ECO2404 Problem Set 1

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Question 1

From Tables 2, 1 we can see both methods behave similarly in computing the the larger root (in absolute value) of the quadratic equation. However the solutions for the smaller root (in absolute value) behave much differently as n increases. Since b is very large, as c approaches 0 computing the smaller root (in absolute value) using the quadratic formula results in taking $-b + \sqrt{b^2 - 4ac} \approx -b + \sqrt{b^2} \approx 0$. From Table 1 the quadratic formula results in a root which is exactly 0 for $n \geq 7$, however since c is always positive the smaller root should never be precisely 0. In contrast the alternative method provides much precise computations for the smaller root (in absolute value) as it avoid the the numerical error in computing $-b + \sqrt{b^2 - 4ac}$ and clearly results in more numerically stable and precise solutions as n increases.

Table 1: Quadratic Formula Roots

10^{-n}	Large Root (In absolute value)	Small root (In absolute value)
1	-99999.99999899999	-1.00000034e-6
2	-99999.9999999	-1.00000761e-7
3	-99999.99999999	-9.99716576e-9
4	-99999.999999999	-1.00408215e-9
5	-99999.9999999999	-1.01863407e-10
6	-100000.0	-7.27595761e-12
7	-100000.0	0.0
8	-100000.0	0.0

Table 2: Alternative method

10^{-n}	Large Root (In absolute value)	Small root (In absolute value)
1	-99999.99999899999	-1.00000000e-6
2	-99999.9999999	-1.00000000e-7
3	-99999.99999999	-1.00000000e-8
4	-99999.999999999	-1.00000000e-9
5	-99999.9999999999	-1.00000000e-10
6	-100000.0	-1.00000000e-11
7	-100000.0	-1.00000000e-12
8	-100000.0	-1.00000000e-13

Question 2.)

a)

One such mathematical definition of the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$ can be seen as the solution to the quadric equation:

$$\phi^2 = 1 - \phi \tag{1}$$

Multiplying both side of the quadratic equation by ϕ^{n-1} results in the recursion:

$$\phi^{n+1} = \phi^{n-1} - \phi^n \tag{2}$$

b)

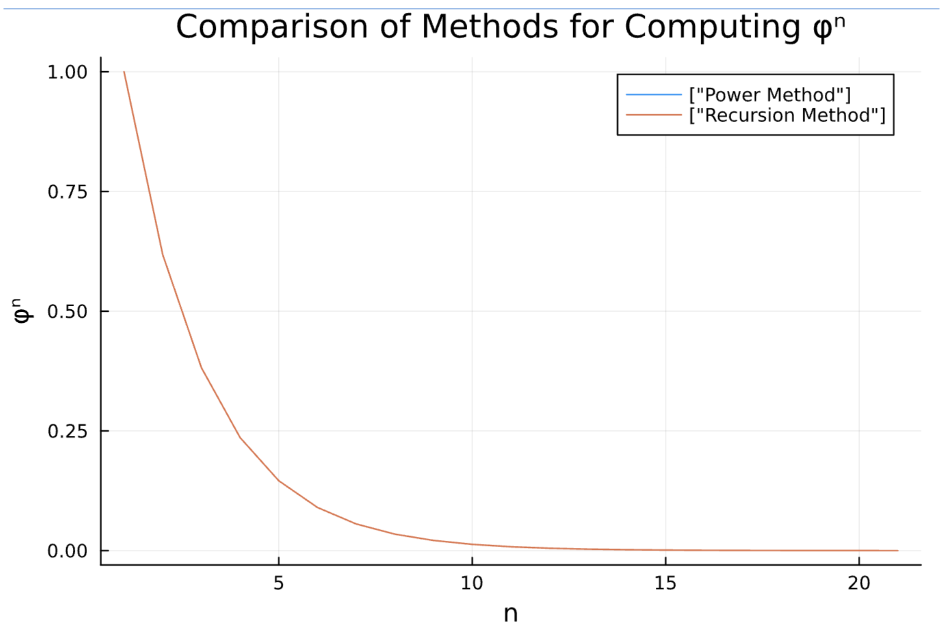

Figure 1: Comparison of Methods: Computing ϕ^n

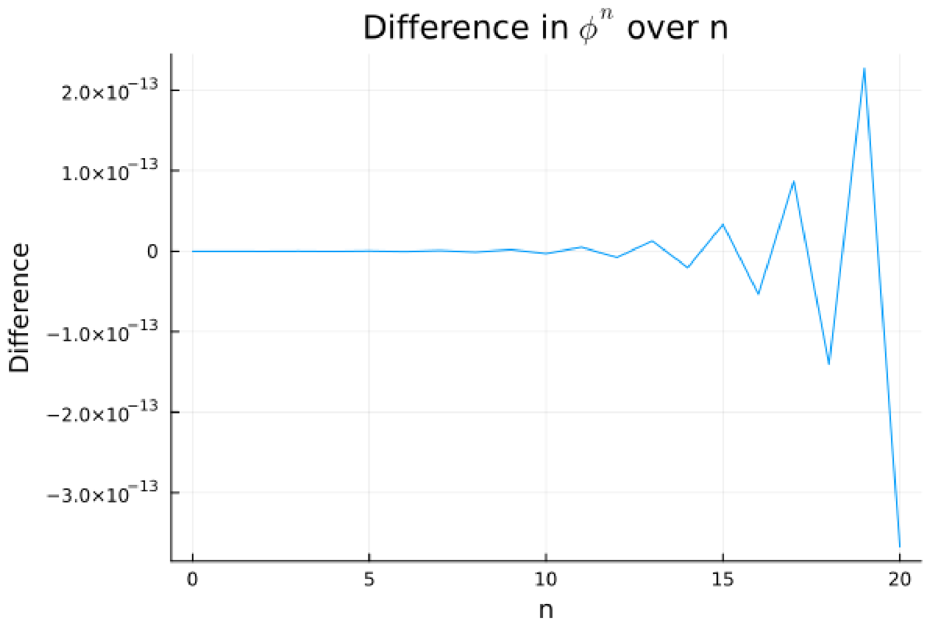

Figure 2: Difference in ϕ^n between the two methods

After several iterations, the values of ϕ^n between the two methods begin to diverge. This is evidently caused by the truncation error, ϕ^n has an infinite decimal but only a finite amount of digits can be stored in the floating point variable. The full value of the power method results in a likewise infinite decimal which is shortened by the computer. The plot shows the difference between the two arrays, with the difference accelerating towards greater values.

Table 3: ϕ Recursion vs. Power Methods

Iterations	Recursion	Power
18	0.0001...18735	0.0001...23962
19	0.0001...46011	0.0001...34356
20	0.00006....41338	0.00006...89609

Question 3.)

(a)

i.

Table 4: Number of iterations until VFI convergence

Number of grid points (N)	Iterations until convergence
11	11
101	31
1001	52

ii.

From the below plots we can see $V(k), g(k)$ are both concave and monotonically increasing in k. $V(k)$ being monotonic reflects that higher capital results in higher lifetime utility, and the concavity of $V(k)$ reflects the diminishing marginal effect of capital on lifetime utility. The fact that $g(k)$ is monotonically increasing and concave reflects that higher capital results in higher savings, and the concavity of $g(k)$ reflects the diminishing marginal benefit of saving. Since k is a discretized grid for small values of N both functions appear as step functions, as the grid of capital becomes finer and N increase the both functions appear as smooth curves.

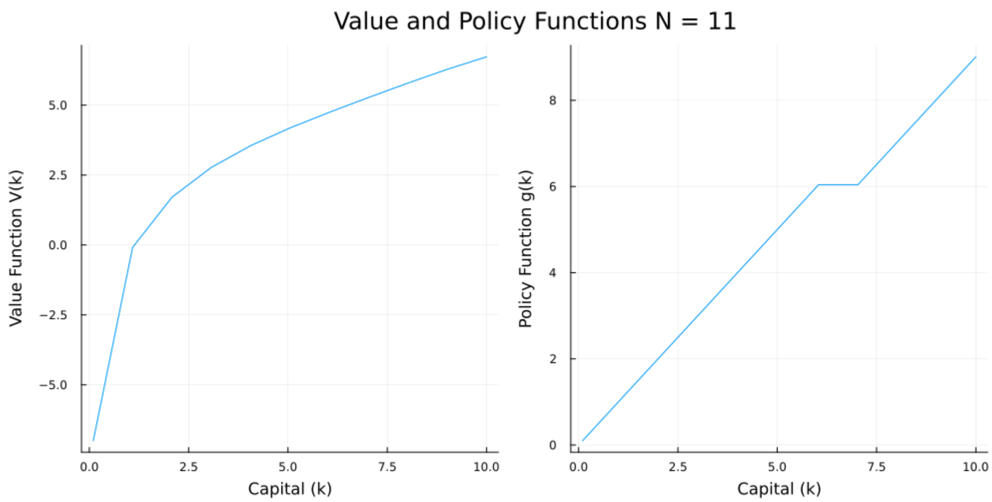

Figure 3: Value and policy functions, $N = 11$.

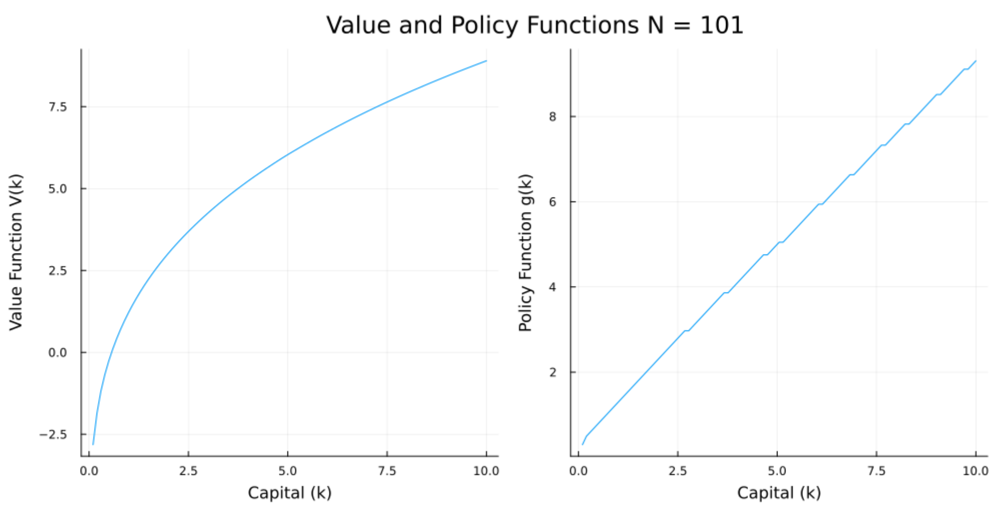

Figure 4: Value and policy functions, $N = 101$.

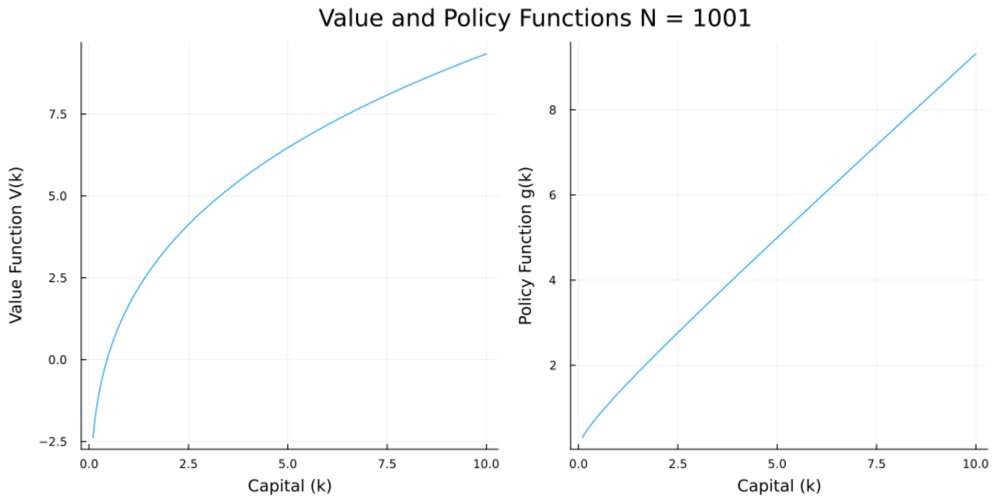

Figure 5: Value and policy functions, $N = 1001$.

(b)

i.

Table 5: Number of iterations until VFI convergence

Number of grid points (N)	Tolerance	Iterations until convergence
11	1e-5	11
101	1e-5	31
1001	1e-5	52
11	1e-6	11
101	1e-6	31
1001	1e-6	52

ii.

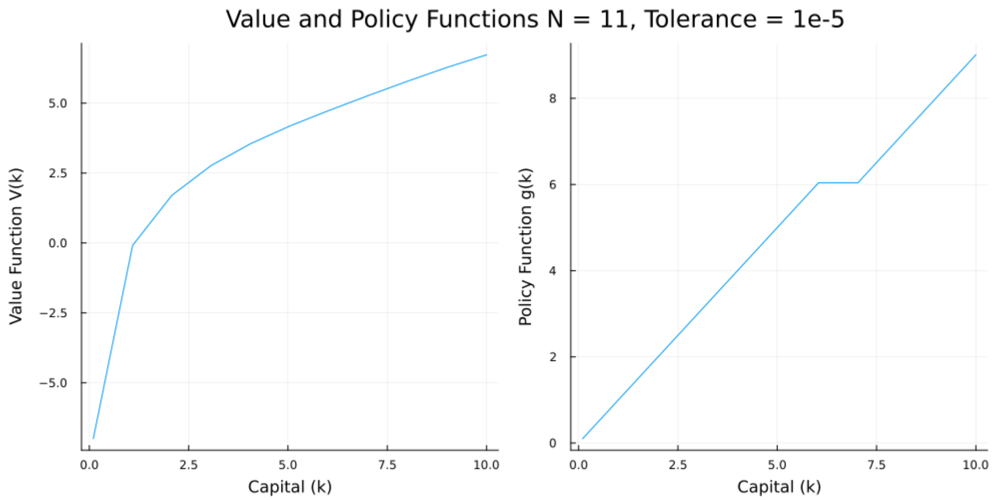

Figure 6: Value and policy functions, $N = 11$, tolerance 10^{-5}.

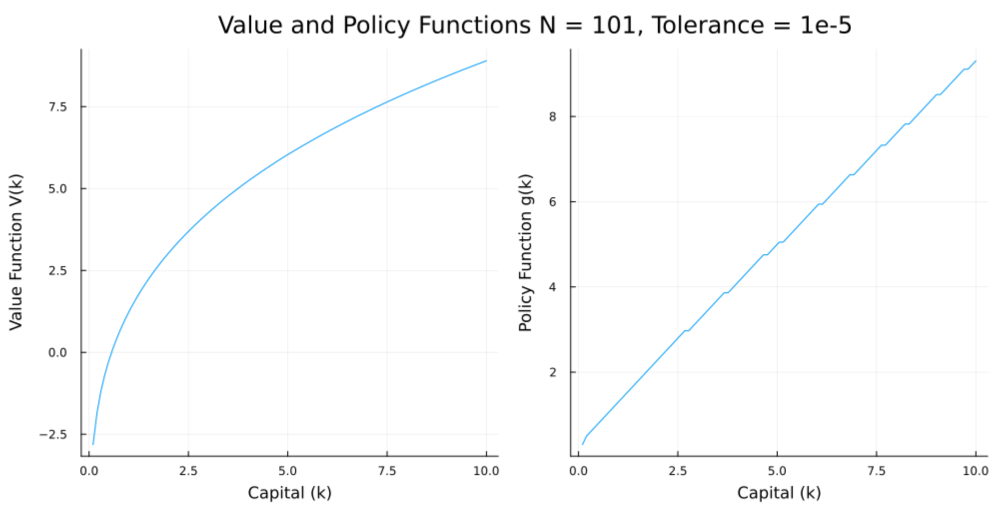

Figure 7: Value and policy functions, $N = 101$, tolerance 10^{-5}.

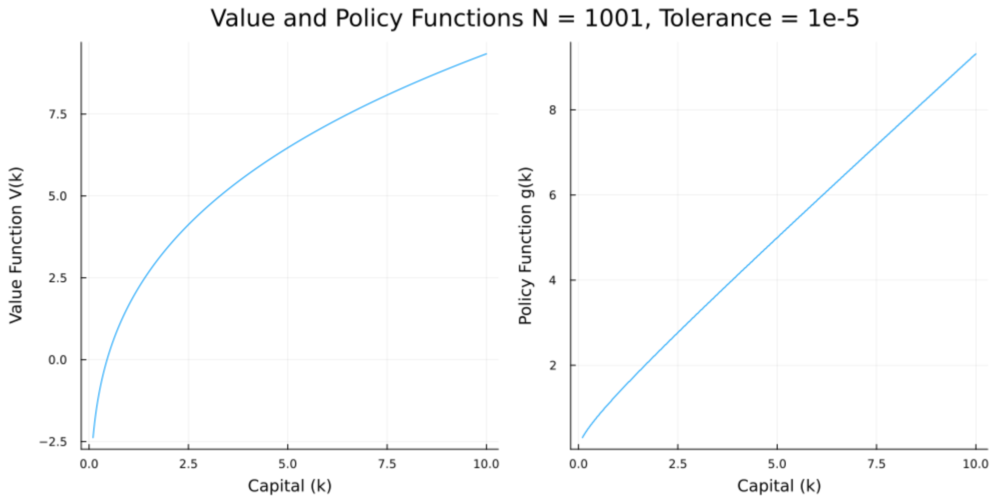

Figure 8: Value and policy functions, $N = 1001$, tolerance 10^{-5}.

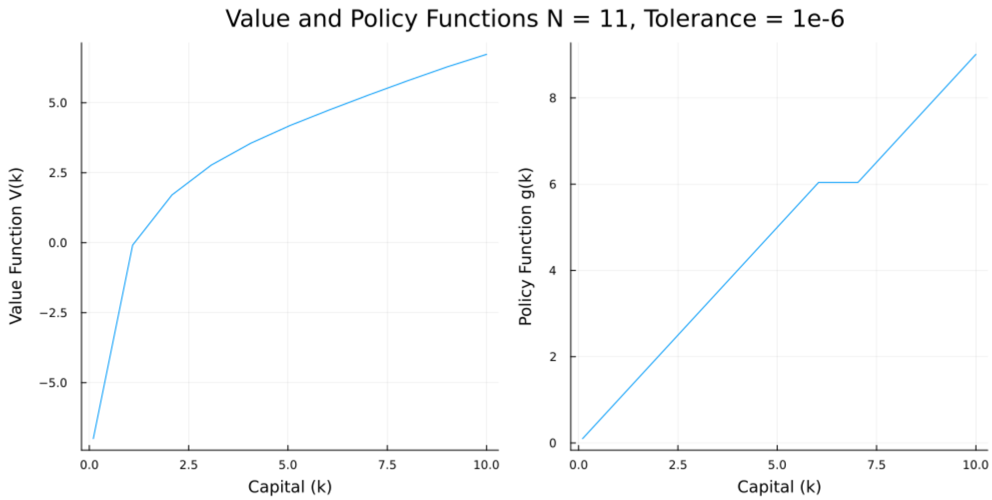

Figure 9: Value and policy functions, $N = 11$, tolerance 10^{-6}.

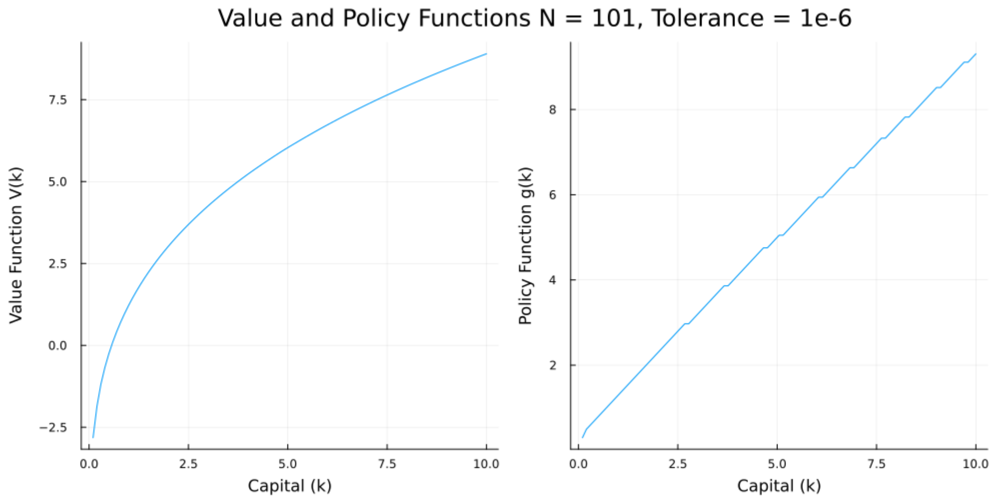

Figure 10: Value and policy functions, $N = 101$, tolerance 10^{-6}.

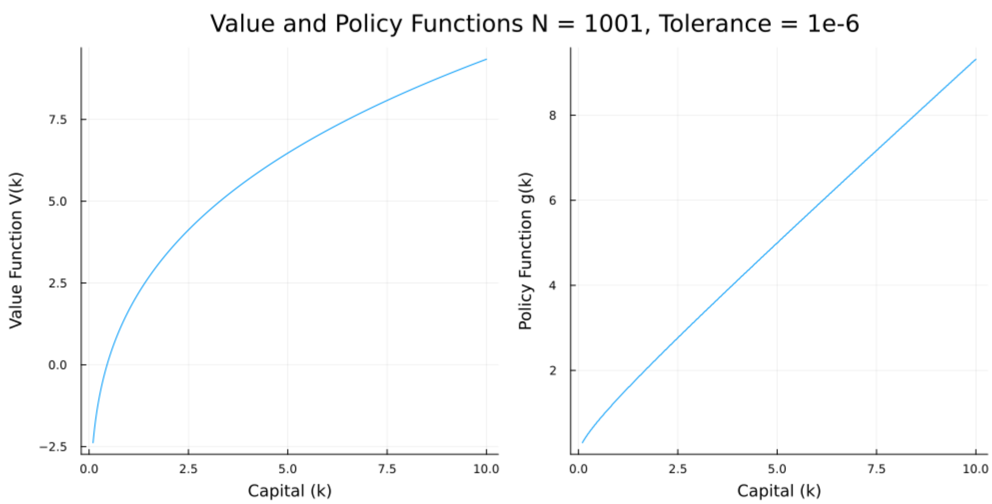

Figure 11: Value and policy functions, $N = 1001$, tolerance 10^{-6}.

From the above plots running the VFI algorithm with the convergence tolerance set to either 10^{-5} or 10^{-6} we see virtually no change in the resulting functions $V(k)$ and $g(k)$ when N is changed from $11, 101, 1001$. As in part a) both the value function and policy function appear as step functions when the gid size N is small and appears much much smoother as N increases. Additionally from Table 5 the number of iterations until convergence does not change when the tolerance is set to either 10^{-5} or 10^{-6}. We would expect that the plots of $V(k)$ and $g(k)$ and the number of iterations until convergence should change as tolerance changes however this is now what we observe. This behaviour occurs as the gid defining k is stored as a discrete set of values (when in reality it is continuous). If the gird of k is not made fine enough we will observe that if optimal capital in the next period $k^{'}_{n+1}(k)$ is slightly higher than $k^{'}_{n}(k)$ but not large enough so that it falls on the next level of capital in the grid, the VFI algorithm will set $k^{'}_{n+1}(k) = k^{'}_{n}(k)$ resulting in.

$$\max_{k \in K} |k^{'}_{n+1}(k) - k^{'}_{n}(k)| = \max_{k \in K} |k^{'}_{n}(k) - k^{'}_{n}(k)| = 0 < tolerance$$

That is the VFI algorithm stops regardless of the tolerance chosen provided the capital grid is not made fine enough.

(c)

i.

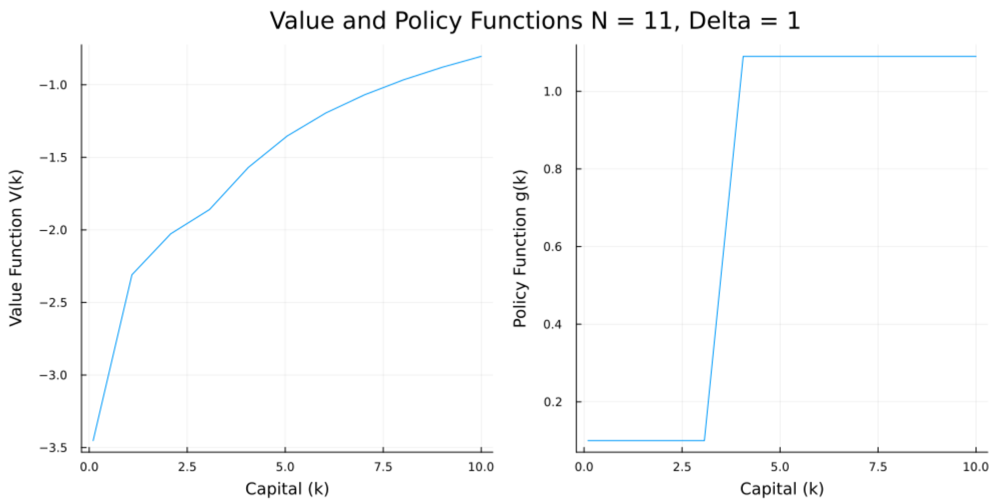

Figure 12: Value and policy functions, $N = 11$, $\delta = 1$.

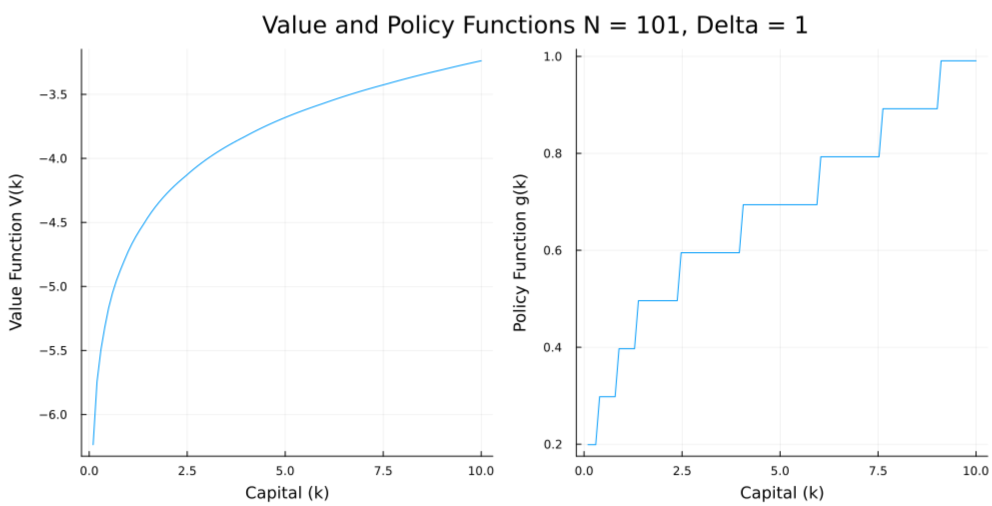

Figure 13: Value and policy functions, $N = 101$, $\delta = 1$.

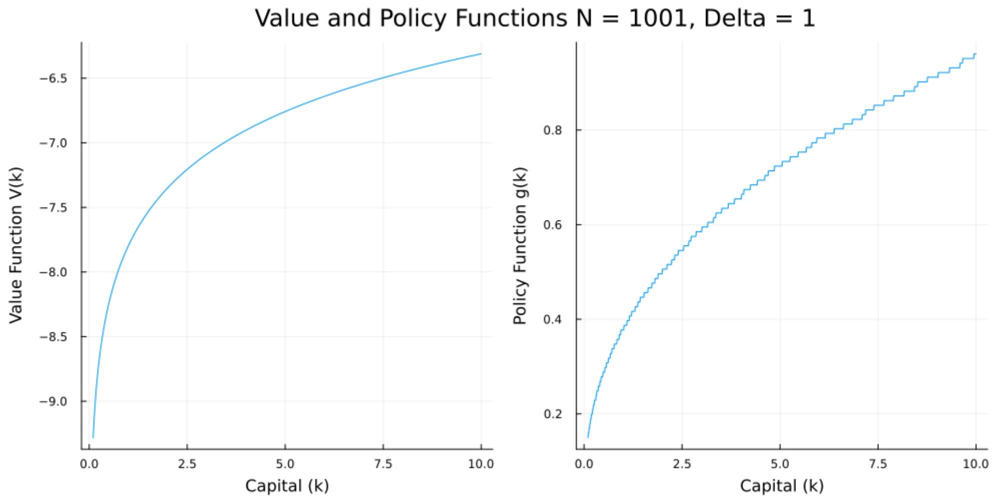

Figure 14: Value and policy functions, $N = 1001$, $\delta = 1$.

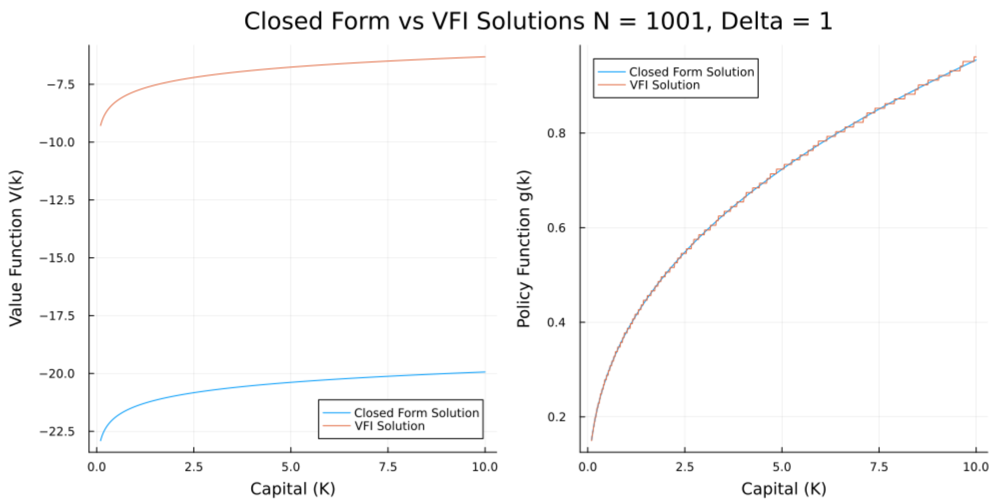

Figure 15: Closed-form vs VFI solutions, $N = 1001$, $\delta = 1$.

ii.

Above we plot the true analytical solutions for $V(k)$ and $g(k)$ along with the numerical solutions provided by the VFI algorithm for $N = 1001$. Here we can see that the policy function appears very close to its analytical solution and the only between the two curves is captured by on how fine the capital grid is made. In contrast we see a lage difference between the VFI solution and the analytical solution for $V(k)$. Here we can see that although both estimates of $V(k)$ are roughly the same shape (log-linear) the VFI solution appears vertically shifted upward relative to the analytical solution. This behaviour is observed based on the stopping criteria of the VFI algorithm. Since we set our stopping criteria based on the policy function rather than the value function, that is

$$\max_{k \in K} |k'_{n+1}(k) - k'_n(k)| < tolerance$$

stopping the VFI algorithum when the estimated policy function is close enough to its analytical solution does not guarantee that the value function $V(k)$ is sufficiently close to its analytical solution.

Table 6: Number of iterations until VFI convergence across all questions

Section	N	Tolerance	δ	Iterations
(a)	11	1e-8	0.1	11
	101	1e-8	0.1	31
	1001	1e-8	0.1	52
(b)	11	1e-5	0.1	11
	101	1e-5	0.1	31
	1001	1e-5	0.1	52
	11	1e-6	0.1	11
	101	1e-6	0.1	31
	1001	1e-6	0.1	52
(c)	11	1e-8	1.0	2
	101	1e-8	1.0	4
	1001	1e-8	1.0	8