2D Shear: Analyses of Least-Squares Objective Functions

Velocity Objective

Given a target simulation's end state $\boldsymbol{U}(T)$, consider an objective functional given by

$$H[\boldsymbol{u}(T)] = (\boldsymbol{u}(T) - \boldsymbol{U}(T)) \cdot (\boldsymbol{u}(T) - \boldsymbol{U}(T)) = (\boldsymbol{u}(T) - \boldsymbol{U}(T))^2$$

To determine the adjoint problem's initial ($t = T$) state, we take

$$\begin{aligned}
\frac{\delta H}{\delta \boldsymbol{u}(T)} &= \lim_{\delta \boldsymbol{u}(T) \to 0} \left[\frac{H[\boldsymbol{u}(T) + \delta \boldsymbol{u}(T)] - H[\boldsymbol{u}(T)]}{\delta \boldsymbol{u}(T)} \right] \\
&= \lim_{\delta \boldsymbol{u}(T) \to 0} \left[\frac{(\boldsymbol{u}(T) + \delta \boldsymbol{u}(T) - \boldsymbol{U}(T))^2 - (\boldsymbol{u}(T) - \boldsymbol{U}(T))^2}{\delta \boldsymbol{u}(T)} \right] \\
&= \lim_{\delta \boldsymbol{u}(T) \to 0} \left[\frac{2\delta \boldsymbol{u}(T)(\boldsymbol{u}(T) - \boldsymbol{U}(T))}{\delta \boldsymbol{u}(T)} \right] \\
&= 2(\boldsymbol{u}(T) - \boldsymbol{U}(T))
\end{aligned}$$

Note: as we take $T \to 0$, we have that $\boldsymbol{u}(0) \to \boldsymbol{u}(T)$ and $\boldsymbol{U}(0) \to \boldsymbol{U}(T)$. From this, it follows that the adjoint system is initialized as

$$\boldsymbol{\mu}(T) = 2(\boldsymbol{u}(0) - \boldsymbol{U}(0)) \quad \to \quad \boldsymbol{\mu}(0)$$

Therefore

$$\frac{\delta H}{\delta \boldsymbol{u}(0)} \to 2(\boldsymbol{u}(0) - \boldsymbol{U}(0))$$

Which is the shortest direction to the optimized state in L_2 space.

Vorticity Objective

Next, we define

$$\begin{aligned} \omega &= \boldsymbol{\nabla} \times \boldsymbol{u} \cdot \hat{\boldsymbol{k}} \equiv \boldsymbol{\nabla}^{\perp} \cdot \boldsymbol{u} = \partial_x v - \partial_y u \\ W &= \boldsymbol{\nabla} \times \boldsymbol{U} \cdot \hat{\boldsymbol{k}} \equiv \boldsymbol{\nabla}^{\perp} \cdot \boldsymbol{U} = \partial_x V - \partial_y U \end{aligned}$$

and consider an objective functional given by

$$\begin{aligned} H[\boldsymbol{u}(T)] &= (\omega(T) - W(T))^2 \\ &= (\partial_x v(T) - \partial_y u(T) - W(T))^2 \end{aligned}$$

Note: if we take

$$\frac{\delta H}{\omega(T)} = 2(\omega(T) - W(T)),$$

we recover the same result as before. It's more interesting to consider

$$\begin{aligned} \frac{\delta H}{\delta u(T)} &= \lim_{\delta u(T)\to 0} \Big[\frac{(\partial_x v(T) - \partial_y u(T) - \partial_y \delta u(T) - W(T))^2 - (\partial_x v(T) - \partial_y u(T) - W(T))^2}{\delta u(T)} \Big] \\ &= \lim_{\delta u(T)\to 0} \Big[\frac{-\partial_y \delta u(T) \big(\partial_x v(T) - \partial_y u(T) - W(T)\big)}{\delta u(T)} \Big] \end{aligned}$$

integrating by parts gives

$$= \lim_{\delta u(T)\to 0} \Big[\frac{-\partial_y \Big[\delta u(T) \big(\partial_x v(T) - \partial_y u(T) - W(T)\big)\Big] + \delta u(T) \partial_y \Big[\partial_x v(T) - \partial_y u(T) - W(T)\Big]}{\delta u(T)} \Big]$$

The first term in the numerator cancels when imprenetrable boundary conditions are employed, leaving

$$= \lim_{\delta u(T)\to 0} \Big[\frac{\delta u(T) \partial_y \Big[\partial_x v(T) - \partial_y u(T) - W(T)\Big]}{\delta u(T)} \Big]$$

$$= \partial_y \Big[\omega(T) - W(T)\Big]$$

Applying the same process to $\frac{\delta H}{\delta v(T)}$, we find

$$\frac{\delta H}{\delta \boldsymbol{u}(T)} = \boldsymbol{\nabla}^{\perp} \Big[\omega(T) - W(T)\Big]$$

Streamfunction objective

We define the streamfunctions $\psi(x, y)$ and $\phi(x, y)$ satisfying

$$\begin{aligned}\boldsymbol{\nabla}^{\perp}\psi &= \boldsymbol{u} \\ \boldsymbol{\nabla}^{\perp}\phi &= \boldsymbol{U}\end{aligned}$$

Note that

$$\begin{aligned}\boldsymbol{\nabla}^{\perp} \cdot \boldsymbol{\nabla}^{\perp}\psi &= \nabla^2\psi = \boldsymbol{\nabla}^{\perp} \cdot \boldsymbol{u} = \omega \\ \boldsymbol{\nabla}^{\perp} \cdot \boldsymbol{\nabla}^{\perp}\phi &= \nabla^2\phi = \boldsymbol{\nabla}^{\perp} \cdot \boldsymbol{U} = W\end{aligned}$$

Consider the original velocity objective

$$\begin{aligned}H[\boldsymbol{u}(T)] &= (\boldsymbol{u}(T) - \boldsymbol{U}(T)) \cdot (\boldsymbol{u}(T) - \boldsymbol{U}(T)) \\ &= (\boldsymbol{u}(T) - \boldsymbol{U}(T))^2 \\ &= (\boldsymbol{\nabla}^{\perp}\psi(T) - \boldsymbol{\nabla}^{\perp}\phi(T))^2\end{aligned}$$

We then take

$$\begin{aligned}\frac{\delta H}{\delta\psi(T)} &= \lim_{\delta\psi(T)\to 0} \frac{(\boldsymbol{\nabla}^{\perp}(\psi(T) + \delta\psi(T)) - \boldsymbol{\nabla}^{\perp}\phi(T))^2 - (\boldsymbol{\nabla}^{\perp}\psi(T) - \boldsymbol{\nabla}^{\perp}\phi(T))^2}{\delta\psi(T)} \\ &= \lim_{\delta\psi(T)\to 0} \frac{2\delta\psi(T)(\boldsymbol{\nabla}^{\perp}\psi(T) - \boldsymbol{\nabla}^{\perp}\phi(T))}{\delta\psi(T)} \\ &= 2(\boldsymbol{\nabla}^{\perp}\psi(T) - \boldsymbol{\nabla}^{\perp}\phi(T)) \\ &= 2(\boldsymbol{u}(T) - \boldsymbol{U}(T))\end{aligned}$$

This is the same thing as before! To get a new adjoint ic, we need to take

$$H[\psi(T)] = (\psi(T) - \phi(T))^2$$

along with

$$\frac{\delta H}{\delta\boldsymbol{u}(T)} = \frac{\delta}{\delta\boldsymbol{u}(T)}[(\psi(T) - \phi(T))^2]$$

Thus, by applying the appropriate boundary conditions, we can solve for ψ given a flow field.

$$\begin{aligned}\psi &= \boldsymbol{\nabla}^{-2}\omega \\ \phi &= \boldsymbol{\nabla}^{-2}W\end{aligned}$$

Recall that we are interested in a streamfunction objective's variational derivative with respect to velocity, as differentiating the least-squares functional $(\psi - \phi)^2$ wrt ψ gives us the same thing as before!

$$\begin{aligned}H[\psi(T)] &= (\psi - \phi)^2 \\ &= (\boldsymbol{\nabla}^{-2}\omega - \boldsymbol{\nabla}^{-2}W)^2 \\ &= (\boldsymbol{\nabla}^{-2}\omega)^2 - 2\boldsymbol{\nabla}^{-2}\omega\boldsymbol{\nabla}^{-2}W + (\boldsymbol{\nabla}^{-2}W)^2\end{aligned}$$

Taking the variation

$$\begin{aligned}
\frac{\delta H}{\delta u} &= \lim_{\delta u \to 0} \frac{2\boldsymbol{\nabla}^{-2}\boldsymbol{\nabla}^{\perp} \cdot \boldsymbol{u}\boldsymbol{\nabla}^{-2}\boldsymbol{\nabla}^{\perp} \cdot \delta \boldsymbol{u} - 2\boldsymbol{\nabla}^{-2}\boldsymbol{\nabla}^{\perp} \cdot \delta \boldsymbol{u}\boldsymbol{\nabla}^{-2}W}{\delta u} \\
&= \lim_{\delta u \to 0} \frac{2\boldsymbol{\nabla}^{-2}\boldsymbol{\nabla}^{\perp} \cdot \delta \boldsymbol{u}(\boldsymbol{\nabla}^{-2}\boldsymbol{\nabla}^{\perp} \cdot \boldsymbol{u} - \boldsymbol{\nabla}^{-2}W)}{\delta u} \\
&= 2(\psi - \phi) \lim_{\delta u \to 0} \frac{\boldsymbol{\nabla}^{-2}\boldsymbol{\nabla}^{\perp} \cdot \delta \boldsymbol{u}}{\delta u}
\end{aligned}$$