MRI Notes

Momentum Equation: Nonlinear Terms

Navier-Stokes (Verbatim from Jeff Oishi's "MRI prefers" paper):

$$\frac{D\boldsymbol{u'}}{Dt} + f\hat{\boldsymbol{z}} \times \boldsymbol{u'} + Su'_x\hat{\boldsymbol{y}} + \boldsymbol{\nabla}p' + \nu\boldsymbol{\nabla} \times \boldsymbol{\omega}' = B_0\partial_z\boldsymbol{b'}$$

where f is the corriolis parameter, S is the background shearing rate, and $B_0\hat{\boldsymbol{z}}$ is a uniform background magnetic field. The equation is linearized wrt perturbations so the material derivative goes like

$$\begin{aligned}\frac{D}{Dt} &\equiv \partial_t + \overline{\boldsymbol{u}} \cdot \boldsymbol{\nabla} \\ &= \partial_t + Sx\partial_y\end{aligned}$$

due to the background velocity $\overline{\boldsymbol{u}} = Sx\hat{\boldsymbol{y}}$. In the nonlinear case we have

$$= \partial_t + (Sx\hat{\boldsymbol{y}} + \boldsymbol{u'}) \cdot \boldsymbol{\nabla}$$

From inspection and stuff, the irrotational momentum equation goes like

$$\frac{D\boldsymbol{u}}{Dt} + \boldsymbol{\nabla}p + \nu \times \boldsymbol{\omega} = \boldsymbol{b} \cdot \boldsymbol{\nabla}\boldsymbol{b}$$

Next we generalize $\boldsymbol{u} = \boldsymbol{u'} + Sx\hat{\boldsymbol{y}}$ and $\boldsymbol{b} = \boldsymbol{b'} + B_0\hat{\boldsymbol{z}}$, giving

$$\underline{\partial_t\boldsymbol{u'} + \boldsymbol{u'} \cdot \boldsymbol{\nabla}\boldsymbol{u'} + Sx\partial_y\boldsymbol{u'} + Su'_x\hat{\boldsymbol{y}}} + \boldsymbol{\nabla}p + \nu\boldsymbol{\nabla} \times \boldsymbol{\omega} = B_0\partial_z\boldsymbol{b'} + \boldsymbol{b'} \cdot \boldsymbol{\nabla}\boldsymbol{b'}$$

where the material derivative $\frac{D\boldsymbol{u}}{Dt}$ consists of the underlined terms. Note this definition differs from that of the associated script

Induction Equation: Nonlinear Terms

The MHD induction equation (*Fluid Mechanics of Planets and Stars, 2019*) is given by

$$\partial_t\boldsymbol{b} = \underline{\boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{b})} + \eta\nabla^2\boldsymbol{b}.$$

We expand the underlined term using the following identity

$$\boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{b}) = \boldsymbol{u}\boldsymbol{\nabla} \cdot \boldsymbol{b} - \boldsymbol{b}\boldsymbol{\nabla} \cdot \boldsymbol{u} + \boldsymbol{b} \cdot \boldsymbol{\nabla}\boldsymbol{u} - \boldsymbol{u} \cdot \boldsymbol{\nabla}\boldsymbol{b}$$

where the first two terms on the RHS vanish due to incompressibility and the abscence of magnetic monopoles. Therefore

$$\partial_t\boldsymbol{b} + \boldsymbol{u} \cdot \boldsymbol{\nabla}\boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{\nabla}\boldsymbol{u} + \eta\nabla^2\boldsymbol{b}.$$

Substituting the decompositions for $\boldsymbol{u}$ and $\boldsymbol{b}$ as above yields

$$\partial_t(B_0\hat{\boldsymbol{z}} + \boldsymbol{b'}) + (Sx\hat{\boldsymbol{y}} + \boldsymbol{u'}) \cdot \boldsymbol{\nabla}(B_0\hat{\boldsymbol{z}} + \boldsymbol{b'}) = (B_0\hat{\boldsymbol{z}} + \boldsymbol{b'}) \cdot \boldsymbol{\nabla}(Sx\hat{\boldsymbol{y}} + \boldsymbol{u'}) + \eta\nabla^2(B_0\hat{\boldsymbol{z}} + \boldsymbol{b'})$$
$$\partial_t\boldsymbol{b'} + (Sx\hat{\boldsymbol{y}} + \boldsymbol{u'}) \cdot \boldsymbol{\nabla}\boldsymbol{b'} = (B_0\hat{\boldsymbol{z}} + \boldsymbol{b'}) \cdot \boldsymbol{\nabla}(Sx\hat{\boldsymbol{y}} + \boldsymbol{u'}) + \eta\nabla^2\boldsymbol{b'}$$

dropping the $'$s and taking the x, y, and z components of the above yields

$$\partial_t b_x + Sx\partial_y b_x + \boldsymbol{u} \cdot \boldsymbol{\nabla} b_x = B_0\partial_z u_x + \boldsymbol{b} \cdot \boldsymbol{\nabla} u_x + \eta\nabla^2 b_x$$
$$\partial_t b_y + Sx\partial_y b_y + \boldsymbol{u} \cdot \boldsymbol{\nabla} b_y = B_0\partial_z u_y + \boldsymbol{b} \cdot \boldsymbol{\nabla} u_y + Sb_x + \eta\nabla^2 b_y$$
$$\partial_t b_z + Sx\partial_y b_z + \boldsymbol{u} \cdot \boldsymbol{\nabla} b_z = B_0\partial_z u_z + \boldsymbol{b} \cdot \boldsymbol{\nabla} u_z + \eta\nabla^2 b_z$$

Note that the nonlinear operator on the LHS is identical for each scalar equation. Accordingly, we redefine D_t to be this operator, i.e.

$$D_t A \equiv \partial_t A + Sx\partial_y A + \boldsymbol{u} \cdot \boldsymbol{\nabla} A$$

NOTE: the material derivative substitution implemented in `Dedalus` excludes the nonlinear term.

We continue by taking ∂_z of the b_y equation

$$\partial_z D_t b_y = \partial_z\Big[\partial_t b_y + Sx\partial_y b_y + \boldsymbol{u} \cdot \boldsymbol{\nabla} b_y\Big] = \partial_z B_0\partial_z u_y + \partial_z[\boldsymbol{b} \cdot \boldsymbol{\nabla} u_y] + S\partial_z b_x + \eta\nabla^2\partial_z b_y$$
$$D_t\partial_z b_y + \partial_z\boldsymbol{u} \cdot \boldsymbol{\nabla} b_y = B_0\partial_z^2 u_y + \partial_z\boldsymbol{b} \cdot \boldsymbol{\nabla} u_y + \boldsymbol{b} \cdot \boldsymbol{\nabla}\partial_z u_y + S\partial_z b_x + \eta\nabla^2\partial_z b_y$$

and the ∂_y of the b_z equation

$$\partial_y D_t b_z = \partial_y\Big[\partial_t b_z + Sx\partial_y b_z + \boldsymbol{u} \cdot \boldsymbol{\nabla} b_z\Big] = B_0\partial_z\partial_y u_z + \partial_y[\boldsymbol{b} \cdot \boldsymbol{\nabla} u_z] + \eta\nabla^2\partial_y b_z$$
$$D_t\partial_y b_z + \partial_y\boldsymbol{u} \cdot \boldsymbol{\nabla} b_z = B_0\partial_z\partial_y u_z + \partial_y\boldsymbol{b} \cdot \boldsymbol{\nabla} u_z + \boldsymbol{b} \cdot \boldsymbol{\nabla}\partial_y u_z + \eta\nabla^2\partial_y b_z$$

Recall that the current density is given by

$$\boldsymbol{j} = j_x\hat{\boldsymbol{x}} + j_y\hat{\boldsymbol{y}} + j_z\hat{\boldsymbol{z}} = \boldsymbol{\nabla} \times \boldsymbol{b}$$
$$j_x = \partial_y b_z - \partial_z b_y$$
$$D_t j_x = D_t\partial_y b_z - D_t\partial_z b_y$$

Therefore

$$D_t j_x + \partial_y\boldsymbol{u} \cdot \boldsymbol{\nabla} b_z - \partial_z\boldsymbol{u} \cdot \boldsymbol{\nabla} b_y = B_0\partial_z\omega_x + \boldsymbol{b} \cdot \boldsymbol{\nabla}\omega_x - S\partial_z b_x + \eta\nabla^2 j_x + \partial_y\boldsymbol{b} \cdot \boldsymbol{\nabla} u_z - \partial_z\boldsymbol{b} \cdot \boldsymbol{\nabla} u_y$$

Expanding the material derivative and grouping linear terms on the LHS gives

$$\boxed{\begin{aligned}\partial_t j_x + Sx\partial_y j_x - B_0\partial_z\omega_x + S\partial_z b_x - \eta\nabla^2 j_x = {} & \boldsymbol{b} \cdot \boldsymbol{\nabla}\omega_x - \boldsymbol{u} \cdot \boldsymbol{\nabla} j_x \\ & + \partial_y\boldsymbol{b} \cdot \boldsymbol{\nabla} u_z - \partial_z\boldsymbol{b} \cdot \boldsymbol{\nabla} u_y \\ & - \partial_y\boldsymbol{u} \cdot \boldsymbol{\nabla} b_z + \partial_z\boldsymbol{u} \cdot \boldsymbol{\nabla} b_y\end{aligned}}$$

Let's check that. Using the following identity

$$\begin{aligned}
\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{b}) &= \boldsymbol{\nabla} \times \big(\boldsymbol{u}(\boldsymbol{\nabla} \cdot \boldsymbol{b}) - \boldsymbol{b}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) + (\boldsymbol{b} \cdot \boldsymbol{\nabla})\boldsymbol{u} - (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{b}\big) \\
&= \boldsymbol{\nabla} \times \big((\boldsymbol{b} \cdot \boldsymbol{\nabla})\boldsymbol{u} - (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{b}\big) \\
&= \boldsymbol{\nabla} \times \left(\begin{bmatrix} \boldsymbol{b} \cdot \boldsymbol{\nabla} u_x \\ \boldsymbol{b} \cdot \boldsymbol{\nabla} u_y \\ \boldsymbol{b} \cdot \boldsymbol{\nabla} u_z \end{bmatrix} - \begin{bmatrix} \boldsymbol{u} \cdot \boldsymbol{\nabla} b_x \\ \boldsymbol{u} \cdot \boldsymbol{\nabla} b_y \\ \boldsymbol{u} \cdot \boldsymbol{\nabla} b_z \end{bmatrix} \right)
\end{aligned}$$

Taking the x-component gives

$$\begin{aligned}
\hat{\boldsymbol{x}} \cdot \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{b}) &= \partial_y(\boldsymbol{b} \cdot \boldsymbol{\nabla} u_z - \boldsymbol{u} \cdot \boldsymbol{\nabla} b_z) - \partial_z(\boldsymbol{b} \cdot \boldsymbol{\nabla} u_y - \boldsymbol{u} \cdot \boldsymbol{\nabla} b_y) \\
&= \partial_y \boldsymbol{b} \cdot \boldsymbol{\nabla} u_z + \boldsymbol{b} \cdot \boldsymbol{\nabla} \partial_y u_z - \partial_y \boldsymbol{u} \cdot \boldsymbol{\nabla} b_z - \boldsymbol{u} \cdot \boldsymbol{\nabla} \partial_y b_z \\
&\quad - \partial_z \boldsymbol{b} \cdot \boldsymbol{\nabla} u_y - \boldsymbol{b} \cdot \boldsymbol{\nabla} \partial_z u_y + \partial_z \boldsymbol{u} \cdot \boldsymbol{\nabla} b_y + \boldsymbol{u} \cdot \boldsymbol{\nabla} \partial_z b_y \\
&= \boldsymbol{b} \cdot \boldsymbol{\nabla} \omega_x - \boldsymbol{u} \cdot \boldsymbol{\nabla} j_x \\
&\quad + \partial_y \boldsymbol{b} \cdot \boldsymbol{\nabla} u_z - \partial_z \boldsymbol{b} \cdot \boldsymbol{\nabla} u_y + \partial_z \boldsymbol{u} \cdot \boldsymbol{\nabla} b_y - \partial_y \boldsymbol{u} \cdot \boldsymbol{\nabla} b_z
\end{aligned}$$

Letting $B_0 = S = 0$, the expanded current-density equation is given by

$$\begin{aligned}
\partial_t j_x &= \eta \nabla^2 j_x \\
&\quad + \boldsymbol{b} \cdot \boldsymbol{\nabla} \omega_x - \boldsymbol{u} \cdot \boldsymbol{\nabla} j_x \\
&\quad + \partial_y \boldsymbol{b} \cdot \boldsymbol{\nabla} u_z - \partial_z \boldsymbol{b} \cdot \boldsymbol{\nabla} u_y + \partial_z \boldsymbol{u} \cdot \boldsymbol{\nabla} b_y - \partial_y \boldsymbol{u} \cdot \boldsymbol{\nabla} b_z \\
&= \eta \nabla^2 j_x + \hat{\boldsymbol{x}} \cdot \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{b}).
\end{aligned}$$

by symmetry we have

$$\partial_t \boldsymbol{j} = \eta \boldsymbol{\nabla}^2 \boldsymbol{j} + \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{b})$$

We can "uncurl" this equation, and if we assume the unknown scalar potential-gradient term $\boldsymbol{\nabla}\phi = \boldsymbol{0}$, we have

$$\partial_t \boldsymbol{b} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{b}) + \eta \boldsymbol{\nabla}^2 \boldsymbol{b}$$

which is the usual form of the induction equation

Navier-Stokes:

$$\partial_t \boldsymbol{u} + Sx\partial_y \boldsymbol{u} + f\hat{z} \times \boldsymbol{u} + [S\boldsymbol{u} \cdot \hat{x}]\hat{y} + \boldsymbol{\nabla} p - \nu\nabla^2 \boldsymbol{u} - \boldsymbol{B} \cdot \boldsymbol{\nabla} \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{\nabla} \boldsymbol{b} - \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u}$$

Induction:

$$\partial_t \boldsymbol{A} - \boldsymbol{\nabla}\phi - \eta\nabla^2 \boldsymbol{A} - \boldsymbol{u} \times \boldsymbol{B} - Sx\hat{y} \times \boldsymbol{b} = \boldsymbol{u} \times \boldsymbol{b}$$

Incompressibility and Coulomb gauge

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = \boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$$

$\boldsymbol{A}$ definition

$$\boldsymbol{b} = \boldsymbol{\nabla} \times \boldsymbol{A}$$

We define the Lagrangian with multipliers $\boldsymbol{\mu}, \boldsymbol{\Lambda}, \pi, \alpha, \boldsymbol{\beta}$

$$\mathcal{L} \equiv \int_0^T \Big\langle \boldsymbol{\mu} \cdot \Big[\partial_t \boldsymbol{u} + Sx\partial_y \boldsymbol{u} + f\hat{z} \times \boldsymbol{u} + [S\boldsymbol{u} \cdot \hat{x}]\hat{y} + \boldsymbol{\nabla} p - \nu\nabla^2 \boldsymbol{u} - \boldsymbol{B} \cdot \boldsymbol{\nabla} \boldsymbol{b} - \boldsymbol{b} \cdot \boldsymbol{\nabla} \boldsymbol{b} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u}\Big]\Big\rangle dt$$
$$+ \int_0^T \Big\langle \boldsymbol{\Lambda} \cdot \Big[\partial_t \boldsymbol{A} - \boldsymbol{\nabla}\phi - \eta\nabla^2 \boldsymbol{A} - \boldsymbol{u} \times \boldsymbol{B} - Sx\hat{y} \times \boldsymbol{b} - \boldsymbol{u} \times \boldsymbol{b}\Big]\Big\rangle dt$$
$$+ \int_0^T \Big\langle \pi \boldsymbol{\nabla} \cdot \boldsymbol{u}\Big\rangle dt + \int_0^T \Big\langle \alpha \boldsymbol{\nabla} \cdot \boldsymbol{A}\Big\rangle dt + \int_0^T \Big\langle \boldsymbol{\beta} \cdot \Big[\boldsymbol{b} - \boldsymbol{\nabla} \times \boldsymbol{A}\Big]\Big\rangle dt$$

The variation with respect to $\boldsymbol{u}$

$$\frac{\delta \mathcal{L}}{\delta \boldsymbol{u}} \delta \boldsymbol{u} =$$
$$\int_0^T \Big\langle \delta \boldsymbol{u} \cdot \Big[- \partial_t \mu - Sx\partial_y \boldsymbol{\mu} + f\hat{z} \times \boldsymbol{\mu} + [S\boldsymbol{\mu} \cdot \hat{y}]\hat{x} + (\boldsymbol{\nabla} \boldsymbol{u})^T \cdot \boldsymbol{\mu} - \boldsymbol{u} \cdot \boldsymbol{\mu} - \nu\nabla^2 \boldsymbol{\mu} - \boldsymbol{\Lambda} \times \boldsymbol{B} - \boldsymbol{\Lambda} \times \boldsymbol{b} - \boldsymbol{\nabla}\pi\Big]\Big\rangle dt$$