Linearity of Expectation

- For RVs X & Y, $c \in \mathbb{R}$

$$\mathbb{E}[cX + Y] = c\mathbb{E}[X] + \mathbb{E}[Y]$$

Thm: Let $X : \Omega \to \mathbb{R}$ be a RV. If $\mathbb{E}[X] = \mu < \infty$. Then there is some $\omega \in \Omega$ s.t. $X(\omega) \geq \mu$. There's also some $\omega' \in \Omega$ s.t. $X(\omega') \leq \mu$.

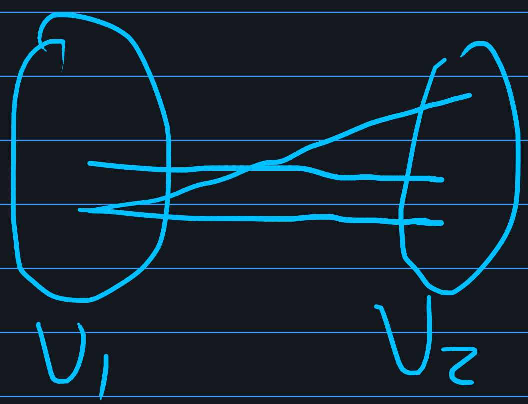
Pf: Suppose $\mathbb{E}[X] = \mu$ but $X(\omega) < \mu$ $\forall \omega \in \Omega$. Derive contradiction by showing $\mathbb{E}[X] < \mu$. $\boxtimes$

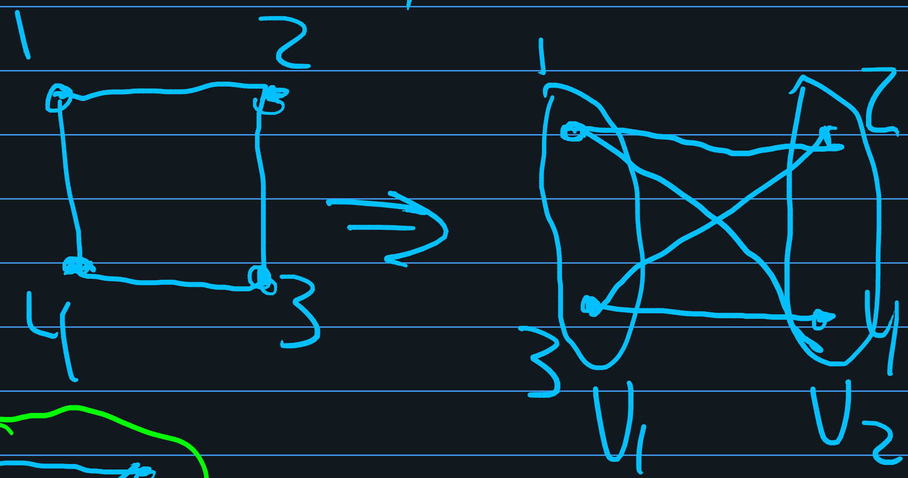
Really powerful. Lets you prove existence results with probabilistic techniques.

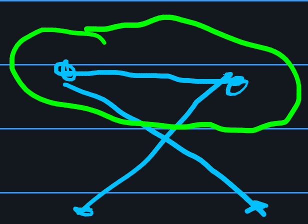
Ex: Let $G = (V, E)$ be a graph w/ n vertices & e edges.

Then G has a bipartite subgraph with $\geq e/2$ edges.

Def: A graph is bipartite if its vertex set can be partitioned $V = V_1 \sqcup V_2$ s.t. all edges have one vert. in V_1 & another in V_2







Pf: Let T be a random subset of V where each vertex has prob. $1/2$ of appearing in T independently.

- Let $U = V \setminus T$. $\Rightarrow V = T \sqcup U$
- Let X be the number of crossings; edges between T & U
- Key step: write X as a sum of indicator RVs.

$$X = \sum_{xy \in E} X_{xy}, \text{ where } X_{xy} = \begin{cases} 1 & \text{if } xy \text{ crosses} \\ 0 & \text{else.} \end{cases}$$

xy is an edge in E

$x, y \in V$

indicator

expected # of crossings:

$$\mathbb{E}[X] = \mathbb{E}\Big[\sum_{xy \in E} X_{xy}\Big] = \sum_{xy \in E} \mathbb{E}[X_{xy}].$$

$$\mathbb{E}[X_{xy}] = 1 \cdot \mathbb{P}[X_{xy} = 1] + 0 \cdot \mathbb{P}[X_{xy} = 0]$$
$$= 1/2.$$

$$\mathbb{P}[X_{xy}=1] = 1/2$$

$= \mathbb{P}[$one of x,y is in T, other in $U]$

$= \mathbb{P}[$two coin tosses come up different$]$

$= 1/2$

$$\Rightarrow \mathbb{E}[X] = \sum_{xy\in E} \mathbb{E}[X_{xy}] = \sum_{xy \in E} 1/2 = \frac{1}{2}\sum_{xy\in E} 1$$

$$= e/2$$

by Thm, there is some partition

$V = T \sqcup U$ w/ $\geq e/2$ crossings.

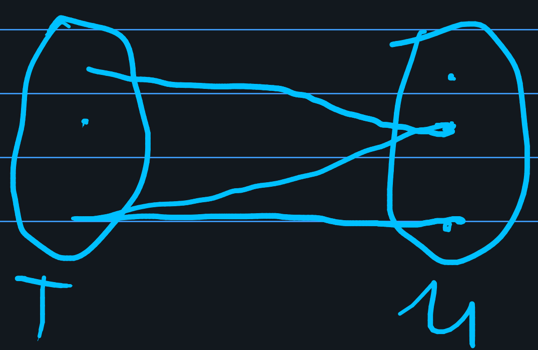
delete all edges inside T & U

$\Rightarrow (T \sqcup U, \text{crossings})$ is bipartite

w/ $\geq e/2$ edges.

$A \sqcup B$ = union where A & B disjoint

$\square$



5. Choose a random inscribed cube & let X = # blue vertices

write X as sum of indicators

use linearity of expectation to compute $\mathbb{E}[X]$.

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4. 4. Suppose you are given a polynomial $P(x)$ and its degree is unknown. There is an oracle that can give you the value of the polynomial at any value that you wish. In addition, suppose that all the coefficients of the polynomial are non-negative integers. Show that you can recover the polynomial with only two queries to the oracle.

You give the oracle a number t & it gives you $P(t)$.

Prove that you can determine $P(x)$ from $P(t_1)$ & $P(t_2)$ for some t_1 & t_2.

Binary expansion: any integer n can be written uniquely

$n = a_0 + a_1 \cdot 2^1 + a_2 \cdot 2^2 + \cdots + a_k 2^k$ for some $a_i \in \{0, 1\}$

e.g. $6 = 0 \cdot 2^0 + 1 \cdot 2^1 + 1 \cdot 2^2$

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can do the same for any base $b \geq 2$

$n = a_0 + a_1 \cdot b^1 + a_2 \cdot b^2 + \cdots + a_k \cdot b^k$

$0 \leq a_i < b \quad \forall i$

How can you get an upper bound on the size of the coefficients of P?

Ask for $P(1)$.

if $P(x) = a_0 + a_1 x + \cdots + a_n x^n$

$P(1) = a_0 + a_1 + a_2 + \cdots + a_n$

$\geq \max_i a_i$.

$$\mathbb{P}[X_{xy}=1] = \mathbb{P}\left[\text{edge } xy \text{ has one end in } T \text{ \& other in } \mathcal{U}\right]$$

$$= \mathbb{P}[\text{one of } x, y \text{ is in } \mathcal{U}, \text{ other is in } T]$$

$$= \mathbb{P}[x \in T, y \in \mathcal{U}] + \mathbb{P}[x \in \mathcal{U}, y \in T]$$

$$= \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2}$$

$$= 1/2 .$$