Math 13 - Week 7: Number Theory

1. Prove that a number is a perfect square if and only if it has an odd number of divisors.

2. A set S and a relation $\sim$ on S is given. For each example, check if $\sim$ is reflexive, symmetric, and/or transitive. If the relation is an equivalence relation, describe the equivalence classes.

   (a) $S = \mathbb{R}$ where $a \sim b$ if and only if $a \leq b$.

   (b) $S = \mathbb{R}$ where $a \sim b$ if and only if $|a| = |b|$.

   (c) $S = \mathbb{Z}$ where $a \sim b$ if and only if $a \mid b$.

   (d) $S = 2^{\mathbb{N}}$ where $A \sim B$ if and only if $A \subseteq B$.

3. Is the operation $a \oplus b = a^b$ well-defined for $a, b \in \mathbb{Z}_n$?

4. Let $S = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$. Define the relation $\sim$ on S where $(a, b) \sim (c, d)$ if and only if $ad = bc$.

   (a) Is $(1, 5) \sim (-3, -15)$?

   (b) Is $(-1, 1) \sim (2, 3)$?

   (c) Prove that $\sim$ is an equivalence relation.