0.1 Problem Setup

Consider the non-dissipative incompressible MHD equations in a rotating frame

$$\begin{aligned}\partial_t \boldsymbol{v} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} + f\hat{\boldsymbol{z}}\times\boldsymbol{v} + \boldsymbol{\nabla}p &= \boldsymbol{b}\cdot\boldsymbol{\nabla}\boldsymbol{b}\\ \boldsymbol{\nabla}\cdot\boldsymbol{v} &= 0\\ \partial_t \boldsymbol{b} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{b} &= \boldsymbol{b}\cdot\boldsymbol{\nabla}\boldsymbol{v}\\ \boldsymbol{\nabla}\cdot\boldsymbol{b} &= 0\end{aligned}$$

With axisymmetry (constant in y) we consider linear perturbations (with x and z dependence of the form)

$$\begin{aligned}\boldsymbol{v} &= [v_0(x) + v(x,z)]\hat{\boldsymbol{y}} - \hat{\boldsymbol{y}}\times\boldsymbol{\nabla}\psi(x,z)\\ \boldsymbol{b} &= b(x,z)\hat{\boldsymbol{y}} - \hat{\boldsymbol{y}}\times\boldsymbol{\nabla}\left(a_0(x) + a(x,z)\right)\end{aligned}$$

such that

$$\begin{aligned}v_0(x) &= Sx - \frac{Sd}{2}\\ a_0(x) &= Bx - \frac{Bd}{2}\end{aligned}$$

with $0 < x < d$. S is the shear rate of the background azimuthal flow (which decreases in x). B is the constant poloidal background magnetic field (in the z direction).

0.2 Displacement Equation

The Eulerian displacement vector satisfies

$$\begin{aligned}\boldsymbol{v} &= \partial_t\boldsymbol{\xi} + v_0(x)\hat{\boldsymbol{y}}\cdot\boldsymbol{\nabla}\boldsymbol{\xi} - \boldsymbol{\xi}\cdot\boldsymbol{\nabla}(v_0(x)\hat{\boldsymbol{y}})\\ &= \partial_t\boldsymbol{\xi} - S\xi_x\hat{\boldsymbol{y}}\end{aligned}$$

with

$$\boldsymbol{b} = \boldsymbol{\nabla}\times(\boldsymbol{\xi}\times\boldsymbol{B}).$$

Substituting these expressions into the linearized induction equation gives

$$\begin{aligned} IND &= \partial_t \boldsymbol{b} - \boldsymbol{\nabla} \times (\boldsymbol{v_0} \times \boldsymbol{b} + \boldsymbol{v} \times \boldsymbol{B}) \\ &= \boldsymbol{\nabla} \times (\partial_t \boldsymbol{\xi} \times \boldsymbol{B}) - \boldsymbol{\nabla} \times (\boldsymbol{v_0} \times \boldsymbol{b} + \partial_t \boldsymbol{\xi} \times \boldsymbol{B} - S\xi_x \hat{\boldsymbol{y}} \times \boldsymbol{B}) \\ &= -\boldsymbol{\nabla} \times (\boldsymbol{v_0} \times \boldsymbol{b} - S\xi_x \hat{\boldsymbol{y}} \times \boldsymbol{B}) \\ &= -\boldsymbol{\nabla} \times (Sx\hat{\boldsymbol{y}} \times (\boldsymbol{\nabla} \times (\boldsymbol{\xi} \times \boldsymbol{B})) - S\xi_x \hat{\boldsymbol{y}} \times \boldsymbol{B}). \end{aligned}$$

Notice

$$\begin{aligned} \boldsymbol{\nabla} \times (\boldsymbol{\xi} \times \boldsymbol{B}) &= \boldsymbol{\xi}(\boldsymbol{\nabla} \cdot \boldsymbol{B}) - \boldsymbol{B}(\boldsymbol{\nabla} \cdot \boldsymbol{\xi}) + (\boldsymbol{B} \cdot \boldsymbol{\nabla})\boldsymbol{\xi} - (\boldsymbol{\xi} \cdot \boldsymbol{\nabla})\boldsymbol{B} \\ &= -\boldsymbol{B}(\boldsymbol{\nabla} \cdot \boldsymbol{\xi}) + (\boldsymbol{B} \cdot \boldsymbol{\nabla})\boldsymbol{\xi} \\ &= B\partial_z \boldsymbol{\xi} \end{aligned}$$

implies

$$IND = -SB\boldsymbol{\nabla} \times (x\hat{\boldsymbol{y}} \times \partial_z \boldsymbol{\xi} - \xi_x \hat{\boldsymbol{y}} \times \hat{\boldsymbol{z}}).$$

However,

$$\begin{aligned} \boldsymbol{\nabla} \times (x\hat{\boldsymbol{y}} \times \partial_z \boldsymbol{\xi}) &= x\hat{\boldsymbol{y}}(\boldsymbol{\nabla} \cdot \partial_z \boldsymbol{\xi}) - \partial_z \boldsymbol{\xi}(\boldsymbol{\nabla} \cdot x\hat{\boldsymbol{y}}) + (\partial_z \boldsymbol{\xi} \cdot \boldsymbol{\nabla})x\hat{\boldsymbol{y}} - (x\hat{\boldsymbol{y}} \cdot \boldsymbol{\nabla})\partial_z \boldsymbol{\xi} \\ &= \partial_z \xi_x \hat{\boldsymbol{y}} \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{\nabla} \times (\xi_x \hat{\boldsymbol{y}} \times \hat{\boldsymbol{z}}) &= \xi_x \hat{\boldsymbol{y}}(\boldsymbol{\nabla} \cdot \hat{\boldsymbol{z}}) - \hat{\boldsymbol{z}}(\boldsymbol{\nabla} \cdot \xi_x \hat{\boldsymbol{y}}) + (\hat{\boldsymbol{z}} \cdot \boldsymbol{\nabla})\xi_x \hat{\boldsymbol{y}} - (\xi_x \hat{\boldsymbol{y}} \cdot \boldsymbol{\nabla})\hat{\boldsymbol{z}} \\ &= \partial_z \xi_x \hat{\boldsymbol{y}}. \end{aligned}$$

Therefore, we automatically satisfy $IND = 0$.

With the induction equation satisfied, we turn our attention to the momentum equation. Combining the displacement relation with the momentum equation gives the second-order boundary value problem

$$0 = -\xi_x'' + k_z^2 \left(1 + \frac{\Omega(-B^2 k_z^2 S + \gamma^2(\Omega - S))}{(B^2 k_z^2 + \gamma^2)^2}\right) \xi_x.$$

The growth rate $\gamma \to 0$ for marginal-stability

$$0 = -\xi_x'' + \left(k_z^2 - \frac{S\Omega}{B^2}\right) \xi_x.$$

Thus, for finite k_z, we obtain a marginally-stable mode when

$$\sqrt{\frac{S\Omega}{B^2} - k_z^2} = \frac{d}{\pi}.$$

We define the potential

$$V \equiv -\frac{S\Omega}{B^2}.$$

Next we multiply the x-displacement equation by ξ_x^* and integrate over the domain, yielding a functional

$$\mathcal{F} \equiv \int_0^d |\xi_x'|^2 + k_z^2|\xi_x|^2 + V|\xi_x|^2 dx.$$

The displacement $\xi_x = A\sin(\pi x/d)$

$$\mathcal{F} = A^2\left(\frac{\pi^2}{2d} + k_z^2\frac{d}{2} + V\frac{d}{2}\right).$$

Saturation occurs when

$$\mathcal{F} = -\delta\mathcal{F}$$

where

$$\delta\mathcal{F} = \int_0^d \delta V|\xi_x|^2 dx.$$

The potential varies due to changes in the background magnetic field and shear parameter, i.e.

$$\delta V = -\Omega B^{-2}\delta S + 2S\Omega B^{-3}\delta B$$

0.3 Eigenvalue Problem

Substitution yields

$$\begin{aligned}
\partial_t b + J(\psi, b) &= J(a, v) + B\partial_z v - S\partial_z a,\\
\partial_t a + J(\psi, a) &= B\partial_z\psi,\\
\partial_t v + J(\psi, v) - (f+S)\partial_z\psi &= J(a, b) + B\partial_z b,\\
\partial_t\nabla^2\psi + J(\psi, \nabla^2\psi) + f\partial_z v &= J(a, \nabla^2 a) + B\partial_z\nabla^2 a
\end{aligned}$$

where the Jacobian

$$J(p, q) \equiv \partial_x p\partial_z q - \partial_z p\partial_x q$$

We denote linear perturbations with apostrophes, e.g. $v(x,z,t) = v_0(x) + v'(x,z,t)$. We discard the Jacobian terms to obtain the linearized system

$$\begin{aligned}
\partial_t b' &= B\partial_z v' - S\partial_z a', \\
\partial_t a' &= B\partial_z \psi', \\
\partial_t v' - (f+S)\partial_z \psi' &= B\partial_z b', \\
\partial_t \nabla^2 \psi' + f\partial_z v' &= B\partial_z \nabla^2 a'.
\end{aligned}$$

We isolate the nonlinear feedback by taking the z-average of the nonlinear equations. For velocity, we rewrite the y-component equation as

$$\begin{aligned}
\partial_t \langle v_y \rangle &= \langle \boldsymbol{b} \cdot \boldsymbol{\nabla} b_y - \boldsymbol{v} \cdot \boldsymbol{\nabla} v_y \rangle \\
&= \langle \boldsymbol{\nabla} \cdot (b_y \boldsymbol{b} - v_y \boldsymbol{v}) \rangle \\
&= \partial_x \langle -b_y \partial_z a + v_y \partial_z \psi \rangle .
\end{aligned}$$

For the induction equation, we use the scalar potential

$$\begin{aligned}
\partial_t \langle a \rangle &= - \langle J(\psi, a) \rangle \\
&= \langle \partial_z \psi \partial_x a - \partial_x \psi \partial_z a \rangle \\
&= - \langle u_x \partial_x a + u_z \partial_z a \rangle \\
&= - \langle u_x \partial_x a - a \partial_z u_z \rangle .
\end{aligned}$$

The velocity has no divergence and no y-dependence, meaning $\partial_x u_x + \partial_z u_z = 0$ such that

$$\begin{aligned}
&= - \langle u_x \partial_x a + a \partial_x u_x \rangle \\
&= -\partial_x \langle a u_x \rangle \\
&= \partial_x \langle a \partial_z \psi \rangle
\end{aligned}$$

The background state is characterized by the shear parameter S as well as the vertical magnetic field B. We eliminate the time derivatives by manipulating the linear equations. For the velocity, we have

$$\begin{aligned}
\partial_t \langle v_y \rangle &= \partial_x \left\langle -b_y \partial_z a + B^{-1} v_y \partial_t a \right\rangle \\
&= \partial_x \left\langle a \partial_z b_y + B^{-1} v_y \partial_t a \right\rangle \\
&= B^{-1} \partial_x \langle a \partial_t v_y - (f+S) a \partial_z \psi + v_y \partial_t a \rangle \\
&= B^{-1} \partial_x \left\langle a \partial_t v_y - \frac{f+S}{B} a \partial_t a + v_y \partial_t a \right\rangle \\
&= B^{-1} \partial_x \left\langle \partial_t (a v_y) - \frac{f+S}{2B} \partial_t \left(a^2\right) \right\rangle \\
&= \partial_t \left(B^{-1} \partial_x \left\langle a v_y - \frac{f+S}{2B} a^2 \right\rangle \right) \\
\therefore \ \langle v_y \rangle &= Sx + \partial_x \left\langle \frac{1}{B} a v_y - \frac{f+S}{2B^2} a^2 \right\rangle
\end{aligned}$$

For the magnetic potential, we have

$$
\begin{aligned}
\partial_t \langle a \rangle &= B^{-1} \partial_x \langle a \partial_t a \rangle \\
&= \partial_t \left(\frac{1}{2B} \partial_x \langle a^2 \rangle \right) \\
\therefore \ \langle a \rangle &= Bx + \frac{1}{2B} \partial_x \langle a^2 \rangle
\end{aligned}
$$

0.4 Amplitude Estimate

The feedback terms from the previous subsection allow us to compute the variations of the background quantities

$$
\begin{aligned}
\delta S(x) &= \partial_x^2 \left\langle \frac{1}{B} a v_y - \frac{f+S}{2B^2} a^2 \right\rangle \\
\delta B(x) &= \frac{1}{2B} \partial_x^2 \langle a^2 \rangle
\end{aligned}
$$

Once the x-displacement ξ_x is known, we can compute the other eigenfunctions

$$
\begin{aligned}
a &= -B\xi_x \\
v_y &= \frac{\left(\gamma^2 k_z^{-1} - B^2 k_z\right)\left(k_z^2 - \pi^2 d^{-2}\right)}{2\Omega k_z} \xi_x = \kappa \xi_x .
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\delta S(x) &= \partial_x^2 \left\langle \frac{1}{B} a v_y - \frac{2\Omega+S}{2B^2} a^2 \right\rangle \\
\delta B(x) &= \frac{1}{2B} \partial_x^2 \langle a^2 \rangle
\end{aligned}
$$

Recall,

$$
\begin{aligned}
\delta V &= -\Omega B^{-2} \delta S + 2S\Omega B^{-3} \delta B \\
&= -\Omega B^{-2} \partial_x^2 \left\langle \frac{1}{B} a v_y - \frac{2\Omega+S}{2B^2} a^2 \right\rangle + S\Omega B^{-4} \partial_x^2 \langle a^2 \rangle .
\end{aligned}
$$

Then, writing everything in terms of the x-displacement

$$
\begin{aligned}
&= \Omega B^{-2} \partial_x^2 \left\langle \kappa \xi_x^2 + \frac{2\Omega+S}{2} \xi_x^2 \right\rangle + S\Omega B^{-2} \partial_x^2 \langle \xi_x^2 \rangle \\
&= \Omega B^{-2} \left(\kappa + \Omega + \frac{3S}{2} \right) \partial_x^2 \langle \xi_x^2 \rangle
\end{aligned}
$$

implies

$$\begin{aligned}\delta\mathcal{F} &= \int_0^d |\xi_x|^2(\Omega B^{-2}\left(\kappa+\Omega+\frac{3S}{2}\right)\partial_x^2\left\langle\xi_x^2\right\rangle)dx \\ &= A^4\frac{3\pi^2}{8d}\Omega B^{-2}\left(\kappa+\Omega+\frac{3S}{2}\right).\end{aligned}$$

Therefore, the overall amplitude estimate

$$A = \sqrt{\frac{\left(\frac{\pi^2}{2d}+k_z^2\frac{d}{2}+V\frac{d}{2}\right)}{\frac{3\pi^2}{8d}\Omega B^{-2}\left(\kappa+\Omega+\frac{3S}{2}\right)}}$$

0.5 Cylindrical Geometry

First we will implement the vector-potential formulation. Let the magnetic field $\boldsymbol{b} = b_r\hat{\boldsymbol{r}} + b_\theta\hat{\boldsymbol{\theta}} + b_z\hat{\boldsymbol{z}}$ be the curl of some divergence-free vector potential $\boldsymbol{A} = A_r\hat{\boldsymbol{r}} + A_\theta\hat{\boldsymbol{\theta}} + A_z\hat{\boldsymbol{z}}$, such that

$$\begin{aligned}\boldsymbol{b} &= \boldsymbol{\nabla}\times\boldsymbol{A} \\ b_r &= r^{-1}\partial_\theta A_z - \partial_z A_\theta \\ b_\theta &= \partial_z A_r - \partial_r A_z \\ b_z &= \partial_z A_\theta - r^{-1}\partial_\theta A_r + r^{-1}A_\theta\end{aligned}$$

Assuming we have axisymmetry ($\partial_\theta \to 0$)

$$\begin{aligned}b_r &= -\partial_z A_\theta \\ b_\theta &= \partial_z A_r - \partial_r A_z \\ b_z &= \partial_z A_\theta + r^{-1}A_\theta\end{aligned}$$

Thus, at $t = 0$, we set

$$A_\theta = rb_z|_{t=0} = rB_1\frac{1+4(r/r_1)^5}{5(r/r_1)^3}$$

For $B_1 = 1$ and $r_1 = 1$,

$$A_\theta = \frac{1+4r^5}{5r^2}$$

0.5.1 Scalar Potential

First, define the velocity and magnetic flux as follows

$$\begin{aligned}\boldsymbol{u} &= -\partial_z\psi\hat{\boldsymbol{r}} + v\hat{\boldsymbol{\theta}} + \partial_r\psi\hat{\boldsymbol{z}} \\ \boldsymbol{b} &= -\partial_z a\hat{\boldsymbol{r}} + b\hat{\boldsymbol{\theta}} + \partial_r a\hat{\boldsymbol{z}}.\end{aligned}$$

We define the following scalar operators:

$$
\begin{aligned}
J[P,Q] &\equiv \partial_r P \partial_z Q - r^{-1}\partial_z P \partial_r (rQ) \\
J_n[P,Q] &\equiv \partial_r P \partial_z Q - \partial_z P \partial_r Q \\
\boldsymbol{\nabla}_h^2[A] &\equiv \partial_r^2 A + \partial_z^2 A \\
\boldsymbol{\nabla}^2[A] &\equiv \partial_r^2 A + r^{-1}\partial_r A + \partial_z^2 A \\
\boldsymbol{\nabla}_r^2[A] &\equiv \partial_r^2 A + r^{-1}\partial_r A - r^{-2} A + \partial_z^2 A.
\end{aligned}
$$

The scalar-potential/streamfunction form is given by

$$
\begin{aligned}
\partial_t b - \eta \boldsymbol{\nabla}_r^2 b &= -J_n[\psi, b] + J_n[a, v] \\
\partial_t a - \eta \boldsymbol{\nabla}^2 a &= -J_n[\psi, a] \\
\partial_t v - \nu \boldsymbol{\nabla}_r^2 v &= -J[\psi, v] + J[a, b] \\
\partial_t \boldsymbol{\nabla}_h^2 \psi - \nu \boldsymbol{\nabla}^2 \boldsymbol{\nabla}_h^2 \psi + \nu r^{-2} \boldsymbol{\nabla}_h^2 \psi &= J_n[\boldsymbol{\nabla}_h^2 \psi, \psi] - J_n[\boldsymbol{\nabla}_h^2 a, a] + 2r^{-1}\left(b \partial_z b - v \partial_z v\right)
\end{aligned}
$$