Math 13 - Week 9: Equivalence Relations

Let's review the superdense coding protocol from section 2.3. Here, Alice wants to send a 2-bit message to Bob. They start by preparing two qubits in the following entangled state,

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}. \tag{1}$$

Alice takes the first qubit and Bob the second. Now if Alice wants to send Bob the message "00", she does nothing to her qubit. If she wants to send "01" she applies Z, for "10" she applies X and for "11" she applies iY.

Question 1. Aren't I, X, Y and Z operators on *single* qubits? If so, how can Alice apply them to the *two* qubit state, $|\psi\rangle$?

Solution. Alice isn't applying these single qubit operators to the *whole* state, but to just her single qubit. If she were to apply the operator X to just her qubit (assume it's the first one), then the *whole* state becomes

$$(X \otimes I)\,|\psi\rangle\,.$$

The I above represents Bob doing nothing to his qubit. □

Alice's actions are encoded in the following display.

$$\begin{aligned}
00 &: |\psi\rangle \mapsto (I \otimes I)\,|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \\
01 &: |\psi\rangle \mapsto (Z \otimes I)\,|\psi\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}} \\
10 &: |\psi\rangle \mapsto (X \otimes I)\,|\psi\rangle = \frac{|10\rangle + |01\rangle}{\sqrt{2}} \\
11 &: |\psi\rangle \mapsto (iY \otimes I)\,|\psi\rangle = \frac{|10\rangle - |10\rangle}{\sqrt{2}}.
\end{aligned} \tag{2}$$

It's easy to check that these states are all mutually orthogonal. Remember that $|00\rangle$ is shorthand for $|0\rangle \otimes |0\rangle$ and that the inner product on the tensor product is

$$\big(a\,|v_1\rangle \otimes |w_1\rangle\,, b\,|v_2\rangle \otimes |w_2\rangle\,\big) = a^* b \langle v_1|v_2\rangle\langle w_1|w_2\rangle.$$