

MRI Notes

Momentum Equation: Nonlinear Terms

Verbatim from Jeff Oishi's MRI paper:

$$\frac{D\mathbf{u}'}{Dt} + f\hat{\mathbf{z}} \times \mathbf{u}' + Su'_x\hat{\mathbf{y}} + \nabla p' + \nu \nabla \times \boldsymbol{\omega}' = B_0 \partial_z \mathbf{b}'$$

where f is the coriolis parameter, S is the background shearing rate, and $B_0\hat{\mathbf{z}}$ is a uniform background magnetic field. This equation is linearized wrt perturbations. Accordingly, the material derivative goes like

$$\begin{aligned} \frac{D}{Dt} &\equiv \partial_t + \mathbf{u} \cdot \nabla \\ &= \partial_t + Sx\partial_y \end{aligned}$$

due to the background velocity $\bar{\mathbf{u}} = Sx\hat{\mathbf{y}}$. In the nonlinear case we have

$$= \partial_t + (Sx\hat{\mathbf{y}} + \mathbf{u}') \cdot \nabla$$

From inspection and stuff, the irrotational momentum equation goes like

$$\frac{D\mathbf{u}}{Dt} + \nabla p + \nu \nabla \times \boldsymbol{\omega} = \mathbf{b} \cdot \nabla \mathbf{b}$$

Next we generalize $\mathbf{u} = \mathbf{u}' + Sx\hat{\mathbf{y}}$ and $\mathbf{b} = \mathbf{b}' + B_0\hat{\mathbf{z}}$, giving

$$\underline{\partial_t \mathbf{u}' + \mathbf{u}' \cdot \nabla \mathbf{u}' + Sx\partial_y \mathbf{u}' + Su'_x \hat{\mathbf{y}} + \nabla p + \nu \nabla \times \boldsymbol{\omega}} = B_0 \partial_z \mathbf{b}' + \mathbf{b}' \cdot \nabla \mathbf{b}'$$

where the material derivative $\frac{D\mathbf{u}}{Dt}$ consists of the underlined terms. Note this definition differs from that of the associated script

Induction Equation: Nonlinear Terms

The MHD induction equation (as given by *Fluid Mechanics of Planets and Stars, 2019*) is given by

$$\partial_t \mathbf{b} = \underline{\nabla \times (\mathbf{u} \times \mathbf{b})} + \eta \nabla^2 \mathbf{b}.$$

We expand the underlined term using the following identity

$$\nabla \times (\mathbf{u} \times \mathbf{b}) = \mathbf{u} \nabla \cdot \mathbf{b} - \mathbf{b} \nabla \cdot \mathbf{u} + \mathbf{b} \cdot \nabla \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{b}$$

where the first two terms on the RHS vanish due to incompressibility and the absence of magnetic monopoles. Therefore

$$\partial_t \mathbf{b} + \mathbf{u} \cdot \nabla \mathbf{b} = \mathbf{b} \cdot \nabla \mathbf{u} + \eta \nabla^2 \mathbf{b}.$$

Substituting the decompositions for \mathbf{u} and \mathbf{b} as above yields

$$\begin{aligned}\partial_t(B_0\hat{\mathbf{z}} + \mathbf{b}') + (Sx\hat{\mathbf{y}} + \mathbf{u}') \cdot \nabla(B_0\hat{\mathbf{z}} + \mathbf{b}') &= (B_0\hat{\mathbf{z}} + \mathbf{b}') \cdot \nabla(Sx\hat{\mathbf{y}} + \mathbf{u}') + \eta\nabla^2(B_0\hat{\mathbf{z}} + \mathbf{b}') \\ \partial_t\mathbf{b}' + (Sx\hat{\mathbf{y}} + \mathbf{u}') \cdot \nabla\mathbf{b}' &= (B_0\hat{\mathbf{z}} + \mathbf{b}') \cdot \nabla(Sx\hat{\mathbf{y}} + \mathbf{u}') + \eta\nabla^2\mathbf{b}'\end{aligned}$$

dropping the 's and taking the x, y , and z components of the above yields

$$\begin{aligned}\partial_t b_x + Sx\partial_y b_x + \mathbf{u} \cdot \nabla b_x &= B_0\partial_z u_x + \mathbf{b} \cdot \nabla u_x + \eta\nabla^2 b_x \\ \partial_t b_y + Sx\partial_y b_y + \mathbf{u} \cdot \nabla b_y &= B_0\partial_z u_y + \mathbf{b} \cdot \nabla u_y + Sb_x + \eta\nabla^2 b_y \\ \partial_t b_z + Sx\partial_y b_z + \mathbf{u} \cdot \nabla b_z &= B_0\partial_z u_z + \mathbf{b} \cdot \nabla u_z + \eta\nabla^2 b_z\end{aligned}$$

Note that the nonlinear operator on the LHS is identical for each scalar equation. Accordingly, we redefine D_t to be this operator, i.e.

$$D_t A \equiv \partial_t A + Sx\partial_y A + \mathbf{u} \cdot \nabla A$$

NOTE: the material derivative substitution implemented in Dedalus excludes the nonlinear term.

We continue by taking ∂_z of the b_y equation

$$\begin{aligned}\partial_z D_t b_y &= \partial_z \left[\partial_t b_y + Sx\partial_y b_y + \mathbf{u} \cdot \nabla b_y \right] = \partial_z B_0\partial_z u_y + \partial_z [\mathbf{b} \cdot \nabla u_y] + S\partial_z b_x + \eta\nabla^2 \partial_z b_y \\ D_t \partial_z b_y + \partial_z \mathbf{u} \cdot \nabla b_y &= B_0\partial_z^2 u_y + \partial_z \mathbf{b} \cdot \nabla u_y + \mathbf{b} \cdot \nabla \partial_z u_y + S\partial_z b_x + \eta\nabla^2 \partial_z b_y\end{aligned}$$

and the ∂_y of the b_z equation

$$\begin{aligned}\partial_y D_t b_z &= \partial_y \left[\partial_t b_z + Sx\partial_y b_z + \mathbf{u} \cdot \nabla b_z \right] = B_0\partial_z \partial_y u_z + \partial_y [\mathbf{b} \cdot \nabla u_z] + \eta\nabla^2 \partial_y b_z \\ D_t \partial_y b_z + \partial_y \mathbf{u} \cdot \nabla b_z &= B_0\partial_z \partial_y u_z + \partial_y \mathbf{b} \cdot \nabla u_z + \mathbf{b} \cdot \nabla \partial_y u_z + \eta\nabla^2 \partial_y b_z\end{aligned}$$

Recall that the current density is given by

$$\begin{aligned}\mathbf{j} &= j_x \hat{\mathbf{x}} + j_y \hat{\mathbf{y}} + j_z \hat{\mathbf{z}} = \nabla \times \mathbf{b} \\ j_x &= \partial_y b_z - \partial_z b_y \\ D_t j_x &= D_t \partial_y b_z - D_t \partial_z b_y\end{aligned}$$

Therefore

$$D_t j_x + \partial_y \mathbf{u} \cdot \nabla b_z - \partial_z \mathbf{u} \cdot \nabla b_y = B_0\partial_z \omega_x + \mathbf{b} \cdot \nabla \omega_x - S\partial_z b_x + \eta\nabla^2 j_x + \partial_y \mathbf{b} \cdot \nabla u_z - \partial_z \mathbf{b} \cdot \nabla u_y$$

Expanding the material derivative and grouping linear terms on the LHS gives

$$\begin{aligned}\partial_t j_x + Sx\partial_y j_x - B_0\partial_z \omega_x + S\partial_z b_x - \eta\nabla^2 j_x &= \mathbf{b} \cdot \nabla \omega_x - \mathbf{u} \cdot \nabla j_x \\ &\quad + \partial_y \mathbf{b} \cdot \nabla u_z - \partial_z \mathbf{b} \cdot \nabla u_y \\ &\quad - \partial_y \mathbf{u} \cdot \nabla b_z + \partial_z \mathbf{u} \cdot \nabla b_y\end{aligned}$$

Let's check that. Using the following identity

$$\begin{aligned}
\nabla \times \nabla \times (\mathbf{u} \times \mathbf{b}) &= \nabla \times (\mathbf{u}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{u}) + (\mathbf{b} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{b}) \\
&= \nabla \times ((\mathbf{b} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{b}) \\
&= \nabla \times \left(\begin{bmatrix} \mathbf{b} \cdot \nabla u_x \\ \mathbf{b} \cdot \nabla u_y \\ \mathbf{b} \cdot \nabla u_z \end{bmatrix} - \begin{bmatrix} \mathbf{u} \cdot \nabla b_x \\ \mathbf{u} \cdot \nabla b_y \\ \mathbf{u} \cdot \nabla b_z \end{bmatrix} \right)
\end{aligned}$$

Taking the x -component gives

$$\begin{aligned}
\hat{x} \cdot \nabla \times \nabla \times (\mathbf{u} \times \mathbf{b}) &= \partial_y (\mathbf{b} \cdot \nabla u_z - \mathbf{u} \cdot \nabla b_z) - \partial_z (\mathbf{b} \cdot \nabla u_y - \mathbf{u} \cdot \nabla b_y) \\
&= \partial_y \mathbf{b} \cdot \nabla u_z + \mathbf{b} \cdot \nabla \partial_y u_z - \partial_y \mathbf{u} \cdot \nabla b_z - \mathbf{u} \cdot \nabla \partial_y b_z \\
&\quad - \partial_z \mathbf{b} \cdot \nabla u_y - \mathbf{b} \cdot \nabla \partial_z u_y + \partial_z \mathbf{u} \cdot \nabla b_y + \mathbf{u} \cdot \nabla \partial_z b_y \\
&= \mathbf{b} \cdot \nabla \omega_x - \mathbf{u} \cdot \nabla j_x \\
&\quad + \partial_y \mathbf{b} \cdot \nabla u_z - \partial_z \mathbf{b} \cdot \nabla u_y + \partial_z \mathbf{u} \cdot \nabla b_y - \partial_y \mathbf{u} \cdot \nabla b_z
\end{aligned}$$

Letting $B_0 = S = 0$, the expanded current-density equation is given by

$$\begin{aligned}
\partial_t j_x &= \eta \nabla^2 j_x \\
&\quad + \mathbf{b} \cdot \nabla \omega_x - \mathbf{u} \cdot \nabla j_x \\
&\quad + \partial_y \mathbf{b} \cdot \nabla u_z - \partial_z \mathbf{b} \cdot \nabla u_y + \partial_z \mathbf{u} \cdot \nabla b_y - \partial_y \mathbf{u} \cdot \nabla b_z \\
&= \eta \nabla^2 j_x + \hat{x} \cdot \nabla \times \nabla \times (\mathbf{u} \times \mathbf{b}).
\end{aligned}$$

by symmetry we have

$$\partial_t \mathbf{j} = \eta \nabla^2 \mathbf{j} + \nabla \times \nabla \times (\mathbf{u} \times \mathbf{b})$$

We can “uncurl” this equation, and if we assume the unknown scalar potential-gradient term $\nabla \phi = \mathbf{0}$, we have

$$\partial_t \mathbf{b} = \nabla \times (\mathbf{u} \times \mathbf{b}) + \eta \nabla^2 \mathbf{b}$$

which is the usual form of the induction equation