MRI Notes

Induction Equation: Potential Form

The induction equation is given by

$$\partial_t \boldsymbol{b} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{b}) + \eta \boldsymbol{\nabla}^2 \boldsymbol{b}$$

Using the following identity

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{f} = \boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \boldsymbol{f} - \boldsymbol{\nabla}^2 \boldsymbol{f}$$

and assumming η to be constant, we use $\boldsymbol{\nabla} \cdot \boldsymbol{b} = 0$, giving

$$\partial_t \boldsymbol{b} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{b}) - \eta \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{b}.$$

Then we define a vector potential $\boldsymbol{\nabla} \times \boldsymbol{A} \equiv \boldsymbol{b}$, yielding

$$\partial_t \boldsymbol{\nabla} \times \boldsymbol{A} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{b}) - \boldsymbol{\nabla} \times (\eta \boldsymbol{\nabla} \times \boldsymbol{b})$$
$$\partial_t \boldsymbol{A} = \boldsymbol{u} \times \boldsymbol{b} - \eta \boldsymbol{\nabla} \times \boldsymbol{b} + \boldsymbol{\nabla}\phi$$

where ϕ is a scalar potential arizing from "uncurling" the equation. We must then provide an additional constraint to fix ϕ: the Coulomb gauge $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$. Therefore

$$-\boldsymbol{\nabla} \times \boldsymbol{b} = -\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{A} = \boldsymbol{\nabla}^2 \boldsymbol{A}.$$

Next we decompose $\boldsymbol{u} = \boldsymbol{u_0} + \boldsymbol{u}'$ and $\boldsymbol{b} = \boldsymbol{b_0} + \boldsymbol{b}'$. We assume the mean quantities $\boldsymbol{u_0}$ and $\boldsymbol{b_0}$ are themselves solutions to the original problem. If we consider only the 0th mode of $\boldsymbol{b}$, i.e. $\boldsymbol{b} \cdot \hat{\boldsymbol{e}_i} \sim e^{i0}$ then clearly $\boldsymbol{\nabla}^2 \boldsymbol{b} = \boldsymbol{0}$ and therefore

$$\partial_t \boldsymbol{b}' = \boldsymbol{\nabla} \times (\boldsymbol{u_0} \times \boldsymbol{b}') + \boldsymbol{\nabla} \times (\boldsymbol{u}' \times \boldsymbol{b_0}) + \boldsymbol{\nabla} \times (\boldsymbol{u}' \times \boldsymbol{b}').$$

Using another identity

$$\boldsymbol{\nabla} \times (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{A}\boldsymbol{\nabla} \cdot \boldsymbol{B} - \boldsymbol{B}\boldsymbol{\nabla} \cdot \boldsymbol{A} + \boldsymbol{B} \cdot \boldsymbol{\nabla}\boldsymbol{A} - \boldsymbol{A} \cdot \boldsymbol{\nabla}\boldsymbol{B}$$

Momentum Equation: Nonlinear Terms

Verbatim from Jeff Oishi's MRI paper:

$$\frac{D\boldsymbol{u}'}{Dt} + f\hat{\boldsymbol{z}} \times \boldsymbol{u}' + Su'_x\hat{\boldsymbol{y}} + \boldsymbol{\nabla}p' + \nu\boldsymbol{\nabla} \times \boldsymbol{\omega}' = B_0\partial_z \boldsymbol{b}'$$

where f is the corriolis parameter, S is the background shearing rate, and $B_0\hat{\boldsymbol{z}}$ is a uniform background magnetic field. This equation is linearized wrt perturbations. Accordingly, the material derivative goes like

$$\frac{D}{Dt} \equiv \partial_t + \boldsymbol{u} \cdot \boldsymbol{\nabla}$$
$$= \partial_t + Sx\partial_y$$

due to the background velocity $\overline{\boldsymbol{u}} = Sx\hat{\boldsymbol{y}}$. In the nonlinear case we have

$$= \partial_t + (Sx\hat{\boldsymbol{y}} + \boldsymbol{u}') \cdot \boldsymbol{\nabla}$$

From inspection and stuff, the irrotational momentum equation goes like

$$\frac{D\boldsymbol{u}}{Dt} + \boldsymbol{\nabla}p + \nu \times \boldsymbol{\omega} = \boldsymbol{b} \cdot \boldsymbol{\nabla}\boldsymbol{b}$$

Next we generalize $\boldsymbol{u} = \boldsymbol{u}' + Sx\hat{\boldsymbol{y}}$ and $\boldsymbol{b} = \boldsymbol{b}' + B_0\hat{\boldsymbol{z}}$, giving

$$\underline{\partial_t\boldsymbol{u}' + \boldsymbol{u}' \cdot \boldsymbol{\nabla}\boldsymbol{u}' + Sx\partial_y\boldsymbol{u}' + Su_x'\hat{\boldsymbol{y}}} + \boldsymbol{\nabla}p + \nu\boldsymbol{\nabla} \times \boldsymbol{\omega} = B_0\partial_z\boldsymbol{b}' + \boldsymbol{b}' \cdot \boldsymbol{\nabla}\boldsymbol{b}'$$

where the material derivative $\frac{D\boldsymbol{u}}{Dt}$ consists of the underlined terms. Note this definition differs from that of the associated script

Induction Equation: Nonlinear Terms

The MHD induction equation (as given by *Fluid Mechanics of Planets and Stars, 2019*) is given by

$$\partial_t\boldsymbol{b} = \underline{\boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{b})} + \eta\nabla^2\boldsymbol{b}.$$

We expand the underlined term using the following identity

$$\boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{b}) = \boldsymbol{u}\boldsymbol{\nabla} \cdot \boldsymbol{b} - \boldsymbol{b}\boldsymbol{\nabla} \cdot \boldsymbol{u} + \boldsymbol{b} \cdot \boldsymbol{\nabla}\boldsymbol{u} - \boldsymbol{u} \cdot \boldsymbol{\nabla}\boldsymbol{b}$$

where the first two terms on the RHS vanish due to incompressibility and the abscence of magnetic monopoles. Therefore

$$\partial_t\boldsymbol{b} + \boldsymbol{u} \cdot \boldsymbol{\nabla}\boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{\nabla}\boldsymbol{u} + \eta\nabla^2\boldsymbol{b}.$$

Substituting the decompositions for $\boldsymbol{u}$ and $\boldsymbol{b}$ as above yields

$$\partial_t(B_0\hat{\boldsymbol{z}} + \boldsymbol{b}') + (Sx\hat{\boldsymbol{y}} + \boldsymbol{u}') \cdot \boldsymbol{\nabla}(B_0\hat{\boldsymbol{z}} + \boldsymbol{b}') = (B_0\hat{\boldsymbol{z}} + \boldsymbol{b}') \cdot \boldsymbol{\nabla}(Sx\hat{\boldsymbol{y}} + \boldsymbol{u}') + \eta\nabla^2(B_0\hat{\boldsymbol{z}} + \boldsymbol{b}')$$
$$\partial_t\boldsymbol{b}' + (Sx\hat{\boldsymbol{y}} + \boldsymbol{u}') \cdot \boldsymbol{\nabla}\boldsymbol{b}' = (B_0\hat{\boldsymbol{z}} + \boldsymbol{b}') \cdot \boldsymbol{\nabla}(Sx\hat{\boldsymbol{y}} + \boldsymbol{u}') + \eta\nabla^2\boldsymbol{b}'$$

dropping the $'$s and taking the $x, y,$ and z components of the above yields

$$\partial_t b_x + Sx\partial_y b_x + \boldsymbol{u} \cdot \boldsymbol{\nabla}b_x = B_0\partial_z u_x + \boldsymbol{b} \cdot \boldsymbol{\nabla}u_x + \eta\nabla^2 b_x$$
$$\partial_t b_y + Sx\partial_y b_y + \boldsymbol{u} \cdot \boldsymbol{\nabla}b_y = B_0\partial_z u_y + \boldsymbol{b} \cdot \boldsymbol{\nabla}u_y + Sb_x + \eta\nabla^2 b_y$$
$$\partial_t b_z + Sx\partial_y b_z + \boldsymbol{u} \cdot \boldsymbol{\nabla}b_z = B_0\partial_z u_z + \boldsymbol{b} \cdot \boldsymbol{\nabla}u_z + \eta\nabla^2 b_z$$

Note that the linear operator on the LHS is identical for each scalar equation. Accordingly, we redefine D_t to be this operator, i.e.

$$D_t A \equiv \partial_t A + Sx\partial_y A + \boldsymbol{u} \cdot \boldsymbol{\nabla} A.$$

Recall that the current density is given by

$$\boldsymbol{j} = j_x\hat{\boldsymbol{x}} + j_y\hat{\boldsymbol{y}} + j_z\hat{\boldsymbol{z}} = \boldsymbol{\nabla} \times \boldsymbol{b}$$
$$j_x = \partial_y b_z - \partial_z b_y$$

We then rewrite the laplacian term as follows

$$\eta\nabla^2\boldsymbol{b} = -\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{b} = -\boldsymbol{\nabla} \times \boldsymbol{j}.$$

Therefore the x-component of the induction equation is given by

$$D_t b_x - B_0\partial_z u_x + \eta(\partial_y j_z - \partial_z j_y) = \boldsymbol{b} \cdot \boldsymbol{\nabla} u_x$$

Finally, we combine the x and y components of the induction equation. Taking the material derivative of the x-component of the current density gives

$$D_t j_x = \partial_y D_t b_z - \partial_z D_t b_y.$$

We have equations for that stuff!

$$D_t j_x = \partial_y(B_0\partial_z u_z + \boldsymbol{b} \cdot \boldsymbol{\nabla} u_z + \eta\nabla^2 b_z) - \partial_z(B_0\partial_z u_y + \boldsymbol{b} \cdot \boldsymbol{\nabla} u_y + Sb_x + \eta\nabla^2 b_y)$$
$$= B_0\partial_z\omega_x + \boldsymbol{b} \cdot \boldsymbol{\nabla}\omega_x + \eta\nabla^2 j_x - S\partial_z b_x$$

Rearranging and expanding gives

$$\partial_t j_x + Sx\partial_y j_x - B_0\partial_z\omega_x + S\partial_z b_x - \eta\nabla^2 j_x = \boldsymbol{b} \cdot \boldsymbol{\nabla}\omega_x - \boldsymbol{u} \cdot \boldsymbol{\nabla} j_x$$