0.1 Problem Setup

Consider the non-dissipative incompressible MHD equations in a rotating frame

$$\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} + f\hat{\boldsymbol{z}} \times \boldsymbol{v} + \boldsymbol{\nabla} p = \boldsymbol{b} \cdot \boldsymbol{\nabla} \boldsymbol{b}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$$

$$\partial_t \boldsymbol{b} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{\nabla} \boldsymbol{v}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{b} = 0$$

With axisymmetry (constant in y) we consider linear perturbations (with x and z dependence of the form)

$$\boldsymbol{v} = [v_0(x) + v(x, z)]\hat{\boldsymbol{y}} - \hat{\boldsymbol{y}} \times \boldsymbol{\nabla} \psi(x, z)$$

$$\boldsymbol{b} = b(x, z)\hat{\boldsymbol{y}} - \hat{\boldsymbol{y}} \times \boldsymbol{\nabla} \left(a_0(x) + a(x, z) \right)$$

such that

$$v_0(x) = Sx - \frac{Sd}{2}$$

$$a_0(x) = Bx - \frac{Bd}{2}$$

with $0 < x < d$. S is the shear rate of the background azimuthal flow (which decreases in x). B is the constant poloidal background magnetic field (in the z direction).

0.2 Displacement Equation

The Eulerian displacement vector satisfies

$$\boldsymbol{v} = \partial_t \boldsymbol{\xi} + v_0(x)\hat{\boldsymbol{y}} \cdot \boldsymbol{\nabla} \boldsymbol{\xi} - \boldsymbol{\xi} \cdot \boldsymbol{\nabla}(v_0(x)\hat{\boldsymbol{y}})$$

$$= \partial_t \boldsymbol{\xi} - S\xi_x \hat{\boldsymbol{y}}$$

with

$$\boldsymbol{b} = \boldsymbol{\nabla} \times (\boldsymbol{\xi} \times \boldsymbol{B}).$$

Substituting these expressions into the linearized induction equation gives

$$
\begin{aligned}
IND &= \partial_t \boldsymbol{b} - \boldsymbol{\nabla} \times (\boldsymbol{v_0} \times \boldsymbol{b} + \boldsymbol{v} \times \boldsymbol{B}) \\
&= \boldsymbol{\nabla} \times (\partial_t \boldsymbol{\xi} \times \boldsymbol{B}) - \boldsymbol{\nabla} \times (\boldsymbol{v_0} \times \boldsymbol{b} + \partial_t \boldsymbol{\xi} \times \boldsymbol{B} - S\xi_x \hat{\boldsymbol{y}} \times \boldsymbol{B}) \\
&= -\boldsymbol{\nabla} \times (\boldsymbol{v_0} \times \boldsymbol{b} - S\xi_x \hat{\boldsymbol{y}} \times \boldsymbol{B}) \\
&= -\boldsymbol{\nabla} \times (Sx\hat{\boldsymbol{y}} \times (\boldsymbol{\nabla} \times (\boldsymbol{\xi} \times \boldsymbol{B}))) - S\xi_x \hat{\boldsymbol{y}} \times \boldsymbol{B}.
\end{aligned}
$$

Notice

$$
\begin{aligned}
\boldsymbol{\nabla} \times (\boldsymbol{\xi} \times \boldsymbol{B}) &= \boldsymbol{\xi}(\boldsymbol{\nabla} \cdot \boldsymbol{B}) - \boldsymbol{B}(\boldsymbol{\nabla} \cdot \boldsymbol{\xi}) + (\boldsymbol{B} \cdot \boldsymbol{\nabla})\boldsymbol{\xi} - (\boldsymbol{\xi} \cdot \boldsymbol{\nabla})\boldsymbol{B} \\
&= -\boldsymbol{B}(\boldsymbol{\nabla} \cdot \boldsymbol{\xi}) + (\boldsymbol{B} \cdot \boldsymbol{\nabla})\boldsymbol{\xi} \\
&= B\partial_z \boldsymbol{\xi}
\end{aligned}
$$

implies

$$
IND = -SB\boldsymbol{\nabla} \times (x\hat{\boldsymbol{y}} \times \partial_z \boldsymbol{\xi} - \xi_x \hat{\boldsymbol{y}} \times \hat{\boldsymbol{z}}).
$$

However,

$$
\begin{aligned}
\boldsymbol{\nabla} \times (x\hat{\boldsymbol{y}} \times \partial_z \boldsymbol{\xi}) &= x\hat{\boldsymbol{y}}(\boldsymbol{\nabla} \cdot \partial_z \boldsymbol{\xi}) - \partial_z \boldsymbol{\xi}(\boldsymbol{\nabla} \cdot x\hat{\boldsymbol{y}}) + (\partial_z \boldsymbol{\xi} \cdot \boldsymbol{\nabla})x\hat{\boldsymbol{y}} - (x\hat{\boldsymbol{y}} \cdot \boldsymbol{\nabla})\partial_z \boldsymbol{\xi} \\
&= \partial_z \xi_x \hat{\boldsymbol{y}}
\end{aligned}
$$

and

$$
\begin{aligned}
\boldsymbol{\nabla} \times (\xi_x \hat{\boldsymbol{y}} \times \hat{\boldsymbol{z}}) &= \xi_x \hat{\boldsymbol{y}}(\boldsymbol{\nabla} \cdot \hat{\boldsymbol{z}}) - \hat{\boldsymbol{z}}(\boldsymbol{\nabla} \cdot \xi_x \hat{\boldsymbol{y}}) + (\hat{\boldsymbol{z}} \cdot \boldsymbol{\nabla})\xi_x \hat{\boldsymbol{y}} - (\xi_x \hat{\boldsymbol{y}} \cdot \boldsymbol{\nabla})\hat{\boldsymbol{z}} \\
&= \partial_z \xi_x \hat{\boldsymbol{y}}.
\end{aligned}
$$

Therefore, we automatically satisfy $IND = 0$.

With the induction equation satisfied, we turn our attention to the momentum equation. Combining the displacement relation with the momentum equation gives the second-order boundary value problem

$$
0 = -\xi_x'' + k_z^2 \left(1 + \frac{\Omega(-B^2 k_z^2 S + \gamma^2(\Omega - S))}{(B^2 k_z^2 + \gamma^2)^2} \right) \xi_x.
$$

The growth rate $\gamma \to 0$ for marginal-stability

$$
0 = -\xi_x'' + \left(k_z^2 - \frac{S\Omega}{B^2} \right) \xi_x.
$$

Thus, for finite k_z, we obtain a marginally-stable mode when

$$
\sqrt{\frac{S\Omega}{B^2} - k_z^2} = \frac{d}{\pi}.
$$

We define the potential

$$V \equiv -\frac{S\Omega}{B^2}.$$

Next we multiply the x-displacement equation by ξ_x^* and integrate over the domain, yielding a functional

$$\mathcal{F} \equiv \int_0^d |\xi_x'|^2 + k_z^2 |\xi_x|^2 + V|\xi_x|^2 dx.$$

The displacement $\xi_x = A\sin(\pi x/d)$

$$\mathcal{F} = A^2 \left(\frac{\pi^2}{2d} + k_z^2 \frac{d}{2} + V\frac{d}{2} \right).$$

Saturation occurs when

$$\mathcal{F} = -\delta\mathcal{F}$$

where

$$\delta\mathcal{F} = \int_0^d \delta V |\xi_x|^2 dx.$$

The potential varies due to changes in the background magnetic field and shear parameter, i.e.

$$\delta V = -\Omega B^{-2}\delta S + 2S\Omega B^{-3}\delta B$$

0.3   Eigenvalue Problem

Substitution yields

$$\partial_t b + J(\psi, b) = J(a, v) + B\partial_z v - S\partial_z a,$$
$$\partial_t a + J(\psi, a) = B\partial_z \psi,$$
$$\partial_t v + J(\psi, v) - (f + S)\partial_z \psi = J(a, b) + B\partial_z b,$$
$$\partial_t \nabla^2 \psi + J(\psi, \nabla^2 \psi) + f\partial_z v = J(a, \nabla^2 a) + B\partial_z \nabla^2 a$$

where the Jacobian

$$J(p, q) \equiv \partial_x p \partial_z q - \partial_z p \partial_x q$$

We denote linear perturbations with apostrophes, e.g. $v(x, z, t) = v_0(x) + v'(x, z, t)$. We discard the Jacobian terms to obtain the linearized system

$$\partial_t b' = B\partial_z v' - S\partial_z a',$$
$$\partial_t a' = B\partial_z \psi',$$
$$\partial_t v' - (f + S)\partial_z \psi' = B\partial_z b',$$
$$\partial_t \nabla^2 \psi' + f\partial_z v' = B\partial_z \nabla^2 a'.$$

We isolate the nonlinear feedback by taking the z-average of the nonlinear equations. For velocity, we rewrite the y-component equation as

$$\partial_t \langle v_y \rangle = \langle \boldsymbol{b} \cdot \boldsymbol{\nabla} b_y - \boldsymbol{v} \cdot \boldsymbol{\nabla} v_y \rangle$$
$$= \langle \boldsymbol{\nabla} \cdot (b_y \boldsymbol{b} - v_y \boldsymbol{v}) \rangle$$
$$= \partial_x \langle -b_y \partial_z a + v_y \partial_z \psi \rangle.$$

For the induction equation, we use the scalar potential

$$\partial_t \langle a \rangle = - \langle J(\psi, a) \rangle$$
$$= \langle \partial_z \psi \partial_x a - \partial_x \psi \partial_z a \rangle$$
$$= - \langle u_x \partial_x a + u_z \partial_z a \rangle$$
$$= - \langle u_x \partial_x a - a\partial_z u_z \rangle.$$

The velocity has no divergence and no y-dependence, meaning $\partial_x u_x + \partial_z u_z = 0$ such that

$$= - \langle u_x \partial_x a + a\partial_x u_x \rangle$$
$$= -\partial_x \langle a u_x \rangle$$
$$= \partial_x \langle a\partial_z \psi \rangle$$

The background state is characterized by the shear parameter S as well as the vertical magnetic field B. We eliminate the time derivatives by manipulating the linear equations. For the velocity, we have

$$\partial_t \langle v_y \rangle = \partial_x \langle -b_y \partial_z a + B^{-1} v_y \partial_t a \rangle$$
$$= \partial_x \langle a\partial_z b_y + B^{-1} v_y \partial_t a \rangle$$
$$= B^{-1}\partial_x \langle a\partial_t v_y - (f + S)a\partial_z \psi + v_y \partial_t a \rangle$$
$$= B^{-1}\partial_x \left\langle a\partial_t v_y - \frac{f + S}{B}a\partial_t a + v_y \partial_t a \right\rangle$$
$$= B^{-1}\partial_x \left\langle \partial_t (av_y) - \frac{f + S}{2B}\partial_t \left(a^2\right) \right\rangle$$
$$= \partial_t \left(B^{-1}\partial_x \left\langle av_y - \frac{f + S}{2B}a^2 \right\rangle \right)$$
$$\therefore \langle v_y \rangle = Sx + \partial_x \left\langle \frac{1}{B}av_y - \frac{f + S}{2B^2}a^2 \right\rangle$$

For the magnetic potential, we have

$$\partial_t \langle a \rangle = B^{-1} \partial_x \langle a \partial_t a \rangle$$
$$= \partial_t \left(\frac{1}{2B} \partial_x \langle a^2 \rangle \right)$$
$$\therefore \quad \langle a \rangle = Bx + \frac{1}{2B} \partial_x \langle a^2 \rangle$$

0.4 Amplitude Estimate

The feedback terms from the previous subsection allow us to compute the variations of the background quantities

$$\delta S(x) = \partial_x^2 \left\langle \frac{1}{B} a v_y - \frac{f+S}{2B^2} a^2 \right\rangle$$
$$\delta B(x) = \frac{1}{2B} \partial_x^2 \langle a^2 \rangle$$

Once the x-displacement ξ_x is known, we can compute the other eigenfunctions

$$a = -B\xi_x$$
$$v_y = \frac{\left(\gamma^2 k_z^{-1} - B^2 k_z \right) \left(k_z^2 - \pi^2 d^{-2} \right)}{2\Omega k_z} \xi_x = \kappa \xi_x.$$

Therefore

$$\delta S(x) = \partial_x^2 \left\langle \frac{1}{B} a v_y - \frac{2\Omega + S}{2B^2} a^2 \right\rangle$$
$$\delta B(x) = \frac{1}{2B} \partial_x^2 \langle a^2 \rangle$$

Recall,

$$\delta V = -\Omega B^{-2} \delta S + 2S\Omega B^{-3} \delta B$$
$$= -\Omega B^{-2} \partial_x^2 \left\langle \frac{1}{B} a v_y - \frac{2\Omega + S}{2B^2} a^2 \right\rangle + S\Omega B^{-4} \partial_x^2 \langle a^2 \rangle.$$

Then, writing everything in terms of the x-displacement

$$= \Omega B^{-2} \partial_x^2 \left\langle \kappa \xi_x^2 + \frac{2\Omega + S}{2} \xi_x^2 \right\rangle + S\Omega B^{-2} \partial_x^2 \langle \xi_x^2 \rangle$$
$$= \Omega B^{-2} \left(\kappa + \Omega + \frac{3S}{2} \right) \partial_x^2 \langle \xi_x^2 \rangle$$

implies

$$\delta\mathcal{F} = \int_0^d |\xi_x|^2 (\Omega B^{-2} \left(\kappa + \Omega + \frac{3S}{2} \right) \partial_x^2 \langle \xi_x^2 \rangle) dx$$
$$= A^4 \frac{3\pi^2}{8d} \Omega B^{-2} \left(\kappa + \Omega + \frac{3S}{2} \right).$$

Therefore, the overall amplitude estimate

$$A = \sqrt{\frac{\left(\frac{\pi^2}{2d} + k_z^2 \frac{d}{2} + V \frac{d}{2} \right)}{\frac{3\pi^2}{8d} \Omega B^{-2} \left(\kappa + \Omega + \frac{3S}{2} \right)}}$$