1 MRI Notes

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1.1.1 Potential Form

The induction equation is given by

$$\partial_t \boldsymbol{b} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{b}) + \eta \boldsymbol{\nabla}^2 \boldsymbol{b}$$

Using the following identity

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{f} = \boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \boldsymbol{f} - \boldsymbol{\nabla}^2 \boldsymbol{f}$$

and assumming η to be constant, we use $\boldsymbol{\nabla} \cdot \boldsymbol{b} = 0$, giving

$$\partial_t \boldsymbol{b} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{b}) - \eta \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{b}.$$

Then we define a vector potential $\boldsymbol{\nabla} \times \boldsymbol{A} \equiv \boldsymbol{b}$, yielding

$$\begin{gathered} \partial_t \boldsymbol{\nabla} \times \boldsymbol{A} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{b}) - \boldsymbol{\nabla} \times (\eta \boldsymbol{\nabla} \times \boldsymbol{b}) \\ \partial_t \boldsymbol{A} = \boldsymbol{u} \times \boldsymbol{b} - \eta \boldsymbol{\nabla} \times \boldsymbol{b} + \boldsymbol{\nabla}\phi \end{gathered}$$

where ϕ is a scalar potential arizing from "uncurling" the equation. We must then provide an additional constraint to fix ϕ: the Coulomb gauge $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$. Therefore

$$-\boldsymbol{\nabla} \times \boldsymbol{b} = -\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{A} = \boldsymbol{\nabla}^2 \boldsymbol{A}.$$

Next we decompose $\boldsymbol{u} = \boldsymbol{u_0} + \boldsymbol{u'}$ and $\boldsymbol{b} = \boldsymbol{b_0} + \boldsymbol{b'}$. We assume the mean quantities $\boldsymbol{u_0}$ and $\boldsymbol{b_0}$ are themselves solutions to the original problem. If we consider only the 0th mode of $\boldsymbol{b}$, i.e. $\boldsymbol{b} \cdot \boldsymbol{\hat{e}_i} \sim e^{i0}$ then clearly $\boldsymbol{\nabla}^2 \boldsymbol{b} = \boldsymbol{0}$ and therefore

$$\partial_t \boldsymbol{b'} = \boldsymbol{\nabla} \times (\boldsymbol{u_0} \times \boldsymbol{b'}) + \boldsymbol{\nabla} \times (\boldsymbol{u'} \times \boldsymbol{b_0}) + \boldsymbol{\nabla} \times (\boldsymbol{u'} \times \boldsymbol{b'}).$$

Using another identity

$$\boldsymbol{\nabla} \times (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{A}\boldsymbol{\nabla} \cdot \boldsymbol{B} - \boldsymbol{B}\boldsymbol{\nabla} \cdot \boldsymbol{A} + \boldsymbol{B} \cdot \boldsymbol{\nabla}\boldsymbol{A} - \boldsymbol{A} \cdot \boldsymbol{\nabla}\boldsymbol{B}$$

Momentum Equation

Verbatim from Jeff Oishi's MRI paper:

$$\frac{D\boldsymbol{u'}}{Dt} + f\boldsymbol{\hat{z}} \times \boldsymbol{u'} + Su'_x\boldsymbol{\hat{y}} + \boldsymbol{\nabla}p' + \nu\boldsymbol{\nabla} \times \boldsymbol{\omega}' = B_0\partial_z\boldsymbol{b'}$$

where f is the corriolis parameter, S is the background shearing rate, and $B_0\hat{\boldsymbol{z}}$ is a uniform background magnetic field. This equation is linearized wrt perturbations. Accordingly, the material derivative goes like

$$\begin{aligned}\frac{D}{Dt} &\equiv \partial_t + \boldsymbol{u}\cdot\boldsymbol{\nabla}\\ &= \partial_t + Sx\partial_y\end{aligned}$$

due to the background velocity $\overline{\boldsymbol{u}} = Sx\hat{\boldsymbol{y}}$. In the nonlinear case we have

$$= \partial_t + (Sx\hat{\boldsymbol{y}} + \boldsymbol{u'})\cdot\boldsymbol{\nabla}$$

From inspection and stuff, the irrotational momentum equation goes like

$$\frac{D\boldsymbol{u}}{Dt} + \boldsymbol{\nabla}p + \nu\times\boldsymbol{\omega} = \boldsymbol{b}\cdot\boldsymbol{\nabla}\boldsymbol{b}$$

Next we generalize $\boldsymbol{u} = \boldsymbol{u'} + Sx\hat{\boldsymbol{y}}$ and $\boldsymbol{b} = \boldsymbol{b'} + B_0\hat{\boldsymbol{z}}$, giving

$$\underline{\partial_t\boldsymbol{u'} + \boldsymbol{u'}\cdot\boldsymbol{\nabla}\boldsymbol{u'} + Sx\partial_y\boldsymbol{u'} + Su'_x\hat{\boldsymbol{y}}} + \boldsymbol{\nabla}p + \nu\boldsymbol{\nabla}\times\boldsymbol{\omega} = B_0\partial_z\boldsymbol{b'} + \boldsymbol{b'}\cdot\boldsymbol{\nabla}\boldsymbol{b'}$$

where the material derivative $\frac{D\boldsymbol{u}}{Dt}$ consists of the underlined terms