Math 13 - Week 5: Binomial Coefficients

1. Let n be an integer with $n \geq 2$. Prove that

$$\binom{n}{2} = 1 + 2 + 3 + \cdots + (n-1) = \sum_{k=1}^{n-1} k.$$

   Try to come up with two different ways to prove it.

2. For integers n and k with $0 \leq k \leq n$, prove that

$$\binom{n}{k} = \binom{n}{n-k}.$$

3. Consider a grid that is l units long and w units wide. Starting from the lowest-left square, how many paths are there to the upper-right square in which each step either goes one unit to the right or one unit vertically?

4. Use the binomial theorem to prove

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots \pm \binom{n}{n} = 0,$$

   provided $n > 0$. Move all the negative terms to the right-hand side to deduce

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots.$$

   Give a description of what this means in terms of subsets of an n-element set and prove it using counting.

5. Prove that a set with n elements has exactly 2^n subsets. Give as many proofs as you can for this.