[f(x)g(x)]^{(0)} = f(x)g(x)$

$$\sum_{k=0}^{0} \binom{0}{k} f^{(k)} g^{(n-k)} = fg \checkmark$$

base $n=1$: product rule

Suppose $[fg]^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}$

$$[fg]^{(n+1)} = \left([fg]^{(n)}\right)'$$

$$\overset{\text{inductive}}{=} \left(\sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)} \right)'$$

$$= \sum_{k=0}^{n} \binom{n}{k} f^{(k+1)} g^{(n-k)} + \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n+1-k)}$$

let $k' = k+1$

$$= \sum_{k=1}^{n+1} \binom{n}{k-1} f^{(k)} g^{(n+1-k)} + \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n+1-k)}$$

$$= \binom{n}{n} f^{(n+1)} g + \sum_{k=1}^{n} \binom{n}{k-1} f^{(k)} g^{(n+1-k)}$$

$$+ \binom{n}{0} f g^{(n+1)} + \sum_{k=1}^{n} \binom{n}{k} f^{(k)} g^{(n+1-k)}$$

$$= f^{(n+1)} g + \sum_{k=1}^{n} \left[\underbrace{\binom{n}{k-1} + \binom{n}{k}}_{\binom{n+1}{k}} \right] f^{(k)} g^{(n+1-k)} + f g^{(n+1)}$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(k)} g^{(n+1-k)} \qquad \blacksquare$$

3. Show that the Fibonacci sequence F_n satisfies

$$F_{5n+2} > 10^n$$

for all $n \geq 1$.

Binet: $F_n = \dfrac{\alpha^n - \beta^n}{\sqrt{5}}$, where

$\alpha = \dfrac{1+\sqrt{5}}{2}$, $\beta = \dfrac{1-\sqrt{5}}{2}$

Induction: base: $n=0$, $n=1$

$F_2 = 1+1 = 2 > 10^0 = 1$

$F_7 = 13 > 10^1 = 10$

Suppose $F_{5n+2} > 10^n$

$F_{5(n+1)+2} = F_{5n+7} = F_{5n+6} + F_{5n+5}$

$= (F_{5n+5} + F_{5n+4}) + (F_{5n+4} + F_{5n+3}) = F_{5n+5} + 2F_{5n+4} + F_{5n+3}$

$\therefore = 8F_{5n+2} + 5F_{5n+1}$

$$= 8F_{5n+2} + 2F_{5n+1} + 3F_{5n+1}$$

$$= 8F_{5n+2} + 2F_{5n+1} + 3F_{5n} + 3F_{5n-1}$$

$$> 8F_{5n+2} + 2F_{5n+1} + 2F_{5n}$$

$$= 10F_{5n+2} > 10 \cdot 10^n = 10^{n+1}$$

$$\blacksquare$$