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Math 274 - Homework 3

Problem 1. Prove that for any positive integer $k > 1$ there is a $c = c(k)$ so that for any collection of subsets $A_1, \ldots, A_k \subseteq \{0,1\}^n$ that that satisfy $|A_i| \geq 2^n/k$ for all i, there are points $v_i \in A_i$ such that any pair of points v_i, v_j, $i \neq j$, differ in at most $c\sqrt{n}$ coordinates.

Problem 2. Prove that if M is an $n \times n$ matrix over some finite field $\mathbb{F}$ with $per(M) \neq 0$, then for every vector $b \in \mathbb{F}^n$ there exists $x \in \{0,1\}^n$ for which every coordinate i in Mx is distinct from b_i.

Problem 3. Let $H = (V, E)$ be a hypergraph where each edge is of size t and each vertex has degree at most t. Show that

$$disc(H) = O(\sqrt{t \log t}).$$

Problem 4. Fix $n \in \mathbb{N}$. We say that $P(n)$ is true if for any $a_1, \ldots, a_{2n-1} \in \mathbb{Z}$, there is an $I \subseteq [2n-1]$ with $\sum_{i \in I} a_i \equiv 0 \pmod{n}$ and $|I| = n$. Show that if $P(n)$ and $P(m)$ are true, then so is $P(nm)$.

Problem 5. A 1-factorization in a hypergraph $H = (V, E)$ is a collection of edge-disjoint perfect matchings that cover all the edges of H. Let K_n^k denote the complete k-uniform hypergraph on n vertices. Our goal is to prove the following theorem.

Theorem 0.0.1. *Let k and n be two positive integers for which n is divisible by k. Then the complete k-uniform hypergraph on n vertices admits a 1-factorization.*

(a) Prove the following lemma.

Lemma 0.0.2. *For any real $m \times n$ matrix M with integer row and column sums, there is an integer $m \times n$ matrix M' having the same row and column sums as M and satisfying*

$$|m_{ij} - m'_{ij}| < 1, \quad \forall i, j.$$