Linear Partial Differential Equations

1. Consider the **diffusion equation**

$$\partial_t u = a\partial_x^2 u \tag{1}$$

on the domain $x \in [0, 2\pi)$.

Derive the solution to this equation, assumming the initial state is given by an arbitrary Fourier mode, i.e.

$$u(x, 0) = \sin(nx) \tag{2}$$

where n is a natural number ($n \in \mathbb{N}$)

.

Hint: assume the solution is of the form

$$u(x, t) = \sin(nx)f(t), \tag{3}$$

then substitute (3) into (1) and solve for $f(t)$. The initial condition (2) places a constraint on $f(0)$, so your solution should be unique (there shouldn't be any undetermined constants or coefficients).

If you can get this solution for a single arbitrary Fourier mode, then all you have to do for a more general initial state $u(x, 0)$ is find its Fourier coefficients and treat them each individually. This is the magic of linearity!

Solutions go like

$$u(x, t) = e^{-an^2 t}\sin(nx)$$