1 Astrodynamics and Space Applications Qualifying Exam Notes

gps

orb

quals

sad

stm

2 Satellite Navigation Past Problems

3 Orbit Mechanics Past Problems

3.1 Problem 0

3.1.1 Problem statement

In Keplerian mechanics, several important types of orbital maneuvers are noncoplanar. For example, the capability to change both the ascending node and the inclination with only one maneuver is efficient and can widen the launch window.

Assume that the orbital elements for an Earth orbit are given. If the orbit is circular both initially and after the maneuver, let $i_0 = 30°$, $i_f = 90°$, $\Omega_o = 0°$, $\Omega_f = 60°$, where o reflects the original orbit and f indicates a value in the final orbit.

1. Determine the appropriate maneuver location in each orbit.
2. If the circular orbit possesses a radius of $4R_\oplus$, determine the magnitude of the required single impulse to accomplish the goal.

3.1.2 Solution

We'll define the "location" of the maneuver in the initial and final orbits with the argument of latitude θ_o and θ_f, the angle between the ascending node and the spacecraft's position vector. Because the orbits are circular, we can't really use the true anomaly. We can then form a spherical triangle with side lengths $\Omega_f - \Omega_o$ along the equator, and then θ_i extending upwards from the left at an angle of i_0, and θ_f extending upwards from the right at an angle of i_f.

Using the spherical law of sines, we can solve for θ_o:

$$\frac{\sin \theta_o}{\sin i_f} = \frac{\sin(\Omega_f - \Omega_o)}{\sin (180° - i_0 - i_f)}$$
$$\sin \theta_o = \frac{\sin i_f \sin(\Omega_f - \Omega_o)}{\sin (180° - i_0 - i_f)}$$

Plugging in values, we find:

$$\sin \theta_o = \frac{\sin 90° \sin(60° - 0°)}{\sin (180° - 30° - 90°)}$$
$$= \frac{\sin 60°}{\sin 60°}$$
$$= 1$$
$$\theta_o = 90°$$

And similarly for θ_f:

$$\frac{\sin \theta_f}{\sin i_o} = \frac{\sin(\Omega_f - \Omega_o)}{\sin (180° - i_0 - i_f)} = 1$$
$$\theta_f = i_o = 30°$$

The magnitude of the required impulse is given by the law of cosines, where we know that the angle between the initial and final position vectors is $i_f - i_o = 60°$, the interior angle of the spherical triangle at the point of intersection. The circular velocity in the initial orbit is given by:

$$v_c = \sqrt{\frac{\mu_\oplus}{r}}$$
$$= \sqrt{\frac{\mu_\oplus}{4R_\oplus}}$$

And the magnitude of the required impulse is given by:

$$\frac{\Delta v}{2v_c} = \sin\left(\frac{60°}{2}\right)$$

$$= \frac{1}{2}$$

$$\Delta v = v_c$$

$$= \sqrt{\frac{\mu_\oplus}{4R_\oplus}}$$

3.2   Problem 1

3.2.1   Problem statement

Consider a hyperbolic flyby of a planet

1. Determine the values of the periapsis flyby radius r_p and hyperbolic excess speed v_∞ that yield the *maximum possible* magnitude of the equivalent Δv_{eq} for the spacecraft due to the flyby. Express your answer for r_p in terms of the planet radius r_s; include the constraint that $r_p \geq r_s$.
2. Determine this maximum Δv_{eq} in terms of v_s, the circular speed at the surface of the planet. Also determine the numerical values for the corresponding turn angle δ and the hyperbolic eccentricity e.

3.2.2   Solution

We know that the angle between the incoming and outgoing hyperbolic asymptotes is given by:

$$\delta = 2\sin^{-1}\left(\frac{1}{e}\right)$$

$$= 2\sin^{-1}\left(\frac{\Delta v_{eq}}{2v_\infty}\right)$$

We'll use these two expressions for δ to solve for the conditions that maximize Δv. First, we have to find a way to introduce r_p into the equation. We know that the distance from the attracting focus to the center of the hyperbola is given by:

$$ae = r_p + a$$

$$e = \frac{r_p}{a} + 1$$

We also know that by conservation of energy at $r = \infty$, we can express the semi-major axis a in terms of the hyperbolic excess speed v_∞:

$$\frac{v_\infty^2}{2} = \frac{\mu}{2a}$$

$$a = \frac{\mu}{v_\infty^2}$$

Substituting this into the expression for e:

$$e = \frac{r_p}{\mu/v_\infty^2} + 1$$

$$= \frac{r_p v_\infty^2}{\mu} + 1$$

Such that we can equate the two expressions for δ:

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$$2\sin^{-1}\left(\frac{\Delta v_{eq}}{2v_\infty}\right) = 2\sin^{-1}\left(\frac{1}{\frac{r_p v_\infty^2}{\mu}+1}\right)$$

$$\frac{\Delta v_{eq}}{2v_\infty} = \frac{1}{\frac{r_p v_\infty^2}{\mu}+1}$$

$$\Delta v_{eq} = \frac{2v_\infty}{\frac{r_p v_\infty^2}{\mu}+1}$$

This tells us that for any given v_∞, minimizing r_p will maximize Δv_{eq}. The minimum value of r_p is r_s, the radius of the planet. Solving for the v_∞ that corresponds to this minimum r_p requires taking the derivative of the Δv_{eq} expression with respect to v_∞ and looking for critical points:

$$\frac{\partial \Delta v_{eq}}{\partial v_\infty} = \frac{2}{\frac{r_p v_\infty^2}{\mu}+1} - \frac{2v_\infty}{\left(\frac{r_p v_\infty^2}{\mu}+1\right)^2}\frac{2r_p v_\infty}{\mu}$$

$$= \frac{\frac{2r_p v_\infty^2}{\mu}+2-\frac{4r_p v_\infty^2}{\mu}}{\left(\frac{r_p v_\infty^2}{\mu}+1\right)^2}$$

$$= \frac{2-\frac{2r_p v_\infty^2}{\mu}}{\left(\frac{r_p v_\infty^2}{\mu}+1\right)^2}$$

We notice that the denominator is always positive, so we can simply set the numerator to zero:

$$2-\frac{2r_p v_\infty^2}{\mu} = 0$$

$$\frac{2r_p v_\infty^2}{\mu} = 2$$

$$v_\infty^2 = \frac{\mu}{r_p}$$

$$v_\infty = \sqrt{\frac{\mu}{r_p}}$$

This is an interesting result! We have found that the hyperbolic excess velocity for maximum Δv_{eq} is equal to the circular velocity at the surface of the planet. Solving for the corresponding value of Δv_{eq}:

$$\Delta v_{eq} = \frac{2v_\infty}{\frac{r_p v_\infty^2}{\mu}+1}$$

$$= \frac{2\sqrt{\frac{\mu}{r_p}}}{\frac{r_p\left(\sqrt{\frac{\mu}{r_p}}\right)^2}{\mu}+1}$$

$$= \frac{2\sqrt{\frac{\mu}{r_p}}}{\frac{\mu}{\mu}+1}$$

$$= \frac{2\sqrt{\frac{\mu}{r_p}}}{2}$$

$$= \sqrt{\frac{\mu}{r_p}}$$

We can also solve for the corresponding values of δ:

$$\delta = 2\sin^{-1}\left(\frac{1}{e}\right)$$

$$= 2\sin^{-1}\left(\frac{\Delta v_{eq}}{2v_\infty}\right)$$

$$= 2\sin^{-1}\left(\frac{\sqrt{\frac{\mu}{r_p}}}{2\sqrt{\frac{\mu}{r_p}}}\right)$$

$$= 2\sin^{-1}\left(\frac{1}{2}\right)$$

$$= 60°$$

And e:

$$e = \frac{r_p}{a} + 1$$

$$= \frac{r_p}{\frac{\mu}{v_\infty^2}} + 1$$

$$= \frac{r_p v_\infty^2}{\mu} + 1$$

$$= \frac{\mu}{\mu} + 1$$

$$= 2$$

4 Attitude Dynamics Past Problems

5 Orbit Determination Past Problem