

2D Shear: Analyses of Least-Squares Objective Functions

Velocity Objective

Given a target simulation's end state $\mathbf{U}(T)$, consider an objective functional given by

$$H[\mathbf{u}(T)] = (\mathbf{u}(T) - \mathbf{U}(T)) \cdot (\mathbf{u}(T) - \mathbf{U}(T)) = (\mathbf{u}(T) - \mathbf{U}(T))^2$$

To determine the adjoint problem's initial ($t = T$) state, we take

$$\begin{aligned} \frac{\delta H}{\delta \mathbf{u}(T)} &= \lim_{\delta \mathbf{u}(T) \rightarrow 0} \left[\frac{H[\mathbf{u}(T) + \delta \mathbf{u}(T)] - H[\mathbf{u}(T)]}{\delta \mathbf{u}(T)} \right] \\ &= \lim_{\delta \mathbf{u}(T) \rightarrow 0} \left[\frac{(\mathbf{u}(T) + \delta \mathbf{u}(T) - \mathbf{U}(T))^2 - (\mathbf{u}(T) - \mathbf{U}(T))^2}{\delta \mathbf{u}(T)} \right] \\ &= \lim_{\delta \mathbf{u}(T) \rightarrow 0} \left[\frac{2\delta \mathbf{u}(T)(\mathbf{u}(T) - \mathbf{U}(T))}{\delta \mathbf{u}(T)} \right] \\ &= 2(\mathbf{u}(T) - \mathbf{U}(T)) \end{aligned}$$

Note: as we take $T \rightarrow 0$, we have that $\mathbf{u}(0) \rightarrow \mathbf{u}(T)$ and $\mathbf{U}(0) \rightarrow \mathbf{U}(T)$. From this, it follows that the adjoint system is initialized as

$$\boldsymbol{\mu}(T) = 2(\mathbf{u}(0) - \mathbf{U}(0)) \quad \rightarrow \quad \boldsymbol{\mu}(0)$$

Therefore

$$\frac{\delta H}{\delta \mathbf{u}(0)} \rightarrow 2(\mathbf{u}(0) - \mathbf{U}(0))$$

Which is the shortest direction to the optimized state in L_2 space.

Vorticity Objective

Next, we define

$$\begin{aligned}\omega &= \nabla \times \mathbf{u} \cdot \hat{\mathbf{k}} \equiv \nabla^\perp \cdot \mathbf{u} = \partial_x v - \partial_y u \\ W &= \nabla \times \mathbf{U} \cdot \hat{\mathbf{k}} \equiv \nabla^\perp \cdot \mathbf{U} = \partial_x V - \partial_y U\end{aligned}$$

and consider an objective functional given by

$$\begin{aligned}H[\mathbf{u}(T)] &= (\omega(T) - W(T))^2 \\ &= (\partial_x v(T) - \partial_y u(T) - W(T))^2\end{aligned}$$

Note: if we take

$$\frac{\delta H}{\omega(T)} = 2(\omega(T) - W(T)),$$

we recover the same result as before. It's more interesting to consider

$$\begin{aligned}\frac{\delta H}{\delta u(T)} &= \lim_{\delta u(T) \rightarrow 0} \left[\frac{(\partial_x v(T) - \partial_y u(T) - \partial_y \delta u(T) - W(T))^2 - (\partial_x v(T) - \partial_y u(T) - W(T))^2}{\delta u(T)} \right] \\ &= \lim_{\delta u(T) \rightarrow 0} \left[\frac{-\partial_y \delta u(T) (\partial_x v(T) - \partial_y u(T) - W(T))}{\delta u(T)} \right]\end{aligned}$$

integrating by parts gives

$$= \lim_{\delta u(T) \rightarrow 0} \left[\frac{-\partial_y \left[\delta u(T) (\partial_x v(T) - \partial_y u(T) - W(T)) \right] + \delta u(T) \partial_y \left[\partial_x v(T) - \partial_y u(T) - W(T) \right]}{\delta u(T)} \right]$$

The first term in the numerator cancels when impenetrable boundary conditions are employed, leaving

$$\begin{aligned}&= \lim_{\delta u(T) \rightarrow 0} \left[\frac{\delta u(T) \partial_y \left[\partial_x v(T) - \partial_y u(T) - W(T) \right]}{\delta u(T)} \right] \\ &= \partial_y \left[\omega(T) - W(T) \right]\end{aligned}$$

Applying the same process to $\frac{\delta H}{\delta v(T)}$, we find

$$\frac{\delta H}{\delta \mathbf{u}(T)} = \nabla^\perp \left[\omega(T) - W(T) \right]$$

Streamfunction objective

We define the streamfunctions $\psi(x, y)$ and $\phi(x, y)$ satisfying

$$\begin{aligned}\nabla^\perp \psi &= \mathbf{u} \\ \nabla^\perp \phi &= \mathbf{U}\end{aligned}$$

Note that

$$\begin{aligned}\nabla^\perp \cdot \nabla^\perp \psi &= \nabla^2 \psi = \nabla^\perp \cdot \mathbf{u} = \omega \\ \nabla^\perp \cdot \nabla^\perp \phi &= \nabla^2 \phi = \nabla^\perp \cdot \mathbf{U} = W\end{aligned}$$

Consider the original velocity objective

$$\begin{aligned}H[\mathbf{u}(T)] &= (\mathbf{u}(T) - \mathbf{U}(T)) \cdot (\mathbf{u}(T) - \mathbf{U}(T)) \\ &= (\mathbf{u}(T) - \mathbf{U}(T))^2 \\ &= (\nabla^\perp \psi(T) - \nabla^\perp \phi(T))^2\end{aligned}$$

We then take

$$\begin{aligned}\frac{\delta H}{\delta \psi(T)} &= \lim_{\delta \psi(T) \rightarrow 0} \frac{(\nabla^\perp(\psi(T) + \delta \psi(T)) - \nabla^\perp \phi(T))^2 - (\nabla^\perp \psi(T) - \nabla^\perp \phi(T))^2}{\delta \psi(T)} \\ &= \lim_{\delta \psi(T) \rightarrow 0} \frac{2\delta \psi(T)(\nabla^\perp \psi(T) - \nabla^\perp \phi(T))}{\delta \psi(T)} \\ &= 2(\nabla^\perp \psi(T) - \nabla^\perp \phi(T)) \\ &= 2(\mathbf{u}(T) - \mathbf{U}(T))\end{aligned}$$

This is the same thing as before! To get a new adjoint ic, we need to take

$$H[\psi(T)] = (\psi(T) - \phi(T))^2$$

along with

$$\frac{\delta H}{\delta \mathbf{u}(T)} = \frac{\delta}{\delta \mathbf{u}(T)} [(\psi(T) - \phi(T))^2]$$

Thus, by applying the appropriate boundary conditions, we can solve for ψ given a flow field.

$$\begin{aligned}\psi &= \nabla^{-2} \omega \\ \phi &= \nabla^{-2} W\end{aligned}$$

Recall that we are interested in a streamfunction objective's variational derivative with respect to velocity, as differentiating the least-squares functional $(\psi - \phi)^2$ wrt ψ gives us the same thing as before!

$$\begin{aligned}H[\psi(T)] &= (\psi - \phi)^2 \\ &= (\nabla^{-2} \omega - \nabla^{-2} W)^2 \\ &= (\nabla^{-2} \omega)^2 - 2\nabla^{-2} \omega \nabla^{-2} W + (\nabla^{-2} W)^2\end{aligned}$$

Taking the variation

$$\begin{aligned}
\frac{\delta H}{\delta u} &= \lim_{\delta u \rightarrow 0} \frac{2\nabla^{-2}\nabla^\perp \cdot \mathbf{u} \nabla^{-2}\nabla^\perp \cdot \delta \mathbf{u} - 2\nabla^{-2}\nabla^\perp \cdot \delta \mathbf{u} \nabla^{-2}W}{\delta u} \\
&= \lim_{\delta u \rightarrow 0} \frac{2\nabla^{-2}\nabla^\perp \cdot \delta \mathbf{u} (\nabla^{-2}\nabla^\perp \cdot \mathbf{u} - \nabla^{-2}W)}{\delta u} \\
&= 2(\psi - \phi) \lim_{\delta u \rightarrow 0} \frac{\nabla^{-2}\nabla^\perp \cdot \delta \mathbf{u}}{\delta u}
\end{aligned}$$