Math 130B - Homework 1

1. Two fair dice are rolled. Let X be the smallest and Y be the largest value obtained on the dice.

   (a) Find the probability mass functions of X and Y.

   (b) Find the joint probability mass function of X and Y.

   (c) Are X and Y independent?

2. A box of 8 light bulbs is known to contain 2 that are defective. The bulbs are tested one at a time until the defective ones are found. Let N_1 be the number of tests done until the first defective bulb is found, and let N_2 be the same for the second bulb.

   (a) Find the joint probability mass function of N_1 and N_2.

   (b) Are N_1 and N_2 independent?

3. Let $X_1, X_2, \ldots$ be a sequence of independent uniform $[0,1]$ random variables. For a fixed constant $c \in [0,1]$, define the random variable N by

$$N = \min\{n : X_n > c\}.$$

   (a) Explain in just a few words how N relates to the X_i's and c.

   (b) Is N independent of X_N? Give an intuitive explanation as well as a rigorous one.

4. The joint probability density function of X and Y is given by

$$f(x,y) = c(y^2 - x^2)e^{-y} \qquad -y \le x \le y,\ 0 < y < \infty.$$

   (a) Find c.

   (b) Find the marginal densities of X and Y.

   (c) Are X and Y independent?

   (d) Find $\mathbb{E}[X]$.

5. The joint density function of X and Y is

$$f(x,y) = x + y \qquad 0 < x < 1,\ 0 < y < 1.$$

   (a) Are X and Y independent?

   (b) Find $\Pr[X + Y < 1]$.

6. Consider a stick of unit length. Break the stick at two points chosen uniformly and independently at random. What is the probability that you can form a triangle from the three pieces that remain?

7. Consider independent trials, each of which results in outcome i, where i can be $0, 1, \ldots, k$, with probability p_i.

   Let N denote the number of trials needed to obtain an outcome not equal to 0 and let X be that outcome.

   (a) Find $\Pr[N = n]$, $n \geq 1$.

   (b) Find $\Pr[X = j]$, $j = 1, \ldots, k$.

   (c) Show that $\Pr[N = n, X = j] = \Pr[N = n] \Pr[X = j]$.

   (d) Is it intuitive that X and N are independent?

8. (a) A biased coin with head probability p is flipped n times. Let X be the number of heads and Y the number of tails. Are X and Y independent?

   (b) Suppose instead that the coin is tossed a *random* number of times N, where $N \sim Poisson(\lambda)$. Show that X and Y are independent.

9. Let X_1 and X_2 be independent exponential random variables with respective parameters λ_1 and λ_2.

   (a) Find the distribution of $Z = X_1/X_2$.

   (b) Compute $\Pr[X_1 < X_2]$.

10. Suppose that P and Q are independent random variables, both uniformly distributed on $(0, 1)$.

    (a) What is the joint cumulative distribution function of P and Q?

    (b) What is the probability that all of the roots of the equation $x^2 + Px + Q = 0$ are real?