1. A biased coin with head probability p is flipped n times. Let X be the number of heads and Y the number of tails. Clearly, X and Y are not independent since $Y = n - X$ (in particular, knowing X lets us compute Y). Suppose instead that the coin is tossed a random number of times N, where $N \sim Poisson(\lambda)$. Show that now X and Y are independent.

Need to show that for any $h, t \geq 0$

$$\Pr[X=h, Y=t] = \Pr[X=h]\Pr[Y=t]$$

well, $\Pr[X=h, Y=t]$

$$= \Pr[X=h, Y=t, N=t+h]$$

$$= \Pr[X=h, Y=t \mid N=t+h]\Pr[N=t+h]$$

$= \ldots =$ keep going.

1. Let X be a random variable that takes on values between 0 and c. Show that

$$\mathrm{Var}[X] \leq \frac{c^2}{4}.$$

One way to do this is to first show that $E[X^2] \leq cE[X]$ and then show that

$$\mathrm{Var}[X] \leq c^2[\alpha(1-\alpha)],$$

where $\alpha = \frac{E[X]}{c}$.

$0 \leq X \leq c$

$$\mathrm{Var}[X] = E[X^2] - E[X]^2$$

$$= E[X \cdot X] - E[X]^2$$

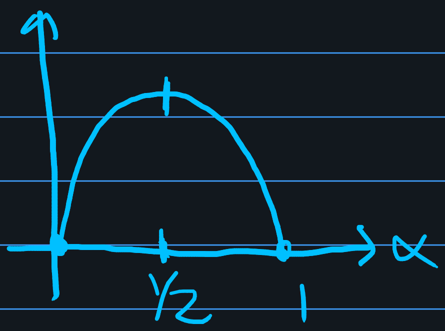
$$\leq E[cX] - E[X]^2$$

$$= c\,\mathbb{E}[X] - \mathbb{E}[X]^2 = \mathbb{E}[X](c - \mathbb{E}[X])$$

$$= c^2 \frac{\mathbb{E}[X]}{c}\left(1 - \frac{\mathbb{E}[X]}{c}\right)$$

$$\Rightarrow \mathrm{Var}[X] \leq c^2 \alpha(1-\alpha), \text{ where } \alpha = \frac{\mathbb{E}[X]}{c}$$

maximize this



max'd when $\alpha = 1/2$

$$\leq c^2 \frac{1}{2}\left(1 - \frac{1}{2}\right) = \frac{c^2}{4}$$

$\boxed{E}$

This bound is sharp: consider

$$X = \begin{cases} 0 & \text{w/ prob } 1/2 \\ c & \text{w/ prob } 1/2 \end{cases}$$

$$\Rightarrow \mathrm{Var}[X] = c^2/4$$

More on HW 2 problem 1.

$$\Pr[X=h, Y=t \mid N=t+h] \Pr[N=t+h]$$

$$= \binom{t+h}{h} p^h (1-p)^t e^{-\lambda} \frac{\lambda^{(t+h)}}{(t+h)!}$$

$$\ldots = \Pr[X=h] \Pr[Y=t]$$

HW Problem 4

(b) Suppose we are told at least one of the children is a boy. What is the probability that both of them are boys?

$$\Pr[BB \mid \geq 1 B] = \frac{\Pr[B,B]}{\Pr[\geq 1 B]} = \frac{1/4}{3/4}$$

$$= 1/3$$

2. Say you want to write a computer program that needs to simulate a continuous random variable X whose distribution function is F. You don't know how to simulate X directly, but you can simulate uniform random variables just fine. Assuming the distribution function F is strictly increasing, describe a way to simulate X.

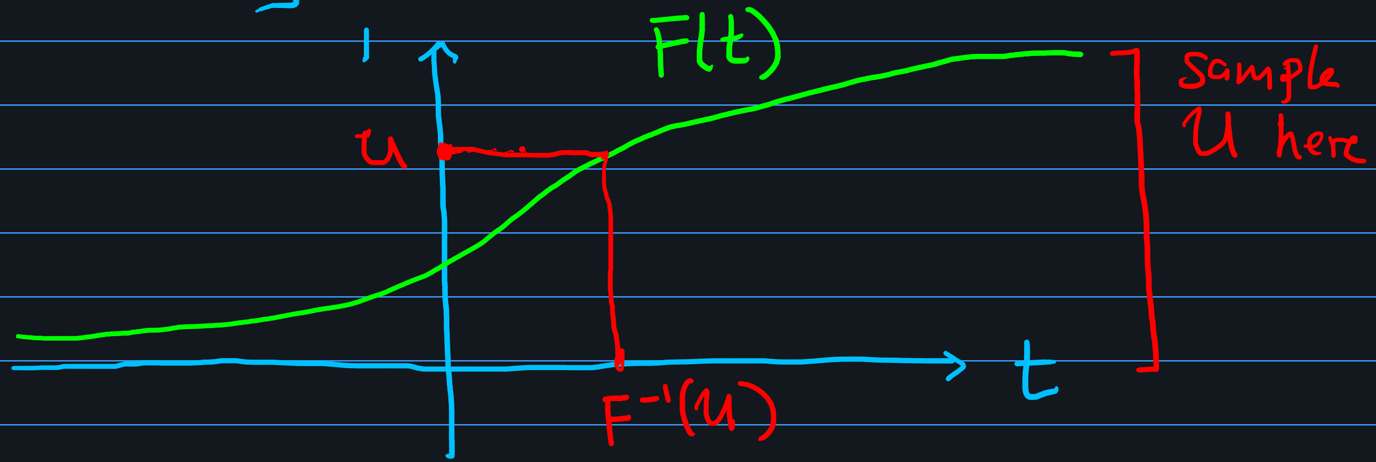
The idea is to sample uniformly on $[0,1]$ & use F^{-1} to turn this into a sample of X.

- Formally, let $U \sim \mathrm{Unif}([0,1])$. Then $\Pr[U \leq t] = t \quad \forall t \in [0,1]$.

- Since $F : \mathbb{R} \to (0,1)$ is strictly increasing, it has an inverse $F^{-1} : (0,1) \to \mathbb{R}$, where $F^{-1}(u)$ is the unique $x \in \mathbb{R}$ s.t. $F(x) = u$.

- $$\Pr[F^{-1}(U) \leq t] \overset{F \text{ invertible}}{=} \Pr[U \leq F(t)] = F(t).$$

So $F^{-1}(U)$ has the same distribution as X, so we can simulate X by simulating $F^{-1}(U)$.



3. If X is an exponential random variable with mean $1/\lambda$, show that

$$E[X^k] = \frac{k!}{\lambda^k}, \quad k = 1, 2, \ldots$$

X has density $f(x) = \lambda e^{-\lambda x}$ $\quad x \geq 0$

$$E[X^k] = \int_0^\infty x^k f(x)\,dx = \int_0^\infty \lambda x^k e^{-\lambda x}\,dx.$$

Use integration by parts + induction.

4. If X is an exponential random variable with parameter $\lambda = 1$¡ compute the probability density function of the random variable Y defined by $Y = \log X$.

supposed to be a comma

X has density $f(x) = e^{-x}$.

$\Rightarrow X$ has distribution $F(t) = \int_0^t f(x)\,dx = 1 - e^{-t}$

Let G be the distribution of Y. Then

$$G(t) = \Pr[Y \leq t] = \Pr[\log X \leq t]$$

$$= \Pr[X \leq e^t] = 1 - \exp(-e^t).$$

derivative of distribution is density, so

Y has density

$$g(x) = G'(x) = -\exp(-e^t)(-e^t)$$

$$= \exp(t - e^t)$$

5. The time in hours required to repair a machine is an exponentially distributed random variable with parameter $\lambda = \frac{1}{2}$. What is

(a) The probability that a repair time exceeds 2 hours?

(b) The conditional probability that a repair takes at least 10 hours, given that that its duration exceeds 9 hours?

Let T be the repair time.

then T has distribution

$$F(t) = \Pr[T \le t] = 1 - e^{-t/2}.$$

a) $\Pr[T > 2] = 1 - \Pr[T \le 2]$

$$= 1 - (1 - e^{-1}) = 1/e$$

b) $\Pr[T > 10 \mid T > 9] = \dfrac{\Pr[T > 10 \mid T > 9]}{\Pr[T > 9]}$

$$= \frac{\Pr[T > 10]}{\Pr[T > 9]} = \frac{1 - \Pr[T \le 10]}{1 - \Pr[T \le 9]} = \frac{e^{-5}}{e^{-9/2}} = e^{-1/2}$$

<u>or</u> use memoryless property of the exponential distribution:

$$\Pr[T \geq 10 \mid T \geq 9] = \Pr[T \geq 9+1 \mid T \geq 9]$$

$$= \Pr[T \geq 1] = 1 - \Pr[T \le 1]$$

$$= e^{-1/2}$$

$\boxtimes$