Math 130B - Expectation

1. Let X and Y be jointly continuous random variables.

   (a) Show that if X and Y are independent, then $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.

   (b) Give an example of two jointly continuous random variables X and Y for which $\mathbb{E}[XY] \neq \mathbb{E}[X]\mathbb{E}[Y]$.

2. Consider the continuous iid random variables $T_1, \ldots, T_n$. For any $k = 1, 2, \ldots, n$, find

$$\mathbb{E}\left(\frac{T_1 + T_2 + \cdots + T_k}{T_1 + T_2 + \cdots + T_n}\right).$$

3. Let $G = (V, E)$ be a graph with n vertices and e edges. Show that you can partition the vertices of G into two subsets A and B in such a way that half of the edges in G cross between A and B.

4. Let σ be a random permutation of $\{1, 2, \ldots, n\}$, chosen uniformly. Let X be the number of fixed points of σ (recall that i is a fixed point of σ if $\sigma(i) = i$). Find $\mathbb{E}[X]$.

5. A *tournament* with n players is a special kind of directed graph. Imagine all n players play two-person games with each other. Draw an arrow from player u to player v if player u beat player v in their game (assume there are no draws).

   A *path* is a sequence of edges $v_1 \to v_2 \to \ldots v_k$ where $v_i \to v_{i+1}$ is a directed edge and the vertices are all distinct. The path is called *Hamiltonian* if it contains all vertices in the graph.

   Show that there is a tournament T with n players and at least $n! \cdot 2^{-(n-1)}$ Hamiltonian paths.