$f(x) = \sin^2(x)$

$= \frac{1}{2} - \frac{1}{2}\cos(2x)$

6. Let $f : \mathbb{R} \to \mathbb{R}$ be 2π-periodic and Riemann integrable on $[-\pi, \pi]$. Prove that $\lim_{x\to 0} \left(\int_{-\pi}^{\pi} |f(x+t) - f(t)|^2 \, dt\right)^{1/2} = 0$.

claim: $\varphi_x(t) := \varphi(x+t)$ & φ cts

$\varphi_x \xrightarrow[x\to 0]{} \varphi$ on $[-\pi, \pi]$

$\Rightarrow |\varphi_x - \varphi|^2 \rightrightarrows 0$

$C[-\pi, \pi]$ $\qquad$ $R[-\pi, \pi]$

prob 5

$\forall f \in R[-\pi, \pi] \Rightarrow \forall \varepsilon > 0, \exists \varphi \in C_{2\pi}([-\pi, \pi])$ s.t. $\int |f - \varphi|^2 < \varepsilon$

$\|f_x - f\|_2 \leq \|f_x - \varphi_x\|_2 + \|\varphi_x - \varphi\|_2 + \|\varphi - f\|_2$

$\boxtimes$

1. Find Fourier series for

$$|\sin x|$$

- $|\sin x|$ is even
  $\Rightarrow$ only cosine terms

- π-periodic $\left(|\sin(x/2)| \right)$
  $\Rightarrow$ $\underline{\cos[2nx]}$ terms
  (exercise)

- $$a_k = \frac{2}{\pi} \int_0^{\pi} \underline{\sin x \cos(2nx)}\, dx$$

$\circ$ —2 +

$$= \frac{1}{\pi} \int_0^{\pi} \left[\sin[(2n+1)x] + \sin[(1-2n)x] \right]$$

$\cos\left(2n\left(x - \frac{\pi}{2}\right)\right) = \cos(2nx - \pi n) = C \cdot \sin(n \quad dx$

$$\Longrightarrow \boxed{|\sin x| \sim \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2nx)}{4n^2 - 1}}$$

b) find the F. series for $f(x-\alpha)$ in terms of the series for f.

use this to get series for $\cos x$

$$a_n^{(\alpha)} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-\alpha)\cos(nx)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos[nt+n\alpha]\,dt$$

trig

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\{\cos(nx)\cos(n\alpha) - \sin(nx)\sin(n\alpha)\}\,dx$$

$$= \frac{\cos(n\alpha)}{\pi}a_k - \frac{\sin(n\alpha)}{\pi}b_k$$

do $b_n^{(\alpha)}$ same way.

(b) show $g \to 0$ as $t \to \infty$

Let $f \in R[0, 2\pi]$

Show $g(t) = \int_0^{2\pi} f(x) \sin(tx)\, dx$

is unif'ly cts on $\mathbb{R}$.

Pf: $|g(s) - g(t)|$ is small when $|s - t|$ small

$$|g(s) - g(t)| = \left| \int_0^{2\pi} f(x) \left[\sin(sx) - \sin(tx) \right] dx \right|$$

$|\sin x - \sin y| \leq |x - y|$

$$\leq \|f\|_\infty \cdot 2\pi |s - t|$$

$\Longrightarrow$ Lipschitz.

$$f(x) = \sum_{n \geq 1} \frac{x \sin(n^2 x)}{n^2}$$

converges ptwise to a cts fn.