28. Let $f_n : \mathbb{R} \to \mathbb{R}$ be continuous, and suppose that (f_n) converges uniformly on $\mathbb{Q}$. Show that (f_n) actually converges uniformly on all of $\mathbb{R}$. [Hint: Show that (f_n) is uniformly Cauchy.]

Show (f_n) unifly Cauchy.

$\forall n, m$

$$|f_n(x) - f_m(x)| \leq$$

for any $q_x \in \mathbb{Q}$

$$\leq |f_n(x) - f_n(q_x)| + |f_n(q_x) - f_m(q_x)| + |f_m(q_x) - f_m(x)|$$

- Second term: $f_n \rightrightarrows f$ on $\mathbb{Q}$

$\Rightarrow$ unifly cauchy on $\mathbb{Q}$

$\Rightarrow \forall n, m > N,$

$|f_n(q) - f_m(q)| < \varepsilon$

$\forall q \in \mathbb{Q}$

now take $n, m > N$

- choose q_x s/t first & third terms are small (by density of $\mathbb{Q}$ in $\mathbb{R}$)

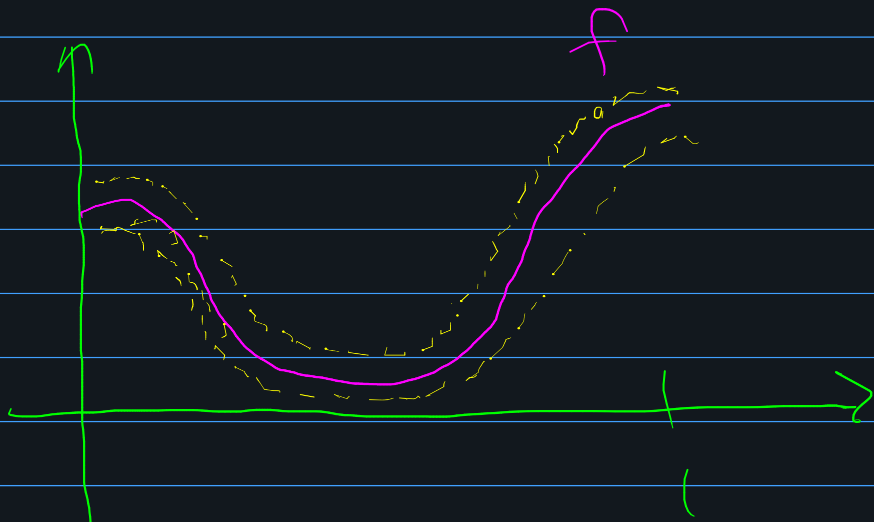
- even though this q_x depends on x, n, & m, we can always find such a q_x so that $|f_n(x) - f_m(x)| < \varepsilon$

1. Show that $U = \{ f \in C[0,1] : f > 0 \}$ is open.

$\boxed{\text{idea}}$ neighborhood $\subseteq U$

Pf.

Picture



- Since f cts on compact set, it attains its min, say $\delta > 0$
- Consider the $\delta/2$-ball around f.

$\boxtimes$

Alternatively: say $f_n \rightrightarrows f$, $f_n \in \mathcal{U}^c$

f cts since unif limit of cts

fns is cts.

- for any x, $f_n(x) \to f(x)$

$\{f_n(x)\}$ is a convergent seq

of real #s ≤ 0, so its

limit, $f(x)$ is ≤ 0.

Q: where is compactness of

$[0,1]$ used here?

(note: first approach fails

if not on a cpt set)

2. Show that $S = \{\sin(nx) : n \in \mathbb{Z}\}$

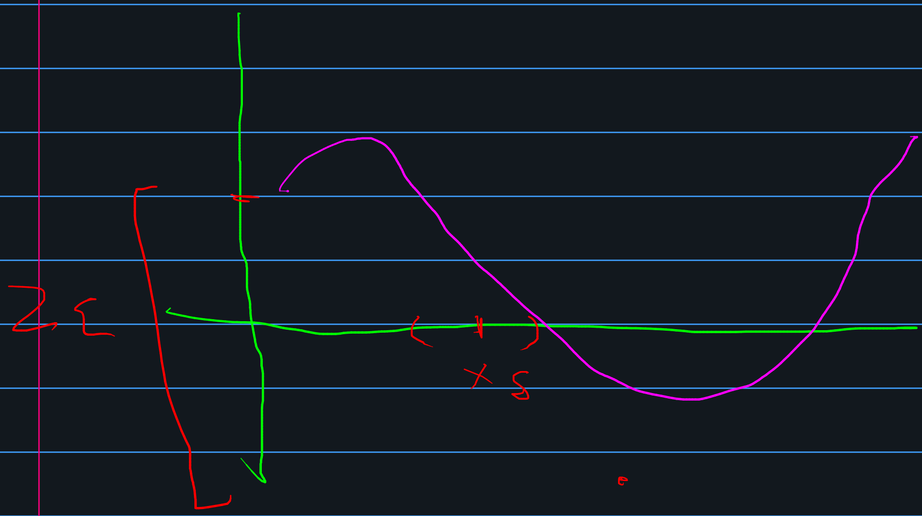
is not equicts on $[-\pi, \pi]$.

Pf: to be equicts, need:

$\forall \varepsilon > 0, \exists \delta$ st $|x-y| < \delta$

$\Rightarrow |f(x) - f(y)| < \varepsilon \quad \forall f \in S$

Say this is true



- choose n st: $n \gg \pi/\delta$
- this way, $\sin(nx)$ completes a full period over <u>any</u> interval of length $\geq \delta$.
- over any such interval, $\sin(nx)$ attains $+1$ & -1

$\Rightarrow |\sin(nx) - \sin(ny)|$ attains the value 2 on any such interval

$\square$

3. Consider

$$\mathcal{F} = \left\{ F(x) = \int_0^x f(t)\,dt : \begin{array}{l} f \in C[0,1] \\ \|f\|_\infty \le 1 \end{array} \right\}$$

a) Show $\mathcal{F}$ is bdd & equicts

Pf: bdd take $F \in \mathcal{F}$

$$\|F\|_\infty = \sup_{x\in[0,1]} \left| \int_0^x f(t)\,dt \right|$$

$$\le \sup_{x\in[0,1]} \int_0^x |f(t)|\,dt$$

$$\le \sup_{x\in[0,1]} \left(x \, \|f\|_\infty \right)$$

$$\le \|f\|_\infty \le 1$$

equicontinuity: given ε, need to find δ st. $|x-y|<\delta$

$\Rightarrow |F(x) - F(y)| < \varepsilon \quad \forall F \in \mathcal{F}$.

for any $x, y \in [0,1]$

$$|F(x) - F(y)| = \left| \int_x^y f(t)\,dt \right|$$

$$\le |x-y| \cdot \|f\|_\infty$$

$$\le |x-y| \quad (\text{indep of } F)$$

so if $|x-y| < \varepsilon$

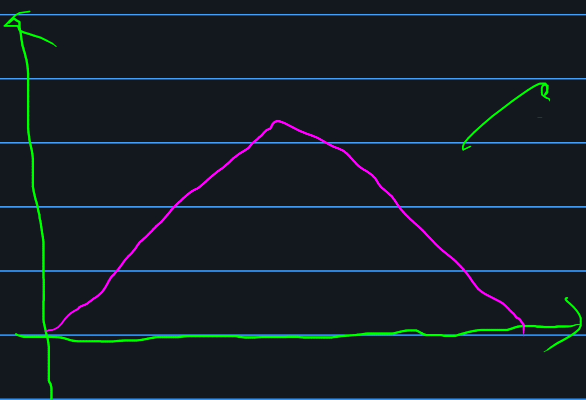
$$\Rightarrow |F(x) - F(y)| < \varepsilon$$

b) $\mathcal{F}$ is not closed.

Idea: note that by FTC,

$$\mathcal{F} \subseteq C'[0,1]$$

should $\overline{\mathcal{F}} \subseteq C'[0,1]$?



$$\begin{cases} x & \text{for } x \le \frac{1}{2} \\ 1-x & \text{for } x \ge \frac{1}{2} \end{cases}$$

c) show $\overline{\mathcal{F}} = \left\{ f \in C[0,1] : \begin{array}{l} f(0) = 0 \\ f \text{ Lip} \\ \text{w/ const } 1 \end{array} \right\}$

Consider $T : C[0,1] \to C[0,1]$

$$(Tf)(x) = \int_0^x f(t)\,dt$$

we showed $T\left(\overline{B,(0)}\right)$ is not closed

claim: $\overline{T(B,(0))}$ is compact

i.e. T is a compact operator