Homework 1/Math205A

- Real numbers and related subjects

1. Prove $\sqrt{3}$ is not a rational number

2. If $r \neq 0$ is a rational number and q is a irrational number (not rational real number). Prove $r+q$ and rq are irrational.

3. Let $\mathbb{Q}^c = \mathbb{R} \setminus \mathbb{Q}$ be the set of all irrational numbers. Prove $\mathbb{Q}^c$ is dense in $\mathbb{R}$.

4. Let A be a nonempty set of real numbers which is bounded from below. Let $-A = \{-x : x \in A\}$. Prove $\inf A = -\sup(-A)$.

5. Fix $b > 1$. Complete the following problems:
(a) If $m, n, p, q \in \mathbb{Z}$ (integers) with $n, q > 0$ and $r = m/n = p/q$, prove

$$(b^m)^{1/n} = (b^p)^{1/q}.$$

Hence it makes sense to define $b^r = (b^m)^{1/n}$.
(b) Prove that $b^{r+s} = b^r b^s$ if r and s are rational.
(c) If x is real, define $B(x) = \{b^t : t \text{ is rational and } t \leq x\}$. Prove that $b^r = \sup B(r)$ when r is rational. Hence it makes sense to define

$$b^x = \sup B(x), \quad x \in \mathbb{R}$$

(d) Prove $b^{x+y} = b^x b^y$ for all $x, y \in \mathbb{R}$.

6. For $b > 1$ and $y > 0$. Prove that there is a unique $x \in \mathbb{R}$ so that $b^x = y$.

7. If x and y are complex numbers. Prove that

$$\Big||x| - |y|\Big| \leq |x - y|$$

8. Use mathematics induction to prove: If $z_1, \cdots, z_n$ are complex number, then

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|.$$

9. Prove that

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

if $x = (x_1, \cdots, x_k), y = (y_1, \cdots, y_k) \in \mathbb{R}^k$. Interpret this geometrically, as a statement about parallelograms.