HomeWork 1 Math 271C, Spring 2020.

1. Find the solution of

$$\begin{aligned} dX_t^{(1)} &= X_t^{(2)} dt + \sigma^{(1)} d\beta^{(1)}, \\ dX_t^{(2)} &= X_t^{(1)} dt + \sigma^{(2)} d\beta^{(2)}, \\ X_0^{(1)} &= 1, \quad X_0^{(2)} = 0, \end{aligned}$$

with β correlated Brownian motions: $d\left\langle \beta^{(1)}, \beta^{(2)} \right\rangle = \rho$ and $\sigma^{(j)}$ constants (problem can be interpreted as modeling a vibrating string subject to a random force).

2. Consider *geometric Brownian motion* X solving

$$X_t = x_0 + \mu \int_0^t X_s ds + \sigma \int_0^t X_s d\beta_s,$$

with $x_0 > 0$.

Show that we have a strong solution satisfying:

(i) if $\nu < \sigma^2/2$ then, $\lim_{t\to\infty} X_t = 0, \sup_{0\le t<\infty} X_t < \infty$, a.s.

(ii) if $\nu > \sigma^2/2$ then, $\inf_{0\le t<\infty} X_t > 0, \lim_{t\to\infty} X_t = \infty$, a.s.

(iii) if $\nu = \sigma^2/2$ then, $\inf_{t\to\infty} X_t = 0, \sup_{0\le t<\infty} X_t = \infty$, a.s.

3. Let $u(t, x)$ be the smooth solution of the terminal PDE

$$\partial_t u(t,x) + \frac{1}{2}\sigma^2(x)\partial_x^2 u(t,x) + \mu(x)\partial_x u(t,x) + c(x)u(t,x) = 0, t < T, x \in \mathbb{R}, \quad u(T,x) = h(x),$$

with μ, σ, c, h bounded and smooth. Let X_t be the Itô process defined by

$$X_t = x + \int_0^t \mu(X_s) ds + \int_0^t \sigma(X_s) d\beta_s.$$

Show that

$$Y_t = \exp[\int_0^t c(X_s) ds] u(t, X_t),$$

is a martingale. Deduce that

$$u(t, X_t) = \mathbb{E}\left\{ \exp[\int_t^T c(X_s) ds] f(X_T) \mid \mathcal{F}_t \right\}.$$

Remark: This can be written

$$u(t, x) = \mathbb{E}\left\{ \exp[\int_t^T c(X_s) ds] f(X_T) \mid X_t = x \right\}.$$

which is the Markov property for X_t, more about this shortly.

4. Do Problem 5.15 in Øksendal.