Problems in Extremal Graph and Hypergraph Theory

Liam Hardiman

December 10, 2020

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- If $X \subseteq V$, then the **subgraph induced by** X, $G[X]$, is the graph with vertex set X and all edges from E that have both ends in X.

Example

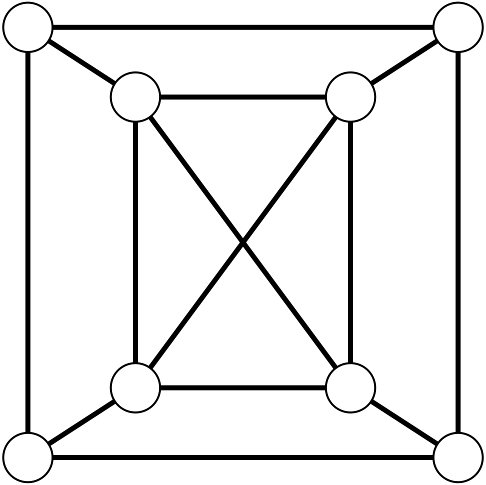

Figure: A graph

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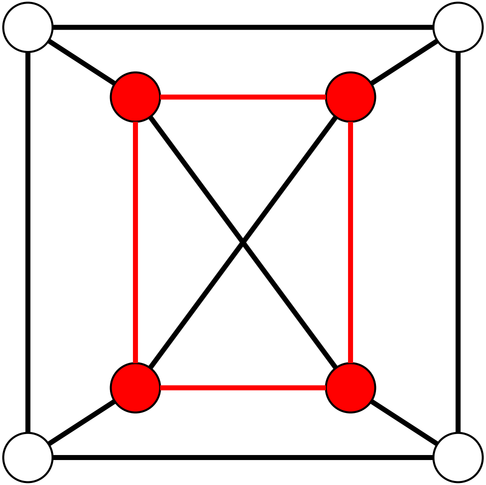

Figure: A subgraph

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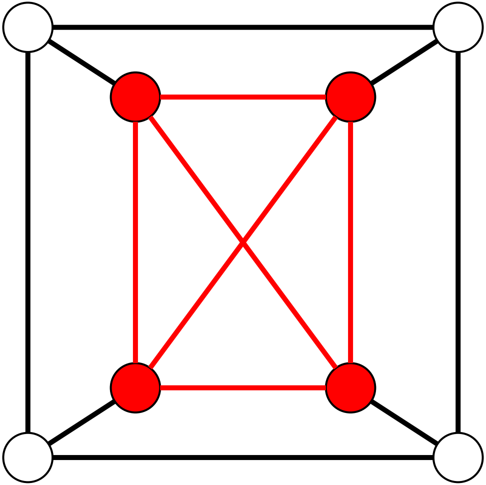

Figure: An induced subgraph

An existence question

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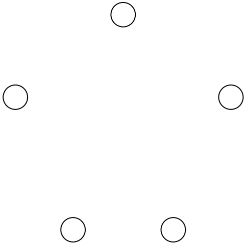

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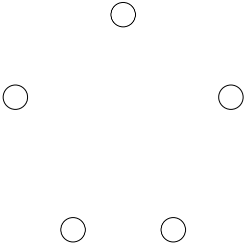

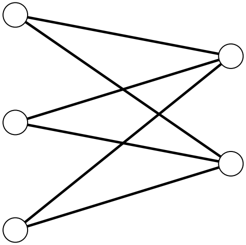

Guaranteeing the existence of a subgraph

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If G is a graph on n vertices with more than $\lfloor n/2 \rfloor \lceil n/2 \rceil$ edges, then G contains a triangle.

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Theorem (P. Turán - 1941)

If G is a graph on n vertices with more than $(1 - \frac{1}{r})\frac{n^2}{2}$ edges, then G contains a copy of K_{r+1}.

What about other graphs?

- Given a graph H, the **extremal number** (or **Turán number**) $\mathrm{ex}(n, H)$ is the largest number of edges in an H-free graph on n vertices.

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Theorem (P. Erdős, A. Stone - 1946)

*Let H be a nonempty graph and let r be the **chromatic number** of H (the smallest number of colors needed to color the vertices of H so that no two adjacent vertices are of the same color). Then*

$$ex(n, H) = \left(1 - \frac{1}{r-1}\right)\binom{n}{2} + o(n^2).$$

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- Problems related to embedding subgraphs based on the number of edges in the host graph are called **Turán-type problems**.

Larger substructures

- Suppose we have 16 sports teams. Due to travel restrictions, some teams cannot play each other. How many pairs of teams that can play each other do we need to ensure that we can pair off all teams?

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- We can't do this if there's only one road into any particular city.

Larger substructures

Definition

Let $G = (V, E)$ be a graph. A subset of edges $\mathcal{M} \subseteq E$ is called a **matching** if its edges are vertex-disjoint. The matching $\mathcal{M}$ is **perfect** if the vertices that comprise it cover all of V.

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Definition

Let $G = (V, E)$ be a graph and let $\mathcal{C}$ be a sequence of edges. Then $\mathcal{C}$ is a **Hamiltonian cycle** if it is a cycle that visits every vertex in G exactly once.

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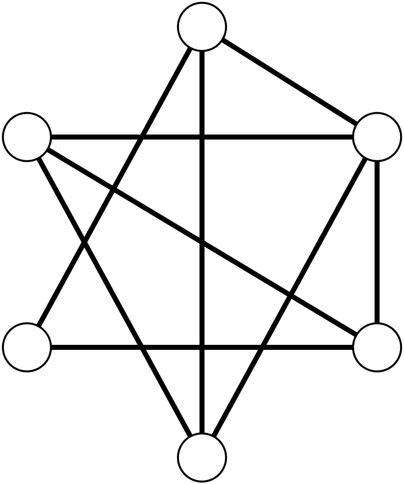

Figure: A graph

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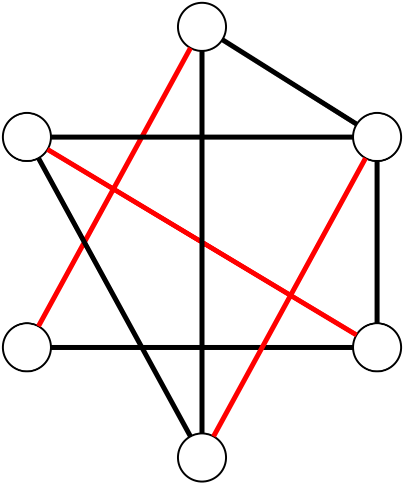

Figure: A perfect matching

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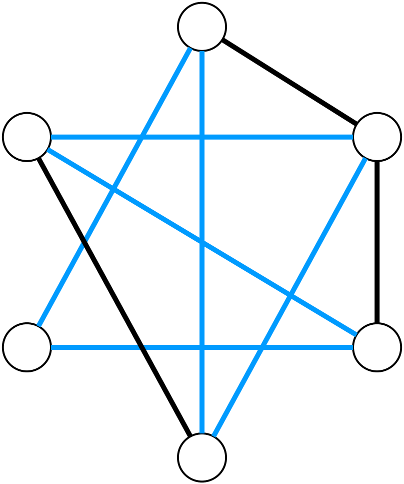

Figure: A Hamiltonian cycle

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Theorem (P. Hall - 1935)

Let G be a bipartite graph with parts X and Y (all edges in G have one end in X and one end in Y). Then G contains a matching that saturates X if and only if every subset W of X has at least $|W|$ neighbors in Y.

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- Apparently, some structures aren't simply related to the number of edges in the host graph, but how they're distributed.
- Graph embedding problems relating to the degree of the host graph are called **Dirac-type problems**.

From one to many

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- The **Erdős-Renyi random graph**, $\mathcal{G}(n, p)$, is the random variable that outputs a graph on n vertices, any two of which are independently connected with probability p. It's one of the most well-studied random structures.

From one to many - intuition from random graphs

- Idea: if $p > 1/2$, then $G \sim \mathcal{G}(n, p)$ is Dirac whp. Maybe we can use probabilistic tools to estimate the number of perfect matchings/Hamiltonian cycles in G. Does this tell us anything about deterministic Dirac graphs?

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- By Markov's inequality, $G \sim \mathcal{G}(n, p)$ doesn't have "too many" Hamiltonian cycles (perfect matchings) whp.
- Glebov and Krivelevich showed that if $p = \frac{\log n + \log\log n + \omega(1)}{n}$, then the number of Hamiltonian cycles in $G \sim \mathcal{G}(n, p)$, up to a sub-exponential factor, concentrates about its mean.

From one to many - intuition from random graphs

Theorem (B. Cuckler, J. Kahn - 2009)

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$$\#\{\textit{perfect matchings in } G\} \geq (1 - o(1))^n \cdot \frac{n!}{2^n (n/2)!}$$

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- Edge disjoint cycles? Covering? Directed graphs?

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- Sounds like a graph, except edges are triples!
- Kirkman's problem is asking us to find something resembling a collection of perfect matchings.

A generalization - hypergraphs

- A **hypergraph** $H = (V, E)$ consists of a set of vertices V and a set of edges $E \subseteq 2^V$. We say that H is k-uniform if every edge consists of exactly k vertices (note that a graph is a 2-uniform hypergraph).

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- We can talk about Turán or Dirac-type problems in this setting too.
- More generalized notion of degree: if S is any subset, then $d(S)$ is the number of edges that contain S. $\delta_t(H)$ is the minimum of $d(S)$ over all subsets of size t.

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- Matchings seem fine.
- Cycles are a bit less clear since the edges in graph cycles overlap at the ends. How can we account for this in hypergraphs?
- We say that a k-uniform hypergraph H contains a **Hamiltonian ℓ-cycle** if there is a cyclic ordering of the vertices of H such that the edges of the cycle are segments of length k in this ordering and any two consecutive edges f_i, f_{i+1} share exactly ℓ vertices.

Example

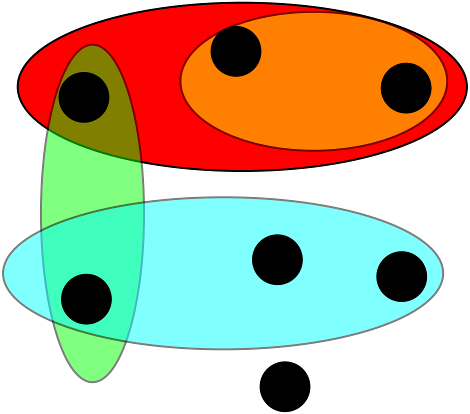

Figure: A hypergraph

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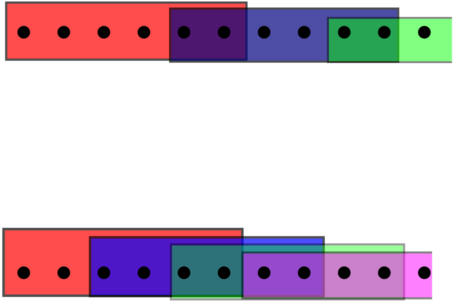

Figure: Top: $k = 6$, $\ell = 2$. Bottom: $k = 6$, $\ell = 4$.

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- Notice that if $\ell = 0$, then a Hamiltonian ℓ-cycle is a perfect matching.

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- If $\ell > k/2$, *three* consecutive edges in an ℓ-cycle overlap. This ostensibly makes counting these more nuanced.
- For $\ell \leq k/2$, the number of Hamiltonian ℓ-cycles we expect to see in a random hypergraph with edge probability p is

$$(n-1)! \cdot \frac{k-\ell}{2} \cdot \left(\frac{p}{\ell!(k-2\ell)!} \right)^{\frac{n}{k-\ell}} .$$

Matchings and cycles in hypergraphs

Theorem (A. Ferber, M. Krivelevich, B. Sudakov - 2016)

Let $0 \le \ell < k/2$ and let $1/2 < p \le 1$. If $(k-\ell) \mid n$ and $\delta_{k-1} \ge pn$, then the number of Hamiltonian ℓ-cycles in the k-uniform hypergraph H is at least

$$(1-o(1))^n \cdot n! \cdot \left(\frac{p}{\ell!(k-2\ell)!}\right)^{\frac{n}{k-\ell}}.$$

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- Notice that the minimum **co-degree**, $\delta_{k-1}(H)$ plays a role analogous to the minimum degree in a graph. A **Dirac hypergraph** on n vertices is one whose minimum co-degree is at least $n/2$.

Matchings and cycles in hypergraphs - current work

- Extending the work of Ferber, Krivelevich and Sudakov to the case $\ell < k-1$. Joint with A. Mond (University of Cambridge)

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- Extending the work of Ferber, Krivelevich and Sudakov to the case $\ell < k-1$. Joint with A. Mond (University of Cambridge)
- Future work: a tight (up to subexponential factor) bound for tight ($\ell = k-1$) cycles; number of *disjoint* ℓ-cycles.

A motivating problem

Exercise (L. Lovász, T. Gallai - 1979)

Assume that at each vertex of a graph there is a light that is turned on. Toggling any light toggles all of its neighbors as well. Turn off all of the lights.

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- Solution: find some $X \subseteq V(G)$ where all degrees in $G[X]$ are even and every $v \notin X$ has odd degree into X. Then toggle every light in X.

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- Every light in $V \setminus X$ flips an odd number of times (each of its neighbors in X) and every light in X flips an odd number of times (each of its neighbors in X and itself).

Even degrees

Theorem (L. Lovász, T. Gallai - 1979)

(Stated as an exercise in Lovász' book). Let $G = (V, E)$ be any graph. Then G admits a partitioning of its vertex set into two parts, $V = V_1 \cup V_2$, so that each vertex in $G[V_1]$ and each vertex in $G[V_2]$ has even degree. In particular, any graph on n vertices has an even subgraph of order at least $n/2$.

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- Use this theorem to prove that our solution is possible.
- This theorem contains two statements: one about a large induced subgraph and another about partitioning into particular subgraphs.

Odd subgraphs

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Theorem (A. Scott - 1992)

Every graph $G(V, E)$ on n vertices, none of which are isolated, contains a set $W \subseteq V(G)$ such that $|W| \geq \frac{n}{900 \log n}$ and $G[W]$ has all degrees odd.

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Conjecture (A. Scott - 2001)

There exists some constant $c > 0$ such that every graph $G(V, E)$ on n vertices, none of which are isolated, contains a set $W \subseteq V(G)$ such that $|W| \geq cn$ and $G[W]$ has all degrees odd.

Other moduli and remainders

Theorem (A. Scott - 1992)

Let $G \sim \mathcal{G}(n, 1/2)$. Then with high probability, G has an induced subgraph on at least $0.7729n$ vertices with all degrees odd.

Other moduli and remainders

Theorem (A. Scott - 1992)

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Conjecture (A. Scott - 2001)

For any positive integer q at least 2, there exists a constant c_q so that every graph on n vertices without isolated vertices has an induced subgraph on at least $c_q n$ vertices with all degrees $1 \pmod{q}$.

Other moduli and Remainders

Theorem (A. Ferber, H., M. Krivelevich - 2020+)

Let $q \geq 2$ and let r be an integer. Then there exists a constant c_q such that, with high probability, the random graph $G \sim \mathcal{G}(n, 1/2)$ has an induced subgraph on at least $c_q n$ vertices with all degrees $r \pmod{q}$.

Key idea behind proof

- If $G \sim \mathcal{G}(n, 1/2)$, then its adjacency matrix M ($M_{ij} = 1$ if and only vertices i and j are connected and $M_{ij} = 0$ otherwise) is a random symmetric $n \times n$ matrix whose above-diagonal entries are iid Bern$(1/2)$ random variables.

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Observation

G contains an induced subgraph with all degrees $r \pmod q$ if and only if its adjacency matrix contains a principal submatrix B satisfying $B\mathbf{1} \equiv r\mathbf{1} \pmod q$.

Proof (sketch) of theorem

Theorem (Chebyshev's Inequality)

Let X be a nonnegative integer-valued random variable with finite variance. Then

$$\Pr[X > 0] \geq 1 - \frac{Var[X]}{(\mathbb{E}[X])^2}.$$

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- Show that c_q can be chosen so that $\mathrm{Var}[X_k] = o(E[X_k]^2)$. Then G has an induced subgraph of size $c_q n$ with high probability.

Key computation

Fix a positive integer $q \geq 2$ and let r be an integer. Let $x_1, \ldots, x_t$ be iid Bern$(1/2)$ random variables. If $\delta_0(x) = 1$ if and only if $x \equiv 0 \pmod{q}$ and $\delta_0(x) = 0$ otherwise, then

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&= \frac{1}{q} \sum_{\ell \in \mathbb{Z}_q} e^{-2\pi i r \ell/q} \left(\frac{1 + e^{2\pi i \ell/q}}{2}\right)^t .
\end{aligned}$$

Key computation

- Isolate the $\ell \equiv 0$ term, apply the triangle inequality and Euler's identity.

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$$\left|\Pr[x_1 + \cdots + x_t \equiv r] - \frac{1}{q}\right| \le \frac{q-1}{q} e^{-2t/q^2}.$$

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- The distribution of a random Bernoulli sum modulo q is asymptotically uniform!
- Using this, we can show that if M is the adjacency matrix of a random graph, then $M\mathbf{1} \pmod q$ is (asymptotically) distributed uniformly over all possible values it can take.

Partitioning

Theorem (L. Lovász, T. Gallai - 1979)

Let $G = (V, E)$ be any graph. Then G admits a partitioning of its vertex set into two parts, $V = V_1 \cup V_2$, so that each vertex in $G[V_1]$ and each vertex in $G[V_2]$ has even degree. In particular, any graph on n vertices has an even subgraph of order at least $n/2$.

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Theorem (A. Ferber, H., M. Krivelevich, 2020+)

For $q > 1$ and $r \leq q$, $G \sim \mathcal{G}(n, 1/2)$ admits a partition into $q + 1$ classes such that the degrees in each class are congruent to $r \pmod q$ with high probability.

Future work - discrete Fourier analysis

Theorem (A. Ferber - 2020)

Let M_n denote an $n \times n$ symmetric ± 1 matrix chosen uniformly from all such matrices. Let $p(n)$ denote the probability that M_n is singular. Then there exists some constant $C > 0$ for which

$$p(n) = O\left(\frac{\log^C n}{\sqrt{n}}\right).$$

Future work - discrete Fourier analysis

Theorem (R. Meshulam - 1990)

Let G be a finite abelian group and let $s(G)$ denote the maximal s for which there exists a sequence $a_1, \ldots, a_s \in G$ such that $\sum_{i \in I} a_i \neq 0$ for any nonempty $I \subseteq [s]$. Then if m is the maximum order of elements in G, we have

$$s(G) \leq m\left(1 + \log \frac{|G|}{m}\right).$$

Future work - discrete Fourier analysis

Conjecture (N. Alon, N. Linial, R. Meshulam - 1988)

For every prime p, there exists a constant c_p such that for any $n \geq 1$, the union of any c_p linear bases for $\mathbb{Z}_p$ forms an additive basis for $\mathbb{Z}_p$.

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- The best known bound is at most $c_p \log n$ many linear bases.

Thank you.