Liam Hardiman *January 27, 2020*

271B - Homework 1

Problem 1. The standard Ornstein-Uhlenbeck process X_t is a Gaussian process with mean zero and auto-covariance $C(t,s) = \mathbb{E}[X_t X_s] = \exp(-|t-s|)/2$. Let N_t be the standard Poisson process and define the process $Y_t = \zeta(-1)^{N_t}$, where ζ is a random variable independent of the Poisson process that takes values ± 1 with probability $1/2$.

Show that X_t and $Z_t = Y_{t/2}/\sqrt{2}$ are both stationary in the strong sense and have the same covariance. Does Y_t satisfy the Kolmogorov continuity condition? Are these processes stochastically continuous?

Solution. First we'll show that X_t is stationary. Let $\tau \in \mathbb{R}$. Since X_t is Gaussian with mean zero for all t, so is $X_{t+\tau}$. For any s and t we also have that

$$\mathbb{E}[X_{s+\tau} X_{t+\tau}] = \frac{1}{2} e^{-|(s+\tau)-(t+\tau)|} = \frac{1}{2} e^{-|s-t|} = \mathbb{E}[X_s X_t].$$

Since a Gaussian process is determined by its mean and covariance, we have that X_t and $X_{t+\tau}$ are equal in distribution, so the process is stationary. $\square$