Math 205A - Week 2

1. Can you find two different sequences (a_n) and (b_n) that are subsequences of one another?

2. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies

$$\lim_{h \to 0} [f(x+h) - f(x-h)] = 0$$

   for every $x \in \mathbb{R}$. Does this imply that f is continuous?

3. Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies the functional equation

$$f(u+v) = f(u) + f(v) \quad \text{for all} \quad u, v \in \mathbb{R}.$$

   (a) Prove that $f(mx) = mf(x)$ for all $x \in \mathbb{R}$ and $m \in \mathbb{Z}$.

   (b) Prove that $f(x) = cx$ for all $x \in \mathbb{Q}$ where $c = f(1)$.

   (c) Deduce that if f is continuous on $\mathbb{R}$, then $f(x) = cx$ for all $x \in \mathbb{R}$.

4. Prove that a countable, closed and bounded subset of $\mathbb{R}$ cannot be perfect. *Hint: Suppose* $X = \{x_1, x_2, \ldots\}$ *and find a decreasing family* X_n *of closed nonempty subsets of* X *with* $x_n \notin X_n$.

5. Let $E \subseteq \mathbb{R}$ be uncountable. Prove that there exists an x in $\mathbb{R}$ such that $E \cap (-\infty, x)$ and $E \cap (x, \infty)$ are both uncountable.