# M2 English Seminar Written Report on the works of Yves D'Angelo, Remi Catellier and Laurent Monasse on Branching Dynamical Networks.

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May 3, 2019

# Contents

# 1 The Fisher or KPP equation

## 1.1 Preliminary

Our starting point is the diffusion equation:

$$\partial_t u = \Delta u \tag{1}$$

In addition to diffusion, let's consider models where the growth rate of $u$ also depends on density $u$.
We obtain the following equation:

$$\partial_t u = \Delta u + F(u) \tag{2}$$

where F is sufficiently smooth. It is often natural in models to consider $F(u)$ proportional to $u$ for small $u$ ("growth"), and as $u$ becomes close to 1, the growth $F(v)$ ceases: $F(1) = 0$ ("saturation"). These kinds of models were first introduced and examined closely by the works of Fisher[1] and Kolmogorov, Petrovsky and Piscounuv[2]. An exemple of such an equation is:

$$\partial_t u = \Delta u + u(1-u) \tag{3}$$

that is worked with in the 1 dimensional (scalar) case $u = u(x,t)$.

## 1.2 Reaction

When looking at space independent solutions $u(x,t) = v(t)$ in (**??**), the one-dimensional ordinary differential equation (ODE)

$$\partial_t v = v - v^2 = F(v) \tag{4}$$

is obtained. There are two equilibriums ($F(v) = 0$)) for $v = 0$ and $v = 1$. $F'(0) > 0$ shows that $v = 0$ is unstable and $F'(1) < 0$ shows that $v = 1$ is asymptotically stable.

## 1.3 Reaction-Diffusion

In the space $X = C^0_{b,unif}(\mathbb{R},\mathbb{R})$ of bounded and uniformly continuous functions, there is local existence and uniqueness of the solutions for the Fisher-KPP equation. Due to maximum principle, there is also global existence and uniqueness of solutions.

**Theorem 1.** *:*
*Existence and Unicity of the solution in X: Let $U_0 \in X$. There exists a unique solution of the Fisher KPP eqution $U \in C([0,\infty[, X)$ with initial condition $U_0$.*

**Theorem 2.** *:*
*Maximum principle: Let $u_1$ and $u_2$ be two solutions of (**??**). If there exists a $t_0$ such that $u_1(x,t_0) < u_2(x,t_0)$ for all $x$ then $u_1(x,t) < u_2(x,t)$ for all $x$ and $t > t_0$*

## 1.4 Front solutions

We look for front solutions of (**??**) (or propagation waves) linking the equilibrium states $u = 1$ (at $-\infty$) and $u = 0$ (at $+\infty$)
Let $u(x,t) = h(x-ct) = h(y)$ with $y = x - ct$ in (**??**).
The obtained equations on $h$ are

$$\begin{cases} h''(y) + ch'(y) + F(h(y)) = 0 \\ h(-\infty) = 1 \\ h(+\infty) = 0 \end{cases} \tag{5}$$

which is an elliptic non linear equation. Thus, the problem is to find $c$ and $h \in C^2$ such as (**??**) is verified.

**Theorem 3.** *:*
*Suppose $F \in C^1([0,1])$ such as $F(0) = F(1) = 0$ and $F$ is non negative. There exists a critical speed $c_*$ such as $c_*^2 \geq 4F'(0)$ and:*

*i) For all $c \geq c_*$, the equation (**??**) has a solution $h_c : \mathbb{R} \to ]0,1[$ of class $C^3$ This solution is unique modulo translations.*

*ii) For all $c < c_*$ the equation (**??**) has no solutions $h : \mathbb{R} \to [0,1]$*

Remark : in the second case there are soltions but not confined in [0,1], which do not make sense in population studies / densities.

## 2 Branching Dynamical Network Growth

In this section we will study the following model on the growth of a dynamical branching nework, for exemple a fungus, proposed by Rémi Catellier, Yves D'Angelo and Cristiano Ricci, with adequate rescaling:

$$\begin{cases} \partial_t\mu + \nabla(\mu v) = f(C)(\mu+\rho) - \mu\rho \\ \partial_t(\mu v) + \nabla(\mu v \times v) + T\nabla\mu = -\lambda\mu v + \mu\nabla C - \mu v\rho \\ \partial_t\rho = F(v)\mu \\ \partial_t C = -b\rho C \end{cases} \tag{6}$$

The unkownn $\mu$ represents the density of the apices of the fungus.
The unknown $\rho$ represents the density of the network.
The unknown $v$ represents the speed of the apices.
The unknown $C$ represents the concentration of nutrient.
The parameters $T$, $\lambda$ and $b$ are scalar constants that represent temperature, fluid damping on the speed of the apexes, and the rate of consumption of the nutrients by the network.
The function $f$ indicates the influence of the concentration of nutrient on the growth of the fungus. Usualy, to have a stationary state on the growth of the fungus, we need $f(0) = 0$ and $f(x)/x$ in $L^1$ near 0.
The function $F$ represents the inverse of the average time spent by apexes in a given point, and is given by the expression:

$$F(V) = (\frac{1}{2\pi T})^{\frac{d}{2}} \int_{\mathbb{R}^d} |v| \exp\left(-\frac{|v-V|^2}{2T}\right) dv \tag{7}$$

where d is the dimension of the problem. This model is often simplified by substituting $F(V)$ with a constant: $F(V) = F_0$ .

## 2.1 Explanation of the terms in equation (??)

The fungus is a branching dynamical network that can be studied in two parts: the apexes (tips of the newtwork) and the hyphen (branches of the network).

Looking at each line of equation (**??**) seperately, the model has:

i) The first line is the mass balance equation on the apexes with classical left term $\partial_t\mu+\nabla(\mu v)$. The right term is composed of $f(C)(\mu+\rho)$ corresponding to a growth of the number of apexes depending on the concentration of nutrient and the existing mass of apexes and hyphen, and a term $-\mu\rho$ which corresponds to anastomosis : a tip that encounters a branch will merge with it and be destroyed. There is a growth term and a saturation term like the KPP model.

ii) The second line is the momentum balance equation of the apexes with classical left term $\partial_t(\mu v) + \nabla(\mu v \times v)$. The term $T\nabla\mu$ represents a brownian motion followed by the apexes. The term $-\lambda\mu v$ represents a fluid damping in the physics of the problem. The term $+\mu\nabla C$ represents of proponency of the apexes to go where the nutrient concentation is dense. The term $-\mu v\rho$ represents the loss of momentum due to anastomosis.

iii) The third line describes the relationship between apexes and hyphen: the trail of the apexes are the branches.

iv) The fourth line describes the evolution of the nutrient concentration: it is eaten by the hyphen with rate $bC$.

## 2.2 Front wave equations

Looking for front wave solutions, let $c$ be the wave's speed and let $\xi = x - ct$.
In the limit $T \to \infty$, $\lambda \to \infty$, $\frac{T}{\lambda}$ constant, the following equations for front waves are obtained:

$$\begin{cases} -c\mu' - \frac{T}{\lambda}\mu'' = f(C)(\mu+\rho) - \mu\rho \\ -c\rho' = F_0\mu \\ C' = \frac{b\rho C}{c} \end{cases} \tag{8}$$

The stationary states here are: $(\mu, \rho, C) = \begin{cases} (0, 0, C_0) \\ (0, \rho_\infty, 0) \end{cases}$

## 2.3 Near $(0, 0, C_0)$

Near $(0, 0, C_0)$ let $f(C_0) = f_0$, the result is :

$$\begin{cases} -c\mu' - \frac{T}{\lambda}\mu'' = f_0(\mu+\rho) \\ -c\rho' = F_0\mu \end{cases} \tag{9}$$

which becomes:

$$\rho''' + \frac{c\lambda}{T}\rho'' + \frac{f_0\lambda}{T}\rho' - \frac{\lambda F_0 f_0}{Tc}\rho = 0 \tag{10}$$

of characteristic polynomial:

$$P(X) = X^3 + \frac{c\lambda}{T}X^2 + \frac{f_0\lambda}{T}X - \frac{\lambda F_0 f_0}{Tc} \tag{11}$$

For $c < 0$, $P(0) > 0$. Thus P has a negative root $r_1$ .
In order that P has two other real roots $r_3 > r_2 > r_1$ we need (equivalent proposition) that P' has two roots and that the discriminant $\Delta$ of P be positive.

### 2.3.1 First condition: P' has two real roots:

$P'(X) = 3X^2 + 2\frac{c\lambda}{T}X + \frac{f_0\lambda}{T}$ has for discriminent: $\Delta' = 4(\frac{\lambda}{T})^2(c^2 - 3\frac{T}{\lambda}f_0)$ which gives the condition:

$$\boxed{c^2 > 3\frac{T}{\lambda}f_0} \tag{12}$$

### 2.3.2 Second condition: $\Delta > 0$:

For a general 3 order polynomial of the form $P = aX^3+bX^2+cX+d$ we have $\Delta = b^2c^2 + 18abcd - 27a^2d^2 - 4ac^3 - 4b^3d$ which in our case gives:

$$\begin{aligned}
\Delta &= \frac{\lambda^4}{T^4}f_0^2c^2 - 18\frac{\lambda^3 f_0^2 F_0}{T^3} - 27\frac{\lambda^2 F_0^2 f_0^2}{T^2c^2} - 4\frac{f_0^3\lambda^3}{T^3} + 4\frac{\lambda^4 F_0 f_0 c^2}{T^4} \\
&= c^2\frac{\lambda^4 f_0(f_0 + 4F_0)}{T^4} - \frac{\lambda^3 f_0^2(18F_0 + 4)}{T^3} - \frac{27\lambda^2 F_0^2 f_0^2}{T^2} * \frac{1}{c^2} \\
&= \frac{\lambda^4 f_0}{T^4c^2}[(f_0 + 4F_0)c^4 - \frac{Tf_0(18F_0 + 4)}{\lambda}c^2 - 27\frac{T^2F_0^2 f_0}{\lambda^2}]
\end{aligned}$$

It's sign is the same as the sign of the 2 order polynomial in $c^2$

$$D(c^2) = (f_0 + 4F_0)c^4 - \frac{Tf_0(18F_0 + 4)}{\lambda}c^2 - 27\frac{T^2F_0^2 f_0}{\lambda^2} \tag{13}$$

of discriminent $d$:

$$\begin{aligned}
d &= \Big(\frac{Tf_0(18F_0 + 4)}{\lambda}\Big)^2 + 108(f_0 + 4F_0)\frac{T^2F_0^2 f_0}{\lambda^2} \\
&= \frac{T^2 f_0}{\lambda^2}(f_0(18F_0 + 4)^2 + 108(f_0 + 4F_0)F_0^2) > 0
\end{aligned}$$

Thus we obtain the condition on the positivity of $\Delta$:

$$\boxed{c^2 > \frac{Tf_0(18F_0 + 4) + T\sqrt{f_0(f_0(18F_0 + 4)^2 + 108(f_0 + 4F_0)F_0^2)}}{2\lambda(f_0 + 4F_0)}} \tag{14}$$

### 2.3.3 Sign of the roots

We have $r_3 < 0$. As $r_1 r_2 r_3 < 0$, $r_2$ and $r_1$ have the same sign.
Moreover $P'$ has a symetry of axis $X = -\frac{c\lambda}{3T} > 0$. Because $c < 0$, $P$ has a local minimum (of negative value) in a positive point. Thus $P$ has a positive root.
Thus $r_1 > r_2 > 0$:
Under the conditions (**??**) and (**??**), $P$ has two positive roots and one negative root.

## 2.4 Near $(0, \rho_\infty, 0)$

Near $(0, \rho_\infty, 0)$:

$$\begin{cases} -c\mu' - \frac{T}{\lambda}\mu'' = f(C)\rho_\infty - \mu\rho_\infty \\ C' = \frac{b\rho_\infty C}{c} \end{cases} \tag{15}$$

the second lign gives

$$C = K \exp\left(\frac{b\rho_\infty}{c} t\right) \tag{16}$$

and the first lign is a second order ODE in $\mu$ with source $f(C)\rho_\infty$ and of homogeneous polynômial:

$$Q(X) = X^2 + \frac{c\lambda}{T} X - \frac{\rho_\infty \lambda}{T} \tag{17}$$

which posseses always two real roots of opposite sign.

# 3 References


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