1 MRI Notes

1.1 Daniel (8/6/2021)

1.1.1 Potential Form

The induction equation is given by

$$\partial_t \boldsymbol{b} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{b}) + \eta \boldsymbol{\nabla}^2 \boldsymbol{b}$$

Using the following identity

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{f} = \boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \boldsymbol{f} - \boldsymbol{\nabla}^2 \boldsymbol{f}$$

and assumming η to be constant, we use $\boldsymbol{\nabla} \cdot \boldsymbol{b} = 0$, giving

$$\partial_t \boldsymbol{b} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{b}) - \eta \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{b}.$$

Then we define a vector potential $\boldsymbol{\nabla} \times \boldsymbol{A} \equiv \boldsymbol{b}$, yielding

$$\begin{gathered}
\partial_t \boldsymbol{\nabla} \times \boldsymbol{A} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{b}) - \boldsymbol{\nabla} \times (\eta \boldsymbol{\nabla} \times \boldsymbol{b}) \\
\partial_t \boldsymbol{A} = \boldsymbol{u} \times \boldsymbol{b} - \eta \boldsymbol{\nabla} \times \boldsymbol{b} + \boldsymbol{\nabla} \phi
\end{gathered}$$

where ϕ is a scalar potential arizing from "uncurling" the equation. We must then provide an additional constraint to fix ϕ: the Coulomb gauge $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$. Therefore

$$-\boldsymbol{\nabla} \times \boldsymbol{b} = -\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{A} = \boldsymbol{\nabla}^2 \boldsymbol{A}.$$

Next we decompose $\boldsymbol{u} = \boldsymbol{u_0} + \boldsymbol{u'}$ and $\boldsymbol{b} = \boldsymbol{b_0} + \boldsymbol{b'}$. We assume the mean quantities $\boldsymbol{u_0}$ and $\boldsymbol{b_0}$ are themselves solutions to the original problem. If we consider only the 0th mode of $\boldsymbol{b}$, i.e. $\boldsymbol{b} \cdot \boldsymbol{\hat{e}_i} \sim e^{i0}$ then clearly $\boldsymbol{\nabla}^2 \boldsymbol{b} = \boldsymbol{0}$ and therefore

$$\partial_t \boldsymbol{b'} = \boldsymbol{\nabla} \times (\boldsymbol{u_0} \times \boldsymbol{b'}) + \boldsymbol{\nabla} \times (\boldsymbol{u'} \times \boldsymbol{b_0}) + \boldsymbol{\nabla} \times (\boldsymbol{u'} \times \boldsymbol{b'}).$$

Using another identity

$$\boldsymbol{\nabla} \times (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{A}\boldsymbol{\nabla} \cdot \boldsymbol{B} - \boldsymbol{B}\boldsymbol{\nabla} \cdot \boldsymbol{A} + \boldsymbol{B} \cdot \boldsymbol{\nabla}\boldsymbol{A} - \boldsymbol{A} \cdot \boldsymbol{\nabla}\boldsymbol{B}$$