Real Analysis
Math 205B/H140B, HW#3

Chapter 11, exercises 38, 55, 60, 63, and the following problems:

Problem 1.

TRUE or FALSE:

Suppose $\{p_n\} \subset C[0,1]$ is a sequence of polynomials of degree at most 2021, and $p_n \rightrightarrows f$ on $[0,1]$. Then f is also a polynomial of degree at most 2021.

Problem 2.

Let $\mathcal{F} \subset C[0,1]$ be a family of polynomials of degree at most 2021, such that for any $x \in [0,1]$ and any $p \in \mathcal{F}$ we have $|p(x)| \le 2021$. Is $\mathcal{F}$ equicontinuous? Is $\mathcal{F}$ compact?

Problem 3.

TRUE or FALSE:

For any $f \in C[0,1]$ and any $\varepsilon > 0$ there exists a polynomial $p(x)$ such that $p'(0) = 2$, and for any $x \in [0,1]$ we have $|p(x) - f(x)| < \varepsilon$.

Problem 4.

Suppose $\mathcal{F} \subset C[0,1]$ is an equicontinuous family, and is also a linear subspace in $C[0,1]$. Prove that every $f \in \mathcal{F}$ must be constant. Can we replace $C[0,1]$ in this statement by $C(X)$ with an arbitrary compact metric space X?

Problem 5.

Is the family $\{f_n\}_{n\in\mathbb{N}}$, where $f_n(x) = x^n$, equicontinuous? Explain your answer.

Problem 6.

Prove that for any continuous function $f : \mathbb{R} \to \mathbb{R}$ there are C^∞-functions $g_1, g_2 : \mathbb{R} \to \mathbb{R}$ such that for any $x \in \mathbb{R}$

$$g_1(x) < f(x) < g_2(x),$$

and

$$\int_{-\infty}^{+\infty} (g_2(x) - g_1(x))dx < \infty.$$