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270A - Homework 1

1. (a) Let $\mathcal{F}$ be the family of all finite subsets of Ω and their complements. Is $\mathcal{F}$ a σ-algebra?

   Solution. If Ω is finite then $\mathcal{F}$ is simply the power set of Ω, which is definitely a σ-algebra. However, if $\mathcal{F}$ is infinite, then $\mathcal{F}$ is never a σ-algebra. To see this, let (x_n) be a countable sequence of distinct elements in Ω and consider the set of even-indexed terms

   $$F = \{x_n : n = 2k,\ k \in \mathbb{N}\}.$$

   This set is a countable union of singletons and all singletons belong to $\mathcal{F}$. F is clearly infinite, but so is its complement, which contains the (infinite) set of odd-indexed terms. We conclude that F is neither finite nor co-finite, so $\mathcal{F}$ is not closed under countable unions when Ω is an infinite set. □

   (b) Let $\mathcal{F}$ be the family of all countable subsets of Ω and their complements. Is $\mathcal{F}$ a σ-algebra?

   Solution. $\mathcal{F}$ is indeed a σ-algebra. The empty set is clearly countable, and $\Omega^C = \emptyset$. Let F_n be a countable collection of sets in $\mathcal{F}$ and consider their union, $F = \cup_{n=1}^{\infty} F_n$. If each F_n is countable, then F is just a countable union of countable sets: countable. If one of the F_n's, say F_k, were co-countable, then $F^C \subseteq F_k^C$, which is countable, so F is co-countable. Since $\mathcal{F}$ contains the empty set and Ω and is closed under countable unions and complements, it is a σ-algebra. □

   (c) Let $\mathcal{F}$ and $\mathcal{G}$ be two σ-algebras of subsets of Ω. Is $\mathcal{F} \cap \mathcal{G}$ always a σ-algebra?

   Solution. $\mathcal{F}$ is a σ-algebra. Since $\mathcal{F}$ and $\mathcal{G}$ both contain $\emptyset$ and Ω, so does their intersection. Let E_n be a countable collection of sets in $\mathcal{F} \cap \mathcal{G}$. Since $\mathcal{F}$ and $\mathcal{G}$ are both σ-algebras, the union $E = \cup_{n=1}^{\infty} E_n$ is in both $\mathcal{F}$ and $\mathcal{G}$ and each E_n^C is in both $\mathcal{F}$ and $\mathcal{G}$ as well. □

   (d) Let $\mathcal{F}$ and $\mathcal{G}$ be two σ-algebras of subsets of Ω. Is $\mathcal{F} \cup \mathcal{G}$ always a σ-algebra?

   Solution. The union need not be a σ-algebra. Let $\Omega = \{1, 2, 3, 4\}$, $\mathcal{F} = \{\emptyset, \Omega, \{1\}, \{2, 3, 4\}\}$, and $\mathcal{G} = \{\emptyset, \Omega, \{2\}, \{1, 3, 4\}\}$. $\mathcal{F}$ and $\mathcal{G}$ are σ-algebras, but the set $\{1\} \cup \{2\} = \{1, 2\}$ is not in their union. □

2. A subset $A \subset \mathbb{N}$ is said to have asymptotic density if

   $$\lim_{n\to\infty} \frac{|A \cap \{1, \ldots, n\}|}{n}$$

   exists. Let $\mathcal{F}$ be the collection of subsets of $\mathbb{N}$ for which the asymptotic density exists. Is $\mathcal{F}$ a σ-algebra?

   Solution. □