1 What is NP?

NP is a class of languages that have polynomial time verifiers

1.1 Verifiers

A verifier for a language A is an algorithm V where
$A = \{w|$ V accepts $\langle w, c\rangle$for some string c $\}$
The time it takes to run a verifier is measured in terms of the length of w. In this system, c is the certificate

1.2 The Big Cheng 5: Proving a problem in NP the right way!

1. Show that Q is a decision problem. This is usually quite obvious from the way Q's question is asked, so merely stating that Q is a decision problem will often be sufficient.

2. Describe what a certificate would be for a Yes instance of Q. Most of the time this is also quite obvious from the way Q's question is asked. If the question asks, "is there a blah?", (i.e., it essentially asks for the existence of something), then usually the certificate is simply the blah (the thing about whose existence is asked by the question).

3. Explain why the certificate is polysize with respect to the input size. Frequently the certificate is no larger than the input. In such cases the certificate size is at most linear with respect to the input size.

4. Describe what a verification algorthm would do. The certificate is commonly something that satisfies some set of criteria. So the algorithm needs to check (or verify) that these criteria are satisfied.

5. Explain why the verification algorithm runs in polytime. Typically we would only need to give running times for the checks that the algorithm performs

This is the official, immutable way to show that a problem is in NP, this is the only way to do it, and there are no other ways. We will henceforth follow them religiously.

2 What is NP closed under? (Assignment 3 Q2 solution)

2.1 Prove that NP is closed under union, intersection, concaterntaion, and Kleene star

2.1.1 Let $A, B \in$ NP, show $A \cup B \in$ NP

If A,B are in NP, then A and B have polynomial time verifiers verifyA and verifyB. Let L=$A \cup B$. Now, we wish to argue that $L \in$ NP

1. Since A and B are decision questions, asking if an input is in L must also be a decision question

2. A certificate would be $\langle c_1, c_2\rangle$ where c_1 is a certificate for verifyA, and c_2 is a certificate for verifyB

3. The certificate would be polysize with respect to input size of L since c_1 and c_2 are polysize with respect to the inputs to A and B.

4. A verification algorithm would pass the input to verifyA and verifyB with the appropriate certificates. If either verifyA or verifyB returns true, then return true, else false

5. Our algorithm is polytime since it only runs two polytime algorithms, verifyA and verifyB

2.1.2 WTS: Let $A, B \in$ NP, show $A \cap B \in$ NP

If A,B are in NP, then A and B have polynomial time verifiers verifyA and verifyB. Let L=$A \cap B$. Now, we wish to argue that $L \in$ NP

1. Since A and B are decision questions, asking if an input is in L must also be a decision question

2. A certificate would be $\langle c_1, c_2\rangle$ where c_1 is a certificate for verifyA, and c_2 is a certificate for verifyB

3. The certificate would be polysize with respect to input size of L since c_1 and c_2 are polysize with respect to the inputs to A and B.

4. A verification algorithm would pass the input to verifyA and verifyB with the appropriate certificates. If verifyA and verifyB returns true, then return true, else false

5. Our algorithm is polytime since it only runs two polytime algorithms, verifyA and verifyB

2.1.3 WTS: Let $A, B \in$ NP, show $AB \in$ NP

If A,B are in NP, then A and B have polynomial time verifiers verifyA and verifyB. Let L=AB. Now, we wish to argue that $L \in$ NP

1. Since A and B are decision questions, asking if an input is in L must also be a decision question
2. A certificate would be $\langle c_1, c_2 \rangle$ where c_1 is a certificate for verifyA, and c_2 is a certificate for verifyB
3. The certificate would be polysize with respect to input size of L since c_1 and c_2 are polysize with respect to the inputs to A and B.
4. VerifyL
5. If n=$|w|$, then, worst case, verifyA will be run n times, and verifyB will be run n times. This is n times some polynomial, which itself, is a polynomial. Therefore, this algorithm runs in polytime.

VerifyL$\langle w, \langle c_1, c_2 \rangle \rangle$:

for $i = 0 \rightarrow |w|$:

if verifyA($w[0..i], c_1$) and verifyB($w[i..], c_2$):

return true;

return false;

2.1.4 WTS: Let $A \in$ NP, show $A^\star \in$ NP

If A is in NP, then A has a polynomial time verifier verifyA. Let L=$A^\star$. Now, we wish to argue that $L \in$ NP

1. This is a decision problem
2. A certificate $\langle c \rangle$ would be the same as a certificate to A
3. The certificate is polysize with respect to input size since its polysize with respect to inputs to A, which are less than or equal to inputs to L
4. A verifier would iterate through the input, running verifyA on $\langle w\,[0..i]\,, c \rangle$. If verifyA returns true, return true, else false
5. This is polytime with respect to $|w| = n$, since the input is of size n, and verifyA, which is polynomial, will get run n times, which is polytime

2.2 Discuss the closure of NP under complement and homomorphisms

Define a new language operation and prove whether NP is closed under it
Is NP closed under "inverses" of the above operations, so if $L_1 \cup L_2 \in$ NP, Does it follow that $L_2 \in$ NP?
What about all the others?
Is co-NP closed under any of the above language operations?

3 NP-COMPLETENESS

NP-Complete problems are problems that are both in NP and NP-hard. An NP-Hard problem is a problem that is at least as hard as all other NP-hard problems. Putting these toegether, if one NP-complete problem has a polytime solver, they all do. But, if one NP-complete problem can be shown to have no polytime solver, none of them do

3.1 The grandfather of NP-Complete problems: SAT

SAT = $\{\langle \phi \rangle \,| \phi$ is a satisfiable boolean formula$\}$
A satisfiable boolean formula is a boolean formula such that some assignment of variables makes the formula output true. Truth assignments are often labeled τ, and assigning the variables in ϕ with the assignments in τ is $\tau\,(\phi)$
This proof looks too big to put here

4 Polytime reductions

Language A is polytime reducible to language B, $A \leq_p B$ if a TOTAL, POLYTIME, and COMPUTABLE function $f : \Sigma^\star \rightarrow \Sigma^\star$ exists with the property that $w \in A \iff f(w) \in B$.
So to reduce A to B, create a function which maps yes instances of A to yes instances of B, and no instances of A to no instances of B, this is the contrapositive.

4.1 How do polytime reductions relate to time complexity?

If $A \leq_p B$, or A polytime reduces to B, and B is in P, then, here's a polytime solver for A

solveA$\langle w \rangle$:

$w_b = f(w)$
return solveB(w_b)

Since f is a polytime computable total function that maps yes instances of A to yes instances of B, and solveB is polytime, if solveB accepts, then w must be in A, and if solveB rejects, w must NOT be in A. Additionally, since B and f are polytime computable, solveA must be polytime computable. This is the core of polytime computability

4.2 Polytime reductions and it's incomprehensible notation

$\leq_p$ is one of the most confusing and poorly implemented operators I have ever seen. Typically, symbols notate a relation between two objects. The "justification" for $\leq_p$ is that if $A \leq_p B$, A is "Less than or equal to B" in terms of hardness. This is complete gibberish. This is an "operator" which communicates the thing one can prove using the existence of a relation between two objects. Imagine if we wrote all notation this way! There would be COMPLETE ANARCHY!!!! Not only that, but this convention encourages un-mathematical, un-creative, instrumentalist applications of mathematical reasoning. This is wholly unfit for a theoretical course. This is unlike all other notation, and the person who invented it should be candidate #2 for NASA's man-on-the-sun space mission right after Elon Musk.

4.2.1 How to remember this difficult notation

Rule 1: read right to left: if $A \leq_p B$, then a polytime reduction FROM A, TO B exists, I.e. inputs in A, go TO, B. The english phraseology gets this exactly correct. A polytime reduction goes from A to B, and the inputs of A, go TO B.

4.3 Polytime reduction definition of NP-complete

Using our new approach, we can define NP-completeness to be
"A language B is NP-complete if it satisfies two conditions:
1. B is in NP
2. if A∈NP, then $A \leq_p B$ for any A in NP
This works because if B has a polytime solver, then A must have a polytime solver. This means that B's solvability in polynomial time makes a claim about every single NP-complete problem's solvability in polytime, which is the point of the NP-complete class

5 The first reduction: SAT$\leq_p$3SAT

PROVE 3SAT is in NP!
3SAT = $\{\phi | \phi$ is a 3-cnf formula which is satisfiable$\}$
A 3-cnf formula, or "conjunctive normal form formula with 3 variables in each clause" is composed of clauses as follows: $(x \vee y \vee \neg x) \wedge (z \vee p \vee \neg x) \wedge \cdots$. Notably, these formulas have no constants. An easy way to remember is that cnf formulas are "sad" $({}^{\odot}\vee^{\odot})$
WTS: SAT$\leq_p$3SAT. Showing SAT Polytime reduces to 3SAT will show that if 3SAT is in P, then SAT is in P. This proves that 3SAT is NP-complete
The details of this reduction are not super important, since finding an algorithm to take any formula composed of literals, ands, or's and nots and transforming it into a 3-cnf formula is very boring and not required knowledge. But an appropriate algorithm would have the following passes:
1. Transform literals into tautologies, $\neg x \vee x = 1$, $\neg x \wedge x = 0$
2. Use de morgan's law to make all nots go directly in front of variables
3. probably many others
Intuitively, such an algorithm makes sense since we are used to transforming logical statements into other equivalent statements, since this was tested in a67 and many other courses.
By the principle of arm waiving, we now have a function $f(\phi) \rightarrow \phi'$st if ϕ is a boolean formula, ϕ' is a 3-cnf formula. and ϕ is equivalent to ϕ'.
Now, for the proof:
If $\phi \in$ SAT, then ϕ' is satisfiable, which implies since it is 3-cnf, it is in 3SAT
(using the contraposition)
If $\phi \notin$ SAT, then ϕ' is not satisfiable, which implies it is not in 3SAT
QED

6 The first REAL reduction: 3SAT$\leq_p$CLIQUE, Showing CLIQUE is NP-complete

CLIQUE = $\{\langle G, k \rangle \mid$ G is a graph which contains a clique of size k$\}$

6.1 Proof that CLIQUE∈NP

Here's a polytime verifier for CLIQUE:
Let the certificate C be a set of vertices that form a clique, and let G be a graph

VerifyCLIQUE$\langle\langle G,k\rangle, C\rangle$:

if $|C| < k$:
  return false
for each vertex in C:
  check that it connects to every other vertex in c
  if it doesn't return false
return true

Worst case, G=C, so at worst, this algorithm runs in n^2 time. This is sufficient justification to show that CLIQUE∈NP

6.2 Proof that CLIQUE is NP-complete

A clique is a set of vertices where each vertex in the clique is connected to every other vertex in the clique with an edge.
The way we will go about reducing these is to come up with some equivalent structure to clauses, and conjunctions in graphs, often called widgets or gadgets. Since some have argued that graphs are a "universal" data structure capable of storing all other data structures, this should be easy!
The steps to make this construction go as follows, Given 3-cnf formula ϕ, output G and k as follows:
1. For each variable in ϕ, add a corresponding vertex in Gnamed after the variable
2. Connect every vertex to every other vertex in G with an edge
3. Remove all edges between "contradictory" vertexes like $\neg x$ and x
4. Remove all edges between vertexes that appear in the same clause (so if $(c_1 \vee c_2 \vee c_3)$ appears in ϕ, no edge should exist from $c_1$to c_2 to c_3)
5. set k to be the number of clauses in ϕ
Now, to argue that $\langle\phi\rangle \in$ 3SAT $\iff$ $\langle G,k\rangle \in$ CLIQUE
If $\langle\phi\rangle \in$ 3SAT, then a satisfiable truth assignment τ exists such that every clause in ϕ returns true. This means at least one variable in each triplet evaluates to true. If one puts these three variables (henceforth vertices) into a set, these vertices will form a k-clique.
Now, for the reverse direction, suppose that G contains a k-clique:
We know the elements of this clique must all appear in distinct clauses since there are no edges to vertices of the same clause. Similarly, we know none of these vertices have edges to contradictory labels by the given construction above. Now, create a truth assignment τ with the labels from the vertices inside the k-clique. Therefore, this truth assignment must satisfy ϕ. Therefore, if $\langle G,k\rangle \in$ CLIQUE, then $\langle\phi\rangle \in$ 3SAT

6.3 What is this?

What are the gadgets?
The core idea is that a clique must contain k vertices who's variables can ALL BE TRUE at the same time, AND are of different clauses. This is a corollary of satisfiability. This is why the two restrictions on edges exist. In short, the reasoning is that "If k variables can be true at the same time (SAT), then an edge exists between all of them", and "if k vertices form a clique (CLIQUE), then they can all be true at the same time, and are of different clauses

6.4 Time Complexity Analysis

I argue that our mapping reduction is polytime because

1. It iterates over ϕ once to create G's adjacency list: $\mathcal{O}(n)$
2. It iterates over G's adjacency list once to connect all edges: $\mathcal{O}\left(n^2\right)$
3. It iterates twice more to ensure a proper construction: $\mathcal{O}\left(2n^2\right)$

Therefore, this is a polytime construction

6.5 The big result?

CLIQUE is NP-complete

7 Vertex Cover is NP Complete

VERTEX-COVER=$\{\langle G,k\rangle$ G is an undirected graph which has a k-node vertex cover$\}$
A vertex cover VC is a set of vertices such that all edges either start or end at a node in VC

7.1 Proof that VERTEX-COVER∈NP

Here's a polynomial time verifier for VERTEX-COVER, where C is a set of nodes which are in the vertex cover

VerifyVERTEX-COVER$\langle\langle G,k\rangle, C\rangle$:

if $|C| < k$:

return false

For each vertex in C:

mark each of its adjacent edges as covered

for each edge in G:

if the edge is not marked as covered:

return false

return true

This verifier clearly runs in polynomial time since it goes through every edge of G once, which serves a maximum complexity of $\mathcal{O}\left(n^2\right)$, where n is the number of vertices, and n^2 is the number of edges in a fully connected graph.

7.2 Proof that VERTEX-COVER is NP-Complete

While it might seem reasonable to reduce from CLIQUE $\leq_p$ VERTEX-COVER, we are instead going to choose 3SAT
WTS: 3SAT $\leq_p$ VERTEX-COVER
So, our reduction takes inputs in 3SAT, and transforms them into inputs into VERTEX-COVER, namely a graph G, and a size k.
Our input is a 3-cnf formula ϕ
To make this reduction, we will
1. Take every variable in ϕ, and add a vertex in G.
2. If x_i and $\bar{x}_i$appear in G, add an edge between x_i and $\bar{x}_i$.
3. If 3 variables occur in a clause of ϕ, add a new copy of each variable into G, and add an edge between each new vertice to every other vertice (GRAPH NEEDED)
4. Connect all x_i's to other x_i's and all $\bar{x}_i$to every other $\bar{x}_i$
5. set k to be the number of variables in ϕ + 2 times the number of clauses in ϕ
Now, for the justification:
If $\langle\phi\rangle \in$ 3SAT, then, select a satisfiable assignment τ, and put all the variable widgets that are assigned true in τ into our vertex cover. Now, take 1 variable that was assigned true from each clause, and put the other two variables in its clause into the vertex cover. Since our vertex cover has 1 variable per variable widget in the VC, and 2 clause widget variables per clause in the VC, it must have k vertices in the vertex-cover. Therefore, a vertex cover of size k exists.
If $\langle G,k\rangle \in$ VERTEX-COVER, then our Vertex Cover VC, must select 1 and only 1 variable per variable widget, since if it selected 2, it would have to skimp out on including clause variable widgets, which wouldn't form a full vertex cover. In addition, the vertex cover must select 2 nodes from each clause, where the unselected node is set to true, since if it didn't it would miss the edge from a node which was set to false to the variable widget. This forms a VC of size k, and the selected variables in the vertex clause must form satisfiable truth assignment since one unselected-but-true edge must exist in each clause.

8 HAMPATH is NP-Complete

A hamiltonian path is a path that visits each vertex of a graph exactly once. A hamiltonian path from s to t consists of a hamiltonian path with its head at s and tail at t. (heads and tails since these are directed graphs)
Formally: HAMPATH = $\{\langle G,s,t\rangle \mid$ G contains a hamiltonian path from s to t$\}$

8.1 HAMPATH∈NP

Here is a polytime verifier for HAMPATH. Let C, the certificate, be a hamiltonian path (a set of vertexes and edges) from s to t.

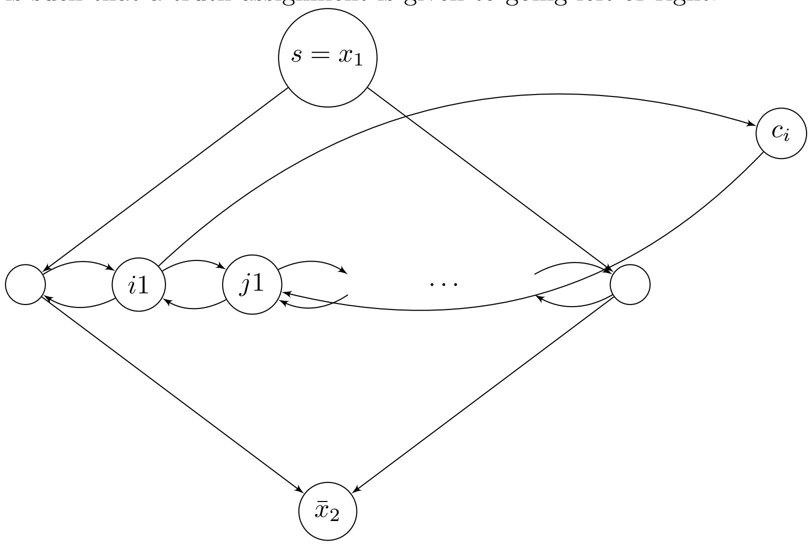
VerifyHAMPATH$\langle\langle G,s,t\rangle, C\rangle$:

Trace along the path C: and mark edges G which were visited. If an edge that was previously visited is visited again, reject
check if every vertex in G, if all were visited, accept
else reject (or loop?)

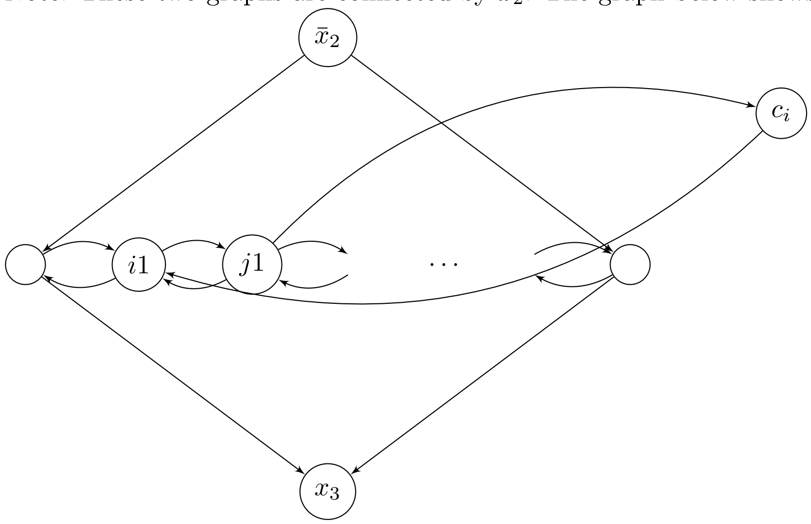
This verifier goes through the path C, which is $\mathcal{O}(n)$. Then, checking every vertex is visited takes $\mathcal{O}(n)$. Therefore, this verifier is polynomial time and takes $\mathcal{O}(n)$.

8.2 HAMPATH is NP-Complete

WTS: 3SAT $\leq_p$ HAMPATH
To do this, we need to take 3-cnf formulas ϕ and turn them into paths which contain a hamiltonian path if and only if ϕ is satisfiable. The way we do this is with two primary structures: The variable widget and the clause widget. The structure of the variable widget is such that a truth assignment is given to going left or right.



Note: These two graphs are connected by $\bar{x}_2$. The graph below shows what happens when a negated variable occurs



The number of these "vertices" in the middle is equal to 3 times the number of clauses. 3 times allows for 1 buffer node between pairs If a variable x_i is in clause c_j, then add an edge from the jth pair in the ith diamond to the jth clause node. But, if a variable $\bar{x}_i$ appears in clause c_j, then add two edges going in the reverse direction.
Finally, let G = the graph we have just constructed, and let s be the first variable widget, and t be the last variable widget. This is how we construct G,s, and t.
How this works: One can only assign 1 value (left/right or true/false) to each variable widget, and each clause widget must be gone through at least once by a single variable. This simulates a truth assignment, and that that truth assignment assigns true to each clause iff there is a satisfiable truth assignment for ϕ.

8.2.1 If $\langle \phi \rangle \in$ 3SAT

Then ϕ has a satisfiable truth assignment τ. Following the variable widget convention, we will go left/right according to what the truth assignment tells us to do. When we encounter the option to visit a clause, we will take it unless the clause has already been satisfied (i.e. visited). Since this formula is satisfiable, all clause nodes will be hit, as well as the top and bottom of each variable widget, and all nodes inbetween. This implies that $\langle G, s, t \rangle \in$ HAMPATH.

8.2.2 If $\langle G, s, t \rangle \in$ HAMPATH

Then G has a hamiltonian path from s to t, where s is the first variable, and t is the last variable in a truth assignment τ. This hamilton path must cross through each clause widget once, and must cross through each variable widget once. This implies that there must be a truth assignment such that every clause is satisfied, which implies that τ is a satisfiable truth assignment for ϕ. Therefore, $\langle \phi \rangle \in$ 3SAT

9 SUBSET-SUM is NP-Complete

9.1 Proof that SUBSET-SUM$\in$ NP

9.2 Proof that SUBSET-SUM is NP-Hard

WTS: 3SAT$\leq_p$SUBSET-SUM

9.2.1 The construction

Given a 3-cnf ϕ, an input to 3SAT, with l variables, and i clauses. Make a table as follows:

	x_1	x_2	$\cdots$	x_l	c_1	c_2	$\cdots$	c_i
t_1	1	0	0	0				
f_1	1	0	0	0				
t_2	0	1	0	0				
f_2	0	1	0	0				
$\vdots$	0	0	$\vdots$	0				
$\vdots$	0	0	$\vdots$	0				
t_l				1				
f_l				1				
t	1	1	$\cdots$	1				

We are going to make the rows our subset, and the row t will be our sum. If row t_1 is selected then make variable x_1 true, and if row f_1 is selected, make variable x_1 false. Since t contains a 1 in each column with a variable, this means that every variable can only be assigned true or false.

Next, modify the table. If t_j is selected, mark each clause in the row t_j 1 if x_j appears in the clause like so:

	x_1	x_2	$\cdots$	x_l	c_1	c_2	$\cdots$	c_i
t_1	1	0	0	0	1	0	$\cdots$	0
f_1	1	0	0	0	0	1	$\cdots$	1
t_2	0	1	0	0		$\vdots$		
f_2	0	1	0	0				
$\vdots$	0	0	$\vdots$	0				
$\vdots$	0	0	$\vdots$	0				
t_l				1				
f_l				1				
t	1	1	$\cdots$	1				

Now, we have to ensure that at least 1 variable in each clause is marked as true. The problem is that one clause, two clauses, and three clauses could be true. The way we control for this is adding two extra rows per clause, which both contain a 1, like so:

	x_1	x_2	$\cdots$	x_l	c_1	c_2	$\cdots$	c_i
t_1	1	0	0	0	1	0	$\cdots$	0
f_1	1	0	0	0	0	1	$\cdots$	1
t_2	0	1	0	0		$\vdots$		
f_2	0	1	0	0				
$\vdots$	0	0	$\vdots$	0				
$\vdots$	0	0	$\vdots$	0				
t_l				1				
f_l				1	1	0	$\cdots$	1
h_1	0	0	0	0	1			
g_1	0	0	0	0	1			
h_2	0	0	$\vdots$	0		1		
g_2	0	0	$\cdots$	0		1		
h_3	0	0	0	0			$\cdots$	
g_3	0	0	0	0			$\cdots$	
h_4	0	0	0	0				1
g_4	0	0	0	0				1
t	1	1	$\cdots$	1	3	3	3	3

Now, let t be the t row, and the subset S be all the other rows of the table.

9.2.2 Given that $\phi \in$ 3SAT, show $\langle S, t\rangle \in$ SUBSET-SUM

If ϕ is in 3SAT, that means there is a truth assignment τ which satisfies ϕ. This ϕ must also satisfy at least one variable in each clause. By the above construction, that implies that there must be some subset which sums to t.

9.2.3 Given that $\langle S, t\rangle \in$ SUBSET-SUM, show $\langle \phi \rangle \in$ 3SAT

We know S has some subset which sums to t. Using the variable construction, we know this subset must select either t_i or f_i for $1 \leq i \leq l$, where l is the number of rows (variables) in this section of the construction. Let our truth assignment τ assign true and false accordingly (true if t_i is selected, false if f_i is selected). In the second quadrent of the table, the clause section, if there is a subset sum which sums to t, then that subset must contain at least a 1 for each clause. By the above construction, this implies τ sets each clause to true. This implies that ϕ is satisfiable, which implies that $\langle \phi \rangle \in$ 3SAT

9.2.4 Justification that this reduction is polytime.

To constructing the structure (dimensions and layout) of the table, which is assumed to be constant time, filling out the variables section takes $\mathcal{O}(l)$ time, and filling out the clause section takes $3 \cdot \mathcal{O}(l)$ time, where l is the number of variables. If n is the number of variables in ϕ, this reduction takes $\mathcal{O}(l)$ time, which is polytime.

10 3-COL is NP-Complete

lec: CSCC63_20221101 , 42 mins in
3-col:
Input: A graph G
Question: Is there a coloring of G with 3 colors such that no two adjacent vertices share the same color.

10.1 Proving that 3-COL is in NP

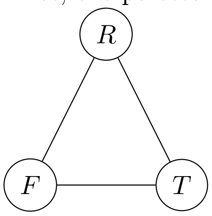
1. 3-COL asks us if an object exists or not. This means its a decision problem
2. A certificate would be a function which maps vertices to colorings.
3. For 3-COL, the input size is measured as the number of vertices in G. This is linear to a map which takes the vertices of G as input, for obvious reasons
4. The verification algorithm would iterate over each vertex, and check that none of its neighbors are colored with the same color
5. For a worst-case fully connected graph, this verifier would have to check each node and its n neighbors. This means it runs in n^2 time

10.2 Showing that 3-COL is NP-Hard

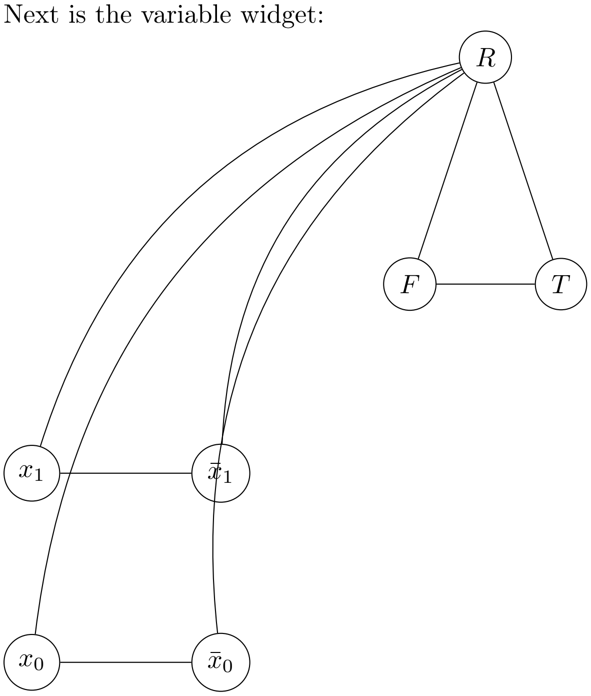
WTS: 3SAT$\leq_p$3COL

10.2.1 The construction

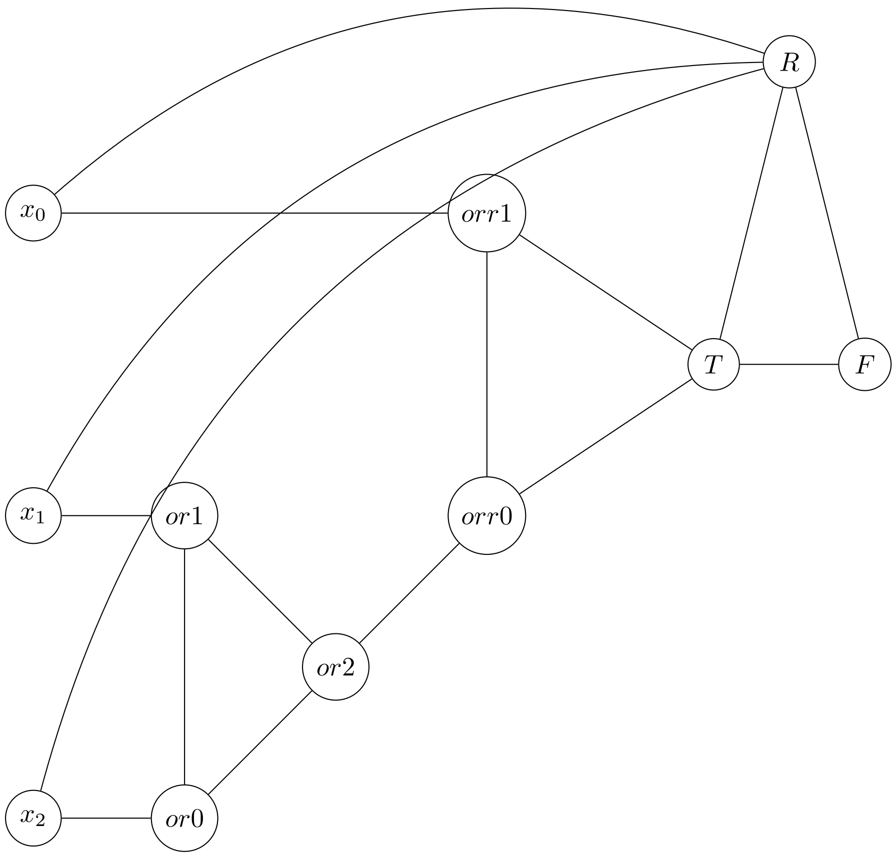
We need three core technologies to complete this reduction: The Variable Widget, the Clause Widget, and the Palette, as well as our three colors: True, False, and Red.
First, the palette looks like:



In order for this graph to be 3-colorable, this palette widget must have 3 distinct colors on each vertex. With out loss of generality, let these vertexes be named True, False, and Red, with the assumption that a 3-coloring of this graph will color these vertices appropriatly.
Next is the variable widget:



This widget assures that variables can only be assigned the true color or the false color (which will correspond with their truth assignments). In addition, these widgets assure that a variable and its complement cannot be assigned to the same color.
Finally, and most complicated: The clause widget:



10.2.2 Assume $\langle \phi \rangle \in$ 3SAT, show $\langle G \rangle \in$3COL

10.2.3 Assume $\langle G \rangle \in$3COL, show $\langle \phi \rangle \in$3SAT

10.2.4 The reduction is polytime

11 Assignment question: 2b

11.1 Prove that DISJOINT-CLIQUE is in NP

11.2 Show DISJOINT-CLIQUE is NP-Complete

We will reduce from CLIQUE $\leq_p$ DISJOINT-CLIQUE
To transform an input to CLIQUE $\langle G, k \rangle$ to an input $\langle G', k' \rangle$ to DISJOINT-CLIQUE, do as follows:
1. Put all vertices with k-1 or more edges into a group C
2. Let $G' = C + C'$where $C'$is all of the edges and vertices in C but with an added $'$ to the vertex names
3. Let $k' = k$
Pf that this reduciton works:

11.2.1 Assume $\langle G, k \rangle \in$ CLIQUE,

If G has a clique of size k, then all edges in the clique must have k-1 or more edges leading out of it. This implies that all these edges are cloned into G', which implies that G'has two cliques of size k (or more). This implies that $\langle G', k' \rangle \in$ DISJOINT-CLIQUE

11.2.2 Assume $\langle G', k' \rangle \in$ DISJOINT-CLIQUE

If G' has at least two cliques of size k, then suppose for purposes of contradiction that G has no clique of size k. If G has no clique of size k, then no subset of C' must form a clique of size k. This implies that none of the components of C form a clique of size k. This implies that G', which equals $C + C'$, has no cliques of size k, which is a contradiction to the assumption that G' has at least two cliques of size k. Therefore, if $\langle G', k' \rangle \in$ DISJOINT-CLIQUE $\implies$ $\langle G, k \rangle \in$ CLIQUE

11.2.3 This reduction is polytime

Finding all the vertices with $k - 1$ or more edges takes $\mathcal{O}\left(n^2\right)$ time, so this reduction is polytime

11.3 Find a non-decision version of DISJOINT-CLIQUE

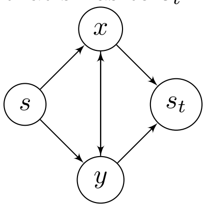
MAX-DISJOINT-CLIQUE
Input: A graph G
Output: A set of vertices V_1, V_2 such that V_1 and V_2 are disjoint cliques, and there are no two disjoint cliques of greater size in G
I did not want to prove this

12 DOUBLE-HAMPATH is NP-complete

12.1 Show that DOUBLE-HAMPATH∈NP

12.2 Show that DOUBLE-HAMPATH is NP-Complete

WTS: HAMPATH $\leq_p$DOUBLE-HAMPATH
To construct this, if $\langle G, s, t\rangle$ is an input to HAMPATH, we must map it to an input $\langle G', s', t'\rangle$ to DOUBLE-HAMPATH
The way we will do this is with a "decision widget": Add 3 vertices: x, y, s_t. Add two edges, one from s to x, one from s to y. Next, add an edge from x to y and an edge from y to x. Then, add an edge from x and y to s_t. Now, move all inbound and outbound edges that s has to s_t. This looks as follows:



Finally, let G' equal our graph with this new widget, let $s' = s$ and $t' = t$
Now, assume that $\langle G, s, t\rangle \in$ HAMPATH. Because of the nature of our decision widget, we can either go up then down through it, or down then up(zig then zag or zag then zig?). Then, the rest of the path remains the same since the rest of the graph remains untouched. Since the rest of the graph contains a hampath, this implies that G' contains two hampaths from s' to t'.
Now, using the contrapositive, assume that $\langle G, s, t\rangle \notin$ HAMPATH. This implies that no hamiltonian path exists from s to t. This implies that no hamiltonian path exists from s_t to t. Therefore, there is no hamiltonian path from $s = s'$ to $t = t'$. Therefore, G' contains no double-hampath from s' to t'.
WHY POLYTIME

12.3 Find a non-decision version