MRI Notes

Induction Equation: Potential Form

The induction equation is given by

$$\partial_t \boldsymbol{b} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{b}) + \eta \boldsymbol{\nabla}^2 \boldsymbol{b}$$

Using the following identity

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{f} = \boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \boldsymbol{f} - \boldsymbol{\nabla}^2 \boldsymbol{f}$$

and assumming η to be constant, we use $\boldsymbol{\nabla} \cdot \boldsymbol{b} = 0$, giving

$$\partial_t \boldsymbol{b} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{b}) - \eta \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{b}.$$

Then we define a vector potential $\boldsymbol{\nabla} \times \boldsymbol{A} \equiv \boldsymbol{b}$, yielding

$$\partial_t \boldsymbol{\nabla} \times \boldsymbol{A} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{b}) - \boldsymbol{\nabla} \times (\eta \boldsymbol{\nabla} \times \boldsymbol{b})$$
$$\partial_t \boldsymbol{A} = \boldsymbol{u} \times \boldsymbol{b} - \eta \boldsymbol{\nabla} \times \boldsymbol{b} + \boldsymbol{\nabla}\phi$$

where ϕ is a scalar potential arizing from "uncurling" the equation. We must then provide an additional constraint to fix ϕ: the Coulomb gauge $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$. Therefore

$$-\boldsymbol{\nabla} \times \boldsymbol{b} = -\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{A} = \boldsymbol{\nabla}^2 \boldsymbol{A}.$$
$$\partial_t \boldsymbol{A} = \boldsymbol{u} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) - \eta \boldsymbol{\nabla}^2 \boldsymbol{A} + \boldsymbol{\nabla}\phi$$

Next we decompose $\boldsymbol{u} = \boldsymbol{u_0} + \boldsymbol{u'}$ and $\boldsymbol{b} = \boldsymbol{b_0} + \boldsymbol{b'} \quad \rightarrow \quad \boldsymbol{A} = \boldsymbol{A_0} + \boldsymbol{A'}$. We assume the mean quantities $\boldsymbol{u_0}$ and $\boldsymbol{A_0}$ are themselves solutions to the original problem. If we consider only the 0th mode of $\boldsymbol{b}$, i.e. $\boldsymbol{b} \cdot \hat{\boldsymbol{e}}_{\boldsymbol{i}} \sim e^{i0}$ then clearly $\boldsymbol{\nabla}^2 \boldsymbol{b} = \boldsymbol{0}$ and therefore

$$\partial_t \boldsymbol{b'} = \boldsymbol{\nabla} \times (\boldsymbol{u_0} \times \boldsymbol{b'}) + \boldsymbol{\nabla} \times (\boldsymbol{u'} \times \boldsymbol{b_0}) + \boldsymbol{\nabla} \times (\boldsymbol{u'} \times \boldsymbol{b'}).$$

Using another identity

$$\boldsymbol{\nabla} \times (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{A}\boldsymbol{\nabla} \cdot \boldsymbol{B} - \boldsymbol{B}\boldsymbol{\nabla} \cdot \boldsymbol{A} + \boldsymbol{B} \cdot \boldsymbol{\nabla}\boldsymbol{A} - \boldsymbol{A} \cdot \boldsymbol{\nabla}\boldsymbol{B}$$

Let's check that. Using the following identity

$$
\begin{aligned}
\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{b}) &= \boldsymbol{\nabla} \times \big(\boldsymbol{u}(\boldsymbol{\nabla} \cdot \boldsymbol{b}) - \boldsymbol{b}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) + (\boldsymbol{b} \cdot \boldsymbol{\nabla})\boldsymbol{u} - (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{b}\big) \\
&= \boldsymbol{\nabla} \times \big((\boldsymbol{b} \cdot \boldsymbol{\nabla})\boldsymbol{u} - (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{b}\big) \\
&= \boldsymbol{\nabla} \times \left(\begin{bmatrix} \boldsymbol{b} \cdot \boldsymbol{\nabla} u_x \\ \boldsymbol{b} \cdot \boldsymbol{\nabla} u_y \\ \boldsymbol{b} \cdot \boldsymbol{\nabla} u_z \end{bmatrix} - \begin{bmatrix} \boldsymbol{u} \cdot \boldsymbol{\nabla} b_x \\ \boldsymbol{u} \cdot \boldsymbol{\nabla} b_y \\ \boldsymbol{u} \cdot \boldsymbol{\nabla} b_z \end{bmatrix} \right)
\end{aligned}
$$

Taking the x-component gives

$$
\begin{aligned}
\hat{\boldsymbol{x}} \cdot \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{b}) &= \partial_y(\boldsymbol{b} \cdot \boldsymbol{\nabla} u_z - \boldsymbol{u} \cdot \boldsymbol{\nabla} b_z) - \partial_z(\boldsymbol{b} \cdot \boldsymbol{\nabla} u_y - \boldsymbol{u} \cdot \boldsymbol{\nabla} b_y) \\
&= \partial_y \boldsymbol{b} \cdot \boldsymbol{\nabla} u_z + \boldsymbol{b} \cdot \boldsymbol{\nabla} \partial_y u_z - \partial_y \boldsymbol{u} \cdot \boldsymbol{\nabla} b_z - \boldsymbol{u} \cdot \boldsymbol{\nabla} \partial_y b_z \\
&\qquad - \partial_z \boldsymbol{b} \cdot \boldsymbol{\nabla} u_y - \boldsymbol{b} \cdot \boldsymbol{\nabla} \partial_z u_y + \partial_z \boldsymbol{u} \cdot \boldsymbol{\nabla} b_y + \boldsymbol{u} \cdot \boldsymbol{\nabla} \partial_z b_y \\
&= \boldsymbol{b} \cdot \boldsymbol{\nabla} \omega_x - \boldsymbol{u} \cdot \boldsymbol{\nabla} j_x \\
&\qquad + \partial_y \boldsymbol{b} \cdot \boldsymbol{\nabla} u_z - \partial_z \boldsymbol{b} \cdot \boldsymbol{\nabla} u_y + \partial_z \boldsymbol{u} \cdot \boldsymbol{\nabla} b_y - \partial_y \boldsymbol{u} \cdot \boldsymbol{\nabla} b_z
\end{aligned}
$$

Letting $B_0 = S = 0$, the expanded current-density equation is given by

$$
\begin{aligned}
\partial_t j_x &= \eta \nabla^2 j_x \\
&\qquad + \boldsymbol{b} \cdot \boldsymbol{\nabla} \omega_x - \boldsymbol{u} \cdot \boldsymbol{\nabla} j_x \\
&\qquad + \partial_y \boldsymbol{b} \cdot \boldsymbol{\nabla} u_z - \partial_z \boldsymbol{b} \cdot \boldsymbol{\nabla} u_y + \partial_z \boldsymbol{u} \cdot \boldsymbol{\nabla} b_y - \partial_y \boldsymbol{u} \cdot \boldsymbol{\nabla} b_z \\
&= \eta \nabla^2 j_x + \hat{\boldsymbol{x}} \cdot \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{b}).
\end{aligned}
$$

by symmetry we have

$$
\partial_t \boldsymbol{j} = \eta \boldsymbol{\nabla}^2 \boldsymbol{j} + \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{b})
$$

We can "uncurl" this equation, and if we assume the unknown scalar potential-gradient term $\boldsymbol{\nabla}\phi = \boldsymbol{0}$, we have

$$
\partial_t \boldsymbol{b} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{b}) + \eta \boldsymbol{\nabla}^2 \boldsymbol{b}
$$

which is the usual form of the induction equation

Navier-Stokes:

$$\partial_t \boldsymbol{u} + Sx\partial_y \boldsymbol{u} + f\hat{z} \times \boldsymbol{u} + [S\boldsymbol{u} \cdot \hat{x}]\hat{y} + \boldsymbol{\nabla} p - \nu \nabla^2 \boldsymbol{u} - \boldsymbol{B} \cdot \boldsymbol{\nabla} \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{\nabla} \boldsymbol{b} - \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u}$$

Induction:

$$\partial_t \boldsymbol{A} - \boldsymbol{\nabla} \phi - \eta \nabla^2 \boldsymbol{A} - \boldsymbol{u} \times \boldsymbol{B} - Sx\hat{y} \times \boldsymbol{b} = \boldsymbol{u} \times \boldsymbol{b}$$

Incompressibility and Coulomb gauge

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = \boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$$

$\boldsymbol{A}$ definition

$$\boldsymbol{b} = \boldsymbol{\nabla} \times \boldsymbol{A}$$

We define the Lagrangian with multipliers $\boldsymbol{\mu}, \boldsymbol{\Lambda}, \pi, \alpha, \boldsymbol{\beta}$

$$\mathcal{L} \equiv \int_0^T \Big\langle \boldsymbol{\mu} \cdot \Big[\partial_t \boldsymbol{u} + Sx\partial_y \boldsymbol{u} + f\hat{z} \times \boldsymbol{u} + [S\boldsymbol{u} \cdot \hat{x}]\hat{y} + \boldsymbol{\nabla} p - \nu \nabla^2 \boldsymbol{u} - \boldsymbol{B} \cdot \boldsymbol{\nabla} \boldsymbol{b} - \boldsymbol{b} \cdot \boldsymbol{\nabla} \boldsymbol{b} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u}\Big]\Big\rangle dt$$
$$+ \int_0^T \Big\langle \boldsymbol{\Lambda} \cdot \Big[\partial_t \boldsymbol{A} - \boldsymbol{\nabla} \phi - \eta \nabla^2 \boldsymbol{A} - \boldsymbol{u} \times \boldsymbol{B} - Sx\hat{y} \times \boldsymbol{b} - \boldsymbol{u} \times \boldsymbol{b}\Big]\Big\rangle dt$$
$$+ \int_0^T \Big\langle \pi \boldsymbol{\nabla} \cdot \boldsymbol{u} \Big\rangle dt + \int_0^T \Big\langle \alpha \boldsymbol{\nabla} \cdot \boldsymbol{A} \Big\rangle dt + \int_0^T \Big\langle \boldsymbol{\beta} \cdot \Big[\boldsymbol{b} - \boldsymbol{\nabla} \times \boldsymbol{A}\Big]\Big\rangle dt$$

The variation with respect to $\boldsymbol{u}$

$$\frac{\delta \mathcal{L}}{\delta \boldsymbol{u}} \delta \boldsymbol{u} =$$
$$\int_0^T \Big\langle \delta \boldsymbol{u} \cdot \Big[- \partial_t \mu - Sx\partial_y \boldsymbol{\mu} + f\hat{z} \times \boldsymbol{\mu} + [S\boldsymbol{\mu} \cdot \hat{y}]\hat{x} + (\boldsymbol{\nabla} \boldsymbol{u})^T \cdot \boldsymbol{\mu} - \boldsymbol{u} \cdot \boldsymbol{\mu} - \nu \nabla^2 \boldsymbol{\mu} - \boldsymbol{\Lambda} \times \boldsymbol{B} - \boldsymbol{\Lambda} \times \boldsymbol{b} - \boldsymbol{\nabla} \pi \Big]\Big\rangle dt$$