18. If A is either open or closed, show that bdry(A) is nowhere dense in M. Is the same true of any set A ?

Counterex: $\mathbb{Q} \subseteq \mathbb{R}$ $\quad$ $\mathbb{Q}$ not open/closed

$\partial \mathbb{Q} = \mathbb{R}$ dense

- say A open/closed
- say $U \subseteq \text{int}(\partial A)$ open
- $\forall x \in \partial A$, $\exists$ ball that is not contained in ∂A

(since this fails for $\mathbb{Q}$, have to use open/closedness of A to justify)

recall: $\boxed{\overline{A} = A^{\circ} \sqcup \partial A}$

- if A open, $\overline{A} = A \sqcup \partial A$

if $\exists U$ open in ∂A, $U \cap A = \emptyset$

explain why this $\Rightarrow U = \emptyset$

- same(ish) idea for $A = \overline{A}$

Prob: say $f \in C^1[0,1]$ $\left(\begin{matrix} f \in C[0,1] \\ f' \in C[0,1] \end{matrix}\right)$

show, $\exists$ seq of polynomials p_n

s.t. $p_n \rightrightarrows f$ on $[0,1]$ AND

$p_n' \rightrightarrows f'$.

Pf: Trick! Start w/ f'

- Weierstrass says $\exists$ polys

$q_n \rightrightarrows f'$

- let $p_n(x) = \int_0^x q_n(t)\,dt + f(0)$

since $f \in C^1$, $f(x) = f(0) + \int_0^x f'(t)\,dt$

$$|p_n(x) - f(x)| = \left|\int_0^x (q_n(t) - f'(t))\,dt\right|$$

$$\leq \int_0^x \underbrace{|q_n(t) - f'(t)|}_{\leq \varepsilon \text{ for all } t,\ n \text{ large by unif conv.}}\,dt$$

n large

$$\leq \varepsilon \cdot x \leq \varepsilon \text{ for all } x$$

$x \in [0,1]$ $\Longrightarrow p_n \rightrightarrows f$.

Challenge: let $f \in C^\infty[0,1]$, show $\exists p_n$ polys s.t. $p^{(k)} \rightrightarrows f^{(k)}$ $\forall k \geq 0$

Q: $f : [0,1] \to \mathbb{R}$ cts suppose

$$\int_0^1 f(x) x^n \, dx = 0 \quad \forall n \geq 0$$

T/F: $f \equiv 0$?

Weierstrass might help.

$\exists$ plys $p_n \rightrightarrows f$ by Weierstrass.

$$f(x) p_n(x) \rightrightarrows f^2(x)$$

hypothesis

$$0 = \int_0^1 f(x) p_n(x) dx \longrightarrow \int_0^1 f^2(x) \, dx$$

$$\Rightarrow \int_0^1 f^2(x) dx = 0$$

use continuity of f to show $f^2 \equiv 0$
$\Rightarrow f \equiv 0$.

$\boxtimes$

Say $f: [0,\infty) \to \mathbb{R}$ cts

& $\lim_{x\to\infty} f(x) = 0$ Show

we can uniformly approximate

$f(x)$ by functions of the form

$$\sum_{n=0}^{N} c_n \underbrace{e^{-nx}}$$

Pf: Consider $g(x) = f(-\log x)$

on $[0,1]$ (so $g(0)=0$)

$$\lim_{x\to 0} g(x) = \lim_{x\to\infty} f(x) = 0$$

Weierstrass 2 ⇒ ∃ P_n Polys

$P_n \rightrightarrows g$ on $[0,1]$

$P_n(e^{-x}) \rightrightarrows f(x)$ on $[0,\infty)$

$$x = -\log y$$

$$|P_n(e^{-x}) - f(x)| = |P_n(y) - f(-\log y)|$$