**Lesson 1 – Index notation**

**1. Vectors and vector operations**

Our standard form for the notation of a vector is . If we designate the components of this vector with a set of numerical indices: ; ; 

and a similar notation for the basis vectors, ; ; and . Then the vector can be represented in index notation as .

In the above notation *i* is called the *dummy index* since it shows up twice in the same term. In index notation, anytime an index appears twice in the result, it is equivalent to a dot product and is always going to be summed over. As a consequence of this, we will drop the summation sign knowing that it is always implied over repeated indices. The vector above can then be represented in the nice compact form



The convention of dropping the summation symbol over indices that occur multiple times is called the *Einstein summation convention* or summation convention for short. Differential operators also can be described in this notation. If we specify the individual components of the gradient operator as the following

;  

Then the gradient operator takes the index form



It is best to think about this type of notation in taking the gradient or divergence. In index notation, the divergence of the vector  is expressed as



The gradient of a scalar function  is expressed as



**Kronecker Delta function:**

A function often used in index notation is the Kronecker delta function, , which is defined as follows: 



The Kronecker delta function is an *isotropic tensor* that allows us to convert or contract indices.

**Alternating tensor:**

Another important tensor in index notation is the alternating tensor (also known as the “epsilon tensor”)  which has the following properties:



A very important identity is the “epsilon-delta” relation:



Conversion of indices

For a vector , application of the Kronecker delta allows us to convert from the *i* to the *j* index:



In the above equation, *i* is a dummy index and *j* is the free index.

Contraction of indices

For the vectors  and , the direct product of these two vector is a second order tensor but application of the Kronecker delta function leads to a direct product:



**Example:**

In chapter 5 we will learn about the stress tensor which is defined as



Currently, indices *i* and *j* are *free indices* in that they do not repeat in any of the terms. Now take the dot product of the above equation with the unit vector,  , then



Now the *j* index is a *dummy index* and *i* is a free index. Application of the definition of the Kronecker delta function allows us to simplify the first term in the above expression:



With a coordinate system in 3-dimensions, a vector may be presented by an ordered set of components with each having a projection on the coordinate axes 1,2,3:



Three commonly used coordinate systems are rectangular, cylindrical, and spherical. The vector  can be represented by the sum of the magnitudes of the projections on the orthogonal axes:



The unit vectors  are  in Cartesian (rectangular) coordinate system,  in cylindrical coordinate system, and  in spherical coordinate system.

Thus equivalent notation:





But



The magnitude of the vector is given as



Addition and subtraction of vectors in index notation is easy:

.

Multiplication by a scalar, *s* gives:



The dot product of two vectors gives:



The cross product of two vectors gives:



The del operator  is given as



Thus the partial derivative of a scalar  (gradient) is represented:



The Laplacian of a vector is the following:





Divergence of a vector :



Notice that *i* repeats in the divergence term, implying a dot product. Contrast this with the **gradient of a vector field**, which is a tensor with 9 elements:



Curl of a vector :



**2. Tensors.**

A tensor is an ordered array of nine components



The transpose of a Tensor, , simply interchanges the elements of each element

.

The **dyadic** product (not to be confused with the dot product) of two vectors results in a tensor:



Thus the gradient of a vector can be represented as the following.

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When the vector above is the velocity vector, the resulting velocity gradient is a two-dimensional tensor that determines the distortion of a fluid element as it moves. The velocity gradient can be separated into symmetric  and antisymmetric components:

,

where



and

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The symmetric component,  defines the *rate of strain* or *deformation* tensor acting on the fluid element. The diagonal components of *e* characterize the rate of stretching (aka longitudinal deformation, or variations of velocity in the direction of motion). For example, describes the rate of change of the east-west motion in the east-west direction, . All three components of stretching give the *trace* of *e* and the divergence of the velocity field:

,

which represents the rate of increase in volume(or dilation) of the fluid element. The off-diagonal elements of *e* describe the rate of *shear*, or transverse deformation (orthogonal to the motion).

The antisymmetric component of the velocity gradient, , is called the rotation tensor. It has no diagonal elements and describes relative motion, not deformation. It contains only three independent components and is related to the vorticity vector:



Tensors are easily added, in a manner similar to vectors:



The **double dot product** of two tensors results in a scalar:



And the dot product of a vector with a tensor is:

