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Math 274 - Homework 1

Problem 1. Let $\{(A_i, B_i) : 1 \leq i \leq m\}$ be a family of pairs of subsets of the set of integers such that $|A_i| = k$ for all i, $|B_i| = \ell$ for all i, $A_i \cap B_i = \emptyset$ for all i, and $(A_i \cap B_j) \cup (A_j \cap B_i) \neq \emptyset$ for all $i \neq j$. Show that $m \leq (k+\ell)^{k+\ell}/(k^k \ell^\ell)$.

Proof. Let U be the union of the A_i and the B_i for all i. We randomly sample a set $X \subseteq U$ by including each $x \in U$ independently with probability p, to be determined later. Now let E_i be the event that X contains all of A_i and contains no element of B_i. Note that these events are disjoint. Indeed, if E_i and E_j both occur, then X contains both A_i and A_j and contains nothing from B_i or B_j. But at least one of $A_i \cap B_j$ and $A_j \cap B_i$ is nonempty, so this cannot happen.

Now since $|A_i| = k$ and $|B_i| = \ell$, we have $\Pr[E_i] = p^k(1-p)^\ell$ for all i. From this we deduce

$$1 \geq Pr[\cup_{i=1}^m E_i] = \sum_{i=1}^m \Pr[E_i] = mp^k(1-p)^\ell,$$

so $m \leq p^{-k}(1-p)^{-\ell}$. This function of p attains its minimum at $p = \frac{k}{k+\ell}$, which gives the desired bound. □

Problem 2. Suppose there are m red clubs $R_1, \ldots, R_m$ and m blue clubs $B_1, \ldots, B_m$ in a town of n citizens. Define the $m \times m$ matrix M as follows

$$M_{ij} = |A_i \cap B_j|.$$

Show that if M is non-singular, then $m \leq n$.

Proof. Identify the citizens with the set $[n]$ and each club R_i, B_i with its indicator vector in $\mathbb{R}^n$. If we let R and B be the $m \times n$ matrices whose rows are the R_i and B_i, respectively, then $M = RB^T$. Since M is nonsingular, its rank is m and we have

$$\begin{aligned}
m &= \operatorname{rank}(M) \\
&= \operatorname{rank}(RB^T) \\
&\leq \min(\operatorname{rank}(R), \operatorname{rank}(B)) \\
&\leq n.
\end{aligned}$$

□

Problem 3. Let p be a prime. Let $A \subseteq \mathbb{Z}_p$ be a set such that $|A| < p^{2/3}$. Prove that there are $x, y \in \mathbb{Z}_p$ such that $A \cap (A+x) \cap (A+y) = \emptyset$.

Proof. We pick x and y uniformly and independently at random from $\mathbb{Z}_p$. Notice that $a \in A \cap (A+x) \cap (A+y)$ if and only if $a - a_1 = x$ and $a - a_2 = y$ for some a_1, a_2 in A. With this in mind, consider the (directed) (multi) graph whose vertex set is A, where we draw a red edge from a to a' if $a' - a = x$ and a blue edge if $a' - a = y$. Now a is in $A \cap (A+x) \cap (A+y)$ if and only if it has an outward-directed red edge and an outward-directed blue edge. There are $3\binom{|A|}{3}$ ways to choose three elements of A with one chosen "center". Since x and y were chosen uniformly at random, the probability that any particular (ordered) pair is a red edge is $1/p$, and the same holds for blue edges. Thus, the probability that any "centered" triple has a red edge and a blue edge emanating from its center is $2/p^2$. The expected number of such properly colored triples is then

$$\frac{6}{p^2}\binom{|A|}{3} \leq |A|^3/p^2.$$

If this quantity is less than 1, then there is a pair (x, y) that yields no good triples, so the set $A \cap (A+x) \cap (A+y)$ is empty. This occurs when $A < p^{2/3}$. $\square$

Problem 5. Let $m(n, s)$ denote the maximum number of points in $\mathbb{R}^n$ such that their pairwise distances take at most s values. Prove that

$$\binom{n+1}{s} \leq m(n,s) \leq \binom{n+s+1}{s}.$$

Proof. We start with the upper bound. Suppose the pairwise distances among the points $P^{(1)}, \ldots, P^{(m)}$ in $\mathbb{R}^n$ take at most s values, say $d_1, \ldots, d_s$. Consider the polynomials $f_1, \ldots, f_m$ in $x = (x_1, \ldots, x_n)$ given by

$$f_i(x) = \left(\|x - P^{(i)}\|^2 - d_1^2\right) \cdots \left(\|x - P^{(i)}\|^2 - d_s^2\right).$$

These polynomials are linearly independent since $f_i(P^{(j)}) = c\delta_{ij}$, where $c = (-1)^s(d_1 \cdots d_s)^2$. We can then upper bound m by the dimension of any subspace that the polynomials $f_1, \ldots, f_m$ live in. While each polynomial here has degree $2s$, expanding the expression $\|x - P^{(i)}\|^2$ reveals that it can be thought of as a polynomial of degree at most s in the $(n+1)$ variables $x_1, \ldots, x_n$ and $\sum_{k=1}^n x_k^2$. This space has dimension $\binom{n+s+1}{s}$, the desired upper bound.

As for the lower bound, consider a regular n simplex in $\mathbb{R}^n$. For concreteness, label its vertices $v_1, \ldots, v_{n+1}$. Consider the set of centers of mass of every subset of s vertices, that is,

$$S = \left\{ \frac{1}{s}\sum_{j=1}^{s} v_{i_j} : \{v_{i_1}, \ldots, v_{i_s}\} \subseteq \{v_1, \ldots, v_{n+1}\} \right\}.$$

We claim that S is a set of $\binom{n+1}{s}$ points whose pairwise distances take at most s values. Indeed, suppose two collections of s vertices, $v_{i_1}, \ldots, v_{i_s}$ and $v_{j_1}, \ldots, v_{j_s}$ have the same center of mass. Then

$$\sum_{k=1}^{s} v_{i_k} = \sum_{k=1}^{s} v_{j_k}.$$

By the affine independence of the vertices of the simplex, this implies that these two subsets are in fact identical, hence, S consists of $\binom{n+1}{s}$ points. Next, we claim that the distance between the centers of mass of $S_i = \{v_{i_1}, \dots, v_{i_s}\}$ and $S_j = \{v_{j_1}, \dots, v_{j_s}\}$ is determined by the size of $|S_i \cap S_j|$. Suppose the size of this intersection is t and that the first t elements of S_i and S_j are the same. The squared distance between their centers of mass is then (up to a factor of $1/s^2$)

$$\left\| \sum_{k=t+1}^{s} v_{i_k} - \sum_{k=t+1}^{s} v_{j_k} \right\|^2 = \sum_{k=t+1}^{s} \|v_{i_k} - v_{j_k}\|^2 + 2 \sum_{t<k<\ell\leq s} \langle (v_{i_k} - v_{j_k}), (v_{i_\ell} - v_{j_\ell}) \rangle.$$

Since the vertices of the simplex are equidistant from one another and any two pairs vertices have the same inner product, this quantity is independent of which subsets of s vertices with t common vertices we pick. $\square$

$$F(s) = \int_0^\infty f(t)e^{-st}\, dt = \oint f \cdot dr = \det(A - \lambda I) = p(\lambda) = e^{\Theta n}$$