nvs X complete iff its

closed unit ball is complete.

Assume closed unit ball complete.

Consider a seq (x_n) in X.

s.t. $a := \sum \|x_n\| < \infty$

(Idea: X complete iff $\sum \|x_n\| < \infty$
$\Rightarrow \sum x_n \in X$)

Consider $\underline{\left(x_n/a\right)}$

$\forall n$ $\frac{x_n}{a} \in B(0,1)$ (since $a \geq \|x_n\|$ $\forall n$)

$$\sum \left\|\frac{x_n}{a}\right\| = \frac{1}{a} \sum \|x_n\| = 1 < \infty$$

$$\Rightarrow \sum \frac{x_n}{a} \in \overline{B(0,1)}$$

$$\Rightarrow \sum x_n \in X$$

7.30 Wts (M,d) complete then any open $G \subseteq M$ is homeo to a complete space.

note: opens themselves might not be complete e.g. $(0,1) \subseteq \mathbb{R}$.

30. If (M, d) is complete, prove that every open subset G of M is homeomorphic to a complete metric space. [Hint: Let $F = M \setminus G$ and consider the metric $\rho(x, y) = d(x, y) + \left|(d(x, F))^{-1} - (d(y, F))^{-1}\right|$ on G.]

put this metric ρ on G & show its equivalent to d. This gives $(G,d) \cong (G,\rho)$

this □ complete

idea: G might not be complete if it has sequences that converge outside of G

$d(x,y) = |x-y|$

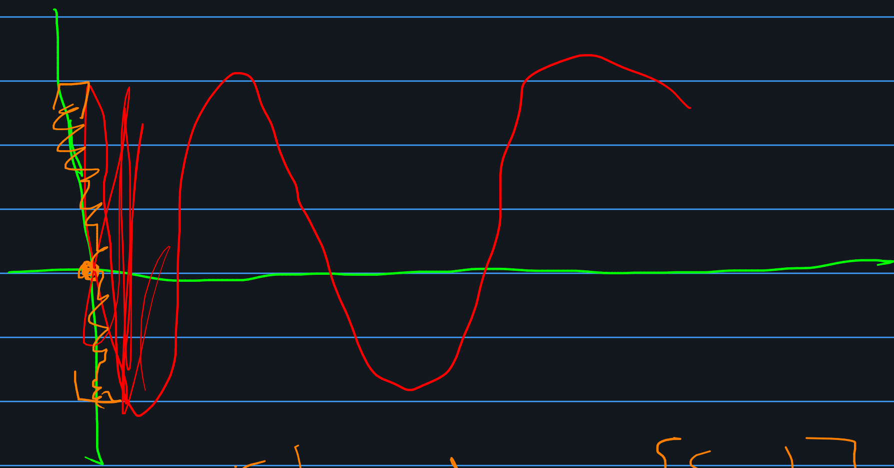
$\tilde{d}(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|$

Problem 1.

Prove that an open set $\mathbb{U} \subset \mathbb{R}^n$ is connected if and only if it is path connected. Can one replace "open set" by "closed set" in this statement?

Nah

Topologist's sine curve



closed on $[\delta, 1]$ $\forall \delta > 0$

graph of $\sin(\frac{1}{x})$ on $(0,1]$

$\cup \left(\{0\} \times [-1,1]\right)$

Give an ex of a closed & bdd subset of l_∞ that is not totally bdd.

$S = \{e_n : n = 1, 2, \ldots\}$

where $e_n = (0, \ldots, 1, 0, \ldots)$

S is bdd since $\|e_n\|_\infty = 1$
$\forall n$

but no finite collection of
balls w/ radius $1/2$ covers
S, since $\|e_m - e_n\|_\infty = 1$ $\forall m \neq n$

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2. put a metric on $\mathbb{R}$
so that it's not complete
anymore.

$$d(x,y) = |\arctan(x) - \arctan(y)|$$

eg $x_n = n$ is cauchy wrt
d but it doesn't converge in
$(\mathbb{R}, d)$

3. T/F of $(X,d) \cong (Y,\rho)$

& X complete then so

is Y.

N-th iterate

4. Let M be complete & say

cts

$f: M \to M$ is s.t. f^N is

a contraction for some N.

Then f has a unique fixed pt.

You can find it by iterating

f on any $x_0 \in M$.

The fixed pt will be

$f^{N-1}(x_0)$ for any x_0

Banach $\Rightarrow f^N(a) = a$ some a.

$f(a) = f(f^N(a)) = f^N(f(a))$

$\Rightarrow f(a) = a$

unless otherwise stated, the metric on $C[a,b]$ is the sup metric.

$T: C[0,1] \to C[0,1]$

$$(Tf)(x) = \int_0^x f(t)\,dt$$

not a contraction, but $T \circ T$ is

(find the fixed point)

T isn't a contraction since ...

consider $f(x) = 1$. $\|f\|_\infty = 1$

$Tf = x$ $\quad \|x\|_\infty = 1$

$$\|T^2 f\|_\infty = \left\| T \int_0^t f(s)\,ds \right\|_\infty$$

$$= \sup_{0 \le x \le 1} \left| \int_0^x \int_0^t f(s)\,ds\,dt \right|$$

$$\leq \sup_{0\leq x\leq 1} \int_0^x \int_0^t |f(s)|\,ds\,dt$$

$$\leq \sup_{0\leq x\leq 1} \int_0^x \|f\|_\infty\, t\, dt$$

$$\leq \|f\|_\infty \sup_{0\leq x\leq 1} \int_0^x t\,dt$$

$$\leq \frac{1}{2} \|f\|_\infty$$