Boundary Integral Methods
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Project #2
Singular Volterra Integral Equation

The Volterra integral equation to be solved is:

$$x(s) = \frac{1}{(1+s)^{1/2}} + \frac{\pi}{8} - \frac{1}{4}\sin^{-1}\left(\frac{1-s}{1+s}\right) - \frac{1}{4}\int_0^s \frac{x(t)}{(s-t)^{1/2}}dt \tag{1}$$

We will solve it by defining a quadrature rule for the singular integral. Here, $0 \leq s \leq 1$. Also given that the exact solution is $x_e(s) = 1/(1+s)^{1/2}$.

1 Determining the Quadrature Rule

We first find a quadrature rule of the following form to deal with the singular integral,

$$\boxed{\int_0^{s_i} (s_i - t)^{-1/2} x(t)dt = \sum_{j=0}^{i} w_{ij} x(t_j)} \tag{2}$$

We consider the quadrature points $s_i = t_i = ih$, $i = 0, 1, 2..., N$. The constant step size is taken to be $h = \frac{1}{N}$. So, we have

$$I_i = \int_0^{s_i} (s_i - t)^{-1/2} x(t)dt = \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} (s_i - t)^{-1/2} x(t)dt \tag{3}$$

For the rule to be exact when $x(t)$ is a first degree polynomial, we linearly interpolate $x(t)$ in the interval $[t_j, t_{j+1}]$. Thus, we have

$$x(t) \approx \left(\frac{t_{j+1} - t}{h}\right) x(t_j) + \left(\frac{t - t_j}{h}\right) x(t_{j+1}) + \mathcal{O}(h^2) \tag{4}$$

Using (4) in (3), we get,

$$I_i = \frac{1}{h}\sum_{j=0}^{i-1}\left(x(t_j)\int_{t_j}^{t_{j+1}} (s_i - t)^{-1/2}(t_{j+1} - t)dt + x(t_{j+1})\int_{t_j}^{t_{j+1}} (s_i - t)^{-1/2}(t - t_j)dt\right)$$

$$\Rightarrow \sum_{j=0}^{i} w_{ij} x(t_j) = \frac{1}{h}\sum_{j=0}^{i-1}\left(x(t_j)\int_{t_j}^{t_{j+1}} \frac{(t_{j+1} - t)}{(s_i - t)^{1/2}}dt + x(t_{j+1})\int_{t_j}^{t_{j+1}} \frac{(t - t_j)}{(s_i - t)^{1/2}}dt\right)$$

Comparing the coefficients of $x(t_j)$ for each j, we get

$$w_{i0} = \frac{1}{h}\int_0^{t_1}(t_1 - t)(s_i - t)^{-1/2}dt \tag{5}$$

$$w_{ii} = \frac{1}{h}\int_{t_{i-1}}^{t_i}(t - t_{i-1})(s_i - t)^{-1/2}dt \tag{6}$$

$$w_{ij} = \frac{1}{h}\left\{\int_{t_j}^{t_{j+1}}(t_{j+1} - t)(s_i - t)^{-1/2}dt + \int_{t_{j-1}}^{t_j}(t - t_{j-1})(s_i - t)^{-1/2}dt\right\} \tag{7}$$

Each of the above weights can be found analytically, using the following two integrals:

$$I = \int \frac{1}{\sqrt{a-t}}dt = -2\sqrt{a-t} \tag{8}$$

$$J = \int \frac{t}{\sqrt{a-t}}dt = \frac{-2}{3}(2a+t)\sqrt{a-t} \tag{9}$$

Replacing s_i and t_i by ih in equations (5), (6) and (7), we get the weights as

$$w_{i0} = \frac{1}{h}\int_0^{h}(h - t)(ih - t)^{-1/2}dt$$

$$w_{ii} = \frac{1}{h}\int_{(i-1)h}^{ih}(t - (i-1)h)(ih - t)^{-1/2}dt$$

$$w_{ij} = \frac{1}{h}\left\{\int_{jh}^{(j+1)h}((j+1)h - t)(ih - t)^{-1/2}dt + \int_{(j-1)h}^{jh}(t - (j-1)h)(ih - t)^{-1/2}dt\right\}$$

Taking $a = s_i = ih$ in equations (8) and (9), we can write

$$w_{i0} = \frac{1}{h}\left[hI - J\right]_0^h$$

$$w_{ii} = \frac{1}{h}\left[J - (i-1)hI\right]_{(i-1)h}^{ih}$$

$$w_{ij} = \frac{1}{h}\left\{\left[(j+1)hI - J\right]_{jh}^{(j+1)h} + \left[J - (j-1)hI\right]_{(j-1)h}^{jh}\right\}$$

Simplifying these, we can finally write

$$\boxed{w_{i0} = \frac{2\sqrt{h}}{3}\Big(2(i-1)^{3/2} - \sqrt{i}(2i-3)\Big)} \tag{10}$$

$$\boxed{w_{ii} = \frac{4\sqrt{h}}{3}} \tag{11}$$

$$\boxed{w_{ij} = \frac{4\sqrt{h}}{3}\left((i-j-1)^{3/2} - 2(i-j)^{3/2} + (i-j+1)^{3/2}\right)} \tag{12}$$

Using these quadrature weights, and the corresponding quadrature rule, the Volterra Equation (1) was solved numerically, as shown in the next section.

2 Numerical Results

We first let $f(s) = \frac{1}{(1+s)^{1/2}} + \frac{\pi}{8} - \frac{1}{4}\sin^{-1}\left(\frac{1-s}{1+s}\right)$. Rewriting (1) for the quadrature points, we have,

$$x(s_i) = f(s_i) - \frac{1}{4}\int_0^{s_i} \frac{x(t)}{(s_i - t)^{1/2}} dt$$
$$\Rightarrow x(s_i) = f(s_i) - \frac{1}{4}\sum_{j=0}^{i} w_{ij} x(t_j)$$

Since $s_i = t_i$, we can rewrite the last equation as,

$$x(s_i) = f(s_i) - \frac{1}{4}\sum_{j=0}^{i} w_{ij} x(s_j) \tag{13}$$

We define the following quantities,

$$\vec{x} = (x(s_0), x(s_1), ...x(s_N))^t$$
$$\vec{f} = (f(s_0), f(s_1), ...f(s_N))^t$$
$$W = (W(i,j))_{ij} = \begin{cases} w_{(i-1)(j-1)}/4, & 1 \leq j \leq i \leq N+1, \\ 0, & j > i \end{cases}$$

Thus, equation (3) can be written in vector form as follows,

$$\vec{x} = \vec{f} - W\vec{x}$$
$$(I + W)\vec{x} = \vec{f}$$
$$\vec{x} = (I + W)^{-1}\vec{f}$$

The plot acquired after the numerical simulation is given below and the numerical solution agrees with the exact solution,

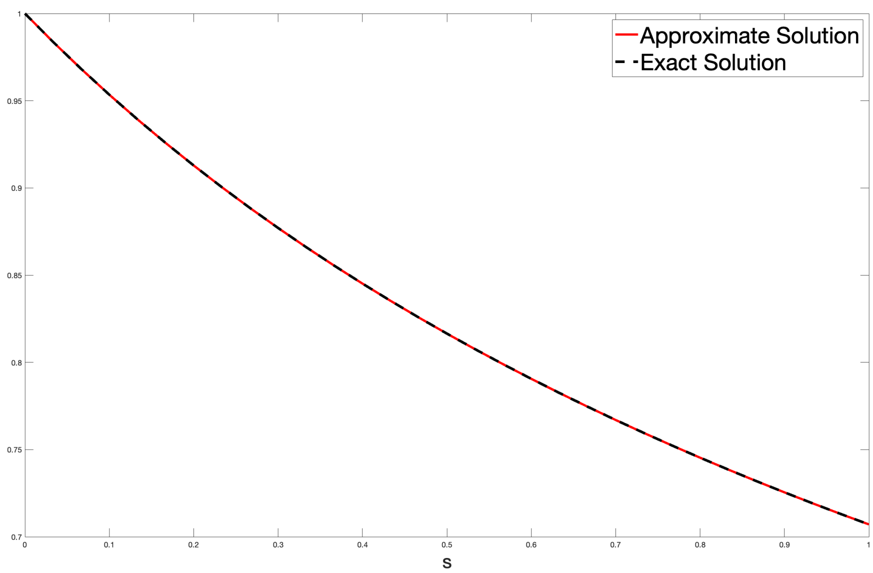

Figure 1: Comparison of Numerical and Exact solutions. The number of sub-intervals was taken to be 30 ($N = 30$).

3 Error Analysis

The quadrature error for our problem is,

$$e = ||x - x_e||_\infty = \max\{|x(s_i) - x_e(s_i)|, 1 \leq i \leq N\}$$

Given below is the table of recorded errors for different values of N,

N	10	20	40	80	160
$\lvert\lvert x - x_e \rvert\rvert_\infty$	0.8282×10^{-4}	0.2112×10^{-4}	0.0536×10^{-4}	0.0135×10^{-4}	0.0034×10^{-4}

Let the error and mesh size relation be given by, $e_h \propto h^p = N^{-p}$. From the above table, we make the following observation,

$$\frac{e_{2h}}{e_h} \approx 4$$
$$\Rightarrow 2^p = 4$$
$$\Rightarrow p = 2$$

Thus, the order of convergence is 2. This can also be observed from the following log − log plot.

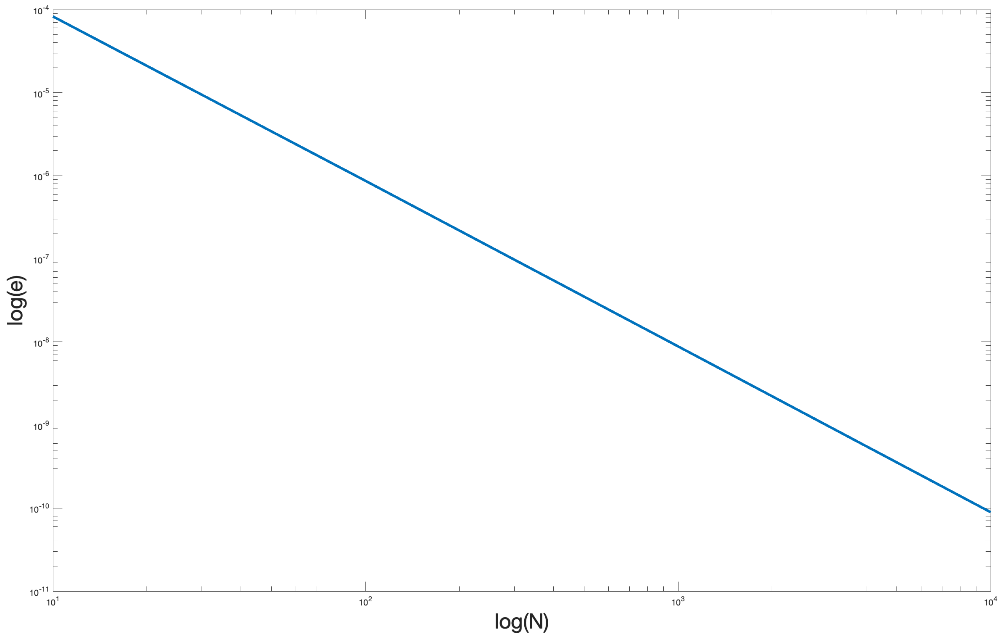

Figure 2: Four values of N were considered, $N = 10, 100, 1000, 10000$. The observed slope was -2.

Since the observed slope was -2, we can conclude from here that $\log(e) \propto -2\log(N)$, i.e. $e \propto N^{-2} = h^2$.
Let $e_h = c/N^2$. Then, by observation we had for $N = 2$, $e \sim 0.0022$. So,

$$0.0022 = \frac{c}{4}$$
$$\Rightarrow c = 0.0088$$

Now, for 6-decimal place accuracy, we consider

$$\begin{aligned} e_N &< 10^{-7} \\ \Rightarrow \frac{0.0088}{N^2} &< 10^{-6} \\ \Rightarrow N^2 &> 8.8 \times 10^3 \\ \Rightarrow N &> 93.8 \end{aligned}$$

Indeed, for $\boxed{N = 94}$ we get an error of order 10^{-7} and for $N = 93$ we get an error of order 10^{-6}. So to achieve 6 decimal place accuracy, we should take $N \geq 94$.