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270A - Homework 4

Problem 1. Let $E_1, E_2, \ldots$ be events on the same probability space. Assume that

$$\mathbb{P}[E_n] \to 0 \quad \text{and} \quad \sum_n \mathbb{P}[E_n \cap E_{n-1}^c] < \infty.$$

Show that

$$\mathbb{P}[E_n \text{ occur infinitely often}] = 0.$$

Proof. By the continuity of measure we have

$$\begin{aligned}\mathbb{P}[\limsup E_n^c] &= \lim_{N\to\infty} \mathbb{P}\left[\bigcup_{n=N}^{\infty} E_n^c\right] \\ &\geq \lim_{N\to\infty} \mathbb{P}[E_n^c] \\ &= 1,\end{aligned}$$

where the last equality follows from the hypothesis that $\mathbb{P}[E_n] \to 0$. Now by Borel-Cantelli, $\mathbb{P}[E_n \cap E_{n-1}^c \text{ i.o.}] = 0$. Since $\mathbb{P}[E_{n-1}^c \text{ i.o.}] = 1$ by the above discussion, we have

$$\mathbb{P}[E_n \text{ i.o.}] = \mathbb{P}[E_n \cap E_{n-1}^c \text{ i.o.}] = 0.$$

□

Problem 2. Let $X_1, X_2, \ldots$ be iid random variables with the standard exponential distribution,

$$\mathbb{P}[X_i > x] = e^{-x}, \quad x \geq 0.$$

(a) Show that

$$\limsup_n \frac{X_n}{\log n} = 1 \text{ a.s.}$$

Proof. For any positive t we have

$$\sum_{n=1}^{\infty} \mathbb{P}[X_n > t\log n] = \sum_{n=1}^{\infty} \frac{1}{n^t} = \begin{cases} C_t < \infty, & \text{if } t \leq 1 \\ \infty, & \text{if } t > 1 \end{cases}.$$

By Borel-Cantelli, we then have

$$\mathbb{P}\left[\frac{X_n}{\log n} > t \text{ infinitely often}\right] = \begin{cases} 1, & \text{if } t \leq 1 \\ 0, & \text{if } t > 1 \end{cases}.$$

In particular, we have that $\frac{X_n}{\log n} > 1$ infinitely often almost surely, but $\frac{X_n}{\log n} > t$ only finitely often almost surely for any $t > 1$. We conclude that $\limsup \frac{X_n}{\log n} = 1$ almost surely. □

(b) Let $M_n = \max_{1\le k\le n} X_k$. Show that

$$\limsup_n \frac{M_n}{\log n} = 1 \text{ a.s.}$$

Proof. Fix $t > 0$ and let E_n be the event given by $E_n = \{M_n > t\log n\}$. By L'Hôpital's rule, we have

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{P}[E_n] &= \lim_{n\to\infty} \mathbb{P}[X_k > t\log n \text{ for at least one } 1 \le k \le n] \\
&= \lim_{n\to\infty} 1 - \mathbb{P}[X_k \le t\log n \text{ for each } 1 \le k \le n] \\
&= \lim_{n\to\infty} 1 - (1 - \mathbb{P}[X_1 > t\log n])^n \quad \text{(since the } X_k\text{'s are independent)} \\
&= \lim_{n\to\infty} 1 - \left(1 - \frac{1}{n^t}\right)^n \\
&= 0 \text{ if and only if } t > 1.
\end{aligned}$$

Now let's compute $\mathbb{P}[E_n \setminus E_{n-1}]$.

$$\begin{aligned}
\mathbb{P}[E_n \setminus E_{n-1}] = \mathbb{P}[&X_k > t\log n \text{ for at least one } 1 \le k \le n \\
&\text{AND } X_k \le t\log(n-1) \text{ for each } 1 \le k \le n-1].
\end{aligned}$$

The only way $X_k \le t\log(n-1)$ can hold for each $1 \le k \le n-1$ while still having at least one of $1 \le k \le n$ satisfy $X_k > t\log n$ is for $X_n > t\log n$ to hold. We then have

$$\sum \mathbb{P}[E_n \setminus E_{n-1}] = \sum \mathbb{P}[X_n > t\log n] = \sum \frac{1}{n^t} < \infty \text{ if and only if } t > 1.$$

By problem 1, we then have $\mathbb{P}[M_n > t\log n \text{ infinitely often}] = 0$ if and only if $t > 1$. By the same reasoning we used in part (a), we have that $\limsup \frac{M_n}{\log n} = 1$ almost surely. $\square$

Problem 3. Let

$$\psi(x) = \begin{cases} x^2 & \text{if } |x| \le 1 \\ |x| & \text{if } |x| \ge 1 \end{cases}.$$

Let $X_1, X_2, \ldots$ be independent mean zero random variables. Show that if $\sum \mathbb{E}[\psi(X_n)] < \infty$, then $\sum X_n$ converges almost surely.

Proof. By Markov's inequality we have

$$\sum \mathbb{P}[|X_n| > 1] \le \sum \mathbb{E}[|X_n| \cdot \mathbb{1}_{|X_n|>1}] < \infty.$$

By Borel-Cantelli, $|X_n| \le 1$ eventually almost surely. In other words, $X_n = X_n \cdot \mathbb{1}_{|X_n|\le 1}$ eventually almost surely. In particular, we have that $\sum X_n$ converges a.s. if and only if $\sum X_n \cdot \mathbb{1}_{|X_n|\le 1}$ converges a.s.

Now let's look at $\sum \mathbb{E}[X_n \cdot \mathbb{1}_{|X_n|\leq 1}]$. Since $\mathbb{E}[X_n] = 0$, we have $\mathbb{E}[X_n \cdot \mathbb{1}_{|X_n|\leq 1}] = -\mathbb{E}[X_n \cdot \mathbb{1}_{|X_n|>1}]$. This gives

$$\sum |\mathbb{E}[X_n \cdot \mathbb{1}_{|X_n|\leq 1}]| = \sum |\mathbb{E}[X_n \cdot \mathbb{1}_{|X_n|\geq 1}]| \leq \sum \mathbb{E}[|X_n| \cdot \mathbb{1}_{|X_n|\geq 1}] \leq \sum \mathbb{E}[\psi(X_n)] < \infty.$$

The variances of $X_n \cdot \mathbb{1}_{|X_n|\leq 1}$ are also summable:

$$\sum \mathrm{Var}[X_n \cdot \mathbb{1}_{|X_n|\leq 1}] \leq \sum \mathbb{E}[X_n^2 \cdot \mathbb{1}_{|X_n|\leq 1}] \leq \sum \mathbb{E}[\psi(X_n)] < \infty.$$

By Kolmogorov's two-series theorem, we have that $\sum X_n \cdot \mathbb{1}_{|X_n|\leq 1}$ converges almost surely, which shows that $\sum X_n$ converges almost surely by our earlier discussion. □

Problem 4. Construct a sequence of independent mean zero random variables $X_1, X_2, \ldots$ such that

$$\frac{1}{n}\sum_{k=1}^{n} X_k \to \infty \text{ a.s.}$$

Why does this example not contradict the strong law of large numbers?

Solution. Consider the probability triplet $(\Omega, \mathcal{F}, \mathbb{P})$ where Ω is the unit interval $[0, 1]$, $\mathcal{F}$ is the Borel σ-algebra, and $\mathbb{P}$ is the Lebesgue measure. Define the sequence of random variables X_k by

$$X_k = \begin{cases} k(1-k), & \text{if } 0 \leq x \leq \frac{1}{k} \\ k, & \text{if } \frac{1}{k} < x \leq 1. \end{cases}$$

Each X_k has zero mean since

$$\mathbb{E}[X_k] = \frac{1}{k} \cdot k(1-k) + \left(1 - \frac{1}{k}\right) \cdot k = 0.$$

Now for any $\omega \in [0, 1]$ we have $\frac{1}{n} < \omega \leq 1$ for all n larger than some N_ω. We then have for n larger than N_ω,

$$\frac{1}{n}\sum_{k=1}^{n} X_k(\omega) = \frac{1}{n}\left(\sum_{k=1}^{N_\omega} k(k-1) + \sum_{k=N_\omega+1}^{n} k\right) = \frac{1}{n}(C_\omega + \Omega(n^2)) \to \infty,$$

for some constant C_ω. This doesn't contradict the strong law of large numbers, since the strong law requires that the variables $X_1, X_2, \ldots$ be iid, whereas the variables X_k defined above are not identically distributed. □

Problem 5. Suppose disasters occur at random times X_i apart from each other. Precisely, the k-th disaster occurs at time $T_k = X_1 + \cdots + X_k$, where the X_i are iid random variables taking positive values with finite mean μ. Let

$$N(t) = \max\{n : T_n \leq t\}$$

be the number of disasters that have occurred by time t. Prove that

$$N(t) \to \infty \quad \text{and} \quad \frac{N(t)}{t} \to \frac{1}{\mu}$$

almost surely as $t \to \infty$.

Proof. First, we claim that $N(t) < n$ if and only if $t < T_n$. This is true since

$$N(t) < n = \max(m : T_m \leq t) < n \iff T_n > t.$$

From this, we can deduce that $t < T_{N(t)+1}$ since $N(t) < N(t) + 1$. Similarly, since $N(t) \leq N(t)$, we have $T_{N(t)} \leq t$. Putting these together gives

$$T_{N(t)} \leq t < T_{N(t)+1}. \tag{1}$$

Now by the strong law of large numbers we have that $\frac{T_n}{n} \to \mu$ almost surely. Fix $\epsilon > 0$. For n sufficiently large, we have that $|\frac{T_n}{n} - \mu| \leq \epsilon$ a.s. From this we deduce that $T_n \leq n(\mu + \epsilon)$ a.s. Since $T_n \leq t$ if and only if $N(t) \geq n$, we have

$$N(n(\mu + \epsilon)) \geq n$$

for n large. Taking n to infinity and using the fact that $N(t)$ is nondecreasing, we have that $N(t) \to \infty$ as $t \to \infty$. Dividing (1) through by $N(t)$ gives

$$\frac{T_{N(t)}}{N(t)} \leq \frac{t}{N(t)} \leq \frac{T_{N(t)+1}}{N(t)} = \frac{T_{N(t)} + X_{N(t)+1}}{N(t)}.$$

By the strong law of large numbers and the fact that $N(t) \to \infty$, we have that $T_{N(t)}/N(t) \to \mu$ a.s. as $t \to \infty$. Since the X_k's are identically distributed with finite mean, we have that $X_{N(t)+1}/N(t) \to 0$ a.s. Both sides of the above inequality then tend to μ a.s., so $\frac{N(t)}{t} \to \frac{1}{\mu}$ a.s. $\square$

Problem 6. Let $X_1, X_2, \ldots$ be independent random variables. Show that $\sum X_n$ converges in probability if and only if $\sum X_n$ converges almost surely.

Proof. Almost sure convergence always implies convergence in probability, so it just remains to show the converse. To this end, let $S_n = \sum_{k=1}^n X_k$ be the n-th partial sum. Let's show that S_n is Cauchy a.s. Fix n and some N and apply Etemadi's inequality to the variables $X_{n+1}, X_{n+1}, \ldots, X_N$:

$$\mathbb{P}\left[\max_{n+1 \leq m \leq N} |X_{n+1} + \cdots + X_m| > 3\epsilon\right] \leq 3 \cdot \max_{n+1 \leq m \leq N} \mathbb{P}[|X_{n+1} + \cdots + X_m| > \epsilon].$$

Letting $N \to \infty$, we have

$$\mathbb{P}\left[\sup_{m > n} |X_{n+1} + \cdots + X_m| > 3\epsilon\right] \leq 3 \cdot \sup_{m > n} \mathbb{P}[|X_{n+1} + \cdots + X_m| > \epsilon].$$

Now since $\sum X_n$ converges in probability, its partial sums are Cauchy in probability. Consequently, as we take n to infinity, the right-hand side of the above inequality tends to zero as $n \to \infty$. We have then shown that $\sup_{m>n} |S_m - S_n| \to 0$ in probability. Since this quantity is decreasing in n, it must converge a.s. as well. Since the partial sums are a.s. Cauchy, we have that $\sum X_n$ converges a.s. $\square$

Problem 7. Let $X_1, X_2, \ldots$ be iid random variables taking non-negative values, such that $\mathbb{P}[X_i > 0] > 0$. Prove that

$$\sum X_n = \infty \text{ a.s.}$$

Proof. Since $\mathbb{P}[X_i > 0] > 0$, we can find δ and ϵ both positive such that $\mathbb{P}[X_i > \delta] > \epsilon$. We then have

$$\sum \mathbb{P}[X_i > \delta] > \sum \epsilon = +\infty.$$

By Borel-Cantelli, we then have that $\mathbb{P}[X_i > \delta \text{ infinitely often}] = 1$. Let $A = \{X_i > \delta \text{ infinitely often}\}$. For any $\omega \in A$, we have that $\sum X_n(\omega)$ is a sum that contains infinitely many terms of size at least δ. Since each X_n takes only nonnegative values, this sum must diverge at ω. Since $\mathbb{P}[A] = 1$, we have that $\sum X_n = \infty$ almost surely. □

Problem 8. Call a number $x \in [0, 1]$ badly approximable by rationals if there exists $c(x) > 0$ and $\epsilon(x) > 0$ such that for any $p, q \in \mathbb{N}$ we have

$$\left| x - \frac{p}{q} \right| > \frac{c}{q^{2+\epsilon}}.$$

Prove that almost all numbers in $[0, 1]$ are badly approximable.

Proof. Fix any positive ϵ and c. Let E_q be the set of rationals that are *not* badly approximable by a rational with denominator q:

$$E_q = \left\{ x \in [0,1] : \left| x - \frac{p}{q} \right| \leq \frac{c}{q^{2+\epsilon}} \text{ for some } 0 \leq p \leq q \right\} = \bigcup_{p=0}^{q} \left\{ x \in [0,1] : \left| x - \frac{p}{q} \right| \leq \frac{c}{q^{2+\epsilon}} \right\}.$$

The set of not badly approximable numbers is the union of the E_q's. Let's compute the measure of this set, $E_{c,\epsilon} = \cup_q E_q$

$$\begin{aligned} m(E_{c,\epsilon}) &= \sum_q \sum_{p=0}^{q} m \left\{ x \in [0,1] : \left| x - \frac{p}{q} \right| \leq \frac{c}{q^{2+\epsilon}} \right\} \\ &\leq \sum_q \sum_{p=0}^{q} \frac{2c}{q^{2+\epsilon}} \\ &= 2c \sum_q \frac{q+1}{q^{2+\epsilon}} \\ &< \infty. \end{aligned}$$

By Borel-Cantelli, we have that for any fixed $\epsilon, c > 0$, almost every $x \in [0, 1]$ satisfies $|x - p/q| \leq C/q^{2+\epsilon}$ for only finitely many p and q. □

Problem 9. Let $X_1, X_2, \ldots$ be iid random variables with finite mean μ. Prove that

$$\frac{1}{\ln n} \sum_{k=1}^{n} \frac{X_k}{k} \to \mu \text{ a.s.}$$

Proof. When we proved the strong law of large numbers in class, we crucially used the fact that if a sequence of numbers x_n converges to x, then it converges in mean: $\frac{1}{n}\sum_{k=1}^{n} x_k$ converges to x as well. In this problem, we'll show that if $x_n \to x$, then $\frac{1}{\ln n}\sum_{k=1}^{n}\frac{x_k}{k} \to x$. This fact along with minor modifications to our proof from class will establish the claim.

From the monotonicity of $f(x) = \frac{1}{x}$, we have that

$$\ln n + \frac{1}{n} \leq \sum_{k=1}^{n} \frac{1}{k} \leq \ln n + 1.$$

Dividing through by $\ln n$ and taking the limit establishes

$$\frac{1}{\ln n}\sum_{k=1}^{n}\frac{1}{k} \to 1.$$

Now suppose a sequence of real numbers x_n converges to x. By the above reasoning, we have

$$\frac{1}{\ln n}\sum_{k=1}^{n}\frac{x}{k} = x\left(\frac{1}{\ln n}\sum_{k=1}^{n}\frac{1}{k}\right) \to x.$$

Fix $\epsilon > 0$ and choose N such that $|x_n - x| < \epsilon$ for all $n > N$. If $n > N$ we have

$$\begin{aligned}\left|\frac{1}{\ln n}\sum_{k=1}^{n}\frac{x_k}{k} - \frac{1}{\ln n}\sum_{k=1}^{n}\frac{x}{k}\right| &\leq \frac{1}{\ln n}\sum_{k=1}^{N}\frac{|x_k - x|}{k} + \frac{1}{\ln n}\sum_{k=N+1}^{n}\frac{|x_k - x|}{k} \\ &\leq \frac{1}{\ln n}\sum_{k=1}^{N}\frac{|x_k - x|}{k} + \epsilon \cdot \frac{1}{\ln n}\sum_{k=1}^{n}\frac{1}{k}.\end{aligned}$$

The first sum has only finitely many terms in it, so as $n \to \infty$ it vanishes. The second sum approaches ϵ as n tends to infinity. We've then shown that the terms of $\frac{1}{\ln n}\sum_{k=1}^{n}\frac{x_k}{k}$ and $\frac{1}{\ln n}\sum_{k=1}^{n}\frac{x}{k}$ become arbitrarily close to one another. Since the latter sequence limits to x, we must then have

$$\frac{1}{\ln n}\sum_{k=1}^{n}\frac{x_k}{k} \to x. \tag{2}$$

Returning to the problem at hand, by splitting $X = X^+ - X^-$, we can assume without loss of generality that $X_k \geq 0$. Let's start by truncating the X_k's and define $Y_k = X_k \cdot \mathbb{1}_{\{|X_k| \leq k\}}$ and $T_n = \sum_{k=1}^{n} Y_k/k$. We claim that it suffices to prove that $T_n/\ln n \to \mu$ a.s. The idea is that $X_k = Y_k$ eventually almost surely. To see this, consider the sum

$$\sum \mathbb{P}[|X_k| > k] \leq \int_0^\infty \mathbb{P}[|X_1| > t]\ dt = \mathbb{E}[|X_1|] < \infty.$$

By Borel-Cantelli, we have that $|X_k| \leq k$ eventually almost surely. If $|X_k| \leq k$, then by definition we have $X_k = Y_k$. Consequently, the difference $\sum_{k=1}^{n}\frac{1}{k}|X_k(\omega) - Y_k(\omega)| < C_\omega < \infty$ for all n.

Let $\epsilon > 0$ be arbitrary and let $k(n) = \lceil 2^{(1+\epsilon)^n} \rceil$. Let $\delta > 0$. By Chebyshev's inequality we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \mathbb{P}[|T_{k(n)} - \mathbb{E}[T_{k(n)}]| > \delta \ln k(n)] &\leq \frac{1}{\delta^2} \sum_n \frac{\mathrm{Var}[T_{k(n)}]}{(1+\epsilon)^{2n}} \\
&= \frac{1}{\delta^2} \sum_n (1+\epsilon)^{-2n} \sum_{m=1}^{k(n)} \mathrm{Var}\left[\frac{Y_m}{m}\right] \\
&= \frac{1}{\delta^2} \sum_{m=1}^{\infty} \frac{1}{m^2} \mathrm{Var}[Y_m] \sum_{n:k(n)>m} (1+\epsilon)^{-2n} \\
&\leq \frac{C_\epsilon}{\delta^2} \sum_m \frac{1}{m^2} \mathrm{Var}[Y_m].
\end{aligned}$$

Now we showed in class that $\sum \mathrm{Var}[Y_m]/m^2 < \infty$, so the above sum is finite. Since δ was arbitrary, we have that

$$\frac{T_{k(n)} - \mathbb{E}[T_{k(n)}]}{\ln k(n)} \to 0 \text{ a.s.} \tag{3}$$

By dominated convergence we have that $\mathbb{E}[Y_k] \to \mathbb{E}[X_1] = \mu$. By our earlier discussion and (2), we have

$$\frac{\mathbb{E}[T_{k(n)}]}{\ln k(n)} = \frac{1}{\ln k(n)} \sum_{k=1}^{k(n)} \frac{\mathbb{E}[Y_k]}{k} \to \mu.$$

By (3), we must have that $\frac{T_{k(n)}}{\ln k(n)} \to \mu$ a.s. Now we interpolate: suppose $k(n) \leq m < k(n+1)$. Since we're assuming that $Y_k \geq 0$ for all k, we have

$$\frac{T_{k(n)}}{\ln k(n+1)} \leq \frac{T_m}{\ln m} \leq \frac{T_{k(n+1)}}{\ln k(n)}. \tag{4}$$

Now $\ln k(n+1)/\ln k(n) \to (1+\epsilon)$, so after taking limits we have

$$\frac{1}{1+\epsilon}\mu \leq \liminf_{m\to\infty} \frac{T_m}{\ln m} \leq \limsup_{m\to\infty} \frac{T_m}{\ln m} \leq (1+\epsilon)\mu.$$

Taking $\epsilon \to 0$ establishes $\frac{T_m}{\ln m} \to \mu$. By our previous discussion, this implies that $\frac{1}{\ln n} \sum_{k=1}^n \frac{X_k}{k} \to \mu$ a.s. $\square$