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# Math 274 - Homework 2

**Problem 1.** Let $v_1 = (x_1, y_1), \ldots, v_n = (x_n, y_n)$ be $n$ vectors in $\mathbb{Z}^2$, where each $x_i$ and each $y_i$ is a positive integer that does not exceed $\frac{2^{n/2}}{10\sqrt{n}}$. Show that there exist two disjoint nonempty subsets $I, J \subseteq [n]$ such that $\sum_{i \in I} v_i = \sum_{j \in J} v_j$.

*Proof.* Consider the random sum $V = (X, Y) = \sum_{i=1}^{n} \epsilon_i v_i$ where each $\epsilon_i$ is an iid Bernoulli random variable with success probability $1/2$. We'll show that a sizeable proportion of the possible $V$ live in an axis-aligned box centered about the mean of $V$. If the size of this box is smaller than the number of assignments of the $\epsilon_i$'s that make $V$ land in this box, then there must be two assignments of the $\epsilon_i$'s that give the same realization of $V$.

Since $X$ is a sum of independent scaled Bernoulli random variables, its variance is easily computed:

$$\mathrm{Var}[X] = \sum_{i=1}^{n} x_i^2 \cdot \mathrm{Var}[\epsilon_i] \leq \frac{2^n}{400}.$$

Note that the $y$-coordinate has the same variance. Now by Chebyshev we have

$$\Pr\left[\left|X - \frac{2^n}{400}\right| \leq t\right] \geq 1 - \frac{\mathrm{Var}[X]}{t^2}.$$

□