TOPOLOGY

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Contents

0. Some Theorems and Definitions

Here I prove several small theorems that I use in proving the exercises.

Theorem (0.1). *Every infinite set has a finite subset of cardinality n for all $n \in \mathbb{N}$.*

Proof. Let S be an infinite set. There exists an element $x \in S$ since S is nonempty. Then $\{x\} \subset S$. So S contains a finite subset of cardinality 1. Suppose S contains a finite subset E of size n. Then there exists an element $x \in S - E$ otherwise S would have a cardinality of n and be finite. Then the subset $E \cup \{x\}$ is a finite subset of cardinality $(n+1)$. This concludes the induction. Hence, S has a finite subset of cardinality n for all $n \in \mathbb{N}$. □

Theorem (0.2). *Suppose B is an infinite set. If there is an injection $f\colon B \to \mathbb{N}$, then B is countably infinite.*

Proof. Suppose B is an infinite set and there is an injection $f\colon B \to \mathbb{N}$. Let us define $g\colon B \to im(f)$ where $g(b) = f(b)$ for all $b \in B$ so that g is a bijection. Now suppose $im(f)$ is finite. Then there exists some bijection $h\colon im(f) \to J_n$ where $J_n = \{1, 2, ..., n\}$ for some $n \in \mathbb{N}$. Since h and g are bijections, the composite $h \circ g\colon B \to J_n$ is a bijection. It follows B is finite which is a contradiction to the initial hypothesis of the theorem. It follows $im(f)$ is infinite. But since $im(f) \subset N$, by Theorem 1.8 we know that $im(f)$ must have the same cardinality as $\mathbb{N}$. And since g is a bijection between B and $im(f)$, it follows B must also have the same cardinality as $\mathbb{N}$. Hence, B is countably infinite. □

Theorem (0.3). *Suppose B is an infinite set and S is a countably infinite set. If there is an injection $f\colon B \to S$, then B is countably infinite.*

Proof. Suppose B is an infinite set, S is countably infinite, and there is an injection $f\colon B \to S$. Since S is countably infinite there exists a bijection $g\colon S \to \mathbb{N}$. It follows the composite $g \circ f\colon B \to \mathbb{N}$ is an injection. By Theorem 0.3 it follows B is countably infinite. □

Theorem (0.4). *Suppose B is an infinite set. If there exists a surjection $f\colon \mathbb{N} \to B$, then B is countably infinite.*

Proof. Suppose B is an infinite set and there is a surjection $f\colon \mathbb{N} \to B$. Then define $g\colon B \to \mathbb{N}$ as $g(b)$ equals the minimum of the set $f^{-1}(b)$. Now it is shown that g is an injection. Suppose $b_1 \neq b_2$. Then the sets $f^{-1}(b_1)$ and $f^{-1}(b_2)$ are disjoint, otherwise there exists an $n \in \mathbb{N}$ such that $f(n) = b_1$ and $f(n) = b_2$ yet $b_1 \neq b_2$. Since these sets are disjoint, they cannot share a common minimum. Therefore $g(b_1) \neq g(b_2)$. Thus $g\colon B \to \mathbb{N}$ is injective. Then by Theorem 0.2 the set B is countably infinite. □

Theorem (0.5). *Suppose B is an infinite set and S is countably infinite. If there exists a surjection $f\colon S \to B$, then B is countably infinite.*

Proof. Since S is countably infinite there exists a bijection $g\colon \mathbb{N} \to S$. We know B is infinite and there is a surjection $f\colon S \to B$. Then the composite $f \circ g\colon \mathbb{N} \to B$ is surjective. Then by Theorem 0.5, it follows B is countably infinite. □

Theorem (0.6). *Let either A or B (or both) be countably infinite. Then the cartesian product $A \times B$ is countably infinite. If A and B are both finite, then the cartesian product $A \times B$ is finite.*

Proof. First suppose A and B are countably infinite. Then we have bijections $f\colon A \to \mathbb{N}$ and $g\colon B \to \mathbb{N}$. Then define $h\colon A \times B \to \mathbb{N}$ as follows

$$h(a,b) = 2^{f(a)}3^{g(b)}$$

We show h is injective. Suppose $h(a,b) = h(a',b')$. Then $2^{f(a)}3^{g(b)} = 2^{f(a')}3^{g(b')}$. But since 2 and 3 are prime it follows $f(a) = f(a')$ and $g(b) = g(b')$. And since f and g are injective it follows $a = a'$ and $b = b'$ so then $(a,a') = (b,b')$. Thus h is injective and by Theorem 0.2 it follows $A \times B$ is countably infinite.

Without a loss of generality if A is countably infinite and B is finite then we would have bijections $f\colon A \to \mathbb{N}$ and $g\colon B \to \{1,2,...,n\}$ in place of the ones defined above. By the same argument above $A \times B$ would be countably infinite.

Suppose both A and B are finite. Then there there exist bijections $f\colon A \to \{1,2,...,m\}$ and $g\colon B \to \{1,2,...,n\}$. Then define $h\colon A \times B \to \mathbb{N}$ as

$$h(a,b) = 2^{f(a)}3^{g(b)}$$

The image of $A \times B$ under h ($im(h)$)is bounded below and above by $(2^1)(3^1)$ and $(2^m)(3^n)$ respectively. Therefore $im(h)$ is some subset of $\{1,2,...,(2^m)(3^n)\}$ which is finite. Thus $im(h)$ is finite. So there is a bijection from $A \times B$ to $im(h)$ which is finite, hence $A \times B$ is finite.

□

Theorem (0.7). *If a finite cartesian product contains at least one set which is countably infinite, then the product is countably infinite.*

Proof. We prove by induction. For the base case suppose we have at least one countably infinite sets, A_1 and A_2. Then the cartesian product $A_1 \times A_2$ is countably infinite by Theorem 0.6. For the inductive step suppose we have $n \geq 2$ sets and we know that either the cartesian product

$$B = A_1 \times A_2 \times \cdots \times A_n$$

or A_{n+1} is countably infinite. Then the $B \times A_{n+1}$ is a cartesian product of at least one countably infinite set which is countable by Lemma 0.6. Now define the set C as follows

$$C = A_1 \times A_2 \times \cdots A_n \times A_{n+1}$$

Then the define the bijection $f\colon B \times A_{n+1} \to C$ as $f((a_1,a_2,...,a_n),a_{n+1}) = (a_1,a_2,...,a_n,a_{n+1})$. Since $B \times A_{n+1}$ is countably infinite and there is a bjiection between $B \times A_{n+1}$ and C, it follows C is countably infinite. This concludes the induction. □

Theorem (0.8). $\mathbb{N} \times \mathbb{N}$ *is countably infinite.*

Proof. Since $\mathbb{N} \times \mathbb{N}$ is a finite cartesian product of two countably infinite sets, it follows by Theorem 0.6 that $\mathbb{N} \times \mathbb{N}$ is countably infinite. □

Here is a ongoing list of all definitions for the course.

Definition. Suppose X is a set. Then $\mathcal{T}$ is a **topology** on X if and only if $\mathcal{T}$ is a collection of subsets of X such that

(1) $\varnothing \in \mathcal{T}$
(2) $X \in \mathcal{T}$
(3) if $U, V \in \mathcal{T}$ then $U \cap V \in \mathcal{T}$
(4) if $\{U_\alpha\}_{\alpha\in\lambda}$ is a collection of sets of $\mathcal{T}$, then $\cup_{\alpha\in\lambda} U_\alpha \in \mathcal{T}$

Definition. A **topological space** is an ordered pair $(X, \mathcal{T})$, where X is a set and $\mathcal{T}$ is a topology on X.

Definition. A set $U \subset X$ is an **open set** in $(X, \mathcal{T})$ if and only if $U \in \mathcal{T}$.

Definition. An open set containing point p is a **neighborhood** of p.

Definition. The **standard topology** $\mathcal{T}_{\text{std}}$ on $\mathbb{R}^n$ is defined as follows: a subset U of $\mathbb{R}^n$ belongs to $\mathcal{T}_{\text{std}}$ if and only if for each point p of U there is an $\epsilon_p > 0$ such that $B(p, \epsilon_p) \subset U$ where $B(p, \epsilon_p)$ is defined as

$$B(p, \epsilon)_p = \{q | d(p, q) < \epsilon_p\}$$

where $d(p, q)$ is defined as

$$d(p, q) = \left(\sum_{i=1}^{n} (p_i - q_i)^2\right)^{1/2}$$

Definition. In the **metric topology** a set U is open if and only if for every point $p \in U$ there exists some $r \in \mathbb{R}$ such that $r > 0$ and $B(p, r) \subset U$ and d is a metric. That is $d(p, p) = 0$, $d(p, q) > 0$, $d(p, q) = d(q, p)$, and $d(p, q) \leq d(p, r) + d(r, q)$.

Example. In the **discrete topology** $\mathcal{T} = \{2^X\}$.

Example. In the **indiscrete topology** $\mathcal{T} = \{\varnothing, X\}$.

Example. In the **finite complement** (or **cofinite**) **topology**, a subset U of X is open if and only if $X - U$ is finite or $U = \varnothing$.

Example. In the **countable complement** or (**cocountable**) **topology** a subset U of X is an open set if and only if $X - U$ is countable or $U = \varnothing$.

Definition. Let $(X, \mathcal{T})$ be a topological space, A a subset of X, and p is a point in X. Then p is a **limit point** of A if and only if for each open set U containing p, $(U - \{p\}) \cap A \neq \varnothing$.

Definition. Let $(X, \mathcal{T})$ be a topological space, A a subset of X, and p a point in X. If $p \in A$ but p is not a limit point of A, then p is an **isolated point** of A.

Definition. Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. Then the **closure** of A in X, denoted $\overline{A}$, is the set A together with all its limit points in X.

Definition. Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. The subset A is **closed** if and only if $\overline{A} = A$, in other words, if A contains all its limit points.

Definition. The **interior** of a set A in a topological space X, denoted A°, is defined as:

$$A^\circ = \bigcup_{U \subset A, U \in \mathcal{T}} U$$

Points of A° are called **interior points** of A.

Definition. The **boundary** of A, denoted ∂A, is defined as $\overline{A} \cap \overline{X - A}$.

Definition. A **sequence** in a topological space X is a function from $\mathbb{N}$ to X. The image of i under this function is a point of X denoted x_i, and we traditionally write the sequence by listing its images: $x_1, x_2, x_3, ...$ or in shorter form: $(x_i)_{i \in \mathbb{N}}$.

Definition. A point $p \in X$ is a **limit of the sequence** $(x_i)_{i \in \mathbb{N}}$, or equivalently $(x_i)_{i \in \mathbb{N}}$ **converges** to p (written $x_i \to p$), if and only if for every open set U containing p, there is an $N \in \mathbb{N}$ such that for all $i > N$, the point x_i is in U.

Definition. Let $\mathcal{T}$ be a topology on a set X, and let $\mathcal{B} \subset \mathcal{T}$. Then $\mathcal{B}$ is a **basis** for the topology $\mathcal{T}$ if and only if every open set in $\mathcal{T}$ is the union of elements of $\mathcal{B}$. If $B \in \mathcal{B}$, we say B is a **basis element** or **basic open set**.

Example. The **lower limit topology** on $\mathbb{R}$ is generated by a basis consisting of all sets of the form $[a, b)$. Denote this space by $\mathbb{R}_{\text{LL}}$.

Definition. Suppose $\mathcal{T}$ and $\mathcal{T}'$ are two topologies one the same underlying set X. If $\mathcal{T} \subset \mathcal{T}'$, then $\mathcal{T}'$ is **finer** than $\mathcal{T}$. Equivalently, $\mathcal{T}$ is courser than $\mathcal{T}'$. We say **strictly coarser** or **strictly finer** if additionally $\mathcal{T} \neq \mathcal{T}'$.

Definition. Let $(X, \mathcal{T})$ be a topological space and $\mathcal{S}$ be a collection of subsets of X. Then $\mathcal{S}$ is a **subbasis** for $\mathcal{T}$ if and only if the collection $\mathcal{B}$ of all finite intersections of sets in $\mathcal{S}$ is a basis for $\mathcal{T}$. An element of $\mathcal{S}$ is called a *subbasis* element or a **subbasic open set**.

Definition. A **total order** $\leq$ on a set X is a binary relation with the following properties

(1) $x \leq x$ for all $x \in X$
(2) if $x \leq y$ and $y \leq z$ then $x \leq z$ for all $x, y, z \in X$
(3) if $x \leq y$ and $y \leq x$ then $x = y$ for all $x, y \in X$
(4) for all $x, y \in X$ either $x \leq y$ or $y \leq x$

Definition. Let X be a set totally ordered by $\leq$. Let $\mathcal{B}$ be the collection of all subsets of X that are any of the following forms:

$$\{x \in X \mid x < a\} \text{ or } \{x \in X \mid a < x\} \text{ or } \{x \in X \mid a < x < b\}$$

for $a, b \in X$. Then $\mathcal{B}$ is a basis for a topology $\mathcal{T}$, called the **order topology** on X.

Definition. Given the sets A and B, their **product** (or **Cartesian product**) $A \times B$ is the set of all ordered pairs (a, b) such $a \in A$ and $b \in B$. If A and B are totally ordered by $\leq_A$ and $\leq_B$, respectively, then the **dictionary order** or **lexicographic order** $\leq$ on $A \times B$ is specified by defining $(a_1, b_1) < (a_2, b_2)$ if $a_1 <_A a_2$, or if $a_1 = a_2$ and $b_1 <_B b_2$.

1. Cardinality: To Infinity and Beyond

Exercise (1.1). For sets $A_1, A_2 \subset X$ it is shown that $X - (A_1 \cup A_2) = (X - A_1) \cap (X - A_2)$

Proof. First it is shown that $X - (A_1 \cup A_2) \subset (X - A_1) \cap (X - A_2)$. Let $x \in X - (A_1 \cup A_2)$. Then by definition of the complement we have $x \in X$ but $x \notin (A_1 \cup A_2)$. This means $x \notin A_1$ and $x \notin A_2$. Since $x \in X$ but $x \notin A_1$ we have $x \in X - A_1$. Similarly, $x \in X$ but $x \notin A_2$ implies $x \in X - A_2$. Combining these last two statements and by the definition of intersection we have $x \in (X - A_1) \cap (X - A_2)$.

Second it is shown $(X - A_1) \cap (X - A_2) \subset X - (A_1 \cup A_2)$. Let $x \in (X - A_1) \cap (X - A_2)$. Then by definition of intersection we have $x \in X - A_1$ and $x \in X - A_2$. Then $x \in X$ but $x \notin A_1$ and $x \notin A_2$. Then $x \notin A_1 \cup A_2$. But since $x \in X$ we have $x \in X - (A_1 \cup A_2)$. It follows the original statement has been proven.

□

Exercise (1.2). Let X be a set, and let $\{A_k\}_{k=1}^{N}$ be a finite collection of sets such that $A_k \subset X$ for each $k = 1, 2, ..., N$. Then

$$X - \Big(\bigcup_{k=1}^{N} A_k\Big) = \bigcap_{k=1}^{N} (X - A_k) \tag{1}$$

and

$$X - \Big(\bigcap_{k=1}^{N} A_k\Big) = \bigcup_{k=1}^{N} (X - A_k) \tag{2}$$

Theorem (1.6). *Show* $|2\mathbb{N}| = |\mathbb{N}|$

Proof. An appropriate bijection $f \colon \mathbb{N} \to 2\mathbb{N}$ would be $f(n) = 2n$ for all $n \in \mathbb{N}$. It is shown f is injective. Suppose $f(n_1) = f(n_2)$. Then $2n_1 = 2n_2$ which implies $n_1 = n_2$. Now it is shown f is surjective. Suppose $b \in 2\mathbb{N}$. By definition of $2\mathbb{N}$ it follows there is some $n \in \mathbb{N}$ such that $b = 2n$. It follows f is surjective. □

Theorem (1.7). *Show* $|\mathbb{Z}| = |\mathbb{N}|$

Proof. An appropriate bijection $f \colon \mathbb{N} \to \mathbb{Z}$

$$f(n) = \begin{cases} \frac{n}{2} & \text{n is even} \\ \frac{1-n}{2} & \text{n is odd} \end{cases}$$

We show that f is injective. Suppose $n_1 \neq n_2$. For the first case suppose n_1 and n_2 are both even. Then we have $f(n_1) = n_1/2$ and $f(n_2) = n_2/2$. It follows $f(n_1) \neq f(n_2)$ otherwise we would have $n_1/2 = n_2/2$ which implies $n_1 = n_2$, a contradiction. Now suppose either n_1 is even and n_2 is odd or n_1 is odd and n_2 is even. Then without a loss of generality suppose n_1 is even and n_2 is odd. Then $f(n_1) = n_1/2$ and $f(n_2) = (1 - n_2)/2$. For all $n \in \mathbb{N} = \{1, 2, ...\}$, if n is even then $f(n) = n/2$ is a positive. For all $n \in \mathbb{N}$, if n is odd then $f(n) = (1 - n)/2 \leq 0$ or is not positive. It follows the image of the even natural numbers under f and the image of the odd numbers under f are disjoint. Therefore, if n_1 is even and n_2 is odd, then $f(n_1) \neq f(n_2)$. For the third case, suppose both n_1 and n_2 are odd and $n_1 \neq n_2$. Then we have $f(n_1) = (1 - n_1)/2$ and $f(n_2) = (1 - n_2)/2$. It follows $f(n_1) \neq f(n_2)$ otherwise we would have $(1 - n_1)/2 = (1 - n_2)/2$ which by algebra implies $n_1 = n_2$, a contradiction.

Now it is shown that f is surjective. Suppose $b \in \mathbb{Z}$. Then either $b > 0$ or $b \leq 0$. In the former case there let $x = 2b$. It follows x is an even natural number is in the domain of f. The value of f evaluated at x gives $f(x) = 2b/2 = b$. Hence, for any $b > 0$ there exists an $x \in \mathbb{N}$ such that $f(x) = b$. Now suppose the latter case. Let $b \leq 0$. Then let $x = 1 - 2b$. Since $b \leq 0$ it follows $x > 0$. Then x is in the domain of f. And f evaluated at x gives $f(x) = (1-(1-2b))/2 = b$. It follows for any $b \leq 0$ there exists an $x \in \mathbb{N}$ such that $f(x) = b$. Thus f is surjective.

□

Theorem (1.8). *Show that every subset of* $\mathbb{N}$ *is either finite or has the same cardinality as* $\mathbb{N}$.

Proof. Let S be a subset of $\mathbb{N}$. Then S is either finite or infinite. If S is finite we are done. Then suppose S is infinite. We then define a function $f \colon \mathbb{N} \to S$ as follows: Let $f(1)$ be equal to the smallest element of S. This is well-defined since $S \subset \mathbb{N}$ and $\mathbb{N}$ is well-ordered. We then define f recursively. Specifically, $f(n)$ is equal to the smallest element of the set $S - \{f(1), f(2), ..., f(n-1)\}$ for $n \in \mathbb{N}$ and $n > 1$. We now show f is injective. Suppose $n \neq m$. Then without a loss of generality let $n > m$. Then we have $n = m + k$ for some $k \in \mathbb{N}$. Then we have

$$f(n) = \text{ the smallest element of the set } S - \{f(1), f(2), ..., f(m), ..., f(m+k-1)\}$$

It follows $f(n) \neq f(m)$ since

$$f(m) \notin S - \{f(1), f(2), ..., f(m), ..., f(m+k-1)\}$$

Thus f is injective. Now we show f is surjective. Let $im(f)$ denote the image of $\mathbb{N}$ under f, or the image set of f. Then let $T = S - im(f)$. We show that $T = \varnothing$. For sake of contradiction, suppose $T \neq \varnothing$. Then T is nonempty and therefore has a smallest element x by the well ordering property. And $x \neq f(1)$ since $f(1) \in im(f)$. Then define the set T' as

$$T' = \{y \in S | y < x\}$$

We know T' is finite since $T' \subset \mathbb{N}$ and is bounded above. By Lemma 1.8.1 T' has a largest element denoted as x'. And $x' \in im(f)$ since $x' < x$. Hence $x' = f(k)$ for some $k \in \mathbb{N}$. Since x' is the largest element in S less than x, it follows that $f(k+1) = x$. Hence $x \in im(f)$ but $x \in S - im(f)$. This is a contradiction which implies f is surjective. □

Lemma. *Suppose* $T \subset \mathbb{N}$, T *is finite, and* T *is nonempty.Then there exists a largest element in* T.

Proof. Let $T \subset \mathbb{N}$ and let T be finite. We prove by induction. Let $|T| = 1$. Then let $x \in T$. Then the largest element is x. Now suppose there is a largest element for a finite subset of $\mathbb{N}$ of cardinality n. Let $|T| = n + 1$. Then we can partition T into two subsets, T_1 and T_2, where T_1 contains only the smallest element, denoted as y,which is well-defined since T is well-ordered. Let $T_2 = T - T_1$. Then $|T_2| = n$ and by the inductive hypothesis T_2 has a largest element, denoted by x. It follows x is the largest element of T since x is the largest element of T_2 and $x > y$ since y is the smallest element. This concludes the inductive step. It follows there exists a largest element for any finite subset of $\mathbb{N}$ of cardinality $n \in \mathbb{N}$. □

Theorem (1.9). *Show that every infinite set has a countable infinite subset.*

Proof. Let A be an infinite set. By Theorem 0.1 we know that for any infinite set there exists a finite subset A_n of cardinality n for all $n \in \mathbb{N}$. Now let us construct the set B as

$$B = \bigcup_{n=1}^{\infty} A_n$$

We show that B is countably infinite which proves the theorem. Let $J_n = \{1, 2, ..., n\}$. Then for each A_n there exists a one-to-one correspondece $f_n \colon J_n \to A_n$. Let $a' \in A_1$. We then define the $g \colon \mathbb{N} \times \mathbb{N} \to B$ as

$$g(n, m) = \begin{cases} f_n(m) & m \leq n \\ a' & m > n \end{cases}$$

We show g is surjective. Suppose $b \in B$. Then by definition of union we have $b \in A_{n'}$ for some $n' \in \mathbb{N}$. If $b \in A_{n'}$ then there exists some $m' \in J_{n'}$ such that $f_{n'}(m') = b$. Moreover, since $m' \in J_{n'}$ it follows $m' \leq n$. Hence, there exists some $(n', m') \in \mathbb{N} \times \mathbb{N}$ where $m' \leq n'$ such that $f_{n'}(m') = b$. So g is surjective. Since there is a surjection $g \colon \mathbb{N} \times \mathbb{N} \to B$ and we know $\mathbb{N} \times \mathbb{N}$ is countably infinite by Theorem 0.8, it follows that B must also be countably infinite by Theorem 0.5. □

Theorem (1.10). *Prove that a set is infinite if and only if there is an injection from the set into a proper subset of itself.*

Proof. We prove the forward direction first. Let A be an infinite set. By Theorem 1.9 we know there exists some countably infinite $B \subset A$. If B is countably infinite then there exists some bijection $g \colon \mathbb{N} \to B$. Let us define the $h \colon \mathbb{N} \to \mathbb{N}$ where $h(n) = n + 1$ for all $n \in \mathbb{N}$. It follows h is injective since if $h(m) = h(n)$ then we have $m + 1 = n + 1$ which implies $m = n$. Since h is injective and g is bijective it follows $g \circ h \circ g^{-1} \colon B \to B$ is injective. Suppose $g(h(g^{-1}(n))) = g(h(g^{-1}(m)))$. Since g is bijective it follows $h(g^{-1}(n)) = h(g^{-1}(m))$. Since h is injective it follows $g^{-1}(n) = g^{-1}(m)$ and because g is bijective it follows $n = m$. So $g \circ h \circ g^{-1} \colon B \to B$ is in fact injective. Moreover, $g \circ h \circ g^{-1}$ is an injection from B into a proper subset of itself, namely the set $B - \{g(1)\}$ due to h be defined as $h(n) = n + 1$. Let $f \colon A \to A$ be a bijection defined as $f(x) = x$ for all $x \in A$. Then define the bijection $j \colon (A - B) \to (A - B)$ as $j(x) = f(x)$ for all $x \in (A - B)$. Then define $k \colon A \to A$ as

$$k(x) = \begin{cases} g \circ h \circ g^{-1}(x) & x \in B \\ j(x) & x \in A - B \end{cases}$$

We know $j(x)$ and $g \circ h \circ g^{-1}$ are injective. Moreover, we know $g \circ h \circ g^{-1}$ is an injection from B to a proper subset of B since there does not exist $x \in B$ such that $(g \circ h \circ g^{-1})(x) = g(1) \in B$. It follows k is an injection from A to a proper subset of A.

Now we prove the backwards direction. Suppose we have the set A and there exists an injection from A into a proper subset of itself. We show A must be infinite. For the sake of contradiction suppose A is finite. Then $|A| = n$ for some $n \in \mathbb{N}$. We assume A is nonempty since if it were empty then it has no proper subset so there would not be an injective mapping. Now suppose there is an injection $f \colon A \to B$ where B is a proper subset of A which we denote as $B \subsetneq A$. Then let the image of A under f be equal to C or $C = im(f)$. Note that $C \subset B \subsetneq A$.Then define $g \colon A \to C$ as $g(x) = f(x)$ for all $x \in A$. It follows g is a bijection. Thus A and C must have the same cardinality. But $C \subsetneq A$ so $|C| \neq |A|$. This is a contradiction. Hence, A must be infinite. □

Theorem (1.11). *Prove the union of two countable sets is countable.*

Proof. There are three cases: (1) A and B are countably infinite, (2) one of the sets is countably infinite and the other is finite, and (3) both are finite. For case (1) let A and B be countably infinite sets. Then there exist bijections $f\colon \mathbb{N} \to A$ and $g\colon \mathbb{N} \to B$. Let $h\colon \mathbb{N} \to (A \cup B)$ be defined as

$$h(n) = \begin{cases} f\left(\frac{n+1}{2}\right) & \text{n is odd} \\ g\left(\frac{n}{2}\right) & \text{n is even} \end{cases}$$

We show h is surjective. Suppose $y \in (A \cup B)$. Then either $y \in A$ or $y \in B$. If $y \in A$ then there exists some x such that $f(x) = y$ where $x \in \mathbb{N}$ since f is bijective. Now we show there exists some odd natural number n such that $x = (n+1)/2$. Since n is an odd natural number. By doing some algebra we get $n = 2x - 1$ where $x \in \mathbb{N}$. Since $2x$ is even $2x - 1$ is odd. Moreover, $2x - 1 \geq 1$ for all $x \in \mathbb{N}$. Thus, we have found an appropriate n. Now we evaluate h at n to show n maps to y under h.

$$\begin{aligned} h(n) &= h(2x-1) \\ &= f\left(\frac{(2x-1)+1}{2}\right) \\ &= f(x) \\ &= y \end{aligned}$$

If $y \in B$ then there exists some x such that $g(x) = y$ since g is bijective. Now we show there exists some even natural number n such that $x = n/2$. Solving for x we have $n = 2x$ and since $x \in \mathbb{N}$ it follows $n = 2x$ is the desired even natural number. Now we evaluate h at n to show n maps to y under h.

$$\begin{aligned} h(n) &= h(2x) \\ &= g\left(\frac{2x}{2}\right) \\ &= g(x) \\ &= y \end{aligned}$$

It follows h is surjective. By Theorem 0.4, it follows $A \cup B$ is countably infinite.

For case (2) without a loss of generality let A by finite and B be countably infinite. Then there exist bijections $f\colon J_m \to A$ and $g\colon \mathbb{N} \to B$ where $J_m = \{1, 2, ..., m\}$. Then let the $h\colon \mathbb{N} \to (A \cup B)$ be defined as follows

$$h(n) = \begin{cases} f(n) & 1 \leq n \leq m \\ g(n-m) & n > m \end{cases}$$

Now we show h is surjective. Suppose $y \in (A \cup B)$. Then either $y \in A$ or $y \in B$. Suppose $y \in A$. Then since $f\colon J_m \to A$ is a surjection it follows there is some $n \in \mathbb{N}$ and $1 \leq n \leq m$ such that $f(n) = y$. But since $n \in J_m$ it follows $h(n) = f(n) = y$. Now suppose $y \in B$. tThen there exists some $x \in \mathbb{N}$ such that $g(x) = y$ since g is surjective. Then let $x = n - m$. It follows $n = x + m \in \mathbb{N}$ and therefore is in the domain of h. Since $x > 0$ it follows $n > m$ so then evaluating h at x gives $g(n) = g(n - m) = g(x) = y$. Thus h is surjective. Then by Theorem 0.4 it follows $A \cup B$ is countable.

For case (3) suppose both sets A and B are finite. Then there exists bijections $f\colon J_m \to A$ and $g\colon J_r \to B$. Then let $h\colon J_{m+r} \to (A \cup B)$ be defined as follows

$$h(n) = \begin{cases} f(n) & 1 \leq n \leq m \\ g(n-m) & n > m \end{cases}$$

It is shown h is surjective. Suppose $y \in (A \cup B)$. Then either $y \in A$ or $y \in B$. Let $y \in A$. Then since f is surjective there is some $n \in J_m$ such that $f(n) = y$. It follows $n \in J_{m+r}$ so evaluating h at n gives $h(n) = f(n) = y$. Now suppose $y \in B$. Then there is some $x \in J_r$ such that $g(x) = y$. Let $x = n - m$. Then $n = x + m \leq r + m$. It follows n is in the domain of h. Note since $x > 0$ so $n > m$. Then evaluating h at n gives $h(n) = g(n-m) = g(x) = y$. It follows h is surjective. Then by Theorem 0.4 $(A \cup B)$ is countable. □

Theorem (1.12). *Show that the union of countably many countable sets is countable.*

Proof. There are two cases. Either we have (1) a finite union of countable sets or (2) a countably infinite union of countable sets. We first prove case (1). Then we show

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$$

is countable for some $n \in \mathbb{N}$ and where A_i is countable for all $i \in \{1, 2, ..., n\}$. We prove this by inducting on n. For the base case let $n = 1$. Since A_1 is countable we are done. Now assume $\cup_{i=1}^{n} A_i$ and A_{n+1} are countable. Since both sets are countable, by Theorem 1.11 their union is countable. But $(\cup_{i=1}^{n} A_i) \cup A_{n+1}$ is equivalent to $\cup_{i=1}^{n+1} A_i$. It follows

$$\bigcup_{i=1}^{n+1} A_i = A_1 \cup A_2 \cup \cdots \cup A_{n+1}$$

is also countable. This concludes the induction. Thus, a finite union of countable sets is countable.

Now we prove case (2). Suppose we have a countably infinite union of countable sets denoted by S.

$$S = \bigcup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup \cdots$$

Our first goal is to construct the surjection $f \circ h\colon \mathbb{N} \to S$ which is to be defined later. First, for each A_i there is an associated bijection g_i^* with $\mathbb{N}$ or $\{1, 2, ..., n\}$. If A_i is finite then $g_i^*\colon \{1, ..., n\} \to A_i$. If A_i is countably infinite then $g_i^*\colon \mathbb{N} \to A_i$. Then for each A_i we construct an associated surjection $g_i\colon \mathbb{N} \to A_i$. If A_i is finite then $g_i\colon \mathbb{N} \to A_i$ is defined as

$$g_i(m) = \begin{cases} g_i^*(m) & 1 \leq m \leq n \\ g_i^*(1) & m > n \end{cases}$$

If A_i is countably infinite then $g_i\colon \mathbb{N} \to A_i$ is defined as $g_i(m) = g_i^*(m)$ for all $m \in \mathbb{N}$. Then we construct the surjection $f\colon \mathbb{N} \times \mathbb{N} \to S$ defined as $f(i, m) = g_i(m)$. By Theorem 0.8 we know $\mathbb{N} \times \mathbb{N}$ is countably infinite. Therefore, there exists a bijection $h\colon \mathbb{N} \to \mathbb{N} \times \mathbb{N}$. It follows the composite $f \circ h\colon \mathbb{N} \to S$ is a surjection. Now our goal is to construct an injection from $f \circ h$ and then a bijection.

Now we define $k\colon S \to \mathbb{N}$ as

$$k(x) = \text{ minimum of } (f \circ h)^{-1}(x)$$

Moreover, k is well-defined for all values of $x \in S$ since $f \circ h$ is surjective, and the minimum is well-defined since $\mathbb{N}$ (and any subset of it) is well-ordered. Now we show k is injective. Suppose $x_1 \neq x_2$. For the sake of contradiction assume the minimum of $(f \circ h)^{-1}(x_1)$ and minimum of $(f \circ h)^{-1}(x_2)$ are equal and denote this value as x'. Then we have $x_1 = (f \circ h)(x')$ and $x_2 = (f \circ h)(x')$ but we assumed $x_1 \neq x_2$. It follows $k(x_1) \neq k(x_2)$. Hence, k is injective. However, k is not necessarily surjective. But if we restrict the range of k to the image set of k and define a new function k' with the restricted range, then k' is surjective. Thus, define $k' \colon S \to im(k)$ as $k'(x) = k(x)$ for all $x \in S$. Since k is injective k' is also injective. Thus k' is a bijection. Moreover, $im(k) \subset \mathbb{N}$, and it follows from Theorem 1.8 that $im(k)$ is either finite or has the same cardinality as $\mathbb{N}$. In either case $im(k)$ is countable. Since k' is a bijection between S and a countable set $im(k)$, it follows S must also be countable. □

Theorem (1.13). *The set $\mathbb{Q}$ is countable.*

Proof. Define the function $f \colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Q}$ as follows

$$f(n,m) = \begin{cases} n/m & m \neq 0 \\ 0 & m = 0 \end{cases}$$

It is shown f is surjective. Suppose $q \in \mathbb{Q}$. Then there $q = \dfrac{n}{m}$ for some $n, m \in \mathbb{Z}$. Since $n, m \in \mathbb{Z}$ it follows $(n,m) \in \mathbb{Z} \times \mathbb{Z}$. Since (n,m) is in the domain of f we can evaluate f at (n,m) which gives

$$f(n,m) = \frac{n}{m} = q$$

It follows f is surjective. Since $\mathbb{Z} \times \mathbb{Z}$ is a finite cartesian product of countably infinite sets, by Theorem 0.7 it follows $\mathbb{Z} \times \mathbb{Z}$ is countably infinite. Thus we have a surjection from a countably infinite set to $\mathbb{Q}$ which implies $\mathbb{Q}$ is countably infinite by Theorem 0.5. □

Theorem (1.14). *The set of all finite subsets of a countable set is countable.*

Proof. Let A be a countably infinite set. Then there is a bijection $f \colon \mathbb{N} \to A$. Let A_k be the set of all finite subsets of cardinality k. Then we define $g_k \colon \mathbb{Z}^k \to A_k$ as

$$g_k(x_1, x_2, ..., x_k) = \{f(x_1), f(x_2), ..., f(x_k)\}$$

We show that g_k is surjective. Suppose $Y \in A_k$. Then $Y = \{y_1, y_2, ..., y_k\}$ where $y_i \in A$ for all $1 \leq i \leq k$. Since f is a bijection is follows for each $y_i \in A$ there exists some $x_i \in \mathbb{N}$ such that $f(x_i) = y_i$. It follows there exists some $(x_1, x_2, ..., x_k)$ such that $g(x_1, x_2, ..., x_k) = Y$. Note $\mathbb{Z}^k$ is countably infinite by Theorem 0.7. Since g_k is surjective and $\mathbb{Z}^k$ is countably infinite, by Theorem 0.5 it follows A_k is countably infinite. Let B be the set of all finite subsets of the countably infinite set A. Then $B = \cup_{k=1}^{\infty} A_k$. Then B is a countable union of countable sets. Therefore by Theorem 1.12 it follows B is countable.

Now suppose A is finite. If a set is finite, then it has cardinality $|A| = n$ for some $n \in \mathbb{N}$. By Theorem 1.18 it follows the set of all finite subsets (the power set) has cardinality 2^n. Thus the set of all finite subsets of A has a bijection $f \colon 2^n \to 2^A$ and is therefore finite. □

Exercise (1.15). Suppose a submarine is moving in the plane along a straight line at a constant speed such that at each hour, the submarine is at a lattice point, that is, a point whose two coordinates are both integers. Suppose at each hour you can explode one depth charge at a lattice point that will hit the submarine if it is there. You do not know the submarine's direction, speed, or its current position. Prove that you can explode one depth charge each hour in such a way that you will be guaranteed to eventually hit the submarine.

Proof. Say the submarine is in the $\mathbb{R}^2$ plane. Since we don't know the direction, speed, or position of the submarine in the plane, the general form of the position of the submarine with respect to time is

$$(x_0 + at, y_0 + bt)$$

where x_0 is the initial x-coordinate, a is the horizontal velocity, y_0 is the initial y-coordinate, b is the vertical velocity, and t is the time past the starting time. We have $x_0, a, y_0, b \in \mathbb{Z}$ and $t \in \mathbb{N}$. We know $\mathbb{Z}^4$ is a finite cartesian product of countably infinite sets and is therefore countable by Theorem 0.7. Then there is the bijection $f\colon \mathbb{N} \to \mathbb{Z}^4$ defined as

$$f(t) = (f_x(t), f_a(t), f_y(t), f_b(t))$$

Since f is a bijection it follows for any choice of x_0, a, y_0, and b there exists some $t \in \mathbb{N}$ such that $f(t) = (x_0, a, y_0, b)$. Since $\mathbb{Z}^4$ is countable it follows we can check every possible direction, speed, and original position of the submarine. Suppose we want to hit the submarine with initial horizontal position x_0, horizontal velocity a, initial vertical position y_0, and vertical velocity b. Then at time $t_0 = f^{-1}(x_0, a, y_0, b)$ we explode a depth charge at coordinate

$$(x, y) = (f_x(t_0) + f_a(t_0)t_0, f_y(t_0) + f_b(t_0)t_0)$$

It follows we can be guaranteed to eventually hit the submarine. □

Theorem (1.16). *(Cantor's Theorem). The cardinality of the set of natural numbers is not the same as the cardinality of the set of real numbers. That is, the set of real numbers is uncountable.*

Proof. Let us restrict have a list of real numbers on the interval $[0, 1]$ such as

$$\begin{gathered} 0.0000... \\ 0.9999... \\ 0.2934... \\ 0.1298... \\ 0.0209... \end{gathered}$$

If we take the 1st digit of the 1st number, 2nd digit of the 2nd number, and forever we would get 0.999... and we could change each digit in the expansion so that our created digit does not have the same decimal expansion as any other. For example, we could change 0.999... to 1.000.... However, 1.000... is the same value as 0.999... even though they have different decimal expansions. Therefore we cannot guarantee that our created digit would not be in an infinite list of real numbers. Now we remove the real numbers with multiple decimal expansions from our inteverl $[0, 1]$. If a real number has multiple decimal expansion in base 10 then there is a pair of two expansions: one with trailing zeroes and one with trailing nines. Let A_n be the set of all decimal expansions within the interval $[0, 1]$ that have trailing nines starting at the decimal place corresponding to 10^{-1}. For example $A_1 = \{0.\overline{9}\}$ and $A_2 = \{0.0\overline{9}, 0.1\overline{9}, ..., 0.8\overline{9}\}$. Generally, we have $|A_n| = 10^{n-1} - 1$ for $n \geq 2$ and $|A_1| = 1$. And for each A_n where $n \in \mathbb{N}$ each element in A_n corresponds to the unique value of a real number. And the union of A_n for all $n \in \mathbb{N}$ corresponds to all real numbers with pairs of decimal expansions since a real number with multiple decimal expansions between the interval $[0, 1]$ must contain a decimal expansion with trailing nines past the decimal point. Let $S = \cup_{i=1}^{\infty} A_i$ which is a countable union of countable sets. By Theorem 1.12, it follows S is countable. Now suppose we consider the set $T = [0, 1] - S$. Then we list out a series of real numbers in this set below

$$\begin{array}{cc} 1 & 0.0102... \\ 2 & 0.9992... \\ 3 & 0.2934... \\ 4 & 0.1298... \\ 5 & 0.0209... \\ \vdots & \vdots \end{array}$$

Now if we create a new decimal by the algorithm described above then we create the decimal 0.9999... which is not in our list since we have removed all real numbers with multiple decimal expansions. It follows $1 = 0.9999...$ is not in our list. Moreover, since we have associated each natural number with some element in our set $T \subset [0, 1]$ it follows we have defined a function $f \colon \mathbb{N} \to [0, 1]$. And since $0.9999... \notin T$ it follows f is not a surjection. Generally, if we have any list of real numbers from the set T and create a new decimal with the algorithm, then we will find that the new decimal is not in the list. Therefore there exists no surjection from $\mathbb{N}$ to $[0, 1] \in \mathbb{R}$ and therefore there exists no surjection from $\mathbb{N} \to \mathbb{R}$. Hence, $\mathbb{R}$ is uncountable. □

Exercise (1.17). Suppose $A = \{a, b, c\}$. Explicitly write out 2^A, the power set of A.

$2^A = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

Theorem (1.18). *If a set A is finite, then the power set of A has cardinality $2^{|A|}$, that is, $|2^A| = 2^{|A|}$.*

Proof. We prove by induction. Suppose A has a cardinality of 0. Then $A = \varnothing$. Then $2^A = \{\varnothing\}$.Then $\{\varnothing\}$ has a bijection with $\{1\} \subset \mathbb{N}$. Thus 2^A is finite and $|2^A| = 2^0$. This concludes the basis step and now we move to the inductive step. Now suppose we know that if a set A is finite with cardinality of $n \in \mathbb{N}$. It follows there is a bijection $f \colon 2^n \to 2^A$. We then define $h \colon 2^{n+1} \to 2^{\{A \cup \{x\}\}}$ where $x \notin A$ as follows

$$h(i) = \begin{cases} f(i) & 1 \leq i \leq 2^n \\ f(i - 2^n) \cup \{x\} & 2^n + 1 \leq i \leq 2^{n+1} \end{cases}$$

We first show h is surjective. Let $A^* \in 2^{A \cup \{x\}}$. We show there exists $i \in \{1, 2, ..., 2^{n+1}\}$ such that $h(i) = A^*$. There are two cases: either (1) $x \in A^*$ or (2) $x \notin A^*$. Suppose we have case (1). Then let $B = A^* - \{x\}$. Since $A^* \subset A \cup \{x\}$ and we know $A \cap \{x\} = \varnothing$, it follows $A^* - \{x\} \subset A$ so then $B \subset A$. Then B is in the range of f which is bijective. Therefore, there exists $i \in \{1, ..., 2^n\}$ such that $f(i) = B$. Let $j = i + 2^n$. Since $1 \leq i \leq 2^n$ it follows $2^n + 1 \leq j \leq 2^{n+1}$. Then evaluating h at j gives

$$h(j) = f(j - 2^n) \cup \{x\} = f(i + 2^n - 2^n) \cup \{x\} = f(i) \cup \{x\} = B \cup \{x\} = A^*$$

Now suppose we have case (2). We know $A^* \subset A \cup \{x\}$ and $A \cap \{x\} = \varnothing$ and since $x \notin A^*$ it follows $A^* \subset A$. Thus, there exists $i \in \{1, ..., 2^n\}$ such that $f(i) = A^*$ where $1 \leq i \leq 2^n$.

Then evaluating h at i gives
$$h(i) = f(i) = A^*$$
Thus for any $A^* \in 2^{A\cup\{x\}}$ there exists $i \in \{1, ..., 2^{n+1}\}$ such that $h(i) = A^*$. Hence, h is a surjection.

We show h is injective. Let $a, b \in \{1, 2, ..., 2^{n+1}\}$ and $a \neq b$. There are three cases: (1) $1 \leq a, b \leq 2^n$, (2) $2^n + 1 \leq a, b \leq 2^{n+1}$, or (3) either $1 \leq a \leq 2^n$ and $2^n + 1 \leq b \leq 2^{n+1}$ or vice versa. Suppose we have case (1). Then we have $h(a) = f(a)$ and $h(b) = f(b)$ but $f(a) \neq f(b)$ since f is bijective. So $h(a) \neq h(b)$. Suppose we have case (2). Then we have $h(a) = f(a - 2^n) \cup \{x\}$ and $h(b) = f(b - 2^n) \cup \{x\}$. We know $f(a - 2^n) \neq f(b - 2^n)$ since f is bijective. Moreover, we know $x \notin f(i)$ for all $i \in \{1, ..., 2^n\}$, thus $f(a - 2^n) \cup \{x\} \neq f(b - 2^n) \cup \{x\}$. Now suppose we have case (3). Without a loss of generality let $1 \leq a \leq 2^n$ and $2^n + 1 \leq b \leq 2^{n+1}$. Then $x \notin h(a)$ but $x \in h(b)$. Thus $h(a) \neq h(b)$. Thus h is injective. Since we have shown h is both surjective and injective, it follows h is bijective. Since there is a bijection from 2^{n+1} and $2^{A\cup\{x\}}$ it follows the set $A \cup \{x\}$ with cardinality $|A \cup \{x\}| = n + 1$ has a power set of cardinality $2^{|A\cup\{x\}|} = 2^{n+1}$. This concludes the induction. ☐

Theorem (1.19). *For any set A, there is an injection from A into 2^A.*

Theorem. *For a set A, let P be the set of all functions from A to the two-point set $\{0, 1\}$. Then $|P| = |2^A|$.*

Theorem (1.21). *There is a one-to-one correspondence between $2^{\mathbb{N}}$ and the set of all infinite sequences $0's$ and $1's$.*

Proof. Suppose we have a sequence $a_1a_2a_3\cdots$ where for each a_i and $i \in \mathbb{N}$ we have $a_i = 0$ or $a_i = 1$. Then the bijection from the set of all possible sequences to $2^{\mathbb{N}}$ is defined as
$$f(a_1a_2a_3\cdots) = \{i \in \mathbb{N} | a_i = 1\}$$
We show f is injective. Suppose we have $a = a_1a_2a_3\cdots \neq b_1b_2b_3\cdots$. Then there exists some $i \in \mathbb{N}$ such that $a_i \neq b_i$. Without a loss of generality suppose $a_i = 1$ and $b_i = 0$. Then $i \in f(a_1a_2\cdots)$ but $i \notin f(b_1b_2\cdots)$. It follows $f(a_1a_2\cdots) \neq f(b_1b_2\cdots)$. Now we show f is surjective. Suppose $A \in 2^{\mathbb{N}}$. For we define a_i for each $i \in \mathbb{N}$ as follows
$$a_i = \begin{cases} 1 & i \in A \\ 0 & i \notin A \end{cases}$$
Then $f(a_1a_2\cdots)$ gives A since for each $i \in A$ we defined $a_i = 1$ which then implies $i \in f(a_1a_2\cdots)$ and if $i \notin A$ we defined $a_i = 0$ which implies $i \notin f(a_1a_2\cdots)$. It follows f is surjective. Since f is both injective and surjective, it follows there is a one-to-one correspondence between $2^{\mathbb{N}}$ and the set of all infinite sequences of 0's and 1's. ☐

Theorem (1.22). *(Cantor's Power Set Theorem). There is no surjection from set A onto 2^A. Thus for any set A, the cardinality of A is not the same as the cardinality of its power set. In other words, $|A| \neq |2^A|$.*

Proof. Suppose there is a surjection $f: A \to 2^A$. Define the set B as
$$B = \{a \in A \mid a \in A - f(a)\}$$
B is then some subset of A. It follows $B \in 2^A$. Since f is surjective, it follows there exists some $a_0 \in A$ such that $f(a_0) = B$. Since $B \subset A$ either $a_0 \in B$ or $a_0 \in A - B$. If $a_0 \in B$ then $a_0 \in A - f(a_0)$ so $a_0 \in A - B$ which is a contradiction. Now suppose $a_0 \notin B$. Then

$a_0 \in A - f(a_0)$ so then $a_0 \in B$ which is a contradiction. It follows, there does not exist a surjection f. $\square$

Problem. Let $\mathbb{Q}[x]$ denote the set of polynomials with coefficients in $\mathbb{Q}$. Let

$$\mathbb{A} = \{\alpha \in \mathbb{C} | f(\alpha) = 0 \text{ for some } f \in \mathbb{Q}[x]\}$$

Prove that $\mathbb{A}$ is countable.

Proof. The set of all nth degree polynomial with coefficients in $\mathbb{Q}$ corresponds to $\mathbb{Q}^n$ where each tuple represents the coefficients of the polynomial. For each nth degree polynomial, there is at most n complex roots. If every nth degree polynomial had exactly n distinct complex roots then the Cartesian product $\mathbb{Q}^n \times \{1, 2, ..., n\}$ would be a bijection with the set of complex roots of all nth degree polynomials in $\mathbb{Q}[x]$. But since different nth degree polynomials in $\mathbb{Q}[x]$ may share complex roots and not every nth degree polynomial has n complex roots, it follows $\mathbb{Q}^n \times \{1, 2, ..., n\}$ has at least the same cardinality as A_n where A_n is defined as

$$A_n = \{\alpha \in \mathbb{C} | f(\alpha) = 0 \text{ for some } f \in \mathbb{Q}[x] \text{ and } f \text{ is degree n}\}$$

So then $|\mathbb{Q}^n \times \{1, 2, ..., n\}| \geq |A_n|$. So then we have

$$|\bigcup_{n=1}^{\infty}(\mathbb{Q}^n \times \{1, 2, ..., n\})| \geq |\bigcup_{n=1}^{\infty} A_n|$$

But we know $\mathbb{Q}^n$ is a finite product of countably infinite sets so it is countably infinite by Theorem 0.7 But $\mathbb{Q}^n \times \{1, 2, ..., n\}$ is a finite Cartesian product of at least one countable infinite set ($\mathbb{Q}^n$) so then by Theorem 0.7 again we know $\mathbb{Q}^n \times \{1, 2, ..., n\}$ must also be countably infinite. Then

$$\bigcup_{n=1}^{\infty}(\mathbb{Q}^n \times \{1, 2, ..., n\})$$

is a countably infinite union of countably infinite sets so by Theorem 1.12 we know that this countably infinite union is also countably infinite. This means there is a bijection between $\bigcup_{n=1}^{\infty} (\mathbb{Q}^n \times \{1, 2, ..., n\})$ and $\mathbb{N}$ and we know by Theorem 1.8 that every subset of $\mathbb{N}$ is countable. Moreover, we know $\bigcup_{n=1}^{\infty} A_n = \mathbb{A}$. Then

$$|\bigcup_{n=1}^{\infty}(\mathbb{Q}^n \times \{1, 2, ..., n\})| = |\mathbb{N}| \geq |\bigcup_{n=1}^{\infty} A_n| = |\mathbb{A}|$$

It follows $\mathbb{A}$ is countable. $\square$

Theorem (1.25). *If A and B are sets such that there exist injections f from A into B and g from B into A, then $|A| = |B|$.*

Proof. Define $S_0 = A - g(B)$. Then define S_n recursively as $S_n = g(f(S_{n-1}))$ for $n = 1, 2, ...$ Then define the set S as $S = \cup_{n=0}^{\infty} S_n$. Using set S we construct the function $h \colon A \to B$ as follows

$$h(x) = \begin{cases} g^{-1}(x) & x \in A - S \\ f(x) & x \in S \end{cases}$$

First we show that the function h is well-defined. A possible problem is g^{-1} is not defined on the entire set $A - S$. However, by definition of S we know $A - g(B) \subset S$ so then $A - S \subset g(B)$. Hence, g^{-1} is well defined on the set $A - S$. Now we show that h is surjective. Suppose $y \in B$. We must show either $y \in f(S)$ or $y \in g^{-1}(A - S)$. Suppose $y \notin f(S)$. Since g is injective it follows $g(y) \notin g(f(S))$. But $g(f(S)) = \cup_{n=1}^{\infty} S_n$ so then $g(f(S)) = S - S_0$. So we have $g(y) \notin S - S_0$. But $g(B) \cap S_0 = \varnothing$ so then $g(y) \notin S$ or $g(y) \in A - S$. It follows $y \in g^{-1}(A - S)$. This implies h is surjective. Now we show h is injective. There are three cases: (1) both $x, y \in A - S$, (2) both $x, y \in S$, or (3) one is in $A - S$ and the other is in S. Suppose we have case (1). Then $x \neq y$ and $x, y \in A - S$. Then $h(x) = g^{-1}(x)$ and $h(y) = g^{-1}(y)$. We know g^{-1} is injective since g is a function. Thus if $x \neq y$ then $g^{-1}(x) \neq g^{-1}(y)$. Now suppose we have case (2). Since f is injective, it follows $f(x) \neq f(y)$ if $x \neq y$. Now suppose we have case (3). Without a loss of generality, let $x \in A - S$ and $y \in S$. For the sake of contradiction, suppose $g^{-1}(x) = f(y)$. Since $f(y) \in f(S)$ then $g^{-1}(x) \in f(S)$. It follows $x \in g(f(S))$ so then $x \in S$. But this is a contradiction. It follows h is injective. Since h is both surjective and injective, it follows h is a bijection between A and B. Thus $|A| = |B|$. $\square$

Theorem (1.27). $|\mathbb{R}| = |(0,1)| = |\,[0,1]\,|$.

Proof. We construct injections between $\mathbb{R}$ and $(0,1)$. First, define the injection $f\colon \mathbb{R} \to (0,1)$ as $f(x) = \frac{1}{\pi}\arctan(x) + \frac{1}{2}$. Then construct the injection $g\colon (0,1) \to \mathbb{R}$ defined as $g(x) = x$. Then by the Schroeder-Bernstein theorem it follows $|\mathbb{R}| = |(0,1)|$. Now we construct injections between $\mathbb{R}$ and $[0,1]$. Define the injection $f\colon \mathbb{R} \to [0,1]$ as $f(x) = \frac{1}{\pi}\arctan(x) + 0.5$. Then define the injection $g\colon [0,1] \to \mathbb{R}$ defined as $g(x) = x$. Again, by the Schroeder-Bernstein theorem it follows $|\mathbb{R}| = |\,[0,1]\,|$. Now we construct injections between $(0,1)$ and $[0,1]$. Define the injection $f\colon (0,1) \to [0,1]$ as $f(x) = x$. Then define the injection $g\colon [0,1] \to (0,1)$ as $g(x) = \frac{x}{2} + \frac{1}{4}$. Then by the Schroeder-Bernstein theorem it follows $|(0,1)| = |\,[0,1]\,|$. $\square$

Theorem (1.30). $|\mathbb{R}| = |2^{\mathbb{N}}|$.

Proof. Since we proved that $|\,[0,1]\,| = \mathbb{R}$ in Theorem 1.27 if suffices to show that $|2^{\mathbb{N}}| = |\,[0,1]\,|$. Moreover, we proved in the Lemma below that the set of all sequences consisting of 0's and 1's has the same cardinality as $2^{\mathbb{N}}$. Let the set of all sequences of 0's and 1's be denoted as S. Then it is sufficient to show $|S| = |\,[0,1]\,|$. We now construct injections between these two sets. First we construct injection $f\colon [0,1] \to S$ defined as follows. Let $x \in [0,1]$ and be the binary representation of the real number. If there are two such representations such as $1.0000\cdots = 0.1111\cdots$ then choose the one with trailing 0's. Then let the ith digit of the binary representation be the ith digit of the sequence in S. For example, if $x = 0.101\cdots$ then $x_1 = 0, x_2 = 1, x_3 = 0, x_4 = 1$ and then the sequence $s = f(x) \in S$ is $s = f(x) = 0101\cdots$ so $s_1 = 0, s_2 = 1, s_3 = 0$, and generally $s_i = x_i$. Now we show f is injective. Suppose we have two different real numbers $x, y \in [0,1]$ and $x \neq y$. Then they must have different sequences in S otherwise if they did have the same sequence we would have $x = y$ which is a contradiction. Thus, f is injective. Now we define the injection $g\colon S \to [0,1]$ as

$$g(x) = \begin{cases} \frac{x}{2} + \frac{1}{2} & \text{x has a trailing sequence of 1's} \\ \frac{x}{2} & \text{otherwise} \end{cases}$$

Note that x is a sequence so when we say $x/2$ this means $x_{i+1} := x_i$. and $x_1 = 0$. In other words, dividing by $x/2$ for the sequence x is equivalent to shifting the digits to the right by one place value. And adding $1/2$ to some sequence x means adding $x = x_1x_2x_3\cdots$ to $01000\cdots$ after converting them to the associated real number in base 2. So we are adding

$$x + 1/2 = x_1.x_2x_3\cdots + 0.1000\cdots$$

Now it is shown g is injective. Suppose we have two sequences $x, y \in S$ such that their corresponding real numbers are the same such as 0111... and 1000... Then one must have a trailing sequence of 1's and the other a trailing sequence of 0's. Without a loss of generality let x have a trailing sequence of 1's and y have a trailing sequence of 0's. Then $g(x) = x/2+1/2$ and $g(y) = y/2$. It follows $g(x) \neq g(y)$ since the real values $x/2$ and $y/2$ are equal $x/2 = y/2$ so then $g(x) = x/2 + 1/2 \neq y/2 = g(y)$. Now suppose x and y are two sequences associated with two different real numbers. Then there are three cases: (1) both have a trailing sequence of 1's, (2) both do not have a trailing sequence of 1's, and (3) one has a trailing sequence of 1's and the other does not. If we have case (1) then we use $g(x) = x/2 + 1/2$ but the function $x/2 + 1/2$ is injective. If we have case (2) then we use $g(x) = x/2$ which is also injective. Now consider case (3). Suppose we define functions $h\colon [0, 1] \to [0, 1]$ and $r\colon [0, 1] \to [0, 1]$ as $h(x) = \frac{x}{2} + \frac{1}{2}$ and $r(x) = \frac{x}{2}$. Then $im(h) = \left[\frac{1}{2}, 1\right]$ and $im(r) = \left[0, \frac{1}{2}\right]$. Then $im(h) \cap im(r) = \frac{1}{2}$. It follows that it is sufficient to show that if we have two binary sequences x, y with different real numbers and satisfy case (3), then it is impossible for such a pair to satisfy $g(x) = \frac{1}{2} = g(y)$. Without a loss of generality let x have a trailing sequence of 1's and y does not. Then we must have $g(y) = y/2 = 1/2$. The only possible sequence y is 1000.... However, there does not exist any sequence x such that $x/2 + 1/2 = 1/2$ since the only possible sequence is 0000.... but x was defined to be a sequence of trailing 1's. It follows g is injective for all three cases. Since we have injections g and f, by the Schroeder-Bernstein Theorem it follows $|2^{\mathbb{N}}| = |S| = |\,[0, 1]\,| = |\mathbb{R}|$. □

Lemma (1.30). $2^{\mathbb{N}}$ *has the same cardinality as the set of all sequences consisting of 0's and 1's.*

Proof. Suppose we have a sequence of 0's and 1's. Such a sequence can be represented by a list of digits

$$x_1x_2x_3\cdots$$

where the x_i digit is either 0 or 1. Let the set of all binary sequences be denoted by the set S. Now we construct a bijection between S and $2^{\mathbb{N}}$. Define the bijection $f\colon S \to 2^{\mathbb{N}}$ as

$$f(x) = \{i \in \mathbb{N} | x_i = 1\}$$

Now we show that f is surjective. Suppose $A \in 2^{\mathbb{N}}$. Then A is some subset of the natural numbers. Then construct a binary sequence x such that $x_i = 1$ if $i \in A$ and $x_i = 0$ if $i \notin A$. It follows $x \in S$ and $f(x) = A$ so f is surjective. Now we show f is injective. Suppose we have two distinct binary sequences x and y. If $x \neq y$ then for some $i \in \mathbb{N}$ we have $x_i \neq y_i$. Without a loss of generality let $x_i = 1$ and $y_i = 0$. Then $i \in f(x)$ but $i \notin f(y)$ so then $f(x) \neq f(y)$. Hence, f is injective. Since f is both surjective and injective, it follows f is a bijection. Thus, the cardinality of S and $2^{\mathbb{N}}$ are the same. □

Definition. Suppose X is a set. Then $\mathcal{T}$ is a topology on X if and only if $\mathcal{T}$ is a collection of subsets of X such that

(1) $\varnothing \in \mathcal{T}$
(2) $X \in \mathcal{T}$
(3) if $U \in \mathcal{T}$ and $V \in \mathcal{T}$, then $U \cap V \in \mathcal{T}$
(4) if $\{U_\alpha\}_{\alpha\in\lambda}$ is any collection of sets of $\mathcal{T}$, then $\bigcup_{\alpha\in\lambda} U_\alpha \in \mathcal{T}$

Definition. A topological space is an ordered pair $(X, \mathcal{T})$, where X is a set and $\mathcal{T}$ is a topology on X. We use the word space to mean topological space unless otherwise noted.

Definition. A set $U \subset X$ is called an open set in $(X, \mathcal{T})$ if and only if $U \in \mathcal{T}$.

Theorem (2.1). *Let $\{U_i\}_{i=1}^n$ be a finite collection of open sets in a topological space $(X, \mathcal{T})$. Then $\bigcap_{i=1}^n U_i$ is open.*

Proof. Let $\{U_i\}_{i=1}^n$ be a finite collection of open sets in $(X, \mathcal{T})$. We show $\bigcap_{i=1}^n U_i$ is open by induction. For the base case, suppose we have a collection of one open set U_1. Then $\bigcup_{i=1}^1 U_1 = U_1$ is open so we are done. Now suppose we know for any finite collection $\{V_i\}_{i=1}^n$ that $\cap_{i=1}^n V_i$ is an open set in $(X, \mathcal{T})$. Now suppose we have a finite collection of open sets $\{U_i\}_{i=1}^{n+1}$. Then we know

$$(\bigcap_{i=1}^{n+1} U_i) = (\bigcap_{i=1}^{n} U_i) \cap U_{n+1}$$

which a intersection of two sets. We know the left set is open by the inductive hypothesis and the right is open by assumption. And we know by definition of a topology that the intersection of two open sets must be open. Hence $\cap_{i=1}^{n+1} U_i$ is open. This concludes the induction. $\square$

Exercise (2.2). Why does your proof not prove the false statement that the infinite intersection of open sets is necessarily open?

We can supply a counterexample. Suppose you had the standard topology on the real numbers, $\mathbb{R}_{std}$. Let $I_n = (-1/n, 1/n)$. Then the infinite intersection $\cap_{i=1}^{\infty} = \{0\}$ which is not open in $\mathbb{R}_{std}$.

Theorem (2.3). *A set U is open in a topological space $(X, \mathcal{T})$ if and only if for every point $x \in U$, there exists an open set U_x such that $x \in U_x \subset U$.*

Proof. First we prove the forward direction. Suppose U is an open set in a topological space $(X, \mathcal{T})$. Then for every $x \in U$ we know U is an open set such that $x \in U \subset U$. Thus choose $U_x := U$. Now we prove the reverse direction. Suppose for every point $x \in U$, there exists an open set U_x such that $x \in U_x \subset U$. Let $G = \cup_{x\in U} U_x$. Since any arbitrary union of open sets is open it follows G is open. We show $G = U$. First we show $G \subset U$. Let $p \in G$. Then $p \in U_x$ for some $x \in U$. But we know $U_x \subset U$ so then $p \in U$. Thus $G \subset U$. Now we show $U \subset G$. Let $p \in U$. Then $p \in U_p$. Since $G = \cup_{p\in U} U_p$ it follows $p \in G$. Thus $U \subset G$. Since $G \subset U$ and $G \subset U$ it follows $G = U$. Since G is open it follows U is also open. $\square$

Exercise (2.5). Verify that the discrete, indiscrete, finite complement, and countable complement topologies are indeed topologies for any set X.

Proof. First we show the discrete topology is a topology on any set X. The discrete topology is the collection 2^X so it contains all subsets of X including $\varnothing$ and X. Thus $\varnothing, X \in \mathcal{T}_{\text{discrete}}$. Suppose $U \in \mathcal{T}_{\text{discrete}}$ and $V \in \mathcal{T}_{\text{discrete}}$. Then U and V are subsets of X. It follows $U \cap V$ is also a subset of X. So then $U \cap V \in \mathcal{T}_{\text{discrete}}$. Now suppose we have a collection of subsets of X denoted as $\{U_\alpha\}_{\alpha\in\lambda}$. Then the union over the collection $\cup_{\alpha\in\lambda}U_\alpha$ is also a subset of X. For sake of contradiction, suppose the union over the collection was not a subset of X. Then there exists some $x \in \cup_{\alpha\in\lambda}U_\alpha$ such that $x \notin X$. Then $x \in U_\alpha$ for some α. But if $x \notin X$ then $U_\alpha \not\subset X$. But this contradicts the assumption that we have a collection of subsets of X. Thus the union $\cup_{\alpha\in\lambda}U_\alpha \subset X$ and thus $\cup_{\alpha\in\lambda}U_\alpha \in 2^X$ and is open. It follows the discrete topology is a topology on X.

Now we show the indiscrete topology is a topology on any set X. It follows $\mathcal{T} = \{\varnothing, X\}$ so then $\varnothing \in \mathcal{T}$ and $X \in \mathcal{T}$. Also there is only one intersection $X \cap \varnothing = \varnothing \in \mathcal{T}$. And there is only one union $X \cup \varnothing = X \in \mathcal{T}$. Thus the indiscrete topology is a topology on any set X.

Now we show the finite complement (cofinite) topology is a topology on X. By definition of a cofinite topology we have $\varnothing \in \mathcal{T}_{\text{cofinite}}$. Since $X - X = \varnothing$ which is finite, it follows $X \in \mathcal{T}_{\text{cofinite}}$. Now suppose $U \in \mathcal{T}_{\text{cofinite}}$ and $V \in \mathcal{T}_{\text{cofinite}}$. Then $X - U$ and $X - V$ are both finite. Then $X - (U \cap V) = (X - U) \cup (X - V)$ which is a finite union of finite sets. This is finite. It follows $X - (U \cap V)$ is finite. Thus $U \cap V \in \mathcal{T}_{\text{cofinite}}$. Now suppose we have a collection of open sets $\{U_\alpha\}_{\alpha\in\lambda}$. If we take the union over the collection we have $\cup_{\alpha\in\lambda}U_\alpha$. If we take the complement we get $X - \cup_{\alpha\in\lambda}U_\alpha$ which by DeMorgan's Law is $\cap_{\alpha\in\lambda}(X - U_\alpha)$. Let $\beta \in \lambda$. Then we know $X - U_\beta$ is finite. Since $\cap_{\alpha\in\lambda}(X - U_\alpha) \subset X - U_\beta$ it follows $\cap_{\alpha\in\lambda}(X - U_\alpha)$ must be finite. Thus $\cup_{\alpha\in\lambda}U_\alpha \in \mathcal{T}_{\text{cofinite}}$. It follows the finite complement topology is a topology on X.

Now we show the countable complement topology is a topology on X. Since $X - X = \varnothing$ which is countable it follows $X \in \mathcal{T}$. By definition of the countable complement topology we know $\varnothing \in \mathcal{T}$. Suppose $U, V \in \mathcal{T}$. Then $X - (U \cap V) = (X - U) \cup (X - V)$. Since $U, V \in \mathcal{T}$ we know $X - U$ and $X - V$ are countable. Then $(X - U) \cup (X - V)$ is a countable union of countable sets which we proved in the last homework (Theorem 1.12). Thus, $X - (U \cap V)$ is countable. Hence, $U \cap V \in \mathcal{T}$. Now suppose we have a collection of open sets denoted as $\{U_\alpha\}_{\alpha\in\lambda}$. Then taking the complement and applying DeMorgan's Law we get $X - \{U_\alpha\}_{\alpha\in\lambda} = \cap_{\alpha\in\lambda}(X - U_\alpha)$. Let $\beta \in \lambda$. We know $X - U_\beta$ is countable since $U_\beta \in \mathcal{T}$. And we know $\cap_{\alpha\in\lambda}(X - U_\alpha) \subset X - U_\beta$. It follows $\cap_{\alpha\in\lambda}(X - U_\alpha)$ is at most countable. Thus $\cup_{\alpha\in\lambda}U_\alpha \in \mathcal{T}$. It follows the countable complement topology is a topology on X. ☐

Exercise (2.8). Let $X = \mathbb{R}$ and $A = (1, 2)$. Verify that 0 is a limit point of A in the indiscrete topology and the finite complement topology, but not in the standard topology nor the discrete topology on $\mathbb{R}$.

Proof. We first show 0 is a limit point of A in the indiscrete topology. Suppose U is an open set such that $0 \in U$. Since $\mathcal{T}$ is the indiscrete topology we only have two sets in $\mathcal{T} = \{\varnothing, \mathbb{R}\}$. Since $0 \notin \varnothing$ and $0 \in \mathbb{R}$ we only have consider the open set $U = \mathbb{R}$. If $U = \mathbb{R}$ then $(U - \{0\}) \cap (1, 2) = (1, 2) \neq \varnothing$. Thus 0 is a limit point in the indiscrete topology on $\mathbb{R}$.

Now we show 0 is a limit point of A in the finite complement topology. Suppose we have an open set U such that $0 \in U$. For the sake of contradiction, suppose $U \cap (1, 2) = \varnothing$. It follows $(1, 2) \subset \mathbb{R} - U$ so then $\mathbb{R} - U$ must be infinite. But if $U \in \mathcal{T}$ then $X - U$ must be finite. This is a contradiction. It follows $U \cap (1, 2) \neq \varnothing$. Moreover, since $\{0\} \cap (1, 2) = \varnothing$ it follows $(U - \{0\}) \cap (1, 2) \neq \varnothing$. Thus 0 is a limit point in the finite complement topology on $\mathbb{R}$.

Now we show 0 is not a limit point in the standard topology. Let us choose the open interval $(-1, 1)$. It follows $((-1, 1) - \{0\}) \cap (1, 2) = \varnothing$. Thus 0 is not a limit point of $\mathbb{R}_{\text{std}}$.

Now we show 0 is not a limit point in the discrete topology. Let us choose the open interval $\{0\}$. Then $(\{0\} - \{0\}) \cap (1, 2) = \varnothing$. Thus 0 is not a limit point of the discrete topology on $\mathbb{R}$. □

Theorem (2.9). *Suppose $p \notin A$ in a topological space $(X, \mathcal{T})$. Then p is not a limit point of A if and only if there exists a neighborhood U of p such that $U \cap A = \varnothing$.*

Proof. Suppose $p \notin A$ in a topological space $(X, \mathcal{T})$. We prove the forward direction first. Let p not be a limit point of A. Then by definition of a limit point there exists some open set U such that $(U - \{p\}) \cap A = \varnothing$. But since $p \notin A$ we have $U \cap A = \varnothing$.

Now we prove the reverse direction. Suppose there exists some neighborhood U of p such that $U \cap A = \varnothing$. Since $U - \{p\} \subset U$ it follows $(U - \{p\}) \cap A = \varnothing$. Hence, p cannot be a limit point of A by definition of a limit point. □

Exercise (2.11). Give examples of sets A in various topological spaces $(X, \mathcal{T})$ with

(1) a limit point of A that is an element of A
(2) a limit point of A that is not an element of A
(3) an isolated point of A
(4) a point not in A that is not a limit point of A

1. (a) Consider the standard topology on $\mathbb{R}^2$. Then let
$$A = \{(x_1, x_2) \in \mathbb{R}^2 | (x_1^2 + x_2^2)^{1/2} \leq 1\}$$
The the point $p = (1, 0)$ is a limit point of A but also $p \in A$.
(b) Consider the standard topology on $\mathbb{R}$. Then let
$$A = \{\frac{1}{n} | n \in \mathbb{N} = \{1, 2, ...\}\} \cup \{0\}$$
The point $p = 0$ is a limit point of A and $p \in A$.
2. (a) Consider the standard topology on $\mathbb{R}^2$. Then let
$$A = \{(x_1, x_2) \in \mathbb{R}^2 | (x_1^2 + x_2^2)^{1/2} < 1$$
The the point $p = (1, 0)$ is a limit point of A but $p \notin A$.
(b) Consider the standard topology on $\mathbb{R}$. Then let
$$A = \{\frac{1}{n} | n \in \mathbb{N} = \{1, 2, ...\}\}$$
The point $p = 0$ is a limit point of A but $p \notin A$.
3. (a) Consider the standard topology on $\mathbb{R}^2$. Let
$$A = \{(0, 0)\}$$
Then $(0, 0)$ is an isolated point of A.
(b) Consider the standard topology on $\mathbb{R}$. Let
$$A = \{\frac{1}{n} | n \in \mathbb{N} = \{1, 2, ...\}\}$$
The point $p = 1$ is an isolated point of A.

4. (a) Consider the standard topology on $\mathbb{R}^2$. Let
$$A = \{(x_1, x_2) \in \mathbb{R}^2 | 0 \leq x_1, x_2 \leq 1\}$$
Then the point $p = (2, 2)$ is not a point in A and is not a limit point of A.
(b) Consider the standard topology on $\mathbb{R}$. Let
$$A = \{\frac{1}{n} | n \in \mathbb{N} = \{1, 2, ...\}\}$$
The point $p = -1$ is not a point of A and is not a limit point of A.

Exercise 2.12.

(1) Which sets are closed in a set X with the discrete topology?
Let $A \subset X$. We show that A is closed by showing that A has no limit points. Let $p \in X$. In the discrete topology $\{p\}$ is a neighborhood of the point p. So then the intersection of the neighborhood $\{p\}$ with A has at most one point which is p. In other words $\{p\} - \{p\} \cap A = \varnothing$. So then p is not a limit point. So then $\overline{A} = A$ since A does not have any limit points. Hence, all subsets of X are closed.

(2) Which sets are closed in a set X with the indiscrete topology? The empty set is closed vacuously. The whole set X is a closed subset of X. Now suppose we have some nonempty proper subset $A \subsetneq X$. Then let $p \in X$. Then there is only one neighborhood of p in the indiscrete topology which is X. Then p is a limit point if $X - \{p\} \cap A \neq \varnothing$. Then all points $p \in X$ are limit points of A yet $A \neq X$ so then $A \neq \overline{A}$. It follows the only subsets that are closed are empty and the whole set in the indiscrete topology.

(3) Which sets are closed in a set X with the finite complement topology? Suppose X is a finite set. It follows all subsets of X are finite and their complements are finite. Then all subsets of X are open. Now suppose we have some subset $A \subset X$. Then let $p \in X$. Then let $U = (X - A) \cup p$. Then $U - \{p\} \cap A = \varnothing$. It follows all points in X are not limit points of A so then A has no limits points and is closed.
Now suppose X is an infinite set. Let A be a finite subset of X. Then let $p \in X$. Then define U as $X - U = A - \{p\}$ so then $U \in \mathcal{T}$. Then $U = (X - A) \cup \{p\}$. Then U is a neighborhood of p. Then $U - \{p\} \cap A = (X - A \cup \{p\}) - \{p\} \cap A = (X - A) \cap A = \varnothing$. It follows all points $p \in X$ are not limit points of A. Then A is a closed subset of X. It follows all finite subsets of X are closed.
Now suppose X is an infinite set and A is an infinite subset of X. Then let

(4) Which sets are closed in a set X with the countable complement topology?

Exercise (2.13). For any topological space $(X, \mathcal{T})$ and $A \subset X$, the set $\overline{A}$ is closed. That is, for any set A in a topological space, $\overline{\overline{A}} = \overline{A}$.

Proof. We show $\overline{\overline{A}} = \overline{A}$. First we show $\overline{A} \subset \overline{\overline{A}}$. Let $p \in \overline{A}$. Then by definition, p is in the closure of $\overline{A}$ so $\overline{A} \subset \overline{\overline{A}}$.

Now we show $\overline{\overline{A}} \subset \overline{A}$. Let $p \in \overline{\overline{A}}$. Then $p \in \overline{A}$ or p is a limit point of $\overline{A}$. In the former case we are done. Now suppose we have the latter case. We assume $p \notin A$ because if $p \in A$ then $p \in \overline{A}$. If p is a limit point of $\overline{A}$ then by definition we have for all neighborhoods that contain point p that $U - \{p\} \cap \overline{A} \neq \varnothing$. Then let $q \in U - \{p\} \cap \overline{A}$. Then we have two cases: either (1) $q \in A$ or (2) $q \notin A$ and q is a limit point of A. If (1) $q \in A$ then $U - \{p\} \cap A \neq \varnothing$ since q is in this intersection. Now suppose we have case (2). Then $U - \{q\} \cap A \neq \varnothing$. Since we assume $p \notin A$ and $q \notin A$ we have $U - \{p\} \cap A \neq \varnothing$. Then in both cases (1) and (2) we have $U - \{p\} \cap A \neq \varnothing$. It follows this is true for all open neighborhoods U of p. Thus, $p \in A$ which implies $p \in \overline{A}$. This means $\overline{\overline{A}} \subset \overline{A}$.

Since $\overline{\overline{A}} \subset \overline{A}$ and $\overline{A} \subset \overline{\overline{A}}$ it follows $\overline{\overline{A}} = \overline{A}$. □

Theorem (2.14). *Let $(X, \mathfrak{T})$ be a topological space. Then the set A is closed if and only if $X - A$ is open.*

Proof. Let A be a closed subset of X. We show $X - A$ is open. Let $p \in X - A$. Since A is closed and $p \notin A$ it follows p is not a limit point of A. By definition of limit point, it follows there exists some open neighborhood U_p of p such that $U_p - \{p\} \cap A = \varnothing$. It follows $U_p \subset X - A$ since U_p and A are disjoint. Then we have $p \in U_p \subset X - A$. Moreover, U_p is an open subset of $X - A$. It follows $X - A$ is open.

Now we prove the reverse direction. Suppose $X - A$ is open. Let p be a limit point of A. We show p must be in A. By definition of a limit point for every open neighborhood U of p we have $U - \{p\} \cap A \neq \varnothing$. Suppose $p \in X - A$. then there exists some open neighborhood V such that $p \in V \subset X - A$. Then $V \cap A = \varnothing$. This is a contradiction. Therefore, $p \in A$. It follows A contains all of its limit points. Hence, A is closed. □

Theorem (2.15). *Let $(X, \mathfrak{T})$ be a topological space, and let U be an open set and A a closed subset of X. Then the set $U - A$ is open and the set $A - U$ is closed.*

Proof. Let U be an open subset and A be a closed subset of X. We first show $U - A$ is open. The set $U - A$ is the same as $U \cap (X - A)$ and we know from Theorem 2.14 that $(X - A)$ must be open since A is closed. It follows we have an intersection of two open sets which is open in any topological space. Thus $U \cap (X - A) = U - A$ is open.

Now we show $A - U$ is closed. Equivalently we can show $X - (A - U)$ is open by Theorem 2.14. $A - U$ is the same as $A \cap (X - U)$ so then the complement of $A - U$ is $(X - A) \cup U$. So we have a union of open sets which is open in any topological space. It follows $X - (A - U)$ is open and therefore $A - U$ is closed. □

Theorem (2.16). *Let $(X, \mathfrak{T})$ be a topological space. Then:*

(1) *$\varnothing$ is closed.*
(2) *X is closed.*
(3) *The union of finitely many closed sets is closed.*
(4) *Let $\{A_\alpha\}_{\alpha \in \lambda}$ be a collection of closed subsets in $(X, \mathfrak{T})$. Then $\bigcap_{\alpha \in \lambda} A_\alpha$ is closed.*

Proof. (1) is true vacuously. The empty set has no limit points so its closure is equal to itself, namely empty. Now we prove (2). Any point p that is a limit point of X is in X so $\overline{X} \subset X$. And $X \subset \overline{X}$ by definition of closure. So $\overline{X} = X$. Now we prove (3). Suppose we have a finite set of closed sets U_i where $i \in \{1, 2, ..., n\}$. Then the complements are open by Theorem 2.14. Then the intersection $\bigcap_{i=1}^{n} (X - U_i)$ is open by definition of a topology. Taking the complement of this intersection results in the closed set $\bigcup_{i=1}^{n} U_i$. It follows a finite union of closed sets is closed. Now we prove (4). Suppose we have a collection $\{A_\alpha\}_{\alpha \in \lambda}$ of closed sets. Then the union $\bigcup_{\alpha \in \lambda} (X - A_\alpha)$ is open since it is an arbitrary union of open sets. Then taking the complement gives the closed set $\bigcap_{\alpha \in \lambda} A_\alpha$. It follows an arbitrary intersection of closed sets is closed. □

Theorem (2.20). *For any set A in a topological space X, the closure of A equals the intersection of all closed sets containing A, that is,*

$$\overline{A} = \bigcap_{B \supset A, B \in \mathcal{C}} B$$

where $\mathcal{C}$ is the collection of all closed sets in X.

Proof. First we show $\overline{A} \subset \bigcap_{B \supset A, B \in \mathcal{C}} B$. Let the point $p \in \overline{A}$. There are two cases: either (1) $p \in A$ or (2) $p \notin A$ but p is a limit point of A. If we have case (1) then $p \in \bigcap_{B \supset A, B \in \mathcal{C}} B$ since for every $B \in \mathcal{C}$ we have $A \subset B$. Now suppose we have case (2). Then p is a limit point of A. Then for all open neighborhoods U of p we have $U - \{p\} \cap A \neq \varnothing$. Let $B' \in \mathcal{C}$ and $B' \supset A$. Since $A \subset B'$, it follows $U - \{p\} \cap B' \neq \varnothing$ for all U. It follows B' contains all the limit points of A. Hence, $p \in B'$. It follows $p \in \bigcap_{B \supset A, B \in \mathcal{C}} B$. Therefore, $\overline{A} \subset \bigcap_{B \supset A, B \in \mathcal{C}} B$.

Now we show $\bigcap_{B \supset A, B \in \mathcal{C}} B \subset \overline{A}$. Since $\overline{A} \supset A$ and $\overline{A} \in \mathcal{C}$ it follows $\overline{A} \supset \bigcap_{B \supset A, B \in \mathcal{C}} B$. This concludes the proof. □

Theorem (2.22). *Let A and B be subsets of a topological space $(X, \mathcal{T})$. Then:*

(1) *$A \subset B$ implies $\overline{A} \subset \overline{B}$.*
(2) *$\overline{A \cup B} = \overline{A} \cup \overline{B}$.*

Proof. We first prove $A \subset B$ implies $\overline{A} \subset \overline{B}$. Let $p \in \overline{A}$ and $A \subset B$. We know $B \subset \overline{B}$ by definition of closure. Then $A \subset B \subset \overline{B}$ so $A \subset \overline{B}$. Hence, if $p \in A$ then $p \in \overline{B}$. Now suppose $p \in \overline{A}$ but $p \notin A$. Then p must be a limit point of A. Then for all neighborhoods U of p we have $U - \{p\} \cap A \neq \varnothing$. Since $A \subset B$ it follows $U - \{p\} \cap B \neq \varnothing$. Thus, $p \in \overline{B}$. It follows $\overline{A} \subset \overline{B}$.

Now we prove $\overline{A \cup B} = \overline{A} \cup \overline{B}$. We first show $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$. Let $p \in \overline{A \cup B}$. Then either $p \in A \cup B$ or p is a limit point of $A \cup B$.

If we have the former case, $p \in A \cup B$, then either $p \in A$ or $p \in B$. It follows either $p \in \overline{A}$ or $p \in \overline{B}$. Hence, $p \in \overline{A} \cup \overline{B}$.

If we have the latter case, then p is a limit point of $A \cup B$. For sake of contradiction suppose $p \notin \overline{A} \cup \overline{B}$. Then p is not a limit point of neither A nor B. Then there exist open neighborhoods of p called U and V such that $U - \{p\} \cap A = \varnothing$ and $V - \{p\} \cap B = \varnothing$. By definition of a topology we know $U \cap V$ is also an open set. Moreover, $U \cap V$ contains p since both U and V contain p. Then we have

$$\begin{aligned} ((U \cap V) - \{p\}) \cap (A \cup B) &= (((U \cap V) - \{p\}) \cap A) \cup (((U \cap V) - \{p\}) \cap B) \\ &= \varnothing \cup \varnothing \\ &= \varnothing \end{aligned}$$

It follows p is not a limit point of $A \cup B$. But this is a contradiction. It follows $p \in \overline{A} \cup \overline{B}$. So then $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$.

Now we show $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$. Let $p \in \overline{A} \cup \overline{B}$. If $p \in A$ or $p \in B$ then $p \in A \cup B$ and then $p \in \overline{A \cup B}$ by definition of closure. Now suppose p is a limit point of A or p is a limit point of B. Since $A \subset A \cup B$ and $B \subset A \cup B$ it follows $\overline{A} \subset \overline{A \cup B}$ and $\overline{B} \subset \overline{A \cup B}$ which we proved above. Thus, $p \in \overline{A \cup B}$. It follows $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$.

Since $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$ and $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$ it follows $\overline{A} \cup \overline{B} = \overline{A \cup B}$ □

Proof. We provide an alternative proof using theorem 2.20. First we prove part (1). Note that

$$\overline{A} = \cap_{U \supset A, U \in \mathcal{C}} U$$

And $\overline{B}$ contains A and is closed. Therefore, $\overline{A} \subset \overline{B}$.

Now we prove part (2). First we show $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$. Note that

$$\overline{A \cup B} = \cap_{U \supset A \cup B, U \in \mathcal{C}} U$$

We know $\overline{A} \cup \overline{B} \supset A \cup B$ so then $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$.

Now we show $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$. Note that $A \subset \overline{A \cup B}$ and $B \subset \overline{A \cup B}$ so then $\overline{A} \subset \overline{A \cup B}$ and $\overline{B} \subset \overline{A \cup B}$ so then $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$. Hence $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

$\square$

Exercise (2.24). In $\mathbb{R}^2$ with the standard topology, describe the limit points and closure of each of the following two sets:

(1) $S = \left\{ \left(x, \sin\left(\dfrac{1}{x}\right)\right) \mid x \in (0,1) \right\}$
(2) $C = \{(x,0) \mid x \in [0,1]\} \cup \bigcup_{n=1}^{\infty} \left\{ \left(\frac{1}{n}, y\right) \mid y \in [0,1] \right\}$

Let S be defined as

$$S = \left\{ \left(x, \sin\left(\frac{1}{x}\right)\right) \mid x \in (0,1) \right\}$$

First we find the limit points of S. First, the continuity of $\sin(1/x)$ on $\mathbb{R} - \{0\}$ implies $S \cup \{(1, \sin(1))\} \subset \overline{S}$. Now we show the set

$$A = \{(0,a) \mid a \in [-1,1]\} \subset \overline{S}$$

Let $(0,a) \in A$. We show $(0,a)$ is a limit point of S in the standard topology on $\mathbb{R}^2$. It suffices to show that for any $\epsilon > 0$ that $(B((0,a),\epsilon) - \{(0,a)\}) \cap S \neq \varnothing$. We know there exists a real number $1/x$ such that $0 < 1/\epsilon < 1/x$ and $\sin(1/x) = a$ since the function sin is periodic. Then we have the point $(x, \sin(1/x)) \in S$ where $0 < x < \epsilon$. Then

$$\begin{aligned}
\sqrt{(0-x)^2 + (a - \sin(1/x))^2} &= \sqrt{x^2 + (a - \sin(1/x))^2} \\
&= \sqrt{x^2 + (a-a)^2} \\
&= \sqrt{x^2} \\
&= |x|
\end{aligned}$$

and $|x| < \epsilon$ since $0 < x < \epsilon$. It follows the point $(x, \sin(1/x)) \in B((0,a),\epsilon)$ and $(x, \sin(1/x)) \neq (0,a)$ so $(B((0,a),\epsilon) - \{(0,a)\}) \cap S \neq \varnothing$. Thus $(0,a)$ is a limit point of S. Hence, $A \subset \overline{S}$. Then the closure of S is

$$\overline{S} = S \cup A \cup \{(1, \sin(1)\}$$

In addition to B and $(1, \sin(1))$ being limit points of S, all points of S are limit points of S since $\sin(1/x)$ is continuous on $\mathbb{R} - \{0\}$.

Let C be defined as

$$C = \{(x,0) \mid x \in [0,1]\} \cup \bigcup_{n=1}^{\infty} \left\{ \left(\frac{1}{n}, y\right) \mid y \in [0,1] \right\}$$

By definition of closure C is in its closure. $C \subset \overline{C}$. Now we show the line segment $L = \{(0,\ell) \in \mathbb{R}^2 \mid \ell \in [0,1]\}$ is in the closure. $L \subset \overline{C}$. Let $(0,\ell) \in L$. Then take some open set in $\mathbb{R}^2_{\text{std}}$ $B((0,\ell),\epsilon)$ where $\epsilon > 0$. Then there exists some natural number n such that $1/n < \epsilon$ by the archimdean property. Then choose the point $(1/n,\ell)$ which is in C since $\ell \in [0,1]$. Then the distance between the the point $(0,\ell)$ and $(1/n,\ell)$ is $\sqrt{(0-1/n)^2+(\ell-\ell)^2} = |1/n|$ and $|1/n| < \epsilon$ since $0 < 1/n < \epsilon$. It follows $(1/n,\ell) \in B((0,\ell),\epsilon) \cap C$ and since $(1/n,\ell) \neq (0,\ell)$ it follows $(B((0,\ell),\epsilon) - \{(0,\ell)\}) \cap C \neq \varnothing$. Thus $(0,\ell)$ is a limit point. Hence, $L \subset \overline{C}$. Thus the closure of C is $\overline{C} = C \cup L$.

Exercise (2.25). In the standard topology on $\mathbb{R}$, there exists a non-empty subset C of the closed unit interval $[0,1]$ that is closed, contains no non-empty open interval, and where no point of C is an isolated point.

Proof. We show the cantor set satisfies the criteria above. We construct the cantor set. Let $C_0 = [0,1]$. Then C_{n-1} consists of 2^n closed intervals. For each interval, take out the middle third. Then this is defined to be C_n. For example,

$$\begin{aligned} C_0 &= [0,1] \\ C_1 &= [0,1/3] \cup [2/3,1] \\ C_2 &= [0,1/9] \cup [2/9,3/9] \cup [6/9,7/9] \cup [8/9,1] \end{aligned}$$

Generally we have

$$C_n = I_n^1 \cup I_n^2 \cup I_n^3 \cup \cdots \cup I_n^{2^n}$$

Where if $j < k$ then $x < y$ for all $x \in I_i^j, y \in I_i^k$ for all $j,k \in \{1,2,3,...,2^i\}$ and all $i \in \mathbb{N} \cup \{0\}$. Then the cantor set is defined to be $C = \cap_{i=1}^{\infty} C_i$. We now show C is a subset of the interval $[0,1]$, closed, contains no non-empty open interval, and no point of C is an isolated point.

Since $[0,1] \supset C_0 \supset C_1 \supset \cdots$ it follows $C \subset [0,1]$. Moreover, C is an infinite intersection of closed sets which is closed by Theorem 2.16. Now we show that C contains no non-empty open interval. For sake of contradiction, suppose that C contains a non-empty open interval (s,t) where $s < t$ and $s,t \in \mathbb{R}$.

Let $P(n)$ be the statement "Let $a,b \in \mathbb{R}$ and $a < b$. If $(a,b) \subset C_n$ then $|a-b| \leq 1/3^n$ for some non-negative integer n." We show that $p(n)$ is true for all $n \geq 0$ by induction. For the base case, consider $n = 0$. Then suppose $(a,b) \subset C_0 = [0,1]$. Then we have $0 \leq a < b \leq 1$. It follows $|a-b| \leq 1 = 1/3^0$. It follows $P(0)$ is true.

Now we move on to the inductive step. For the inductive hypothesis we assume $P(m)$ is true. Now assume $(a,b) \subset C_{m+1}$. Since $C_{m+1} \subset C_m$ it follows $(a,b) \subset C_m$. Then C_m has 2^m disjoint closed intervals.

$$C_m = I_m^1 \cup I_m^2 \cup I_m^3 \cup \cdots \cup I_m^{2^m}$$

where I_w^z is the zth closed interval for C_w. It follows (a,b) is in exactly one of these intervals. Then let $(a,b) = I_m^k$ for some $k \in \{1,2,...,2^m\}$. Then $I_m^k = [c,d]$ where $c,d \in \mathbb{R}$. Since $(a,b) \subset [c,d]$ we have $c \leq a < b \leq d$. It follows $|a-b| \leq d-c$. For the sake of contradiction suppose $d-c > 1/3^m$. Then there exists $p,q \in \mathbb{R}$ such that $(p,q) \subset C_m$ and $|p-q| > 1/3^m$; however, since $(p,q) \subset C_m$ we must have $|p-q| \leq 1/3^m$ by the inductive hypothesis. This is a contradiction. It follows $d-c \leq 1/3^m$.

Since $(a,b) \subset [c,d] \subset C_m$ and by assumption $(a,b) \subset C_{m+1}$, it follows (a,b) falls in the left or right third of $[c,d]$. Specifically, $(a,b) \subset \left[c, c+\frac{1}{3}(d-c)\right]$ or $(a,b) \subset \left[c+\frac{2}{3}(d-c), d\right]$.

Then we have

$$c \leq a < b \leq c + \frac{1}{3}(d-c)$$

or

$$c + \frac{2}{3}(d-c) \leq a < b \leq d$$

In either case we have $|a-b| \leq (d-c)/3$. But $d-c \leq 1/3^m$ so then $|a-b| \leq 1/3^{m+1}$. It follows $P(m+1)$ is true. This concludes the induction. Hence, $P(n)$ is true for all $n \geq 0$.

We assumed that $(s,t) \subset C$. Then $(s,t) \subset C_m$ for all $m \in \mathbb{N} \cup \{0\}$. Since we showed $P(n)$ is true for all $n \in \mathbb{N} \cup \{0\}$, it follows $|s-t| \leq 1/3^m$ for all m. Suppose $|s-t| > 0$. Then by the archimedean property there exists a sufficiently large m such that $|s-t| > 1/3^m$. It follows $|s-t| = 0$. Moreover $(s,t) \in [0,1]$ so then $0 < s,t$ which implies $s = t$. But this is a contradiction since this implies (s,t) is an empty open interval. Moreover, we assumed $s < t$. Hence, C contains no non-empty open interval.

Now we show C contains no isolated points. Let $P^*(n)$ be the statement "let $x \in \mathbb{R}$. If $x \in C_n$ there exists $y \in \mathbb{R}$ such that $y \neq x$, $y \in C$, and $|x-y| \leq 1/3^n$ for some non-negative integer n." We show $P^*(n)$ is true for all non-negative integers. Suppose $x \in \mathbb{R}$ and $x \in C_i$ for some $i \in \mathbb{N} \cup \{0\}$. Then

$$C_i = I_i^1 \cup I_i^2 \cup I_i^3 \cup \cdots \cup I_i^{2^i}$$

Then $x \in I_i^j$ for some $j \in \{1,2,3,...,2^i\}$. Then $I_i^j = [c,d]$ where $c,d \in \mathbb{R}$ and $c < d$. Then $c \leq x \leq d$. If $x < d$ then let $y = d$ so then $(x,y) \subset C_i$. Otherwise if $x = d$ then let $y = c$ so then $(y,x) \subset C_i$. In either case, the hypothesis of $P(i)$ (note $P(i)$ not $P^*(i)$) is satisfied so then by the implication of $P(i)$ we have $|x-y| \leq 1/3^i$. Moreover $y \neq x$.

Now we show $y \in C$. Notice $y = d$ or $y = c$ for I_i^j. If suffices to show that if y is an endpoint of some closed interval I_p^q (where $p \in \mathbb{N} \cup \{0\}$ and $q \in \{1,2,3,...,2^p\}$), then $y \in C$.

Suppose y is an endpoint of some interval in C_k where $k \geq 0$. Then

$$y \in I_k^\ell = \left[c_k^\ell, d_k^\ell\right]$$

for some $\ell \in \{1,2,3,...,2^k\}$ and c_k^ℓ and d_k^ℓ are the corresponding endpoints. Since y is an endpoint of I_k^ℓ we have either $y = c_k^\ell$ or $y = d_k^\ell$. If $y = c_k^\ell$ then

$$y \in I_{k+1}^{2\ell-1} = \left[y, d_{k+1}^{2\ell-1}\right] = \left[y, c_k^\ell + (d_k^\ell - c_k^\ell)/3\right]$$

Then $y = c_{k+1}^{2\ell-1}$. If $y = d_k^\ell$ then

$$y \in I_{k+1}^{2\ell} = \left[c_{k+1}^{2\ell}, y\right] = \left[c_k^\ell + 2(d_k^\ell - c_k^\ell)/3, y\right]$$

Then $y = d_{k+1}^{2\ell}$. It follows y is an endpoint of some interval in C_{k+1}. Hence, we have shown if y is an endpoint of some interval in C_k then y is an endpoint of some interval in C_{k+1}.

Recall that we showed that if $x \in C_i$ then there exists $x \neq y \in C_i$ such $|x-y| \leq 1/3^i$. Moreover, we take y as an endpoint in some closed interval I_i^j. It follows y is an endpoint of some interval in C_i. We have shown that if y is an endpoint of some interval in C_k then y is an endpoint of some interval in C_{k+1} for all $k \geq 0$. By induction, it follows y is an endpoint of C_m and therefore $y \in C_m$ for all $m \geq i$. Moreover, we have $C_0 \supset C_1 \supset \cdots \supset C_i$. It follows $y \in C_v$ (but not necessarily an endpoint of) for all $0 \leq v < i$. It follows $y \in C_w$ for all $w \in \mathbb{N} \cup \{0\}$. Thus $y \in C$. It follows $P^*(n)$ is true for all $n \in \mathbb{N} \cup \{0\}$.

Using $P^*(n)$ we show there exists no isolated points in C. Let $x \in C$ and suppose we have the open set $U = (x-\epsilon, x+\epsilon)$ where $\epsilon \in \mathbb{R}$ and $\epsilon > 0$. Then by the archimedean property there exists a sufficiently large $n \in \mathbb{N} \cup \{0\}$ such that $1/3^n < \epsilon$. Then let $m \in \mathbb{N} \cup \{0\}$

such that $1/3^m < \epsilon$. Since $x \in C$ it follows $x \in C_m$. Then by $P^*(m)$ it follows there exists a $y \in C$ such that $|x - y| \leq 1/3^m < \epsilon$. It follows $((x - \epsilon, x + \epsilon) - \{x\}) \cap C \neq \varnothing$ since $y \in ((x - \epsilon, x + \epsilon) - \{x\}) \cap C$ and $y \neq x$. Thus x is a limit point and not an isolated point.

We have shown the cantor set, denoted as C, is a non-empty subset of the closed unit interval $[0, 1]$ that is closed, contains no non-empty open interval, and no isolated points. $\square$

Theorem (2.26). *Let A be the subset of a topological space X. Then p is an interior point of A if and only if there exists an open set U with $p \in U \subset A$.*

Proof. We prove the forward direction first. Let p be an interior point of A. Then by definition, we have $p \in \bigcup_{U \subset A, U \in \mathcal{T}} U$. Then we have $p \in U$ where $U \subset A$ and $U \in \mathcal{T}$. It follows $p \in U \subset A$ where U is an open set.

Now we prove the reverse direction. Suppose $p \in V \subset A$ for some open set V. Then $V \subset \bigcup_{U \subset A, U \in \mathcal{T}} U$. It follows $p \in \bigcup_{U \subset A, U \in \mathcal{T}} U$. Thus p is an interior point of A. $\square$

Theorem (2.28). *Let A be a subset of a topological space X. Then Int(A), Bd(A), and Int(X-A) are disjoint sets whose union is X.*

Proof. First we show the sets are disjoint. For the sake of contradiction suppose there exists some point p such that $p \in \text{Int}(A) \cap \text{Bd}(A)$. Then $p \in \bigcup_{U \subset A, U \in \mathcal{T}} U$ and $p \in \overline{A} \cap \overline{X - A}$. Then $p \in \overline{X - A}$. Either $p \in X - A$ or p is a limit point of $X - A$. We cannot have the former case since this would imply there does not exist an open neighborhood U of p such that $U \subset A$. Thus we assume $p \in A$ and p is a limit point of $X - A$. By definition of a limit point, it follows for every neighborhood U that contains p we have $U - \{p\} \cap (X - A) \neq \varnothing$. Since $p \in \bigcup_{U \subset A, U \in \mathcal{T}} U$, there exists some open set V such that $p \in V \subset A$. So then $V \cap (X - A) = \varnothing$. It follows V is a neighborhood of p such that $V - \{p\} \cap (X - A) = \varnothing$. This is a contradicts the fact that p is a limit point of $X - A$. Therefore, p cannot exist and the sets Int(A) and Bd(A) are disjoint.

Now we show the sets $\text{Bd}(A)$ and $\text{Int}(X - A)$ are disjoint. We know $\text{Bd}(A) = \overline{A} \cap \overline{X - A} = \overline{X - A} \cap \overline{X - (X - A)} = \text{Bd}(X - A)$. It is then sufficient to show that $\text{Bd}(X - A)$ and $\text{Int}(X - A)$ are disjoint. But we have just shown that for any subset B that $\text{Bd}(B)$ and $\text{Int}(B)$ are disjoint. It follows sets $\text{Bd}(A)$ and $\text{Int}(X - A)$ are disjoint.

Now we show the sets $\text{Int}(X - A)$ and Int (A) are disjoint. For the sake of contradiction suppose there exists some point p such that $p \in \text{Int}(X - A) \cap \text{Int}(A)$. Then we have $p \in \bigcup_{U \subset X - A, U \in \mathcal{T}} U$ and $p \in \bigcup_{V \subset A, V \in \mathcal{T}} V$. It follows there exists open sets U and V such that $p \in U \subset X - A$ and $p \in V \subset A$. Then $p \in X - A$ and $p \in A$. But this is a contradiction. It follows the sets $\text{Int}(X - A)$ and Int (A) are disjoint.

Now we show that the union of all three sets A°, ∂A, and $(X - A)^\circ$ is X. First we show $A^\circ \cup \partial A \cup (X - A)^\circ \subset X$. This is true since all open sets in a topology are subsets of X and all points considered are in X. Now we show $X \subset A^\circ \cup \partial A \cup (X - A)^\circ$. For the sake of contradiction, suppose there exists some point p such that $p \in X$ but $p \notin A^\circ \cup \partial A \cup (X - A)^\circ$. Since $p \notin A^\circ$ there does not exist an open neighborhood U of p such that $U \subset A$. Similarly, since $p \notin (X - A)^\circ$ there does not exist a neighborhood V of p such that $V \subset (X - A)$. But p must be in some open set such as X. Let $\mathcal{G}$ be the collection of neighborhoods that contain p. Let $G \in \mathcal{G}$ be a neighborhood that contains p. It follows $G \not\subset A$ and $G \not\subset X - A$. Then $G \cap A \neq \varnothing$ and $G \cap (X - A) \neq \varnothing$. This holds for all neighborhoods of p. Let $p \in A$. Then $G - \{p\} \cap (X - A) \neq \varnothing$ for all $G \in \mathcal{G}$. Then p is a limit point of $X - A$. Then $p \in \overline{X - A}$. Since $p \in A$ then $p \in \overline{A}$. So then $p \in \partial A$ which is a contradiction. Now suppose $p \in X - A$.

Then for all $G \in \mathcal{G}$ we have $G - \{p\} \cap A \neq \varnothing$. It follows $p \in \overline{A}$. Since $p \in X - A$ it follows $p \in \overline{X - A}$. So then $p \in \partial A$. This is another contradiction. But we must have either $p \in A$ or $p \in X - A$. It follows $X \subset A^\circ \cup \partial A \cup (X - A)^\circ$. Since we showed, $A^\circ \cup \partial A \cup (X - A)^\circ \subset X$ and $X \subset A^\circ \cup \partial A \cup (X - A)^\circ$, it follows $A^\circ \cup \partial A \cup (X - A)^\circ \subset X$. Altogether we have shown the sets $A^\circ, \partial A, (X - A)^\circ$ form a partition on X. □

Theorem (2.30). *Let A be a subset of the topological space X, and let p be a point in X. If the set $\{x_i\}_{i \in \mathbb{N}} \subset A$ and $x_i \to p$, then p is in the closure of A.*

Proof. Let U be an open set that contains point p. Since the sequence $(x_i)_{i \in \mathbb{N}}$ converges to p we know that there exists some $n \in \mathbb{N}$ such that $x_i \in U$ for all $i > n$. And since $\{x_i\}_{i \in \mathbb{N}} \subset A$ we know $U \cap A \neq \varnothing$. Now there are two cases. Either (1) $p \in A$ or (2) $p \notin A$. For case (1) if $p \in A$ then $p \in \overline{A}$. If we have case (2) then $U - \{p\} \cap A \neq \varnothing$. So then p is a limit point of A so then $p \in \overline{A}$. Thus p is in the closure of A for either case. □

Main Ideas: (1) use definition of p being a limit of the sequence.

Theorem (2.31). *Suppose we have the metric topology with some metric space (X, d). If p is a limit point of a set $A \subset X$, then there is a sequence of points in A that converges to p.*

Proof. If p is a limit point of the set A, then for any $r > 0$ there exists an infinite number of points $q \neq p$ such that $d(p, q) < r$ and $q \in A$. If this were not true then for some $r > 0$ there would exist a finite number of points q. Then one could take the q with the smallest distance from p, say $r_{\min}$. Then the open ball $B(p, r_{\min}) \cap A = \varnothing$ which contradicts p being a limit point. Now we construct a sequence $\{x_i\}_{i \in \mathbb{N}}$. For some $n \in \mathbb{N}$ and $n \geq 1$ we know there exists some $q \in A$ such that $d(p, q) < 1/2^n$. Let us define the set T as

$$T = \{q \in A | d(p, q) < \frac{1}{2^n} \text{ and } p \neq q\}$$

We know T must be infinite as we showed in the beginning of the proof. By the Axiom of Choice choose a point q in the set T and let $x_n = q$. Now we have defined a sequence recursively. We now show that p is a limit of the sequence.

Let U be an open set that contains p. Since U is an open set in the metric topology, it follows there exists some $r > 0$ such that $B(p, r) \subset U$. Moreover, there exists some $n \in \mathbb{N}$ such that $1/2^n \leq r$. Then we have $x_j \in U$ for all $j \in \mathbb{N}$ and $j \geq n$. By definition of convergence, the sequence $\{x_i\}_{i \in \mathbb{N}}$ converges to p. □

Main ideas: (1) infinite number of points in the intersection. (2) recursively construct a sequence that converges to p. (3) show that it converges to p

Theorem (2.32). *Find an example of a topological space and a convergent sequence in that space for which the limit of the sequence is not unique.*

Consider the sequence $\{x_n\}_{n \in \mathbb{N}}$ where $x_n = 1/n$ for all $n \in \mathbb{N}$ in the indiscrete topology on $\mathbb{R}$, that is the topological space $(\mathbb{R}, \{\varnothing, \mathbb{R}\})$. In this topology, every point in $\mathbb{R}$ is a limit of the sequence since there is only one open set $\mathbb{R}$ and $x_n \in \mathbb{R}$ for all $n \in \mathbb{N}$. It follows the limit of the sequence is not unique.

Theorem (3.1). *Let $(X, \mathcal{T})$ be a topological space, and let $\mathcal{B}$ be a collection of subsets of X. Then $\mathcal{B}$ is a basis for $\mathcal{T}$ if and only if*

(1) *$\mathcal{B} \subset \mathcal{T}$, and*
(2) *for each set U in $\mathcal{T}$ and point p in U there is a set V in $\mathcal{B}$ such that $p \in V \subset U$.*

Proof. We prove the forward direction first. By definition of a basis we require $\mathcal{B} \subset \mathcal{T}$ hence property (1) is true. By definition of a basis we require every open set in $\mathcal{T}$ be a union of elements in $\mathcal{B}$. Let U be an open set in $\mathcal{T}$. Then $U = \bigcup_{V \in \mathcal{B}'} V$ where $\mathcal{B}' \subset \mathcal{B}$. Now let $p \in U$. Then $p \in V_p$ for some $V_p \in \mathcal{B}'$. We know V_p is open since $V \in \mathcal{B}' \subset \mathcal{B} \subset \mathcal{T}$ and $V_p \subset \bigcup_{V \in \mathcal{B}'} V = U$. Therefore, $p \in V_p \subset U$ and $V_p \in \mathcal{B}$ for all points $p \in U$.

Now we prove the reverse direction. In order for $\mathcal{B}$ to be a basis we require two things. We first require $\mathcal{B} \subset \mathcal{T}$. But this is supplied by the property (1) of Theorem 3.1. Second, we require that every open set U in $\mathcal{T}$ is a union of elements in $\mathcal{B}$. We show this holds from property (2) of Theorem 3.1. Let U be an open set in the topological space. Then by property (2) of Theorem 3.1 we know for every point $p \in U$ that there exists an open set $V_p \in \mathcal{B}$ such that $p \in V_p \subset U$. Let $\{\mathcal{V}_p\}_{p \in U}$ be a collection of basis elements that contain point p. Then by the axiom of choice there exists a function $f \colon U \to \bigcup_{p \in U} \mathcal{V}_p$ where $f(p) \in \mathcal{V}_p$. Now we prove

$$U = \bigcup_{p \in U} f(p)$$

First we show $U \subset \bigcup_{p \in U} f(p)$. Let $q \in U$. Then $q \in f(q)$. Hence $q \in \bigcup_{p \in U} f(p)$.

Now we show $\bigcup_{p \in U} f(p) \subset U$. Let $q \in \bigcup_{p \in U} f(p)$. Then $q \in f(p)$ for some $p \in U$. By nature of construction, $f(p) \subset U$. It follows $p \in U$.

Since we have both $U \subset \bigcup_{p \in U} f(p)$ and $\bigcup_{p \in U} f(p) \subset U$ it follows $U = \bigcup_{p \in U} f(p)$. Thus $\mathcal{B}$ satisfies the definition of a basis.

□

Exercise (3.2). Show the following are bases for the standard topology on $\mathbb{R}$.

(1) Let $\mathcal{B}_1 = \{(a, b) \subset \mathbb{R} | a, b \in \mathbb{Q}\}$. Show that $\mathcal{B}_1$ is a basis for the standard topology on $\mathbb{R}$.
(2) Let $\mathcal{B}_2 = \{(a, b) \cup (c, d) \subset \mathbb{R} | a < b < c < d$ are distinct irrational numbers$\}$. Show that $\mathcal{B}_2$ is also a basis for the standard topology on $\mathbb{R}$.

Proof. We prove part (1) here. First we show that any open set in the standard topology on $\mathbb{R}$ is a union of sets in $\mathcal{B}_1$. Suppose we have some open set in the standard topology of $\mathbb{R}$. Then we have some open interval (a, b) where $a, b \in \mathbb{R}$. First we define sets A_i and B_i for all $i \in \mathbb{N}$ as follows

$$A_i = \{x \in \mathbb{Q} \mid a < x < a + \frac{b-a}{2^i}\}$$
$$B_i = \{x \in \mathbb{Q} \mid b + \frac{a-b}{2^i} < x < b\}$$

Moreover, these sets are non-empty by the archimedean property. Then by the axiom of choice, there exist choice functions $p \colon \mathbb{N} \to \cup_{i \in \mathbb{N}} A_i$ and $q \colon \mathbb{N} \to \cup_{i \in \mathbb{N}} B_i$ such that $p(i) \in A_i$ and $q(i) \in B_i$ for all $i \in \mathbb{N}$. Then define a sequence of open intervals $\{I_n\}_{n \in \mathbb{N}}$ as $I_n = (p(n), q(n))$. Note that $p(n), q(n) \in \mathbb{Q}$ for all $n \in \mathbb{N}$ and so $I_n \in \mathcal{B}_1$ for all $n \in \mathbb{N}$.

Now we show $(a,b) = \cup_{n=1}^{\infty} I_n$. First we show $(a,b) \subset \cup_{n=1}^{\infty} I_n$. Let $x \in (a,b)$. Then $a < x < b$. By the archimedean property there exist natural numbers j and k such that

$$a < a + \frac{b-a}{2^j} < x \text{ and } x < b + \frac{a-b}{2^k} < b$$

Then take $m =$ maximum of the set $\{j, k\}$. Then

$$a + \frac{b-a}{2^m} < a + \frac{b-a}{2^j} \text{ and } b + \frac{a-b}{2^k} < b + \frac{a-b}{2^m}$$

It follows

$$a + \frac{b-a}{2^m} < x < b + \frac{a-b}{2^m}$$

Moreover, $p(m) < a + (b-a)/2^m$ and $q(m) > b + (a-b)/2^m$. It follows $p(m) < x < q(m)$. Then $x \in I_m$ so $x \in \cup_{i=1}^{\infty} I_n$. Hence, $(a,b) \subset \cup_{i=1}^{\infty} I_n$.

Now we show $\cup_{i=1}^{\infty} \subset (a,b)$. Let $x \in \cup_{i=1}^{\infty}$. Then $x \in I_m$ for some $m \in \mathbb{N}$. Then we have

$$a < a + \frac{b-a}{2^m} < x < b + \frac{a-b}{2^m} < b$$

It follows $x \in (a,b)$ so then $\cup_{i=1}^{\infty} I_n \subset (a,b)$.

Since $(a,b) \subset \cup_{n=1}^{\infty} I_n$ and $\cup_{i=1}^{\infty} \subset (a,b)$ we have $(a,b) = \cup_{n=1}^{\infty} I_n$. It follows every open set in the standard topology is a union of elements in $\mathcal{B}_1$.

Now we show $\mathcal{B}_1 \subset \mathcal{T}$. Let $(a,b) \in \mathcal{B}_1$. Then $a, b \in \mathbb{Q}$. Then let $p \in (a,b)$. Then we have $a < p < b$. Then take ϵ as $\epsilon =$ minimum of the set$\{p-a, b-p\}$. Moreover, $\epsilon \in \mathbb{R}$. Then we show the interval $(p-\epsilon, p+\epsilon)$ is a subset of (a,b). Let $q \in (p-\epsilon, p+\epsilon)$. Then $p - \epsilon < q < p + \epsilon$. By definition of ϵ we have

$$a \leq p - \epsilon < q < p + \epsilon \leq b$$

It follows $q \in (a,b)$. Thus $(p-\epsilon, p+\epsilon) \subset (a,b)$. Thus (a,b) is in the standard topology. It follows $\mathcal{B}_1 \subset \mathcal{T}$.

Since $\mathcal{B}_1 \subset \mathcal{T}$ and every open set is a union of sets in $\mathcal{B}_1$ it follows $\mathcal{B}_1$ is a basis for the standard topology on $\mathbb{R}$.

Now we prove part (2). That is, we show $\mathcal{B}_2$ is a basis for the standard topology on $\mathbb{R}$. First we show every open set in the standard topology on $\mathbb{R}$ is a union of elements in $\mathcal{B}_2$. We define sets A_i^* and B_i^* for all $i \in \mathbb{N}$ as follows

$$A_i^* = \{x \in \mathbb{R} - \mathbb{Q} \mid a < x < a + \frac{b-a}{2^i}\}$$

$$B_i^* = \{x \in \mathbb{R} - \mathbb{Q} \mid b + \frac{a-b}{2^i} < x < b\}$$

Then by the axiom of choice there exist choice functions $p^* \colon \mathbb{N} \to \cup_{i \in \mathbb{N}} A_i^*$ and $q^* \colon \mathbb{N} \to \cup_{i \in \mathbb{N}} B_i^*$ such that $p^*(i) \in A_i^*$ and $q^*(i) \in B_i^*$ for all $i \in \mathbb{N}$. Then define C_i as

$$C_i = \{x \in \mathbb{R} - \mathbb{Q} \mid q^*(i) < x < b\}$$

Then by the axiom of choice there exist a function $r \colon \mathbb{N} \to \cup_{i \in \mathbb{N}} C_i$ such that $r(i) \in C_i$ for all $i \in \mathbb{N}$. Then define the set D_i as

$$D_i = \{x \in \mathbb{R} - \mathbb{Q} \mid r(i) < x < b\}$$

Again, by the axiom of choice there exists a function $q \colon \mathbb{N} \to \cup_{i \in \mathbb{N}} D_i$ such that $s(i) \in D_i$ for all $i \in \mathbb{N}$.

Then define $J_n \in \mathcal{B}_2$ as $J_n = (p^*(n), q^*(n)) \cup (r(n), s(n))$. Note that $p^*(n) < q^*(n) < r(n) < s(n)$ and $p^*(n), q^*(n), r(n), s(n) \in \mathbb{R} - \mathbb{Q}$ for all $n \in \mathbb{N}$ so in fact J_n is an element of $\mathcal{B}_2$.

We show $(a, b) = \cup_{n=1}^{\infty} J_n$. First we show $(a, b) \subset \cup_{n=1}^{\infty} J_n$. Let $x \in (a, b)$. Then $a < x < b$. By the archimedean property there exist natural numbers j, k such that

$$a < a + \frac{b-a}{2^j} < x \text{ and } x < b + \frac{a-b}{2^k} < b$$

Then let $m =$ maximum of the set $\{j, k\}$. Then we have

$$a + \frac{b-a}{2^m} < x < b + \frac{a-b}{2^m}$$

It follows

$$a < p^*(m) < a + \frac{b-a}{2^m} \text{ and } b + \frac{a-b}{2^m} < q^*(m) < b$$

Then $p^*(m) < x < q^*(m)$. It follows $x \in J_m$ which implies $x \in \cup_{n=1}^{\infty} J_n$. Hence, $(a, b) \subset \cup_{n=1}^{\infty} J_n$.

Now we show $\cup_{n=1}^{\infty} J_n \subset (a, b)$. Let $x \in \cup_{n=1}^{\infty} J_n$. Then we have $x \in (p^*(n), q^*(n)) \cup (r(n), s(n))$ for some $n \in \mathbb{N}$. Then we have $p^*(n) < x < s(n)$. Since $a < p^*(n)$ and $s(n) < b$ it follows $x \in (a, b)$. Thus $\cup_{n=1}^{\infty} J_n \subset (a, b)$.

Since we have both $(a, b) \subset \cup_{n=1}^{\infty} J_n$ and $\cup_{n=1}^{\infty} J_n \subset (a, b)$ it follows $(a, b) = \cup_{n=1}^{\infty} J_n$. Hence, all open sets in the standard topology are a union of elements of $\mathcal{B}_2$.

Now we show $\mathcal{B}_2 \subset \mathcal{T}$. Let $B \in \mathcal{B}_2$. then $B = (a, b) \cup (c, d)$ where $a < b < c < d$ and $a, b, c, d \in \mathbb{R} - \mathbb{Q}$. To show $B \in \mathcal{T}$ we must show that for all points $q \in B$ there exists some $\delta > 0$ such that $(q - \delta, q + \delta) \subset B$. Now let $p \in B$. Without a loss of generality let $p \in (a, b)$. Then we have $a < p < b$. Then take ϵ as $\epsilon =$ minimum of the set $\{p - a, b - p\}$. Then we show the interval $(p - \epsilon, p + \epsilon) \subset B$. Let $x \in (p - \epsilon, p + \epsilon)$. Then we have $p - \epsilon < x < p + \epsilon$. By definition of ϵ we have $a \leq p - \epsilon < x < p + \epsilon \leq b$. It follows $a < x < b$ so then $x \in B$. The same argument applies if $p \in (c, d)$. It follows B is in the standard topology on $\mathbb{R}$. Hence, $\mathcal{B}_2 \subset \mathcal{T}$.

Since $\mathcal{B}_2 \subset \mathcal{T}$ and every open set in $\mathcal{T}$ is a union of sets in $\mathcal{B}_2$, it follows $\mathcal{B}_2$ is a basis for the standard topology on $\mathbb{R}$. □

Theorem (3.3). *Suppose X is a set and $\mathcal{B}$ is a collection of subsets of X. Then $\mathcal{B}$ is a basis for some topology on X if and only if*

(1) *each point of X is in some element of $\mathcal{B}$, and*
(2) *if U and V are sets in $\mathcal{B}$ and p is a point in $U \cap V$, there is a set W in $\mathcal{B}$ such that $p \in W \subset (U \cap V)$.*

Proof. We prove the forward direction first. Since X itself is an open set, by definition of a basis we have $X = \cup_{B \in \mathcal{B}'} B$ where $\mathcal{B}' \subset \mathcal{B}$. Suppose we have a point $p \in X$. It follows $p \in \cup_{B \in \mathcal{B}'} B$. Thus, we have $p \in B \in \mathcal{B}' \subset \mathcal{B}$. It follows property one holds. Now we prove the second property. Suppose we have two basic open sets U and V and a point p such that $p \in U \cap V$. Since $U, V \in \mathcal{B} \subset \mathcal{T}$ it follows U and V are open sets. Then by definition of a topology we have $U \cap V$ is also an open set. Then we have by Theorem 3.1 we have there exists a basic open set W such that $p \in W \subset (U \cap V)$.

Now we prove the reverse direction. Let X be a set and $\mathcal{B}$ be a collection of subsets of X. Suppose each point of X is in some element of $\mathcal{B}$ and if U and V are sets in $\mathcal{B}$ and p is a point in $U \cap V$, there is a set W in $\mathcal{B}$ such that $p \in W \subset (U \cap V)$. We show that $\mathcal{B}$ is a basis for some topology $\mathcal{T}$ on X.

We define $\mathcal{T}$ as the following: U is in $\mathcal{T}$ if and only if $U = \varnothing$ or U is a union of some elements in $\mathcal{B}$. Now we show $\mathcal{T}$ is in fact a topology on X. By construction we have $\varnothing \in \mathcal{T}$.

Now we show $X \in \mathcal{T}$. Define a function $f \colon X \to \mathcal{B}$ where $f(p) = B_p$ where $B_p \in \mathcal{B}$ and $p \in B_p$ which exists since we assumed it to be true. For each point p there may exist an infinite number of appropriate sets in $\mathcal{B}$ but we can choose one by the axiom of choice.

Now we show $X = \cup_{p \in X} f(p)$. First we show $X \subset \cup_{p \in X} f(p)$. Let $q \in X$. Then q is in the domain of f so then $f(q)$ is well-defined and by definition of f we have $q \in f(q)$. It follows $q \in \cup_{p \in X} f(p)$. Hence, $X \subset \cup_{p \in X} f(p)$.

Now we show $\cup_{p \in X} f(p) \subset X$. Let $q \in \cup_{p \in X} f(p)$. Then $q \in f(p)$ for some $p \in X$. We have $f(p) \in \mathcal{B}$ and we assumed that $\mathcal{B}$ is a collection of subsets of X so $f(p) \subset X$. It follows $q \in X$. Hence, $\cup_{p \in X} f(p) \subset X$.

Since we have $X \subset \cup_{p \in X} f(p)$ and $\cup_{p \in X} f(p) \subset X$ we must have $X = \cup_{p \in X} f(p)$. Since X is a union of elements of $\mathcal{B}$ it follows $X \in \mathcal{T}$.

Now we show if U and V are in $\mathcal{T}$, then $U \cap V$ is in $\mathcal{T}$. If either is empty, then $U \cap V = \varnothing$ which is in $\mathcal{T}$. Thus, we assume both U and V are non-empty. Note that we assumed that for all $p \in U \cap V$ there exists some set $W \in \mathcal{B}$ such that $p \in W \subset U \cap V$. Then let $\mathcal{W}_p$ be a collection of sets defined as

$$\mathcal{W}_p = \{W \in \mathcal{B} \mid p \in W \subset U \cap V\}$$

for all $p \in U \cap V$. By the axiom of choice there exists a function $g \colon U \cap V \to \bigcup_{p \in U \cap V} \mathcal{W}_p$ such that $g(p) \in \mathcal{W}_p$ for all $p \in U \cap V$.

Now we show $U \cap V = \cup_{p \in U \cap V} g(p)$. First we show $U \cap V \subset \cup_{p \in U \cap V} g(p)$. Let $q \in U \cap V$. Then q is in the domain of g so then $g(q)$ is well-defined. By nature of construction, $q \in g(q)$. It follows $q \in \cup_{p \in U \cap V} g(p)$. Thus, $U \cap V \subset \cup_{p \in U \cap V} g(p)$.

Now we show $\cup_{p \in U \cap V} g(p) \subset U \cap V$. Let $q \in \cup_{p \in U \cap V} g(p)$. Then $q \in g(p)$ for some $p \in U \cap V$. By definition of g we have $g(p) \subset U \cap V$. It follows $q \in U \cap V$. Thus $\cup_{p \in U \cap V} g(p) \subset U \cap V$.

Since $U \cap V \subset \cup_{p \in U \cap V} g(p)$ and $\cup_{p \in U \cap V} g(p) \subset U \cap V$, it follows $U \cap V = \cup_{p \in U \cap V} g(p)$. Then $U \cap V$ is a union of elements in $\mathcal{B}$ since $g(p) \in \mathcal{B}$ for all $p \in U \cap V$. Thus $U \cap V \in \mathcal{T}$.

Now suppose we have a collection of sets $U_{\alpha \alpha \in \lambda}$ such that $U_\alpha \in \mathcal{B}$ for all $\alpha \in \lambda$. We show that $\cup_{\alpha \in \lambda} U_\alpha \in \mathcal{T}$. But $\cup_{\alpha \in \lambda} U_\alpha$ is a union of sets that are in $\mathcal{B}$ so then by definition of $\mathcal{T}$ we have $\cup_{\alpha \in \lambda} U_\alpha \in \mathcal{T}$.

It follows $\mathcal{T}$ satisfies all the four required properties of a topology so then $\mathcal{T}$ is in fact a topology on X. Now we show that $\mathcal{B}$ is a basis for $\mathcal{T}$. Let $B \in \mathcal{B}$. Then $B = B \cup B$ so then B is a union of sets in $\mathcal{B}$. Hence $B \in \mathcal{T}$ and $\mathcal{B} \subset \mathcal{T}$. By how we defined $\mathcal{T}$ it follows every set $U \in \mathcal{T}$ is a union of sets in $\mathcal{B}$. Then by definition, $\mathcal{B}$ is a basis for $\mathcal{T}$. □

Example. Define an alternative topology on $\mathbb{R}$, called the **lower limit topology**, generated by a basis consisting of all sets of the form $[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$. Denote this space by $\mathbb{R}_{\text{LL}}$. It is sometimes called the **Sorgenfrey line** or $\mathbb{R}^1_{\text{bad}}$.

Exercise (3.4). Check that the basis proposed above for the lower limit topology is in fact a basis.

Let $\mathcal{B}$ be the collection of sets of the form

$$[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$$

We show that the basis proposed above $\mathcal{B}$ is a basis on $\mathbb{R}$. We use Theorem 3.3 for this. First we show every real number is in some element of $\mathcal{B}$. Afterwards, we show that for any two sets $U, V \in \mathcal{B}$, any point $p \in U \cap V$, there exists a set $W \in \mathcal{B}$ such that $p \in W \subset U \cap V$.

Let $x \in \mathbb{R}$. Then $x \in [x, x+1)$. And $[x, x+1) \in \mathcal{B}$. It follows all elements of $\mathbb{R}$ are in some element of $\mathcal{B}$.

Now suppose we have two elements U and V in $\mathcal{B}$ and there exists some real number x such that $x \in U \cap V$. Let $U = [a, b)$ and $V = [c, d)$. Then we have $a \leq x < b$ and $c \leq x < d$. Let $s =$ maximum of $\{a, c\}$ and $t =$ minimum of $\{b, d\}$. Then we construct the half-open interval $[s, t)$. It follows $x \in [s, t)$. Moreover $[s, t) \in \mathcal{B}$. Now we show $[s, t) \subset U \cap V$. Let $p \in [s, t)$. It follows $s < p < t$. Since $a \leq s$, $c \leq s$, $b \geq t$, and $d \geq t$ it follows $a < p < b$ and $c < p < d$. Then $p \in U$ and $p \in V$ so then $p \in U \cap V$. Hence, $[s, t) \subset U \cap V$. It follows $x \in [s, t) \subset U \cap V$ where $[s, t) \in \mathcal{B}$.

We proved two things. First, we have shown that every real number is in some element of $\mathcal{B}$. Second, we showed for any two sets $U, V \in \mathcal{B}$ and for any point $p \in U \cap V$, there exists some set $W \in \mathcal{B}$ such that $p \in W \subset U \cap V$. It follows from Theorem 3.3 that $\mathcal{B}$ is the basis for some topology on $\mathbb{R}$.

Theorem (3.5). *Every open set in $\mathbb{R}_{std}$ is an open set in $\mathbb{R}_{LL}$, but not vice versa.*

Proof. Let (a, b) be some open interval of $\mathbb{R}_{\text{std}}$ where $a < b$. Then define I_1 as

$$I_1 = \left[\frac{a+b}{2}, b\right)$$
$$I_2 = \left[\frac{3a+b}{4}, b\right)$$

and more generally

$$\begin{aligned} I_n &= \left[\frac{(2^n-1)a+b}{2^n}, b\right) \\ &= \left[a + \frac{b-a}{2^n}, b\right) \end{aligned}$$

We show $(a, b) = \cup_{j=1}^{\infty} I_j$. First we show $\cup_{j=1}^{\infty} I_j \subset (a, b)$. Let $p \in \cup_{j=1}^{\infty} I_j$. Then $p \in I_n$ for some $n \in \mathbb{N}$. Then we have $a + \frac{b-a}{2^n} < p < b$ and we know $a < a + \frac{b-a}{2^n}$ so then $a < p < b$. It follows $p \in (a, b)$ so then $\cup_{j=1}^{\infty} I_j \subset (a, b)$.

Now we show $(a, b) \subset \cup_{j=1}^{\infty} I_j$. Let $p \in (a, b)$. Since $a < p$ we know for a sufficiently large $n \in \mathbb{N}$ that $a + \frac{b-a}{2^n} < p$. And we know $p < b$ so then $a + \frac{b-a}{2^n} < p < b$ for some n. It follows $p \in I_n$ for some $n \in \mathbb{N}$. Thus $p \in \cup_{j=1}^{\infty} I_j$. Hence, $(a, b) \subset \cup_{j=1}^{\infty} I_j$.

Since we have $\cup_{j=1}^{\infty} I_j \subset (a, b)$ and $(a, b) \subset \cup_{j=1}^{\infty} I_j$ it follows $(a, b) = \cup_{j=1}^{\infty} I_j$. It follows every open set in $\mathbb{R}_{\text{std}}$ is an open set in $\mathbb{R}_{\text{LL}}$.

Now we show that there are open sets in $\mathbb{R}_{\text{LL}}$ that are not in $\mathbb{R}_{\text{std}}$. Specifically, we show that the set $[0, 1)$ is an open set in $\mathbb{R}_{\text{LL}}$ but not in $\mathbb{R}_{\text{std}}$. Since $[0, 1)$ is of the form $[a, b)$ where $a, b \in \mathbb{R}$, it follows $[0, 1)$ is an open set in $\mathbb{R}_{\text{LL}}$.

For the sake of contradiction, suppose that $[0, 1)$ is an open set in $\mathbb{R}_{\text{std}}$. Let $\mathcal{B}$ be a collection of sets defined as

$$\{x \in \mathbb{R} \mid a < x < b\}$$

where $a, b \in \mathbb{R}$. By Lemma 3.5 we know that $\mathcal{B}$ is a basis for $\mathbb{R}_{\text{std}}$. Then by definition of a basis, it follows any open set in $\mathbb{R}_{\text{std}}$ is a union of basic open sets. Since we assumed $[0, 1)$ is an open set we have $[0,1) = \cup_{B\in\mathcal{B}'}B$ where $\mathcal{B}' \subset \mathcal{B}$. Since $0 \in [0,1)$ it follows $0 \in \cup_{B\in\mathcal{B}'}B$. Then $0 \in B$ for some $B \in \mathcal{B}'$. Then $c < 0 < d$ for some $c,d \in \mathbb{R}$. It follows $c < c/2 < 0$ so then $c/2 \in (c,d) = B$. Then $c/2 \in \cup_{B\in\mathcal{B}'}B$. Since $[0,1) = \cup_{B\in\mathcal{B}'}B$ we must have $c/2 \in [0,1)$ but $c/2 < 0$. This is a contradiction. It follows $[0,1)$ is not an open set in $\mathbb{R}_{\text{std}}$. ☐

Lemma (3.5). *A basis for $\mathbb{R}_{std}$ is $\mathcal{B}$ where $U \in \mathcal{B}$ if and only if it is of the form*

$$\{x \in \mathbb{R} \mid a < x < b\}$$

for some $a,b \in \mathbb{R}$.

Proof. We first show $\mathcal{B} \subset \mathcal{T}_{\text{std}}$. Let $B \in \mathcal{B}$. Then B is of the form (a,b). To show $B \in \mathcal{T}_{\text{std}}$ it is sufficient to demonstrate that for all points $p \in (a,b)$ there exist $\delta > 0$ such that $(p-\delta, p+\delta) \subset (a,b)$. So let $x \in (a,b)$. Then take $\epsilon =$ minimum of $\{x-a, b-x\}$. Then we have the interval $(x-\epsilon, x+\epsilon)$. Suppose $q \in (x-\epsilon, x+\epsilon)$. Then $x-\epsilon < q < x+\epsilon$. By definition of ϵ we have $a \le x-\epsilon < q < x+\epsilon \le b$. It follows $q \in (a,b)$. Thus $(x-\epsilon, x+\epsilon) \subset (a,b)$. It follows $(a,b) \in \mathcal{T}_{\text{std}}$. Hence, $\mathcal{B} \subset \mathcal{T}_{\text{std}}$.

Second we show for any open set $U \in \mathcal{T}_{\text{std}}$ and for any point $p \in U$, there exists some set $V \in \mathcal{B}$ such that $p \in V \subset U$. Let U be an open set in the standard topology on $\mathbb{R}$. Suppose p is a point in U. Then by definition of the standard topology, there exists some $\epsilon > 0$ such that $(p-\epsilon, p+\epsilon) \in U$. Then fix ϵ. Since $p-\epsilon, p+\epsilon \in \mathbb{R}$ then $(p-\epsilon, p+\epsilon) \in \mathcal{B}$. Then let $V = (p-\epsilon, p+\epsilon)$. It follows $p \in V \subset U$ and $V \in \mathcal{B}$.

By Theorem 3.1, it follows we meet the two necessary conditions for $\mathcal{B}$ to be a basis for the standard topology on $\mathbb{R}$. ☐

Exercise (3.6). Give an example of two topologies on $\mathbb{R}$ such that neither is finer than the other, that is, the two topologies are not comparable.

The standard topology on $\mathbb{R}$ and the cocountable topology on $\mathbb{R}$. The set $(0,1)$ is open in the standard topology but not open in the cocountable topology since $\mathbb{R} - (0,1)$ is uncountable. Then define the set H as

$$H = \{1/n \mid n \in \mathbb{N}\} = \{1, 1/2, 1/3, ...\}$$

Then the set $\mathbb{R} - H$ is in the cocountable topology since its complement is H which is countable. However, it is not open in the standard topology since there does not exist an open interval around 0 that is in $\mathbb{R} - H$. That is, there does not exist an $\epsilon > 0$ such that $(0-\epsilon, 0+\epsilon) \subset \mathbb{R} - H$.

Exercise (3.7). Check that the collection of sets that we specify as a basis in the Double Headed Snake actually forms the basis for a topology.

Proof. Let us denote the Double Headed Snake as $\mathbb{R}_{+00}$. We show that all points in $\mathbb{R}_{+00}$ are in some basic open set. Let $p \in \mathbb{R}_{+00} = \mathbb{R}_+ \cup \{0'\} \cup \{0''\}$. Either p is a non-negative real number or $p \in \{0'\} \cup \{0''\}$. If p is a non-negative real number then $p \in (0, p+1)$ which is a basic open set. If $p \in \{0'\} \cup \{0''\}$ then either $p \in (0,b) \cup \{0'\}$ or $p \in (0,b) \cup \{0''\}$ where $b \in \mathbb{R}_+$. It follows p is in some basic open set.

Now we show that if $U, V \in \mathcal{B}$ and $p \in U \cap V$ then there exists some basic open set W such that $p \in W \subset U \cap V$ where $\mathcal{B}$ the basis that generates the topology. There are three cases: both U and V are of the form (a,b), one is of the form (a,b), and neither is of the form (a,b).

First suppose both U and V are non-disjoint open intervals within $\mathbb{R}_+$. Then let $U = (a, b)$ and $V = (c, d)$ where $a < b$, $c < d$, and a, b, c, and d are non-negative real numbers. Let $p \in (a, b) \cap (c, d)$. It follows $p > a$, $p > c$, $p < b$, and $p < d$. Let $W = (\max\{a, c\}, \min\{b, d\})$. Then $p \in W$. Since $\max\{a, c\}$ and $\min\{b, d\}$ are both in $\mathbb{R}_+$ it follows the interval W is a basic open set. Now we show $W \subset U \cap V$. Let $q \in W$. By definition of W it follows $q > a$, $q > c$, $q < b$, and $q < d$. It follows $q \in (a, b)$ and $q \in (c, d)$. Thus, $q \in U \cap V$. Hence, $W \subset U \cap V$. Then we have $p \in W \subset U \cap V$ where W is a basic open set. It follows by Theorem 3.3 that the specified basis in the Double Headed Snake forms a basis for a topology. ☐

Example. Let $\mathbb{R}_{\text{har}}$ be the set $\mathbb{R}$ with a topology whose basis is all sets of the form (a, b) or $(a, b) - H$, where $H = \{1/n\}_{n \in \mathbb{N}}$ is the harmonic sequence, and $a, b \in \mathbb{R}$.

Exercise (3.9). There are three parts.

(1) In the topological space $\mathbb{R}_{\text{har}}$, what is the closure of the set $H = \{1/n\}_{n \in \mathbb{N}}$?
(2) In the topological space $\mathbb{R}_{\text{har}}$, what is the closure of the set $H^- = \{-1/n\}_{n \in \mathbb{N}}$?
(3) Is it possible to find disjoint open sets U and V in $\mathbb{R}_{\text{har}}$ such that $0 \in U$ and $H \subset V$?

We answer part (1) first. The closure is the set H and all of its limit points. Let $p \in \mathbb{R} - H$. Then there exists some neighborhood U of the form $(a, b) - H$ where $a, b \in \mathbb{R}$. Since $U \cap H = \varnothing$ it follows $(U - \{p\}) \cap H = \varnothing$. So then p is not a limit point. Now let $p \in H$. Then $p = 1/n$ for some natural number n. Then we want to find some open interval $I = (1/n - \epsilon, 1/n + \epsilon)$ such that $I \cap H = \varnothing$. Let $\epsilon = 1/n - 1/(n+1) = 1/(n^2 + n)$. Then $I = (1/(n+1), (n+2)/(n^2+n))$ and $I \cap H = \{1/n\}$. Then $(I - \{1/n\}) \cap H = \varnothing$. Then p is not a limit point. It follows H has no limit points and hence $\overline{H} = H$.

Now we answer part (2). We show that there is only one limit point in $\mathbb{R} - H^-$. Let $p \in \mathbb{R} - H^-$. If $p < -1$ then there exists $\epsilon > 0$ such that $p + \epsilon < -1$. Then $(p - \epsilon, p + \epsilon) \cap H^- = \varnothing$. Now suppose $p > 0$. Then $(p, p+1) \cap H^- = \varnothing$. Now suppose $-1 \leq p < 0$. Note $p \notin H^-$ so $-1 < p < 0$. Define the set T as

$$T = \left\{-\frac{1}{n} \mid n \in \mathbb{N} \text{ and } -\frac{1}{n} \leq p\right\}$$

T is non-empty since $-1/1 = -1 \leq p$. T is finite since there exists some $N \in \mathbb{N}$ such that $-1/N > p$ by the archimedean property and $-1/n > p$ for all $n \geq N$ where $n \in \mathbb{N}$ since $-1/n$ is monotonically increasing on the interval $[-1, 0)$. Moreover, because T is finite, there is a greatest element $-1/m \in T$ where $m \in \mathbb{N}$. Then set ϵ as

$$\epsilon = \min\{p - (-1/m), -1/(m+1) - p\}$$

Note that $p - (-1/m)$ and $-1/(m+1) - p$ are both positive since $-1/m < p < -1/(m+1)$. Then we have $(p - \epsilon, p + \epsilon) \cap H^- = \varnothing$ so then p is not a limit point of H^-. Now we show 0 is a limit point of H^-. Suppose we have an open set $(a, b) \ni 0$ such that $a < 0 < b$. By the archimedean property there exists a sufficiently large natural number n such that $0 > -1/n > a$. It follows $((a, b) - \{0\}) \cap H^- \neq \varnothing$. The same argument applies for all open sets of the form $(a, b) - H$ since $h > 0$ for all $h \in H$. Moreover, since $H \cap H^- = \varnothing$ it follows if $((a, b) - \{0\}) \cap H^- \neq \varnothing$ then $(((a, b) - \{0\}) - H) \cap H^- \neq \varnothing$ for all $a, b \in \mathbb{R}$. It follows the closure of H^- is $\overline{H^-} = H^- \cup \{0\}$.

Now we answer part (3). Suppose U and V are open sets such that $0 \in U$ and $H \subset V$. Then U is of the form $(-\epsilon, a)$ or $(-b, \epsilon)$ where $a, b, \epsilon > 0$. Let $m = \min\{a, \epsilon\}$. Then there exists a natural number n such that $0 < 1/n < m$ by the archimedean property. So then $U \cap H \neq \varnothing$. Since $H \subset V$ it follows $U \cap V \neq \varnothing$. Thus, U and V cannot be disjoint.

Example. Let $\mathbb{H}_{\text{bub}}$ be the upper half-plane $\{(x, y) \mid x, y \in \mathbb{R}, y \geq 0\}$ with a topology whose basis consists of

(1) all balls $B((x, y), r)$, where $0 < r \leq y$, and
(2) all sets $B((x, y), r) \cup \{(x, 0)\}$, where $r = y > 0$.

Exercise (3.10). There are five parts.

(1) In $\mathbb{H}_{\text{bub}}$, what is the closure of the set of rational points on the $x-axis$?
(2) In $\mathbb{H}_{\text{bub}}$, which subsets of the x-axis are closed sets?
(3) In $\mathbb{H}_{\text{bub}}$, let A be a countable set on the x-axis and let z be a point on the x-axis not in A. Show that there exist disjoint open sets U and V such that $A \subset U$ and $z \in V$.
(4) In $\mathbb{H}_{\text{bub}}$, let A and B be countable sets on the x-axis such that A and B are disjoint. Show that there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$.
(5) In $\mathbb{H}_{\text{bub}}$, let A be the rational numbers, and let B be the irrational numbers. Do there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$?

Proof. We answer parts (1) and (2) simultaneously. Suppose we have some subset, A, of the x-axis. Let the set of points on the x-axis be defined by the set $T = \{(a, 0) \in \mathbb{R}^2 \mid a \in \mathbb{R}\}$. Then suppose we have a point $p = (x, 0) \in T - A$. Then $B((x, y), y) \cup \{(x, 0)\} \ni p$ and $(B((x, y)y) \cup \{(x, 0)\} - p) \cap A = \varnothing$ since the only intersection with $B((x, y), y) \cup \{(x, 0)\}$ and the x-axis is point $p = (x, 0)$ itself. It follows p is not a limit point of A. Thus there does not exist any limit points outside of the set A. Hence $\overline{A} = A$. It follows all subsets of the x-axis are closed. This proves part (2). Since the set of rational points on the x-axis is a subset of the x-axis, it follows the closure of this set is itself which proves part (1).

Now we prove part (3). Let A be a countable set on the x-axis and let z be a point on the x-axis not in A. Now we construct open sets U and V such that $A \subset U$ and $z \in V$. Take $z \in \mathbb{R}^2$ as $z = (z_0, 0)$. Then define the open set V as $V = B((z_0, r), r) \cup \{(z_0, 0)\}$ for some $r \in \mathbb{R}$ and $r > 0$. Choose some $r = r_0$ so then $V = B((z_0, r_0), r_0) \cup \{(z_0, 0)\}$. It follows $z \in V$.

Since A is countable, we can enumerate the elements of A such as $(a_1, 0), (a_2, 0)$, etc. So then $A = \{(a_1, 0), (a_2, 0), ...\}$ Then we construct a neighborhood U_i for a_i for all $i \in \mathbb{N}$. Let $U_i = B((a_i, r_i), r_i) \cup \{(a_i, 0)\}$ where r_i is defined as

$$r_i = \frac{(z_0 - a_i)^2}{4r_0}$$

Then define U as $U = \cup_{i=1}^{\infty} U_i$ which is open since arbitrary unions of open sets is open by definition of a topology. It follows $(a_i, 0) \in U$ since $(a_i, 0) \in U_i \subset U$ for all $i \in \mathbb{N}$. It follows $A \subset U$.

Now we show U and V are disjoint. For the sake of contradiction, suppose U and V are not disjoint. Then there exist some point $p \in U \cap V$. By definition of union, it follows there exists some $i \in \mathbb{N}$ such that $p \in U_i \cap V$. Let $p = (x, y)$. Since $p \in U_i$ we have $d((a_i, r_i), (x, y)) < r_i$

and since $p \in V$ we have $d((z_0, r_0), (x, y)) < r_0$. Then by the triangle inequality we have

$$\begin{aligned}
d((a_i, r_i), (z_0, r_0)) &< d((a_i, r_i), (x, y)) + d((x, y), (z_0, r_0)) \\
\sqrt{(a_i - z_0)^2 + (r_i - r_0)^2} &< r_i + r_0 \\
(a_i - z_0)^2 + (r_i - r_0)^2 &< (r_i + r_0)^2 \\
(a_i - z_0)^2 &< 4r_i r_0
\end{aligned}$$

by definition of r_i we have

$$\begin{aligned}
(a_i - z_0)^2 &< 4\Big(\frac{(z_0 - a_i)^2}{4r_0}\Big)r_0 \\
(a_i - z_0)^2 &< (z_0 - a_i)^2
\end{aligned}$$

which is a contradiction. It follows U and V are disjoint.

Now we prove part (4). Let A and B be countable sets on the x-axis such that A and B are disjoint. Since A and B are countable, there exists bijections $f\colon \mathbb{N} \to A$ and $g\colon \mathbb{N} \to B$. We then construct the bijection $h\colon \mathbb{N} \to A \cup B$ defined as

$$h(n) = \begin{cases} (n+1)/2 & \text{n is odd} \\ (n/2) & \text{n is even} \end{cases}$$

Then for all $i \in \mathbb{N}$ we construct open sets U_i such that $h(i) \subset U_i$ and $U_j \cap U_k = \varnothing$ for all $j, k \in \mathbb{N}$ where $j \neq k$. Then we construct U_i as follows. Note $h(i) = (x_i, 0)$. Let $U_1 = B((x_1, r_1), r_1) \cup \{x_1, 0\}$ where we choose $r_1 \in \mathbb{R}$ arbitrarily with the condition $r_1 > 0$.

Now we construct U_n for $n > 1$. First let T_n be defined as

$$T_n = \Big\{\frac{(x_i - x_n)^2}{4r_i} \in \mathbb{R} \mid i \in \mathbb{N} \text{ and } 1 \leq i < n\Big\}$$

Since T_n is non-empty and finite we take the smallest element $\min(T_n)$ and then let $r_n = \min(T_n)$. Then we define U_n as

$$U_n = B((x_n, r_n), r_n) \cup \{x_n, 0\}$$

It follows every point $p \in A \cup B$ has an associated open set $U_k \ni p$. Specifically, $p \in U_{h^{-1}(p)}$.

Now we show $U_j \cap U_k = \varnothing$ for all $j, k \in \mathbb{N}$ such that $j \neq k$. For the sake of contradiction, let $p \in U_j \cap U_k$. Let $p = (x, y)$. Without a loss of generality, let $k > j$. Then T_k is defined as

$$T_k = \Big\{\frac{(x_i - x_k)^2}{4r_i} \in \mathbb{R} \mid i \in \mathbb{N} \text{ and } 1 \leq i < k\Big\}$$

And since $1 \leq j < k$, it follows $r_k = \min(T_k) \leq (x_j - x_k)^2/(4r_j)$. Now consider the sets U_j and U_k

$$\begin{aligned}
U_j &= B((x_j, r_j), r_j) \cup \{x_j, 0\} \\
U_k &= B((x_k, r_k), r_k) \cup \{x_k, 0\}
\end{aligned}$$

Note $x_j \neq x_k$ since $A \cap B = \varnothing$ so we only consider the balls. Recall $p = (x, y)$. Since $p \in U_j \cap U_k$ we then have

$$d((x_j, r_j), (x, y)) < r_j \text{ and } d((x_k, r_k), (x, y)) < r_k$$

Then by the triangle inequality we have

$$\begin{aligned} d((x_j, r_j), (x_k, r_k)) &< d((x_j, r_j), (x, y)) + d((x, y), (x_k, r_k)) \\ (x_j - x_k)^2 + (r_j - r_k)^2 &< (r_j + r_k)^2 \\ (x_j - x_k)^2 &< 4r_j r_k \end{aligned}$$

then by substitution of $r_k = (x_j - x_k)^2/(4r_j)$ we have

$$(x_j - x_k)^2 < (x_j - r_k)^2$$

This is a contradiction. It follows U_j and U_k are disjoint. Again, $h(i) \in U_i$ for all $i \in \mathbb{N}$ since $U_i = B((x_i, r_i), r_i) \cup h(i)$. Now we define U and V. Note h is a bijection and $A \cap B = \varnothing$ so the following definition is well-defined. $U = \cup_{p \in A} U_{h^{-1}(p)}$ and $V = \cup_{p \in B} U_{h^{-1}(p)}$. U and V are open since arbitrary unions of open sets are open. It follows $A \subset U$ and $B \subset V$.

We did not answer part (5). It was turned into an optional exercise. □

Example. Let $\mathbb{Z}_{\text{arith}}$ be the set $\mathbb{Z}$ with a topology whose basis elements are arithmetic progressions, i.e., sets of the form $\{az + b \mid z \in \mathbb{Z}\}$ for $a, b \in \mathbb{Z}, a \neq 0$.

Exercise (3.11). Check that the arithmetic progressions form a basis for a topology on $\mathbb{Z}$.

Proof. Define the set B_z as the set $\{az + b \mid z \in \mathbb{Z}\}$ for $a, b \in \mathbb{Z}$ and $a \neq 0$. Then define $\mathcal{B}$ as the set U is in $\mathcal{B}$ if and only if $U = B_z$ for some $z \in \mathbb{Z}$. We show that $\mathcal{B}$ forms a basis for a topology on $\mathbb{Z}$.

Let $k \in \mathbb{Z}$. We show that k is in some element of $\mathcal{B}$. Since $k = (1)(k) + (0)$ it follows $k \in \{az+b \mid z \in \mathbb{Z}\}$ where $a = 1$ and $b = 0$. So $k \in \{z \mid z \in \mathbb{Z}\} \in \mathcal{B}$. It follows every integer is in some element of $\mathcal{B}$.

Now we show that given U and V are both in $\mathcal{B}$ and there is some point k such that $k \in U \cap V$, then there exists some set W in $\mathcal{B}$ such that $k \in W \subset U \cap V$. Now suppose we have two sets $U, V \in \mathcal{B}$ and some integer $k \in U \cap V$. Define U and V as

$$U = \{az + b \mid z \in \mathbb{Z}\}$$
$$V = \{cz + d \mid z \in \mathbb{Z}\}$$

Then $k = ax + b$ and $k = cy + d$ for some integers a, x, b, c, y, and d where $a \neq 0 \neq c$. Now we construct the set W as

$$W = \{(ac)z + k \mid z \in \mathbb{Z}\}$$

Now $k \in W$ since $(ac)(0) + (k) = k$. And $W \in \mathcal{B}$ since $ac, k \in \mathbb{Z}$ and $ac \neq 0$ since both a and c are non-zero. Now we show $W \subset U \cap V$. Let $p \in W$. Then $p = acz + k$ for some integer $z \in \mathbb{Z}$. Note that a, c, k have already been defined. Then by definition of k we have $p = acz + ax + b = a(cz + x) + b$ where $cz + x \in \mathbb{Z}$ since integers are closed under addition and multiplication. It follows $p \in U$. Now we show $p \in V$. Since $p = acz + k$ and $k = cy + d$ we have $p = acz + cy + d = c(az + y) + d$ where $az + y \in \mathbb{Z}$. Then $p \in V$. It follows $p \in U \cap V$. It follows $W \subset U \cap V$.

Since for any $U, V \in \mathcal{B}$ and for all $p \in U \cap V$ there exists $W \in \mathcal{B}$ such that $p \in W \subset U \cap V$ and all integers are in some element of $\mathcal{B}$, it follows by Theorem 3.3 that $\mathcal{B}$ is a basis for some topology on $\mathbb{Z}$. □

Exercise (3.12). Use $\mathbb{Z}_{\text{arith}}$ to show there are infinitely many primes.

Proof. For the sake of contradiction, suppose there are a finite number of primes. Then there are n total primes where $n \in \mathbb{N}$. Then consider the sets

$$P_1 = \{p_1 z \mid z \in \mathbb{Z}\}$$
$$P_2 = \{p_2 z \mid z \in \mathbb{Z}\}$$
$$\vdots$$
$$P_n = \{p_n z \mid z \in \mathbb{Z}\}$$

where p_i is the ith prime and $1 \le i \le n$. Since every natural number has a prime factorization, it follows every natural number is in at least one of these sets. In other words we have $\mathbb{N} \subset \cup_{i=1}^{n} P_i$. We show this leads to a contradiction. First we define the set P' as

$$P' = \{p_1 p_2 \cdots p_n z + 1 \mid z \in \mathbb{Z}\}$$

Let $x \in P'$ and $x > 0$. Then $x \equiv 1 \pmod{p_i}$ for all $i \in \{1, 2, ..., n\}$. It follows $x \notin P_i$ for all i. But $x \in \mathbb{N} \subset \cup_{i=1}^{n} P_i$. It follows $\mathbb{N} \not\subset \cup_{i=1}^{n} P_i$ which is a contradiction. Hence, there must be an infinite number of primes. $\square$

Theorem (3.15). *Let $(X, \mathcal{T})$ be a topological space, and let $\mathcal{S}$ be a collection of subsets of X. Then $\mathcal{S}$ is a subbasis for $\mathcal{T}$ if and only if*

(1) $\mathcal{S} \subset \mathcal{T}$
(2) *for each set U is $\mathcal{T}$ and point p in U there is a finite collection $\{V_i\}_{i=1}^{n}$ of elements of $\mathcal{S}$ such that*

$$p \in \bigcap_{i=1}^{n} V_i \subset U$$

Proof. Let $\mathcal{S}$ be a collection of subsets of X. We prove the forward direction first. If $\mathcal{S}$ is a subbasis for $\mathcal{T}$, then by definition of a subbasis, the set of all intersections, which we dneote as $\mathcal{B}$, is a basis for $\mathcal{T}$. Take $S \in \mathcal{S}$. Then $S \in \mathcal{B} \subset \mathcal{T}$ since $S \cap S = S$ is a finite intersection. It follows $S \in \mathcal{T}$. Now suppose we have the set U and some point p in U. By definition of a basis, there exists some basic open set B such that $p \in B \subset U$. By definition of a subbasis, B must be a finite intersection of elements in $\mathcal{S}$. Then $B = \cap_{i=1}^{n} S_i$ where $S_i \in \mathcal{S}$ for all $i \in \{1, 2, ..., n\}$. Altogether, we have

$$p \in \cap_{i=1}^{n} S_i \subset U$$

This proves the forward direction. Now we prove the reverse direction. Let $\mathcal{B}$ be the set of all finite intersections of elements in $\mathcal{S}$. Note for any open set U and for any point p in U we have there exists some set $B \in \mathcal{B}$ such that $p \in B \subset U$. Now we show for all $B \in \mathcal{B}$ that $B \in \mathcal{T}$. Since we know that $\mathcal{S} \subset \mathcal{T}$ it follows any finite intersection of elements in $\mathcal{S}$ is also open. It follows all elements in $\mathcal{B}$ are open sets of $\mathcal{T}$. It follows by Theorem 3.1 that $\mathcal{B}$ is in fact a basis for $\mathcal{T}$. Moreover, $\mathcal{B}$ is the set of all finite intersections of elements in $\mathcal{S}$. It follows, by definition, that $\mathcal{S}$ is a subbasis for $\mathcal{T}$. $\square$

Theorem (3.16). *Suppose X is a set and $\mathcal{S}$ is a collection of subsets of X. Then $\mathcal{S}$ is a subbasis for some topology on X if and only if every point of X is in some element of $\mathcal{S}$.*

Proof. We prove the forward direction first. If $\mathcal{S}$ is a subbasis for some topology on X. Then X is an open set in the topology. Then X is a union of basic open sets. If $p \in X$, then p is in some basic open set B. And B is a finite intersection of elements in $\mathcal{S}$ so them p is in some element of $\mathcal{S}$.

Now we prove the reverse direction. Suppose every point p in X is in some element of S. We show that S is a subbasis for some topology. Let $\mathcal{B}$ be the set of all finite intersection of elements in $\mathcal{S}$. We show $\mathcal{B}$ is a basis for some topology on X by using Theorem 3.3.

First, we show that every point p in X is in some element of $\mathcal{B}$. We know every point p is in some element of $V \in \mathcal{S}$. And $V \in \mathcal{B}$ since $V \cap V = V$. So then every point p is in some element of $\mathcal{B}$.

Second, we show that for every two sets $B_1, B_2 \in \mathcal{B}$ and point $p \in B_1 \cap B_2$, there exists a set $W \in \mathcal{B}$ such that $p \in W \subset B_1 \cap B_2$. Since B_1 and B_2 are finite intersections it follows $B_1 \cap B_2$ is also a finite intersection. Hence, $B_1 \cap B_2 \in \mathcal{B}$. Then we have $p \in B_1 \cap B_2 \subset B_1 \cap B_2$. It follows by Theorem 3.3 that $\mathcal{B}$ is a basis for some topology on X. Since $\mathcal{B}$ is the set of all finite intersections of elements in $\mathcal{S}$, it follows $\mathcal{S}$ is a subbasis. $\square$

Exercise (3.18)**.** Let X be a set totally ordered by $\leq$. Let $\mathcal{S}$ be the collection of sets of the form

$$\{x \in X \mid x < a\} \text{ or } \{x \in X \mid a < x\}$$

for $a \in X$. Then $\mathcal{S}$ forms a subbasis for the order topology on X.

Let $\mathcal{B}'$ be the set of all finite intersections of elements in $\mathcal{S}$. Let $\mathcal{B}$ be the basis for the order topology. We show that $\mathcal{B}'$ is a basis for the order topology on X.

Exercise (3.19)**.** Verify that the order topology on $\mathbb{R}$ with the usual $\leq$ order is the standard topology on $\mathbb{R}$.

Exercise (3.20).

Exercise (3.21)**.** In the lexicographically ordered square find the closures of the following subsets:

(1) $A = \left\{ \left(\frac{1}{n}, 0 \right) \mid n \in \mathbb{N} \right\}$
(2) $B = \left\{ \left(1 - \frac{1}{n}, \frac{1}{2} \right) \mid n \in \mathbb{N} \right\}$
(3) $C = \{(x, 0) \mid 0 < x < 1\}$
(4) $D = \left\{ \left(x, \frac{1}{2} \right) \mid 0 < x < 1 \right\}$
(5) $E = \left\{ \left(\frac{1}{2}, y \right) \mid 0 < y < 1 \right\}$

We have $\overline{A} = A$. Then $\overline{B} = B$. Then $\overline{C} = C \cup \{(1,0)\}$. Then $\overline{D} = D$. Then $\overline{E} = E \cup \{(1/2, 0), (1/2, 1)\}$.

Exercise (3.24). In the lexicographically ordered square find the closures of the following subsets:

$$A = \left\{ \left(\frac{1}{n}, 0 \right) \mid n \in \mathbb{N} \right\}$$
$$B = hi$$

Theorem (3.25). *Let $(X, \mathcal{T})$ be a topological space and $Y \subset X$. Then the collection of sets $\mathcal{T}_Y$ is in fact a topology on Y.*

Proof. Suppose we have the topological space $(X, \mathfrak{T})$ and $Y \subset X$. Recall $\mathfrak{T}_Y$ is defined as

$$\mathfrak{T}_Y = \{U \mid U = V \cap Y \text{ for some } V \in \mathfrak{T}\}$$

To show $\mathfrak{T}_Y$ is a topology on Y, we show $\mathfrak{T}_Y$ satisfies the four properties of a topology. First, $\varnothing \in \mathfrak{T}$. Then $\varnothing \cap Y = \varnothing$ so then $\varnothing \in \mathfrak{T}_Y$. Also, $X \in \mathfrak{T}$ so then $X \cap Y = Y$ which implies $Y \in \mathfrak{T}_Y$. Now suppose we have two sets $U, V \in \mathfrak{T}_Y$. Then by construction of $\mathfrak{T}_Y$, there exists open sets U' and V' in $\mathfrak{T}$ such that $U = U' \cap Y$ and $V = V' \cap Y$. Since $U', V' \in \mathfrak{T}$ it follows $U' \cap V' \in \mathfrak{T}$. Then $Y \cap (U' \cap V') \in \mathfrak{T}_Y$. But we know

$$\begin{aligned} Y \cap (U' \cap V') &= (Y \cap U') \cap (Y \cap V') \\ &= U \cap V \end{aligned}$$

It follows $U \cap V \in \mathfrak{T}_Y$. Now suppose we have a collection of sets $\{U_\alpha\}_{\alpha \in \lambda}$ where $U_\alpha \in \mathfrak{T}_Y$ for all $\alpha \in \lambda$. Then there exists U'_α such that $U_\alpha = U'_\alpha \cap Y$ for all $\alpha \in \lambda$. We know $\cup_{\alpha \in \lambda} U'_\alpha$ is an open set in $\mathfrak{T}$ so then $Y \cap (\cup_{\alpha \in \lambda} U_\alpha) \in \mathfrak{T}_Y$ which by DeMorgan's Law is

$$\cup_{\alpha \in \lambda} Y \cap U'_\alpha = \cup_{\alpha \in \lambda} U_\alpha$$

It follows $\cup_{\alpha \in \lambda} U_\alpha$ is an open set in $\mathfrak{T}_Y$. Since $\mathfrak{T}_Y$ satisfies all four properties, $\mathfrak{T}_Y$ is a topology on Y. □

Exercise (3.26). Consider $Y = [0, 1)$ as a subspace of $\mathbb{R}_{\text{std}}$. In Y, is the set $[1/2, 1)$ open, closed, neither, or both?

We show that there are no limits points p in the set $Y - [1/2, 1)$. Let $p \in Y - [1/2, 1)$. Then $0 \leq p < 1/2$. Then we can choose an ϵ such that $p < \epsilon < 1/2$. Then the $(-\epsilon, \epsilon) \in \mathbb{R}_{\text{std}}$. Then $(-\epsilon, \epsilon) \cap Y = [0, \epsilon)$ and then $[0, \epsilon) \cap [1/2, 1) = \varnothing$. Hence, p is not a limit point. It follows the set $[1/2, 1)$ is closed.

Exercise (3.27). Consider a subspace Y of the topological space X. Is every subset $U \subset Y$ that is open in Y also open in X.

This is not always possible. Take Y to be a subset of X that is not open. Then let $U = Y$. Then $U \subset Y$ is an open set in the subspace topology $(Y, \mathfrak{T}_Y$ but U is not open in the topological space $(X, \mathfrak{T})$.

Theorem (3.28). *Let $(Y, \mathfrak{T}_Y)$ be a subspace of $(X, \mathfrak{T})$. A subset $C \subset Y$ is closed in $(Y, \mathfrak{T}_Y)$ if and only if there is a set $D \subset X$, closed in $(X, \mathfrak{T}$, such that $C = D \cap Y$.*

Proof. Let C be a closed subset of Y in $(Y, \mathfrak{T}_Y)$. We prove the forward direction first. We show there is a set $D \subset X$, closed in $(X, \mathfrak{T})$, such that $C = D \cap Y$. Since C is closed, then we have $Y - C$ is open. By definition of a subspace topology, there exists some open set $U \in \mathfrak{T}$ such that $U \cap Y = Y - C$. Then let $D = X - U$. D is closed since U is open. We now show $D \cap Y = C$.

$$\begin{aligned} D \cap Y &= (X - U) \cap Y \\ &= Y - U \end{aligned}$$

since $U \cap Y = Y - C$ we have

$$\begin{aligned} &= Y - (Y - C) \\ &= C \end{aligned}$$

Now we prove the reverse direction. Suppose we have a subset D of X closed in $(X, \mathcal{T})$ such that $D \cap Y = C$. We show C is closed in $(Y, \mathcal{T})_Y$. Since D is closed, then $X - D$ is open. Then $Y \cap (X - D)$ is open in $(Y, \mathcal{T}_Y)$. Since $Y \subset X$ it follows $Y \cap (X - D) = Y - D$ which is open. But $Y - D = Y - (Y \cap D)$ so then $Y \cap D$ must be closed. It follows C is closed. $\square$

Corollary (3.29). *Let $(Y, \mathcal{T}_Y)$ be a subspace of $(X, \mathcal{T})$. A subset $C \subset Y$ is closed in $(Y, \mathcal{T}_Y)$ if and only if $Cl_X(C) \cap Y = C$.*

Proof. Let $(Y, \mathcal{T}_Y)$ be a subspace of $(X, \mathcal{T})$. We prove the forward direction first. Let $C \subset Y$ be a closed subset in $(Y, \mathcal{T}_Y)$. By Theorem 3.28, we know there exists a closed subset $D \subset X$ in $\mathcal{T}$ such that $D \cap Y = C$. Moreover, we know by Theorem 2.20 that $\mathrm{Cl}_X(C)$ is the intersection of all closed sets in $\mathcal{T}$ that contain C. That is

$$\mathrm{Cl}_X(C) = \bigcap_{B \supset C, B \in \mathcal{C}} B$$

where $\mathcal{C}$ is the collection of all closed sets in $\mathcal{T}$. Since $D \supset C$ and $D \in \mathcal{C}$, it follows $\mathrm{Cl}_X(C) \subset D$. It follows $\mathrm{Cl}_X(C) \cap Y \subset D \cap Y = C$. Hence $\mathrm{Cl}_X(C) \cap Y \subset C$. Moreover, $C \subset Y$ be definition of C and $C \subset \mathrm{Cl}_X(C)$ be definition of closure. Thus, $C \subset \mathrm{Cl}_X(C) \cap Y$. It follows $\mathrm{Cl}_X(C) \cap Y = C$.

Now we prove the reverse direction. So we have $\mathrm{Cl}_X(C) \cap Y = C$. Moreover, we know $\mathrm{Cl}_X(C)$ is closed since we proved all closures of sets are closed in Exercise 2.13. It follows from Theorem 3.28 that C is closed. $\square$

Theorem (3.30). *Let $(X, \mathcal{T})$ be a topological space, and let $(Y, \mathcal{T}_Y)$ be a subspace. If $\mathcal{B}$ is a basis for $\mathcal{T}$, then $\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$ is a basis for $\mathcal{T}_Y$.*

Proof. We use the definition of a basis to show that $\mathcal{B}_Y$ is a basis for $\mathcal{T}_Y$. First, we show that $\mathcal{B}_Y \subset \mathcal{T}_Y$. If $B \in \mathcal{B}_Y$, then there exists some set $B' \in \mathcal{B}$ such that $B = B' \cap Y$. And we know all basic open sets are open so then by definition of $\mathcal{T}_Y$ we know $B \in \mathcal{T}_Y$. It follows $\mathcal{B}_Y \subset \mathcal{T}_Y$.

Now we show that all open sets $U \in \mathcal{T}_Y$ are a union of elements in $\mathcal{B}_Y$. Let U be a open set in $\mathcal{T}_Y$. Then there exists a set $U' \in \mathcal{T}$ such that $U = U' \cap Y$. Since $\mathcal{B}$ is a basis for $\mathcal{T}$ it follows U' is a union of elements in $\mathcal{B}$. $U' = \cup_{A \in \mathcal{B}'} A$ where $\mathcal{B}' \subset \mathcal{B}$. Then $U = Y \cap U' = Y \cap \cup_{A \in \mathcal{B}'} A = \cup_{A \in \mathcal{B}'} A \cap Y$. Moreover, $A \cap Y \in \mathcal{B}_Y$ for all $A \in \mathcal{B}'$. It follows U is a union of elements in $\mathcal{B}_Y$. Hence, by definition of a basis, $\mathcal{B}_Y$ is a basis for $\mathcal{T}_Y$. $\square$

Exercise (3.31)**.** Consider the following subspaces of the lexicographically ordered square:

(1) $D = \left\{ \left(x, \frac{1}{2} \right) \mid 0 < x < 1 \right\}$
(2) $E = \left\{ \left(\frac{1}{2}, y \right) \mid 0 < y < 1 \right\}$
(3) $F = \{(x, 1) \mid 0 < x < 1\}$

As sets they are all lines. Describe the relative topologies, especially noting any connections to topologies you have seen already.

For subspace D, we have all singleton sets are open since for any $(x, 1/2)$ we can take the open set $U = \{(x, y) \mid 0 < y < 1\}$ and then $D \cap U = \{x, 1/2\}$. We can take arbitrary unions of singleton sets and still have an open set. It follows all subsets of D are open. It follows we have a discrete topology for $\mathcal{T}_D$.

Now consider set E. Consider the open set $U \in \mathcal{T}$ where $U = \{(1/2, y) \mid a < y < b\}$ where $0 \leq a < b \leq 1$. Then $U \cap E = U$. Then all sets of the form $\{(1/2, y) \mid a < y < b\}$ are open in E. This topology on E is similar to $\mathbb{R}_{\text{std}}$ since we have these open sets in $\mathcal{T}_E$ that are similar to open intervals in $\mathbb{R}_{\text{std}}$.

Now consider the set F. We have "half open intervals" in this topology. Consider the set $U \{(x, y) \mid (a_1, b_1) < (x, y) < (a_2, b_2)\} \in \mathcal{T}$ where $a_1 < a_2$ and $b_1 < 1$. Then

$$U \cap F = \{(x, 1) \mid a_1 \leq x < a_2\}$$

where $0 < a_1 < a_2 \leq 1$. Moreover, consider the set $V = \{(x, y) \mid (a_3, 1) < (x, y) < (a_4, b_4)\}$ where $0 \leq a_3 < a_4 \leq 1$. Then we have

$$V \cap F = \{(x, 1) \mid a_3 < x < a_4\}$$

It follows we have two types of basic open sets. It is somewhat similar to the lower limit topology on $\mathbb{R}$, denoted as $\mathbb{R}_{\text{LL}}$.

Theorem (4.1)**.** *A space (X, T) is T_1 if and only if every point in X is a closed set.*

Proof. We prove the reverse direction first. Let a and b be two distinct points. Since every point in X is closed, $X - a$ and $X - b$ are both open. Then $a \in X - b$ but $b \notin X - b$ and $b \in X - a$ but $a \notin X - a$. Then (X, T) is a T_1 space.

Now we prove the forward direction. Let (X, T) be a T_1 space. Suppose we have some point $b \in X$. Since we have a T_1 space, for all points $p \neq b$ there exists an open set U_p such that $p \in U_p$ and $b \notin U_p$. Then for each p choose some U_p. Then let U be defined as

$$U = \bigcup_{p \in X - \{b\}} U_p$$

We show $U = X - b$. Let $q \in U$. Then $q \in U_\alpha$ for some $\alpha \in X - \{b\}$. Since $b \notin U_\alpha$ it follows $q \in U_\alpha \subset X - b$. Hence, $q \in X - b$ for all $q \in U$ so then $U \subset X - b$. Now suppose $q \in X - \{b\}$. Then there is some associated open set U_q. Since $U_q \subset \cup_{p \in X - \{b\}} U_p$ it follows $q \in U_q \subset U$. Hence $X - b \subset U$.

Since $X - b = U$ and U is an arbitrary union of open sets, it follows $X - b$ is open and b is closed. Our choice of $b \in X$ was arbitrary. It follows every point in X is a closed set. □

Exercise (4.2)**.** Let X be a space with the finite complement topology. Show that X is T_1.

Closed sets in the finite complement topology are finite sets. Since singleton sets are closed, it follows every point in X is a closed set. Then by Theorem 4.1, it follows the space (X, T) is T_1.

Exercise (4.3)**.** Show that $\mathbb{R}_{\text{std}}$ is Hausdorff.

We show $\mathbb{R}_{\text{std}}$ is Hausdorff. More generally, we show all metric spaces are Hausdorff. Let (X, d) be a metric space and let $p, q \in X$ such that $p \neq q$. Then take $\varepsilon = d(p, q)/2$. Then $U = B(p, \varepsilon)$ and $V = B(q, \varepsilon)$. Then $p \in U$ and $q \in V$ and U and V are open since we proved that open balls are open. Now we show U and V are disjoint. For the sake of contradiction, suppose $U \cap V \neq \varnothing$. Then let $w \in U \cap V$. Then $d(p, w) < \varepsilon$ and $d(w, q) < \varepsilon$. Then by the triangle inequality, we have

$$d(p, q) \leq d(p, w) + d(w, q) < 2\varepsilon = d(p, q)$$

which is a contradiction. It follows U and V are disjoint. Hence, all metric spaces are Hausdorff. Since $\mathbb{R}_{\text{std}}$ is a metric space, it follows $\mathbb{R}_{\text{std}}$ is Hausdorff.

Theorem. *Metric spaces are regular*

Proof. We show all metric spaces are regular. Suppose we have some metric space (X, d). Let p be a point in X and A be a closed set in X that does not contain p. Since A is closed $X - A$ is open. Since $p \in X - A$ and $X - A$ is open, there exists some $\varepsilon > 0$ such that $B(p, \varepsilon) \subset X - A$ or equivalently $B(p, \varepsilon) \cap A = \varnothing$. It follows $d(p, a) \geq \varepsilon$ for all $a \in A$. Let $U = B(p, \varepsilon/2)$ and $V = \cup_{\alpha \in A} B(\alpha, \varepsilon/2)$. It follows $p \in U$ and $A \subset V$. Moreover, U and V are open. Now we show U and V are disjoint. For the sake of contradiction, suppose there exists some point q such that $q \in U \cap V$. Then $d(p, q) < \varepsilon/2$ and $d(q, a) < \varepsilon/2$ for some $a \in A$. Moreover, we know that $d(p, \alpha) \geq \varepsilon$ for all $\alpha \in A$. Then by the triangle inequality, we have

$$\varepsilon \leq d(p, a) \leq d(p, q) + d(q, a) < \varepsilon$$

This is a contradiction. It follows U and V are disjoint. Hence, all metric spaces are regular. $\square$

Exercise (4.4)**.** Show that $\mathbb{H}_{\text{bub}}$ is regular.

Let $p = (p_x, p_y) \in X$ and A be a closed subset in $\mathbb{R}^2$ such that $p \notin A$. Then we consider two cases: either $p_y > 0$ or $p_y = 0$. We first consider the case when $p_y > 0$. We construct sets disjoint open sets U and V such that $p \in U$ and $A \subset V$. We first construct U. Define d as

$$d = \inf\{d(p,a) \mid a \in A\}$$

Note $d > 0$ since if $d = 0$ then for any $\varepsilon > 0$ we have $(B(p,\varepsilon) - \{p\}) \cap A \neq \varnothing$ which would imply $p \in A$. Then take $r =$ minimum of $\{p_y, d/2\}$. Then let $U = B(p,r)$.

Now we construct the open set V. For each $\alpha = (\alpha_x, \alpha_y) \in A$ we construct the open set V_α. There are two cases.

If $\alpha_y > 0$ then take $r_\alpha =$ minimum of $\{\alpha_y, d/2\}$ and then let $V_\alpha = B(\alpha, r_\alpha)$. Now we show V_α and U are disjoint. For the sake of contradiction, let $q \in U \cap V_\alpha$. Since $U = B(p,r)$ and $V_\alpha = B(\alpha, r_\alpha)$ it follows $d(p,q) < r$ and $d(q,\alpha) < r_\alpha$. Then by the triangle inequality we have

$$d(p,\alpha) \leq d(p,q) + d(q,\alpha) < r + r_\alpha \leq d/2 + d/2 = d$$

Then $d(p,\alpha) < d$ for some $\alpha \in A$ which is a contradiction since d is the infimum of distances between p and all points in A. It follows U and V_α are disjoint.

Now consider the case $\alpha_y = 0$. Consider the set $S = B(\alpha, d/2)$. By the same reasoning as in the paragraph above, it follows S and U are disjoint. Since S is not open (since it intersects the lower half of the plane) we find an open set V_α such that $\alpha \in V_\alpha \subset S$. Let $V_\alpha = B((\alpha_x, d/4), d/4) \cup \{(\alpha_x, \alpha_y)\}$. By definition of $\mathbb{H}_{\text{bub}}$ it follows V_α is an open set. Moreover, it contains α. Now we show $V_\alpha \subset S$. Since $\{\alpha_x, \alpha_y\} \in S$ it is sufficient to show $B((\alpha_x, d/4), d/4) \in B(\alpha, d/2) = S$.

Let $q \in B((\alpha_x, d/4), d/4)$. Then by the triangle inequality we have

$$d((\alpha_x,\alpha_y), q) \leq d((\alpha_x,\alpha_y), (\alpha_x, d/4)) + d((\alpha_x, d/4), q)$$

since $d((\alpha_x, d/4), q) < d/4$ we get a strictly less than

$$\begin{aligned} d((\alpha_x,\alpha_y), q) &< \sqrt{(\alpha_x - \alpha_x)^2 + (0 - d/4)^2} + d/4 \\ &< d/2 \end{aligned}$$

Hence, $q \in S$ so then $V_\alpha \subset S$. It follows $\alpha \in V_\alpha \subset S$. Since S and U are disjoint, it follows S and V_α are disjoint.

In both cases where $\alpha_y > 0$ and $\alpha_y = 0$ we found an open set V_α such that $\alpha \in V_\alpha$ and $V_\alpha \cap U = \varnothing$. Then let $V = \cup_{\alpha\in A} V_\alpha$. By definition of a topology, V is open. It follows $p \in U$ and $A \subset V$ and $U \cap V = \varnothing$. Hence, we have separated A and point p by open sets when $p_y > 0$. Next, we consider the case where $p_y = 0$.

Suppose we have a point p' and a closed set A'. Suppose $p'_y = 0$. Again, we define $d' = \inf\{d(p', a) \mid a \in A'\}$. Then let $r' = d'/2$. Then let $U' = B(p', r')$. Then let V'_α be constructed as in the prior method for the case $p_y > 0$. Then let $V' = \cup_{\alpha\in A'} V'_\alpha$. By the same reasoning as before, we have $U' \cap V' = \varnothing$ and $p' \in U'$ and $A' \subset V'$. Moreover, V' is open, however U' is not since it intersects the lower half of the $\mathbb{R}^2$ plane. But, we find an open set W' such that $p' \in W' \subset U'$.

Let $W' = B((p'_x, d'/4), d'/4) \cup \{(p'_x, p'_y)\}$. Now we show $W' \subset U'$. Since $p' = \{p'_x, p'_y\} \in U'$ we show specifically show $B((p'_x, d'/4), d'/4) \subset U'$. Then let $q \in B((p'_x, d'/4), d'/4)$. Then we know $d(p', (p'_x, d/4)) = d'/4$ and $d((p'_x, d/4), q) < d'/4$. Then by the triangle inequality, we have

$$d(p, q) \leq d(p, (p_x, d'/4)) + d((p_x, d'/4), q) < d'/4 + d'/4 = d'/2$$

Hence, $d(p, q) < d'/2$ so then $q \in U'$. It follows $W' \subset U'$. Then we have open sets W' and V' such that $p' \in W'$, $A' \subset V'$, and $W' \cap V' = \varnothing$.

It follows in both cases where $p_y = 0$ and $p_y > 0$ that we can separate point p from the closed set A. Hence, $\mathbb{H}_{\text{bub}}$ is regular.

Exercise (4.5)**.** Show that $\mathbb{R}_{\text{LL}}$ is normal.

Proof. Let A and B be disjoint closed sets in $\mathbb{R}_{\text{LL}}$. Then for all $a \in A$, the point a is a not a limit point of B so then there exists some basic open set $U_a \ni a$ such that $U_a \cap B = \varnothing$. Similarly, for all points $b \in B$ there exists some basic open set $V_b \ni b$ such that $V_b \cap A = \varnothing$. Then take $U = \cup_{a \in A} U_a$ and $V = \cup_{b \in B} V_b$. Then $A \subset U$ and $B \subset V$.

However, $U \cap V$ are not necessarily disjoint. Yet, for every pair of sets that intersect, say U_α and V_β, then we can remove their intersection. Specifically, let $U_\alpha = [c_\alpha, d_\alpha)$ and $V_\beta = [c_\beta, d_\beta)$ with some nonempty intersection. Note $\alpha \in U_\alpha$ and $\beta \in V_\beta$. Without a loss of generality, let $c_\alpha < c_\beta < d_\alpha < d_\beta$. We know $\alpha < c_\beta$ otherwise $\alpha \geq c_\beta$ implying $\alpha \in V_\beta$. Similarly, we must have $\beta \geq d_\alpha$. It follows $c_\alpha \leq \alpha < c_\beta$ and $d_\alpha \leq \beta < d_\beta$. Then take $U' = [c_\alpha, c_\beta)$ and $V' = [d_\alpha, d_\beta)$. Then $\alpha \in U'$ and $\beta \in V'$. Moreover, U' and V' are basic open sets. It follows for any U_α and V_β who have a nonempty intersection, we can always new basic open sets such that U'_α and V'_β are disjoint. Hence, there exists open sets U and V such that $A \subset U$ and $B \subset V$. □

Exercise (4.6)**.** There are four parts

(1) Consider $\mathbb{R}^2$ with the standard topology. Let $p \in \mathbb{R}^2$ be a point not in a closed set A. Show that $\inf\{d(a, p) \mid a \in A\} > 0$.
(2) Show that $\mathbb{R}^2$ with the standard topology is regular.
(3) Find two disjoint closed subsets A and B of $\mathbb{R}^2$ with the standard topology such that

$$\inf\{d(a, b) \mid a \in A \text{ and } b \in B\} = 0$$

(4) Show that $\mathbb{R}^2$ with the standard topology is normal.

Proof. We prove part 1. Since A is closed and p is not in A, it follows p is not a limit point of A. Then there must exists some $\varepsilon > 0$ such that $(B(p, \varepsilon) - \{p\}) \cap A = \varnothing$. Moreover, since $p \notin A$ we have $B(p, \varepsilon) \cap A = \varnothing$. It follows for all $a \in A$ that $d(p, a) \geq \varepsilon$. Then $\inf\{d(p, a) \mid a \in A\} \geq \varepsilon > 0$. This concludes the proof for part 1.

We prove part 2. We show that $\mathbb{R}^2_{\text{std}}$ is regular. Suppose we have some point p and closed set A such that $p \notin A$. We construct open sets U and V such that $p \in U$, $A \subset V$, and $U \cap V = \varnothing$. By part one we know $\inf\{d(a, p) \mid a \in A\} > 0$. Then take

$$\varepsilon = (\inf\{d(a, p) \mid a \in A\})/2$$

Then let $U = B(p, \varepsilon)$ and let $V = \cup_{a \in A} B(a, \varepsilon)$. It follows $p \in U$, $A \subset V$, and U and V are open. Now we show U and V are disjoint. For the sake of contradiction, suppose there exists some point q such that q is in the intersection of U and V. Then $d(p, q) < \varepsilon$ and

$d(q,a) < \varepsilon$ for some $a \in A$. Moreover, we have $2\varepsilon \leq d(p,a)$ for all $a \in A$. Then by the triangle inequality, we have

$$2\varepsilon \leq d(p,a) \leq d(p,q) + d(q,a) < 2\varepsilon$$

which gives $2\varepsilon < 2\varepsilon$, a contradiction. It follows $U \cap V = \varnothing$. Hence, $\mathbb{R}^2_{\text{std}}$ is regular.

We prove part 3. Let the sets A and B be defined as

$$A = \{(x,0) \in \mathbb{R}^2 \mid 1 \leq x\}$$
$$B = \{(x,1/x) \in \mathbb{R}^2 \mid 1 \leq x\}$$

Then these sets are closed and $\inf\{d(a,b) \mid a \in A \text{ and } b \in B\} = 0$.

We prove part 4. We show that $\mathbb{R}^2_{\text{std}}$ is normal. Let A and C be closed disjoint subsets of $\mathbb{R}^2_{\text{std}}$. For some point p and set W define the function $f\colon \mathbb{R}^2 \times 2^{\mathbb{R}} \to \mathbb{R}$ as

$$f(p,W) = \inf\{d(p,w) \mid w \in W\}$$

Then for each $a \in A$ let $U_a = B(a, f(a,C)/2)$ and for each $c \in C$ let $V_c = B(c, f(c,A)/2)$. Then let $U = \cup_{a\in A} U_a$ and $V = \cup_{c \in C} V_c$. It follows U and V are open sets such that $A \subset U$ and $C \subset V$. Now we show U and V are disjoint. For the sake of contradiction, suppose there exists some point $p \in U \cap V$. Then $p \in U_\alpha$ for some $\alpha \in A$ and $p \in V_\gamma$ for some $\gamma \in C$. Then $d(\alpha,p) < f(\alpha,C)/2$ and $d(\gamma,p) < f(\gamma,A)/2$. Moreover, $f(\alpha,C)/2 \leq d(\alpha,\gamma)/2$ and $f(\gamma,A)/2 \leq d(\alpha,\gamma)/2$. It follows $d(\alpha,p) < d(\alpha,\gamma)/2$ and $d(\gamma,p) < d(\alpha,\gamma)/2$. Then by the triangle inequality

$$d(\alpha,\gamma) \leq d(\alpha,p) + d(p,\gamma) < d(\alpha,\gamma)$$

which is a contradiction. It follows U and V are disjoint and so $\mathbb{R}^2_{\text{std}}$ is normal. □

Theorem (4.7)**.** *There are three parts*

(1) *A T_2-space is a T_1-space*
(2) *A T_3-space is a T_2-space*
(3) *A T_4-space is a T_3-space*

Proof. Suppose X is a T_2 space. Let a and b be two distinct points in X. Since X is Hausdorff there exists disjoint open sets U and V such that $a \in U$ and $b \in V$. Since U and V are disjoint, it follows $a \notin V$ and $b \notin U$. Hence, X is a T_1 space.

Now suppose X is a T_3 space. Note a T_3 space is both regular and T_1. Let a and b be distinct points in X. Since X is T_1 there exists open an set U such that $a \in U$ and $b \notin U$. Then $X - U$ is a closed set that contains b. Since $a \notin X - U$, $X - U$ is closed, and X is regular, there exists disjoint open sets G and W such that $a \in G$ and $X - U \subset W$. Since $b \in X - U$ it follows $b \in W$. Altogether we have open sets G and W such that $a \in G$, $b \in W$, and $G \cap W = \varnothing$. Hence, X is a T_2 space.

Now suppose X is a T_4 space. We show that it must be a T_3 space. Note a T_4 space is normal and T_1. We show X is regular. Let p be a point in X and A be a closed set in X such that $p \notin A$. Since X is T_1 it follows from Theorem 4.1 that $\{p\}$ is a closed set. Since $\{p\}$ and A are closed sets and X is normal, it follows there exists disjoint open sets U and V such that $p \in U$ and $A \subset V$. Hence, X is T_3. □

Theorem (4.8)**.** *A topological space X is regular if and only if for each point p in X and open set U containing p there exists an open set V such that $p \in V$ and $\overline{V} \subset U$.*

Proof. We prove the forward direction first. Let p be a point in X and U be an open set that contains p. Then $X - U$ is a closed set that does not contain p. Since X is regular, it follows there exists disjoint open sets V and W such that $p \in V$ and $X - U \subset W$. Then $X - W \subset U$ and $X - W$ is closed. And since V and W are disjoint, it follows $V \subset X - W$. Note that $\overline{V}$ is the smallest closed set that contains V. Then $\overline{V} \subset X - W$. Hence, we have $p \in V \subset \overline{V} \subset X - W \subset U$. This concludes the forward direction.

Now we prove the reverse direction. Let p be a point in X and A be a closed set that does not contain p. Then $p \in X - A$ which is open. Then, there exists an open set V such that $p \in V$ and $\overline{V} \subset X - A$. It follows $A \subset X - \overline{V}$ which is open. Now we show V and $X - \overline{V}$ are disjoint. Note that $X - \overline{V} \subset X - V$. Since $V \cap (X - V) = \varnothing$ it follows $V \cap (X - \overline{V}) = \varnothing$. Hence, V and $X - \overline{V}$ are disjoint open sets such that $p \in V$ and $A \subset X - \overline{V}$. Thus, X is regular. $\square$

Theorem (4.9)**.** *A topological space X is normal if and only if for each closed set A in X and open set U containing A there exists an open set V such that $A \subset V$ and $\overline{V} \subset U$.*

Proof. We prove the forward direction first. Let X be normal and A be a closed set with U being an open set that contains A. Then $X - U$ is a closed set. Since A and $X - U$ are disjoint closed sets and X is normal, it follows there exist disjoint open sets V and W such that $A \subset V$ and $X - U \subset W$. Since $X - U \subset W$ then $X - W \subset U$ and note $X - W$ is closed. Moreover, $V \cap W = \varnothing$ so then $V \subset X - W$. Then $X - W$ is a closed set that contains V, but $\overline{V}$ is the smallest closed set that contains V so the $\overline{V} \subset X - W$. Altogether, we have

$$p \in V \subset \overline{V} \subset X - W \subset U$$

Now we prove the reverse direction. Suppose we have two disjoint closed sets A and B. Then $X - B$ is an open set that contains A. We know that there exists some open set G such that $A \subset G$ and $\overline{G} \subset X - B$. Since $\overline{G} \subset X - B$ it follows $B \subset X - \overline{G}$ and $X - \overline{G}$ is open. Now we show G and $X - \overline{G}$ are disjoint. Note $G \subset \overline{G}$ so then $X - \overline{G} \subset X - G$. Since $G \cap (X - G) = \varnothing$ it follows $G \cap (X - \overline{G}) = \varnothing$. Hence, G and $X - \overline{G}$ are disjoint open sets such that $A \subset G$ and $B \subset X - \overline{G}$. It follows X is normal. $\square$

Theorem (4.10)**.** *A topological space X is normal if and only if for each pair of disjoint closed sets A and B, there are disjoint open sets U and V such that $A \subset U$, $B \subset V$, and $\overline{U} \cap \overline{V} = \varnothing$.*

Proof. We prove the forward direction first. Let A and B be disjoint closed sets. Since X is normal, there exist disjoint open sets U' and V' such that $A \subset U'$ and $B \subset V'$. By Theorem 4.9, there exists open sets U and V such that $A \subset U$ and $\overline{U} \subset U'$ and $B \subset V$ and $\overline{V} \subset V'$. Moreover, since $U' \cap V' = \varnothing$ it follows $\overline{U} \cap \overline{V} = \varnothing$.

Now we prove the reverse direction. Let A and B be disjoint closed sets and suppose there exist open sets U and V such that $A \subset U$ and $B \subset V$ and $\overline{U} \cap \overline{V} = \varnothing$. It follows $U \cap V = \varnothing$. Then X is normal. $\square$

Exercise (4.12)**.** There are three parts

(1) Describe an example of a topological space that is T_1 but not T_2.
(2) Describe an example of a topological space that is T_2 but not T_3.
(3) Describe an example of a topological space that is T_3 but not T_4.

The double headed snake topology $\mathbb{R}_{+00}$ is a T_1 but not T_2. Let $\mathbb{R}_+$ denote the set of all positive real numbers. Recall the basis for the double headed snake topology consists of sets of the form (a, b) or $(0, b) \cup \{0'\}$ or $(0, b) \cup \{0''\}$ for all $a, b \in \mathbb{R}_+$. Now we show $\mathbb{R}_{+00}$ is T_1. There are three cases. First consider when we have to distinct points $p, q \in \mathbb{R}_+$. Without a loss of generality let $p < q$. Let $\varepsilon = (q-p)/2$. Then let $U = (p/2, p+\varepsilon)$ and $V = (q-\varepsilon, q+1)$. Then $p \in U$, $q \in V$, $q \notin U$, and $p \notin V$. Now consider the case where $p = \mathbb{R}_+$ and $q = 0'$. Then let $U = (p/2, 2p)$ and $V = (0, p/2) \cup \{0'\}$. Then $p \in U$, $0' \in V$, $p \notin V$, and $0' \notin U$. A similar case holds when $q = 0''$. Third, consider the case where $p = 0'$ and $q = 0''$. Then take $U = (0, 1) \cup \{0'\}$ and $V = (0, 1) \cup \{0''\}$. Then $p \in U$, $q \in V$, $p \notin V$, and $q \notin U$. It follows $\mathbb{R}_{+00}$ is T_1.

Now we show that $\mathbb{R}_{+00}$ is not Hausdorff. We show that there does not exist disjoint open sets U and V such that $0' \in U$ and $0'' \in V$. Consider the basic open set $A = (0, a) \cup \{0'\}$ and the basic open set $B = (0, b) \cup \{0''\}$. For any $a, b \in \mathbb{R}_+$ we have $A \cap B \neq \varnothing$. Specifically, $A \cap B = (0, m)$ where $m = \min\{a, b\}$. It follows there does not exists two open sets that contain $0'$ and $0''$ and are disjoint. Hence, R_{+00} is not T_2.

We show $\mathbb{R}_{\text{har}}$ is T_2 but not T_3. Recall that a basis for this topology consists of sets of the form (a, b) and $(a, b) - H$ where $H = \{1/n\}_{n \in \mathbb{N}}$ and $a, b \in \mathbb{R}$. Now take $p, q \in \mathbb{R}$ such that $p \neq q$. Then take $\varepsilon = |p - q|/2$. Then let $U = (p - \varepsilon, p + \varepsilon)$ and $V = (q - \varepsilon, q + \varepsilon)$. It follows $p \in U$, $q \in V$, and $U \cap V = \varnothing$. Hence, $\mathbb{R}_{\text{har}}$ is Hausdorff which also implies this topology is T_1.

Now we show $\mathbb{R}_{\text{har}}$ is not regular. Take the point 0 and the closed set $H = \{1/n\}_{n \in \mathbb{N}}$. We know H is closed since we proved this in exercise 3.9. Now let U and V be open sets such that $0 \in U$ and $H \subset V$. We must have $(0, a) \subset V$ for some $a \in \mathbb{R}$ since for any $\varepsilon > 0$ there exists some point $h \in H$ such that $0 < h < \varepsilon$. Moreover, for an open set to contain 0 it must be of the form $U = (a, b)$ or $U = (a, b) - H$ where $a < 0 < b$. In either case we have $U \cap V \neq \varnothing$. Hence, 0 and H cannot be separated by disjoint open sets. It follows $\mathbb{R}_{\text{har}}$ is not regular and is not T_3.

$\mathbb{H}_{\text{bub}}$ is regular but not normal which is the result of Exercise 4.4 and 4.14. Now we show $\mathbb{H}_{\text{bub}}$ is T_1 by proving $\mathbb{H}_{\text{bub}}$ is Hausdorff which implies T_1. We show for any points p and q in the upper half plane of $\mathbb{R}^2$ that there exist open sets U and V such that $p \in U$ and $q \in V$ and $U \cap V = \varnothing$. There are three cases.

First consider the p and q are points on the x-axis. Then take $r_p = r_q = d(p, q)/2$. Then let $U = B((p_1, r_p), r_p) \cup \{p\}$ and let $V = B((q_1, r_q), r_q) \cup \{q\}$. It follows $p \in U$ and $q \in V$ and $U \cap V = \varnothing$.

Second consider when p is on the x-axis and q is not on the x-axis. Then take $r_p = d(p, q)/4$ and $r_q = \min\{d(p, q)/2, q_2\}$. Then let $U = B((p_1, r_p), r_p) \cup \{p\}$ and let $V = B(q, r_q)$. It follows $p \in U$ and $q \in V$ and $U \cap V = \varnothing$.

Third, consider when both p and q are not on the x-axis. Then take $r_p = \min\{d(p, q)/2, p_2\}$ and $r_q = \min\{d(p, q)/2, q_2\}$. Then let $U = B(p, r_p)$ and $V = B(q, r_q)$. It follows $p \in U$, $q \in V$, and $U \cap V = \varnothing$. Hence $\mathbb{H}_{\text{bub}}$ is Hausdorff which implies it is also T_1. As mentioned before, we proved that $\mathbb{H}_{\text{bub}}$ is regular but not normal from exercises 4.4 and 4.14. Hence $\mathbb{H}_{\text{bub}}$ is T_3 but not T_4.

Exercise (4.13)**.** Here is a chart that tabulates various topological spaces with their separation properties.

Topology	T_1	Hausdorff	regular	normal
$\mathbb{R}_{\text{std}}$				
$\mathbb{R}^n_{\text{std}}$				
indiscrete topology				
discrete topology				
cofinite topology				
cocountable topology				
$\mathbb{R}_{\text{LL}}$				
$\mathbb{R}_{+00}$				
$\mathbb{R}_{\text{har}}$				
$\mathbb{H}_{\text{bub}}$				
$\mathbb{R}_{\text{std}}$				
$\mathbb{Z}_{\text{arith}}$				
lexicographically ordered square				
2^X				

Exercise (4.14)**.** Show that $\mathbb{H}_{\text{bub}}$ is not normal.

Proof. □

Lemma. *Let $K_n \in \mathbb{R}$ be a closed set for all $n \in \mathbb{N}$. Let $int \cup_{n\in\mathbb{N}} K_n \neq \varnothing$. Then there exists some n such that $int(K_n) \neq \varnothing$.*

Proof. Note that $\text{int}(\cup K_n) = \bigcup_{V\subset \cup K_n, V\in\mathfrak{T}} V$. Then if the interior of the union of all K_n is nonempty, it follows there is a nonempty open interval I that is contained in $\text{int}(\cup K_n)$. Then we have $I \subset \cup K_n$. It follows $|I| \leq |\cup K_n|$. For the sake of contradiction, suppose that $\text{int}(K_n) = \varnothing$ for all n. Then there are no open intervals contained in $\text{int}(K_n)$.

Now we show that $G = \text{int}(K_n)$ must be countable. We know G can be partitioned into a set of rationals and set of irrationals. The rational set will be countable. Now we show the set of irrationals must be countable. Since the interior of G is empty, the set of irrationals within K_n must not be interior points. Moreover, they are either limit points or isolated points. Suppose there is an uncountable number of irrational limit points in K_n. This would imply that we would have a subset of irrational limit points that is dense in $\mathbb{R}$ and since K_n is closed, we would have an open interval contained in K_n. This is a contradiction. Then we must have a countable number of irrational limit points. Moreover, the number of irrational isolated points would be countable. Then in total K_n would be countable. It follows $\text{int}(K_n)$ would be a countable union of countable sets which is countable. But $|I| \leq |\cup K_n|$ and $|I|$ is uncountable. This is a contradiction. Hence, there exists some n such that $\text{int}(K_n) \neq \varnothing$. □

Theorem (4.15)**.** *Order topologies are T_1, Hausdorff, regular, and normal.*

Proof. Let X be a totally ordered set and $\mathfrak{T}$ be an ordered topology on X. Let a and b be distinct points in X. We show there exists disjoint open sets U and V that contain a and b. There are two cases: either the set $\{x \in X \mid a < x < b\}$ is empty or is non-empty.

If $\{x \in X \mid a < x < b\}$ is empty than take $U = \{x \in X \mid x < b\}$ and $V = \{x \in X \mid a < x\}$. Then $a \in U$, $b \in V$, and $U \cap V = \varnothing$ since there does not exist a point p in X such that $a < p < b$.

If $\{x \in X \mid a < x < b\}$ is non-empty, take $c \in \{x \in X \mid a < x < b\}$. Then $a < c < b$. Then let $U = \{x \in X \mid x < c\}$ and $V = \{x \in X \mid c < x\}$. It follows $a \in U$ and $b \in V$. Moreover, $U \cap V = \varnothing$ since there does not a point p such that $p < c$ and $p > c$ since X is totally-ordered. It follows the order topology on X is Hausdorff and also T_1.

Now we show the order topology on X is regular. Recall that the order topology on X has a basis $\mathcal{B}$ that consist of sets of the form

$$\{x \in X \mid a < x\} \text{ and } \{x \in X \mid a < x < b\} \text{ and } \{x \in X \mid x < b\}$$

where a and b are in X. Now we use Theorem 4.8 to show order topologies are regular. Let p be a point in X and U be an open set in $\mathcal{T}$ that contains p. By Theorem 3.1 it follows there exists a basic open set V such that $p \in V \subset U$. Without a loss of generality, let V be of the form $\{x \in X \mid a < x < b\}$ for some $a, b \in X$.

Consider the closure of V. At most, the closure of V contains a and b. Suppose that the closure of V contained some other point q where $q < a$. Then consider the open neighborhood A of q, that is, $A = \{x \in X \mid x < a\}$. Then $A \cap V = \varnothing$. Hence, q is not a limit point of V. A similar argument holds when $q > b$. Hence, the only possible limit points outside of V are a and b.

If the closure of V contains neither a nor b, then $\overline{V} = V$. Then we have $p \in V = \overline{V} \subset U$. Then we are done. Now consider the case where $\overline{V}$ may include a or b. Suppose $\overline{V}$ includes a. Then for every open set $W \ni a$ we have $(W - \{a\}) \cap V \neq \varnothing$. Then consider the open set $\{x \in X \mid x < p\}$. It follows $(\{x \in X \mid x < p\} - \{a\}) \cap V \neq \varnothing$. It follows there exists $c \in X$ such that $a < c < p$. Then take $y_1 = c$ otherwise if $a \notin \overline{V}$ take $y_1 = a$. Similarly, if $\overline{V}$ includes b there exists $d \in X$ such that $p < d < b$. Then take $y_2 = d$ otherwise if $b \notin \overline{V}$ take $y_2 = b$. Let $G = \{x \in X \mid y_1 < x < y_2\}$. It follows $\overline{G} \subset V$ and since $V \subset U$ it follows $\overline{G} \subset U$. Moreover, $p \in G$. Altogether, we have $p \in G$ and $\overline{G} \subset U$.

Note we considered the case where V is of the form $\{x \in X \mid a < x < b\}$. However, similar arguments hold when V is of the form $\{x \in X \mid a < x\}$ or $\{x \in X \mid x < b\}$. It follows order topologies are regular. □

Theorem (4.16). *Let X and Y be Hausdorff. Then $X \times Y$ is Hausdorff.*

Proof. Let $\mathcal{T}_X$ denote topology on X, $\mathcal{T}_Y$ denote the topology on Y, and $\mathcal{T}$ denote the topology on $X \times Y$. Let (a_1, b_1) and (a_2, b_2) be in $X \times Y$ such that $(a_1, b_1) \neq (a_2, b_2)$. Without a loss of generality, consider the case where $a_1 \neq a_2$ and $b_1 \neq b_2$. Since X and Y are Hausdorff, there exist open sets $U_1, U_2 \in \mathcal{T}_X$ and $V_1, V_2 \in \mathcal{T}_Y$ such that $U_1 \cap U_2 = \varnothing$, $V_1 \cap V_2 = \varnothing$, $a_1 \in U_1$, $a_2 \in U_2$, $b_1 \in V_1$, and $b_2 \in V_2$. It follows $(a_1, b_1) \in U_1 \times V_1$ and $(a_2, b_2) \in U_2 \times V_2$. Moreover, since $U_1 \cap U_2 = \varnothing$ and $V_1 \cap V_2 = \varnothing$ it follows $U_1 \times V_1$ and $U_2 \times V_2$ are disjoint. Moreover, $U_1 \times V_1$ and $U_2 \times V_2$ are open sets in the product topology $\mathcal{T}$ since U_1, U_2 and V_1, V_2 are open sets in the topologies $\mathcal{T}_X$ and $\mathcal{T}_Y$ respectively. It follows $X \times Y$ is Hausdorff. □

Theorem (4.17). *Let X and Y be regular. Then $X \times Y$ is regular.*

Proof. Let $(a, b) \in X \times Y$. Since the product topology has a basis that consists of sets of the form $C \times D$ where $C \in \mathcal{T}_X$ and $D \in \mathcal{T}_Y$, it follows there exists some basic open set $U \times V$ such that $(a, b) \in U \times V$. Since $U \times V$ is a basic open set in the product topology, we have $a \in U \in \mathcal{T}_X$ and $b \in V \in \mathcal{T}_Y$. Since X and Y are regular, there exists open sets $U' \in \mathcal{T}_X$ and $V' \in \mathcal{T}_Y$ such that $a \in U'$, $\overline{U'} \subset U$ and $b \in V'$, $\overline{V'} \subset V$. It follows $U' \times V'$ is an open set in the product topology and $(a, b) \in U' \times V'$.

Now we show that $\overline{U' \times V'} \subset U \times V$. If $(c,d) \in U' \times V'$ then $(c,d) \in U \times V$. Now suppose $(c,d) \notin U' \times V'$ but is a limit point of $U' \times V'$. Without a loss of generality, consider the case where $c \notin U'$ and $d \notin V'$. Since (c,d) is a limit point of $U' \times V'$, for any open set $G \times W \ni (c,d)$ in the product topology, we have $((G \times W) - \{(c,d)\}) \cap (U' \times V') \neq \varnothing$. It follows for any open set $A \in \mathcal{T}_X$ we have $(A - \{c\}) \cap U' \neq \varnothing$ and for any open set $B \in \mathcal{T}_Y$ we have $(B - \{d\}) \cap V' \neq \varnothing$. Then c is a limit point of U' and d is a limit point of V', so then $c \in \overline{U'}$ and $d \in \overline{V'}$. Moreover, we know $\overline{U'} \subset U$ and $\overline{V'} \subset V$. It follows $(c,d) \in U \times V$. A similar argument holds for the other two cases where $c \in U'$ but $d \notin V'$ and $c \notin U'$ but $d \in V'$. It follows $\overline{U' \times V'} \subset U \times V$. Hence, $X \times Y$ is regular. $\square$

Theorem (6.5)**.** *A space X is compact if and only if every collection of closed sets with the finite intersection property has a non-empty intersection.*

Proof. We prove the forward direction first. Let $\{C_\alpha\}_{\alpha\in\lambda}$ be a collection of closed sets with the finite intersection property. For the sake of contradiction suppose $\cap_{\alpha\in\lambda}C_\alpha = \varnothing$. It follows $X \subset \cup_{\alpha\in\lambda}C_\alpha^c$. Since X is compact it follows

$$X \subset C_{\alpha_1}^c \cup \cdots \cup C_{\alpha_n}^c$$

But this is equivalent to

$$C_{\alpha_1} \cap \cdots \cap C_{\alpha_n} \subset \varnothing$$

But this contradicts our collection $\{C_\alpha\}_{\alpha\in\lambda}$ having the finite intersection property. Hence, $\cap_{\alpha\in\lambda}C_\alpha \neq \varnothing$.

Now we prove the reverse direction. We prove the contrapositive. Suppose X is not compact. Then there is some open cover $\{V_\alpha\}_{\alpha\in\lambda}$ of X which has no finite subcover. Since there exists no finite subcover, for every intersection of a finite subcollection we have

$$X \not\subset V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}$$

It follows $(V_{\alpha_1} \cup \cdots \cup V_{\alpha_n})^c = V_{\alpha_1}^c \cap \cdots \cap V_{\alpha_n}^c \neq \varnothing$. Then consider the collection $\{V_\alpha^c\}_{\alpha\in\lambda}$. We just showed that any intersection over any finite subcollection is nonempty. Hence, $\{V_\alpha^c\}_{\alpha\in\lambda}$ has the finite intersection property. However, we know that $X \subset \cup_{\alpha\in\lambda}V_\alpha$ so then $\cap_{\alpha\in\lambda}V_\alpha^C = \varnothing$. Hence, there exists a collection of closed sets $\{V_\alpha^c\}_{\alpha\in\lambda}$ with the finite intersection property and has an empty intersection. This proves the contrapositive. $\square$

Theorem (6.8)**.** *Let A be a closed subspace of a compact space. Then A is compact.*

Proof. Let X be a compact space and A a closed subspace of X. Let $\{U_\alpha\}_{\alpha\in\lambda}$ be a collection of open sets in $\mathcal{T}_A$ that cover A. Then by the definition of the subspace topology, there exists a corresponding collection of open sets $\{V_\alpha\}_{\alpha\in\lambda}$ in $\mathcal{T}_X$ such that $U_\alpha = V_\alpha \cap A$ for all α. Since $U_\alpha \subset V_\alpha$ for all α it follows $\{V_\alpha\}_{\alpha\in\lambda}$ is an open cover of A. Then $X \subset A^c \cup (\cup_{\alpha\in\lambda}V_\alpha)$. Since X is a compact space there is a finite subcover. Then $X \subset A^c \cup V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}$. Then $A \subset V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}$. Moreover, we have

$$\begin{aligned}
&A \subset A \cap (V_{\alpha_1} \cup \cdots \cup V_{\alpha_n})\\
&A \subset (V_{\alpha_1} \cap A) \cup \cdots \cup (V_{\alpha_n} \cap A)\\
&A \subset U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}
\end{aligned}$$

Hence we have a finite subcover of A. Therefore, A is compact. $\square$

Theorem (6.9)**.** *Let A be a compact subspace of a Hausdorff space X. Then A is closed.*

Proof. We show A^c is open. Let $p \in A^c$. Since X is Hausdorff, for every point $q \in A$ there exists open disjoint sets U_q and V_q such that $q \in U_q$, $p \in V_q$, and $U_q \cap V_q = \varnothing$. It follows $\{U_q\}_{q\in A}$ is an open cover of A. Since A is compact, it follows $A \subset U_{q_1} \cup \cdots \cup U_{q_n}$. Let $U = U_{q_1} \cup \cdots \cup U_{q_n}$. Now let $V = V_{q_1} \cap \cdots \cap V_{q_n}$. Note $U_{q_i} \cap V_{q_i} = \varnothing$ for all i and $V \subset V_{q_i}$ for all i. Hence, $V \cap U = \varnothing$. Since $A \subset U$ and $V \cap U = \varnothing$ it follows $V \subset A^c$. Then we have $p \in V \subset A^c$. It follows A^c is open. Therefore, A is closed. $\square$

Theorem (6.12)**.** *Every compact, Hausdorff space is normal.*

Proof. Let X be a compact and Hausdorff space. Let A and B be two closed disjoint subsets of X. By the lemma we know X is regular. Hence, for every point $p \in A$ there exists disjoint open sets G_p and H_p such that $p \in G_p$ and $B \subset H_p$. It follows $\{G_p\}_{p\in A}$ is an open cover of A. Hence, $X \subset A^c \cup \cup_{p\in A} G_p$. Since X is compact, there is a finite subcover. Therefore we have

$$X \subset A^c \cup G_{p_1} \cup \cdots \cup G_{p_n}$$

Let $G = G_{p_1} \cup \cdots \cup G_{p_n}$ and let $H = H_{p_1} \cap \cdots \cap H_{p_n}$. Since $H_{p_i} \supset B$ for all i it follows $B \subset H$. Moreover, since $G_{p_i} \cap H_{p_i} = \varnothing$ and $H \subset H_{p_i}$ for all i, it follows $G \cap H = \varnothing$. Altogether, we have $A \subset G$, $B \subset H$, and $G \cap H = \varnothing$. Hence, X is normal. □

Lemma. *Every compact, Hausdorff space is regular.*

Proof. Let X be a compact and Hausdorff space. Let A be a closed subset of X and let p be a point in X such that $p \notin A$. Since X is Hausdorff there exists open disjoint open sets U_q and V_q such that $q \in U_q$, $p \in V_q$, and $U_q \cap V_q = \varnothing$. Then $\{U_q\}_{q\in A}$ is an open cover of A. It follows $X \subset A^c \cup \cup_{q\in A} U_q$. Since X is compact, there exists a finite subcover. Then we have $X \subset A^c \cup U_{q_1} \cup \cdots \cup U_{q_n}$. So then we have

$$A \subset U_{q_1} \cup \cdots \cup U_{q_n}$$

Then let $U = U_{q_1} \cup \cdots \cup U_{q_n}$ and let $V = V_{q_1} \cap \cdots \cap V_{q_n}$. Note $V_{q_i} \cap U_{q_i} = \varnothing$ for all i by construction and $V \subset V_{q_i}$ for all i. Hence, $V \cap U = \varnothing$. Moreover, we have $p \in V$ and $A \subset U$. It follows X is regular. □

Theorem (6.13)**.** *Let $\mathcal{B}$ be a basis for a space X. Then X is compact if and only if every open cover of X by basic open sets in $\mathcal{B}$ has a finite subcover.*

Proof. We prove the forward direction first. Suppose we have an open cover of X by basic open sets. Since X is compact, it follows there is a finite subcover. Now we prove the reverse direction. Suppose we have an open cover of X. But an open cover is a union of open sets and every open set is the union of basic open sets. It follows the open cover is also a union of only basic open sets. Then by the hypothesis, it follows there is a finite subcover of basic open sets. Then for every basic open set, choose an open set in the original cover that contains it. Then there is a finite subcover for the original open cover of open sets. □