Math 13 - Week 9: Equivalence Relations

1. Suppose A is an infinite set such that $|A| < |\mathbb{N}|$.

   (a) Argue that there exists an injective function $f : A \to \mathbb{N}$.

   (b) Let the elements in the image of f in increasing order

   $$\text{Im}(f) = \{n_1, n_2, \ldots\}.$$

   Prove that the image is infinite.

   (c) Show that for all $k \in \mathbb{N}$, there is a unique $a_i \in A$ such that $f(a_k) = n_k$.

   (d) Define $g : \mathbb{N} \to A$ by $g(k) = a_k$. Prove that g is a bijection.

   (e) Why do we obtain a contradiction?

2. What, if anything, is wrong with these proofs?

   (a) We prove that the sum of the first n natural numbers is $\frac{n(n+1)}{2}$. For $P(1)$ we have $1 = \frac{1 \cdot 2}{2}$. Now assume that $k = n$ is true. Then we have

   $$1 + 2 + \cdots + n + (n+1) = P(n) + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{(n+1)(n+2)}{2},$$

   so $k = n + 1$ is true.

   (b) Suppose a and b are integers with $a \equiv 1 \pmod 3$ and $b \equiv 2 \pmod 3$. Then $(a + b) \equiv 0 \pmod 3$.

   Proof. Since $a \equiv 1 \pmod 3$, then $a = 3k + 1$ for some integer k. Similarly, $b = 3k + 2$. We then have $(a + b) = (3k + 1) + (3k + 2) = 6k + 3 = 3(2k + 1)$, which is divisible by 3. Hence, $a + b \equiv 0 \pmod 3$. □

   (c) If there's a surjective function f :

   Proof. Assume for the sake of contradiction that $1 \leq \frac{1}{a}$. Since $a > 1$, we can divide both sides of this equation to obtain $1 > 1/a$, which contradicts our assumption that $1 \leq 1/a$. □