

270A - Homework 4

Problem 1. Let E_1, E_2, \dots be events on the same probability space. Assume that

$$\mathbb{P}[E_n] \rightarrow 0 \quad \text{and} \quad \sum_n \mathbb{P}[E_n \cap E_{n-1}^c] < \infty.$$

Show that

$$\mathbb{P}[E_n \text{ occur infinitely often}] = 0.$$

Proof. By the continuity of measure we have

$$\begin{aligned} \mathbb{P}[\limsup E_n^c] &= \lim_{N \rightarrow \infty} \mathbb{P}\left[\bigcup_{n=N}^{\infty} E_n^c\right] \\ &\geq \lim_{N \rightarrow \infty} \mathbb{P}[E_N^c] \\ &= 1, \end{aligned}$$

where the last equality follows from the hypothesis that $\mathbb{P}[E_n] \rightarrow 0$. Now by Borel-Cantelli, $\mathbb{P}[E_n \cap E_{n-1}^c \text{ i.o.}] = 0$. Since $\mathbb{P}[E_{n-1}^c \text{ i.o.}] = 1$ by the above discussion, we have

$$\mathbb{P}[E_n \text{ i.o.}] = \mathbb{P}[E_n \cap E_{n-1}^c \text{ i.o.}] = 0.$$

□

Problem 2. Let X_1, X_2, \dots be iid random variables with the standard exponential distribution,

$$\mathbb{P}[X_i > x] = e^{-x}, \quad x \geq 0.$$

(a) Show that

$$\limsup_n \frac{X_n}{\log n} = 1 \text{ a.s.}$$

Proof. For any positive t we have

$$\sum_{n=1}^{\infty} \mathbb{P}[X_n > t \log n] = \sum_{n=1}^{\infty} \frac{1}{n^t} = \begin{cases} C_t < \infty, & \text{if } t \leq 1 \\ \infty, & \text{if } t > 1 \end{cases}.$$

By Borel-Cantelli, we then have

$$\mathbb{P}\left[\frac{X_n}{\log n} > t \text{ infinitely often}\right] = \begin{cases} 1, & \text{if } t \leq 1 \\ 0, & \text{if } t > 1 \end{cases}.$$

In particular, we have that $\frac{X_n}{\log n} > 1$ infinitely often almost surely, but $\frac{X_n}{\log n} > t$ only finitely often almost surely for any $t > 1$. We conclude that $\limsup \frac{X_n}{\log n} = 1$ almost surely. □

(b) Let $M_n = \max_{1 \leq k \leq n} X_k$. Show that

$$\limsup_n \frac{M_n}{\log n} = 1 \text{ a.s.}$$

Proof. Fix $t > 0$ and let E_n be the event given by $E_n = \{M_n > t \log n\}$. By L'Hôpital's rule, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{P}[E_n] &= \lim_{n \rightarrow \infty} \mathbb{P}[X_k > t \log n \text{ for at least one } 1 \leq k \leq n] \\ &= \lim_{n \rightarrow \infty} 1 - \mathbb{P}[X_k \leq t \log n \text{ for each } 1 \leq k \leq n] \\ &= \lim_{n \rightarrow \infty} 1 - (1 - \mathbb{P}[X_1 > t \log n])^n \quad (\text{since the } X_k \text{'s are independent}) \\ &= \lim_{n \rightarrow \infty} 1 - \left(1 - \frac{1}{n^t}\right)^n \\ &= 0 \text{ if and only if } t > 1. \end{aligned}$$

Now let's compute $\mathbb{P}[E_n \setminus E_{n-1}]$.

$$\mathbb{P}[E_n \setminus E_{n-1}] = \mathbb{P}[X_k > t \log n \text{ for at least one } 1 \leq k \leq n$$

$$\text{AND } X_k \leq t \log(n-1) \text{ for each } 1 \leq k \leq n-1].$$

The only way $X_k \leq t \log(n-1)$ can hold for each $1 \leq k \leq n-1$ while still having at least one of $1 \leq k \leq n$ satisfy $X_k > t \log n$ is for $X_n > t \log n$ to hold. We then have

$$\sum \mathbb{P}[E_n \setminus E_{n-1}] = \sum \mathbb{P}[X_n > t \log n] = \sum \frac{1}{n^t} < \infty \text{ if and only if } t > 1.$$

By problem 1, we then have $\mathbb{P}[M_n > t \log n \text{ infinitely often}] = 0$ if and only if $t > 1$. By the same reasoning we used in part (a), we have that $\limsup \frac{M_n}{\log n} = 1$ almost surely. \square

Problem 3. Let

$$\psi(x) = \begin{cases} x^2 & \text{if } |x| \leq 1 \\ |x| & \text{if } |x| \geq 1 \end{cases}.$$

Let X_1, X_2, \dots be independent mean zero random variables. Show that if $\sum \mathbb{E}[\psi(X_n)] < \infty$, then $\sum X_n$ converges almost surely.

Proof. By Markov's inequality we have

$$\sum \mathbb{P}[|X_n| > 1] \leq \sum \mathbb{E}[|X_n| \cdot \mathbb{1}_{|X_n| > 1}] < \infty.$$

By Borel-Cantelli, $|X_n| \leq 1$ eventually almost surely. In other words, $X_n = X_n \cdot \mathbb{1}_{|X_n| \leq 1}$ eventually almost surely. In particular, we have that $\sum X_n$ converges a.s. if and only if $\sum X_n \cdot \mathbb{1}_{|X_n| \leq 1}$ converges a.s.

Now let's look at $\sum \mathbb{E}[X_n \cdot \mathbb{1}_{|X_n| \leq 1}]$. Since $\mathbb{E}[X_n] = 0$, we have $\mathbb{E}[X_n \cdot \mathbb{1}_{|X_n| \leq 1}] = -\mathbb{E}[X_n \cdot \mathbb{1}_{|X_n| > 1}]$. This gives

$$\sum |\mathbb{E}[X_n \cdot \mathbb{1}_{|X_n| \leq 1}]| = \sum |\mathbb{E}[X_n \cdot \mathbb{1}_{|X_n| \geq 1}]| \leq \sum \mathbb{E}[|X_n| \cdot \mathbb{1}_{|X_n| \geq 1}] \leq \sum \mathbb{E}[\psi(X_n)] < \infty.$$

The variances of $X_n \cdot \mathbb{1}_{|X_n| \leq 1}$ are also summable:

$$\sum \text{Var}[X_n \cdot \mathbb{1}_{|X_n| \leq 1}] \leq \sum \mathbb{E}[X_n^2 \cdot \mathbb{1}_{|X_n| \leq 1}] \leq \sum \mathbb{E}[\psi(X_n)] < \infty.$$

By Kolmogorov's two-series theorem, we have that $\sum X_n \cdot \mathbb{1}_{|X_n| \leq 1}$ converges almost surely, which shows that $\sum X_n$ converges almost surely by our earlier discussion. \square

Problem 4. Construct a sequence of independent mean zero random variables X_1, X_2, \dots such that

$$\frac{1}{n} \sum_{k=1}^n X_k \rightarrow \infty \text{ a.s.}$$

Why does this example not contradict the strong law of large numbers?

Solution. Consider the probability triplet $(\Omega, \mathcal{F}, \mathbb{P})$ where Ω is the unit interval $[0, 1]$, \mathcal{F} is the Borel σ -algebra, and \mathbb{P} is the Lebesgue measure. Define the sequence of random variables X_k by

$$X_k = \begin{cases} k(1-k), & \text{if } 0 \leq x \leq \frac{1}{k} \\ k, & \text{if } \frac{1}{k} < x \leq 1. \end{cases}$$

Each X_k has zero mean since

$$\mathbb{E}[X_k] = \frac{1}{k} \cdot k(1-k) + \left(1 - \frac{1}{k}\right) \cdot k = 0.$$

Now for any $\omega \in [0, 1]$ we have $\frac{1}{n} < \omega \leq 1$ for all n larger than some N_ω . We then have for n larger than N_ω ,

$$\frac{1}{n} \sum_{k=1}^n X_k(\omega) = \frac{1}{n} \left(\sum_{k=1}^{N_\omega} k(k-1) + \sum_{k=N_\omega+1}^n k \right) = \frac{1}{n} (C_\omega + \Omega(n^2)) \rightarrow \infty,$$

for some constant C_ω . This doesn't contradict the strong law of large numbers, since the strong law requires that the variables X_1, X_2, \dots be iid, whereas the variables X_k defined above are not identically distributed. \square

Problem 5. Suppose disasters occur at random times X_i apart from each other. Precisely, the k -th disaster occurs at time $T_k = X_1 + \dots + X_k$, where the X_i are iid random variables taking positive values with finite mean μ . Let

$$N(t) = \max\{n : T_n \leq t\}$$

be the number of disasters that have occurred by time t . Prove that

$$N(t) \rightarrow \infty \quad \text{and} \quad \frac{N(t)}{t} \rightarrow \frac{1}{\mu}$$

almost surely as $t \rightarrow \infty$.

Proof. First, we claim that $N(t) < n$ if and only if $t < T_n$. This is true since

$$N(t) < n = \max(m : T_m \leq t) < n \iff T_n > t.$$

From this, we can deduce that $t < T_{N(t)+1}$ since $N(t) < N(t) + 1$. Similarly, since $N(t) \leq N(t)$, we have $T_{N(t)} \leq t$. Putting these together gives

$$T_{N(t)} \leq t < T_{N(t)+1}. \quad (1)$$

Now by the strong law of large numbers we have that $\frac{T_n}{n} \rightarrow \mu$ almost surely. Fix $\epsilon > 0$. For n sufficiently large, we have that $|\frac{T_n}{n} - \mu| \leq \epsilon$ a.s. From this we deduce that $T_n \leq n(\mu + \epsilon)$ a.s. Since $T_n \leq t$ if and only if $N(t) \geq n$, we have

$$N(n(\mu + \epsilon)) \geq n$$

for n large. Taking n to infinity and using the fact that $N(t)$ is nondecreasing, we have that $N(t) \rightarrow \infty$ as $t \rightarrow \infty$. Dividing (1) through by $N(t)$ gives

$$\frac{T_{N(t)}}{N(t)} \leq \frac{t}{N(t)} \leq \frac{T_{N(t)+1}}{N(t)} = \frac{T_{N(t)} + X_{N(t)+1}}{N(t)}.$$

By the strong law of large numbers and the fact that $N(t) \rightarrow \infty$, we have that $T_{N(t)}/N(t) \rightarrow \mu$ a.s. as $t \rightarrow \infty$. Since the X_k 's are identically distributed with finite mean, we have that $X_{N(t)+1}/N(t) \rightarrow 0$ a.s. Both sides of the above inequality then tend to μ a.s., so $\frac{N(t)}{t} \rightarrow \frac{1}{\mu}$ a.s. \square

Problem 6. Let X_1, X_2, \dots be independent random variables. Show that $\sum X_n$ converges in probability if and only if $\sum X_n$ converges almost surely.

Proof. Almost sure convergence always implies convergence in probability, so it just remains to show the converse. To this end, let $S_n = \sum_{k=1}^n X_k$ be the n -th partial sum. Let's show that S_n is Cauchy a.s. Fix n and some N and apply Etemadi's inequality to the variables $X_{n+1}, X_{n+1}, \dots, X_N$:

$$\mathbb{P} \left[\max_{n+1 \leq m \leq N} |X_{n+1} + \dots + X_m| > 3\epsilon \right] \leq 3 \cdot \max_{n+1 \leq m \leq N} \mathbb{P}[|X_{n+1} + \dots + X_m| > \epsilon].$$

Letting $N \rightarrow \infty$, we have

$$\mathbb{P} \left[\sup_{m > n} |X_{n+1} + \dots + X_m| > 3\epsilon \right] \leq 3 \cdot \sup_{m > n} \mathbb{P}[|X_{n+1} + \dots + X_m| > \epsilon].$$

Now since $\sum X_n$ converges in probability, its partial sums are Cauchy in probability. Consequently, as we take n to infinity, the right-hand side of the above inequality tends to zero as $n \rightarrow \infty$. We have then shown that $\sup_{m > n} |S_m - S_n| \rightarrow 0$ in probability. Since this quantity is decreasing in n , it must converge a.s. as well. Since the partial sums are a.s. Cauchy, we have that $\sum X_n$ converges a.s. \square

Problem 7. Let X_1, X_2, \dots be iid random variables taking non-negative values, such that $\mathbb{P}[X_i > 0] > 0$. Prove that

$$\sum X_n = \infty \text{ a.s.}$$

Proof. Since $\mathbb{P}[X_i > 0] > 0$, we can find δ and ϵ both positive such that $\mathbb{P}[X_i > \delta] > \epsilon$. We then have

$$\sum \mathbb{P}[X_i > \delta] > \sum \epsilon = +\infty.$$

By Borel-Cantelli, we then have that $\mathbb{P}[X_i > \delta \text{ infinitely often}] = 1$. Let $A = \{X_i > \delta \text{ infinitely often}\}$. For any $\omega \in A$, we have that $\sum X_n(\omega)$ is a sum that contains infinitely many terms of size at least δ . Since each X_n takes only nonnegative values, this sum must diverge at ω . Since $\mathbb{P}[A] = 1$, we have that $\sum X_n = \infty$ almost surely. \square

Problem 8. Call a number $x \in [0, 1]$ badly approximable by rationals if there exists $c(x) > 0$ and $\epsilon(x) > 0$ such that for any $p, q \in \mathbb{N}$ we have

$$\left| x - \frac{p}{q} \right| > \frac{c}{q^{2+\epsilon}}.$$

Prove that almost all numbers in $[0, 1]$ are badly approximable.

Proof. Fix any positive ϵ and c . Let E_q be the set of rationals that are *not* badly approximable by a rational with denominator q :

$$E_q = \left\{ x \in [0, 1] : \left| x - \frac{p}{q} \right| \leq \frac{c}{q^{2+\epsilon}} \text{ for some } 0 \leq p \leq q \right\} = \bigcup_{p=0}^q \left\{ x \in [0, 1] : \left| x - \frac{p}{q} \right| \leq \frac{c}{q^{2+\epsilon}} \right\}.$$

The set of not badly approximable numbers is the union of the E_q 's. Let's compute the measure of this set, $E_{c,\epsilon} = \bigcup_q E_q$

$$\begin{aligned} m(E_{c,\epsilon}) &= \sum_q \sum_{p=0}^q m \left\{ x \in [0, 1] : \left| x - \frac{p}{q} \right| \leq \frac{c}{q^{2+\epsilon}} \right\} \\ &\leq \sum_q \sum_{p=0}^q \frac{2c}{q^{2+\epsilon}} \\ &= 2c \sum_q \frac{q+1}{q^{2+\epsilon}} \\ &< \infty. \end{aligned}$$

By Borel-Cantelli, we have that for any fixed $\epsilon, c > 0$, almost every $x \in [0, 1]$ satisfies $|x - p/q| \leq C/q^{2+\epsilon}$ for only finitely many p and q . \square

Problem 9. Let X_1, X_2, \dots be iid random variables with finite mean μ . Prove that

$$\frac{1}{\ln n} \sum_{k=1}^n \frac{X_k}{k} \rightarrow \mu \text{ a.s.}$$

Proof. When we proved the strong law of large numbers in class, we crucially used the fact that if a sequence of numbers x_n converges to x , then it converges in mean. That is, $\frac{1}{n} \sum_{k=1}^n x_k$ converges to x

as well. By splitting $X = X^+ - X^-$, we can assume without loss of generality that $X_k \geq 0$. Let's start by truncating the X_k 's and define $Y_k = X_k \cdot \mathbb{1}_{\{|X_k| \leq k\}}$ and $T_n = \sum_{k=1}^n Y_k/k$. We claim that it suffices to prove that $T_n/\ln n \rightarrow \mu$ a.s. The idea is that $X_k = Y_k$ eventually almost surely. To see this, consider the sum

$$\sum \mathbb{P}[|X_k| > k] \leq \int_0^\infty \mathbb{P}[|X_1| > t] dt = \mathbb{E}[|X_1|] < \infty.$$

By Borel-Cantelli, we have that $|X_k| \leq k$ eventually almost surely. If $|X_k| \leq k$, then by definition we have $X_k = Y_k$. Consequently, the difference $\sum_{k=1}^n \frac{1}{k} |X_k(\omega) - Y_k(\omega)| < C_\omega < \infty$ for all n .

Let $\epsilon > 0$ be arbitrary and let $k(n) = \lceil 2^{(1+\epsilon)n} \rceil$. Let $\delta > 0$. By Chebyshev's inequality we have

$$\begin{aligned} \sum_{n=1}^{\infty} \mathbb{P}[|T_{k(n)} - \mathbb{E}[T_{k(n)}]| > \delta \ln k(n)] &\leq \frac{1}{\delta^2} \sum_n \frac{\text{Var}[T_{k(n)}]}{(1+\epsilon)^{2n}} \\ &= \frac{1}{\delta^2} \sum_n (1+\epsilon)^{-2n} \sum_{m=1}^{k(n)} \text{Var}\left[\frac{Y_m}{m}\right] \\ &= \frac{1}{\delta^2} \sum_{m=1}^{\infty} \frac{1}{m^2} \text{Var}[Y_m] \sum_{n: k(n) > m} (1+\epsilon)^{-2n} \\ &\leq \frac{C_\epsilon}{\delta^2} \sum_m \frac{1}{m^2} \text{Var}[Y_m]. \end{aligned}$$

Now we showed in class that $\sum \text{Var}[Y_m]/m^2 < \infty$, so the above sum is finite. Since δ was arbitrary, we have that

$$\frac{T_{k(n)} - \mathbb{E}[T_{k(n)}]}{\ln k(n)} \rightarrow 0 \text{ a.s.} \quad (2)$$

By dominated convergence we have that $\mathbb{E}[Y_k] \rightarrow \mathbb{E}[X_1] = \mu$. By the same argument one uses to show $x_n \rightarrow x \implies \frac{1}{n} \sum_{k=1}^n x_k \rightarrow x$, we have that $x_n \rightarrow x \implies \frac{1}{\ln n} \sum_{k=1}^n \frac{x_k}{k} \rightarrow x$. Consequently, we have

$$\frac{\mathbb{E}[T_{k(n)}]}{\ln k(n)} \rightarrow \mu \text{ a.s.}$$

By (2), we have that $\frac{T_{k(n)}}{\ln k(n)} \rightarrow \mu$ a.s. Now we interpolate: suppose $k(n) \leq m < k(n+1)$. Since we're assuming that $Y_k \geq 0$ for all k , we have

$$\frac{T_{k(n)}}{\ln k(n+1)} \leq \frac{T_m}{\ln m} \leq \frac{T_{k(n+1)}}{\ln k(n)}$$

□