Solutions of First Order Linear Systems

An n-dimensional first order linear system is a list of n first order, linear, differential equations, in n unknown functions of one variable. It may be written as the matrix/vector equation

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}$$

where $\mathbf{x}$ is an n-dimensional vector containing the unknown functions and A is the $n \times n$ coefficient matrix. For example, the 3-dimensional linear system

$$\begin{aligned} x_1' &= -x_1 + 5x_2 + 3x_3 + e^t \\ x_2' &= x_2 + x_3 \\ x_3' &= -2x_2 - 2x_3 - t^2 \end{aligned}$$

has

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ and } A = \begin{pmatrix} -1 & 5 & 3 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{pmatrix} \text{ and } \mathbf{f} = \begin{pmatrix} e^t \\ 0 \\ -t^2 \end{pmatrix}$$

Note that the book uses $\mathbf{y}$'s instead of $\mathbf{x}$'s in some sections, so you should be comfortable using either letter. We will almost always use t for the variable.

(1) **Homogeneous Solution:** Solve the homogeneous equation

$$\mathbf{x}' = A\mathbf{x}$$

to find n linearly independent vector solutions $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}$, and hence the homogeneous solution $\mathbf{x_h} = c_1\mathbf{x_1} + c_2\mathbf{x_2} + \cdots + c_n\mathbf{x_n}$. You can check whether $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}$ are linearly independent by calculating the Wronksian

$$W(t_0) = \det\begin{pmatrix} \mathbf{x_1}(t_0) & \mathbf{x_2}(t_0) & \ldots & \mathbf{x_n}(t_0) \end{pmatrix}$$

at any point t_0 in the domain of $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}$, and showing that it is non-zero. Note that the Wronskian is formed by first placing the column vector solutions next to each other to form a square matrix, substituting t_0 and then taking the determinant of the matrix.

To find the n linearly independent vector solutions, you first need to find the eigenvalues λ of A, by solving the characteristic equation $\det(A - \lambda I) = 0$ for λ, and then find the corresponding eigenvectors $\mathbf{v}$, by solving the vector equation $(A - \lambda I)\mathbf{v} = \mathbf{0}$ for $\mathbf{v}$. If there are repeated eigenvalues you may need to also find generalized eigenvectors - this case will be explained below.

(a) **Distinct Real Eigenvalues:** If λ is a real eigenvalue which appears once as a root of the characteristic equation, and $\mathbf{v}$ is the corresponding eigenvector then $\mathbf{x} = e^{\lambda t}\mathbf{v}$ is a vector solution of the homogeneous equation.

(b) **Distinct Complex Eigenvalues:** If $\lambda = \alpha + i\beta$ is a complex eigenvalue then so is its complex conjugate $\lambda = \alpha - i\beta$. Each of these eigenvalues will provide *one* vector solution, but since the eigenvalues are closely related, it is probably easier to use $\lambda = \alpha + i\beta$ to find both solutions. Find the corresponding eigenvector $\mathbf{v}$ and split it using vector addition as $\mathbf{v} = \mathbf{v_1} + i\mathbf{v_2}$, where $\mathbf{v_1}$ and $\mathbf{v_2}$ have real entries. Then the two vector solutions are

$$\mathbf{x_1} = e^{\alpha t}(\cos(\beta t)\mathbf{v_1} - \sin(\beta t)\mathbf{v_2})$$

and

$$\mathbf{x_2} = e^{\alpha t}(\sin(\beta t)\mathbf{v_1} + \cos(\beta t)\mathbf{v_2})$$

Example:

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \mathbf{x}$$

The eigenvalues are $1+i$, and $1-i$. Use the eigenvalue $1+i$ to find the eigenvector by solving

$$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \mathbf{v} = \mathbf{0}$$

Then

$$\mathbf{v} = \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \mathbf{v_1} + i\mathbf{v_2}$$

and so the two corresponding vector solutions are

$$\mathbf{x_1} = e^t(\cos(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sin(t) \begin{pmatrix} 0 \\ 1 \end{pmatrix})$$

and

$$\mathbf{x_2} = e^t(\sin(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \cos(t) \begin{pmatrix} 0 \\ 1 \end{pmatrix})$$

(c) **Repeated Eigenvalues:** If an eigenvalue is repeated we need to analyse the matrix A more carefully to find the corresponding vector solutions.

Definition 1. *The Algebraic Multiplicity (AM) of an eigenvalue λ is the number of times it appears as a root of the characteristic equation* $\det(A - \lambda I) = 0$.

For each eigenvalue λ, we need to find the **same number** of vector solutions as its Algebraic Multiplicity. If $AM = 1$, then we are in one of the situations described above so the more difficult (and interesting) case is when $AM > 1$.

Definition 2. *The Geometric Multiplicity (GM) of an eigenvalue λ is the number of linearly independent eigenvectors corresponding to λ (or the dimension of the eigenspace). An equivalent (and probably more helpful) definition is that the Geometric Multiplicity is the number of degrees of freedom in the eigenvector equation* $(A - \lambda I)\mathbf{v} = \mathbf{0}$.

These two numbers always have the property that $1 \leq GM \leq AM$. In the case where $GM = AM$, we have "enough" eigenvectors to produce solutions of the form $\mathbf{x} = e^{\lambda t}\mathbf{v}$ corresponding to λ and we procede as above.
Thus the most interesting case is when $GM < AM$. To see what happens in general, you are referred to the textbook, however to avoid being cast adrift in a sea of confusing abstraction and notation we shall only consider a special case here.

Definition 3. *If λ is a an eigenvalue for A, then* $\mathbf{v}$ *is a corresponding* **generalized** *eigenvector if $(A - \lambda I)^d\mathbf{v} = \mathbf{0}$ for some positive integer d.*

Note that an eigenvector is also a generalized eigenvector (take $d = 1$) but as d increases there will be more solutions which are not eigenvectors.
Consider a real eigenvalue λ for which $AM = 2$ and $GM = 1$, so that we are looking for two vector solutions corresponding to λ but we only have one linearly independent eigenvector $\mathbf{v_1}$, and hence only one vector solution $\mathbf{x_1} = e^{\lambda t}\mathbf{v_1}$.
Solve the equation $(A - \lambda I)^2\mathbf{v} = \mathbf{0}$ which will have two degrees of freedom, so there will be two linearly independent generalized eigenvectors. It is strongly recommended (though not necessary) that you choose the eigenvector $\mathbf{v_1}$ as one of the generalized

eigenvectors, since this gives us the solution $\mathbf{x_1} = e^{\lambda t}\mathbf{v_1}$ which we already know. If $\mathbf{v_2}$ is the other generalized eigenvector then, the second solution corresponding to λ is

$$\mathbf{x_2} = e^{\lambda t}(I + t(A - \lambda I))\mathbf{v_2}.$$

For more details on how this solution arises, see the Appendix on matrix exponentials.

(2) **Particular Solution:** To find the particular solution $\mathbf{x_p}$ for an inhomogeneous first order linear system

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}$$

we use the Variation of Parameters method.

Definition 4. *The Fundamental Matrix*

$$Y = \begin{pmatrix}\mathbf{x_1} & \mathbf{x_2} & \dots & \mathbf{x_n}\end{pmatrix}$$

is the $n \times n$ matrix formed by writing the n column vector solutions of the homogeneous equation next to each other

It is easy to check that $Y' = AY$ and that Y is invertible because its determinant is the Wronskian and hence non-zero. It can then be shown that the particular vector solution is

$$\mathbf{x_p} = Y \int Y^{-1}\mathbf{f}\,\mathrm{dt}$$

You should compute $\mathbf{x_p}$ in the following order:

(a) Find Y^{-1} which is an $n \times n$ matrix.

(b) Find $Y^{-1}\mathbf{f}$ which is a vector.

(c) Find $\int Y^{-1}\mathbf{f}\,\mathrm{dt}$ which is a vector. You compute the integral of a vector by finding an anti-derivative of each entry, and this is a case where you can choose the constants of integration to be 0 because we are only looking for *one* particular solution.

(d) Find $\mathbf{x_p} = Y \int Y^{-1}\mathbf{f}\,\mathrm{dt}$ which is a vector.

(3) **General Solution:** The general solution is the vector

$$\mathbf{x} = \mathbf{x_h} + \mathbf{x_p} = c_1\mathbf{x_1} + c_2\mathbf{x_2} + \cdots + c_n\mathbf{x_n} + \mathbf{x_p}$$

(4) **Initial Value Problems:** An IVP will have the form

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f},\ \mathbf{x}(t_0) = \mathbf{x_0}$$

In this case you can solve for the constants $c_1, c_2, \dots, c_n$. Substitute the initial condition into the general solution, and this will give you n equations in the n unknown constants.

Examples: See if you can verify the solutions to the following problems.

(1) In this example we have a repeated eigenvalue but there are sufficient eigenvectors

$$\mathbf{x}' = \begin{pmatrix} 2 & 2 & -4 \\ 2 & -1 & -2 \\ 4 & 2 & -6 \end{pmatrix} \mathbf{x}$$

The characteristic equation for the matrix is

$$-(\lambda+1)(\lambda+2)^2 = 0$$

so the eigenvalues are -1 and -2. First consider the eigenvalue $\lambda = -2$, which has Algebraic Multiplicity 2, and for which the eigenvector equation is

$$\begin{pmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ 4 & 2 & -4 \end{pmatrix} \mathbf{v} = \mathbf{0}.$$

This equation has two degrees of freedom so the eigenvalue $\lambda = 2$ has Geometric Multiplicity 2 and thus it has two eigenvectors. For example we could choose

$$\mathbf{v_1} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \ \mathbf{v_2} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

The eigenvalue $\lambda = -1$ has Algebraic Multiplicity 1 and a corresponding eigenvector

$$\mathbf{v_3} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}.$$

The general solution of the system is

$$\mathbf{x} = c_1 e^{-2t} \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_3 e^{-t} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

(2) In this example there is a repeated eigenvalue and we need to find a generalized eigenvector.

$$\mathbf{x}' = \begin{pmatrix} -1 & 2 & 1 \\ 0 & -1 & 0 \\ -1 & -3 & -3 \end{pmatrix} \mathbf{x}$$

The characteristic equation for the matrix is (as in the previous example)

$$-(\lambda+1)(\lambda+2)^2 = 0$$

and so the eigenvalues are -1 and -2. First consider the eigenvalue $\lambda = -2$, which has Algebraic Multiplicity 2, and for which the eigenvector equation is

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ -1 & -3 & -1 \end{pmatrix} \mathbf{v} = \mathbf{0}$$

which only has one degree of freedom. We can choose the eigenvector

$$\mathbf{v_1} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

but now we need to also find a generalized eigenvector. Compute

$$(A-\lambda I)^2 = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ -1 & -3 & -1 \end{pmatrix}^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 0 \end{pmatrix}$$

and then solve the equation

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 0 \end{pmatrix} \mathbf{v} = \mathbf{0}$$

which has two degrees of freedom. As always, we should choose the eigenvector $\mathbf{v_1}$ as one solution. The other solution $\mathbf{v_2}$ is a generalized eigenvector and must be linearly independent with $\mathbf{v_1}$, so for example we can choose

$$\mathbf{v_2} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

The two vector solutions of the system are then

$$\mathbf{x_1} = e^{-2t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

and

$$\mathbf{x_2} = e^{-2t}(I + t \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ -1 & -3 & -1 \end{pmatrix}) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = e^{-2t} \begin{pmatrix} 1+t \\ 0 \\ -t \end{pmatrix}$$

The eigenvalue $\lambda = -1$ has Algebraic Multiplicity 1 and a corresponding eigenvector

$$\mathbf{v_3} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

and so the general solution of the system is

$$\mathbf{x} = c_1 e^{-2t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1+t \\ 0 \\ -t \end{pmatrix} + c_3 e^{-t} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

(3) Solve the initial value problem for following system of equations

$$x' = -x + y + t, y' = x - y - t, \ x(0) = 1, y(0) = 1$$

to find formulas for $x(t)$ and $y(t)$.
First put the system of equations into matrix/vector form $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$ where

$$A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \text{ and } \mathbf{f} = \begin{pmatrix} t \\ -t \end{pmatrix}$$

and then find the eigenvalues of A which are 0 and -2. The corresponding eigenvectors are

$$\mathbf{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \mathbf{v_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

so a Fundamental Matrix for the homogeneous system is

$$Y = \begin{pmatrix} 1 & -e^{-2t} \\ 1 & e^{-2t} \end{pmatrix}.$$

Now we compute the particular solution $\mathbf{x_p}$

(a) $Y^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -e^{2t} & e^{2t} \end{pmatrix}$

(b) $Y^{-1}\mathbf{f} = \begin{pmatrix} 0 \\ -e^{2t}t \end{pmatrix}$

(c) $\int Y^{-1}\mathbf{f}\,\mathrm{dt} = \begin{pmatrix} 0 \\ -\frac{1}{2}e^{2t}t + \frac{1}{4}e^{2t} \end{pmatrix}$

(d) $\mathbf{x_p} = Y\int Y^{-1}\mathbf{f}\,\mathrm{dt} = \begin{pmatrix} \frac{t}{2} - \frac{1}{4} \\ -\frac{t}{2} + \frac{1}{4} \end{pmatrix}$

The general solution is

$$\mathbf{x} = \mathbf{x_h} + \mathbf{x_p} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{t}{2} - \frac{1}{4} \\ -\frac{t}{2} + \frac{1}{4} \end{pmatrix}$$

Finally, remember that $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and substitute the initial values to find $c_1 = 1$ and $c_2 = -1/4$. The solutions are

$$\begin{aligned} x &= \frac{3}{4} + \frac{1}{4}e^{-2t} + \frac{t}{2} \\ y &= \frac{5}{4} - \frac{1}{4}e^{-2t} - \frac{t}{2} \end{aligned}$$

Appendix: Matrix Exponentials

Definition 5. *Let A be a square matrix and let t be a variable then*

$$e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots$$

This is the definition but we NEVER (well almost never) use it to actually find e^{tA} because computing an infinite series of matrices is too hard. Usually we want to compute $e^{tA}\mathbf{v}$ since this is a solution of $\mathbf{x}' = A\mathbf{x}$ for any vector $\mathbf{v}$.

Let λ be an eigenvalue of A, and note that $tA = t(\lambda I + A - \lambda I)$. Then, because $t\lambda I$ is a diagonal matrix, we can use the usual law of exponents and so

$$\begin{aligned} e^{tA} &= e^{t\lambda I}e^{t(A-\lambda I)} \\ &= e^{\lambda t}(I + t(A-\lambda I) + \frac{t^2}{2!}(A-\lambda I)^2 + \frac{t^3}{3!}(A-\lambda I)^3 + \cdots) \end{aligned}$$

Now if $\mathbf{v_1}$ is an eigenvector, then $(A-\lambda I)\mathbf{v_1} = \mathbf{0}$, so

$$e^{tA}\mathbf{v_1} = e^{\lambda t}I\mathbf{v_1} = e^{\lambda t}\mathbf{v_1}$$

and if $\mathbf{v_2}$ is a generalized eigenvector, then $(A-\lambda I)^d\mathbf{v_2} = \mathbf{0}$, so

$$e^{tA}\mathbf{v_2} = e^{\lambda t}(I + t(A-\lambda I) + \frac{t^2}{2!}(A-\lambda I)^2 + \frac{t^3}{3!}(A-\lambda I)^3 + \cdots + \frac{t^{d-1}}{(d-1)!}(A-\lambda I)^{d-1})\mathbf{v_2}$$

Finally, if you ever need to compute a matrix exponential, find a Fundamental Matrix Y for the equation $\mathbf{x}' = A\mathbf{x}$ and then use the formula

$$e^{tA} = Y \cdot Y(0)^{-1}$$