Math 13 - Week 9: Equivalence Relations

1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(t) = (-t, t^3)$ and let $g : \mathbb{R}^2 \to \mathbb{R}$ be given by $g(x, y) = \cos(x^2 + y^2)$. Are these claims valid? If not, explain the error.

   (a) The function g surjects onto $\mathbb{R}$ because $\mathbb{R}^2$ is a higher dimensional space than $\mathbb{R}$.

   (b) The function f surjects onto $\mathbb{R}^2$. To see this, take $(-t, t^3) \in \mathbb{R}^2$. Then $(-t, t^3) = f(t)$, so f is surjective.

   (c) The function f surjects because for every b in B there is an a in A such that $f(a) = b$.

   (d) g is injective because $g(x, y) = g(x', y')$ implies that $(x, y) = (x', y')$.

   (e) f is injective because the only way for f to hit $(-1, 1)$ is to plug in $t = 1$.

2. Let $\sim$ be the relation on $\mathbb{R}^2$ defined by $(x, y) \sim (v, w)$ if and only if $x^2 + y^2 = v^2 + w^2$.

   (a) Prove that $\sim$ is an equivalence relation.

   (b) Describe the quotient $\mathbb{R}^2/\sim$. Think geometrically.

3. Consider the following claim and proof.

   Claim 1. If a relation is symmetric and transitive, then it is reflexive.

   Proof. Suppose $\sim$ is a relation on a set S that is symmetric and transitive. Then for any x, y in S, $x \sim y$ implies $y \sim x$ by symmetry. Since $x \sim y$ and $y \sim x$, then $x \sim x$ by transitivity and the relation is reflexive. $\square$

   Is this proof valid? Why or why not?