15. Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(x+y) = f(x) + f(y)$ for every x, $y \in \mathbb{R}$. If f is continuous at some point $x_0 \in \mathbb{R}$, prove that there is some constant $a \in \mathbb{R}$ such that $f(ax) = ax$ for all $x \in \mathbb{R}$. That is, an additive function that is continuous at even one point is linear – and hence continuous on all of $\mathbb{R}$.

want $a = f(1)$

Let $n \in \mathbb{Z}$

$f(nx) = nf(x)$ (induction)

$f(n) = nf(1) \quad \forall n \in \mathbb{Z}$

(this shows the claim for $\mathbb{Z}$)

idea: extend it to $\mathbb{Q}$

$f(\frac{p}{q}) \quad p, q \in \mathbb{Z}, \; q \neq 0.$

" (induction)

$p \, f(\frac{1}{q})$

WTS $f(\frac{1}{q}) = \frac{1}{q} f(1)$

$\Rightarrow f(1) = f(\frac{q}{q}) = q f(\frac{1}{q})$

$\Rightarrow \frac{1}{q} f(1) = f(\frac{1}{q})$

$\Rightarrow f(\frac{p}{q}) = \frac{p}{q} f(1)$

Idea: use continuity at x_0 to get continuity everywhere.

Continuity at x_0: for any $x_n \to x_0$
$$\Rightarrow f(x_n) \to f(x_0)$$

$$f(x+y) = f(x) + f(y) \quad \forall x, y$$

Let $y \in \mathbb{R}$. Let $y_n \to y$

WTS $f(y_n) \to f(y)$

since $y_n \to y$
$$y_n - y + x_0 \to x_0$$

$$\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} f(y_n - y + x_0 + y - x_0)$$
$$= \lim_{n\to\infty} f(y_n - y + x_0) + f(y - x_0)$$
$$= f(x_0) + f(y - x_0)$$
$$= f(y)$$

So far we have continuity everywhere + $f\left(\frac{p}{q}\right) = \frac{p}{q} f(1) \quad \forall p, q \in \mathbb{Z}$

last thing: show $f(\alpha)=\alpha f(1)$

$\forall \alpha \in \mathbb{R}$

16. Let $f : \mathbb{R} \to \mathbb{R}$, and define $G : \mathbb{R} \to \mathbb{R}^2$ by $G(x) = (x, f(x))$, so that the range of G is the *graph* of f. Show that f is continuous if and only if G is continuous if and only if both of the sets $A = \{(x, y) : y \le f(x)\}$ and $B = \{(x, y) : y \ge f(x)\}$ are closed in $\mathbb{R}^2$. In particular, if f is continuous, then the graph of f is closed in $\mathbb{R}^2$.

Idea for ~~⇐~~ is to "use"
the sets A & B
the graph is $A \cap B$

T/F Let (M,d) be a metric space

If $A \subseteq B \subseteq C \subseteq M$ &
A & C are connected, then
so is B.

"with"

Prove that if M is a connected metric space w/ ≥ 2 pts then it's uncountable.

Hint: let $x \neq y \in M$

$d(x,y) > 0$

idea: $d(x,\cdot)$ "should" hit everything in $[0, d(x,y)]$ since M is connected. Since uncountable we win.

say $\exists\, \xi \in [0, d(x,y)]$ s.t. $\nexists$ $z \in M$ s.t. $d(x,z) = \xi$.

$\underbrace{d(x,\cdot)[[0,\xi)]}_{\text{open}} \cup d(x,\cdot)[(\xi,\infty)]$

$= M$

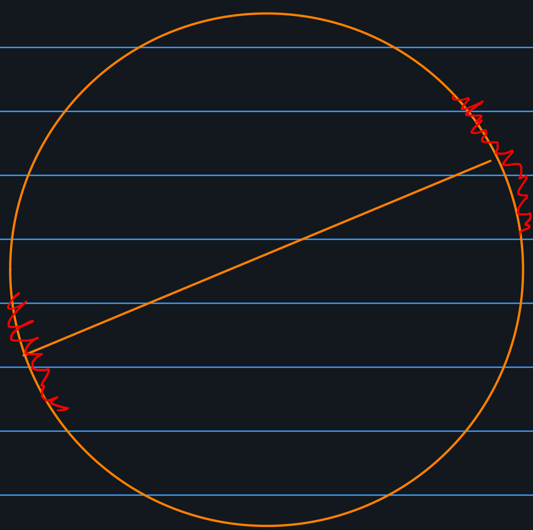
This is a disj union of opens

$\boxed{\checkmark}$

Say M has finite diameter

Let $x, y \in M$ s.t.

$$d(x,y) = \operatorname{diam} M$$



Let $f : (M,d) \longrightarrow (N,\rho)$ be a bijection

Then TFAE

- f open
- f closed
- f^{-1} cts

deduce that f homeo iff f & f^{-1} both open / closed.

- is total boundedness preserved by homeo?

- A totally bdd $\iff \bar{A}$ totally bdd