Marginally-Stable Thermal Equilibria of Rayleigh-Bénard Convection

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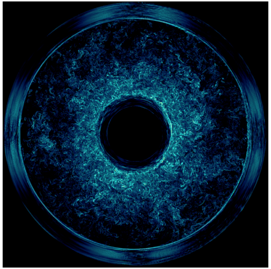
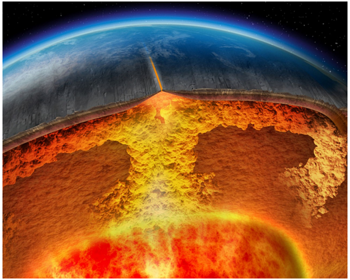
Why are we Still Studying Convection? **Applications**



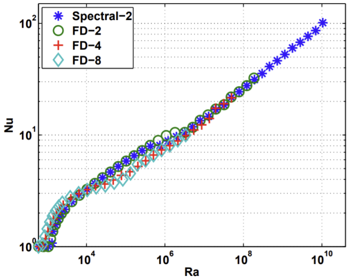
Astrophysics



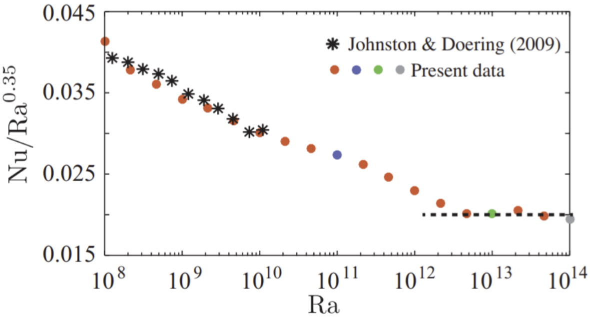
Geophysics

Engineering

Why are we Still Studying Convection? **It's Elusive**



(Johnson and Doering 2009)



(Zhu et al 2019)

Boussinesq Convection: The PDE

- Incompressible kinematics with buoyancy term

$$\nabla \cdot \boldsymbol{u} = 0 \tag{1}$$
$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + T\hat{z} + \mathcal{R}\nabla^2 \boldsymbol{u} \tag{2}$$
$$\frac{\partial T}{\partial t} + \boldsymbol{u} \cdot \nabla T = \mathcal{P}\nabla^2 T \tag{3}$$

where $\boldsymbol{u}$, T, and p and velocity, temperature, and pressure respectively.

- Momentum diffusivity: $\mathcal{R} = \sqrt{\frac{\mathrm{Pr}}{\mathrm{Ra}}}$
- Thermal diffusivity: $\mathcal{P} = \frac{1}{\sqrt{\mathrm{Pr}\,\mathrm{Ra}}}$.

Boussinesq Convection: Miscellaneous Prescriptions

- Nondimensionalized on the freefall timescale
- Domain: $\mathcal{D} = \{(x, z) \mid x \in (0, 4),\ z \in (-1/2, 1/2)\}$
- Prescribed-temperature: $T|_{z=-1/2}(x, t) = 1/2,\ T_{z=1/2}(x, t) = -1/2$

  and no-slip $\boldsymbol{u}|_{z=\pm 1/2}(x, t) = \mathbf{0}$ boundary conditions
- Fixed Prandtl number $\mathrm{Pr} = 1$, varied Rayleigh number $10^6 \leq \mathrm{Ra} \leq 10^9$

Rayleigh's Linear Stability Analysis

- Assume variables can be decomposed like

$$\boldsymbol{u}(x,z,t) = \boldsymbol{u'}(x,z,t) = u'(x,z,t)\hat{x} + w'(x,z,t)\hat{z} \tag{4}$$
$$T(x,z,t) = \overline{T}(z,t) + T'(x,z,t) \tag{5}$$
$$p(x,z,t) = \overline{p}(z,t) + p'(x,z,t). \tag{6}$$

- The Eigenvalue Problem (EVP) is given by

$$\nabla\cdot\boldsymbol{u'} = 0 \tag{7}$$
$$\frac{\partial\boldsymbol{u'}}{\partial t} = -\nabla p' + T'\hat{z} + \mathcal{R}\nabla^2\boldsymbol{u'} \tag{8}$$
$$\frac{\partial T'}{\partial t} + \frac{\partial\overline{T}}{\partial z}w' = \mathcal{P}\nabla^2 T' \tag{9}$$

Solutions to the Linear Problem

Solutions are given by

$$w'(x,z,t) = A\,\Re\left[W(z)\,e^{i(k_x x - st)}\right] \tag{10}$$

$$u'(x,z,t) = A\,\Re\left[U(z)\,e^{i(k_x x - st)}\right] \tag{11}$$

$$T'(x,z,t) = A\,\Re\left[\theta(z)\,e^{i(k_x x - st)}\right] \tag{12}$$

$$p'(x,z,t) = A\,\Re\left[P(z)\,e^{i(k_x x - st)}\right] \tag{13}$$

where A is the (undetermined) mode amplitude, $s = \omega + i\sigma$ is the eigenvalue,

and the allowed wavenumbers $k_x \in \left\{\frac{n\pi}{2} \,\middle|\, n \in \mathbb{N}\right\}$.

The Quasilinear Model

- The 1D quasilinear initial value problem (IVP)

$$\frac{\partial \overline{T}}{\partial t} + \frac{\partial}{\partial z}\langle w'T'\rangle_x = \mathcal{P}\frac{\partial^2 \overline{T}}{\partial z^2} \tag{14}$$

- Evolution due to **Advection**: $\frac{\partial}{\partial z}\langle w'T'\rangle_x$ and **Diffusion**: $\mathcal{P}\frac{\partial^2 \overline{T}}{\partial z^2}$

- Derived by manipulating the nonlinear and linear equations

- Can be solved in conjunction with the EVP provided the amplitude A is known

$\longrightarrow$ **Marginal Stability**

Marginal Stability Constraint

- We assume the perturbations evolve on a shorter timescale than the background state
- We impose **Marginal Stability** at each timestep

$$\max_{k_x}\{\sigma\} = 0. \tag{15}$$

- The growth rate σ is obtained by solving the EVP for some $\overline{T}(z,t)$
- Does not allow us to solve for A directly $\longrightarrow$ discrete root-finding methods

Solving for the Perturbation Amplitude A

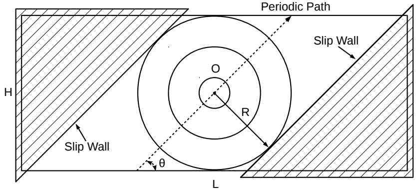
1. Given some guess for the amplitude A and a fixed timestep Δt
2. We evolve the initial (marginally stable) temperature profile $\overline{T}(z,t) \longrightarrow \overline{T}(z,t+\Delta t)$
3. Solve the EVP using $\overline{T}(z,t+\Delta t)$
4. This renders $\sigma(A)$
5. Use Newton's method with finite differences to solve $\sigma(A)=0$

The Amplitude Guess

- Consider advection and diffusion separately on $t \rightarrow t + \Delta t$
- Solve $\dfrac{\partial \overline{T}}{\partial t} + \langle w' T' \rangle_x = 0$, yielding $\overline{T}_{adv} \longrightarrow \sigma_{\mathrm{adv}}$
- Solve $\dfrac{\partial \overline{T}}{\partial t} = \mathcal{P} \dfrac{\partial^2 \overline{T}}{\partial z^2}$, yielding $\overline{T}_{adv} \longrightarrow \sigma_{\mathrm{diff}}$
- For small timesteps, we assume diffusion and advection act independently, i.e.

$$0 = \Delta\sigma \approx A^2 \sigma_{\mathrm{adv}} + \sigma_{\mathrm{diff}} \quad \longrightarrow \quad A^2 \approx -\frac{\sigma_{\mathrm{diff}}}{\sigma_{\mathrm{adv}}} \tag{16}$$

Position animation + Make Table



To play the video, the compiled PDF should be moved out from the "Tmp" directory

$m_x \times m_y$	L_1 error	L_1 order	L_2 error	L_2 order	L_∞ error	L_∞ order
40×20	$3.536\text{e}{-2}$	–	$6.097\text{e}{-2}$	–	$4.105\text{e}{-1}$	–
80×40	$9.113\text{e}{-3}$	1.956	$2.497\text{e}{-2}$	1.288	$1.997\text{e}{-1}$	1.039
160×80	$2.034\text{e}{-3}$	2.163	$6.548\text{e}{-3}$	1.931	$5.236\text{e}{-2}$	1.931
320×160	$5.114\text{e}{-4}$	1.992	$1.640\text{e}{-3}$	1.997	$1.278\text{e}{-2}$	2.035
640×320	$1.287\text{e}{-4}$	1.990	$4.097\text{e}{-4}$	2.001	$3.119\text{e}{-3}$	2.034
1280×640	$3.233\text{e}{-5}$	1.993	$1.024\text{e}{-4}$	2.000	$7.818\text{e}{-4}$	1.996

Ordinary text

- **A 3D, high-resolution, parallelized, gas-solid flow solver**
  - Establishes a numerical framework for the direct simulation of gas-solid flows.
  - Solves coupled and interface-resolved fluid-fluid, fluid-solid, and solid-solid interactions.
  - Addresses shocked flow conditions, irregular and moving geometries, and multibody contact and collisions.
- **Advancement in understanding particle clustering and jetting**
  - Demonstrates a valid statistical dissipative property in solving explosively dispersed granular materials with respect to Gurney velocity.
  - Extends the time range of the velocity scaling law with regard to Gurney energy in the Gurney theory from the steady-state termination phase to the unsteady evolution phase.
  - Proposes an explanation for particle clustering and jetting instabilities to increase the understanding of experimental observations.

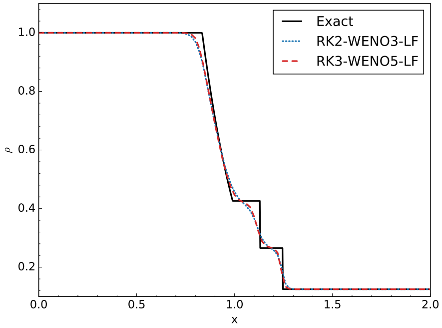
Thank you for your attention!

UNIVERSITY OF WATERLOO

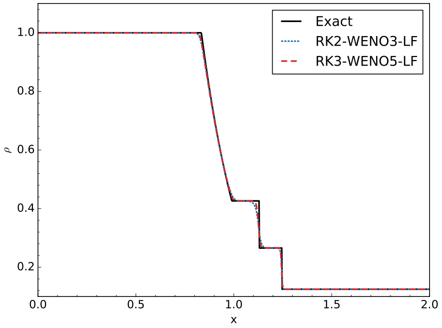
Part I

Appendix

Sod's problem (Sod 1978)



$n = 100$



$n = 500$

$$\begin{array}{llll} \rho = 1; & u = 0; & p = 1 & \text{if } 0 \leq x < 1 \\ \rho = 0.125; & u = 0; & p = 0.1 & \text{if } 1 < x \leq 2 \end{array}$$

References I

Sod, G. A. (1978). “A survey of several finite difference methods for systems of nonlinear hyperbolic conservation laws”. In: J. Comput. Phys. 27.1, pp. 1–31.