1. give an ex of a seq of $f_n \in R[0,1]$ s.t. $f_n \to f$ ptwise but $f \notin R[0,1]$.

e.g. let $\{q_n\}$ be an enum. of $\mathbb{Q}$

$$f_n(x) = \sum_{j=1}^{n} \mathbb{1}_{\{q_j\}}(x)$$

$$= \mathbb{1}_{\bigcup_{j=1}^{n}\{q_j\}}(x)$$

$$\longrightarrow \mathbb{1}_{\mathbb{Q}}(x) \notin R[0,1]$$

If $f_n \in R[0,1]$ &

$f_n \rightrightarrows f \implies f \in R[0,1]$

& $\int f_n \to \int f$

2. consider $f_n : [0,1] \to \mathbb{R}$

$$f_n(x) = \int_0^x \exp\left[(1+t^4)\sin(a_n t^5)\right] dt$$

where $\{a_n\}$ any seq in $\mathbb{R}$.

show f_n has a unif conv. subseq.

Plan: show bdd + equicts then use A-A.

Bdd: $-2 \leq (1+t^4)\sin(a_n t^5) \leq 2$

$$\Rightarrow e^{-2} \leq \exp[\;\;] \leq e^2$$

$$\Rightarrow \left| \int_0^x \exp[\;\;]\, dt \right| \leq \| e^2 x \|_\infty \leq e^2$$

equicts: WL $x, y \in [0,1]$

$$|f_n(x) - f_n(y)| = \left| \int_x^y \exp[\cdots]\,dt \right|$$

$$\leq e^2 |x-y|$$

so given ε, $|x-y| \leq \boxed{\frac{\varepsilon}{e^2}}$

A-A => :)

3. $f \in R[0,1]$. Show

$$\int_0^1 x^n f(x)\,dx \longrightarrow 0$$

Pf:

$$\longrightarrow \begin{cases} 0 & x \neq 1 \\ f(1) & x = 1 \end{cases}$$

ptwise

w/ φ smooth & $\|f-\varphi\|_\infty < \varepsilon$

$u = x^{n+1}$

$du = (n+1)x^n \, dx \Rightarrow x^n dx = \frac{1}{n+1} du$

$$\frac{1}{n+1}\int_0^1 f\left(u^{1/(n+1)}\right) du \longrightarrow 0$$

since $\left\| \frac{f(u^{1/(n+1)})}{n+1} \right\|_\infty \leq \frac{\|f\|_\infty}{n+1} \longrightarrow 0$

$\boxtimes$

$$\left| \int_0^1 f(x) x^n \, dx - \int_0^1 \varphi(x) x^n \, dx \right|$$

$\leq$ small

4. Say $f:[0,1]\to\mathbb{R}$ cts

with $m=\min f$, $M=\max f$

show that for any $c\in[m,M]$

there $\exists$ $\alpha:[0,1]\to\mathbb{R}$ nondec

s.t. $\int_0^1 f(x)\,d\alpha(x)=c$

Indeed, by IVT, $\exists\, x_0\in[0,1]$

st $f(x_0)=c$

$$\alpha(x)=\begin{cases}0 & \text{if } x<x_0\\ 1 & \text{if } x\geq x_0\end{cases}$$

$\boxtimes$

Problem 1.

Use the Stone-Weierstrass Theorem to confirm that piecewise-linear continuous functions are dense in $C[0,1]$.

not a subalg

Lattice version of S-W

Stone–Weierstrass Theorem (lattices). Suppose X is a compact Hausdorff space with at least two points and L is a lattice in $C(X,\mathbf{R})$ with the property that for any two distinct elements x and y of X and any two real numbers a and b there exists an element $f \in L$ with $f(x) = a$ and $f(y) = b$. Then L is dense in $C(X,\mathbf{R})$.

"Polys are dense. p.w. linears are cts, so can find poly close to the pw linear. therefore, p.w. linears are dense"

X an infinite space

$B(X)$ not separable

$\{0,1\}^X$ uncountable

$$\cancel{f_a(x) = \begin{cases} 1 & x = a \\ 0 & \text{else} \end{cases}}$$