

0.1 Problem Setup

Consider the non-dissipative incompressible MHD equations in a rotating frame

$$\begin{aligned}\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + f \hat{\mathbf{z}} \times \mathbf{v} + \nabla p &= \mathbf{b} \cdot \nabla \mathbf{b} \\ \nabla \cdot \mathbf{v} &= 0 \\ \partial_t \mathbf{b} + \mathbf{v} \cdot \nabla \mathbf{b} &= \mathbf{b} \cdot \nabla \mathbf{v} \\ \nabla \cdot \mathbf{b} &= 0\end{aligned}$$

With axisymmetry (constant in y) we consider linear perturbations (with x and z dependence of the form)

$$\begin{aligned}\mathbf{v} &= [v_0(x) + v(x, z)]\hat{\mathbf{y}} - \hat{\mathbf{y}} \times \nabla \psi(x, z) \\ \mathbf{b} &= b(x, z)\hat{\mathbf{y}} - \hat{\mathbf{y}} \times \nabla (a_0(x) + a(x, z))\end{aligned}$$

such that

$$\begin{aligned}v_0(x) &= Sx - \frac{Sd}{2} \\ a_0(x) &= Bx - \frac{Bd}{2}\end{aligned}$$

with $0 < x < d$. S is the shear rate of the background azimuthal flow (which decreases in x). B is the constant poloidal background magnetic field (in the z direction).

0.2 Displacement Equation

The Eulerian displacement vector satisfies

$$\begin{aligned}\mathbf{v} &= \partial_t \boldsymbol{\xi} + v_0(x)\hat{\mathbf{y}} \cdot \nabla \boldsymbol{\xi} - \boldsymbol{\xi} \cdot \nabla (v_0(x)\hat{\mathbf{y}}) \\ &= \partial_t \boldsymbol{\xi} - S\xi_x \hat{\mathbf{y}}\end{aligned}$$

with

$$\mathbf{b} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}).$$

Substituting these expressions into the linearized induction equation gives

$$\begin{aligned}
 IND &= \partial_t \mathbf{b} - \nabla \times (\mathbf{v}_0 \times \mathbf{b} + \mathbf{v} \times \mathbf{B}) \\
 &= \nabla \times (\partial_t \boldsymbol{\xi} \times \mathbf{B}) - \nabla \times (\mathbf{v}_0 \times \mathbf{b} + \partial_t \boldsymbol{\xi} \times \mathbf{B} - S \xi_x \hat{\mathbf{y}} \times \mathbf{B}) \\
 &= -\nabla \times (\mathbf{v}_0 \times \mathbf{b} - S \xi_x \hat{\mathbf{y}} \times \mathbf{B}) \\
 &= -\nabla \times (S x \hat{\mathbf{y}} \times (\nabla \times (\boldsymbol{\xi} \times \mathbf{B})) - S \xi_x \hat{\mathbf{y}} \times \mathbf{B}).
 \end{aligned}$$

Notice

$$\begin{aligned}
 \nabla \times (\boldsymbol{\xi} \times \mathbf{B}) &= \boldsymbol{\xi}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \boldsymbol{\xi}) + (\mathbf{B} \cdot \nabla) \boldsymbol{\xi} - (\boldsymbol{\xi} \cdot \nabla) \mathbf{B} \\
 &= -\mathbf{B}(\nabla \cdot \boldsymbol{\xi}) + (\mathbf{B} \cdot \nabla) \boldsymbol{\xi} \\
 &= B \partial_z \boldsymbol{\xi}
 \end{aligned}$$

implies

$$IND = -SB \nabla \times (x \hat{\mathbf{y}} \times \partial_z \boldsymbol{\xi} - \xi_x \hat{\mathbf{y}} \times \hat{\mathbf{z}}).$$

However,

$$\begin{aligned}
 \nabla \times (x \hat{\mathbf{y}} \times \partial_z \boldsymbol{\xi}) &= x \hat{\mathbf{y}}(\nabla \cdot \partial_z \boldsymbol{\xi}) - \partial_z \boldsymbol{\xi}(\nabla \cdot x \hat{\mathbf{y}}) + (\partial_z \boldsymbol{\xi} \cdot \nabla) x \hat{\mathbf{y}} - (x \hat{\mathbf{y}} \cdot \nabla) \partial_z \boldsymbol{\xi} \\
 &= \partial_z \xi_x \hat{\mathbf{y}}
 \end{aligned}$$

and

$$\begin{aligned}
 \nabla \times (\xi_x \hat{\mathbf{y}} \times \hat{\mathbf{z}}) &= \xi_x \hat{\mathbf{y}}(\nabla \cdot \hat{\mathbf{z}}) - \hat{\mathbf{z}}(\nabla \cdot \xi_x \hat{\mathbf{y}}) + (\hat{\mathbf{z}} \cdot \nabla) \xi_x \hat{\mathbf{y}} - (\xi_x \hat{\mathbf{y}} \cdot \nabla) \hat{\mathbf{z}} \\
 &= \partial_z \xi_x \hat{\mathbf{y}}.
 \end{aligned}$$

Therefore, we automatically satisfy $IND = 0$.

With the induction equation satisfied, we turn our attention to the momentum equation. Combining the displacement relation with the momentum equation gives the second-order boundary value problem

$$0 = -\xi_x'' + k_z^2 \left(1 + \frac{\Omega(-B^2 k_z^2 S + \gamma^2(\Omega - S))}{(B^2 k_z^2 + \gamma^2)^2} \right) \xi_x.$$

The growth rate $\gamma \rightarrow 0$ for marginal-stability

$$0 = -\xi_x'' + \left(k_z^2 - \frac{S\Omega}{B^2} \right) \xi_x.$$

Thus, for finite k_z , we obtain a marginally-stable mode when

$$\sqrt{\frac{S\Omega}{B^2} - k_z^2} = \frac{d}{\pi}.$$

We define the potential

$$V \equiv -\frac{S\Omega}{B^2}.$$

Next we multiply the x -displacement equation by ξ_x^* and integrate over the domain, yielding a functional

$$\mathcal{F} \equiv \int_0^d |\xi'_x|^2 + k_z^2 |\xi_x|^2 + V |\xi_x|^2 dx.$$

The displacement $\xi_x = A \sin(\pi x/d)$

$$\mathcal{F} = A^2 \left(\frac{\pi^2}{2d} + k_z^2 \frac{d}{2} + V \frac{d}{2} \right).$$

Saturation occurs when

$$\mathcal{F} = -\delta\mathcal{F}$$

where

$$\delta\mathcal{F} = \int_0^d \delta V |\xi_x|^2 dx.$$

The potential varies due to changes in the background magnetic field and shear parameter, i.e.

$$\delta V = -\Omega B^{-2} \delta S + 2S\Omega B^{-3} \delta B$$

0.3 Eigenvalue Problem

Substitution yields

$$\begin{aligned} \partial_t b + J(\psi, b) &= J(a, v) + B\partial_z v - S\partial_z a, \\ \partial_t a + J(\psi, a) &= B\partial_z \psi, \\ \partial_t v + J(\psi, v) - (f + S)\partial_z \psi &= J(a, b) + B\partial_z b, \\ \partial_t \nabla^2 \psi + J(\psi, \nabla^2 \psi) + f\partial_z v &= J(a, \nabla^2 a) + B\partial_z \nabla^2 a \end{aligned}$$

where the Jacobian

$$J(p, q) \equiv \partial_x p \partial_z q - \partial_z p \partial_x q$$

We denote linear perturbations with apostrophes, e.g. $v(x, z, t) = v_0(x) + v'(x, z, t)$. We discard the Jacobian terms to obtain the linearized system

$$\begin{aligned}\partial_t b' &= B \partial_z v' - S \partial_z a', \\ \partial_t a' &= B \partial_z \psi', \\ \partial_t v' - (f + S) \partial_z \psi' &= B \partial_z b', \\ \partial_t \nabla^2 \psi' + f \partial_z v' &= B \partial_z \nabla^2 a' .\end{aligned}$$

We isolate the nonlinear feedback by taking the z -average of the nonlinear equations. For velocity, we rewrite the y -component equation as

$$\begin{aligned}\partial_t \langle v_y \rangle &= \langle \mathbf{b} \cdot \nabla b_y - \mathbf{v} \cdot \nabla v_y \rangle \\ &= \langle \nabla \cdot (b_y \mathbf{b} - v_y \mathbf{v}) \rangle \\ &= \partial_x \langle -b_y \partial_z a + v_y \partial_z \psi \rangle .\end{aligned}$$

For the induction equation, we use the scalar potential

$$\begin{aligned}\partial_t \langle a \rangle &= - \langle J(\psi, a) \rangle \\ &= \langle \partial_z \psi \partial_x a - \partial_x \psi \partial_z a \rangle \\ &= - \langle u_x \partial_x a + u_z \partial_z a \rangle \\ &= - \langle u_x \partial_x a - a \partial_z u_z \rangle .\end{aligned}$$

The velocity has no divergence and no y -dependence, meaning $\partial_x u_x + \partial_z u_z = 0$ such that

$$\begin{aligned}&= - \langle u_x \partial_x a + a \partial_x u_x \rangle \\ &= - \partial_x \langle a u_x \rangle \\ &= \partial_x \langle a \partial_z \psi \rangle\end{aligned}$$

The background state is characterized by the shear parameter S as well as the vertical magnetic field B . We eliminate the time derivatives by manipulating the linear equations. For the velocity, we have

$$\begin{aligned}\partial_t \langle v_y \rangle &= \partial_x \langle -b_y \partial_z a + B^{-1} v_y \partial_t a \rangle \\ &= \partial_x \langle a \partial_z b_y + B^{-1} v_y \partial_t a \rangle \\ &= B^{-1} \partial_x \langle a \partial_t v_y - (f + S) a \partial_z \psi + v_y \partial_t a \rangle \\ &= B^{-1} \partial_x \left\langle a \partial_t v_y - \frac{f + S}{B} a \partial_t a + v_y \partial_t a \right\rangle \\ &= B^{-1} \partial_x \left\langle \partial_t (a v_y) - \frac{f + S}{2B} \partial_t (a^2) \right\rangle \\ &= \partial_t \left(B^{-1} \partial_x \left\langle a v_y - \frac{f + S}{2B} a^2 \right\rangle \right) \\ \therefore \langle v_y \rangle &= S x + \partial_x \left\langle \frac{1}{B} a v_y - \frac{f + S}{2B^2} a^2 \right\rangle\end{aligned}$$

For the magnetic potential, we have

$$\begin{aligned}\partial_t \langle a \rangle &= B^{-1} \partial_x \langle a \partial_t a \rangle \\ &= \partial_t \left(\frac{1}{2B} \partial_x \langle a^2 \rangle \right) \\ \therefore \langle a \rangle &= Bx + \frac{1}{2B} \partial_x \langle a^2 \rangle\end{aligned}$$

0.4 Amplitude Estimate

The feedback terms from the previous subsection allow us to compute the variations of the background quantities

$$\begin{aligned}\delta S(x) &= \partial_x^2 \left\langle \frac{1}{B} a v_y - \frac{f+S}{2B^2} a^2 \right\rangle \\ \delta B(x) &= \frac{1}{2B} \partial_x^2 \langle a^2 \rangle\end{aligned}$$

Once the x -displacement ξ_x is known, we can compute the other eigenfunctions

$$\begin{aligned}a &= -B\xi_x \\ v_y &= \frac{(\gamma^2 k_z^{-1} - B^2 k_z)(k_z^2 - \pi^2 d^{-2})}{2\Omega k_z} \xi_x = \kappa \xi_x.\end{aligned}$$

Therefore

$$\begin{aligned}\delta S(x) &= \partial_x^2 \left\langle \frac{1}{B} a v_y - \frac{2\Omega + S}{2B^2} a^2 \right\rangle \\ \delta B(x) &= \frac{1}{2B} \partial_x^2 \langle a^2 \rangle\end{aligned}$$

Recall,

$$\begin{aligned}\delta V &= -\Omega B^{-2} \delta S + 2S\Omega B^{-3} \delta B \\ &= -\Omega B^{-2} \partial_x^2 \left\langle \frac{1}{B} a v_y - \frac{2\Omega + S}{2B^2} a^2 \right\rangle + S\Omega B^{-4} \partial_x^2 \langle a^2 \rangle.\end{aligned}$$

Then, writing everything in terms of the x -displacement

$$\begin{aligned}&= \Omega B^{-2} \partial_x^2 \left\langle \kappa \xi_x^2 + \frac{2\Omega + S}{2} \xi_x^2 \right\rangle + S\Omega B^{-2} \partial_x^2 \langle \xi_x^2 \rangle \\ &= \Omega B^{-2} \left(\kappa + \Omega + \frac{3S}{2} \right) \partial_x^2 \langle \xi_x^2 \rangle\end{aligned}$$

implies

$$\begin{aligned}\delta\mathcal{F} &= \int_0^d |\xi_x|^2 (\Omega B^{-2} \left(\kappa + \Omega + \frac{3S}{2} \right) \partial_x^2 \langle \xi_x^2 \rangle) dx \\ &= A^4 \frac{3\pi^2}{8d} \Omega B^{-2} \left(\kappa + \Omega + \frac{3S}{2} \right).\end{aligned}$$

Therefore, the overall amplitude estimate

$$A = \sqrt{\frac{\left(\frac{\pi^2}{2d} + k_z^2 \frac{d}{2} + V \frac{d}{2} \right)}{\frac{3\pi^2}{8d} \Omega B^{-2} \left(\kappa + \Omega + \frac{3S}{2} \right)}}$$

0.5 Cylindrical Geometry

First we will implement the vector-potential formulation. Let the magnetic field $\mathbf{b} = b_r \hat{\mathbf{r}} + b_\theta \hat{\boldsymbol{\theta}} + b_z \hat{\mathbf{z}}$ be the curl of some divergence-free vector potential $\mathbf{A} = A_r \hat{\mathbf{r}} + A_\theta \hat{\boldsymbol{\theta}} + A_z \hat{\mathbf{z}}$, such that

$$\begin{aligned}\mathbf{b} &= \nabla \times \mathbf{A} \\ b_r &= r^{-1} \partial_\theta A_z - \partial_z A_\theta \\ b_\theta &= \partial_z A_r - \partial_r A_z \\ b_z &= \partial_z A_\theta - r^{-1} \partial_\theta A_r + r^{-1} A_\theta\end{aligned}$$

Assuming we have axisymmetry ($\partial_\theta \rightarrow 0$)

$$\begin{aligned}b_r &= -\partial_z A_\theta \\ b_\theta &= \partial_z A_r - \partial_r A_z \\ b_z &= \partial_z A_\theta + r^{-1} A_\theta\end{aligned}$$

Thus, at $t = 0$, we set

$$A_\theta = r b_z|_{t=0} = r B_1 \frac{1 + 4(r/r_1)^5}{5(r/r_1)^3}$$

For $B_1 = 1$ and $r_1 = 1$,

$$A_\theta = \frac{1 + 4r^5}{5r^2}$$

0.5.1 Scalar Potential

First, define the velocity and magnetic flux as follows

$$\begin{aligned}\mathbf{u} &= -\partial_z \psi \hat{\mathbf{r}} + v \hat{\boldsymbol{\theta}} + \partial_r \psi \hat{\mathbf{z}} \\ \mathbf{b} &= -\partial_z a \hat{\mathbf{r}} + b \hat{\boldsymbol{\theta}} + \partial_r a \hat{\mathbf{z}}.\end{aligned}$$

Inserting this expression for velocity into the θ -component of the momentum equation gives

$$\begin{aligned}\partial_t v + \nu (r^{-2}v - r^{-1}\partial_r v - \partial_r^2 v - \partial_z^2 v) &= r^{-1} (v\partial_z \psi - b\partial_z a) + \partial_z b\partial_r a - \partial_z a\partial_r b + \partial_z \psi\partial_r v - \partial_z v\partial_r \psi \\ \partial_t v + \nu (r^{-2}v - r^{-1}\partial_r v - \partial_r^2 v - \partial_z^2 v) &= r^{-1} (v\partial_z \psi - b\partial_z a) + \partial_z b\partial_r a - \partial_z a\partial_r b + \partial_z \psi\partial_r v - \partial_z v\partial_r \psi\end{aligned}$$

Then, taking the θ -component of the curl of the vector momentum equation gives the following equation for $\nabla^2 \psi$

$$\begin{aligned}\partial_t \nabla^2 \psi + \nu (\partial_z^2 \psi / r^2 - \partial_z^4 \psi - \partial_r \partial_z^2 \psi / r + \partial_r^2 \psi / r^2 - 2\partial_r^2 \partial_z^2 \psi - \partial_r^3 \psi / r - \partial_r^4 \psi) \\ - ((2b\partial_z b)/r) + (2v\partial_z v)/r - \partial_z \psi \partial_r \partial_z^2 \psi - \partial_r a \partial_z^3 a + \partial_r a \partial_r^2 \partial_z a + \partial_r \psi \partial_z^3 \psi + \partial_r \psi \partial_r^2 \partial_z \psi + \partial_z a (\partial_r \partial_z^2 a + \partial_r^3 a) - \partial_z \psi \partial_r^3 \psi\end{aligned}$$