# More on approximating $Lip_K$ functions with $Lip_K$ polynomials.

Sketch: Let $f \in Lip_K[a,b]$

1. We know we can approx $f$ by a polygonal function. Argue that we can make sure it's in $Lip_K$ too.
   $\Rightarrow$ wlog, $f \in Lip_K$ & polygonal
2. Approximate $f \in Lip_K$, polygonal w/ a smooth function that's in $Lip_K$
   $\Rightarrow$ wlog $f \in Lip_K$ is smooth
3. Write $f(x) = f(a) + \int_a^x f'(t)\,dt$.
   go from there.
   Hint: Weierstrass.

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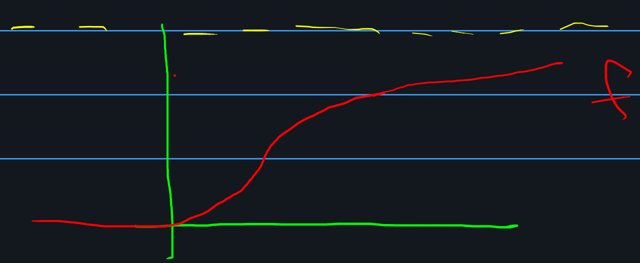
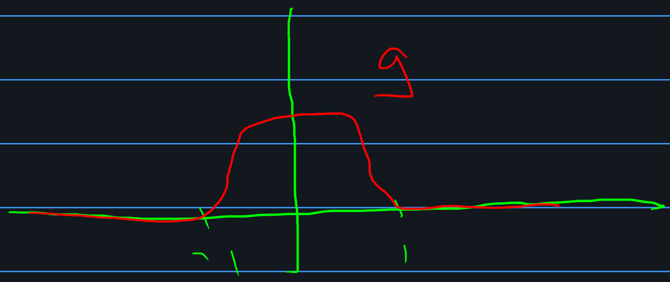
## More on step 2.

in lecture you went through this progression to get smooth approximation:

a) $f(x) = \begin{cases} 0 & x \le 0 \\ e^{-1/x} & x > 0 \end{cases}$ smoothly vanishes as $x \downarrow 0$

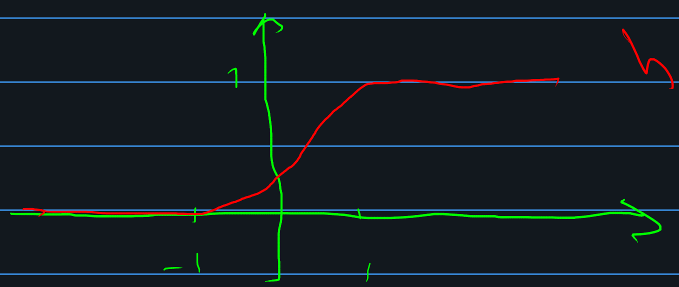


b) $g(x) = f(1+x)f(1-x)$ smooth, goes to 0 as $|x| \to 1$ & "bumps" in the middle.

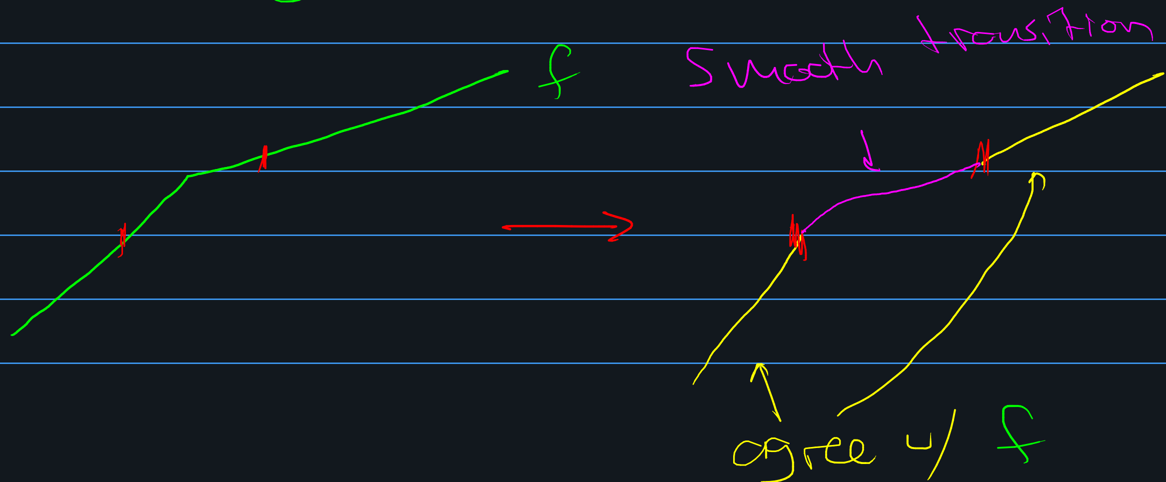



c) $h(x) = \dfrac{1}{\int_{\mathbb{R}} g(t)\,dt} \cdot \displaystyle\int_{-1}^{x} g(t)\,dt$

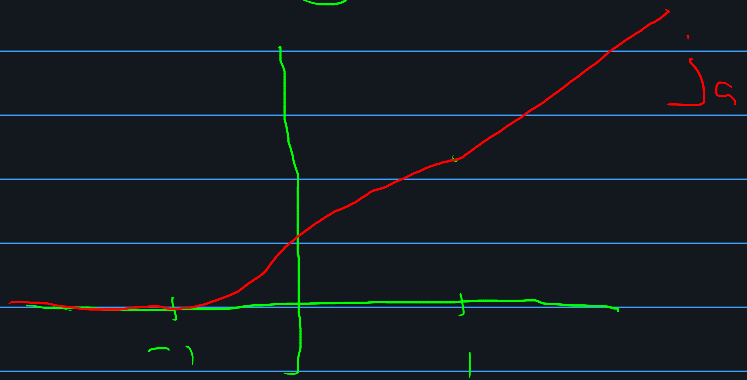
smoothly & monotonically goes from zero on $x \leq -1$ to 1 on $x \geq 1$




Idea: to accomplish step 2, "smooth out" polygonal function, f



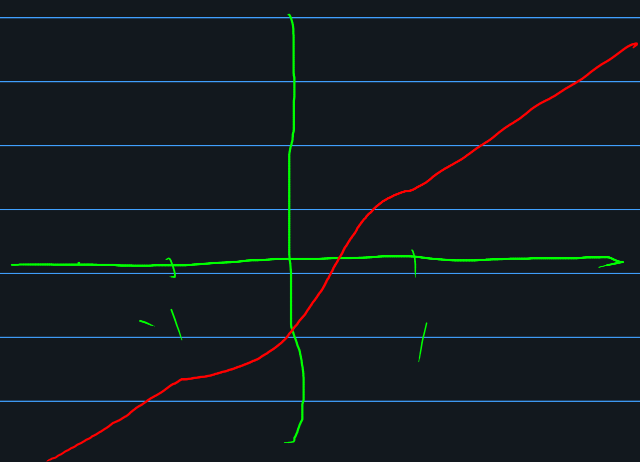
How? Do something with the $h(x)$ from step (c) to build something like this:




$j_a(x)$ should: - smoothly go to zero as $x \to -1$

- smoothly go to some linear function w/ slope $a$ as $x \to 1$

Then use this to build something like




- smoothly goes to linear fn w/ slope $b$ as $x \to -1$

- smoothly & monotonically goes to linear w/ slope $a$ as $x \to 1$.

## Problem 1.

TRUE or FALSE: If $\{f_n\}$, $f_n : [0,1] \to \mathbb{R}$, is a sequence of functions such that for some $M > 0$, $L > 0$ for all $x, y \in [0,1]$ and $n \in \mathbb{N}$ we have

$$|f_n(x) - f_n(y)| \leq L|x-y|, \quad |f(x)| \leq M,$$

then there exists a subsequence that converges in the norm $\|\cdot\|_{BV}$.

- by Arzela-Ascoli, there's a subsequence that converges uniformly (wlog $f_n \rightrightarrows f$)
- <u>if</u> $f_n$ had a subseq that converged in BV, the limit would have to be $f$ as well since $\|\varphi\|_\infty \leq \|\varphi\|_{BV}$
- So to prove the claim false, suffices to find a sequence that converges uniformly, but not in BV.

**6.** We can test several of the inclusions implicit in our discussion up to this point by means of a single family of functions. For $\alpha \in \mathbb{R}$ and $\beta > 0$, set $f(x) = x^\alpha \sin(x^{-\beta})$, for $0 < x \leq 1$, and $f(0) = 0$. Show that:

**(a)** $f$ is bounded if and only if $\alpha \geq 0$.

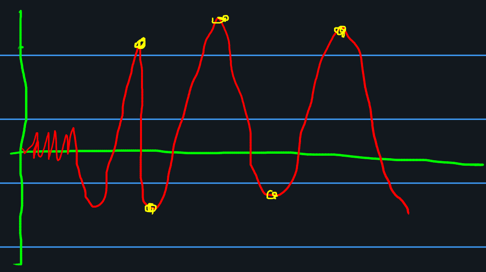
**(b)** $f$ is continuous if and only if $\alpha > 0$.

**(c)** $f'(0)$ exists if and only if $\alpha > 1$.

**(d)** $f'$ is bounded if and only if $\alpha \geq 1 + \beta$.

**(e)** If $\alpha > 0$, then $f \in BV[0,1]$ for $0 < \beta < \alpha$ and $f \notin BV[0,1]$ for $\beta \geq \alpha$.
[Hint: Try a few easy cases first, say $\alpha = \beta = 2$.]

- $\beta \geq \alpha$ : $f$ looks like



pick out the peaks & valleys. $\Rightarrow V_0^1 f$ is at least the sum of the heights from peak to valley.

peaks: when $x^{-\beta} = \frac{\pi}{2} + 2k\pi$

$$\Leftrightarrow x = \left(\frac{\pi}{2} + 2k\pi\right)^{-1/\beta}$$

valleys: when $x = \left(\frac{3\pi}{2} + 2k\pi\right)^{-1/\beta}$

then $V_0^1 f \geq \sum |f(\text{peak}) - f(\text{valley})|$

- show this doesn't converge.

- $\beta < \alpha$ :

Hint: if $|f'| \leq g$, then

$$V_0^1 f \leq \int_0^1 g \quad \text{(why?)}$$