Theorem 1.1 (Main Theorem) Fix $\delta > 0$ and $K > 0$. Then the following hold for all integers $k \geq k_0(\delta, K)$.

(i) For any $N \in \mathbb{N}$ there are at most

$$2^{\delta k}\binom{\frac{1}{2}Kk}{k}N^{\lfloor K+\delta \rfloor}$$

sets $A \subset [N]$ with $|A| = k$ and $|A + A| \leq K|A|$.

(ii) If N is prime there are at most

$$2^{\delta k}\binom{\frac{1}{2}Kk}{k}N^{\lfloor K+\delta \rfloor}$$

sets $A \subset \mathbb{Z}/N\mathbb{Z}$ with $|A| = k$ and $|A + A| \leq K|A|$, provided that $Kk \leq (1 - \delta)N$.

Theorem 2.1 (Main tool used for proving Theorem 1.1) For every $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that the following is true. Let $p > p_0(\epsilon)$ be a sufficiently large prime and let $A \subset \mathbb{Z}/p\mathbb{Z}$ be a set. There is a dilate $A^* = \lambda A$ of A and a prime q, $\epsilon^{-10} \leq q \leq p^{1-\delta}$, such that the following holds. If

$$A_i^* := A^* \cap I_i(q) = A^* \cap \{x \in \mathbb{Z}/p\mathbb{Z} : x/p \in [i/q, (i+1)/q]\}$$

for each $i \leq q$ then, for at least $(1 - \epsilon)q^2$ of the pairs $(i, j) \in [q]^2$,

$$\min(|A_i^*|, |A_j^*|) \leq \epsilon p/q \qquad \text{or} \qquad |A_i^* + A_j^*| \geq (2 - \epsilon)p/q.$$

Definitions

(i) **Balanced Fourier transform.** If $I \subset \mathbb{Z}/p\mathbb{Z}$ is an interval and if $A \subset I$ is a set, then the balanced Fourier transform $\hat{f}_A : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ is defined by

$$\hat{f}_A(\theta) := \sum_{x \in I}(1_A(x) - \alpha)e(x\theta),$$

where $\alpha = |A|/|I|$ and $e(t) = e^{2\pi it}$.

(ii) **ϵ-regularity.** We say that a pair (A, A') of subsets of $\mathbb{Z}/p\mathbb{Z}$ is ϵ-regular if for every $\theta \in \mathbb{R}/\mathbb{Z}$ we have either

$$|\hat{f}_A(\theta)| \leq \epsilon|I| \quad \text{or} \quad |\hat{f}_{A'}(\theta)| \leq \epsilon|I'|$$

and if $\|\theta\| \leq \min(\frac{1}{\epsilon|I|}, \frac{1}{\epsilon|I'|})$ we have both

$$|\hat{f}_A(\theta)| \leq \epsilon|I| \quad \text{and} \quad |\hat{f}_{A'}(\theta)| \leq \epsilon|I'|.$$

Regularity Lemma (Used to prove Theorem 2.1) For every $\epsilon > 0$ there exists $\delta = \delta(\epsilon)$ such that the following holds for every sufficiently large prime p, and every $A \subset \mathbb{Z}/p\mathbb{Z}$. There is a dilate $A^* = \lambda A$ of A and a prime q, $\epsilon^{-10} \leq q \leq p^{1-\delta}$, such that at least $(1 - \epsilon)q^2$ of the pairs (A_i^*, A_j^*) are ϵ-regular.

Counting Lemma (Used with Regularity Lemma to prove Theorem 2.1) Let ϵ, L be positive constants with $L > 16/\epsilon$. Suppose that $I, I' \subset \mathbb{Z}/p\mathbb{Z}$ are intervals with $|I|, |I'| = L + O(1)$. Suppose also that the pair of sets $A \subset I$ and $A' \subset I'$ is ϵ^7-regular and that $|A|, |A'| \geq \epsilon L$. Then $|A + A'| \geq (2 - 8\epsilon)L$.