Problem 2.

Let h be a continuous function on $[0,1]$. Consider $C[0,1]$ - the space of continuous functions on $[0,1]$, equipped with the norm $\|f\| = \max_{x\in[0,1]} |f(x)|$. Let $T : C[0,1] \to \mathbb{R}$ be a linear functional given by

$$T(f) = \int_0^1 h(t)f(t)dt$$

Prove that T is a bounded functional and find $\|T\|$.

a) T is bdd

ie, show $|Tf| \leq C\|f\|_\infty$

$$|Tf| = \left|\int_0^1 h(t)f(t)\,dt\right|$$

$$\leq \left|\|f\|_\infty \int_0^1 h(t)\,dt\right|$$

$$= \|f\|_\infty \left|\int_0^1 h(t)\,dt\right|$$

$$\Rightarrow \|T\| \leq \left|\int_0^1 h(t)\,dt\right|$$

WTS $\|T\| \geq \left|\int_0^1 h(t)\,dt\right|$

Consider $f(t) \equiv 1 \Rightarrow \|f\|_\infty = 1$

$$|Tf| = \left|\int_0^1 h(t)\,dt\right|$$

$$\Rightarrow \|T\| \geq \left|\int_0^1 h(t)\,dt\right|$$

Since $\|T\| = \sup\limits_{\|f\|_\infty = 1} |Tf|$

Problem 3.

Let $K(s,t)$ be a continuous function on $[0,1] \times [0,1]$. Consider $C[0,1]$ - the space of continuous functions on $[0,1]$, equipped with the norm $\|f\| = \max_{x\in[0,1]} |f(x)|$. Let $T : C[0,1] \to C[0,1]$ be a map given by

$$T(f)(s) = \int_0^1 K(s,t)f(t)dt$$

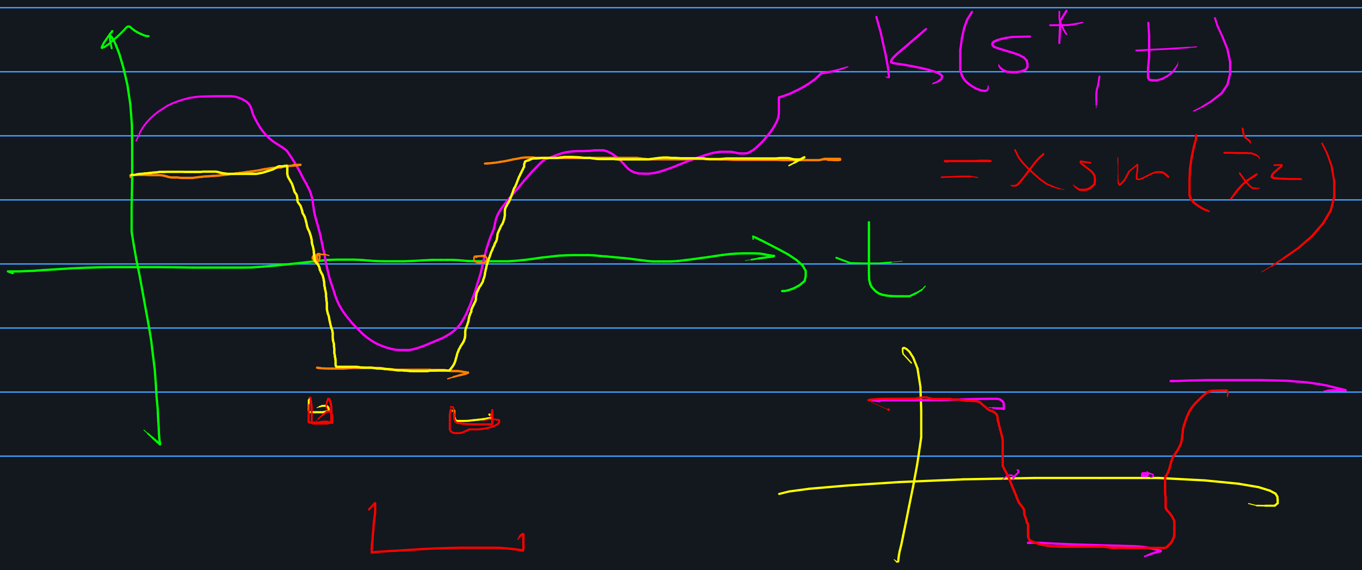
Prove that T is a bounded linear operator and find $\|T\|$.

$$\|Tf\|_\infty = \sup_s \left| \int_0^1 K(s,t) f(t)\,dt \right|$$

$$\le \|f\|_\infty \underbrace{\sup_s \int_0^1 |K(s,t)|\,dt}_{\text{upper bound candidate}}$$

Choose $s^* \in [0,1]$ s.t.

$\int_0^1 |K(s^*,t)|\,dt$ maximized



idea "set" $f(t) = \text{sign}\left(K(s^*, t)\right)$

! not cts! ↓

Let $f_n(x)$ be a seq of cts

1. Show a closed subset of a compact ~~metric~~ space is compact.

Pf: closed subset of complete is complete

subset of totally bdd is totally bdd

$\Rightarrow$ compact.

other proof: superset has decreasing intersection property, so does the subset.

2. Show $\mathbb{R}^2$ is not a countable union of lines.

Pf: · lines are closed + nwd (in $\mathbb{R}^2$)

· by Baire-cat,

$$\mathbb{R}^2 \neq \bigcup_{\text{countable}} (\text{nwd})$$

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a) countable complete metric sp X, has isolated pts

a pt x is isolated if $\{x\}$ is open.

Pf: $X = \bigcup_{x \in X} \{x\}$ Baire-cat

$\Rightarrow$ complete + ≥ 1 pt + no isolated

$\Rightarrow$ uncountable

Let X, Y be Banach spaces & let $\{T_\alpha : \alpha \in A\}$ be a set of bdd linear ops $X \to Y$, satisfying

$$\forall x \in X, \exists K_x < \infty \text{ s.t. } \sup_{\alpha \in A} \|T_\alpha x\| \leq K_x$$

a) Let $A_n = \{x \in X \text{ s.t. } K_x \leq n\}$

Show A_n closed. $\forall n$

Pf: $x \in A_n \iff \sup_{\alpha \in A} \|T_\alpha x\| \leq n$

$$\iff \underbrace{\|T_\alpha x\| \leq n}_{} \quad \forall \alpha$$

$x \mapsto \|T_\alpha x\|$ is cts

$$A_n = \bigcap_{\alpha \in A} \underbrace{\{x : \|T_\alpha x\| \leq n\}}_{\text{closed}}$$

$\triangle$

b) $\exists n_0$ s.t. A_{n_0} has nonempty interior.

Pf: If not, each A_n is nwd

$X = \bigcup_{n \in \mathbb{N}} A_n$ (think)

but X is complete. ↯

c) $\exists L < \infty$ s.t. $\|Tx\| < L$ $\forall x$

Pf: $A_{n_0} \supseteq B_\varepsilon(x_0)$ some $x_0 \in X$, some $\varepsilon > 0$

if $\|x\| \leq 1 \Rightarrow \varepsilon x + x_0 \in A_{n_0}$

$\qquad\qquad\qquad\qquad \cap B_\varepsilon(x_0)$

Let x s.t. $\|x\| \leq 1$

wts $\|Tx\| <$ some L $\forall x$

$$\|T_\alpha x\| = \frac{1}{\varepsilon}\|T_\alpha(\varepsilon x)\|$$

$$= \frac{1}{\varepsilon}\|T_\alpha(\varepsilon x + x_0) - T_\alpha(x_0)\|$$

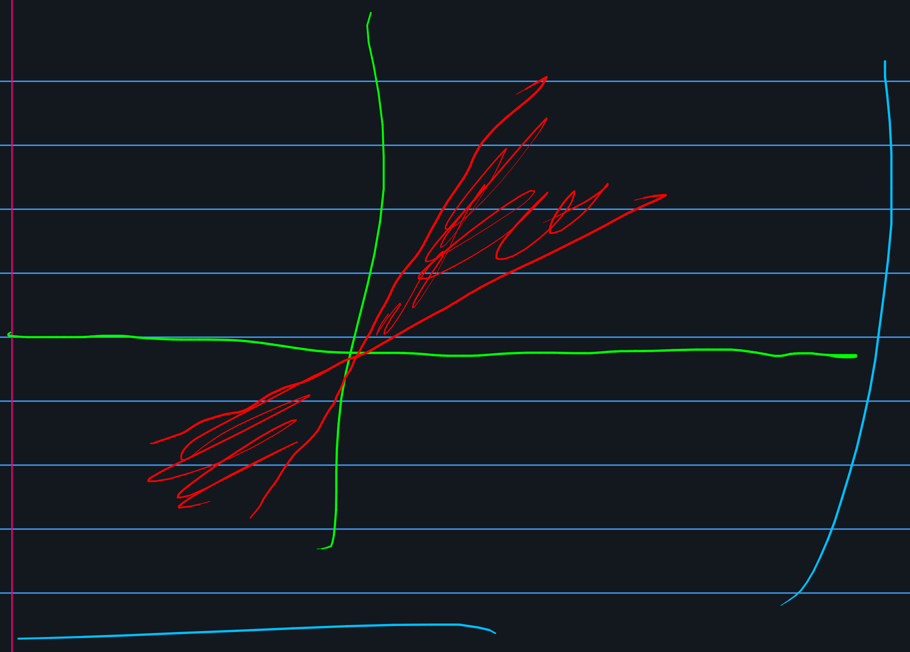
$$\leq \frac{1}{\varepsilon}\left(\|T_\alpha(\varepsilon x + x_0)\| + \|T_\alpha x_0\|\right)$$

$$\leq \frac{1}{\varepsilon}\left(n_0 + n_0\right)$$

$$\leq \frac{2}{\varepsilon} n_0$$

$$\|T_\alpha\| = \sup_{\|x\|\leq 1} \frac{\|T_\alpha x\|}{\|x\|}$$

$$\leq \sup_{\|x\|\leq 1} \frac{K_x}{\|x\|}$$



Let I be an uncountable collection of lines.

& let l be a line not in the countable collection

l has uncountably many pts.

At most one of these points comes from each line in the countable collection