Math 13 - Week 4: More on Induction and the Pigeonhole Principle

1. Prove the following equations by induction. In each case, n is a positive integer.

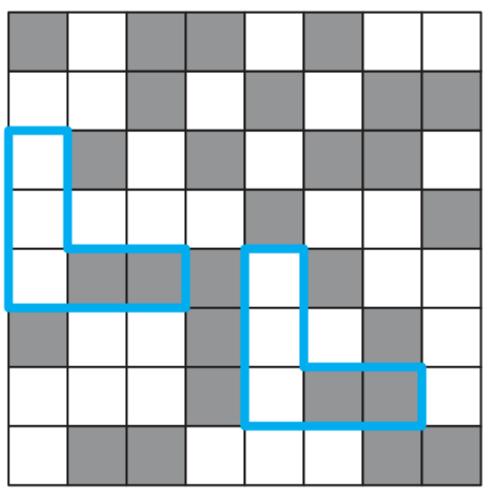
   (a) $1 + 4 + 7 + \cdots + (3n-2) = \frac{n(3n-1)}{2}$.

   (b) $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}$.

   (c) $\lim_{x\to\infty} \frac{x^n}{e^x} = 0$.

2. The squares of an 8×8 chess board are colored black or white. For this problem, an *L-region* is a collection of 5 squares in the shape of a capital L. Such a region includes a square (the lower corner of the L) together with the two squares above and the two squares to the right. Two L-regions are shown in the figure.

   Prove that no matter how we color the chess board, there must be two L-regions that are colored identically (as illustrated by the two L-regions in the figure). (*Hint: How many ways are there to color an L-region?*)



3. Prove that $5^n - 1$ is divisible by 4 for every positive integer n.

4. What is wrong with the following proof?

Theorem 1. For every nonnegative integer n, $5n = 0$.

Proof. We proceed by strong induction on n. For the base case, $n = 0$, we have

$$5n = 5 \cdot 0 = 0. \tag{1}$$

For the inductive step, let k be a nonnegative integer and suppose that $5n = 0$ for all n in the range $0 \leq n \leq k$. Write $k + 1 = i + j$ where i and j are integers satisfying $0 \leq i, j \leq k$. We then apply the induction hypothesis to i and j:

$$5(k + 1) = 5(i + j) = 5i + 5j = 0 + 0 = 0. \tag{2}$$

By induction, $5n = 0$ for all nonnegative integers n. □

5. Recall that the Fibonacci numbers are defined by $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for all $n \geq 1$.

Prove that $F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$ for all $n \geq 1$.