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Math 274 - Homework 3

Problem 1. Prove that for any positive integer $k > 1$ there is a $c = c(k)$ so that for any collection of subsets $A_1, \ldots, A_k \subseteq \{0,1\}^n$ that that satisfy $|A_i| \geq 2^n/k$ for all i, there are points $v_i \in A_i$ such that any pair of points v_i, v_j, $i \neq j$, differ in at most $c\sqrt{n}$ coordinates.

Proof. Recall the following corollary to Azuma's inequality.

Theorem 0.0.1. *Suppose ϵ and λ are positive real numbers satisfying $e^{-\lambda^2/2} = \epsilon$. Then for any $A \subseteq \{0,1\}^n$ of size at least $\epsilon \cdot 2^n$, we have*

$$|B(A, 2\lambda\sqrt{n})| \geq (1-\epsilon)2^n,$$

where $B(S, r)$ is the set of all strings in $\{0,1\}^n$ with Hamming distance at most r from some string in S. From this we deduce that $|B(A_j, 2\lambda\sqrt{n})| \geq (1 - 1/k)2^n$ for all j, where $\lambda = \sqrt{2\log k}$.

We start by showing that $S_1 = \cap_{j\geq 2} B(A_j, 2\lambda\sqrt{n})$ is nonempty (and is, in fact, quite large). From the above discussion and a simple union bound we deduce the following.

$$\left| \bigcup_{j\geq 2} B(A_j, 2\lambda\sqrt{n})^c \right| \leq \sum_{j\geq 2} |B(A_j, 2\lambda\sqrt{n})^c| \leq (1-1/k)2^n.$$

Hence, $|S_1| \geq 2^n/k$ and we can again apply Theorem 0.0.1 to obtain $|B(S_1, 2\lambda\sqrt{n})| \geq (1-1/k)2^n$. Since $|A_1| = 2^n/k$, if A_1 and $B(S_1, 2\lambda\sqrt{n})$ do not intersect, then $A_1^C = B(S_1, 2\lambda\sqrt{n})$ and then A_1 and $B(S_1, 2\lambda\sqrt{n}+1)$ intersect.

By our definition of S_1, there is then some v_1 in A_1 and some v in $\cap_{j\geq 2} B(A_j, 2\lambda\sqrt{n})$ so that $d(v_1, v) \leq 4\lambda\sqrt{n}$. Finally, we may choose v_j in A_j for $j \geq 2$ so that $d(v_j, v) \leq 2\lambda\sqrt{n}$ and

$$d(v_i, v_j) \leq d(v_i, v) + d(v_j, v) \leq 6\lambda\sqrt{n}$$

for all $i \neq j$. □

Problem 2. Prove that if M is an $n \times n$ matrix over some finite field $\mathbb{F}$ with $per(M) \neq 0$, then for every vector $b \in \mathbb{F}^n$ there exists $x \in \{0,1\}^n$ for which every coordinate i in Mx is distinct from b_i.

Proof. For each $b \in \mathbb{F}^n$ consider the polynomial $f_b : \mathbb{F}^n \to \mathbb{F}$ given by

$$f_b(x) = \prod_{i=1}^{n} ((Mx)_i - b_i) = \prod_{i=1}^{n} \left(\sum_{j=1}^{n} M_{ij}x_j - b_i \right).$$

The degree of f_b is n and the coefficient of the term $x_1 \cdots x_n$ is $per(M)$. To see this, note that we obtain this coefficient by summing over all possible ways to pick $A_{ij}x_j$ exactly once from each of the n factors in the product that defines f_b. The desired coefficient is then

$$\sum_{\sigma \in S_n} \prod_{j=1}^{n} M_{j\sigma(j)} = perm(M).$$

Since the coefficient of $x_1^1 \cdots x_n^1$ is nonzero and $|\{0,1\}| = 2$, there is an $x \in \{0,1\}^n$ such that $f_b(x) \neq 0$, which corresponds to a $0/1$ vector for which M_x differs from b in every coordinate. $\square$

Problem 3. Let $H = (V, E)$ be a hypergraph where each edge is of size t and each vertex has degree at most t. Show that

$$disc(H) = O(\sqrt{t \log t}).$$

Proof. $\square$

Problem 4. Fix $n \in \mathbb{N}$. We say that $P(n)$ is true if for any $a_1, \ldots, a_{2n-1} \in \mathbb{Z}$, there is an $I \subseteq [2n-1]$ with $\sum_{i \in I} a_i \equiv 0 \pmod{n}$ and $|I| = n$. Show that if $P(n)$ and $P(m)$ are true, then so is $P(nm)$.

Problem 5. A 1-factorization in a hypergraph $H = (V, E)$ is a collection of edge-disjoint perfect matchings that cover all the edges of H. Let K_n^k denote the complete k-uniform hypergraph on n vertices. Our goal is to prove the following theorem.

Theorem 0.0.2. *Let k and n be two positive integers for which n is divisible by k. Then the complete k-uniform hypergraph on n vertices admits a 1-factorization.*

(a) Prove the following lemma.

Lemma 0.0.3. *For any real $m \times n$ matrix M with integer row and column sums, there is an integer $m \times n$ matrix M' having the same row and column sums as M and satisfying*

$$|m_{ij} - m'_{ij}| < 1, \quad \forall i, j.$$