**27.** If $f \in \mathrm{Lip}_K[a,b]$, show that $f$ can be uniformly approximated by polynomials in $\mathrm{Lip}_K[a,b]$.

one way: our proof of Weierstrass uses Bernstein polys

$$B_n f \rightrightarrows f$$

can show $f \in \mathrm{Lip}_K \Rightarrow B_n f \in \mathrm{Lip}_K$ $\forall n$

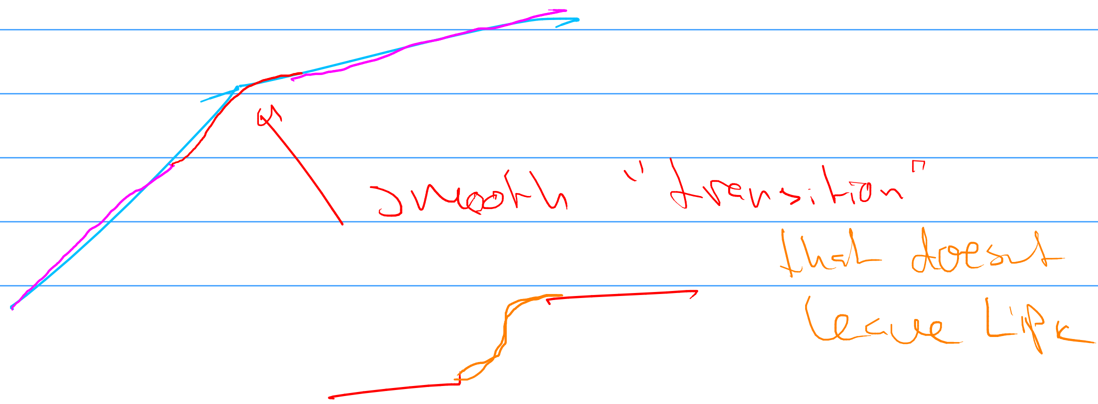
$$|B_n f(x) - B_n f(y)|$$

$$\leq \underbrace{|B_n f(x) - f(x)|}_{\uparrow} + \underbrace{|f(x) - f(y)|}_{\leq K|x-y|} + \underbrace{|f(y) - B_n f(y)|}_{\uparrow}$$

doesn't work.

- can approx w/ p.w. linear that's in $\mathrm{Lip}_K$

- approx the pw-linear thing w/ a smooth fn that's in $\mathrm{Lip}_K$



- this reduces the problem to approxing $\underline{\text{smooth}}$ $Lip_K$'s w/ $Lip_K$ polys. ie wlog $f$ smooth

- ideas: $f(x) = f(0) + \int_0^x f'(t)\,dt$

claim: $|f'| \leq K$

$$|f'(a)| = \lim_{x \to a} \left| \frac{f(x) - f(a)}{x - a} \right|$$

$$\leq \lim_{x \to a} \left| \frac{K|x-a|}{x-a} \right| \leq K$$

$\square$

claim: since $|f'| \leq K$, can approx $f'$ w/ a poly bdd by $K$ (maybe not in $Lip_K$)

claim: $q(x) = p(0) + \int_0^x p(t)\,dt \in Lip_K$

⋮

claim: $q$ approx's $f$, $\square$

Prob: say $f \in B[0,1]$

a) suppose $f \in B[\varepsilon, 1]$ $\forall \varepsilon > 0$

$\overset{?}{\Longrightarrow} f \in B[0,1]$

No. $f(x) = \sin(1/x) \in BV[\varepsilon, 1]$

is $C^1[\varepsilon, 1]$ $\forall \varepsilon$

$\Longrightarrow BV[\varepsilon, 1]$

but $f \notin BV[0,1]$ since (intuitively)

has infinitely many peaks & valleys

of the same height

$\sin(\frac{1}{x}) = 1$ iff $\frac{1}{x} = \frac{\pi}{2} + 2k\pi$

$$\Longleftrightarrow \boxed{\begin{aligned} x &= \frac{1}{\frac{\pi}{2} + 2k\pi} \\ x &= \frac{1}{\frac{3\pi}{2} + 2k\pi} \end{aligned}}$$

$\text{"} = -1$ if

make a partition from these.

**12.** Here is a variation on Exercise 11: If $(f_n)$ is a sequence in $BV[a,b]$, and if $f_n \to f$ pointwise on $[a,b]$, show that $V_a^b f \le \liminf_{n\to\infty} V_a^b f_n$.

b) $f \in BV[\varepsilon, 1]$ $\forall \varepsilon$

$\varepsilon$! $V_\varepsilon^1 f \le M$ $\forall \varepsilon$

$\overset{?}{\Longrightarrow}$ $f \in BV[0,1]$

---

$$\sum^{N} |f(x_j) - f(x_{j-1})|$$

$$= |f(x_1) - f(0)| + \sum_{j=2}^{N} |f(x_j) - f(x_{j-1})|$$

$$\le 2\|f\|_\infty + V_{x_1}^1 f$$

$$\le 2\|f\|_\infty + M$$

c) $f \in BV[\varepsilon, 1]$, $V_\varepsilon^1 f \le M$

$\Longrightarrow V_0^1 f \le M$

maybe $f(x) = \begin{cases} x & \text{for } x > 0 \\ 10 & \text{for } x = 0 \end{cases}$

$|f(0) - f(x_1)| \ge 9$ for $x_1$ small

d) what to add to make

$V_0^1 f \leq M$

maybe ask that $f$ cts at $0$?

---

**12.** Here is a variation on Exercise 11: If $(f_n)$ is a sequence in $BV[a,b]$, and if $f_n \to f$ pointwise on $[a,b]$, show that $V_a^b f \leq \liminf_{n\to\infty} V_a^b f_n$.

for any partition

$$V_a^b f \leq \sum^N |f(x_j) - f(x_{j-1})|$$

$$= \lim_{n\to\infty} \sum^N |f_n(x_j) - f_n(x_{j-1})|$$

$$= \underline{\lim} \sum^N |f_n(x_j) - f_n(x_{j-1})|$$

$$\leq \underline{\lim}\ V_a^b f_n$$

$\boxtimes$

**EXERCISE**

**28.** The polynomials in $z$ obviously separate points in $\mathbb{T}$ and vanish at no point of $\mathbb{T}$. Nevertheless, the polynomials in $z$ (with complex coefficients) are not dense in the space of continuous complex-valued functions on $\mathbb{T}$. To see this, here is a proof that $f(z) = \bar{z}$ cannot be uniformly approximated by polynomials in $z$:

**(a)** If $p(z) = \sum_{k=0}^{n} c_k z^k$, show that $\int_0^{2\pi} \overline{f(e^{it})}\, p(e^{it})\, dt = 0$.

**(b)** Show that $2\pi = \int_0^{2\pi} \overline{f(e^{it})}\, f(e^{it})\, dt = \int_0^{2\pi} \overline{f(e^{it})} \left[ f(e^{it}) - p(e^{it}) \right] dt$.

**(c)** Conclude that $\| f - p \|_\infty \geq 1$ for any polynomial $p$. [Hint: Take absolute values in (b) and note that $|f| = 1$.]

$|w| := \sqrt{w\bar{w}}$

c) $f(z) = \bar{z}$

for $z \in \mathbb{T}$, $|f(z)| = 1$

$\forall z \in \mathbb{T} \Rightarrow z = e^{i\theta}$

$\Rightarrow \bar{z} = \overline{(e^{i\theta})} = e^{-i\theta}$

$|f(z)|^2 = f(z)\overline{f(z)} = e^{-i\theta} e^{i\theta} = e^0 = 1$

$\overline{2\pi = \int_0^{2\pi} \overline{f(e^{i\theta})} \left( f(e^{i\theta}) - p(e^{i\theta}) \right) d\theta}$

$\Rightarrow 2\pi = | \ A \ |$

$\leq \int_0^{2\pi} \left| \cancel{\overline{f(e^{i\theta})}} \right| \left| f(e^{i\theta}) - p(e^{i\theta}) \right|$

$= \int_0^{2\pi} \left| f(e^{i\theta}) - p(e^{i\theta}) \right| d\theta$

$\leq \|f - p\|_\infty \cdot 2\pi$

**6.** We can test several of the inclusions implicit in our discussion up to this point by means of a single family of functions. For $\alpha \in \mathbb{R}$ and $\beta > 0$, set $f(x) = x^{\alpha}\sin(x^{-\beta})$, for $0 < x \le 1$, and $f(0) = 0$. Show that:

**(a)** $f$ is bounded if and only if $\alpha \ge 0$.

**(b)** $f$ is continuous if and only if $\alpha > 0$.

**(c)** $f'(0)$ exists if and only if $\alpha > 1$.

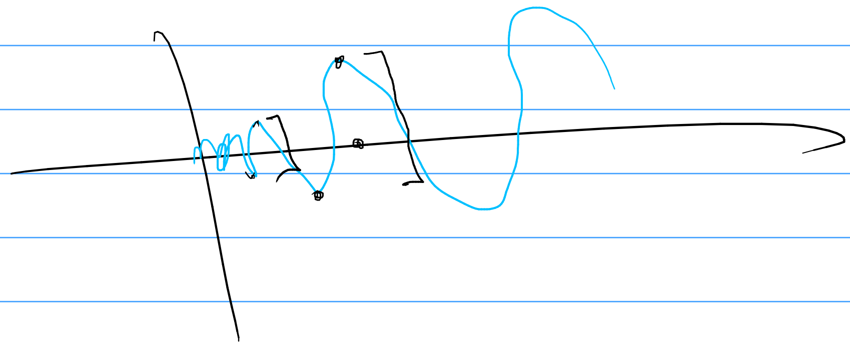
**(d)** $f'$ is bounded if and only if $\alpha \ge 1 + \beta$.

**(e)** If $\alpha > 0$, then $f \in BV[0,1]$ for $0 < \beta < \alpha$ and $f \notin BV[0,1]$ for $\beta \ge \alpha$.
[Hint: Try a few easy cases first, say $\alpha = \beta = 2$.]

easier

e) find a seq of partitions $P_n$

s.t. $V(f, P_n)$ blows up.



idea: choose a partition

that picks out the "peaks"

& "valleys" of $f$.

$$V(f) = \sup_{P} V(f, P)$$

$x_n \quad n \in \mathbb{Z}$

Let $x_{n_k}$ be a monotone subseq (wlog increasing)

let $x_{n_k} < y_k < x_{n_{k+1}}$
st $y_k \neq x_k$ for any $k$

$f$ const on $[y_k, y_{k+1}]$

st $\sum |c_n| < \infty$

$x_{n_1} \; y_1 \; x_{n_2} \; y_2 \; x_{n_3} \; y_3$

$\Rightarrow V(f) \geq |c_{n_1}| + |c_{n_2}| + \cdots$

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$[a,b]$ $(x_n)$ has limit pts inside $[a,b]$

maybe want $\sum |c_k| < M$ along any convergent subseq