Math 130B - Homework 4

1. Let $X_1, X_2, \ldots$ be a sequence of independent random variables with identical distribution. Let their mean be μ and their variance be σ^2. Define $S_n = X_1 + \cdots + X_n$. For any $a > 0$, compute

$$\lim_{n\to\infty} \Pr[\mu \leq S_n \leq \mu + a].$$

2. Let X be a geometric random variable with parameter p.

   (a) Compute the moment generating function, $M(t)$ of X. Be sure to specify for which values of t the moment generating function is finite.

   (b) Use the moment generating function to compute the mean and variance of X.

3. You have a lottery ticket. There are four possible outcomes, each of which has probability $1/4$ of occurring.

   (i) You win nothing;

   (ii) You win \$5 and another lottery ticket;

   (iii) You win two lottery tickets;

   (iv) You win \$10.

   Compute the expected value of your total winnings. Assume that the lottery tickets that can be won in outcomes (ii) and (iii) have the same properties as the one you started with.

4. Let X and Y be independent uniform random variables distribute don $[0, 3]$ and $[1, 3]$. What is the probability that $Y > X^2 - 1$?

5. Consider two components and three types of electrical shock. A type 1 shock causes component 1 to fail. A type 2 shock causes component 2 to fail, and a type 3 shock causes both components 1 and 2 to fail. The times until shocks 1, 2, and 3 occur are independent exponential random variables with respective rates λ_i for $i = 1, 2, 3$. Let X_i denote the time at which component i fails. Find $\Pr[X_1 > s, X_2 > t]$.

6. Let $f(x, y) = \frac{e^{-x/y}e^{-y}}{y}$ for $x, y \geq 0$. Compute the probability that $X > 1$ given that $Y = y$ for some y.

7. (a) Let X be a discrete random variable whose possible values are $1, 2, \ldots$. If $\Pr[X = k]$ is nonincreasing in k, prove that

$$\Pr[X = k] \leq 2\frac{E[X]}{k^2}.$$

   (b) Let X be a nonnegative continuous random variable having a nonincreasing density function. Show that

$$f(x) \leq \frac{2E[X]}{x^2}, \quad \text{for all } x > 0.$$

8. Prove or provide a counterexample.

   (a) If X and Y are two independent random variables with the same distribution, then $\mathrm{Cov}(X+Y, X-Y) = 0$.

   (b) If $X_1, X_2, \ldots$ is a sequence of independent and identically distributed random variables with $E[X_i] = 0$, then

$$\lim_{n\to\infty} \Pr\left[\frac{1}{n}|X_1 + \cdots + X_n| \le 0.01\right] = 1.$$

9. Let $x_i \ge 0$ for $1 \le i \le n$ and assume $x_1 + \cdots + x_n = 1$. Prove that

$$-\sum_{i=1}^{n} x_i \ln(x_i) \le \ln n.$$

   Hint: Introduce a random variable X which takes the values $x_1, \ldots, x_n$ each with probability $1/n$.

10. Let X and Y be independent random variables with $X \sim \mathrm{Pois}(\lambda)$ and $Y \sim \mathrm{Bern}(p)$ with $0 < p < 1$. Define the random variable $Z = XY$.

    (a) Determine the probability mass function p_Z of Z.

    (b) Find the moment generating function $M_Z(t)$ of Z.

    (c) Compute the covariance $\mathrm{Cov}(Y, Z)$ and correlation $\rho(Y, Z)$.