Midterm – Math 130B

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Instructions:

- You must show your work and clearly explain your line of reasoning.
- You have one hour and twenty minutes to complete the exam.

GOOD LUCK!

1. (10 points) Suppose the measurement errors, X_i, from n experiments are uniformly distributed on $[-1, 1]$. Assume that the errors are independent.

   (a) (4 points) Find the probability distribution function for $|X_i|$.

   (b) (6 points) Find the probability density function and probability distribution functions for the largest absolute error. Be sure to clearly state the domains of the PDF and CDF. *[Hint: consider* $Y = \max(|X_1|, \ldots, |X_n|)$*.]*

2. (10 points) Let X and Y be independent random variables with $X \sim \text{Exp}(\lambda)$ and $Y \sim \text{Unif}[a, b]$. Let $U = 2X - 3Y$ and $V = 4X + Y$. Find the joint probability density function for (U, V). Be sure to state the domain of this function.

3. (10 points) Let $X \sim \mathcal{N}(0,1)$ and $Y \sim \mathcal{N}(0,1)$ be independent. Let $Z = X/Y$.

(a) (5 points) Find the conditional probability density $f_{Z|Y}(z \mid Y = y)$.

(b) (5 points) Find the joint probability density $f_{Z,Y}(z, y)$ and use this to find the marginal density $f_Z(z)$. What type of random variable is Z?

4. (10 points) Let X and Y be random variables with joint density

$$f(x,y) = \begin{cases} ce^{-(x+y)}, & \text{if } 0 < x < y < \infty \\ 0, & \text{otherwise.} \end{cases}$$

(a) (2 points) Find the value of c such that f is a density function.

(b) (5 points) What is the conditional density of X given that $Y = y$. Be sure to state the domain of this function.

(c) (3 points) Are X and Y independent? Why or why not?

5. (10 points) True or false? Give a brief justification for each answer.

   (a) Suppose X is a continuous random variable with distribution function F_X. Then there exists a random variable Y with distribution F_Y satisfying $F_Y(y) = 2F_X(y)$.

   (b) Let X and Y be random variables such that $E[XY] = E[X]E[Y]$. Then X and Y are independent.

   (c) If X and Y are independent random variables, then $E[XY] = E[X]E[Y]$.

   (d) If X and Y are independent random variables, then $\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$.

   (e) If the events E_1 and E_2 are independent and the events E_2 and E_3 are independent, then the events E_1 and E_3 are independent.

Discrete Distributions

Distrtibution	PMF	$E[X]$	$\mathrm{Var}(X)$	
Bernoulli(p)	$p(0) = 1 - p$, $p(1) = p$	p	$p(1-p)$	
Binom(n, p)	$p(k) = \binom{n}{k} p^k (1-p)^{n-k}$	np	$np(1-p)$	$(pe^t + 1 - p)^n$
Poisson(λ)	$p(k) = \frac{\lambda^k}{k!} e^{-\lambda}$	λ	λ	$\exp\{\lambda(e^t - 1)\}$
Geometric(p)	$p(k) = (1-p)^{k-1} p$	$1/p$	$\frac{1-p}{p^2}$	

Continuous Distributions

Distrtibution	PDF $f(x)$	CDF $F(x)$	$E[X]$	$\mathrm{Var}(X)$	MGF $M(t)$
Uniform(α, β)	$\frac{1}{\beta - \alpha}$, $\alpha \le x \le \beta$		$\frac{\alpha+\beta}{2}$	$\frac{(\beta-\alpha)^2}{12}$	
$N(\mu, \sigma^2)$	$\frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}$	$\Phi(x)$: see table	μ	σ^2	$\exp(\mu t + \sigma^2 t^2/2)$
Exp(λ)	$\lambda e^{-\lambda x}$, $x \ge 0$	$1 - e^{-\lambda x}$, $x \ge 0$	$1/\lambda$	$1/\lambda^2$	$\frac{\lambda}{\lambda - t}$