1 Astrodynamics and Space Applications Qualifying Exam Notes

gps

orb

quals

sad

stm

2 Satellite Navigation Past Problems

3 Orbit Mechanics Past Problems

3.1 Spring 2021

3.1.1 Problem Statement

Assume a system of four centrobaric bodies that can all move in any spatial dimension.

1. From first principles, derive the vector differential equation governing relative motion. It is not possible to solve the corresponding scalar equations of motion. Why not?
2. Derive expressions for the 10 known integrals of motion associated with this vector differential equation for the 4-body system. What is the physical significance of each?
3. The motion of the Moon relative to the Earth, and influenced by the Sun, is one of the most challenging problems in orbital mechanics. Given the results in (a) and (b), discuss why the first statement is true.

3.1.2 Solution

The vector from the ith body to the jth body is given by:

$$\mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i$$

Taking two derivatives with respect to time:

$$\ddot{\mathbf{r}}_{ij} = \ddot{\mathbf{r}}_j - \ddot{\mathbf{r}}_i$$

The acceleration of the ith body is given by:

$$\ddot{\mathbf{r}}_i = -\sum_{k\neq i} \frac{Gm_k}{|\mathbf{r}_{ki}|^3}\mathbf{r}_{ki}$$

The acceleration of the jth body is given by:

$$\ddot{\mathbf{r}}_j = -\sum_{k\neq j} \frac{Gm_k}{|\mathbf{r}_{kj}|^3}\mathbf{r}_{kj}$$

Such that the relative acceleration of the ith body with respect to the jth body is given by:

$$\ddot{\mathbf{r}}_{ij} = -\sum_{k\neq j} \frac{Gm_k}{|\mathbf{r}_{kj}|^3}\mathbf{r}_{kj} + \sum_{k\neq i} \frac{Gm_k}{|\mathbf{r}_{ki}|^3}\mathbf{r}_{ki}$$

We can pull out the $k = i$ in the first sum and $k = j$ in the second sum:

$$\ddot{\mathbf{r}}_{ij} = -\frac{Gm_i}{|\mathbf{r}_{ji}|^3}\mathbf{r}_{ji} + \frac{Gm_j}{|\mathbf{r}_{ij}|^3}\mathbf{r}_{ij} - \sum_{k\neq j, k\neq i} \frac{Gm_k}{|\mathbf{r}_{kj}|^3}\mathbf{r}_{kj} + \sum_{k\neq i, k\neq j} \frac{Gm_k}{|\mathbf{r}_{ki}|^3}\mathbf{r}_{ki}$$

Combining the first two terms and the last two terms:

$$\ddot{\mathbf{r}}_{ij} = -\frac{G(m_i + m_j)}{|\mathbf{r}_{ji}|^3}\mathbf{r}_{ji} - \sum_{k\neq j, k\neq i} Gm_k \left(\frac{\mathbf{r}_{kj}}{|\mathbf{r}_{kj}|^3} - \frac{\mathbf{r}_{ki}}{|\mathbf{r}_{ki}|^3} \right)$$

Which in the case of the 4-body problem, we take the $i = 1$ and $j = 2$, $k \in [3, 4]$:

$$\ddot{\mathbf{r}}_{12} = -\frac{G(m_1 + m_2)}{|\mathbf{r}_{21}|^3}\mathbf{r}_{21} - \sum_{k=3}^{4} Gm_k \left(\frac{\mathbf{r}_{k2}}{|\mathbf{r}_{k2}|^3} - \frac{\mathbf{r}_{k1}}{|\mathbf{r}_{k1}|^3} \right)$$

Deriving the 10 known integrals of motion begins by first noting that the sum of all forces on the system is zero:

$$\sum_i F = \sum_i \sum_{j \neq i} \left(-\frac{G m_i m_j}{r_{ji}^3} \mathbf{r}_{ji} \right)$$

Because $F_{sys} = m_{sys} a_{sys} = \sum_i m_i \ddot{\mathbf{r}}_i = 0$ for the system as a whole, we can state:

$$\int \sum_i m_i \ddot{\mathbf{r}}_i \, dt = m_{sys} v_{sys} = \sum_i m_i \dot{\mathbf{r}}_i$$

Integrating once more:

$$\int \sum_i m_i \dot{\mathbf{r}}_i \, dt = m_{sys} v_{sys} t + m_{sys} r_{sys} = \sum_i m_i \mathbf{r}_i$$

These two constants of integration $\mathbf{r}_{sys}$ and $\mathbf{v}_{sys}$ are the first two integrals of motion (making up six scalar equations). Specifically, they are solved for by dividing by m_{sys}:

$$\begin{aligned} \mathbf{r}_{sys} &= \frac{1}{m_{sys}} \sum_i m_i \mathbf{r}_i \\ \mathbf{v}_{sys} &= \frac{1}{m_{sys}} \sum_i m_i \dot{\mathbf{r}}_i \end{aligned}$$

The next three integrals of motion are found by taking the summation of the angular momentum of the system. We must develop this by taking the sum of the torques on members of the system. First, we note that the torque on the ith body is given by:

$$\begin{aligned} \tau_i &= \mathbf{r}_i \times \mathbf{F}_i \\ &= \mathbf{r}_i \times \sum_{j \neq i} \left(-\frac{G m_i m_j}{r_{ji}^3} \mathbf{r}_{ji} \right) \end{aligned}$$

Such that the total torque on the system is:

$$\begin{aligned} \sum_i \tau_i &= \sum_i \mathbf{r}_i \times \sum_{j \neq i} \left(-\frac{G m_i m_j}{r_{ji}^3} \mathbf{r}_{ji} \right) \\ &= \sum_i \sum_{j \neq i} \mathbf{r}_i \times \left(-\frac{G m_i m_j}{r_{ji}^3} \mathbf{r}_{ji} \right) \end{aligned}$$

We then note that $\mathbf{r}_i \times \mathbf{r}_{ji} = -\mathbf{r}_{ji} \times \mathbf{r}_i$, such that each term in the summation is annihilated by its counterpart. This leaves us with:

$$\begin{aligned} \sum_i \tau_i &= \mathbf{0} \\ &= \sum_i \mathbf{r}_i \times m_i \ddot{\mathbf{r}}_i \end{aligned}$$

We notice that this summation expansion of the torque is the derivative of another quantity:

$$\sum_i \mathbf{r}_i \times m_i \ddot{\mathbf{r}}_i = \frac{d}{dt} \left(\sum_i \mathbf{r}_i \times m_i \dot{\mathbf{r}}_i \right)$$

Which implies that the integral of the derived quantity is an integral of motion:

$$\sum_i \mathbf{r}_i \times m_i \dot{\mathbf{r}}_i = \mathbf{h}_{sys}$$

Finally, we enforce conservation of energy by first finding the potential function whose gradient is the force on the system:

$$U = \frac{1}{2} G \sum_i \sum_{j \neq i} \frac{m_i m_j}{r_{ji}}$$

Such that the gradient of the potential function is:

$$\begin{aligned} \nabla U &= -G \sum_i \sum_{j \neq i} \frac{m_i m_j}{r_{ji}^3} \mathbf{r}_{ji} \\ &= \sum_i m_i \ddot{\mathbf{r}} \end{aligned}$$

Notice that we can express:

$$\begin{aligned} \sum_i \nabla U &= \sum_i \frac{dU}{d\mathbf{r}_i} \\ &= \sum_i m_i \ddot{\mathbf{r}} \end{aligned}$$

If we multiply both sides by $\dot{\mathbf{r}}$:

$$\begin{aligned} \sum_i m_i \dot{\mathbf{r}}_i \cdot \ddot{\mathbf{r}}_i &= \sum_i \frac{dU}{d\mathbf{r}_i} \frac{d\mathbf{r}_i}{dt} \\ &= \frac{dU}{dt} \end{aligned}$$

We then notice that the left side can also be expressed as the derivative of a quantity:

$$\begin{aligned} \frac{dU}{dt} &= \frac{d}{dt} \left(\sum_i m_i \dot{r}_i^2 \right) \\ U = \sum_i m_i \dot{r}_i^2 + C_2 \end{aligned}$$

Where C_2 is the total system energy. Plugging in our particle system representation of U, we find that the total system energy is given by:

$$C_2 = \sum_i m_i \dot{r}_i^2 - \frac{1}{2} G \sum_i \sum_{j \neq i} \frac{m_i m_j}{r_{ji}}$$

3.2 Fall 2023

Note that this problem was also given in Fall 2021.

3.2.1 Problem Statement

Assuming Keplerian motion, several important types of orbital maneuvers are noncoplanar. For example, the capability to change the ascending node can widen the launch window.

Assume that the orbital elements for an Earth orbit are given.

1. To change only the ascending node, derive an equation (or equations) that, if solved, will identify the location. i.e. the argument of latitude, for the location of the maneuver in the original and final orbits.
2. If the orbit is circular, let $e = 0.0$, $i = 55°$, $\Omega_i = 0°$, $\Omega_f = 45°$, where o reflects the original orbit and f indicates a value in the final orbit. In the relationships from (a), demonstrate that the maneuver location is defined as $\theta_o = 103.36°$. What is the value of θ_f?
3. If the circular orbit possesses a radius of $3R_\oplus$, find the required Δv.

3.2.2 Solution

We can form a spherical triangle with side lengths $\Omega_f - \Omega_o$ along the equator, and then θ_i extending upwards from the left at an angle of i_o, and θ_f extending upwards from the right at an angle of $180° - i_o$. The angle at the top of the triangle is the angle between the initial and final position vectors, which is the angle of the required Δv. We can then use the spherical law of cosines to solve for this angle:

$$\begin{aligned}
\cos a &= \cos b \cos c + \sin b \sin c \cos A \\
\cos A &= -\cos b \cos c + \sin b \sin c \cos a
\end{aligned}$$

Where the lowercase letters are the side lengths and the uppercase letters are the interior angles. Rephrased for our problem, we find the third interior angle a_3:

$$\begin{aligned}
\cos a_3 &= -\cos i_o \cos(180° - i_f) + \sin i_o \sin(180° - i_f) \cos(\Omega_f - \Omega_o) \\
&= \cos^2 55° + \sin^2 55° \cos(45°) \\
&= \cos^2 55° + \frac{\sqrt{2}}{2} \sin^2 55° \\
a_3 &= \cos^{-1}\left(\cos^2 55° + \frac{\sqrt{2}}{2} \sin^2 55°\right) \\
&\approx 37°
\end{aligned}$$

We can then use the spherical law of sines to solve for θ_o:

$$\begin{aligned}
\frac{\sin \theta_o}{\sin i_f} &= \frac{\sin(\Omega_f - \Omega_o)}{\sin a_3} \\
\sin \theta_o &= \frac{\sin i_f \sin(\Omega_f - \Omega_o)}{\sin a_3} \\
\theta_o &= 76.64°
\end{aligned}$$

Notice that the arcsin is also solved by $\theta_o = 180° - 76.64° = 103.36°$. We choose this solution to yield an intersection in the first half of the initial orbit.

Solving for θ_f similarly:

$$\begin{aligned}
\frac{\sin \theta_f}{\sin i_o} &= \frac{\sin(\Omega_f - \Omega_o)}{\sin a_3} \\
\sin \theta_f &= \frac{\sin i_o \sin(\Omega_f - \Omega_o)}{\sin a_3} \\
\theta_f &= 76.64°
\end{aligned}$$

We can then find the magnitude of the required Δv using the law of cosines by recognizing that the magnitude of the velocity is the same for both the initial and final orbits:

$$\begin{aligned}
v_c &= \sqrt{\frac{\mu_\oplus}{r}} \\
&= \sqrt{\frac{\mu_\oplus}{3R_\oplus}}
\end{aligned}$$

And the magnitude of the required Δv is given by:

$$
\begin{aligned}
\frac{\Delta v}{2v_c} &= \sin\left(\frac{a_3}{2}\right) \\
&= \sin\left(\frac{37^\circ}{2}\right) \\
&\approx 0.30 \\
\Delta v &\approx 0.60 v_c \\
&\approx 0.60\sqrt{\frac{\mu_\oplus}{3R_\oplus}} \\
&\approx 2.86\ [km/s]
\end{aligned}
$$

This concludes the derivation of the ten integrals of motion for the n-body problem. The first six scalars are the initial position and velocity of the system, and the next three are the angular momentum of the system. The final scalar is the total energy of the system.

3.3 Problem 0

3.3.1 Problem Statement

In Keplerian mechanics, several important types of orbital maneuvers are noncoplanar. For example, the capability to change both the ascending node and the inclination with only one maneuver is efficient and can widen the launch window.

Assume that the orbital elements for an Earth orbit are given. If the orbit is circular both initially and after the maneuver, let $i_o = 30^\circ$, $i_f = 90^\circ$, $\Omega_o = 0^\circ$, $\Omega_f = 60^\circ$, where o reflects the original orbit and f indicates a value in the final orbit.

1. Determine the appropriate maneuver location in each orbit.
2. If the circular orbit possesses a radius of $4R_\oplus$, determine the magnitude of the required single impulse to accomplish the goal.

3.3.2 Solution

We'll define the "location" of the maneuver in the initial and final orbits with the argument of latitude θ_o and θ_f, the angle between the ascending node and the spacecraft's position vector. Because the orbits are circular, we can't really use the true anomaly. We can then form a spherical triangle with side lengths $\Omega_f - \Omega_o$ along the equator, and then θ_i extending upwards from the left at an angle of i_o, and θ_f extending upwards from the right at an angle of i_f. Note: that in general, a spherical triangle has a sum of interior angles greater than 180°. This means that we must solve for the interior angle at the top of the triangle using the spherical law of cosines:

$$
\begin{aligned}
\cos a &= \cos b \cos c + \sin b \sin c \cos A \\
\cos A &= -\cos b \cos c + \sin b \sin c \cos a
\end{aligned}
$$

Where the lowercase letters are the side lengths and the uppercase letters are the interior angles. Rephrased for our problem, we find the third interior angle a_3:

$$
\begin{aligned}
\cos a_3 &= -\cos i_o \cos i_f + \sin i_o \sin i_f \cos(\Omega_f - \Omega_o) \\
&= -\cos 30^\circ \cos 90^\circ + \sin 30^\circ \sin 90^\circ \cos(60^\circ - 0^\circ) \\
&= -\frac{\sqrt{3}}{2} \cdot 0 + \frac{1}{2} \cdot 1 \cdot \frac{1}{2} \\
&= \frac{1}{4} \\
a_3 &= \cos^{-1}\left(\frac{1}{4}\right) \approx 76^\circ
\end{aligned}
$$

Using the spherical law of sines, we can solve for θ_o:

$$
\begin{aligned}
\frac{\sin\theta_o}{\sin i_f} &= \frac{\sin(\Omega_f - \Omega_o)}{\sin a_3} \\
\sin\theta_o &= \frac{\sin i_f \sin(\Omega_f - \Omega_o)}{\sin a_3}
\end{aligned}
$$

Plugging in values, we find:

$$\begin{aligned}\sin\theta_o &= \frac{\sin 90^\circ \sin(60^\circ)}{\sin 76^\circ} \\ &= \frac{\sin 60^\circ}{\sin 76^\circ} \\ &\approx 0.89 \\ \theta_o &\approx 63^\circ\end{aligned}$$

And similarly for θ_f:

$$\begin{aligned}\frac{\sin\theta_f}{\sin i_o} &= \frac{\sin(\Omega_f - \Omega_o)}{\sin a_3} = 1 \\ \sin\theta_f &= \sin 30^\circ \frac{\sin(60^\circ)}{\sin 76^\circ} \\ &\approx 0.63 \\ \theta_f &\approx 26.5^\circ\end{aligned}$$

The magnitude of the required impulse is given by the law of cosines, where we know that the angle between the initial and final position vectors is $a_3 \approx 76^\circ$, the interior angle of the spherical triangle at the point of intersection. The circular velocity in the initial orbit is given by:

$$\begin{aligned}v_c &= \sqrt{\frac{\mu_\oplus}{r}} \\ &= \sqrt{\frac{\mu_\oplus}{4R_\oplus}}\end{aligned}$$

And the magnitude of the required impulse is given by:

$$\begin{aligned}\frac{\Delta v}{2v_c} &= \sin\left(\frac{76^\circ}{2}\right) \\ &\approx 0.62 \\ \Delta v &\approx 1.23 v_c \\ &\approx 1.23\sqrt{\frac{\mu_\oplus}{4R_\oplus}}\end{aligned}$$

3.4 Fall 2019

3.4.1 Problem Statement

Consider a hyperbolic flyby of a planet

1. Determine the values of the periapsis flyby radius r_p and hyperbolic excess speed v_∞ that yield the *maximum possible* magnitude of the equivalent Δv_{eq} for the spacecraft due to the flyby. Express your answer for r_p in terms of the planet radius r_s; include the constraint that $r_p \geq r_s$.
2. Determine this maximum Δv_{eq} in terms of v_s, the circular speed at the surface of the planet. Also determine the numerical values for the corresponding turn angle δ and the hyperbolic eccentricity e.

3.4.2 Solution

We know that the angle between the incoming and outgoing hyperbolic asymptotes is given by:

$$\begin{aligned}\delta &= 2\sin^{-1}\left(\frac{1}{e}\right) \\ &= 2\sin^{-1}\left(\frac{\Delta v_{eq}}{2v_\infty}\right)\end{aligned}$$

We'll use these two expressions for δ to solve for the conditions that maximize Δv. First, we have to find a way to introduce r_p into the equation. We know that the distance from the attracting focus to the center of the hyperbola is given by:

$$\begin{aligned} ae &= r_p + a \\ e &= \frac{r_p}{a} + 1 \end{aligned}$$

We also know that by conservation of energy at $r = \infty$, we can express the semi-major axis a in terms of the hyperbolic excess speed v_∞:

$$\begin{aligned} \frac{v_\infty^2}{2} &= \frac{\mu}{2a} \\ a &= \frac{\mu}{v_\infty^2} \end{aligned}$$

Substituting this into the expression for e:

$$\begin{aligned} e &= \frac{r_p}{\mu/v_\infty^2} + 1 \\ &= \frac{r_p v_\infty^2}{\mu} + 1 \end{aligned}$$

Such that we can equate the two expressions for δ:

$$\begin{aligned} 2\sin^{-1}\left(\frac{\Delta v_{eq}}{2v_\infty}\right) &= 2\sin^{-1}\left(\frac{1}{\frac{r_p v_\infty^2}{\mu} + 1}\right) \\ \frac{\Delta v_{eq}}{2v_\infty} &= \frac{1}{\frac{r_p v_\infty^2}{\mu} + 1} \\ \Delta v_{eq} &= \frac{2v_\infty}{\frac{r_p v_\infty^2}{\mu} + 1} \end{aligned}$$

This tells us that for any given v_∞, minimizing r_p will maximize Δv_{eq}. The minimum value of r_p is r_s, the radius of the planet. Solving for the v_∞ that corresponds to this minimum r_p requires taking the derivative of the Δv_{eq} expression with respect to v_∞ and looking for critical points:

$$\begin{aligned} \frac{\partial \Delta v_{eq}}{\partial v_\infty} &= \frac{2}{\frac{r_p v_\infty^2}{\mu} + 1} - \frac{2v_\infty}{\left(\frac{r_p v_\infty^2}{\mu} + 1\right)^2} \frac{2r_p v_\infty}{\mu} \\ &= \frac{\frac{2r_p v_\infty^2}{\mu} + 2 - \frac{4r_p v_\infty^2}{\mu}}{\left(\frac{r_p v_\infty^2}{\mu} + 1\right)^2} \\ &= \frac{2 - \frac{2r_p v_\infty^2}{\mu}}{\left(\frac{r_p v_\infty^2}{\mu} + 1\right)^2} \end{aligned}$$

We notice that the denominator is always positive, so we can simply set the numerator to zero:

$$\begin{aligned} 2 - \frac{2r_p v_\infty^2}{\mu} &= 0 \\ \frac{2r_p v_\infty^2}{\mu} &= 2 \\ v_\infty^2 &= \frac{\mu}{r_p} \\ v_\infty &= \sqrt{\frac{\mu}{r_p}} \end{aligned}$$

This is an interesting result! We have found that the hyperbolic excess velocity for maximum Δv_{eq} is equal to the circular velocity at the surface of the planet. Solving for the corresponding value of Δv_{eq}:

$$\begin{aligned}
\Delta v_{eq} &= \frac{2v_\infty}{\frac{r_p v_\infty^2}{\mu}+1} \\
&= \frac{2\sqrt{\frac{\mu}{r_p}}}{\frac{r_p\left(\sqrt{\frac{\mu}{r_p}}\right)^2}{\mu}+1} \\
&= \frac{2\sqrt{\frac{\mu}{r_p}}}{\frac{\mu}{\mu}+1} \\
&= \frac{2\sqrt{\frac{\mu}{r_p}}}{2} \\
&= \sqrt{\frac{\mu}{r_p}}
\end{aligned}$$

We can also solve for the corresponding values of δ:

$$\begin{aligned}
\delta &= 2\sin^{-1}\left(\frac{1}{e}\right) \\
&= 2\sin^{-1}\left(\frac{\Delta v_{eq}}{2v_\infty}\right) \\
&= 2\sin^{-1}\left(\frac{\sqrt{\frac{\mu}{r_p}}}{2\sqrt{\frac{\mu}{r_p}}}\right) \\
&= 2\sin^{-1}\left(\frac{1}{2}\right) \\
&= 60^\circ
\end{aligned}$$

And e:

$$\begin{aligned}
e &= \frac{r_p}{a}+1 \\
&= \frac{r_p}{\frac{\mu}{v_\infty^2}}+1 \\
&= \frac{r_p v_\infty^2}{\mu}+1 \\
&= \frac{\mu}{\mu}+1 \\
&= 2
\end{aligned}$$

4 Attitude Dynamics Past Problems

4.1 Fall 2023

This was the first problem written by Dr. Oguri.

4.1.1 Problem Statement

Let us analyically investigate the attitude motion of a satellite under torque, $\boldsymbol{L} \in \mathbb{R}^3$

The inertial frame and satellite body-fixed frames are represented by $\mathcal{N}$ and $\mathcal{B}$, where $\{\hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2, \hat{\boldsymbol{n}}_3\}$ and $\left\{\hat{\boldsymbol{b}}_1, \hat{\boldsymbol{b}}_2, \hat{\boldsymbol{b}}_3\right\}$ are right-handed bases attached to the $\mathcal{N}$ and $\mathcal{B}$ frames, respectively.

Denote the angular velocity of the $\mathcal{B}$ frame with respect to the $\mathcal{N}$ frame as $\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} = \omega_i \hat{\boldsymbol{b}}_i$ and the inertia tensor dyadic of the stellite about its center of mass (CoM) by $\boldsymbol{I} = I_i \hat{\boldsymbol{b}}_i \hat{\boldsymbol{b}}_i$.

1. Derive the following equation from $\dot{\boldsymbol{H}} = \boldsymbol{L}$, where $\boldsymbol{H}$ represents the body's angular momentum about the CoM.

$$\boldsymbol{I} \cdot \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} = -\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{I} \cdot \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \boldsymbol{L}$$

2. Starting from the above equation, derive a set of differential equations that describe the time derivative of ω_i for $i = 1, 2, 3$, where $\boldsymbol{L} = L_i \hat{\boldsymbol{b}}_i$.
3. Assume that $I_1 < I_2 < I_3$ and $\boldsymbol{L} = 0$. Show that a rotation about $\hat{b}_1$ is a particular solution of the differential equations derived above, and discuss the linear stability of the solution; if it is linearly stable, also discuss whether or not the solution is asymptotically stable with the mathematical reasoning. Finally, qualitatively discuss whether the stability properties changes over time when there is energy dissipation (e.g., due to fuel sloshing).
4. Assume that $I_1 > I_3 > I_2$ and now $\boldsymbol{L}$ represents the gravity gradient (GG) torque in a circular orbit of radius R about a planet of gravitational parameter μ. ?Using an attitude representation of your choice, derive both the Kinematic and dynamic differential equations that govern the satellite motion. Here, an approximate epression of the GG torque, $\boldsymbol{L} \approx \frac{3\mu}{R^5} \boldsymbol{R} \times \boldsymbol{I} \cdot \boldsymbol{R}$ where $\boldsymbol{R} \in \mathbb{R}^3$ is the orbit radius vector, can be used without derivation. Show a particular solution of the derived differential equations and discuss its linear stability.

4.1.2 Solution

Part 1:

Beginning with:

$$\dot{\boldsymbol{H}} = \boldsymbol{L}$$

We recognize that the angular momentum vector $\boldsymbol{H} = \boldsymbol{I}\boldsymbol{\omega}$ such that in the body frame:

$$I\dot{\boldsymbol{\omega}} = \boldsymbol{L}$$

The BKE tells us that the time derivative of a vector in an reference frame is related to the same vector's derivative in the rotating frame by the angular velocity:

$${}^{\mathcal{N}}\dot{\boldsymbol{v}} = {}^{\mathcal{B}}\dot{\boldsymbol{v}} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{v}$$

To be specific, when we say the derivative of a vector expressed in a certain frame, like ${}^{\mathcal{N}}\dot{\boldsymbol{v}}$, we mean the time derivative of the vectors elements when expressed in the basis vectors of that frame. So, for example, if we have a vector $\boldsymbol{v} = v_i \hat{\boldsymbol{n}}_i$ expressed in the $\mathcal{N}$ frame, then ${}^{\mathcal{N}}\dot{\boldsymbol{v}} = \dot{\boldsymbol{v}}_i \hat{\boldsymbol{n}}_i$ as the basis vectors are constant in their frame. Applying this to the angular momentum, we can take its body frame derivative:

$$\begin{aligned} {}^{\mathcal{N}}\dot{\boldsymbol{H}} &= {}^{\mathcal{B}}\dot{\boldsymbol{H}} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{H} \\ &= \boldsymbol{I}\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{I}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \end{aligned}$$

Since we know that ${}^{\mathcal{N}}\dot{\boldsymbol{H}} = \boldsymbol{L}$, we can substitute to yield:

$$\boldsymbol{L} = \boldsymbol{I}\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{I}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$$
$$\boldsymbol{I}\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} = -\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{I}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \boldsymbol{L}$$

Completing the proof.

Part 2:

We can expand the above equation into its components, noting that if the body basis vectors are the principal axes of the body, then the inertia tensor is diagonal and the cross product terms are zero:

$$\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\omega}}_1 \\ \dot{\boldsymbol{\omega}}_2 \\ \dot{\boldsymbol{\omega}}_3 \end{bmatrix} = \begin{bmatrix} \boldsymbol{\omega}_1 \\ \boldsymbol{\omega}_2 \\ \boldsymbol{\omega}_3 \end{bmatrix} \times \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{\omega}_1 \\ \boldsymbol{\omega}_2 \\ \boldsymbol{\omega}_3 \end{bmatrix} + \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix}$$

Such that the scalar EOMs are:

$$\dot{\boldsymbol{\omega}}_1 = \frac{1}{I_1}\left(\omega_2\omega_3\left(I_2 - I_3\right) + L_1\right)$$
$$\dot{\boldsymbol{\omega}}_2 = \frac{1}{I_2}\left(\omega_3\omega_1\left(I_3 - I_1\right) + L_2\right)$$
$$\dot{\boldsymbol{\omega}}_3 = \frac{1}{I_3}\left(\omega_1\omega_2\left(I_1 - I_2\right) + L_3\right)$$

Part 3:

A rotation purely about $\hat{\boldsymbol{b}}_1$ implies that $\boldsymbol{\omega} = \omega_1\hat{\boldsymbol{b}}_1$ with $\omega_2 = \omega_3 = 0$ such that the EOMs become when $\boldsymbol{L} = 0$:

$$\dot{\boldsymbol{\omega}}_1 = \frac{1}{I_1}\left(0 \cdot 0\left(I_2 - I_3\right) + 0\right)$$
$$\dot{\boldsymbol{\omega}}_2 = \frac{1}{I_2}\left(0 \cdot \omega_1\left(I_3 - I_1\right) + 0\right)$$
$$\dot{\boldsymbol{\omega}}_3 = \frac{1}{I_3}\left(\omega_1 \cdot 0\left(I_1 - I_2\right) + 0\right)$$

Which simplify:

$$\dot{\boldsymbol{\omega}}_1 = 0$$
$$\dot{\boldsymbol{\omega}}_2 = 0$$
$$\dot{\boldsymbol{\omega}}_3 = 0$$

This means that the angular velocity is constant in time, and therefore this is a particular solution to the EOMs. To determine the stability of this solution, we can linearize the EOMs about this solution by taking the first order Taylor series expansion of the EOMs about the solution, where we substitute $\boldsymbol{\omega} = \boldsymbol{\omega} + \delta\boldsymbol{\omega}$, discarding any higher order terms in $\delta\boldsymbol{\omega}$:

$$\boldsymbol{\omega} = \begin{bmatrix} \omega_1 + \delta\omega_1 \\ 0 + \delta\omega_2 \\ 0 + \delta\omega_3 \end{bmatrix}$$

$$\dot{\boldsymbol{\omega}}_1 = \frac{1}{I_1}\left(\left(\delta\omega_3 \cdot \delta\omega_2\right)\left(I_2 - I_3\right)\right)$$
$$\dot{\boldsymbol{\omega}}_2 = \frac{1}{I_2}\left(\left(\left(\omega_1 + \delta\omega_1\right) \cdot \delta\omega_3\right)\left(I_3 - I_1\right)\right)$$
$$\dot{\boldsymbol{\omega}}_3 = \frac{1}{I_3}\left(\left(\left(\omega_1 + \delta\omega_1\right) \cdot \delta\omega_2\right)\left(I_1 - I_2\right)\right)$$

Simplifying:

$$\dot{\boldsymbol{\omega}}_1 = 0$$
$$\dot{\boldsymbol{\omega}}_2 = \frac{1}{I_2}\left(\omega_1 \cdot \delta\omega_3 \left(I_3 - I_1\right)\right)$$
$$\dot{\boldsymbol{\omega}}_3 = \frac{1}{I_3}\left(\omega_1 \cdot \delta\omega_2 \left(I_1 - I_2\right)\right)$$

Finding the eigenvalues of this system will determine its linear stability. We can rearrange this as a linear system in terms of the perturbation $\delta\omega_i$:

$$\begin{aligned}\dot{\boldsymbol{\omega}} &= \boldsymbol{A}\delta\omega \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{I_2}\left(\omega_1\left(I_3 - I_1\right)\right) \\ 0 & \frac{1}{I_3}\left(\omega_1\left(I_1 - I_2\right)\right) & 0 \end{bmatrix} \begin{bmatrix} \delta\omega_1 \\ \delta\omega_2 \\ \delta\omega_3 \end{bmatrix}\end{aligned}$$

The eigenvalues of this linear system are given by the solution to:

$$\det\left(\boldsymbol{A} - \lambda\boldsymbol{I}\right) = 0$$

$$\begin{aligned}\det\left(\begin{bmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & \frac{1}{I_2}\left(\omega_1\left(I_3 - I_1\right)\right) \\ 0 & \frac{1}{I_3}\left(\omega_1\left(I_1 - I_2\right)\right) & -\lambda \end{bmatrix}\right) &= 0 \\ \lambda\left(\lambda^2 - \frac{\omega_1^2\left(I_1 - I_2\right)\left(I_3 - I_1\right)}{I_2 I_3}\right) &= 0\end{aligned}$$

Which has solutions:

$$\lambda_1 = 0$$
$$\lambda_2 = \sqrt{\frac{\omega_1^2\left(I_1 - I_2\right)\left(I_3 - I_1\right)}{I_2 I_3}}$$
$$\lambda_3 = -\sqrt{\frac{\omega_1^2\left(I_1 - I_2\right)\left(I_3 - I_1\right)}{I_2 I_3}}$$

Because we are told that $I_1 < I_2 < I_3$, we know that $\left(I_1 - I_2\right) < 0$ and $\left(I_3 - I_1\right) > 0$ such that second and third eigenvalues are purely imaginary as the argument under the square root must be negative. Due to the presence of a zero eigenvalue, the system is marginally stable in the linear sense.

The system is not asymptotically stable as no eigenvalues have negative real parts. This means that the system will not return to the equilibrium solution after a perturbation. This makes intuitive sense as torque-free rigid body motion has no damping capability to return the system to the equilibrium solution.

Let's now turn out attention to the question of energy dissipation. We know that the total kinetic energy of the system is given by:

$$T = \frac{1}{2}\boldsymbol{\omega} \cdot \boldsymbol{I} \cdot \boldsymbol{\omega}$$

We know that as long as no torque is applied to the system due to an external force, the total angular momentum of the system is conserved. Thinking back to the construction of the energy ellipsoid and momentum sphere (expressed in body-frame angular momentum coordinates), losing energy will shrink the energy ellipsoid nonlinearly along all its axes. This could completely change the stability properties of the motion. Shrinking the ellipsoid to the point where are one of its axes has the same magnitude as the angular momentum sphere will create a directrix, resulting in unstable motion about the intermediate axis.

Part 4:

The attitude representation of choice for this writeup is the direction cosine matrix (DCM). We know that the DCM $[\mathcal{BN}]$ is defined as the matrix that takes vectors from the inertial frame to the body frame:

$$
{}^{\mathcal{B}}\boldsymbol{v} = [\mathcal{BN}]\,{}^{\mathcal{N}}\boldsymbol{v}
$$

The kinematic differential equation for the DCM is a relationship between the time derivative of the DCM and the angular velocity of the body frame with respect to the inertial frame. We can derive a relationship between these quantities by taking the $\mathcal{N}$-frame derivative of the $\mathcal{B}$-frame basis vectors:

$$
{}^{\mathcal{N}}\frac{d}{dt}\left(\begin{bmatrix}\hat{\boldsymbol{b}}_1\\ \hat{\boldsymbol{b}}_2\\ \hat{\boldsymbol{b}}_3\end{bmatrix}\right) = {}^{\mathcal{B}}\frac{d}{dt}\left(\begin{bmatrix}\hat{\boldsymbol{b}}_1\\ \hat{\boldsymbol{b}}_2\\ \hat{\boldsymbol{b}}_3\end{bmatrix}\right) + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \left(\begin{bmatrix}\hat{\boldsymbol{b}}_1\\ \hat{\boldsymbol{b}}_2\\ \hat{\boldsymbol{b}}_3\end{bmatrix}\right)
$$

Here we note that the $\mathcal{B}$-frame derivative of the $\hat{\boldsymbol{b}}_i$ unit vectors is zero. Further, we can replace the cross product on the right hand side with the matrix multiplication of the skew symmetric matrix of the angular velocity with the basis vectors:

$$
\begin{aligned}
{}^{\mathcal{N}}\frac{d}{dt}\left(\begin{bmatrix}\hat{\boldsymbol{b}}_1\\ \hat{\boldsymbol{b}}_2\\ \hat{\boldsymbol{b}}_3\end{bmatrix}\right) &= [\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}\times]\begin{bmatrix}\hat{\boldsymbol{b}}_1\\ \hat{\boldsymbol{b}}_2\\ \hat{\boldsymbol{b}}_3\end{bmatrix}\\
&= \begin{bmatrix}0 & -\omega_3 & \omega_2\\ \omega_3 & 0 & -\omega_1\\ -\omega_2 & \omega_1 & 0\end{bmatrix}\begin{bmatrix}\hat{\boldsymbol{b}}_1\\ \hat{\boldsymbol{b}}_2\\ \hat{\boldsymbol{b}}_3\end{bmatrix}
\end{aligned}
$$

We now proceed by computing the effect of the $\mathcal{N}$-frame derivative on each of the $\mathcal{B}$-frame basis vectors, beginning with $\hat{\boldsymbol{b}}_1$:

$$
{}^{\mathcal{N}}\frac{d}{dt}\left(\hat{\boldsymbol{b}}_1\right) = \begin{bmatrix}0 & -\omega_3 & \omega_2\\ \omega_3 & 0 & -\omega_1\\ -\omega_2 & \omega_1 & 0\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\ \omega_3\\ -\omega_2\end{bmatrix}
$$

$$
{}^{\mathcal{N}}\frac{d}{dt}\left(\hat{\boldsymbol{b}}_2\right) = \begin{bmatrix}0 & -\omega_3 & \omega_2\\ \omega_3 & 0 & -\omega_1\\ -\omega_2 & \omega_1 & 0\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}-\omega_3\\ 0\\ \omega_1\end{bmatrix}
$$

$$
{}^{\mathcal{N}}\frac{d}{dt}\left(\hat{\boldsymbol{b}}_3\right) = \begin{bmatrix}0 & -\omega_3 & \omega_2\\ \omega_3 & 0 & -\omega_1\\ -\omega_2 & \omega_1 & 0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}\omega_2\\ -\omega_1\\ 0\end{bmatrix}
$$

At this point, we notice that since the DCM can be expressed as the dot product of the basis vectors of the two frames:

$$
[\mathcal{BN}] = \begin{bmatrix}\hat{\boldsymbol{b}}_1\cdot\hat{\boldsymbol{n}}_1 & \hat{\boldsymbol{b}}_1\cdot\hat{\boldsymbol{n}}_2 & \hat{\boldsymbol{b}}_1\cdot\hat{\boldsymbol{n}}_3\\ \hat{\boldsymbol{b}}_2\cdot\hat{\boldsymbol{n}}_1 & \hat{\boldsymbol{b}}_2\cdot\hat{\boldsymbol{n}}_2 & \hat{\boldsymbol{b}}_2\cdot\hat{\boldsymbol{n}}_3\\ \hat{\boldsymbol{b}}_3\cdot\hat{\boldsymbol{n}}_1 & \hat{\boldsymbol{b}}_3\cdot\hat{\boldsymbol{n}}_2 & \hat{\boldsymbol{b}}_3\cdot\hat{\boldsymbol{n}}_3\end{bmatrix}
$$

Which is really just rows of $\mathcal{B}$-frame vectors expressed in the $\mathcal{N}$-frame:

$$
[\mathcal{BN}] = \begin{bmatrix}{}^{\mathcal{N}}\hat{\boldsymbol{b}}_1\\ {}^{\mathcal{N}}\hat{\boldsymbol{b}}_2\\ {}^{\mathcal{N}}\hat{\boldsymbol{b}}_3\end{bmatrix}
$$

Differentiating:

$$
[\dot{\mathcal{BN}}] = \begin{bmatrix}{}^{\mathcal{N}}\frac{d}{dt}\left(\hat{\boldsymbol{b}}_1\right)\\ {}^{\mathcal{N}}\frac{d}{dt}\left(\hat{\boldsymbol{b}}_2\right)\\ {}^{\mathcal{N}}\frac{d}{dt}\left(\hat{\boldsymbol{b}}_3\right)\end{bmatrix} = \begin{bmatrix}0 & \omega_3 & -\omega_2\\ -\omega_3 & 0 & \omega_1\\ \omega_2 & -\omega_1 & 0\end{bmatrix}\begin{bmatrix}{}^{\mathcal{N}}\hat{\boldsymbol{b}}_1\\ {}^{\mathcal{N}}\hat{\boldsymbol{b}}_2\\ {}^{\mathcal{N}}\hat{\boldsymbol{b}}_3\end{bmatrix}
$$

Such that:

$$[\dot{\mathcal{BN}}] = -\left[\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}\times\right][\mathcal{BN}]$$

Deriving the dynamic differential equation in terms of the DCM requires us to express the orbital radius vector (defined in the orbital frame as $\boldsymbol{R} = R\hat{o}_1$):

$$\begin{aligned}
{}^{\mathcal{B}}\boldsymbol{R} &= [\mathcal{BO}]\begin{bmatrix} R \\ 0 \\ 0 \end{bmatrix} \\
&= R\begin{bmatrix} C_{11} \\ C_{21} \\ C_{31} \end{bmatrix}
\end{aligned}$$

Such that the gravity gradient torque is:

$$\begin{aligned}
\boldsymbol{L} &= \frac{3\mu}{R^5}\boldsymbol{R}\times\boldsymbol{I}\cdot\boldsymbol{R} \\
&= \frac{3\mu}{R^3}\begin{bmatrix} C_{11} \\ C_{21} \\ C_{31} \end{bmatrix}\times\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}\begin{bmatrix} C_{11} \\ C_{21} \\ C_{31} \end{bmatrix} \\
&= \frac{3\mu}{R^3}\begin{bmatrix} C_{21}C_{31}\left(I_3 - I_2\right) \\ C_{31}C_{11}\left(I_1 - I_3\right) \\ C_{11}C_{21}\left(I_2 - I_1\right) \end{bmatrix}
\end{aligned}$$

Plugging this into the EOMs:

$$\begin{gathered}
\boldsymbol{I}\cdot\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} = -\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}\times\boldsymbol{I}\cdot\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \boldsymbol{L} \\
\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}\begin{bmatrix} \dot{\boldsymbol{\omega}}_1 \\ \dot{\boldsymbol{\omega}}_2 \\ \dot{\boldsymbol{\omega}}_3 \end{bmatrix} = \begin{bmatrix} \omega_2\omega_3\left(I_2 - I_3\right) \\ \omega_3\omega_1\left(I_3 - I_1\right) \\ \omega_1\omega_2\left(I_1 - I_2\right) \end{bmatrix} + \frac{3\mu}{R^3}\begin{bmatrix} C_{21}C_{31}\left(I_3 - I_2\right) \\ C_{31}C_{11}\left(I_1 - I_3\right) \\ C_{11}C_{21}\left(I_2 - I_1\right) \end{bmatrix}
\end{gathered}$$

A particular solution of these equations occurs when the body is oriented such that one of its principal axes is in line with the orbital radius vector and body is rotating at the same rate as the orbit (Ω). If we choose to align the largest moment of inertia body axis (I_1) with the orbital radius vector, then we can simplify the above equations by noticing that $C_{11} = 1$, $C_{21} = C_{31} = 0$, while $\omega_3 = \Omega$, $\omega_2 = \omega_1 = 0$:

$$\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}\begin{bmatrix} \dot{\boldsymbol{\omega}}_1 \\ \dot{\boldsymbol{\omega}}_2 \\ \dot{\boldsymbol{\omega}}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This particular solution is expected to be unstable as any small rotation about $\hat{\boldsymbol{b}}_3$ will create an additional torque that will cause large-scale oscillations. This is illustrated by drawing the scenario and showing that any small reorientation will produce a net torque which is not restoring.

5 Orbit Determination Past Problem