Math 13 - Week 4: Induction and the Pigeonhole Principle

1. Let $n \in \mathbb{N}$. Prove that there exist positive integers a and b, with $a \neq b$, such that $n^a - n^b$ is divisible by 10. For example, if $n = 17$, then

$$17^6 - 17^2 = 24137569 - 289 = 24137280.$$

Hint: Consider the numbers $n^1, n^2, \ldots, n^{11}$ and look at the ones digits.

2. Prove that the sum of the first n odd natural numbers is n^2.

3. What is wrong with the following proof?

Theorem 1. $\frac{d}{dx}x^n = 0$ for all $n \geq 0$.

Proof. We proceed by induction on n. For the base case, $n = 0$, we have

$$\frac{d}{dx}x^0 = \frac{d}{dx}1 = 0. \tag{1}$$

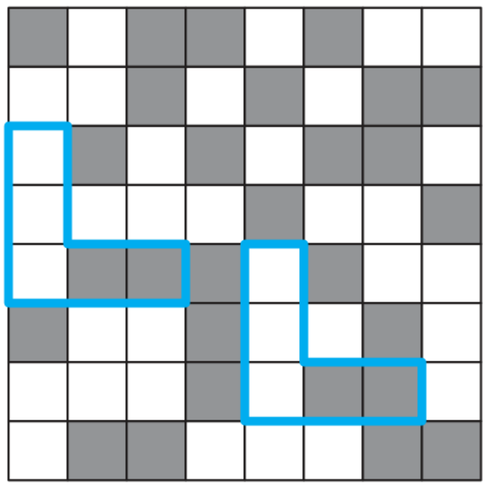
For the inductive step, we assume that $\frac{d}{dx}x^k = 0$ for all $k \leq n$. Then by the product rule,

$$\begin{aligned}
\frac{d}{dx}x^{n+1} &= \frac{d}{dx}(x^n \cdot x) && (2)\\
&= x^n \frac{d}{dx}x^1 + x^1 \frac{d}{dx}x^n && (3)\\
&= x^n \cdot 0 + x^1 \cdot 0 && (4)\\
&= 0. && (5)
\end{aligned}$$

$\square$

4. The squares of an 8×8 chess board are colored black or white. For this problem, an *L-region* is a collection of 5 squares in the shape of a capital L. Such a region includes a square (the lower corner of the L) together with the two squares above and the two squares to the right. Two L-regions are shown in the figure.

   Prove that no matter how we color the chess board, there must be two L-regions that are colored identically (as illustrated by the two L-regions in the figure).



5. (Hard) Let n be a positive integer. Prove that for every square on a $2^n \times 2^n$ chess board, there is a tiling by L-triominoes (three squares that form a (possibly rotated) L-shape) of the remaining $4^n - 1$ squares.