# Homework 4
## Math 270B, Winter 2020, Prof. Roman Vershynin

---

### Problem 1 (Eigenvectors of transition matrices)

Let $(X_n)$ be an irreducible and recurrent Markov chain with (doubly-infinite) transition matrix $P$. Let $\psi : \mathbb{N} \to \mathbb{N}$ be a bounded function satisfying

$$\sum_{j=1}^{\infty} P_{ij}\psi(j) = \psi(i) \quad \text{for all } i \in \mathbb{N}.$$

Show that $\psi$ is a constant function.

*(Hint: check that $\psi(X_n)$ is a bounded martingale, and apply the martingale convergence theorem.)*

### Problem 2 (Stopped $\sigma$-algebra)

Let $S$ and $T$ be stopping times with respect to a filtration $(\mathcal{F}_n)$. Denote by $\mathcal{F}_T$ the collection of events $F$ such that $F \cap \{T \le n\} \in \mathcal{F}_n$ for all $n$.

(a) Show that $\mathcal{F}_T$ is a $\sigma$-algebra.

(b) Show that $T$ is measurable with respect to $\mathcal{F}_T$.

(c) If $E \in \mathcal{F}_S$, show that $E \cap \{S \le T\} \in \mathcal{F}_T$.

(d) Show that if $S \le T$ a.s. then $\mathcal{F}_S \subset \mathcal{F}_T$.

### Problem 3 (Stopped $\sigma$-algebra: reconstruction from the limit)

Let $(X_n)$ be a uniformly bounded martingale with respect to the filtration $(\mathcal{F}_n)$. Let $S$ and $T$ be two stopping times satisfying $S \le T$ a.s. Prove that

$$X_T = \mathbb{E}[X|\mathcal{F}_T] \quad \text{and} \quad X_S = \mathbb{E}[X_T|\mathcal{F}_S]$$

where $X$ is the almost sure limit of $X_n$.

### Problem 4

A die is rolled repeatedly. Which of the following are Markov chains? For those that are, compute the transition matrix.

(a) The largest number $X_n$ shown up to the $n$-th roll.

(b) The number $N_n$ of sixes in $n$ rolls.

(c) At time $r$, the time $C_r$ since the most recent six.

(d) At time $r$, the time $B_r$ until the next six.

## Problem 5 (Reflected random walk)

Let $(S_n)$ be a simple random walk starting at $S_0 = 0$. Show that $X_n = |S_n|$ is a Markov chain.

## Problem 6 (Markov property for stopping times)

Let $(X_n)$ me a Markov chain, and let $T$ be a stopping time with respect to the filtration $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. Show that

$$\mathbb{P}\left\{X_{T+1} = j \mid X_k = x_k \text{ for } 0 \leq k < T,\ X_T = i\right\} = \mathbb{P}\left\{X_{T+1} = j \mid X_T = i\right\}$$

for all $m \geq 0$, $i, j$, and $x_k$.

## Problem 7 (Markov property for stopping times)

Find an example of two Markov chains $(X_n)$ and $(Y_n)$ such that $X_n + Y_n$ is not a Markov chain.

## Problem 8 (Random walk on a cube)

A particle performs a random walk on the vertices of a three-dimensional cube. At each step it remains where it is with probability $1/4$, or moves to one of its neighboring vertices each having probability $1/4$. Compute the mean number of steps until the particle returns to the vertex from which the walk started.

*(Hint: Let $v \to s \to t \to w$ be a path from the original vertex $v$ to the diametrically opposite vertex $w$. Conditioning on the first step and using a symmetry argument, write down a system of linear equations for $\mu_v$, $\mu_s$, $\mu_t$ and $\mu_w$, the mean number of steps to reach $v$ from $v, s, t, w$ respectively.)*

## Problem 9 (Recurrence of a symmetric random walk)

Prove that the symmetric random walk on $\mathbb{Z}^2$ is recurrent, the symmetric random walk on $\mathbb{Z}^3$ is transient.

## Problem 10 (Reversible Markov chains)

Which of the following are reversible Markov chains?

(a) Move from 0 to 1 with probability $p$ and stay at 0 with probability $1 - p$; move from 1 to 0 with probability $q$ and stay at 1 with probability $1 - q$.

(b) A random walk on the vertices of a given triangle: at each time, move to the next vertex counterclockwise with probability $p$ and clockwise with probability $1 - p$.

## Problem 11 (Chess)

A king performs a random walk on a chessboard; at each step it is equally likely to make any one of the available moves (and never stays put). What is the mean recurrence time of a comer square?