Problem 24

$$\limsup (-a_n) = -\liminf (a_n)$$

Hint: suffices to prove

$$\sup(-A) = -\inf A,$$

where $A \subseteq \mathbb{R}$

& $-A = \{-a : a \in A\}$

Worksheet problem 1

a_n has a monotone

subseq a_{n_k} s.t.

$$a_{n_k} \xrightarrow{k\to\infty} \limsup_{n\to\infty} a_n =: a$$

Proof:

Case 1: (a_n) is unbdd above

Let $n_1 = \min\{n : a_n \geq 1\}$

$n_2 = \min\{n > n_1 : a_n \geq 2\}$

$\vdots$

$\Rightarrow a_{n_1} < a_{n_2} < \cdots$

& $a_{n_k} \to \infty$ as $k \to \infty$

Case 2 : (a_n) is bdd above.

$$\limsup a_n = \lim_{n\to\infty} \underbrace{\sup_{k\geq n} a_k}_{\text{decreasing in } n}$$

$$b_n = \sup_{k\geq n} a_k \quad \& \quad b_n \to \limsup a_n$$

~~problem~~: (b_n) might not be a subsequence of (a_n)

ex: $a_n = 1 - \frac{1}{n}$ $\quad n \geq 1$

$$\limsup_{n\to\infty} a_n = 1 \quad \left(\text{since } \lim_{n\to\infty} a_n = 1\right)$$

$$b_n = \sup_{k\geq n}\left(1 - \frac{1}{k}\right) = 1$$

but $\nexists n$ s.t. $a_n = 1$

$\Rightarrow (b_n)$ isn't a subseq. of (a_n)

Idea : first build a subsequence a_{n_k} s.t. $a_{n_k} \xrightarrow{k\to\infty} a$. Then extract a monotone subsequence $a_{n_{k_\ell}}$.

· $\exists\, n_1 < n_2 < \cdots$ s.t. $|a_{n_k} - a| \leq \frac{1}{k}$

Idea $a = \lim_{n\to\infty} \sup_{\ell \geq n} a_\ell$

$\Rightarrow \exists N_k$ s.t. $n \geq N_k$ implies

$$\left| \sup_{\ell \geq n} a_\ell - a \right| \leq \frac{1}{2k}$$

· there are infinitely many ℓ_{N_k} s.t.

$$\left(\sup_{\ell \geq N_k} a_\ell\right) - \frac{1}{2k} \leq a_{\ell_{N_k}} \leq \sup_{\ell \geq N_k} a_\ell$$

· Choose $\ell_{N_k} > \ell_{N_{k-1}}$ (& ℓ_{N_1} arbitrarily)

· then

$$|a_{\ell_{N_k}} - a| \leq \left|a_{\ell_{N_k}} - \sup_{\ell \geq N_k} a_\ell\right| + \left|\sup_{\ell \geq N_k} a_\ell - a\right|$$

$$\leq \frac{1}{2k} + \frac{1}{2k} = \frac{1}{k}$$

Set $n_k = \ell_{N_k}$

· Now extract a monotone subseq from (a_{n_k}) (in general, any sequence has a monotone subsequence)

$\triangle$. Let $S = \{k : a_{n_\ell} > a_{n_k} \ \forall \ell > n\}$ be the set of "valleys" of (a_{n_k})

· if S is infinite, then

$a_{n_{k_1}} < a_{n_{k_2}} < \cdots$ is monotone.

· if S is finite, then $\exists k_1 > k$ for all $k \in S$.

since $k_1 \notin S$, $\exists k_2 > k_1$ s.t. $a_{n_{k_2}} \leq a_{n_{k_1}}$

· since $k_2 \notin S$, $\exists k_3 > k_2$ s.t. ...

& $a_{n_{k_1}} \geq a_{n_{k_2}} \geq \cdots$

Since $\lim_{k \to \infty} a_{n_k} = a$ & $(a_{n_{k_\ell}})$ is a subsequence of (a_{n_k}), $\lim_{\ell \to \infty} a_{n_{k_\ell}} = a$

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HW Problem 4

$$\limsup_{n\to\infty} (a_n+b_n)^{1/n} \overset{?}{=} a+b$$

Try example:

$$a_n = 2^n \quad , \quad b_n = 3^n$$

$$\downarrow$$

$$a = \limsup a_n^{\frac{1}{n}} = 2$$

$$b = \limsup b_n^{1/n} = 3$$

$$(a_n+b_n)^{1/n} = (2^n+3^n)^{\frac{1}{n}}$$

$$= 3\left(1+\left(\frac{2}{3}\right)^n\right)^{\frac{1}{n}}$$

$$\to 3 \neq 2+3. \quad \text{try } \max(a,b)$$

Hint:

$$\max(a_n,b_n) \le a_n+b_n \le 2\max(a_n,b_n)$$

$$\max(a_n,b_n)^{1/n} \le (a_n+b_n)^{1/n} \le 2^{\frac{1}{n}} \max \ldots$$