1. Let $\rho : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be

$$\rho(x,y) = |e^x - e^y|$$

a) Show it's a metric

b) is this equivalent to the std metric? (1-1)

Soln

a) · $\rho \geq 0$? Yes (1-1)

· symmetry? Yes

· $\rho = 0 \iff x = y$?

$\Leftarrow$: If $x = y \Rightarrow \rho(x,y) = |0| = 0$

$\Rightarrow$: If $e^x = e^y \iff x = y$

since e^t is injective

· Δ-ineq: given $x, y, z \in \mathbb{R}$

$$\rho(x,z) = |e^x - e^z| = |e^x - e^y + e^y - e^z|$$

$$\leq |e^x - e^y| + |e^y - e^z|$$

$$= \rho(x,y) + \rho(y,z)$$

□

b) equivalent to $|\cdot|^2$:

$\rho \sim |\cdot|$ is

$$x_n \xrightarrow{|\cdot|} x \iff x_n \xrightarrow{\rho} x$$

suppose $|x_n - x| \to 0$

wts $\rho(x_n, x) \to 0$

$|e^{x_n} - e^x| \to 0$ since $t \mapsto e^t$ is cts

(recall $f: \mathbb{R} \to \mathbb{R}$ is cts iff $x_n \to x \Rightarrow f(x_n) \to f(x)$)

suppose $\rho(x_n, x) \to 0$

wts $|x_n - x| \to 0$

$f(t) = e^t$ has continuous inverse

$f^{-1}(t) = \log t$

$$e^{x_n} \to e^x \Rightarrow f^{-1}(e^{x_n}) \to f^{-1}(e^x)$$

$$x_n \to x$$

$\square$

2. Let (X, ρ) be any metric space

define $\sigma(x,y) = \min(\rho(x,y), 1)$

a) (X, σ) is a metric space with finite diameter

$$\operatorname{diam}(X) = \sup_{x,y} \sigma(x,y)$$

Pf: suppose it is a metric

$$\sup_{x,y} \sigma(x,y) = \sup_{x,y} \min(\rho(x,y), 1)$$

$$\leq 1$$

You show it's a metric

Problem 2.

Suppose d', d'' are two metrics on the same set M. Suppose there exits $C > 1$ such that for any $x, y \in M$ we have

$$C^{-1} d'(x,y) \leq d''(x,y) \leq C d'(x,y).$$

Prove that the metrics d' and d'' are equivalent.

Problem 3.

Suppose d' and d'' are two equivalent metrics on M. Is it true that there exits $C > 1$ such that for any $x, y \in M$ one has

$$C^{-1} d'(x,y) \leq d''(x,y) \leq C d'(x,y) \ ?$$

Prove or give a counterexample.

b) is p ~ σ ?

c) is this true?

(x_n) p-Cauchy

$\Leftrightarrow$

(x_n) σ-Cauchy

3. a) give an example of an open cover of $(0,1)$ that has no finite subcover.

- $\{U_\alpha\}_{\alpha \in \Lambda}$ is an open cover of $(0,1)$ if U_α is open $\forall \alpha$ & $(0,1) \subseteq \bigcup_{\alpha \in \Lambda} U_\alpha$
- if $|\Lambda| < \infty$, it's a finite cover

$U_n = \left(\frac{1}{n}, 1\right)$ $n = 1, \ldots$

This is an open cover since

if $x \in (0,1)$ $\exists n$ s.t.

$$0 < \frac{1}{n} < x$$

(choose $n > \frac{1}{x}$ exists by Archimedean property)

Given any finite subcover, there's a maximal N s.t. U_N is in this cover but $0 < x < \frac{1}{N}$ isn't in the union of the sets in this cover.

$$C = \{U_{n_1}, U_{n_2}, \ldots, U_{n_N}\}$$

$$n_1 < n_2 < \cdots < N$$

If $0 < x < 1/N$, then

$$x \notin \bigcup_{i=1}^{N} U_{n_i}$$

(this shows $(0,1)$ isn't compact)

"Show that any open cover of $[0,1]$ has a finite subcover"

Hint: consider

$$S = \{x : [0,x] \text{ has a finite subcover}\}$$

Nonempty since $0 \in S$

Find $\sup S$ $(= 1)$

In $\mathbb{R}^n$ compact $\Leftrightarrow$ closed + bdd

(Heine-Borel)

If V is a normed vector space

$\overline{B(0,1)} \subseteq V$ is compact

iff $\dim V < \infty$.