HW 1 Math 274

1. Let $\{(A_i, B_i) \mid 1 \le i \le m\}$ be a family of pairs of subsets of the set of integers such that $|A_i| = k$ for all i, $|B_i| = \ell$ for all i, $A_i \cap B_i = \emptyset$ for all i, and $(A_i \cap B_j) \cup (A_j \cap B_i) \neq \emptyset$ for all $i \neq j$. Show that $m \le (k+\ell)^{k+\ell}/(k^k \ell^\ell)$.

2. Suppose there are m red clubs $R_1, \ldots, R_m$ and m blue clubs $B_1, \ldots, B_m$ in a town of n citizens. Define the $m \times m$ matrix M as follows:

   $$M_{ij} = |A_i \cap B_j|.$$

   Show that if M is non-singular, then $m \le n$.

3. Let p be a prime. Let $A \subseteq \mathbb{Z}_p$ be a set such that $|A| < p^{2/3}$. Prove that there are $x, y \in \mathbb{Z}_p$ such that $A \cap (A+x) \cap (A+y) = \emptyset$.

4. Let $X = \{1,2,3\}^n$. A *proper box* of X is a subset $Y \subseteq X$ of the form $Y_1 \times Y_2 \times \ldots \times Y_n$ where each Y_i is a nonempty subset of $\{1,2,3\}$ of size at most 2. Prove that any partition of X into a family of pairwise disjoint proper boxes contains at least 2^n proper boxes.

5. Let $m(n,s)$ denote the maximum number of points in $\mathbb{R}^n$ such that their pairwise distances take at most s values. Prove:

   $$\binom{n+1}{s} \le m(n,s) \le \binom{n+s+1}{S}.$$