HW 3 Math 274

1. Prove that for any positive integer $k > 1$ there is a $c = c(k)$ so that for any collection of subsets $A_1, \dots, A_k \subset \{0,1\}^n$ that satisfy $|A_i| \geq \frac{2^n}{k}$ for all i, there are points $v_i \in A_i$ such that any pair of the points v_i, v_j $(i \neq j)$ differ in at most $c\sqrt{n}$ coordinates.

2. Prove that if M is an $n \times n$ matrix over some finite field $\mathbb{F}$ with $per(M) \neq 0$, then for every vector $b \in \mathbb{F}^n$ there exists $x \in \{0,1\}^n$ for which every coordinate i in Mx is distinct from b_i.

3. Let $H = (V, E)$ be a hypergraph where each edge is of size t and each vertex has degree at most t. Show that
$$disc(H) = O(\sqrt{t \log t}).$$

4. The following will be used in class without a proof: Fix $n \in \mathbb{N}$. We say that $P(n)$ is true if for any $a_1, \dots, a_{2n-1} \in \mathbb{Z}$, there is an $I \subseteq [2n-1]$ with $\sum_{i \in I} a_i = 0 (mod\ n)$ and $|I| = n$. Show that if $P(n)$ and $P(m)$ are true, then also $P(nm)$ is true. HINT: given $a_1, \dots, a_{2mn-1}$, show that there exist disjoint $I_1, \dots, I_{2m-1} \subseteq [2mn-1]$, each of which of size n, such that $\sum_{i \in I_j} a_i = 0 (mod\ n)$ for all j. Then, consider the integers $A_j = k^{-1} \sum_{i \in I_j} a_i$ and think how to proceed.

5. A 1-*factorization* in a hypergraph $H = (V, E)$ is a collection of edge-disjoint perfect matchings that together cover all the edges of H. Let K_n^k denote the *complete* k-uniform hypergraph on n vertices; that is, the vertex set is $V := [n]$ and the edges set consists of all $\binom{n}{k}$ subsets of V of size exactly k.

   Our main goal is to prove the following theorem:

   Theorem 1. *Let k and n be two positive integers for which n is divisible by k. Then, the complete k-uniform hypergraph on n vertices admits a 1-factorization.*

   There are multiple ways to approach this problem, one good option follows:

   (a) Prove the following lemma:

   Lemma 2. *For any real $m \times n$ matrix M with integer row and column sums, there is an integer $m \times n$ matrix M' having the same row and column sums as M and satisfying*
   $$|m_{ij} - m'_{ij}| < 1, \ \forall i, j.$$

   In other words, for every matrix with all row sums and column sums being integers, one can find an integer valued matrix preserving all row sums and column sums, and with every entry of the original matrix being rounded up or down.

(b) Consider a more general problem (which will imply the desired theorem). Given n and nonnegative integers $k_1, \ldots, k_r$, set H_i to be the *complete* k_i-uniform hypergraph on vertex set $[n]$ (that is, its set of edges consists of all subsets of $[n]$ of size k_i). We will answer the following general question:

Question 3. *For which $k_1, \ldots, k_r$ and n is there a partition of $\cup H_i$ into perfect matchings?*

Note that if some k_i's are the same, then the above union is a multiset union.
Suppose we do have such a partition into perfect matchings $\mathcal{F}_1, \ldots, \mathcal{F}_\ell$. For each i, j, set

$$|H_i \cap \mathcal{F}_j| = \alpha_{ij}.$$

That is, α_{ij} is the number of edges from H_i that belong to the matching $\mathcal{F}_i$.
Observe that the α_{ij}'s are all integers and satisfy

$$\sum_j \alpha_{ij} = \binom{n}{k_i} \ \forall i;$$

and

$$\sum_i \alpha_{ij} k_i = n \ \forall j.$$

So, a reasonable guess for answering Question 3 might be the existence of α_{ij} satisfying these two conditions. Indeed, we prove that

Theorem 4. *For any n and $k_1, \ldots, k_r$, if there are nonnegative integers α_{ij} satisfying the above conditions, then there is a partition as in the question (with intersections α_{ij} as defined above).*

Show that Theorem 4 implies Theorem 1.

(c) Prove Theorem 4 by induction on n.