

Math 180B - Diophantine Approximation

1. Show that if d is divisible by a prime congruent to 3 mod 4 then $x^2 - dy^2 = -1$ has no solutions in integers.
2. Prove that there exist irrational numbers α and β such that α^β is rational.
3. Recall that the Farey sequence of order n , \mathbf{F}_n , is the sequence of reduced fractions $0 \leq \frac{a}{b} \leq 1$ arranged in increasing order. Prove that the number of terms in \mathbf{F}_n is $1 + \sum_{j=1}^n \phi(j)$ and that their sum is exactly half this value. Using this, come up with a recursive formula for $|\mathbf{F}_n|$.
4. Consider the Farey sequence

$$\mathbf{F}_n = \left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_k}{b_k} \right),$$

where $b_1 = 1$ and $b_k = 1$. Prove that

$$\sum_{j=1}^{k-1} \frac{1}{b_j b_{j+1}} = 1.$$