1 Setup

The incompressible MHD equations in a local cartesian region in a rotating frame

$$
\begin{aligned}
\partial_t \boldsymbol{v} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} + 2\Omega\hat{\boldsymbol{z}}\times\boldsymbol{v} + \boldsymbol{\nabla}p &= \boldsymbol{b}\cdot\boldsymbol{\nabla}\boldsymbol{b} + \nu\nabla^2\boldsymbol{v} \\
\boldsymbol{\nabla}\cdot\boldsymbol{v} &= 0 \\
\partial_t \boldsymbol{b} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{b} &= \boldsymbol{b}\cdot\boldsymbol{\nabla}\boldsymbol{v} + \eta\nabla^2\boldsymbol{b} \\
\boldsymbol{\nabla}\cdot\boldsymbol{b} &= 0
\end{aligned}
$$

With axisymmetry (constant in y) we consider linear perturbations (with x and z dependence of the form)

$$
\begin{aligned}
\boldsymbol{v} &= [v_0(x) + v(x,z)]\hat{\boldsymbol{y}} - \hat{\boldsymbol{y}}\times\boldsymbol{\nabla}\psi(x,z) \\
\boldsymbol{b} &= b(x,z)\hat{\boldsymbol{y}} - \hat{\boldsymbol{y}}\times\boldsymbol{\nabla}\left(a_0(x) + a(x,z)\right)
\end{aligned}
$$

such that

$$
\begin{aligned}
v_0(x) &= Sx - \frac{Sd}{2} \\
a_0(x) &= Bx - \frac{Bd}{2}
\end{aligned}
$$

with $0 < x < d$. S is the shear rate of the background azimuthal flow (which decreases in x). B is the constant poloidal background magnetic field (in the z direction). Substitution yields

$$
\begin{aligned}
\partial_t b + J(\psi, b) &= J(a, v) + B\partial_z v - S\partial_z a + \eta\nabla^2 b, \\
\partial_t a + J(\psi, a) &= B\partial_z\psi + \eta\nabla^2 a, \\
\partial_t v + J(\psi, v) - (f+S)\partial_z\psi &= J(a, b) + B\partial_z b + \nu\nabla^2 v, \\
\partial_t\nabla^2\psi + J(\psi, \nabla^2\psi) + f\partial_z v &= J(a, \nabla^2 a) + B\partial_z\nabla^2 a + \nu\boldsymbol{\nabla}^4\psi
\end{aligned}
$$

where the Jacobian

$$
J(p,q) \equiv \partial_x p\partial_z q - \partial_z p\partial_x q
$$

The stream function ψ and scalar magnetic potential a are related to their respective vector fields as follows

$$\begin{aligned} vx &= -\partial_z \psi \\ vz &= \partial_x \psi \\ bx &= -\partial_z a \\ bz &= \partial_x a \end{aligned}$$

The Jacobian terms

$$\begin{aligned} J(\psi, \cdot) &= \partial_x \psi \partial_z \cdot - \partial_z \psi \partial_x \cdot \\ &= vz \partial_z \cdot + vx \partial_x \cdot \end{aligned}$$

The vorticity

$$\partial_z vx - \partial_x vz = -\partial_z^2 \psi - \partial_x^2 \psi = -\nabla^2 \psi$$

The coriolis term

$$2\Omega \hat{\boldsymbol{z}} \times \boldsymbol{v} = 2\Omega(-vy\hat{\boldsymbol{x}} + vx\hat{\boldsymbol{y}})$$

whose 2D curl

$$= -f\partial_z vy$$

Next the linearized non-dissipative system

$$\begin{aligned} \partial_t b &= B\partial_z v - S\partial_z a \\ \partial_t a &= B\partial_z \psi \\ \partial_t v - (f+S)\partial_z \psi &= B\partial_z b \\ \partial_t \nabla^2 \psi + f\partial_z v &= B\partial_z \nabla^2 a \end{aligned}$$

Time derivative of the first allows for the elimination of a

$$\partial_t^2 b = B\partial_t \partial_z v - SB\partial_z^2 \psi$$

Rearranging

$$\begin{aligned}\partial_t\partial_z v &= S\partial_z^2\psi + B^{-1}\partial_t^2 b\\ (f+S)B\partial_z^2\psi + B^2\partial_z^2 b &= SB\partial_z^2\psi + \partial_t^2 b\\ fB\partial_z^2\psi + B^2\partial_z^2 b &= \partial_t^2 b\end{aligned}$$

Time derivative of the streamfunction equation

$$\partial_t^2\nabla^2\psi + f(S\partial_z^2\psi + B^{-1}\partial_t^2 b) = B\partial_t\partial_z\nabla^2 a$$

using the second equation for a

$$\partial_t^2\nabla^2\psi + f(S\partial_z^2\psi + B^{-1}\partial_t^2 b) = B^2\partial_z^2\nabla^2\psi$$

and substituting our previous expression for the second order time derivative of b

$$\partial_t^2\nabla^2\psi + f((f+S)\partial_z^2\psi + B\partial_z^2 b)) = B^2\partial_z^2\nabla^2\psi$$

2 Quasilinear Analysis

The vector velocity and magnetic field,

$$U = (-\partial_z\psi, u_y, \partial_x\psi), \qquad B = (-\partial_z A_y, B_y, \partial_x A_y)$$

The full nonlinear set of streamwise equations

$$\begin{aligned}\partial_t u_y - f\partial_z\psi + \boldsymbol{\nabla}\cdot(uu_y - BB_y) &= 0\\ \partial_t A_y + \boldsymbol{\nabla}\cdot(uA_y) &= 0\\ \partial_t B_y + \boldsymbol{\nabla}\cdot(uB_y - Bu_y)\end{aligned}$$

The equation for ψ is more complicated, but we don't need it right now. The initial background parameters satisfy the full streamwise equations. At leading order:

$$U_y = S(x - d/2), \qquad A_y = B_0(x - d/2)$$

and $\psi = B_y = 0$. The linear equations are

$$\begin{aligned}\partial_t u_y &= \partial_z((f-S)\psi + B_0 b_y)\\ \partial_t a_y &= \partial_z(B_0\psi)\\ \partial_t b_y &= \partial_z(B_0 u_y + Sa_y)\end{aligned}$$

We want to see how the linear synamics feeds back onto the z-mean u_y, and a_y fields. These feedbacks represent quadratic-order changes to the "background" parameters. Therefore,

$$\partial_t\langle u_y\rangle + \partial_x\langle u_x u_y - B_x B_y\rangle = 0$$
$$\partial_t\langle\rangle + \partial_x\langle u_x A_y\rangle = 0$$

For any two function, $f(z)$, $g(z)$, a couple of essential properties of the z-average

- $\langle\partial_z f\rangle = 0$
- $\langle f\partial_z g\rangle = -\langle g\partial_z f\rangle$

We want to isolate the feedback from the linear dynamics. Therefore, we can use the linear equations freely in simplifying the quadratic terms and the resulting ∂_t terms tell us how much background change we can expect for a hard day's work.

Therefore, using the definitions of u_x, b_x,

$$\partial_t\langle U_y\rangle = \partial_x\langle u_y\partial_z\psi - b_y\partial_z a_y\rangle$$
$$\partial_t\langle A_y\rangle = \partial_x\langle a_y\partial_z\psi\rangle$$

First, notice that

$$\partial_z\psi = \frac{\partial_t a_y}{B_0}$$

therefore,

$$\langle a_y\partial_z\psi\rangle = \frac{\langle a_y\partial_t a_y\rangle}{B_0} = \frac{\partial_t\langle a_y^2\rangle}{2B_0}$$

Therefore, usng the initial conditions for the background we can pull a time derivative off of both sides of the equation,

$$\langle A_y\rangle = B_0(x - d/2) + \frac{\partial_x\langle a_y^2\rangle}{2B_0}.$$

Next, notice that

$$\partial_t\left\langle U_y\right\rangle = \partial_x\left\langle u_y\partial_z\psi - b_y\partial_z a_y\right\rangle = \partial_x\left\langle u_y\partial_z\psi + a_y\partial_z b_y\right\rangle$$

Then notice

$$\partial_z b_y = \frac{\partial_t u_y}{B_0} - (f - S)\frac{\partial_z\psi}{B_0} = \frac{\partial_t u_y}{B_0} - (f - S)\frac{\partial_t a_y}{B_0^2}$$

Therefore,

$$\langle u_y \partial_z \psi + a_y \partial_z b_y \rangle = \partial_t \left\langle \frac{a_y u_y}{B_0} - (f - S)\frac{a_y^2}{2B_0^2} \right\rangle$$

Finally,

$$\begin{aligned} \langle A_y \rangle &= B_0(x - d/2) + \partial_x \boldsymbol{\Phi} \\ \langle U_y \rangle &= S(x - d/2) + \partial_x \boldsymbol{\mathcal{L}} \end{aligned}$$

where

$$\boldsymbol{\Phi} = \frac{\langle a_y^2 \rangle}{2B_0}, \qquad \boldsymbol{\mathcal{L}} = \frac{\langle 2B_0 a_y u_y - (f - S) a_y^2 \rangle}{2B_0^2}$$

We then express the quasilinear correction terms as variations

$$\begin{aligned} \delta A_y &= \partial_x \boldsymbol{\Phi} \\ \delta U_y &= \partial_x \boldsymbol{\mathcal{L}} \end{aligned}$$

The dynamic shear and magnetic corrections

$$\begin{aligned} \delta B_0 &= \partial_x \delta A_y = \partial_x^2 \boldsymbol{\Phi} \\ \delta S &= \partial_x \delta U_y = \partial_x^2 \boldsymbol{\mathcal{L}} \end{aligned}$$