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271A - Homework 1

1. (a) Let Δ_1, $\Delta_2, \dots$ be independent random variables with mean 0 and variance 1. Let $X_1 = \Delta_1$ and for $n = 1, 2, \dots$ let $X_{n+1} = X_n + \Delta_{n+1} f_n(X_1, \dots, X_n)$ for f_n given bounded deterministic functions. Show that $\{X_n\}$ is a martingale (specify the filtration).

Solution. Let $\{\mathcal{F}_n\}$ be the filtration given by $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$. We have that $\mathbb{E}[|X_1|] = \mathbb{E}[|\Delta_1|] < \infty$, since Δ_1 has finite mean. Suppose now that $\mathbb{E}[|X_n|] < \infty$ for all $n \leq k$ for some k. We then have

$$\begin{aligned}
\mathbb{E}[|X_{k+1}|] &= \mathbb{E}[|X_k + \Delta_{k+1} f_k(X_1, \dots, X_k)|] \\
&\leq \mathbb{E}[|X_k|] + \|f_k\|_{L^\infty} \cdot \mathbb{E}[|\Delta_{k+1}|] \\
&< \infty.
\end{aligned}$$

By induction, each X_n is integrable. Since we're dealing with a discrete stochastic process, it suffices to check the martingale property on consecutive variable-filtration pairs, $\mathbb{E}[X_{n+1}|\mathcal{F}_n]$. Here's a computation.

$$\begin{aligned}
\mathbb{E}[X_{n+1}|\mathcal{F}_n] &= \mathbb{E}[X_{n+1} - X_n + X_n|\mathcal{F}_n] \\
&= \mathbb{E}[\Delta_{n+1} f_n(X_1, \dots, X_n)|\mathcal{F}_n] + X_n \\
&= \mathbb{E}[\Delta_{n+1}|\mathcal{F}_n] \cdot f_n(X_1, \dots, X_n) + X_n \quad (f_n(X_1, \dots, X_n) \text{is } \mathcal{F}_n \text{ measurable}) \\
&= \mathbb{E}[\Delta_{n+1}] \cdot f_n(X_1, \dots, X_n) + X_n \quad (\Delta_{n+1} \text{ is independent of } \mathcal{F}_n) \\
&= X_n.
\end{aligned}$$

Thus, $\{X_n\}$ is a martingale adapted to the filtration $\{\mathcal{F}_n\}$. □

(b) Let $Y_1, \dots$ be independent random variables with mean 0 and variance σ^2. Let $X_n = (\sum_{k=1}^n Y_k)^2 - n\sigma^2$ and show that $\{X_n\}$ is a martingale.

Solution. Let $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$. Let's verify that the variables X_n are integrable.

$$\begin{aligned}
\mathbb{E}[|X_n|] &\leq \sum_{k=1}^n \mathbb{E}[Y_k^2] + 2 \sum_{1 \leq i < j \leq n} \mathbb{E}[Y_i Y_j] + n\sigma^2 \\
&= 2n\sigma^2 \quad (\text{since } Y_i \text{ and } Y_j \text{ are independent for } i \neq j).
\end{aligned}$$

Great. Now let's show that $\mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n$.

$$\begin{aligned}
\mathbb{E}[X_{n+1}|\mathcal{F}_n] &= \mathbb{E}[X_{n+1} - X_n + X_n|\mathcal{F}_n] \\
&= \mathbb{E}\left[\left(\sum_{k=1}^{n+1} Y_k\right)^2 - \left(\sum_{k=1}^{n} Y_k\right)^2 \middle| \mathcal{F}_n\right] - \sigma^2 + X_n \\
&= \mathbb{E}\left[Y_{n+1}\left(2\sum_{k=1}^{n} Y_k + Y_{n+1}\right) \middle| \mathcal{F}_n\right] - \sigma^2 + X_n \\
&= \mathbb{E}[Y_{n+1}^2|\mathcal{F}_n] + 2\sum_{k=1}^{n} \mathbb{E}[Y_{n+1}Y_k|\mathcal{F}_n] - \sigma^2 + X_n \\
&= \sigma^2 + 0 - \sigma^2 + X_n \quad \text{(since the } Y_k\text{'s are independent)} \\
&= X_n.
\end{aligned}$$

□

2. (a) Show that if $X_n \to X$ in L^p, $p \geq 1$, then

$$X_n \to X \text{ in probability.}$$

Solution. Suppose that the random variables X_n are defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. First, we claim that if $X_n \to X$ in L^1. This follows from Hölder's inequality and the finiteness of $\mathbb{P}(\Omega)$.

$$\begin{aligned}
\int_\Omega |X_n - X| \; d\mathbb{P} &\leq \|X_n - X\|_{L^p} \cdot \mathbb{P}(\Omega)^{1/q} \\
&\to 0,
\end{aligned}$$

where q is the Hölder conjugate of p. Now suppose that X_n didn't converge to X in probability. Then for some $\epsilon > 0$, there are infinitely many n such that $\mathbb{P}[E_n > \epsilon] > \epsilon$, where E_n is the event $E_n = \{|X_n - X| > \epsilon\}$. Check this out

$$\begin{aligned}
\int_\Omega |X_n - X| \; d\mathbb{P} &\geq \int_{E_n} |X_n - X| \; d\mathbb{P} \\
&\int_{E_n} \epsilon \; d\mathbb{P} \\
&= \epsilon^2.
\end{aligned}$$

Then X_n *doesn't* converge to X in L^1. We conclude that $X_n \to X$ in probability. □

(b) Construct an example with a sequence X_n of random variables that converges in L^p, but not almost surely.

Solution. Consider the typewriter sequence $f_{n,k}$ given by $f_{n,k}(x) = \chi_{[k2^{-n},(k+1)2^{-n}]}(x)$, where $n = 1, 2, \ldots$ and $k = 0, 1, \ldots 2^n - 1$. Since $f_{n,k}$ is supported on a set of measure 2^{-n}, $f_{n,k} \to 0$ in L^1. But $f_{n,k}(x)$ doesn't converge for any x, since for any fixed n, $f_{n,k}(x) = 1$ for some k. Consequently, $f_{n,k}$ doesn't converge almost surely. $\square$

3. Prove that $B^2(t) - t$ is a martingale, where $B(t)$ is a standard Brownian motion.