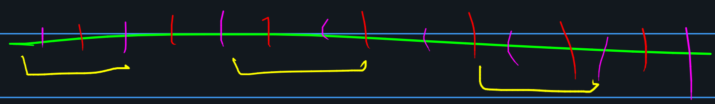
(4) Suppose that $f : [a,b] \to \mathbb{R}$ is a function. We say that f is:

- *absolutely continuous* if, for every $\epsilon > 0$, there is $\delta > 0$ such that, whenever (x_i, y_i), $i = 1, \dots, k$, is a finite sequence of disjoint subintervals of $[a,b]$ with $\sum_{i=1}^{k}(y_i - x_i) < \delta$, we have $\sum_{i=1}^{k} |f(y_i) - f(x_i)| < \epsilon$.
- *of bounded variation* if there is $M > 0$ such that, whenever $P = \{a = x_0, x_1, \dots, x_n = b\}$ is a partition of $[a,b]$, we have $\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \le M$.

(a) Suppose that f is C^1. Prove that f is absolutely continuous.

(b) Suppose that f is absolutely continuous. Prove that f is of bounded variation.



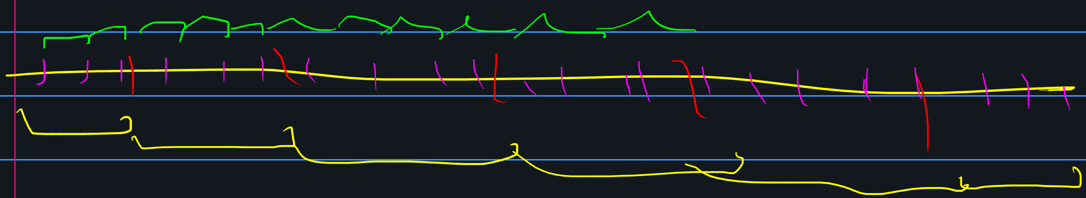
Let $P = \{x_0 = a, x_1, \dots, x_n = b\}$

$$V(f,P) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \le \frac{b-a}{\delta}$$

add $\le \frac{b-a}{\delta}$ pts to P, s.t. $\varepsilon = 1$

look at $P \cup \underbrace{\{\text{equally spaced}, |x_i - x_{i-1}| = \tfrac{\delta}{2}\}}_{\{y_0, \dots, y_n\}}$

$$V(f,P) \le V(f, P \cup P') \le C \cdot n$$



Compute

$$\lim_{n\to\infty} \int_0^1 \underbrace{\sin(nx)\, e^{-x^2}}_{} \, dx$$

if this converges uniformly, can bring limit in

hint: u-sub

$u = nx \qquad dx = \frac{1}{n} du$

$$\lim_{n\to\infty} \int_0^n \frac{1}{n} \sin(u)\, e^{-\frac{u^2}{n^2}} \, du$$

$$= \lim_{n\to\infty} \int_0^\infty \underbrace{\frac{1}{n} \sin(u)\, e^{-\frac{u^2}{n^2}} \chi_{[0,n]}(u)}_{} \, du$$

$\longrightarrow 0$ since

integrand $\longrightarrow 0$ unifly

Let $f \in BV[0, 2\pi]$, $f(0) = f(2\pi)$

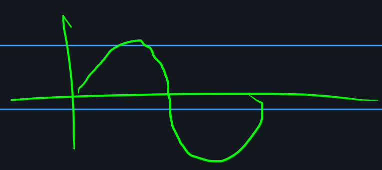
Show $\int_0^{2\pi} \sin(nx) f(x)\,dx$ &

$\int_0^{2\pi} \cos(nx) f(x)\,dx$

exist & both bdd by $\frac{c}{n} V_0^{2\pi} f$

integrand is a product of BV fns.

- $\sin(nx)$



if you integrate it over an interval of length $\frac{2\pi}{n}$ you get 0.

$$\varphi_n(x) = \sum_{k=0}^{n} f\left(k\frac{2\pi}{n}\right) \chi_{\left[k\frac{2\pi}{n},\ (k+1)\frac{2\pi}{n}\right]}(x)$$

$$\int \varphi_n(x) \sin(nx)\,dx = 0$$

$$\left|\int_0^{2\pi} f(x)\sin(nx)\,dx\right|$$

$$= \left|\int_0^{2\pi} f(x)\sin(nx)\,dx - \int_0^{2\pi} \varphi_n(x)\sin(nx)\,dx\right|$$

$$= \left|\int_0^{2\pi} \left(f(x) - \varphi_n(x)\right)\sin(nx)\,dx\right|$$

$$\leq \int_0^{2\pi} \left|f(x) - \varphi_n(x)\right| dx$$

$$= \int_0^{2\pi} \left|f(x) - \sum_{k=0}^{n-1} f\left(k\frac{2\pi}{n}\right)\chi_{I_k}\right| dx$$

$$= \sum_{k=0}^{n-1} \int_{k\frac{2\pi}{n}}^{(k+1)\frac{2\pi}{n}} \underline{\left|f(x) - f\left(k\frac{2\pi}{n}\right)\right|}\,dx$$

$$a < b < c$$

$$V_a^c = V_a^b + V_b^c$$

$$= \sum_{k=0}^{n-1} \frac{2\pi}{n} V_{k\frac{2\pi}{n}}^{(k+1)\frac{2\pi}{n}}(f)$$

$$= \frac{2\pi}{n} V_0^{2\pi} f$$