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271A- Homework 3

Problem 1

Show that the conditions of Kolmogorov's extension/consistency theorem are satisfied for the finite dimensional distributions associated with the Brownian motion paths.

Proof. Let B_t be a standard Brownian motion and let $t_1, \ldots, t_k \in \mathbb{R}$. The associated finite-dimensional distribution $\nu_{t_1,\ldots,t_k}$ is given by

$$\nu_{t_1,\ldots,t_k}(F_1 \times \cdots \times F_k) = \mathbb{P}[B_{t_1} \in F_1, \ldots, B_{t_k} \in F_k],$$

where the F_j's are arbitrary Borel subsets of $\mathbb{R}$. For any $\sigma \in S_k$ we have

$$\begin{aligned}\nu_{t_{\sigma 1},\ldots,t_{\sigma k}}(F_1 \times \cdots \times F_k) &= \mathbb{P}[B_{t_{\sigma 1}} \in F_1, \ldots, B_{t_{\sigma k}} \in F_k] \\ &= \mathbb{P}[B_{t_1} \in F_{\sigma^{-1}(1)}, \ldots, B_{t_k} \in F_{\sigma^{-1}(k)}] \\ &= \nu_{t_1,\ldots,t_k}(F_{\sigma^{-1}(1)} \times \cdots \times F_{\sigma^{-1}(k)}).\end{aligned}$$

This is the first condition required by the Kolmogorov extension theorem. As for the second, we have

$$\begin{aligned}\nu_{t_1,\ldots,t_k,t_{k+1},\ldots,t_n}(F_1 \times \cdots \times F_k \times \mathbb{R} \times \cdots \times \mathbb{R}) &= \mathbb{P}[B_{t_1} \in F_1, \ldots, B_{t_k} \in F_k, B_{t_{k+1}} \in \mathbb{R}, \ldots, B_{t_n} \in \mathbb{R}] \\ &= \mathbb{P}[B_{t_1} \in F_1, \ldots, B_t \in F_k] \\ &= \nu_{t_1,\ldots,t_k}(F_1 \times \cdots \times F_k).\end{aligned}$$

Thus, the finite-dimensional distributions associated with the Brownian motion paths satisfy the conditions of Kolmogorov's extension theorem. □

Problem 2

Let B_t be a two-dimensional Brownian motion and

$$D_r = \{x \in \mathbb{R}^2 : |x| < r\}.$$

Compute $\mathbb{P}[B_t \in D_r]$.

Solution. B_t has components $B_t^{(1)}$ and $B_t^{(2)}$, both independent standard one-dimensional Brownian motions. Let's rewrite the desired probability.

$$\mathbb{P}[B_t \in D_r] = \mathbb{P}[(B_t^{(1)})^2 + (B_t^{(2)})^2 < r^2].$$

With the fundamental theorem of calculus, one can show that if a random variable X has density $f_X(s)$, then X^2 has density

$$f_{X^2}(s) = \frac{1}{2\sqrt{s}}[f_X(\sqrt{s}) + f_X(-\sqrt{s})] \cdot \mathbb{1}_{[0,\infty)}(s).$$

We also know that the density of the sum of two random variables is the convolution of their individual densities. Since $B_t^{(i)}$ has density $f(x) = (2\pi t)^{-1/2}e^{-\frac{x^2}{2t}}$, we convolve this with itself to obtain the density of the sum $X := (B_t^{(1)})^2 + (B_t^{(2)})^2$

$$\begin{aligned}
f_X(x) &= \left[(2\pi t)^{-1/2}e^{-\frac{x^2}{2t}}\right] * \left[(2\pi t)^{-1/2}e^{-\frac{x^2}{2t}}\right] \\
&= \frac{\exp(-\frac{x}{2t})}{2\pi t}\int_0^x \frac{dy}{\sqrt{(x-y)y}} \\
&= \frac{\exp(-\frac{x}{2t})}{2t}.
\end{aligned}$$

Thus, we have that the desired probability is given by

$$\begin{aligned}
\mathbb{P}[X < r^2] &= \frac{1}{2t}\int_0^{r^2} e^{-\frac{x}{2t}}\ dx \\
&= 1 - e^{-\frac{r^2}{2t}}.
\end{aligned}$$

□

Problem 3

Let B_t be a Brownian motion. Show that

(a.) $Y_t = B_T - B_{T-t}$ is a Brownian motion on $[0, T]$.

Proof. We have that $Y_0 = B_T - B_T = 0$. By linearity of expectation we have that

$$\mathbb{E}[Y_t] = \mathbb{E}[B_T] - \mathbb{E}[B_{T-t}] = 0.$$

As for the distributions of the increments we have

$$Y_s - Y_t = B_{T-t} - B_{T-s} \sim \mathcal{N}(0, |s-t|).$$

Suppose now that $0 \le s_1 < t_1 \le s_2 < t_2 \le \cdots \le s_k < t_k \le T$. We then have

$$Y_{t_i} - Y_{s_i} = B_{T-s_i} - B_{T-t_i}.$$

Since $0 < T - T_k \le T - s_k \le \cdots \le T - t_1 \le T - s_1 \le T$ and B is a Brownian motion, the above increments are jointly independent. Finally, since $t \mapsto B_t$ is almost surely continuous, so is $t \mapsto B_T - B_{T-t}$. We conclude that Y is a Brownian motion. □

(b.) Show that $Y'_t = -B_t$ is also a Brownian motion.

Proof. Since $t \mapsto B_t$ is almost surely continuous, so is $t \mapsto -B_t$. We also have $Y'_0 = -B_0 = 0$. We also have by linearity of expectation

$$\mathbb{E}[Y'_t] = -\mathbb{E}[B_t] = 0.$$

As for the increments we have

$$Y'_s - Y'_t = B_t - B_s \sim \mathcal{N}(0, |t-s|).$$

Let $0 \leq s_1 < t_1 \leq \cdots \leq s_k < t_k$. We then have

$$Y'_{t_i} - Y'_{s_i} = B_{s_i} - B_{t_i}.$$

Since B is a Brownian motion, these increments are jointly independent. We conclude that Y' is a Brownian motion. □

(c.) Show that

$$Y''_t = \begin{cases} tB_{1/t}, & 0 < t < \infty \\ 0, & t = 0 \end{cases}$$

is a Brownian motion, assuming that $B_t = o(t)$ as $t \to \infty$ almost surely.

Proof. We're explicitly given that $Y''_0 = 0$ and since $B_t = o(t)$ as $t \to \infty$, we have that Y'' is almost surely continuous at zero. Again, by the linearity of expectation, we have that $\mathbb{E}[Y''_t] = 0$ for all t. As for the variance of the increments, let $s < t$. We have

$$\begin{aligned} \mathrm{Var}[Y''_t - Y''_s] &= \mathrm{Var}[tB_{1/t}] + \mathrm{Var}[sB_{1/s}] - 2 \cdot \mathrm{Cov}(tB_{1/t}, sB_{1/s}) \\ &= t + s - 2st \cdot \min\left\{\frac{1}{t}, \frac{1}{s}\right\} \\ &= t - s. \end{aligned}$$

It remains to show that the increments are independent. Let $0 < s_1 < t_1 \leq \cdots \leq s_k < t_k$. For $i < j$ we have

$$\begin{aligned} &\mathrm{Cov}(Y''_{t_i} - Y''_{s_i}, Y''_{t_j} - Y''_{s_j}) \\ &= t_i t_j \mathrm{Cov}(B_{1/t_i}, B_{1/t_j}) - t_i s_j \mathrm{Cov}(B_{1/t_i}, B_{1/s_j}) - s_i t_j \mathrm{Cov}(B_{1/s_i}, B_{1/t_j}) + s_i s_j \mathrm{Cov}(B_{1/s_i}, B_{1/s_j}) \\ &= 0. \end{aligned}$$

We then have that the increments are uncorrelated, and therefore independent since they are normal. We conclude that $tB_{1/t}$ is a Brownian motion. □

Problem 4

(Kat Dover and Lee Fisher showed me how to do this one.) Let τ_i be iid exponentially distributed random variables with parameter λ. Let $S_0 = 0$ and $S_n = \sum_{i=1}^n \tau_i$, $n \geq 1$. Define

$$N_t = \max(n \geq 0 : S_n \leq t).$$

(a.) Show that the process N_t is right continuous.

Proof. Fix $\epsilon > 0$. We want to show that for all $t \geq 0$, there exists a $\delta_t > 0$ such that

$$\mathbb{P}[N_{t+\delta} > N_t] < \epsilon.$$

Since exponential processes are memoryless, we have

$$\begin{aligned}
\mathbb{P}[N_{t+\delta} - N_t > 0] &= 1 - \mathbb{P}[N_{t+\delta} - N_t = 0] \\
&= 1 - \sum_{k=0}^{\infty} \mathbb{P}[N_{t+\delta} = k,\ N_t = k] \\
&= 1 - \sum_{k=0}^{\infty} \mathbb{P}[N_t = k] \cdot \mathbb{P}[N_{t+\delta} = k \mid N_t = k] \\
&= 1 - \sum_{k=0}^{\infty} \mathbb{P}[N_t = k] \cdot \mathbb{P}[N_\delta = 0].
\end{aligned}$$

We can choose δ small enough so that $\mathbb{P}[N_\delta = 0]$ is small, so the above quantity can be made less than ϵ. We conclude that the process is right-continuous. $\square$

(b.) Show that N_t is a Poisson random variable with parameter λt.

Proof. Let's compute the probability mass function.

$$\mathbb{P}[N_t = n] = \mathbb{P}\left[\sum_{k=0}^{\infty} \mathbb{1}_{\{S_k \leq t\}} = n\right].$$

$\square$