New problems for today are on Canvas.

Theme: Asymptotics

Big O notation

- $f(x) = O(g(x))$ tells you that f grows no faster than g

$\exists C_1$ & C_2 s.t.

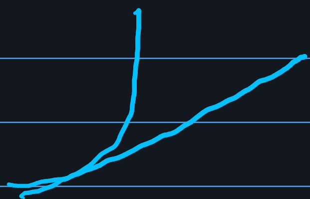
$\forall x \geq C_2$,

$$|f(x)| \leq C_1 |g(x)|$$

f eventually smaller than some multiple of g

Thm: if $\lim_{x \to \infty} \left| \frac{f(x)}{g(x)} \right| < \infty$

$$\Rightarrow f(x) = O(g(x))$$



Applications: Computer Science

esp when talking about algorithms.

Say you have two sorting algorithms.

Alg. 1 sorts a list of length n in $O(n^2)$ steps.

$\Rightarrow$ # steps grows no faster than a multiple of x^2

- Alg 2 takes

$\Theta(n \log n)$ steps.

ie, for n large, takes $\leq$ multiple of $n \log n$ steps.

Expect algorithm 2 to be better

Since $n \log n$ grows <u>more slowly</u> than n^2

$$\lim_{n\to\infty} \frac{n \log n}{n^2}$$

$$= \lim_{n\to\infty} \frac{\log n}{n}$$

$$\text{L'H} = \lim_{n\to\infty} \frac{1/n}{1} = 0$$

$$\Rightarrow n \log n = o(n^2)$$

since $f(x) = o(g(x))$ iff $\lim_{x\to\infty} \left| \frac{f(x)}{g(x)} \right| = 0$.

Show that if $\lim_{x \to \infty} f(x) = \infty$

then $f(x) + \sin x = \Theta(f(x))$.

Pf: $\Theta(f(x))$ means

$O(f(x))$ & $\Omega(f(x))$

grows slower

grows faster

$\Rightarrow$ ie, grows just as quickly as $f(x)$.

show $f(x)+\sin(x) = O(f(x))$

by Thm, show

$$\lim_{x\to\infty}\left|\frac{f(x)+\sin(x)}{f(x)}\right|<\infty$$

$$\left|\frac{f(x)+\sin(x)}{f(x)}\right| = \left|1+\frac{\sin x}{f(x)}\right|$$

$$\leq |1| + \left|\frac{\sin x}{f(x)}\right|$$

$$\leq 1+\frac{1}{|f(x)|} \quad \text{since } |\sin x|\leq 1$$

$$\to 1<\infty \text{ as } x\to\infty$$

Show $f(x)+\sin x = \Omega(f(x))$

$f(x) = \Omega(g(x))$ if $g(x) = O(f(x))$

ie, Show $f(x) = O(f(x)+\sin x)$

$$\overline{\lim_{x\to\infty}} \left| \frac{f(x)}{f(x)+\sin x} \right|$$

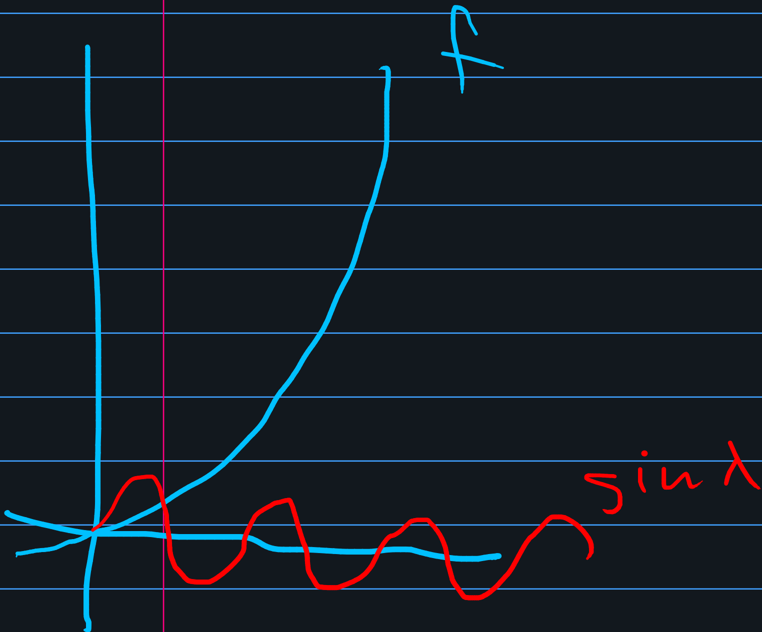
$$= \lim_{x\to\infty} \left| \frac{1}{1+\frac{\sin x}{f(x)}} \right| = 1 < \infty$$

Since $|\sin x| \le 1$, & $f(x) \to \infty$

$$\left|\frac{\sin x}{f(x)}\right| \longrightarrow 0$$

$\Longrightarrow f(x)$ is $O(f(x)+\sin x)$

$\not\mathbb{D}k$


f
sin x


f + sin x

HW: $\lim_{x \to \infty} \frac{(x+2)\cos^2 x}{x}$

DNE but $(x+2)\cos^2 x$

$= O(x)$.

Thm: limit exists

$\Longrightarrow f(x) = O(g(x))$

$\not\Longleftarrow$

$\frac{x+2}{x} \longrightarrow 1$, $\cos^2 x \longrightarrow$ DNE

oscillates

between 0 & 1

$$\frac{(x+2)\cos^2 x}{x} = \frac{x\cos^2 x}{x} + \frac{2\cos^2 x}{x}$$

$$= \cos^2 x + \frac{2\cos^2 x}{x}$$

$\cos^2 x$ doesn't converge as $x \to \infty$

$\frac{2\cos^2 x}{x} \to 0$ as $x \to \infty$

$\Rightarrow$ limit DNE

but $|(x+2)\cos^2 x| \leq |x+2|$

since $|\cos^2 x| \leq 1$

Show $|x+2| \leq C|x|$ for some C

4. you dont know P, but you can ask for $P(1), P(2),$ etc.

* can only ask for $P(a), a \in \mathbb{Q}$

Exercise 1. (a) Let $f(x) = ax + b$ and $g(x) = cx + d$ where $a, c \neq 0$. Show that $f(x) = O(g(x))$.

From the definition, wts $\exists c_1, c_2$

st $\forall x \geq c_2$,

$$|f(x)| \leq c_1 |g(x)|.$$

This inequality holds iff

$$\left|\frac{f(x)}{g(x)}\right| \leq c_1$$

$$\iff \left|\frac{ax+b}{cx+d}\right| \leq c_1$$

The LHS tends to $\left|\frac{a}{c}\right|$ as $x \to \infty$,

so there exists some

c_2 s.t. $x \geq c_2$

$$\Rightarrow \left|\frac{ax+b}{cx+d}\right| \leq 2\left|\frac{a}{c}\right|$$

set $c_1 = |2^a/c|$ $\boxed{\text{Is}}$

(b) Let $f(x) = ax^2 + bx + c$, $a > 0$. Show that $f(x) = \Omega(x^2)$.

from definition, wts $\exists c_1, c_2$

s.t. $\forall x \geq c_2$ $|f(x)| \geq c_1 x^2$

This inequality holds if

$$|ax^2+bx+c| \geq c_1 x^2$$

$$\Rightarrow \frac{|ax^2+bx+c|}{x^2} \geq c_1$$

The LHS tends to $|a|$ as $x \to \infty$, so $\exists c_2$ s.t.

$\forall x \geq c_2$, $\dfrac{|ax^2+bx+c|}{x^2} \geq \frac{1}{2}|a|$

Set $c_1 = \frac{1}{2}|a|$. $\boxtimes$

(c) Show that $\sin x = \Theta(1)$.

Show $\sin x = O(1)$ & $\sin x = \Omega(1)$,

$O(1)$: show $\exists c_1, c_2$ s.t. $\forall x \geq c_2$, $|\sin x| \leq c_2$

since $|\sin x| \leq 1$ $\forall x$,

can set $c_2 = 0$ & $c_1 = 1$

$\Omega(1)$: $\sin x \geq -1$, so

$\sin(x) = \Omega(1)$.

$\Rightarrow \sin(x) = \Theta(1)$. $\boxed{}$

Exercise 2. Prove the following.

(a) $x^2 + \sqrt{x} = O(x^2)$.

$$\lim_{x \to \infty} \left| \frac{x^2 + \sqrt{x}}{x^2} \right| = \lim_{x \to \infty} \left| 1 + \frac{1}{x^{3/2}} \right|$$

$$= 1 < \infty$$

$\Rightarrow x^2 + \sqrt{x} = O(x^2)$

(c) $k^2 2^k = O(e^{2k})$.

write $2^k = \left(e^{\log 2}\right)^k$

$$\Rightarrow \lim_{k\to\infty} \left|\frac{k^2 2^k}{e^{2k}}\right| = \lim_{k\to\infty} \left|\frac{k^2}{\exp\left((2-\log 2)k\right)}\right|$$

$$= 0$$

since $2-\log 2 > 0$. $\square$

(d) $N^{10} 2^N = O(e^N)$.

$2^N = \left(e^{\log 2}\right)^N$

$$\Rightarrow \lim_{N\to\infty} \left|\frac{N^{10} 2^N}{e^N}\right| = \lim_{N\to\infty} \left|\frac{N^{10}}{\exp\left((1-\log 2)N\right)}\right|$$

$= 0$ since $1-\log 2 > 0$

$\square$

Exercise 3. Prove the following.

(a) We often say that a sequence of events E_n happens "with high probability" if $\Pr[E_n] = 1 - o(1)$. Why does this make sense?

a function is $o(1)$ if it tends to 0.

$\mathbb{P}[E_n] = 1 - o(1)$

$\iff \mathbb{P}[E_n] \longrightarrow 1$

Since the probability of E_n occuring tends to 1, it makes sense to say it happens "with high probability."

(b) If $f(x) = o(g(x))$ then $f(x) = O(g(x))$. If $f(x) = \omega(g(x))$ then $f(x) = \Omega(g(x))$. Give examples to show that the converses to these statements are false.

- if $f(x) = o(g(x))$, then

$\lim_{x \to \infty} \left| f(x)/g(x) \right| = 0$, which is finite, so

$\lim_{x \to \infty} \left| \frac{f(x)}{g(x)} \right| < \infty \Rightarrow f(x) = O(g(x))$.

- if $f(x) = \omega(g(x))$, then

$\lim_{x \to \infty} \left| \frac{f(x)}{g(x)} \right| = \infty > 0$, so

$f(x) = \Omega(g(x))$.

- $x^2 = O(x^2)$ since $\frac{x^2}{x^2} = 1 \to 1$,

but $x^2 \neq o(x^2)$ since $\frac{x^2}{x^2} \not\to 0$.

similarly, $x^2 = \Omega(x^2)$, $x^2 \neq \omega(x^2)$.

(c) $k^{300} = o(2^k)$.

L'Hopital + induction

$$\lim_{k\to\infty} \frac{k^{300}}{2^k} = \lim_{k\to\infty} (300)! / (\log 2)^{300} 2^k$$

$$= 0.$$

(d) $k^{0.001} = \omega((\log k)^{375})$.

$$\lim_{k\to\infty} \frac{k^{.001}}{(\log k)^{375}} = \lim_{k\to\infty} \left(\frac{k}{(\log k)^{375000}}\right)^{.001}$$

$$= \left(\lim_{k\to\infty} \frac{k}{(\log k)^{375000}}\right)^{.001}$$

$$= (\infty)^{.001} = \infty$$

$\square$