MRI Notes

Induction Equation: Potential Form

The induction equation is given by

$$\partial_t \boldsymbol{b} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{b}) + \eta \boldsymbol{\nabla}^2 \boldsymbol{b}$$

Using the following identity

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{f} = \boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \boldsymbol{f} - \boldsymbol{\nabla}^2 \boldsymbol{f}$$

and assumming η to be constant, we use $\boldsymbol{\nabla} \cdot \boldsymbol{b} = 0$, giving

$$\partial_t \boldsymbol{b} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{b}) - \eta \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{b}.$$

Then we define a vector potential $\boldsymbol{\nabla} \times \boldsymbol{A} \equiv \boldsymbol{b}$, yielding

$$\begin{aligned} \partial_t \boldsymbol{\nabla} \times \boldsymbol{A} &= \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{b}) - \boldsymbol{\nabla} \times (\eta \boldsymbol{\nabla} \times \boldsymbol{b}) \\ \partial_t \boldsymbol{A} &= \boldsymbol{u} \times \boldsymbol{b} - \eta \boldsymbol{\nabla} \times \boldsymbol{b} + \boldsymbol{\nabla}\phi \end{aligned}$$

where ϕ is a scalar potential arizing from "uncurling" the equation. We must then provide an additional constraint to fix ϕ: the Coulomb gauge $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$. Therefore

$$-\boldsymbol{\nabla} \times \boldsymbol{b} = -\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{A} = \boldsymbol{\nabla}^2 \boldsymbol{A}.$$

Next we decompose $\boldsymbol{u} = \boldsymbol{u_0} + \boldsymbol{u'}$ and $\boldsymbol{b} = \boldsymbol{b_0} + \boldsymbol{b'}$. We assume the mean quantities $\boldsymbol{u_0}$ and $\boldsymbol{b_0}$ are themselves solutions to the original problem. If we consider only the 0th mode of $\boldsymbol{b}$, i.e. $\boldsymbol{b} \cdot \boldsymbol{\hat{e}_i} \sim e^{i0}$ then clearly $\boldsymbol{\nabla}^2 \boldsymbol{b} = \boldsymbol{0}$ and therefore

$$\partial_t \boldsymbol{b'} = \boldsymbol{\nabla} \times (\boldsymbol{u_0} \times \boldsymbol{b'}) + \boldsymbol{\nabla} \times (\boldsymbol{u'} \times \boldsymbol{b_0}) + \boldsymbol{\nabla} \times (\boldsymbol{u'} \times \boldsymbol{b'}).$$

Using another identity

$$\boldsymbol{\nabla} \times (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{A}\boldsymbol{\nabla} \cdot \boldsymbol{B} - \boldsymbol{B}\boldsymbol{\nabla} \cdot \boldsymbol{A} + \boldsymbol{B} \cdot \boldsymbol{\nabla}\boldsymbol{A} - \boldsymbol{A} \cdot \boldsymbol{\nabla}\boldsymbol{B}$$

Momentum Equation: Nonlinear Terms

Navier-Stokes (Verbatim from Jeff Oishi's "MRI prefers" paper):

$$\frac{D\boldsymbol{u'}}{Dt} + f\boldsymbol{\hat{z}} \times \boldsymbol{u'} + Su'_x \boldsymbol{\hat{y}} + \boldsymbol{\nabla}p' + \nu \boldsymbol{\nabla} \times \boldsymbol{\omega}' = B_0 \partial_z \boldsymbol{b'}$$

where f is the corriolis parameter, S is the background shearing rate, and $B_0\boldsymbol{\hat{z}}$ is a uniform background magnetic field. The equation is linearized wrt perturbations so the material derivative goes like

$$\begin{aligned} \frac{D}{Dt} &\equiv \partial_t + \overline{\boldsymbol{u}} \cdot \boldsymbol{\nabla} \\ &= \partial_t + Sx\partial_y \end{aligned}$$

due to the background velocity $\overline{\boldsymbol{u}} = Sx\hat{\boldsymbol{y}}$. In the nonlinear case we have

$$= \partial_t + (Sx\hat{\boldsymbol{y}} + \boldsymbol{u'}) \cdot \boldsymbol{\nabla}$$

From inspection and stuff, the irrotational momentum equation goes like

$$\frac{D\boldsymbol{u}}{Dt} + \boldsymbol{\nabla} p + \nu \times \boldsymbol{\omega} = \boldsymbol{b} \cdot \boldsymbol{\nabla} \boldsymbol{b}$$

Next we generalize $\boldsymbol{u} = \boldsymbol{u'} + Sx\hat{\boldsymbol{y}}$ and $\boldsymbol{b} = \boldsymbol{b'} + B_0\hat{\boldsymbol{z}}$, giving

$$\underline{\partial_t \boldsymbol{u'} + \boldsymbol{u'} \cdot \boldsymbol{\nabla} \boldsymbol{u'} + Sx\partial_y \boldsymbol{u'} + Su'_x\hat{\boldsymbol{y}}} + \boldsymbol{\nabla} p + \nu \boldsymbol{\nabla} \times \boldsymbol{\omega} = B_0 \partial_z \boldsymbol{b'} + \boldsymbol{b'} \cdot \boldsymbol{\nabla} \boldsymbol{b'}$$

where the material derivative $\frac{D\boldsymbol{u}}{Dt}$ consists of the underlined terms. Note this definition differs from that of the associated script

Induction Equation: Nonlinear Terms

The MHD induction equation (*Fluid Mechanics of Planets and Stars, 2019*) is given by

$$\partial_t \boldsymbol{b} = \underline{\boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{b})} + \eta \nabla^2 \boldsymbol{b}.$$

We expand the underlined term using the following identity

$$\boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{b}) = \boldsymbol{u} \boldsymbol{\nabla} \cdot \boldsymbol{b} - \boldsymbol{b} \boldsymbol{\nabla} \cdot \boldsymbol{u} + \boldsymbol{b} \cdot \boldsymbol{\nabla} \boldsymbol{u} - \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{b}$$

where the first two terms on the RHS vanish due to incompressibility and the abscence of magnetic monopoles. Therefore

$$\partial_t \boldsymbol{b} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{\nabla} \boldsymbol{u} + \eta \nabla^2 \boldsymbol{b}.$$

Substituting the decompositions for $\boldsymbol{u}$ and $\boldsymbol{b}$ as above yields

$$\begin{aligned} \partial_t (B_0\hat{\boldsymbol{z}} + \boldsymbol{b'}) + (Sx\hat{\boldsymbol{y}} + \boldsymbol{u'}) \cdot \boldsymbol{\nabla} (B_0\hat{\boldsymbol{z}} + \boldsymbol{b'}) &= (B_0\hat{\boldsymbol{z}} + \boldsymbol{b'}) \cdot \boldsymbol{\nabla} (Sx\hat{\boldsymbol{y}} + \boldsymbol{u'}) + \eta \nabla^2 (B_0\hat{\boldsymbol{z}} + \boldsymbol{b'}) \\ \partial_t \boldsymbol{b'} + (Sx\hat{\boldsymbol{y}} + \boldsymbol{u'}) \cdot \boldsymbol{\nabla} \boldsymbol{b'} &= (B_0\hat{\boldsymbol{z}} + \boldsymbol{b'}) \cdot \boldsymbol{\nabla} (Sx\hat{\boldsymbol{y}} + \boldsymbol{u'}) + \eta \nabla^2 \boldsymbol{b'} \end{aligned}$$

dropping the $'$s and taking the x, y, and z components of the above yields

$$\begin{aligned} \partial_t b_x + Sx\partial_y b_x + \boldsymbol{u} \cdot \boldsymbol{\nabla} b_x &= B_0 \partial_z u_x + \boldsymbol{b} \cdot \boldsymbol{\nabla} u_x + \eta \nabla^2 b_x \\ \partial_t b_y + Sx\partial_y b_y + \boldsymbol{u} \cdot \boldsymbol{\nabla} b_y &= B_0 \partial_z u_y + \boldsymbol{b} \cdot \boldsymbol{\nabla} u_y + Sb_x + \eta \nabla^2 b_y \\ \partial_t b_z + Sx\partial_y b_z + \boldsymbol{u} \cdot \boldsymbol{\nabla} b_z &= B_0 \partial_z u_z + \boldsymbol{b} \cdot \boldsymbol{\nabla} u_z + \eta \nabla^2 b_z \end{aligned}$$

Note that the nonlinear operator on the LHS is identical for each scalar equation. Accordingly, we redefine D_t to be this operator, i.e.

$$D_t A \equiv \partial_t A + Sx\partial_y A + \boldsymbol{u} \cdot \boldsymbol{\nabla} A$$

NOTE: the material derivative substitution implemented in `Dedalus` excludes the nonlinear term.

We continue by taking ∂_z of the b_y equation

$$\begin{aligned}\partial_z D_t b_y = \partial_z\Big[\partial_t b_y + Sx\partial_y b_y + \boldsymbol{u}\cdot\boldsymbol{\nabla} b_y\Big] &= \partial_z B_0 \partial_z u_y + \partial_z[\boldsymbol{b}\cdot\boldsymbol{\nabla} u_y] + S\partial_z b_x + \eta\nabla^2\partial_z b_y\\ D_t\partial_z b_y + \partial_z\boldsymbol{u}\cdot\boldsymbol{\nabla} b_y &= B_0\partial_z^2 u_y + \partial_z\boldsymbol{b}\cdot\boldsymbol{\nabla} u_y + \boldsymbol{b}\cdot\boldsymbol{\nabla}\partial_z u_y + S\partial_z b_x + \eta\nabla^2\partial_z b_y\end{aligned}$$

and the ∂_y of the b_z equation

$$\begin{aligned}\partial_y D_t b_z = \partial_y\Big[\partial_t b_z + Sx\partial_y b_z + \boldsymbol{u}\cdot\boldsymbol{\nabla} b_z\Big] &= B_0\partial_z\partial_y u_z + \partial_y[\boldsymbol{b}\cdot\boldsymbol{\nabla} u_z] + \eta\nabla^2\partial_y b_z\\ D_t\partial_y b_z + \partial_y\boldsymbol{u}\cdot\boldsymbol{\nabla} b_z &= B_0\partial_z\partial_y u_z + \partial_y\boldsymbol{b}\cdot\boldsymbol{\nabla} u_z + \boldsymbol{b}\cdot\boldsymbol{\nabla}\partial_y u_z + \eta\nabla^2\partial_y b_z\end{aligned}$$

Recall that the current density is given by

$$\begin{aligned}\boldsymbol{j} = j_x\hat{\boldsymbol{x}} + j_y\hat{\boldsymbol{y}} + j_z\hat{\boldsymbol{z}} &= \boldsymbol{\nabla}\times\boldsymbol{b}\\ j_x &= \partial_y b_z - \partial_z b_y\\ D_t j_x &= D_t\partial_y b_z - D_t\partial_z b_y\end{aligned}$$

Therefore

$$D_t j_x + \partial_y\boldsymbol{u}\cdot\boldsymbol{\nabla} b_z - \partial_z\boldsymbol{u}\cdot\boldsymbol{\nabla} b_y = B_0\partial_z\omega_x + \boldsymbol{b}\cdot\boldsymbol{\nabla}\omega_x - S\partial_z b_x + \eta\nabla^2 j_x + \partial_y\boldsymbol{b}\cdot\boldsymbol{\nabla} u_z - \partial_z\boldsymbol{b}\cdot\boldsymbol{\nabla} u_y$$

Expanding the material derivative and grouping linear terms on the LHS gives

$$\boxed{\begin{aligned}\partial_t j_x + Sx\partial_y j_x - B_0\partial_z\omega_x + S\partial_z b_x - \eta\nabla^2 j_x = {} & \boldsymbol{b}\cdot\boldsymbol{\nabla}\omega_x - \boldsymbol{u}\cdot\boldsymbol{\nabla} j_x\\ & + \partial_y\boldsymbol{b}\cdot\boldsymbol{\nabla} u_z - \partial_z\boldsymbol{b}\cdot\boldsymbol{\nabla} u_y\\ & - \partial_y\boldsymbol{u}\cdot\boldsymbol{\nabla} b_z + \partial_z\boldsymbol{u}\cdot\boldsymbol{\nabla} b_y\end{aligned}}$$

Let's check that. Using the following identity

$$\begin{aligned}
\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{b}) &= \boldsymbol{\nabla} \times \big(\boldsymbol{u}(\boldsymbol{\nabla} \cdot \boldsymbol{b}) - \boldsymbol{b}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) + (\boldsymbol{b} \cdot \boldsymbol{\nabla})\boldsymbol{u} - (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{b}\big) \\
&= \boldsymbol{\nabla} \times \big((\boldsymbol{b} \cdot \boldsymbol{\nabla})\boldsymbol{u} - (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{b}\big) \\
&= \boldsymbol{\nabla} \times \left(\begin{bmatrix} \boldsymbol{b} \cdot \boldsymbol{\nabla} u_x \\ \boldsymbol{b} \cdot \boldsymbol{\nabla} u_y \\ \boldsymbol{b} \cdot \boldsymbol{\nabla} u_z \end{bmatrix} - \begin{bmatrix} \boldsymbol{u} \cdot \boldsymbol{\nabla} b_x \\ \boldsymbol{u} \cdot \boldsymbol{\nabla} b_y \\ \boldsymbol{u} \cdot \boldsymbol{\nabla} b_z \end{bmatrix} \right)
\end{aligned}$$

Taking the x-component gives

$$\begin{aligned}
\hat{\boldsymbol{x}} \cdot \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{b}) &= \partial_y(\boldsymbol{b} \cdot \boldsymbol{\nabla} u_z - \boldsymbol{u} \cdot \boldsymbol{\nabla} b_z) - \partial_z(\boldsymbol{b} \cdot \boldsymbol{\nabla} u_y - \boldsymbol{u} \cdot \boldsymbol{\nabla} b_y) \\
&= \partial_y \boldsymbol{b} \cdot \boldsymbol{\nabla} u_z + \boldsymbol{b} \cdot \boldsymbol{\nabla} \partial_y u_z - \partial_y \boldsymbol{u} \cdot \boldsymbol{\nabla} b_z - \boldsymbol{u} \cdot \boldsymbol{\nabla} \partial_y b_z \\
&\quad\; - \partial_z \boldsymbol{b} \cdot \boldsymbol{\nabla} u_y - \boldsymbol{b} \cdot \boldsymbol{\nabla} \partial_z u_y + \partial_z \boldsymbol{u} \cdot \boldsymbol{\nabla} b_y + \boldsymbol{u} \cdot \boldsymbol{\nabla} \partial_z b_y \\
&= \boldsymbol{b} \cdot \boldsymbol{\nabla} \omega_x - \boldsymbol{u} \cdot \boldsymbol{\nabla} j_x \\
&\quad\; + \partial_y \boldsymbol{b} \cdot \boldsymbol{\nabla} u_z - \partial_z \boldsymbol{b} \cdot \boldsymbol{\nabla} u_y + \partial_z \boldsymbol{u} \cdot \boldsymbol{\nabla} b_y - \partial_y \boldsymbol{u} \cdot \boldsymbol{\nabla} b_z
\end{aligned}$$

Letting $B_0 = S = 0$, the expanded current-density equation is given by

$$\begin{aligned}
\partial_t j_x &= \eta \nabla^2 j_x \\
&\quad\; + \boldsymbol{b} \cdot \boldsymbol{\nabla} \omega_x - \boldsymbol{u} \cdot \boldsymbol{\nabla} j_x \\
&\quad\; + \partial_y \boldsymbol{b} \cdot \boldsymbol{\nabla} u_z - \partial_z \boldsymbol{b} \cdot \boldsymbol{\nabla} u_y + \partial_z \boldsymbol{u} \cdot \boldsymbol{\nabla} b_y - \partial_y \boldsymbol{u} \cdot \boldsymbol{\nabla} b_z \\
&= \eta \nabla^2 j_x + \hat{\boldsymbol{x}} \cdot \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{b}).
\end{aligned}$$

by symmetry we have

$$\partial_t \boldsymbol{j} = \eta \boldsymbol{\nabla}^2 \boldsymbol{j} + \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{b})$$

We can "uncurl" this equation, and if we assume the unknown scalar potential-gradient term $\boldsymbol{\nabla}\phi = \boldsymbol{0}$, we have

$$\partial_t \boldsymbol{b} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{b}) + \eta \boldsymbol{\nabla}^2 \boldsymbol{b}$$

which is the usual form of the induction equation