HW 2 Math 274

1. Let $v_1 = (x_1, y_1), \ldots, v_n = (x_n, y_n)$ be n vectors in $\mathbb{Z}^2$, where each x_i and each y_i is a positive integer that does not exceed $\frac{2^{n/2}}{10\sqrt{n}}$. Show that there exist two disjoint non-empty subsets $I, J \subset \{1, \ldots, n\}$ such that $\sum_{i \in I} v_i = \sum_{j \in J} v_j$.

2. Let $G = (V, E)$ be a graph on n vertices, and let $0 < \alpha < 1$. Consider the following game on the edge-set of G: there are two players I and II, alternating turns in occupying previously unoccupied edges in E until there are no more edges to claim (with Player I going first). At this point, let $G_1 = (V, E_1)$, where E_1 is the set of all edge occupied by Player I, be the spanning subgraph of G which is spanned by the edges of Player I. Player I wins the game if and only if for every vertex $v \in V$, its degree in G_1 is at least an α-fraction of its original degree in G.

   Show that for large enough n, if G has minimum degree at least (say) $\log^2 n$, then there exists a strategy to win the game with $\alpha \geq \frac{1}{3} - \epsilon$.

3. Prove that for every constant d there exists a constant C so that for any given graph G with $|V(G)| = n$ (where n is arbitrarily large) and $\Delta(G) \leq d$, there exists a 1-1 function $f : V \to [Cn]$ such that the quantities $\{|f(u) - f(v)| : uv \in E(G)\}$ are all distinct.

4. Prove that there exists an integer n_0 such that for every $n > n_0$, any set A of n distinct integers contains two disjoint subsets B_1 and B_2 satisfying $|B_1| = |B_2| > 0.33n$, where each of the sets B_i is sum-free.

5. A *tournament* is a directed graph obtained by orienting all the edges of the complete graph (intuitively, you can think about it as: if x has an ongoing edge to y, then x "beat" y in the tournament). For a tournament $T = (V, E)$ and for two disjoint subsets $A, B \subseteq V$, let $e(A, B) = |\{(a, b) \in E : a \in A, b \in B\}|$ (meaning, the number of edges oriented from a vertex in A into a vertex in B). Prove that there exists an absolute constant $c > 0$ such that for any tournament $T = (V, E)$ on n vertices there are two disjoint subsets $A, B \subseteq V$ such that $e(A, B) - e(B, A) \geq cn^{3/2}$.