Marginally-Stable Thermal Equilibria of Rayleigh-Bénard Convection

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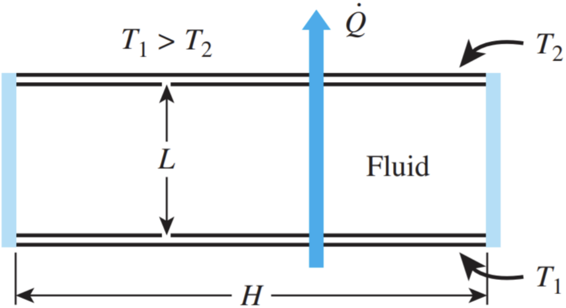
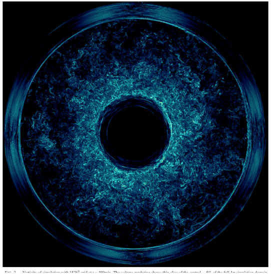
Rayleigh Bénard Convection

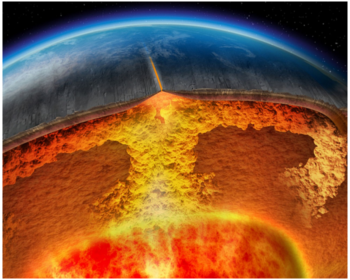


(Cengel 2015)

Why are we Still Studying Convection? **Applications**



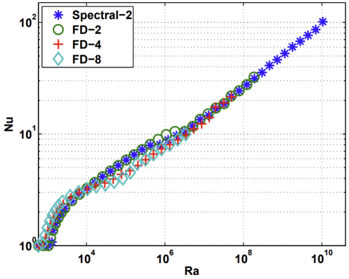
Astrophysics



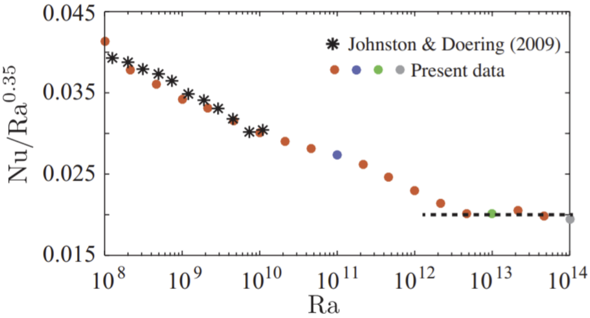
Geophysics

Engineering

Why are we Still Studying Convection? **It's Elusive**



(Johnson and Doering 2009)



(Zhu et al 2019)

Boussinesq Convection: The PDE

- Incompressible kinematics with buoyancy term

$$\nabla \cdot \boldsymbol{u} = 0 \tag{1}$$
$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + T\hat{z} + \mathcal{R}\nabla^2 \boldsymbol{u} \tag{2}$$
$$\frac{\partial T}{\partial t} + \boldsymbol{u} \cdot \nabla T = \mathcal{P}\nabla^2 T \tag{3}$$

where $\boldsymbol{u}$, T, and p and velocity, temperature, and pressure respectively.

- Momentum diffusivity: $\mathcal{R} = \sqrt{\frac{\mathrm{Pr}}{\mathrm{Ra}}}$
- Thermal diffusivity: $\mathcal{P} = \frac{1}{\sqrt{\mathrm{Pr}\,\mathrm{Ra}}}$.

Boussinesq Convection: Miscellaneous Prescriptions

- Nondimensionalized on the freefall timescale
- Domain: $\mathcal{D} = \{(x, z) \mid x \in (0, 4),\ z \in (-1/2, 1/2)\}$
- Prescribed-temperature: $T|_{z=-1/2}(x, t) = 1/2,\ T_{z=1/2}(x, t) = -1/2$

  and no-slip impenetrable $\boldsymbol{u}|_{z=\pm 1/2}(x, t) = \mathbf{0}$ boundary conditions
- Fixed Prandtl number $\mathrm{Pr} = 1$, varied Rayleigh number $10^6 \leq \mathrm{Ra} \leq 10^9$

Rayleigh's Linear Stability Analysis

- Assume variables can be decomposed like

$$\boldsymbol{u}(x,z,t) = \boldsymbol{u'}(x,z,t) = u'(x,z,t)\hat{x} + w'(x,z,t)\hat{z} \tag{4}$$
$$T(x,z,t) = \overline{T}(z,t) + T'(x,z,t) \tag{5}$$
$$p(x,z,t) = \overline{p}(z,t) + p'(x,z,t). \tag{6}$$

- The linear Eigenvalue Problem (EVP) is given by

$$\nabla \cdot \boldsymbol{u'} = 0 \tag{7}$$
$$\frac{\partial \boldsymbol{u'}}{\partial t} = -\nabla p' + T'\hat{z} + \mathcal{R}\nabla^2 \boldsymbol{u'} \tag{8}$$
$$\frac{\partial T'}{\partial t} + \frac{\partial \overline{T}}{\partial z} w' = \mathcal{P}\nabla^2 T' \tag{9}$$

Solutions to the Linear Problem

Solutions are given by

$$w'(x,z,t) = A\,\Re\left[W(z)\,e^{i(k_x x - st)}\right] \tag{10}$$

$$u'(x,z,t) = A\,\Re\left[U(z)\,e^{i(k_x x - st)}\right] \tag{11}$$

$$T'(x,z,t) = A\,\Re\left[\theta(z)\,e^{i(k_x x - st)}\right] \tag{12}$$

$$p'(x,z,t) = A\,\Re\left[P(z)\,e^{i(k_x x - st)}\right] \tag{13}$$

where A is the (undetermined) mode amplitude, $s = \omega + i\sigma$ is the eigenvalue,

and the allowed wavenumbers $k_x \in \left\{\frac{n\pi}{2} \mid n \in \mathbb{N}\right\}$.

The Quasilinear Model

- The 1D quasilinear initial value problem (IVP)

$$\frac{\partial \overline{T}}{\partial t} + \frac{\partial}{\partial z}\langle w' T' \rangle_x = \mathcal{P}\frac{\partial^2 \overline{T}}{\partial z^2} \tag{14}$$

- Evolution due to **Advection**: $\frac{\partial}{\partial z}\langle w' T' \rangle_x$ and **Diffusion**: $\mathcal{P}\frac{\partial^2 \overline{T}}{\partial z^2}$
- Derived by manipulating the nonlinear and linear equations
- Can be solved in conjunction with the EVP provided the amplitude A is known

$\longrightarrow$ **Marginal Stability**

Marginal Stability Constraint

- We assume the perturbations evolve on a shorter timescale than the background state
- We impose **Marginal Stability** at each timestep

$$\max_{k_x}\{\sigma\} = 0. \tag{15}$$

- The growth rate σ is obtained by solving the EVP for some $\overline{T}(z, t)$
- Does not allow us to solve for A directly $\longrightarrow$ discrete root-finding methods

Solving for the Perturbation Amplitude A

1. Given some guess for the amplitude A and a fixed timestep Δt
2. We evolve the initial (marginally stable) temperature profile $\overline{T}(z,t) \xrightarrow{\text{IVP}} \overline{T}(z,t+\Delta t)$
3. Solve the EVP using $\overline{T}(z,t+\Delta t)$
4. This renders $\sigma(A)$
5. Use Newton's method with finite differences to solve $\sigma(A) = 0$

The Amplitude Guess

- Consider advection and diffusion separately on $t \rightarrow t + \Delta t$
- Solve $\dfrac{\partial \overline{T}}{\partial t} + 2\dfrac{\partial}{\partial z}\Re\left[W\theta^*\right] = 0$, yielding $\overline{T}_{adv} \xrightarrow{\text{EVP}} \sigma_{\text{adv}}$
- Solve $\dfrac{\partial \overline{T}}{\partial t} = \mathcal{P}\dfrac{\partial^2 \overline{T}}{\partial z^2}$, yielding $\overline{T}_{adv} \xrightarrow{\text{EVP}} \sigma_{\text{diff}}$
- For small timesteps, we assume diffusion and advection act independently, i.e.

$$0 = \Delta\sigma \approx A^2\sigma_{\text{adv}} + \sigma_{\text{diff}} \quad \longrightarrow \quad A^2 \approx -\frac{\sigma_{\text{diff}}}{\sigma_{\text{adv}}} \tag{16}$$

Multiple Marginally Stable Modes

- Generalize

$$\langle w' T' \rangle_x = \sum_{n=1}^{N} 2A_n^2 \Re\left[W_n \theta_n^*\right] \tag{17}$$

  to accommodate N simultaneously marginally stable modes

- N amplitudes ($\boldsymbol{A} \in \mathbb{R}^N$) to maintain marginal stability in N modes

$$\boldsymbol{\sigma}(\boldsymbol{A}) = \mathbf{0} \tag{18}$$

- Modes with negative amplitudes are not included

Solving for the Amplitude Vector $\boldsymbol{A}$

1. Given some guess for the amplitude vector $\boldsymbol{A}$ and a fixed timestep Δt

2. We construct a Jacobian matrix with finite differences

$$J = \begin{bmatrix} \nabla\sigma_1(A_1, A_2, ..., A_N) \\ \nabla\sigma_2(A_1, A_2, ..., A_N) \\ \vdots \\ \nabla\sigma_N(A_1, A_2, ..., A_N) \end{bmatrix}. \tag{19}$$

3. Use Newton's method to solve $\boldsymbol{\sigma}(\boldsymbol{A}) = \mathbf{0}$

The Amplitude Guess for N modes

- The amplitude vector can be approximated by

$$\boldsymbol{A}^{\mathbf{2}} \approx= -\Sigma_{\mathrm{adv}}^{-1}\boldsymbol{\sigma}_{\mathbf{diff}}. \tag{20}$$

- The advective growth rate matrix Σ_{adv} measures the effect of the jth mode's advection on the ith mode's growth rate

- The diffusive growth rate vector $\boldsymbol{\sigma}_{\mathbf{diff}}$ measures the effect of diffusion on the ith mode's growth rate

Marginally Stable Thermal Equilibria

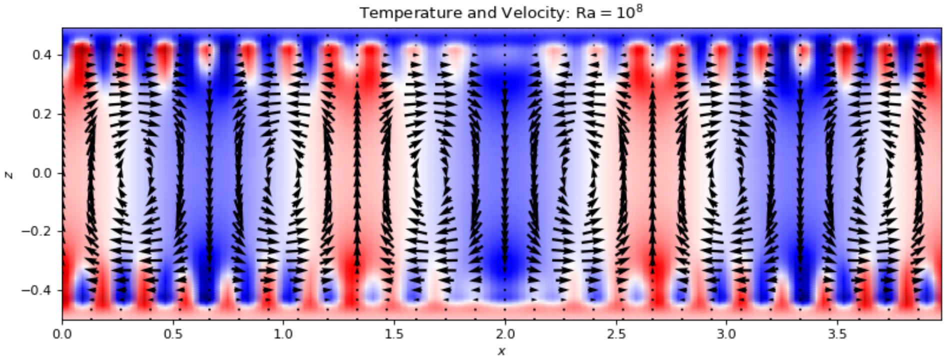
- Evolve until the **advection** cancels **diffusion**

$$\frac{\partial}{\partial z}\langle w' T' \rangle_x = \mathcal{P}\frac{\partial^2 \overline{T}}{\partial z^2} \quad \longrightarrow \quad \frac{\partial \overline{T}}{\partial t} = 0 \tag{21}$$

- Such states are referred to as Marginally Stable Thermal Equilibria (MSTE)

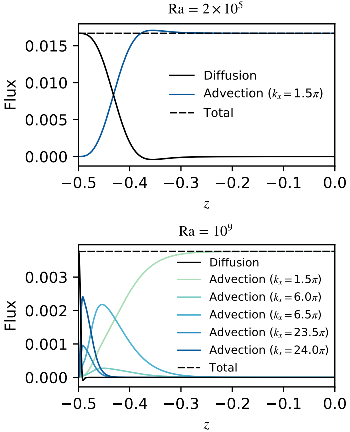
- We compute symmetric MSTE for $10^6 \leq \mathrm{Ra} \leq 10^9$

Marginally Stable Thermal Equilibria (MSTE)

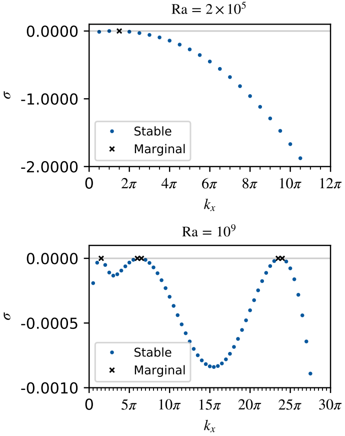


MSTE Flux and Spectra

- High-Ra cases have additional marginal modes at higher wavenumbers
- High wavenumber advections are concentrated near the boundaries, combating the diffusion of their thin boundary layers

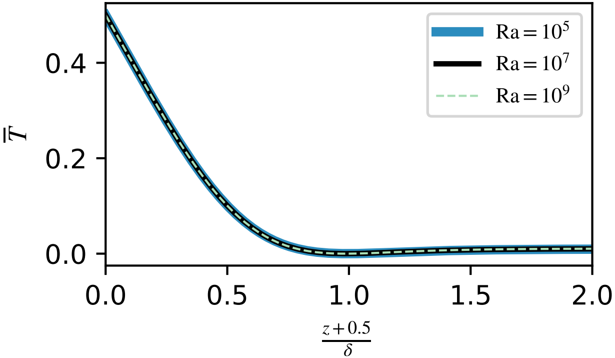


Flux

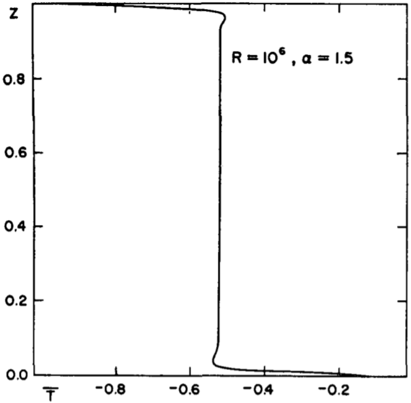


Spectra

Self-Similarity in $\overline{T}$ Near Boundaries



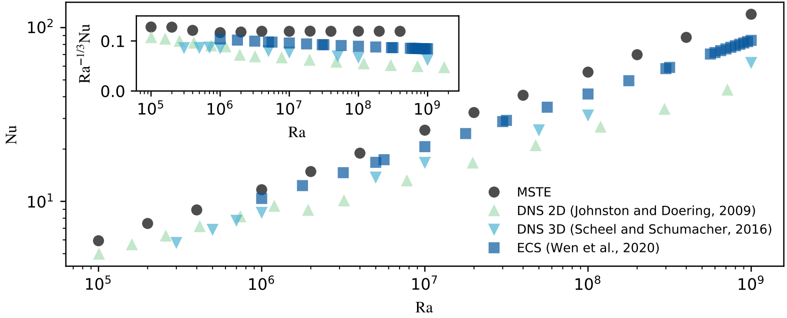
Rescaling according to δ reveals self-similarity in $\overline{T}$ profiles



(Herring 1963)

MSTE Nusselt Numbers: Classical Scaling

- MSTE exhibit “classical” or “Malkus” scaling: $\delta \sim \mathrm{Nu} \sim \mathrm{Ra}^{1/3}$
- MSTE transport heat more efficiently than simulations (DNS) and steady nonlinear solutions (ECS)



MSTE as DNS Initial Conditions

- MSTE are equilibria of the quasilinear equations **not** the full nonlinear equations
- But they ought to be close to the nonlinear system's strange attractor

Linear Initial Conditions

$$T(x,z)\big|_{t=0} = 0.5 - z + \mathcal{N}$$
$$\boldsymbol{u}(x,z)\big|_{t=0} = \mathbf{0} \qquad (22)$$

MSTE Initial Conditions

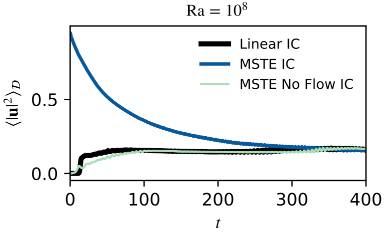
$$T(x,z)\big|_{t=0} = \overline{T}(z) + \sum_{n=1}^{N} A_n \Re\left[\theta_n(z) e^{ik_{x_n}x}\right] + \mathcal{N}$$
$$\boldsymbol{u}(x,z)\big|_{t=0} = \sum_{n=1}^{N} A_n \Re\left[\left(U_n(z)\hat{x} + W_n(z)\hat{z}\right) e^{ik_{x_n}x}\right]$$
$$(23)$$

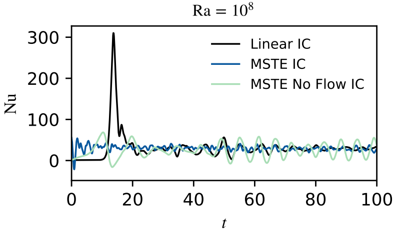
Nonlinear Simulations with MSTE Initial Conditions

No Convective Turnover!

Nonlinear Simulations with MSTE Initial Conditions

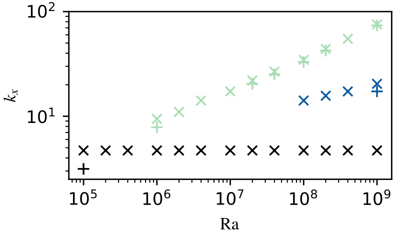
- MSTE simulations do not exhibit characteristic turnover
- They take longer to equilibrate due to exagerrated flows which viscously attenuate

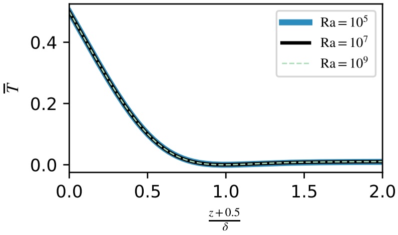




Future Work

- Explore mappings and $\mathrm{Ra} - \mathrm{Pr}$ parameter space
- MSTE in different domains: non-cartesian and different aspect ratios
- Compute high-Ra MSTE using intrinsic $\overline{T}$ profile and δ scaling





Acknowledgments

Co-Authors: Daniel Lecoanet (Northwestern ESAM), Evan Anders (Northwestern CIERA)

Special Thanks: Charlie Doering, Baole Wen, Greg Chini, Kyle Augustson, Emma Kaufman, Mallory Drevline

Supported By: Northwestern's Walter P. Murphy Fellowship for early career graduate students

Thank you!