Lecture 1: Introduction and Dynamic Programming

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Four Components of a Quantitative Project

1. Model specification:
   - Preferences, technology, demographic structure, equilibrium concept, frictions, driving forces, etc.
2. Numerical solution:
   - Programming language, algorithms, accuracy vs speed, etc.
3. Calibration/Estimation:
   - Simulation-based estimation, global optimization
4. Analyzing the solved model:
   - Policy experiments/counterfactuals, welfare analysis, transitions, etc.

This Class: 2 & 3

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   - 1-2 discrete state variables
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3. An **estimation/calibration problem** with 5 to 15 parameters by matching moments
   - where moments can have kinks or jumps in parameters
   - the objective is likely to have multiple local minima (sometimes hundreds of them)

A Word about Programming Languages

- Choice of programming language is critical for successfully solving problems like the one above.
- Three (broad) types of programming languages
  - Low-level/Compiled languages: Fortran, C/C++
  - High level/Interpreted languages: Matlab, Python, R, Stata, etc.
  - High-level language with option to compile: Julia.

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  - High-level language with option to compile: Julia.
- One important difference: **Speed!**
- In scientific disciplines where computational demands are high, compiled languages are much more popular.
- Julia is a great option: A more modern language that can be fast if you know how to optimize it. But it requires work & experience to make use of its speed. (Still not as fast as C/Fortran though)

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- **Important note**: Linux/Mac are much more efficient at memory management than Windows. So, for large problems with *very* large data objects (like large matrices or arrays), your code can run **much faster** using the former.

Dynamic Programming: Introduction

GOAL: Solve the Bellman Equation

$$V(k,z) = \max_{c,k'} \left[u(c) + \beta \mathbb{E}(V(k',z')|z)\right]$$

$$c + k' = (1+r)k + z$$

$$z' = \rho z + \eta \qquad\qquad \eta \overset{i.i.d.}{\sim} F(.)$$

- Solution involves finding unknown functions: $c(k,z), k'(k,z), V(k,z)$

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Three Key Questions:

1. Does a solution exist?
2. If so, is the solution unique?
3. If the answers to (1) and (2) are yes: how do we find this solution?

Contraction Mapping Theorem

- **Definition (Contraction Mapping)** Let (S, d) be a metric space and $T : S \to S$ be a mapping of S into itself. T is a contraction mapping with modulus β, if for some $\beta \in (0, 1)$ we have

$$d(Tv_1, Tv_2) \leq \beta d(v_1, v_2)$$

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- **Contraction Mapping Theorem:** Let (S, d) be a complete metric space and suppose that $T : S \to S$ is a contraction mapping. Then, T has a unique fixed point $v^* \in S$ such that

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- The beauty of CMT is that it is a *constructive* theorem: it not only tells us the existence/uniqueness of v^* but it also shows us how to find it!

Qualitative Properties of v^*

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- **Corollary:** Let (S, d) be a complete metric space and $T : S \to S$ be a contraction mapping with $Tv^* = v^*$.
  a. If $\overline{S}$ is a <u>closed</u> subset of S, and $T(\overline{S}) \subset \overline{S}$, then $v^* \in \overline{S}$.
  b. If, in addition, $T(\overline{S}) \subset \overline{\overline{S}} \subset \overline{S}$, then $v^* \in \overline{\overline{S}}$.

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  b. If, in addition, $T(\overline{S}) \subset \overline{\overline{S}} \subset \overline{S}$, then $v^* \in \overline{\overline{S}}$.
- $\overline{\overline{S}} = \{$continuous, bounded, strictly concave$\}$. Not a complete metric space. $\overline{S} = \{$continuous, bounded, weakly concave$\}$ is.
  - So we need to be able to establish that T maps elements of $\overline{S}$ into $\overline{\overline{S}}$.

A Prototype Problem

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\begin{aligned}
V(k,z) = \max_{c,k'} &\left[u(c) + \beta \int V(k',z') f(z'|z) dz' \right] \\
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\end{aligned}
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- How to evaluate the <u>conditional expectation</u> (integral)?
- How to do <u>constrained optimization</u> (esp. in more than one dimension)?
- There are quick-and-dirty methods that are slow and inaccurate, and advanced methods that are fast and accurate. To do any kind of ambitious work, you will need the latter.

Simple Analytical Example

Let's Start with a Simple Analytical Example

Neoclassical Growth Model

- Consider the special case with log utility, Cobb-Douglas production and full depreciation:

$$\begin{aligned} V(k) &= \max_{c,k'} \{\log c + \beta V(k')\} \\ \text{s.t} \quad c &= Ak^{\alpha} - k' \end{aligned}$$

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- Our goal is to find $V(k)$ and a decision rule g such that $k' = g(k)$

I. Backward Induction (Brute Force)

- If $t = T < \infty$, in the last period we would have: $V_0(k) \equiv 0$ for all k. Therefore:

$$V_1(k) = \max_{k'} \left\{ \log\left(Ak^{\alpha} - k'\right) + \underbrace{\beta V_0(k')}_{\equiv 0} \right\}$$

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$$V_2 = \max_{k'} \left\{ \log\left(Ak^{\alpha} - k'\right) + \beta \underbrace{\left(\log A + \alpha \log k'\right)}_{V_1(k)} \right\}$$

$$\Rightarrow \text{FOC}: \qquad \frac{1}{Ak^{\alpha} - k'} = \frac{\beta\alpha}{k'} \qquad \Rightarrow \qquad k' = \frac{\alpha\beta}{1+\alpha\beta} \times A \times k^{\alpha}$$

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- **Substitute k' to obtain V_2.** We can keep iterating to find the solution...

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II. Guess and Verify (Value Function)

- Let $LHS = a + b \log k$. Plug in the expression for k' into the RHS:

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RHS &= \log\left(Ak^{\alpha} - \frac{\beta b}{1+\beta b}Ak^{\alpha}\right) + \beta\left(a + b\log\left(\frac{\beta b}{1+\beta b}Ak^{\alpha}\right)\right) \\
&= \underbrace{(1+\beta b)\log A + \log\left(\frac{1}{1+\beta b}\right) + a\beta + b\beta\log\left(\frac{\beta b}{1+\beta b}\right)}_{\textbf{CONSTANT!}} \\
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- **Imposing the condition** that $LHS \equiv RHS$ for all k, we find a and b :

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- We have solved the model in one step!

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- Then we can apply the same logic as above and solve for the unknown coefficients, which then gives us the complete solution.
- Many solution methods rely on various versions of this general idea! (perturbation methods, collocation methods, parametrized expectations, Krusell-Smith, etc.).

III. Guess and Verify (Policy Functions)

- Let the policy rule for savings be: $k' = g(k)$. The Euler equation is:

$$\frac{1}{Ak^{\alpha} - g\left(k\right)} - \frac{\beta\alpha A\left(g\left(k\right)^{\alpha-1}\right)}{A\left(g\left(k\right)^{\alpha} - g\left(g\left(k\right)\right)\right)} = 0 \quad \text{for all } k.$$

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- Guess $g(k) = sAk^{\alpha}$, and substitute above:

$$\frac{1}{(1-s)Ak^{\alpha}} = \frac{\beta\alpha A(sAk^{\alpha})^{\alpha-1}}{A((sAk^{\alpha})^{\alpha} - sA(aAk^{\alpha})^{\alpha})}$$

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- As can be seen, k cancels out, and we get $s = \alpha\beta$.

III. Guess and Verify (Policy Functions)

- Let the policy rule for savings be: $k' = g(k)$. The Euler equation is:

$$\frac{1}{Ak^{\alpha} - g(k)} - \frac{\beta\alpha A\left(g(k)^{\alpha-1}\right)}{A(g(k)^{\alpha} - g(g(k)))} = 0 \quad \text{for all } k.$$

  which is a functional equation in $g(k)$.

- Guess $g(k) = sAk^{\alpha}$, and substitute above:

$$\frac{1}{(1-s)Ak^{\alpha}} = \frac{\beta\alpha A(sAk^{\alpha})^{\alpha-1}}{A((sAk^{\alpha})^{\alpha} - sA(aAk^{\alpha})^{\alpha})}$$

- As can be seen, k cancels out, and we get $s = \alpha\beta$.

- By using a very flexible choice of $g()$ this method too can be used for solving very general models.

Numerical Value Function Iteration (VFI)

Standard VFI

- Standard Value Function Iteration is simply the application of the Contraction Mapping Theorem

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Algorithmus 1 : Standard Value Function Iteration

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1. Set $n = 0$. Choose an initial guess $V_0 \in S$.
2. Obtain V_{n+1} by applying the mapping: $V_{n+1} = TV_n$, which entails (i) maximizing the right-hand side of the Bellman equation and (ii) then plugging in the decision rule obtained into the RHS.
3. Stop if convergence criteria satisfied: $|V_{n+1} - V_n| < \texttt{toler}$. Otherwise, increase n and return to step 2.

- We will call (i) the maximization step, and (ii) the evaluation step.

Quick Digression: How To Represent $V(k, z)$ on a Computer?

- A value function with continuous state variables—e.g., $V(k, z)$—is an infinite-dimensional object.

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- Limits number of continuous state variables you can use.
- **Better idea:** Define $V(k, z) := V(k_i, z_j)$ for $i = 1, 2, ..., I$ and $j = 1, 2, ..., J$ + an interpolation method for all off-grid points.

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- Another approach would be to use parametric families of analytical functions:
  - Polynomials (Chebyshev, Hermite, Legendre, etc.), or combinations of power functions, logs, exponentials, etc.

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- We will talk more about this next week (Interpolation).
- **Bottom line:** We will think of all continuous functions below as:
  - (i) a collection of discrete points on a grid +
  - (ii) an interpolation method for all off-grid points.

Now: Apply VFI to Neoclassical Growth Model

Consider the neoclassical growth model:

$$
\begin{aligned}
V(k,z) &= \max_{c,k'} \left\{ \frac{c^{1-\gamma}}{1-\gamma} + \beta \mathbb{E}\left(V(k',z')\,|z\right) \right\} \\
\text{s.t} \quad c + k' &= e^{z} k^{\alpha} + (1-\delta)k \\
z' &= \rho z + \eta', \qquad k' \geq \underline{k}.
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- VFI: **Maximization step:** First, **maximize** the RHS, and solve for the policy rule (for $i = 1, \ldots, I$;and $j = 1, \ldots, J$:

$$\tilde{s}_n(k_i, z_j) = \arg\max_{s \geq \underline{k}} \left\{ \frac{\left(e^{z_j} k_i^{\alpha} + (1-\delta)k_i - s\right)^{1-\gamma}}{1-\gamma} + \beta \mathbb{E}\left(V_n(s,z')\,|z_j\right) \right\}. \tag{1}$$

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- VFI: **Evaluation step:** Plug $\tilde{s}_n(k_i, z_j)$ into eq. (1):

$$V_{n+1}(k_i, z_j) = \frac{\left(e^{z_j} k_i^{\alpha} + (1-\delta)k_i - \tilde{s}_n(k_i, z_j)\right)^{1-\gamma}}{1-\gamma} + \beta \mathbb{E}\left(V_n(\tilde{s}_n(k_i, z_j), z')\,|z_j\right). \tag{2}$$

VFI is (Very!) Slow. How to Speed It Up?

- Recall that for a contraction mapping: $d(Tv_1, Tv_2) \leq \beta d(v_1, v_2)$.
- So when $\beta \approx 1$ (say $\beta = 0.99$ or 0.999), VFI can be **very slow**.
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  3. **Endogenous Grid Method** (EGM).
- In general, basic VFI should never be used without at least one of these add-ons.
  - **EGM is your best bet when it's applicable**. But in certain cases, it's not.
  - In those cases, a combination of Howard's algorithm and MQP can be very useful.

Speed Boost 1: Howard's Policy Iteration

Howard's Policy Iteration

- Howard's Policy iteration follows from **two key observations** about VFI:
  - The maximization step (eq. 1) is **typically much more costly** (in computational time) than the evaluation step (eq. 2).
  - But the latter uses the updated decision rule, $\tilde{s}_n(k_i, z_j)$, **for only one period** (since savings decisions after tomorrow is embedded in V_n on the RHS).
- **Policy Iteration**: Repeat the evaluation step multiple times between each maximization step.
- **Definition:** For a given value function J and a decision rule w, define **Howard's mapping** $\widetilde{T}$ as the operator that "plugs in w_n" into the RHS of the Bellman equation:

$$\widetilde{T}_w J(k_i, z_j) = \frac{\left(e^{z_j} k_i^{\alpha} + (1-\delta)k_i - w(k_i, z_j)\right)^{1-\gamma}}{1-\gamma} + \beta \mathbb{E}\left(J\left(w(k_i, z_j), z'\right) | z_j\right) \tag{3}$$

Howard's Policy Iteration: Cont'd

- We will be interested in applying the Howard mapping repeatedly, so for $m = 1, \ldots, M$, let

$$J_{m+1} \equiv \widetilde{T}_w J_m(k_i, z_j) \tag{4}$$

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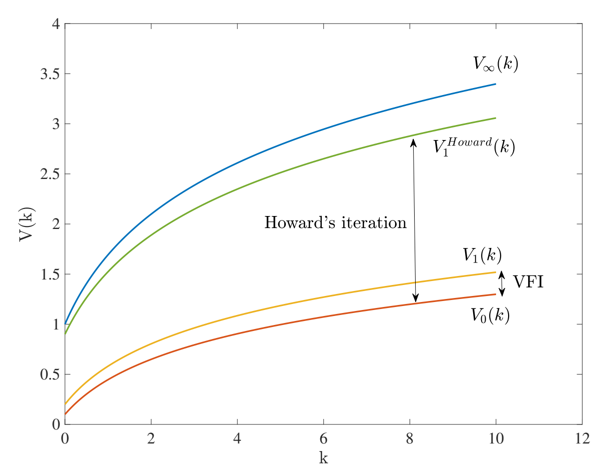
- But it is an operator that is much cheaper to apply. So, it seems worth applying more than once.

Algorithmus 2 : VFI with Policy Iteration Algorithm

1. Set $n = 0$. Choose an initial guess $V_0 \in S$.
2. Maximization Step: Obtain $\tilde{s}_n$ as in (3).
3. **Howard Step**: Set $J_0 \equiv V_n$, and iterate on $J_{m+1} \equiv \widetilde{T}_{\tilde{s}_n} J_m(k_i, z_j)$ for $m = 0, 1, 2, \ldots$ Then, set $V_{n+1} = J_* \equiv \lim_{m\to\infty} J_m$
4. Stop if convergence criteria satisfied: $|V_{n+1} - V_n| < \texttt{toler}$. Otherwise, increase n and return to step 2.

- Note: It is often possible to obtain the fixed point J_* in a finite number of steps.

VFI vs Howard's Algorithm



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- **Good news:** *Potentially* very fast convergence.
- **Bad news:** No more global convergence like Standard VFI (unless state space is discrete)

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- The basic issue is that **Howard's algorithm is a bit too ambitious** (or greedy). With extreme speed comes instability.
- We can alleviate these problems by slightly **modifying the algorithm to tone it down.**

VFI with Modified Policy Iteration (MPI) Algorithm

- Modify *Step 3* of Howard's algorithm above:
  - **Modified Howard Step**: Set $J_0 \equiv V_n$, and iterate on $J_{m+1} \equiv \widetilde{T}_{\tilde{s}_n} J_m(k_i, z_j)$ for $m = 0, 1, 2, \ldots, M < \infty$. Choose a moderate value for M (by experimentation and smaller for more challenging problems). Then, set $V_{n+1} = J_M$.

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  - HW #1 asks you to experiment to see the tradeoffs.
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- The choice of m will be a key decision to make.
  - HW #1 asks you to experiment to see the tradeoffs.
  - We will also see some benchmarking results in Lecture 4 to help guide this choice.
- Note: In some cases we will see later, the iteration will be unstable or will not converge smoothly. In such cases, it will be optimal to **slow down** (or **dampen**) rather than accelerate the Bellman iteration (effectively $m < 1$). This is how $\rightarrow$

Dampened VFI Algorithm

Modify Step 2 of the VFI algorithm as follows:

2*. Obtain J_{n+1} from V_n by applying the standard *Bellman mapping*:

$$J_{n+1} = TV_n,$$

(i.e., maximize RHS of the Bellman equation and evaluate with the new optimal policy.)

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3*. Obtain V_{n+1} by taking a **convex combination** of J_{n+1} and V_n :

$$V_{n+1} = \theta J_{n+1} + (1-\theta) V_n \quad \text{with} \quad \theta \in (0,1].$$

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- Note: VFI corresponds to $\theta = 1$.

Speed Boost 2:

MacQueen-Porteus Bounds

Error Bounds: Background

- In iterative numerical algorithms, we need a stopping rule.
- In dynamic programming, we want to know how far we are from the true solution in each iteration.

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- In iterative numerical algorithms, we need a stopping rule.
- In dynamic programming, we want to know how far we are from the true solution in each iteration.
- **Contraction Mapping Theorem** can be used to show:

$$\|V^* - V_k\|_\infty \leq \frac{1}{1-\beta} \|V_{k+1} - V_k\|_\infty .$$

- So if we want to stop when the value function is ε away from the true solution, our stopping criterion is:

$$\|V_{k+1} - V_k\|_\infty < \varepsilon \times (1-\beta).$$

Two Remarks

1. The CMT bound is for the worst case scenario (sup-norm). If V^* varies over a wide range, this bound will (typically) be misleading—too pessimistic.
   - Consider $u(c) = \frac{c^{1-\alpha}}{1-\alpha}$ with $\alpha = RRA = 10$. V will cover an enormous range of values. Bound will be too pessimistic.

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- One way to address both issues is by defining the stopping rule in the policy function space:
  - It is typically easier to judge what it means to consume or save $x\%$ less than optimal (caution: we will see exceptions!)
  - **Also:** Policy functions converge faster than values, so this typically allows stopping sooner.

MacQueen-Porteus Bounds

Consider this alternative formulation of a dynamic programming problem:

$$V(x_i) = \max_{y \in \Gamma(x_i)} \left[U(x_i, y) + \beta \sum_{j=1}^{J} \pi_{ij}(y) V(x_j) \right], \tag{5}$$

- State space is discrete.
- But choices are continuous.
- Allows for simple modeling of interesting problems.
- Popular formulation in other fields using dynamic programming.
  - See, e.g., Bertsekas and Shreve (1978) which is a wonderful book on DP, or Bertsekas and Ozdaglar (2009) for a more up to date comprehensive treatment.

Theorem 1
[MacQueen-Porteus bounds] *Consider*

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define

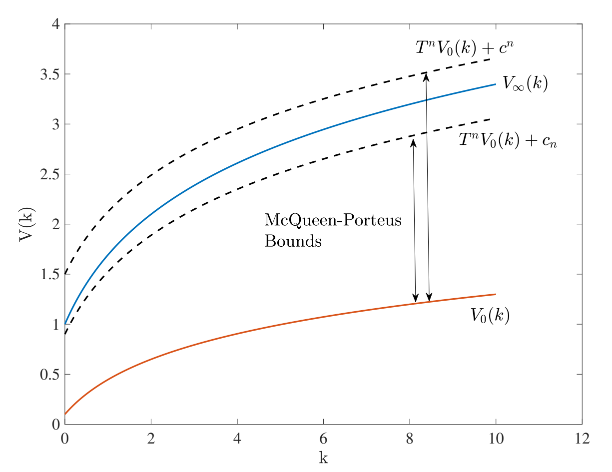
$$\underline{c}_n \equiv \frac{\beta}{1-\beta} \times \min\left[V_n - V_{n-1}\right] \qquad \overline{c}_n \equiv \frac{\beta}{1-\beta} \times \max\left[V_n - V_{n-1}\right] \tag{7}$$

Then, for all $\overline{x} \in X$, we have:

$$T^n V_0(\overline{x}) + \underline{c}_n \leq V^*(\overline{x}) \leq T^n V_0(\overline{x}) + \overline{c}_n. \tag{8}$$

Furthermore, *with each iteration, the two bounds approach the true solution* **monotonically.**

VFI versus McQueen-Porteus Bounds



MQP Bounds: Comments

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- Example: Suppose $V_n(\overline{x}) - V_{n-1}(\overline{x}) = \alpha$ for all $\overline{x}$ and that $\alpha = 100$ (a large number).

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- CMT bound implies: $\|V^* - V_n\|_\infty \leq \frac{1}{1-\beta}\|V_n(\overline{x}) - V_{n-1}(\overline{x})\|_\infty = \frac{\alpha}{1-\beta}$, so we would keep iterating.
- MQP implies $\underline{c}_n = \overline{c}_n = \alpha$, which the then implies

$$\frac{\alpha\beta}{1-\beta} = V^*(\overline{x}) - T^n V_0(\overline{x}) = \frac{\alpha\beta}{1-\beta}.$$

MQP Bounds: Comments

- MQP bounds can be quite tight.
- Example: Suppose $V_n(\overline{x}) - V_{n-1}(\overline{x}) = \alpha$ for all $\overline{x}$ and that $\alpha = 100$ (a large number).
- CMT bound implies: $\|V^* - V_n\|_\infty \leq \frac{1}{1-\beta} \|V_n(\overline{x}) - V_{n-1}(\overline{x})\|_\infty = \frac{\alpha}{1-\beta}$, so we would keep iterating.
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- We find $V^*(\overline{x}) = V_n(\overline{x}) + \frac{\alpha\beta}{1-\beta}$, in one step!
- **MQP** provides both lower and upper bound for signed difference.

VFI Stopping Rule with MQP Bounds

Algorithmus 3 : VFI Stopping Rule with MQP Error Bounds

[*Step 3'*:] Stop when $\overline{c}_n - \underline{c}_n < \texttt{toler}$. Then take the final estimate of V^* to be either the median

$$\tilde{V} = T^n V_0 + \left(\frac{\overline{c}_n + \underline{c}_n}{2}\right)$$

or the mean (i.e., average error bound across states):

$$\hat{V} = T^n V_0 + \frac{\beta}{n(1-\beta)} \sum_{i=1}^{n} \left(T^n V_0(\overline{x}_i) - T^{n-1} V_0(\overline{x}_i)\right).$$

Algorithmus 4 : VFI Acceleration with MQP Error Bounds

[*Step 2'*:] After every m iteration (for e.g., $m = 1, 2, ..., M$) on the VFI algorithm, instead of the usual VFI updating $V_{n+1} = TV_n$, take one "big" MQP step by setting:

$$V_{n+1} = TV_n + \left(\frac{\overline{c}_n + \underline{c}_n}{2} \right)$$

[*Step 3'*:] Same as above (in VFI Stopping Rule with MQP Bound algorithm)

VFI Acceleration with MQP Bounds

Algorithmus 5 : VFI Acceleration with MQP Error Bounds

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[*Step 3'*:] Same as above (in VFI Stopping Rule with MQP Bound algorithm)

- We will experiment with different values of m, including $m = 1$.

MQP: Convergence Rate

- Bertsekas (1987) derives the convergence rate of MQP bounds algorithm
- It is proportional to the **subdominant eigenvalue** of $\pi_{ij}(y^*)$ (the transition matrix evaluated at optimal policy).

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- VFI is proportional to the **dominant** eigenvalue, which is always 1 (because π is a random matrix). Multiplied by β, gives convergence rate.
- Subdominant (2nd largest) eigenvalue ($|\lambda_2|$) is sometimes $\ll 1$ and sometimes not:
  - AR(1) process, discretized: $|\lambda_2| = \rho$ (persistence parameter)
  - More than 1 ergodic set: $|\lambda_2| = 1$.
- When ρ is low, this can lead to substantial improvements in speed.

Benchmarking MQP and MPI: Parameters

- We will consider:
  - $\beta = 0.95, 0.99, 0.999$
  - $RRA = 1, 5$
  - MPI: $m = 0, 50, 500$
  - MPQ: $m = 1$. (MQP update step in every VFI iteration).
  - $V_0 = 0$ (inefficient choice)

β : time discount factor, m : $\#$ of Howard iterations, γ : relative risk aversion.

Table 1: Mc-Queen Porteus Bounds and Policy Iteration

$\beta \to$	0.95			0.99			0.999		
m :	0	50	500	0	50	500	0	50	500
MQP	(RRA) $\gamma = 1$								
No	14.99	1.07	1.00*	26.48	1.28	1.00*	33.29	1.41	1.00*
Yes	0.32	0.60	0.79	0.10	0.23	0.27	0.01	0.03	0.04
	(RRA) $\gamma = 5$								
No	13.03	0.96	1.00*	26.77	1.28	1.00*	33.37	1.45	1.00*
Yes	0.67	0.67	0.69	0.14	0.24	0.30	0.02	0.04	0.06

*Time normalized to 1 for the Howard run with $m = 500$ and without MQP.

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3. The two algorithms are not additive or even always complements:
   - When **MQP** is used, adding **MPI** *slows down* the solution (notice rising times in second rows)
   - When Howard is used, MQP still speeds up solution but less than before: by as low as 1.5 fold for $\beta = 0.95$ but as high as 25 fold for higher β.

Takeaways (Cont'd)

- **Important note:** These numbers are not written in stone! Your mileage will vary depending on the complexity of the problem and other factors.

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- **To sum up**:
  - a combination of Howard and MQP is a good default to use when EGM is not feasible.
  - MQP especially useful when you lack a good initial guess (as in this example, $V_0 = 0$ is a bad initial guess).
  - Even with EGM, MQP and MPI can help further speed up the code.

Speed Boost 3: Endogenous Grid Method

Endogenous Grid Method

- In standard VFI, we have FOC:

$$c^{-\gamma} = \beta \mathbb{E}\left(V_k\left(k', z'\right)|z_j\right).$$

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- This equation can be rewritten (by substituting out consumption using the budget constraint) as

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- **Slow for three reasons:**

  1. This is a **non-linear** equation in k'.
  2. $V(k_i, z_j)$ is stored at grid points, so **for every trial value of k'** in maximization, we need to:
     2.1 **evaluate the conditional expectation** (since k' appears inside the expectation), and
     2.2 **interpolate** to obtain off-grid values $V(k', z_j')$ for each z_j'.

EGM

- **View the problem differently:**

$$
\begin{aligned}
V\left(k, z_j\right) &= \max_{c,k'} \left\{ \frac{c^{1-\gamma}}{1-\gamma} + \beta \mathbb{E}\left(V\left(k_i', z'\right) | z_j\right) \right\} \\
\text{s.t} \quad c + k_i' &= z_j k^{\alpha} + (1-\delta)k \\
\ln z' &= \rho \ln z_j + \eta',
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- Now the same FOC as before:

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but solve for k as a function of k_i' and z_j :

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z_j k^\alpha + (1-\delta)k = \left[\beta \mathbb{E}\left(V_k(k_i', z') \,|z_j\right)\right]^{-1/\gamma} + k_i'.
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- Problem 1 still remains: LHS **still nonlinear** in k.

EGM

- **Trick 2:** Define

$$Y \equiv zk^{\alpha} + (1-\delta)k \tag{11}$$

and rewrite the Bellman equation (without discretization) as:

$$\mathcal{V}(Y,z) = \max_{k'} \left\{ \frac{(Y-k')^{1-\gamma}}{1-\gamma} + \beta \mathbb{E}\left(\mathcal{V}(Y',z')\,|z\right) \right\}$$

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- **Key observation:** Y' is only a function of k_i' and z', so we can write the conditional expectation on RHS as:

$$\mathbb{V}(k_i', z_j) \equiv \beta\mathbb{E}\left(\mathcal{V}\left(Y'(k_i', z'), z'\right)|z_j\right).$$

EGM

- Plug $\mathbb{V}$ back into modified Bellman Equation:

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  - without searching over values of k'—hence avoiding repeated interpolation and integration!
  - without solving a nonlinear equation in k'
- Once $c^*(k_i', z_j)$ is obtained, use the resource constraint to compute today's end-of-period resources: $Y^*(k_i', z_j) = c^*(k_i', z_j) + k_i'$ as well as

$$\mathcal{V}(Y^*(k_i', z_j), z_j) = \frac{(c^*(k_i', z_j))^{1-\gamma}}{1-\gamma} + \mathbb{V}(k_i', z_j)$$

EGM: The Algorithm

0: Set $n = 0$. Construct a grid for tomorrow's capital and today's shock: (k'_i, z_j). Choose an initial guess $\mathbb{V}^0(k'_i, z_j)$.

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$$c^*(k'_i, z_j) = \left(\mathbb{V}^n_k(k'_i, z_j)\right)^{-1/\gamma}.$$

2: Obtain today's end-of-period resources as a function of tomorrow's capital and today's shock:

$$Y^*(k'_i, z_j) = c^*(k'_i, z_j) + k'_i,$$

and today's updated value function,

$$\mathcal{V}^{n+1}\left(Y^*(k'_i, z_j), z_j\right) = \frac{\left(c^*(k'_i, z_j)\right)^{1-\gamma}}{1-\gamma} + \mathbb{V}^n(k'_i, z_j)$$

by plugging in consumption decision into the RHS.

EGM: The Algorithm (Cont'd)

3: Interpolate $\mathcal{V}^{n+1}$ to obtain its values on a grid of tomorrow's end-of-period resources:
$Y' = z'(k_i')^\alpha + (1-\delta)k_i'$.

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$$\mathbb{V}^{n+1}(k_i', z_j) = \beta \mathbb{E}\left(\mathcal{V}^{n+1}\left(Y'(k_i', z'), z'\right)|z_j\right).$$

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5: Stop if convergence criterion is satisfied and obtain beginning-of-period capital, k, by solving the nonlinear equation $Y^{n*}(i,j) \equiv z_j k^{\alpha} + (1-\delta)k$, for all i, j. Otherwise, go to step 1.

Comments

- Whenever EGM can be applied, it should be your default choice. It can easily be 1-2 orders of magnitude faster than VFI with acceleration methods.
- Extensions and Limitations:
  - Two choice variables can be handled with some loss of efficiency. See Barillas and Fernandez-Villaverde (JEDC 2007) and Maliar and Maliar (2013).
  - Two state variables: currently no "simple" solution that keeps accuracy intact.
  - Borrowing constraints: Very easy to deal with.

Is This Worth the Trouble? Yes!

	β			
Utility	0.95	0.98	0.99	0.995
VFI	28.9	74	119	247
VFI + Howard	7.17	18.2	29.5	53
VFI + Howard + MQP	7.17	16.5	26	38
VFI + Howard + MQP +100 grid	2.15	5.2	8.2	12
EGM (expanding grid curv=2)	0.38	0.94	1.92	4

Table 2: Time for convergence (seconds)

- RRA=2; 300 points in capital grid, expanding grid with exponent of 3.