

# Least-Squares Time Reversal of PDEs

Liam O'Connor<sup>1,2</sup>

<sup>1</sup>*Department of Engineering Sciences and Applied Mathematics,  
Northwestern University, Evanston, IL 60208 USA and*

<sup>2</sup>*Center for Interdisciplinary Exploration and Research in Astrophysics,  
Northwestern University, Evanston, IL, 60201 USA*

abstract

## I. INTRODUCTION

Rayleigh-Bénard convection plays a foundational role in astrophysical and geophysical settings. The resulting buoyancy-driven flows regulate heat transfer and generate large-scale vortices [1]. Turbulent convection, which is associated with large Rayleigh numbers  $Ra$ , is difficult to simulate. State of the art simulations performed by [2] have reached  $Ra \sim 10^{14}$  but estimates for the sun's convective zone and earth's interior are  $Ra \sim 10^{16} - 10^{20}$  and  $Ra \sim 10^{20} - 10^{30}$  respectively [3, 4].

## II. GENERALIZED PROBLEM SETUP

Here I'll present a generalized PDE which we want to invert along with a bunch of notation. We'll define our target simulation, least-squares objective, and cite examples where the adjoint looping method has been successfully applied to inverse problems. We'll conclude this discussion by stressing the importance of the objective function's shape in "initial condition space", because ultimately this is what curtails our gradient-based inversion technique.

## III. ANALYTIC ANALYSIS

### A. Linearity and Reversibility

Reversible linear problems are trivial cases of the problems we intend to study. Linear systems' objective functions' shapes do not depend on the target—the target manifests itself as a constant shift in ic space. Thus we might as well consider problems with  $U(0) = U(T) = 0$  for all  $x$ . Reversible wave equations, which conserve energy yield objective functions whose levelsets in ic space are spherical, so the gradient at any given guess always provides the shortest path to the target. In contrast, irreversible diffusive terms distort the spherical levelsets into hyperellipses, whose semi-minimum (maximum) axis is oriented in the direction of the lowest (highest) wavenumber fourier mode's coefficient. Thus, when traversing an arbitrary path of steepest descent toward the target IC, low wavenumber modes are matched with the target first, followed by increasingly higher wavenumber modes.

### B. Nonlinear Reversible Problems

Here the target matters. We consider burger's equation

$$\partial_t u + u \partial_x u = 0$$

where energy is conserved

$$\begin{aligned} 0 &= \partial_t \langle \frac{1}{2} u^2 \rangle + \langle u^2 \partial_x u \rangle \\ &= \partial_t \langle \frac{1}{2} u^2 \rangle + \langle \frac{1}{3} \partial_x u^3 \rangle \\ &= \partial_t \langle \frac{1}{2} u^2 \rangle \end{aligned}$$

Next we consider a guess with a known deviation from the target  $u(x, 0) = U(x, 0) + u'(x, 0)$ . Our objective function is given by  $\langle (u(T) - U(T))^2 \rangle = \langle u'(T)^2 \rangle$ . The forward evolution equation for the perturbation is given by

$$0 = \partial_t u' + u' \partial_x U + U \partial_x u' + u' \partial_x u'$$

then we have that

the evolution of the target is given by

$$\begin{aligned} 0 &= \frac{1}{2} \partial_t \langle u'^2 \rangle + \langle u' \partial_x (u' U) \rangle + \langle \frac{1}{3} \partial_x u'^3 \rangle \\ &= \frac{1}{2} \partial_t \langle u'^2 \rangle + \langle u' \partial_x (u' U) \rangle \\ &= \frac{1}{2} \partial_t \langle u'^2 \rangle + \langle U \partial_x (u'^2) \rangle \end{aligned}$$

Suppose  $u' = \alpha U$ . In this case  $u = U + u' = (1 + \alpha)U$  evolves identically to  $U$  but at a rate which is  $(1 + \alpha)^2$  faster. Thus the objective is given by  $\langle (U((1 + \alpha)^2 T) - U(T))^2 \rangle$

$$\begin{aligned} u'(T) &= u'(0) + \int_0^T \partial_t u' dt \\ &= u'(0) - \int_0^T \partial_x (u' U) + \frac{1}{2} \partial_x (u'^2) dt \end{aligned}$$

Therefore  $\langle u'(T) \rangle = \langle u'(0) \rangle$ . The objective function

$$\begin{aligned} \langle u'(T)^2 \rangle &= \langle u'(0)^2 \rangle - \langle u'(0) \int_0^T \partial_x (u' U) + \frac{1}{2} \partial_x (u'^2) dt \rangle \\ &\quad + \langle \left[ \int_0^T \partial_x (u' U) + \frac{1}{2} \partial_x (u'^2) dt \right]^2 \rangle \end{aligned}$$

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