Problem 2.

Let $\mathcal{F} \subset C[0,1]$ be a family of polynomials of degree at most 2021, such that for any $x \in [0,1]$ and any $p \in \mathcal{F}$ we have $|p(x)| \leq 2021$. Is $\mathcal{F}$ equicontinuous? Is $\mathcal{F}$ compact?

(a) (b)

(b): if ans to (a) is "no"

Arz-Ascoli: if X compact MS & $\mathcal{F} \subseteq C(X)$ then $\mathcal{F}$ compact iff $\mathcal{F}$ is ...
- closed
- (unif) bdd
- equicts

then A-A $\Rightarrow$ not cpct since not equicts.

- if ans to (a) is "yes"

by A-A, to get cpct, need closed + unifly bdd.

bdd? yes; $|p(x)| \leq 2021$.

- closed? : ie, if (p_n) seq polys deg ≤ 2021, bdd by 2021 unif conv, then the limit is also poly deg 2021 bdd by 2021.

· If ans to problem 1 is "true"

then we're good & the family is closed.

Prob 1: T

Fact: any finite dim subspace of a Banach sp is closed.

$P_{2021} = \text{span}(1, x, \dots, x^{2021})$

is finite dim $\Rightarrow$ closed.

· Use the fact that any two norms on a finite dim space are equivalent

$\|\cdot\|_1 \sim \|\cdot\|_2$ iff $\exists\, C > 0$ st. $\forall x$

$$C^{-1}\|x\|_1 \leq \|x\|_2 \leq C\|x\|_1$$

Idea for pf of Fact:

say V is a normed vect sp. &

$U \subseteq V$ st. $U = \langle e_1, \dots, e_n \rangle$.

Say (u_n) is Cauchy in U

$$\Rightarrow u_1 = \boxed{c_1^{(1)}} e_1 + \cdots + \boxed{c_N^{(1)}} e_N$$

$$u_2 = \boxed{c_1^{(2)}} e_1 + \cdots + \boxed{c_N^{(2)}} e_N$$

$$\vdots$$

what should $\lim_{n\to\infty} u_n$ be?

show that $\lim_{n\to\infty} c_i^{(n)}$ exists

for $i = 1, \dots, N$

Hint: $\|\cdot\| \sim \|\cdot\|_{\ell_1}$

where $\|u\|_{\ell_1} := \|c_1 e_1 + \cdots + c_N e_N\|_{\ell_1}$

$$= |c_1| + \cdots + |c_N|$$

Problem 6.

Prove that for any continuous function $f : \mathbb{R} \to \mathbb{R}$ there are C^∞-functions $g_1, g_2 : \mathbb{R} \to \mathbb{R}$ such that for any $x \in \mathbb{R}$

$$g_1(x) < f(x) < g_2(x),$$

and

$$\int_{-\infty}^{+\infty} (g_2(x) - g_1(x))dx < \infty.$$

do we need $\int f < \infty$?

I don't think so?

$\sin(x)$ $\quad \sin(x) + e^{-|x|}$

$\sin(x) - e^{-|x|}$

$$\Rightarrow \int 2e^{-|x|} < \infty$$

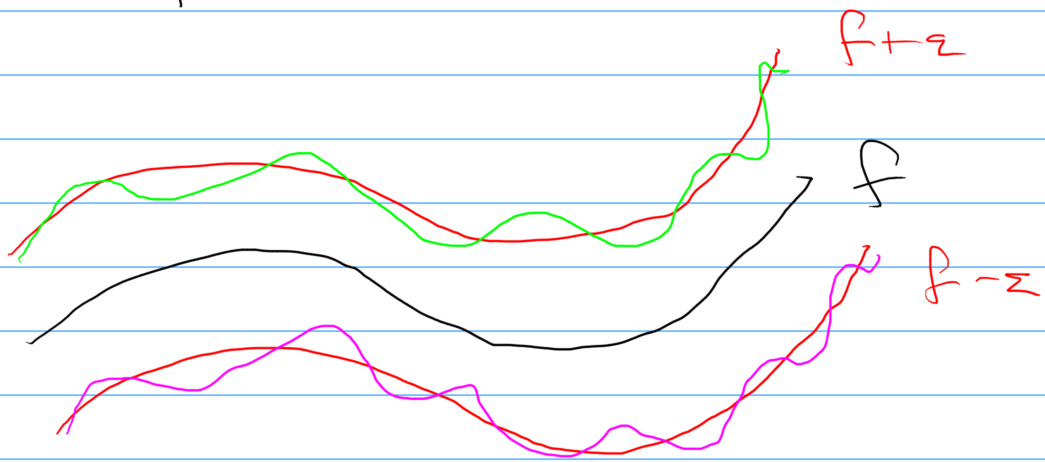
use ~~uniform~~ smooth approx

somehow.

* need to approx smoothly from above/below

unify approximate $f+\varepsilon$

& $f-\varepsilon$



F → ~~F~~ approx $f+\varepsilon$

G → ~~G~~ approx $f-\varepsilon$ constant

Issue $|F-G| \le c$

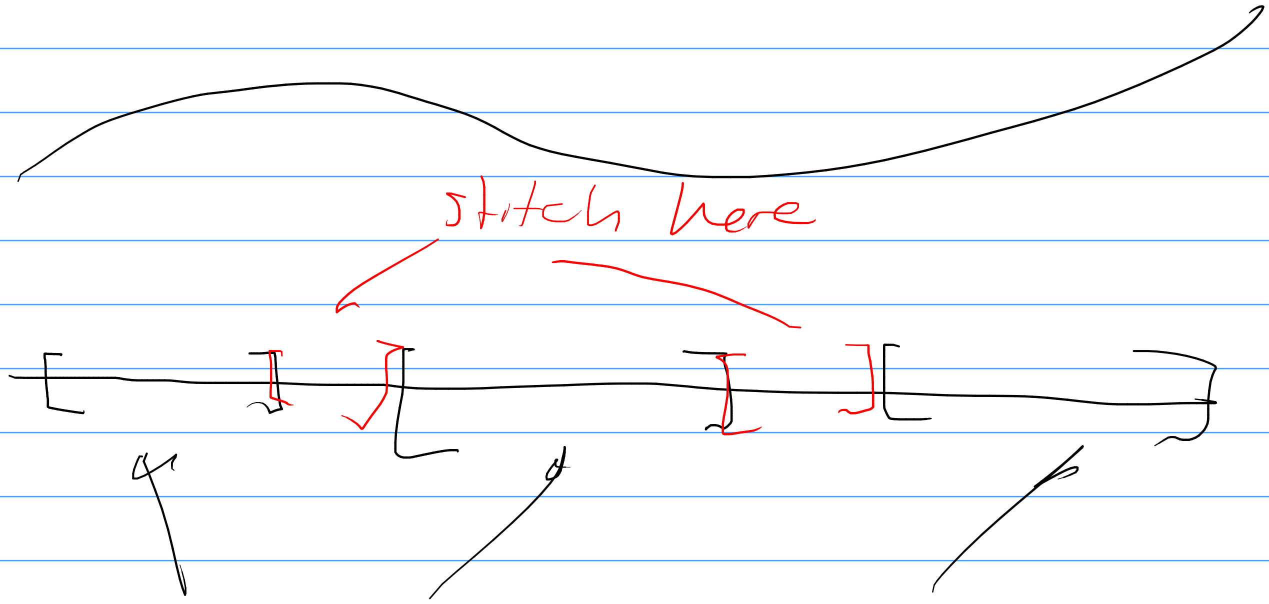
$$\int F-G = \infty$$

!!

maybe uniformly approx $f \pm 2^{-n}$

on $[n, n+1]$ w/ precision $1/2^n$

$[n+1, n+2]$



approx by 2^{-n}

1. Let $\mathcal{F} \subseteq C[0,1]$ closed, bdd, equicts. Prove $\exists g \in \mathcal{F}$ s.t. $$\int_0^1 g(x)\,dx \geq \int_0^1 f(x)\,dx$$ $\forall f \in \mathcal{F}$.

A-A $\Rightarrow$ $\mathcal{F}$ compact.

Claim: $T: C[0,1] \longrightarrow \mathbb{R}$

$$Tf = \int_0^1 f \quad \text{is cts}$$

$\Rightarrow$ let $f, g \in C[0,1]$

$$|Tf - Tg| = \left|\int_0^1 f-g\right|$$

$$\leq \text{length}[0,1] \cdot \|f-g\|_\infty$$

$$= \|f-g\|_\infty$$

$\Longrightarrow$ T is bdd + linear $\Rightarrow$ cts $\blacksquare$

So T is cts on the cpct set $\mathcal{F}$ $\Rightarrow$ achieves max