2D Shear: Analyses of Least-Squares Objective Functions

Velocity Objective

Given a target simulation's end state $\boldsymbol{U}(T)$, consider an objective functional given by

$$H[\boldsymbol{u}(T)] = (\boldsymbol{u}(T) - \boldsymbol{U}(T)) \cdot (\boldsymbol{u}(T) - \boldsymbol{U}(T)) = (\boldsymbol{u}(T) - \boldsymbol{U}(T))^2$$

To determine the adjoint problem's initial ($t = T$) state, we take

$$\begin{aligned}
\frac{\delta H}{\delta \boldsymbol{u}(T)} &= \lim_{\delta \boldsymbol{u}(T) \to 0} \left[\frac{H[\boldsymbol{u}(T) + \delta \boldsymbol{u}(T)] - H[\boldsymbol{u}(T)]}{\delta \boldsymbol{u}(T)} \right] \\
&= \lim_{\delta \boldsymbol{u}(T) \to 0} \left[\frac{(\boldsymbol{u}(T) + \delta \boldsymbol{u}(T) - \boldsymbol{U}(T))^2 - (\boldsymbol{u}(T) - \boldsymbol{U}(T))^2}{\delta \boldsymbol{u}(T)} \right] \\
&= \lim_{\delta \boldsymbol{u}(T) \to 0} \left[\frac{2\delta \boldsymbol{u}(T)(\boldsymbol{u}(T) - \boldsymbol{U}(T))}{\delta \boldsymbol{u}(T)} \right] \\
&= 2(\boldsymbol{u}(T) - \boldsymbol{U}(T))
\end{aligned}$$

Note: as we take $T \to 0$, we have that $\boldsymbol{u}(0) \to \boldsymbol{u}(T)$ and $\boldsymbol{U}(0) \to \boldsymbol{U}(T)$. From this, it follows that the adjoint system is initialized as

$$\boldsymbol{\mu}(T) = 2(\boldsymbol{u}(0) - \boldsymbol{U}(0)) \quad \to \quad \boldsymbol{\mu}(0)$$

Therefore

$$\frac{\delta H}{\delta \boldsymbol{u}(0)} \to 2(\boldsymbol{u}(0) - \boldsymbol{U}(0))$$

Which is the shortest direction to the optimized state in L_2 space.

Vorticity Objective

Next, we define

$$\begin{aligned}\omega &= -\boldsymbol{\nabla}\times\boldsymbol{u}\cdot\hat{\boldsymbol{k}} \equiv \boldsymbol{\nabla}^{\perp}\cdot\boldsymbol{u} = \partial_x v - \partial_y u \\ W &= -\boldsymbol{\nabla}\times\boldsymbol{U}\cdot\hat{\boldsymbol{k}} \equiv \boldsymbol{\nabla}^{\perp}\cdot\boldsymbol{U} = \partial_x V - \partial_y U\end{aligned}$$

and consider an objective functional given by

$$\begin{aligned}H[\boldsymbol{u}(T)] &= (\omega(T) - W(T))^2 \\ &= (\partial_x v(T) - \partial_y u(T) - W(T))^2\end{aligned}$$

Note: if we take

$$\frac{\delta H}{\omega(T)} = 2(\omega(T) - W(T)),$$

we recover the same result as before. It's more interesting to consider

$$\begin{aligned}\frac{\delta H}{\delta u(T)} &= \lim_{\delta u(T)\to 0}\Big[\frac{(\partial_x v(T) - \partial_y u(T) - \partial_y \delta u(T) - W(T))^2 - (\partial_x v(T) - \partial_y u(T) - W(T))^2}{\delta u(T)}\Big] \\ &= \lim_{\delta u(T)\to 0}\Big[\frac{-\partial_y \delta u(T)\big(\partial_x v(T) - \partial_y u(T) - W(T)\big)}{\delta u(T)}\Big]\end{aligned}$$

integrating by parts gives

$$= \lim_{\delta u(T)\to 0}\Big[\frac{-\partial_y\Big[\delta u(T)\big(\partial_x v(T) - \partial_y u(T) - W(T)\big)\Big] + \delta u(T)\partial_y\Big[\partial_x v(T) - \partial_y u(T) - W(T)\Big]}{\delta u(T)}\Big]$$

The first term in the numerator cancels when imprenetrable boundary conditions are employed, leaving

$$= \lim_{\delta u(T)\to 0}\Big[\frac{\delta u(T)\partial_y\Big[\partial_x v(T) - \partial_y u(T) - W(T)\Big]}{\delta u(T)}\Big]$$

$$= \partial_y\Big[\omega(T) - W(T)\Big]$$

Applying the same process to $\frac{\delta H}{\delta v(T)}$, we find

$$\frac{\delta H}{\delta \boldsymbol{u}(T)} = \boldsymbol{\nabla}^{\perp}\Big[\omega(T) - W(T)\Big]$$