MRI Notes

Induction Equation: Potential Form

The induction equation is given by

$$\partial_t \boldsymbol{b} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{b}) + \eta \boldsymbol{\nabla}^2 \boldsymbol{b}$$

Using the following identity

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{f} = \boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \boldsymbol{f} - \boldsymbol{\nabla}^2 \boldsymbol{f}$$

and assumming η to be constant, we use $\boldsymbol{\nabla} \cdot \boldsymbol{b} = 0$, giving

$$\partial_t \boldsymbol{b} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{b}) - \eta \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{b}.$$

Then we define a vector potential $\boldsymbol{\nabla} \times \boldsymbol{A} \equiv \boldsymbol{b}$, yielding

$$\begin{aligned}
\partial_t \boldsymbol{\nabla} \times \boldsymbol{A} &= \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{b}) - \boldsymbol{\nabla} \times (\eta \boldsymbol{\nabla} \times \boldsymbol{b}) \\
\partial_t \boldsymbol{A} &= \boldsymbol{u} \times \boldsymbol{b} - \eta \boldsymbol{\nabla} \times \boldsymbol{b} + \boldsymbol{\nabla}\phi
\end{aligned}$$

where ϕ is a scalar potential arizing from "uncurling" the equation. We must then provide an additional constraint to fix ϕ: the Coulomb gauge $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$. Therefore

$$-\boldsymbol{\nabla} \times \boldsymbol{b} = -\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{A} = \boldsymbol{\nabla}^2 \boldsymbol{A}.$$

Next we decompose $\boldsymbol{u} = \boldsymbol{u_0} + \boldsymbol{u'}$ and $\boldsymbol{b} = \boldsymbol{b_0} + \boldsymbol{b'}$. We assume the mean quantities $\boldsymbol{u_0}$ and $\boldsymbol{b_0}$ are themselves solutions to the original problem. If we consider only the 0th mode of $\boldsymbol{b}$, i.e. $\boldsymbol{b} \cdot \boldsymbol{\hat{e}_i} \sim e^{i0}$ then clearly $\boldsymbol{\nabla}^2 \boldsymbol{b} = \boldsymbol{0}$ and therefore

$$\partial_t \boldsymbol{b'} = \boldsymbol{\nabla} \times (\boldsymbol{u_0} \times \boldsymbol{b'}) + \boldsymbol{\nabla} \times (\boldsymbol{u'} \times \boldsymbol{b_0}) + \boldsymbol{\nabla} \times (\boldsymbol{u'} \times \boldsymbol{b'}).$$

Using another identity

$$\boldsymbol{\nabla} \times (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{A}\boldsymbol{\nabla} \cdot \boldsymbol{B} - \boldsymbol{B}\boldsymbol{\nabla} \cdot \boldsymbol{A} + \boldsymbol{B} \cdot \boldsymbol{\nabla}\boldsymbol{A} - \boldsymbol{A} \cdot \boldsymbol{\nabla}\boldsymbol{B}$$

Momentum Equation: Nonlinear Terms

Verbatim from Jeff Oishi's MRI paper:

$$\frac{D\boldsymbol{u'}}{Dt} + f\boldsymbol{\hat{z}} \times \boldsymbol{u'} + Su'_x\boldsymbol{\hat{y}} + \boldsymbol{\nabla}p' + \nu\boldsymbol{\nabla} \times \boldsymbol{\omega}' = B_0\partial_z\boldsymbol{b'}$$

where f is the corriolis parameter, S is the background shearing rate, and $B_0\boldsymbol{\hat{z}}$ is a uniform background magnetic field. This equation is linearized wrt perturbations. Accordingly, the material derivative goes like

$$\begin{aligned}
\frac{D}{Dt} &\equiv \partial_t + \boldsymbol{u} \cdot \boldsymbol{\nabla} \\
&= \partial_t + Sx\partial_y
\end{aligned}$$

due to the background velocity $\overline{\boldsymbol{u}} = Sx\hat{\boldsymbol{y}}$. In the nonlinear case we have

$$= \partial_t + (Sx\hat{\boldsymbol{y}} + \boldsymbol{u'}) \cdot \boldsymbol{\nabla}$$

From inspection and stuff, the irrotational momentum equation goes like

$$\frac{D\boldsymbol{u}}{Dt} + \boldsymbol{\nabla} p + \nu \times \boldsymbol{\omega} = \boldsymbol{b} \cdot \boldsymbol{\nabla} \boldsymbol{b}$$

Next we generalize $\boldsymbol{u} = \boldsymbol{u'} + Sx\hat{\boldsymbol{y}}$ and $\boldsymbol{b} = \boldsymbol{b'} + B_0\hat{\boldsymbol{z}}$, giving

$$\underline{\partial_t \boldsymbol{u'} + \boldsymbol{u'} \cdot \boldsymbol{\nabla} \boldsymbol{u'} + Sx\partial_y \boldsymbol{u'} + Su'_x\hat{\boldsymbol{y}}} + \boldsymbol{\nabla} p + \nu \boldsymbol{\nabla} \times \boldsymbol{\omega} = B_0 \partial_z \boldsymbol{b'} + \boldsymbol{b'} \cdot \boldsymbol{\nabla} \boldsymbol{b'}$$

where the material derivative $\frac{D\boldsymbol{u}}{Dt}$ consists of the underlined terms

Induction Equation: Nonlinear Terms

The MHD induction equation (as given by *Fluid Mechanics of Planets and Stars, 2019*) is given by

$$\partial_t \boldsymbol{b} = \underline{\boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{b})} + \eta \nabla^2 \boldsymbol{b}.$$

We expand the underlined term using the following identity

$$\boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{b}) = \boldsymbol{u} \boldsymbol{\nabla} \cdot \boldsymbol{b} - \boldsymbol{b} \boldsymbol{\nabla} \cdot \boldsymbol{u} + \boldsymbol{b} \cdot \boldsymbol{\nabla} \boldsymbol{u} - \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{b}$$

where the first two terms on the RHS vanish due to incompressibility and the abscence of magnetic monopoles. Therefore

$$\partial_t \boldsymbol{b} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{\nabla} \boldsymbol{u} + \eta \nabla^2 \boldsymbol{b}.$$

Substituting the decompositions for $\boldsymbol{u}$ and $\boldsymbol{b}$ as above yields

$$\begin{aligned} \partial_t(B_0\hat{\boldsymbol{z}} + \boldsymbol{b'}) + (Sx\hat{\boldsymbol{y}} + \boldsymbol{u'}) \cdot \boldsymbol{\nabla}(B_0\hat{\boldsymbol{z}} + \boldsymbol{b'}) &= (B_0\hat{\boldsymbol{z}} + \boldsymbol{b'}) \cdot \boldsymbol{\nabla}(Sx\hat{\boldsymbol{y}} + \boldsymbol{u'}) + \eta \nabla^2 (B_0\hat{\boldsymbol{z}} + \boldsymbol{b'}) \\ \partial_t \boldsymbol{b'} + (Sx\hat{\boldsymbol{y}} + \boldsymbol{u'}) \cdot \boldsymbol{\nabla} \boldsymbol{b'} &= (B_0\hat{\boldsymbol{z}} + \boldsymbol{b'}) \cdot \boldsymbol{\nabla}(Sx\hat{\boldsymbol{y}} + \boldsymbol{u'}) + \eta \nabla^2 \boldsymbol{b'} \end{aligned}$$

dropping the $'$s and taking the x, y, and z components of the above yields

$$\begin{aligned} \partial_t b_x + Sx\partial_y b_x + \boldsymbol{u} \cdot \boldsymbol{\nabla} b_x &= B_0 \partial_z u_x + \boldsymbol{b} \cdot \boldsymbol{\nabla} u_x + \eta \nabla^2 b_x \\ \partial_t b_y + Sx\partial_y b_y + \boldsymbol{u} \cdot \boldsymbol{\nabla} b_y &= B_0 \partial_z u_y + \boldsymbol{b} \cdot \boldsymbol{\nabla} u_y + Sb_x + \eta \nabla^2 b_y \\ \partial_t b_z + Sx\partial_y b_z + \boldsymbol{u} \cdot \boldsymbol{\nabla} b_z &= B_0 \partial_z u_z + \boldsymbol{b} \cdot \boldsymbol{\nabla} u_z + \eta \nabla^2 b_z \end{aligned}$$

Note that the linear operator on the LHS is identical for each scalar equation. Accordingly, we redefine D_t to be this operator, i.e.

$$D_t A \equiv \partial_t A + Sx\partial_y A + \boldsymbol{u} \cdot \boldsymbol{\nabla} A.$$

Recall that the current density is given by

$$\begin{aligned} \boldsymbol{j} = j_x\hat{\boldsymbol{x}} + j_y\hat{\boldsymbol{y}} + j_z\hat{\boldsymbol{z}} &= \boldsymbol{\nabla} \times \boldsymbol{b} \\ j_x &= \partial_y b_z - \partial_z b_y \end{aligned}$$

We then rewrite the laplacian term as follows

$$\eta\nabla^2\boldsymbol{b} = -\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{b} = -\boldsymbol{\nabla} \times \boldsymbol{j}.$$

Therefore the x-component of the induction equation is given by

$$D_t b_x - B_0\partial_z u_x + \eta(\partial_y j_z - \partial_z j_y) = \boldsymbol{b} \cdot \boldsymbol{\nabla} u_x$$

Finally, we combine the x and y components of the induction equation. Taking the material derivative of the x-component of the current density gives

$$D_t j_x = \partial_y D_t b_z - \partial_z D_t b_y.$$

We have equations for that stuff!

$$\begin{aligned} D_t j_x &= \partial_y(B_0\partial_z u_z + \boldsymbol{b} \cdot \boldsymbol{\nabla} u_z + \eta\nabla^2 b_z) - \partial_z(B_0\partial_z u_y + \boldsymbol{b} \cdot \boldsymbol{\nabla} u_y + Sb_x + \eta\nabla^2 b_y) \\ &= B_0\partial_z\omega_x + \boldsymbol{b} \cdot \boldsymbol{\nabla}\omega_x + \eta\nabla^2 j_x - S\partial_z b_x \end{aligned}$$

Rearranging and expanding gives

$$\partial_t j_x + Sx\partial_y j_x - B_0\partial_z\omega_x + S\partial_z b_x - \eta\nabla^2 j_x = \boldsymbol{b} \cdot \boldsymbol{\nabla}\omega_x - \boldsymbol{u} \cdot \boldsymbol{\nabla} j_x$$