Adjoint MRI Derivation

Definitions

Navier-Stokes:

$$\partial_t \boldsymbol{u} + Sx\partial_y \boldsymbol{u} + f\hat{z} \times \boldsymbol{u} + [S\boldsymbol{u} \cdot \hat{x}]\hat{y} + \boldsymbol{\nabla} p - \nu\nabla^2 \boldsymbol{u} = (\boldsymbol{\nabla} \times \boldsymbol{b}) \times \boldsymbol{b} - \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u}$$

Induction:

$$\partial_t \boldsymbol{b} - \eta\nabla^2 \boldsymbol{b} - \boldsymbol{\nabla} \times (Sx\hat{y} \times \boldsymbol{b}) = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{b})$$

Divergence-free constraints

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = \boldsymbol{\nabla} \cdot \boldsymbol{b} = 0$$

We define the Lagrangian with multipliers $\boldsymbol{\mu}, \pi, \boldsymbol{\beta}, \alpha$

$$\mathcal{L} \equiv \int_0^T \Big\langle \boldsymbol{\mu} \cdot \Big[\partial_t \boldsymbol{u} + Sx\partial_y \boldsymbol{u} + f\hat{z} \times \boldsymbol{u} + [S\boldsymbol{u} \cdot \hat{x}]\hat{y} + \boldsymbol{\nabla} p - \nu\nabla^2 \boldsymbol{u} - \boldsymbol{b} \cdot \boldsymbol{\nabla} \boldsymbol{b} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u}\Big]\Big\rangle dt$$
$$+ \int_0^T \Big\langle \boldsymbol{\beta} \cdot \Big[\partial_t \boldsymbol{b} - \eta\nabla^2 \boldsymbol{b} - \boldsymbol{\nabla} \times (Sx\hat{y} \times \boldsymbol{b}) - \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{b})\Big]\Big\rangle dt$$
$$+ \int_0^T \Big\langle \pi \boldsymbol{\nabla} \cdot \boldsymbol{u}\Big\rangle dt + \int_0^T \Big\langle \alpha \boldsymbol{\nabla} \cdot \boldsymbol{b}\Big\rangle dt$$

Incompressibility

Taking the variation with respect to p gives

$$\begin{aligned}\int_0^T \Big\langle \boldsymbol{\mu} \cdot \boldsymbol{\nabla} \delta p\Big\rangle dt &= \int_0^T \Big\langle \boldsymbol{\nabla} \cdot (\delta p\, \boldsymbol{\mu}) - \delta p \boldsymbol{\nabla} \cdot \boldsymbol{\mu}\Big\rangle dt \\ &= \int_0^T \Big\langle - \delta p \boldsymbol{\nabla} \cdot \boldsymbol{\mu}\Big\rangle dt\end{aligned}$$

Making the adjoint velocity divergenceless, i.e. $\boldsymbol{\nabla} \cdot \boldsymbol{\mu} = 0$.

Navier-Stokes

Taking the variation with respect to $\boldsymbol{u}$, the first term in Navier-Stokes integrand becomes

$$\begin{aligned}
\int_0^T \Big\langle \boldsymbol{\mu} \cdot \partial_t \delta \boldsymbol{u} \Big\rangle dt &= \int_0^T \Big\langle \partial_t(\boldsymbol{\mu} \cdot \delta \boldsymbol{u}) - \delta \boldsymbol{u} \cdot \partial_t \boldsymbol{\mu} \Big\rangle dt \\
&= \Big\langle \int_0^T \partial_t(\boldsymbol{\mu} \cdot \delta \boldsymbol{u}) dt \Big\rangle - \int_0^T \Big\langle \delta \boldsymbol{u} \cdot \partial_t \boldsymbol{\mu} \Big\rangle dt \\
&= \Big\langle \boldsymbol{\mu}(\boldsymbol{x}, T) \cdot \delta \boldsymbol{u}(\boldsymbol{x}, T) - \boldsymbol{\mu}(\boldsymbol{x}, 0) \cdot \delta \boldsymbol{u}(\boldsymbol{x}, 0) \Big\rangle - \int_0^T \Big\langle \delta \boldsymbol{u} \cdot \partial_t \boldsymbol{\mu} \Big\rangle dt
\end{aligned}$$

The second term becomes

$$\begin{aligned}
\int_0^T \Big\langle \boldsymbol{\mu} \cdot Sx \partial_y \delta \boldsymbol{u} \Big\rangle dt &= \int_0^T \Big\langle Sx \left(\partial_y(\boldsymbol{\mu} \cdot \delta \boldsymbol{u}) - \delta \boldsymbol{u} \cdot \partial_y \boldsymbol{\mu} \right) \Big\rangle dt \\
&= - \int_0^T \Big\langle \delta \boldsymbol{u} \cdot (Sx \partial_y \boldsymbol{\mu}) \Big\rangle dt
\end{aligned}$$

The third term becomes

$$\int_0^T \Big\langle \boldsymbol{\mu} \cdot (f \hat{z} \times \delta \boldsymbol{u}) \Big\rangle dt = - \int_0^T \Big\langle \delta \boldsymbol{u} \cdot (f \hat{z} \times \boldsymbol{\mu}) \Big\rangle dt$$

The fourth term becomes

$$\int_0^T \Big\langle \boldsymbol{\mu} \cdot \left(\big[S \delta \boldsymbol{u} \cdot \hat{x} \big] \hat{y} \right) \Big\rangle dt = \int_0^T \Big\langle \delta \boldsymbol{u} \cdot \left(\big[S \boldsymbol{\mu} \cdot \hat{x} \big] \hat{y} \right) \Big\rangle dt$$

The fifth term $\boldsymbol{\nabla} p$ has no variation with respect to $\boldsymbol{u}$. Using Green's vector identity and applying impenetrable and (no-slip or stress-free) boundary conditions, the sixth term becomes

$$\begin{aligned}
\int_0^T \Big\langle \boldsymbol{\mu} \cdot \nu \nabla^2 \delta \boldsymbol{u} \Big\rangle dt &= \int_0^T \Big\langle \delta \boldsymbol{u} \cdot \nu \nabla^2 \boldsymbol{\mu} + \boldsymbol{\nabla} \cdot (\boldsymbol{\mu} \times (\boldsymbol{\nabla} \times \delta \boldsymbol{u}) - \delta \boldsymbol{u} \times (\boldsymbol{\nabla} \times \boldsymbol{\mu})) \Big\rangle dt \\
&= \int_0^T \Big\langle \delta \boldsymbol{u} \cdot \nu \nabla^2 \boldsymbol{\mu} \Big\rangle + \iint (\boldsymbol{\mu} \times (\boldsymbol{\nabla} \times \delta \boldsymbol{u}) - \delta \boldsymbol{u} \times (\boldsymbol{\nabla} \times \boldsymbol{\mu})) \cdot \hat{n} \, dS \, dt \\
&= \int_0^T \Big\langle \delta \boldsymbol{u} \cdot \nu \nabla^2 \boldsymbol{\mu} \Big\rangle dt
\end{aligned}$$

The seventh (lorenz force) term has no variation in $\boldsymbol{u}$. The eighth and final term in Navier-Stokes can be rewritten as

$$\boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} = \frac{1}{2} \boldsymbol{\nabla} |\boldsymbol{u}|^2 - \boldsymbol{u} \times (\boldsymbol{\nabla} \times \boldsymbol{u})$$

Absorbing the magnitude term into the hydrodynamic pressure, we replace the advection term with the cross-curl term which follows

$$\begin{aligned}
\int_0^T \Big\langle \boldsymbol{\mu}\cdot\delta\big(\boldsymbol{u}\times(\boldsymbol{\nabla}\times\boldsymbol{u})\big)\Big\rangle dt &= \int_0^T \Big\langle \boldsymbol{\mu}\cdot\big(\delta\boldsymbol{u}\times(\boldsymbol{\nabla}\times\boldsymbol{u})+\boldsymbol{u}\times(\boldsymbol{\nabla}\times\delta\boldsymbol{u})\big)\Big\rangle dt \\
&= \int_0^T \Big\langle \delta\boldsymbol{u}\cdot\big((\boldsymbol{\nabla}\times\boldsymbol{u})\times\boldsymbol{\mu}\big)+(\boldsymbol{\nabla}\times\delta\boldsymbol{u})\cdot(\boldsymbol{\mu}\times\boldsymbol{u})\Big\rangle dt \\
&= \int_0^T \Big\langle \delta\boldsymbol{u}\cdot\big((\boldsymbol{\nabla}\times\boldsymbol{u})\times\boldsymbol{\mu}\big) \\
&\qquad + \boldsymbol{\nabla}\cdot\big(\delta\boldsymbol{u}\times(\boldsymbol{\mu}\times\boldsymbol{u})\big)+\delta\boldsymbol{u}\cdot\big(\boldsymbol{\nabla}\times(\boldsymbol{\mu}\times\boldsymbol{u})\big)\Big\rangle dt \\
&= \int_0^T \Big\langle \delta\boldsymbol{u}\cdot\big((\boldsymbol{\nabla}\times\boldsymbol{u})\times\boldsymbol{\mu}+\boldsymbol{\nabla}\times(\boldsymbol{\mu}\times\boldsymbol{u})\big)\Big\rangle dt
\end{aligned}$$

The last term in the induction equation becomes

$$\begin{aligned}
\int_0^T \Big\langle \boldsymbol{\beta}\cdot\big(\boldsymbol{\nabla}\times(\delta\boldsymbol{u}\times\boldsymbol{b})\big)\Big\rangle dt &= \int_0^T \Big\langle \boldsymbol{\nabla}\cdot((\delta\boldsymbol{u}\times\boldsymbol{b})\times\boldsymbol{\beta})+(\boldsymbol{\nabla}\times\boldsymbol{\beta})\cdot(\delta\boldsymbol{u}\times\boldsymbol{b})\Big\rangle dt \\
&= \int_0^T \Big\langle (\boldsymbol{\nabla}\times\boldsymbol{\beta})\cdot(\delta\boldsymbol{u}\times\boldsymbol{b})\Big\rangle dt \\
&= \int_0^T \Big\langle \delta\boldsymbol{u}\cdot(\boldsymbol{b}\times(\boldsymbol{\nabla}\times\boldsymbol{\beta}))\Big\rangle dt
\end{aligned}$$

The penultimate term in the Lagrangian becomes

$$\begin{aligned}
\int_0^T \Big\langle \pi\boldsymbol{\nabla}\cdot\delta\boldsymbol{u}\Big\rangle dt &= \int_0^T \Big\langle \boldsymbol{\nabla}\cdot(\pi\delta\boldsymbol{u})-\delta\boldsymbol{u}\cdot\boldsymbol{\nabla}\pi\Big\rangle dt \\
&= \int_0^T \Big\langle -\delta\boldsymbol{u}\cdot\boldsymbol{\nabla}\pi\Big\rangle dt
\end{aligned}$$

Gathering terms which have $\delta\boldsymbol{u}\cdot()$ in common, we obtain the adjoint Navier Stokes Equation

$$\partial_t\boldsymbol{\mu}+Sx\partial_y\boldsymbol{\mu}-\big[S\boldsymbol{\mu}\cdot\hat{x}\big]\hat{y}+f\hat{z}\times\boldsymbol{\mu}+\boldsymbol{\nabla}\pi+\nu\nabla^2\boldsymbol{\mu}=-(\boldsymbol{\nabla}\times\boldsymbol{u})\times\boldsymbol{\mu}-\boldsymbol{\nabla}\times(\boldsymbol{\mu}\times\boldsymbol{u})-\boldsymbol{b}\times(\boldsymbol{\nabla}\times\boldsymbol{\beta})$$

Induction Equation

Taking the variation with respect to $\boldsymbol{b}$, we begin with the Lorenz force in Navier Stokes

$$\begin{aligned}\int_0^T \Big\langle \boldsymbol{\mu}\cdot\delta\big((\boldsymbol{\nabla}\times\boldsymbol{b})\times\boldsymbol{b}\big)\Big\rangle dt &= \int_0^T \Big\langle \boldsymbol{\mu}\cdot\big((\boldsymbol{\nabla}\times\delta\boldsymbol{b})\times\boldsymbol{b}+(\boldsymbol{\nabla}\times\boldsymbol{b})\times\delta\boldsymbol{b}\big)\Big\rangle dt\\ &= \int_0^T \Big\langle (\boldsymbol{\nabla}\times\delta\boldsymbol{b})\cdot(\boldsymbol{b}\times\boldsymbol{\mu})+\delta\boldsymbol{b}\cdot\big(\boldsymbol{\mu}\times(\boldsymbol{\nabla}\times\boldsymbol{b})\big)\Big\rangle dt\\ &= \int_0^T \Big\langle \boldsymbol{\nabla}\cdot\big(\delta\boldsymbol{b}\times(\boldsymbol{b}\times\boldsymbol{\mu})\big)+\delta\boldsymbol{b}\cdot(\boldsymbol{\nabla}\times(\boldsymbol{b}\times\boldsymbol{\mu}))+\delta\boldsymbol{b}\cdot\big(\boldsymbol{\mu}\times(\boldsymbol{\nabla}\times\boldsymbol{b})\big)\Big\rangle dt\\ &= \int_0^T \Big\langle \delta\boldsymbol{b}\cdot\big(\boldsymbol{\nabla}\times(\boldsymbol{b}\times\boldsymbol{\mu})+\boldsymbol{\mu}\times(\boldsymbol{\nabla}\times\boldsymbol{b})\big)\Big\rangle dt\end{aligned}$$

The time-derivative (first term) in the induction equation becomes

$$\begin{aligned}\int_0^T \Big\langle \boldsymbol{\beta}\cdot\partial_t\delta\boldsymbol{b}\Big\rangle dt &= \int_0^T \Big\langle \partial_t(\boldsymbol{\beta}\cdot\delta\boldsymbol{b})-\delta\boldsymbol{b}\cdot\partial_t\boldsymbol{\beta}\Big\rangle dt\\ &= \Big\langle \int_0^T \partial_t(\boldsymbol{\beta}\cdot\delta\boldsymbol{b})dt\Big\rangle - \int_0^T \Big\langle \delta\boldsymbol{b}\cdot\partial_t\boldsymbol{\beta}\Big\rangle dt\\ &= \Big\langle \boldsymbol{\beta}(\boldsymbol{x},T)\cdot\delta\boldsymbol{b}(\boldsymbol{x},T)-\boldsymbol{\beta}(\boldsymbol{x},0)\cdot\delta\boldsymbol{b}(\boldsymbol{x},0)\Big\rangle - \int_0^T \Big\langle \delta\boldsymbol{b}\cdot\partial_t\boldsymbol{\beta}\Big\rangle dt\end{aligned}$$

Repeating the earlier procedure, the diffusive (second) term in the induction equation becomes

$$\begin{aligned}\int_0^T \Big\langle \boldsymbol{\beta}\cdot\eta\nabla^2\delta\boldsymbol{b}\Big\rangle dt &= \int_0^T \Big\langle \delta\boldsymbol{b}\cdot\eta\nabla^2\boldsymbol{\beta}+\boldsymbol{\nabla}\cdot(\boldsymbol{\beta}\times(\boldsymbol{\nabla}\times\delta\boldsymbol{b})-\delta\boldsymbol{b}\times(\boldsymbol{\nabla}\times\boldsymbol{\beta}))\Big\rangle dt\\ &= \int_0^T \Big\langle \delta\boldsymbol{b}\cdot\eta\nabla^2\boldsymbol{\beta}\Big\rangle + \iint (\boldsymbol{\beta}\times(\boldsymbol{\nabla}\times\delta\boldsymbol{b})-\delta\boldsymbol{b}\times(\boldsymbol{\nabla}\times\boldsymbol{\beta}))\cdot\hat{n}\,dS\,dt\\ &= \int_0^T \Big\langle \delta\boldsymbol{b}\cdot\eta\nabla^2\boldsymbol{\beta}\Big\rangle dt\end{aligned}$$

The third term in the induction equation becomes

$$\begin{aligned}\int_0^T \Big\langle \boldsymbol{\beta}\cdot\Big[\boldsymbol{\nabla}\times(Sx\hat{y}\times\delta\boldsymbol{b})\Big]\Big\rangle dt &= \int_0^T \Big\langle \boldsymbol{\nabla}\cdot((Sx\hat{y}\times\delta\boldsymbol{b})\times\boldsymbol{\beta})+(Sx\hat{y}\times\delta\boldsymbol{b})\cdot(\boldsymbol{\nabla}\times\boldsymbol{\beta})\Big\rangle dt\\ &= \int_0^T \Big\langle (Sx\hat{y}\times\delta\boldsymbol{b})\cdot(\boldsymbol{\nabla}\times\boldsymbol{\beta})\Big\rangle dt\\ &= \int_0^T \Big\langle \delta\boldsymbol{b}\cdot(\boldsymbol{\nabla}\times\boldsymbol{\beta})\times(Sx\hat{y})\Big\rangle dt\end{aligned}$$

Using the same procedure, the fourth and final term in the induction equation becomes

$$\int_0^T \Big\langle \boldsymbol{\beta} \cdot \Big[\boldsymbol{\nabla} \times (\boldsymbol{u} \times \delta \boldsymbol{b}) \Big] \Big\rangle dt = \int_0^T \Big\langle \delta \boldsymbol{b} \cdot (\boldsymbol{\nabla} \times \boldsymbol{\beta}) \times \boldsymbol{u} \Big\rangle dt$$

Again repeating one of the earlier procedures,

$$\begin{aligned} \int_0^T \Big\langle \alpha \boldsymbol{\nabla} \cdot \delta \boldsymbol{b} \Big\rangle dt &= \int_0^T \Big\langle \boldsymbol{\nabla} \cdot (\alpha \delta \boldsymbol{b}) - \delta \boldsymbol{b} \cdot \boldsymbol{\nabla} \alpha \Big\rangle dt \\ &= \int_0^T \Big\langle - \delta \boldsymbol{b} \cdot \boldsymbol{\nabla} \alpha \Big\rangle dt \end{aligned}$$

Therefore the adjoint induction equation is given by

$$\partial_t \boldsymbol{\beta} + \boldsymbol{\nabla} \alpha + \eta \nabla^2 \boldsymbol{\beta} + (\boldsymbol{\nabla} \times \boldsymbol{\beta}) \times (Sx\hat{y}) = -\big(\boldsymbol{\nabla} \times (\boldsymbol{b} \times \boldsymbol{\mu}) + \boldsymbol{\mu} \times (\boldsymbol{\nabla} \times \boldsymbol{b}) \big) - (\boldsymbol{\nabla} \times \boldsymbol{\beta}) \times \boldsymbol{u}$$