Math 205A - Week 1

1. Let (a_n) be a sequence of real numbers. Show that (a_n) has two monotone subsequences, one converging to $\limsup_{n\to\infty} a_n$ and another converging to $\liminf_{n\to\infty} a_n$.

2. Sometimes, a limit is informally defined as follows: "As n goes to infinity, a_n gets closer and closer to L." Find as many faults with this definition as you can.

   (a) Can a sequence satisfy this definition and still fail to converge?

   (b) Can a sequence converge yet fail to satisfy this definition?

3. Suppose that $\lim_{n\to\infty} a_n = L$. Show that $\lim_{n\to\infty} \frac{a_1+a_2+\ldots+a_n}{n} = L$.

4. Suppose that (a_n) is a sequence of positive real numbers. Show that $\limsup_{n\to\infty}(a_n^{-1}) = (\liminf a_n)^{-1}$.

5. Prove the monotone sequence theorem: any monotone increasing sequence that's bounded above also converges and any monotone decreasing sequence bounded below also converges. Try proving this using only the least upper bound property of $\mathbb{R}$. Is this theorem true in $\mathbb{Q}$?

6. So far you've seen a few ways in which $\mathbb{R}$ is different from $\mathbb{Q}$.

   (a) The least upper bound property

   (b) The monotone sequence theorem

   (c) The nested interval theorem

   (d) The Bolzano-Weierstrass theorem

   Prove that these are all equivalent.