First Order Optimization Algorithms Bench-marked

Interim Report

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Abstract

In recent years, it has become increasingly important to develop efficient optimization algorithms that can be used to solve a wide range of problems. Among these algorithms, the so-called "first-order" optimization algorithms are particularly popular, due to their ability to achieve good performance using only information about the objective function and its first derivative. One of the most well-known first-order optimization algorithms is gradient descent, which is often used as the basis for more complex optimization methods.[1]

In this project, I will explore and implement five different first-order optimization algorithms, starting with gradient descent and ending with the Adam method, which is one of the most widely-used optimization algorithms in the world. I will benchmark the algorithms against challenging objective functions, and visualize their performance to help us understand how they work and compare them against each other.

One of the key benefits of first-order optimization algorithms is their ability to handle a wide range of different types of objective functions. For example, they can be used to optimize functions that have many local minima, or those that are highly non-convex. Additionally, these algorithms are typically very fast to compute, which makes them particularly useful in applications where speed is a key concern, such as in machine learning.

Overall, the goal in this project is to gain a better understanding of first-order optimization algorithms, and to see how they perform in practice.

Keywords— First Order, Optimization, Convexity, Non-Convexity, Gradient Descent, Momentum Gradient Descent, Nesterov's Accelerated Gradient Method, Adaptive Gradient Descent, Adaptive Moment Estimation

1 Introduction

Gradient descent is one of the most popular algorithms used to perform optimization and has become the most common method for optimizing neural networks. Many state-of-the-art deep learning libraries, such as lasagne, caffe, and keras, include implementations of various algorithms for optimizing gradient descent. However, these algorithms are often used as black-box optimizers, as practical explanations of their strengths and weaknesses are difficult to come by.

The aim of this research paper is to provide intuition about the behavior of different first order algorithms for optimizing gradient descent, help readers put them to use, and explore their practical performance. I will begin by introducing the general idea of gradient descent. I will then briefly introduce the most common optimization algorithms by summarizing the intuition behind their birth and the challenges the algorithm faces. I will then present their performance optimizing challenging objective functions and summarize findings.

In this project I implemented stochastic renditions of Vanilla Gradient Descent, Momentum Descent, Nesterov Accelerated Descent, Adagrad Descent and Adam Descent. To measure the performance and robustness of these algorithms I tested each of these algorithms on five difficult objective functions —— the Booth Function, Beale Function, Rosenbrock Function and Ackley Function.

2 Gradient Descent

Gradient descent is an optimization algorithm that is used to find the values of parameters that minimize a given objective function. It works by iteratively updating the parameters in the direction opposite to the gradient of the objective function with respect to the parameters. A gradient is a vector that indicates the direction of the greatest rate of increase of a function at a given point. In other words, it is a vector of partial derivatives that describes the slope of a function at a given point. The learning rate determines the size of the steps taken to reach the minimum. In other words, gradient descent follows the direction of the slope of the objective function downhill until it reaches a local minimum. It is widely used in machine learning, particularly in training neural networks. There are three variants of gradient descent which differ in how much data is used to compute the gradient of the objective function.

Batch gradient descent calculates the gradient of the cost function with respect to the model's parameters for the entire training dataset. This can be slow and impractical for datasets that do not fit in memory. Batch gradient descent also does not allow for online learning, where new examples can be added on the fly.

Stochastic gradient descent (SGD), on the other hand, performs a parameter update for each training example. This can be much faster and enables online learning. However, the frequent updates in SGD can cause the objective function to fluctuate heavily, which can complicate convergence to the minimum.

Mini-batch gradient descent is a compromise between batch gradient descent and SGD. It uses mini-batches of training examples to update the model's parameters, which can reduce the variance of the

updates and make use of optimized matrix operations common in deep learning libraries. Mini-batch gradient descent is typically the algorithm of choice when training a neural network.

As mentioned previously, in this project I used stochastic implementations of the following algorithms. In the following, $J(\theta)$ represents the objective function, θ_t represents the model's parameters at time step t, and η represents the learning rate.

2.1 Vanilla Gradient Descent

$$\theta_t = \theta_{t-1} - \eta \bigtriangledown_\theta J(\theta_t)$$

Figure 1: Vanilla Gradient Descent Update Rule

Vanilla gradient descent is an optimization algorithm that finds the values of parameters that minimize a given objective function. It does this by iteratively updating the parameters in the direction opposite to the gradient of the objective function with respect to the parameters. Vanilla gradient descent implements this in the most straightforward way possible with just a single hyper-parameter, the learning rate. Learning rate determines the size of the steps taken to reach the minimum. [1]

2.2 Momentum Method

$$\nu_t = \gamma\nu_{t-1} + \eta \bigtriangledown_\theta J(\theta_t)$$

$$\theta_t = \theta_{t-1} - \nu_t$$

Figure 2: Moment Method Update Rule

Momentum descent accelerates gradient descent by adding a fraction of the update vector of the past time step to the current update vector. That is, momentum descent introduces the term ν_t which increases for dimensions whose gradients point in the same directions and reduces updates for dimensions whose gradients change directions. This allows the algorithm to gain momentum in the right direction and dampen oscillations. Momentum descent can converge to the minimum faster and more reliably than vanilla gradient descent, especially when dealing with shallow neural networks or noisy data. However, it can also overshoot the minimum, so the learning rate and momentum hyper-parameters must be carefully chosen. [1]

2.3 Nesterov Accelerated Gradient Method

$$\nu_t = \gamma\nu_{t-1} + \eta \bigtriangledown_\theta J(\theta_t - \gamma\nu_{t-1})$$

$$\theta_t = \theta_{t-1} - \nu_t$$

Figure 3: Nesterov Accelerated Gradient Method Update Rule

The Nesterov Accelerated Gradient (NAG) improves upon standard gradient descent by calculating the gradient of the objective function at a point that is ahead of the current position in the direction of the previous update. This allows the algorithm to take into account the momentum of the previous updates and adjust the current update accordingly. NAG is a way to give our momentum term this kind of prescience. We know that we will use our momentum term to move the parameters, thus computing $\theta - \gamma\nu_{t-1}$ gives us an approximation of the next position of the parameters. That is using the current momentum we can infer a rough location of the next parameters.

2.4 Adagrad Method

Adagrad adapts the learning rate to the parameters, performing larger updates for infrequent and smaller updates for frequent parameters. Thus in our update rule is vectorized to allow for individual parameter updates. G_t is a diagonal matrix composed of the sum of the squares of the gradients up to time t. This allows the algorithm to converge faster and perform better on sparse data compared to vanilla gradient

$$G_t \in \mathbb{R}^{dxd}$$

$$\theta_t = \theta_{t-1} - \frac{\eta}{\sqrt{G_t + \epsilon}} \odot \bigtriangledown_\theta J(\theta_t)$$

Figure 4: Adagrad Method Update Rule

descent. However, Adagrad has a couple of drawbacks. It accumulates the squared gradients in the denominator, which can cause the learning rate to become too small and the algorithm to stop learning. Additionally, Adagrad does not use the momentum term, which can be beneficial for convergence.

2.5 Adam Method

$$m_t = \beta_1 m_{t-1} + (1 - \beta_1) \bigtriangledown_\theta J(\theta_t)$$

$$v_t = \beta_2 v_{t-1} + (1 - /beta_2)(\bigtriangledown_\theta J(\theta_t))^2$$

$$\hat{m_t} = \frac{m_t}{(1 - \beta_1^t)}$$

$$\hat{v_t} = \frac{v_t}{(1 - \beta_2^t}$$

$$\theta_t = \theta_{t-1} - \frac{\nu}{\sqrt{\hat{v_t}}} \hat{m_t}$$

Figure 5: Adagrad Method Update Rule

Adam is an optimization algorithm that is a combination of the Adagrad and momentum descent methods. It uses adaptive learning rates that are calculated per parameter and a momentum term that helps the algorithm converge faster and avoid oscillations. We calculate $\hat{m_t}$ and $\hat{v_t}$ to correct for biasing towards 0 since m_t and v_t are initialized as the zero vector. The creators of the Adam method proposed default values of 0.9 for β_1, 0.999 for β_2, and 10^{-8} for ϵ.

3 Objective Functions

3.1 Booth Function

$$f(x) = (x_1 + 2x_2 - 7)^2 + (2x_1 + x_2 - 5)^2$$

$$\bigtriangledown_x f(x) = \begin{bmatrix} (10x_1 + 8x_2 - 34) \\ (8x_1 + 10x_2 - 38) \end{bmatrix}$$

Figure 6: Booth Function and its gradient vector

The Booth objective function is a mathematical function used in optimization problems. The Booth function has a global minimum at the point (1, 3), where its value is 0. It is often used as a test function in optimization algorithms. One of the main challenges of optimizing the Booth objective function is that it has multiple local minima. This means that the optimization algorithm may get stuck in a local minimum and be unable to find the global minimum at (1, 3), where the value of the function is 0. Additionally, the function is non-convex, which can make it difficult for optimization algorithms to find the global minimum. These challenges make the Booth function a useful test function for evaluating the performance of optimization algorithms. [2]

3.2 Beale Function

The Beale objective function is a mathematical function used in optimization problems. The Beale function has a global minimum at the point (3, 0.5), where its value is 0. It is often used as a test function in optimization algorithms because it has multiple local minima, which can make it challenging to optimize. [2]

$$f(x) = (1.5 - x_1 + x_1x_2)^2 + (2.25 - x_1 + x_1x_2^2)^2 + (2.625 - x_1 + x_1x_2^3)^2$$

$$\bigtriangledown_x f(x) = \begin{bmatrix} 2x_1x_2^6 + 2x_1x_2^4 + 5.25x_2^3 - 4x_1x_2^3 + 4.5x_2^2 - 2x_1x_2^2 + 3x_2 - 4x_1x_2 + 6x_1 - 12.75 \\ 6x_1^2x_2^5 + 4x_1^2x_2^3 - 6x_1^2x_2^2 - 2x_1^2x_2 - 2x_1^2 + 15.75x_1x_2^2 + 9x_1x_2 + 3x_1 \end{bmatrix}$$

Figure 7: Beale Function and its gradient vector

3.3 Rosenbrock Function

$$f(x) = 100 \cdot (x_2 - x_1^2)^2 + (x_1 - 1)^2$$

$$\bigtriangledown_x f(x) = \begin{bmatrix} 202x_1 - 200x_2^2 - 2 \\ 400x_2^3 - 400x_1x_2 \end{bmatrix}$$

Figure 8: Rosenbrock Function and its gradient vector

The Rosenbrock objective function, also referred to as the Valley or Banana function, is a mathematical function used in optimization problems. The Rosenbrock function has a global minimum at the point (1, 1), where its value is 0. One of the main challenges of optimizing the Rosenbrock objective function is that it has a curved, elongated shape that makes it difficult for optimization algorithms to converge. This can cause the algorithm to get stuck in a local minimum and be unable to find the global minimum at (1, 1). Additionally, the function is non-convex, which can make it difficult for optimization algorithms to find the global minimum. These challenges make the Rosenbrock function a useful test function for evaluating the performance of optimization algorithms. [2]

3.4 Ackley Function

$$f(x) = -20 \cdot exp\left(-\frac{1}{5}\sqrt{\frac{x_1^2 + x_2^2}{2}}\right) - exp\left(\frac{\cos 2\pi x_1 + \cos 2\pi x_2}{2}\right) + 20 + exp(1)$$

$$\bigtriangledown_x f(x) = \begin{bmatrix} \frac{2^{1.5} \cdot exp\left(-\frac{(x_1^2+x_2^2)^{0.5}}{5 \cdot 2^{0.5}}\right) x_1 + \pi exp\left(\frac{\cos 2\pi x_1 + \cos 2\pi x_2}{2}\right) \sin(2\pi x_1)(x_1^2+x_2^2)^{0.5}}{(x_1^2+x_2^2)^{0.5}} \\ \frac{2^{1.5} \cdot exp\left(-\frac{(x_1^2+x_2^2)^{0.5}}{5 \cdot 2^{0.5}}\right) x_2 + \pi exp\left(\frac{\cos 2\pi x_1 + \cos 2\pi x_2}{2}\right) \sin(2\pi x_2)(x_1^2+x_2^2)^{0.5}}{(x_1^2+x_2^2)^{0.5}} \end{bmatrix}$$

Figure 9: Ackley Function and its gradient vector

The Ackley function is commonly used to evaluate the performance of optimization algorithms. The Ackley function has a global minimum at the point (0, 0), where its value is 0. In its two-dimensional form, it has a complex shape with a nearly flat outer region and a large hole at the center. This can be challenging for optimization algorithms, particularly hill-climbing algorithms, which may get stuck in one of the function's many local minima. [2]

4 Methodology

In order to test the performance of each optimization algorithm on each of the objective functions, I implemented a function for large-scale cross-validation hyper-parameter tuning, see Appendix A. This allowed me get a general idea of the performance of each algorithm as well as find parameters in the optimal neighborhood. Additionally, this process allowed me to observe the general behavior of each algorithm and gain insights into how they operate. By conducting this experiment, I was able to determine which algorithm was the best performer on each objective function and gain a better understanding of the strengths and limitations of each algorithm.

To deduce the best algorithm for each of objective functions convergence accuracy, algorithmic complexity and convergence time was taking into account. After deducing the best algorithm to use for each of the objective function, I furthered the fine tuning of hyper-parameters find the optimal hyper-parameters.

5 Analysis

5.1 Booth Function

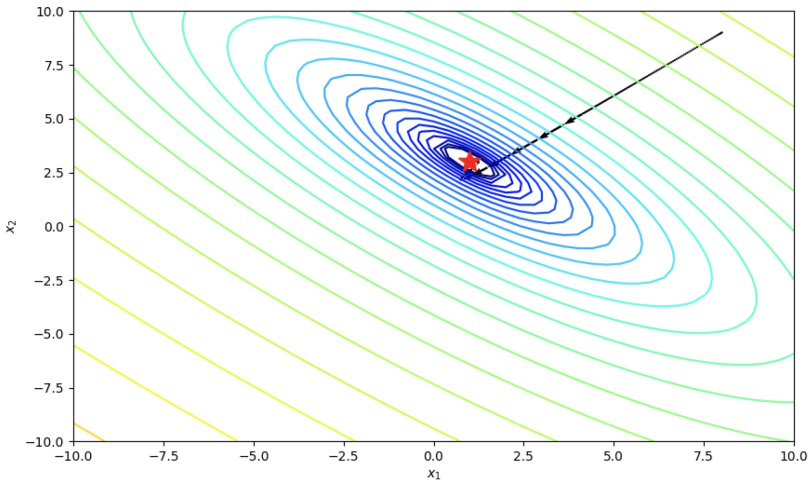


Figure 10: The optimal convergence pathway of the Booth function

For the Booth function the optimal model was achieved using the NAG algorithm, see Appendix A. This final model was able to achieve an ϵ_x of 0.0 and an ϵ_f of 0.0; that is the minimizer loss was 0.0 and the minimum loss was 0.0. On average the model converged in 0.0363 seconds. Although convergence was achieved by all algorithms, the NAG algorithm did it most efficiently, that is it minimized compute complexity and thus convergence time. The Adam and Adagrad methods both had much more initial variance in their respective convergence path ways.

5.2 Beale Function

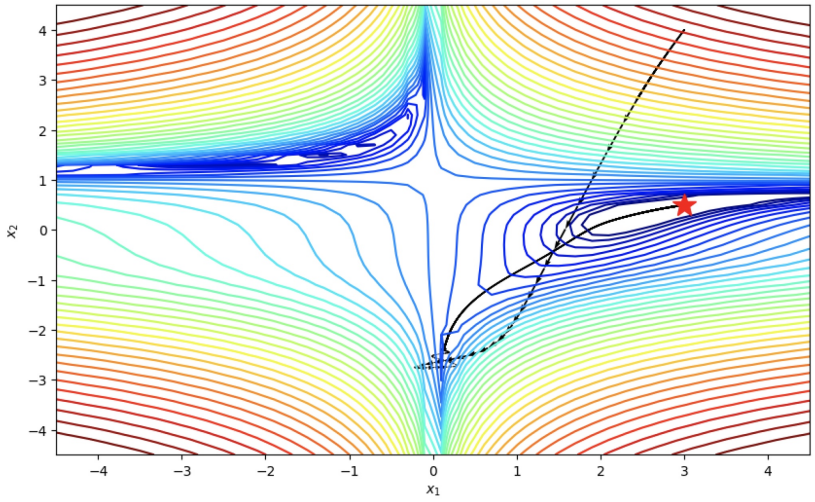


Figure 11: The optimal convergence pathway of the Beale function

For the Beale function the optimal model was achieved by the Momentum algorithm, see Appendix A. This final model was able to achieve an ϵ_x of 3.233018248352212e-15 and an ϵ_f of 1.86430018616684 43e-30. One average the model converged in 0.7654 seconds. Although, it appears the path take by our final model first overshoots the global minimum, the algorithm eventually turns itself around and finds the global minimum. This overshooting behavior is common of the Momentum method but it still was able to converge quicker than any of the other algorithms. Vanilla gradient descent converged in the most direct path but needed a very small learning rate to not get stuck within a local minimum and thus converged slowly. The Adagrad function was not able to converge.

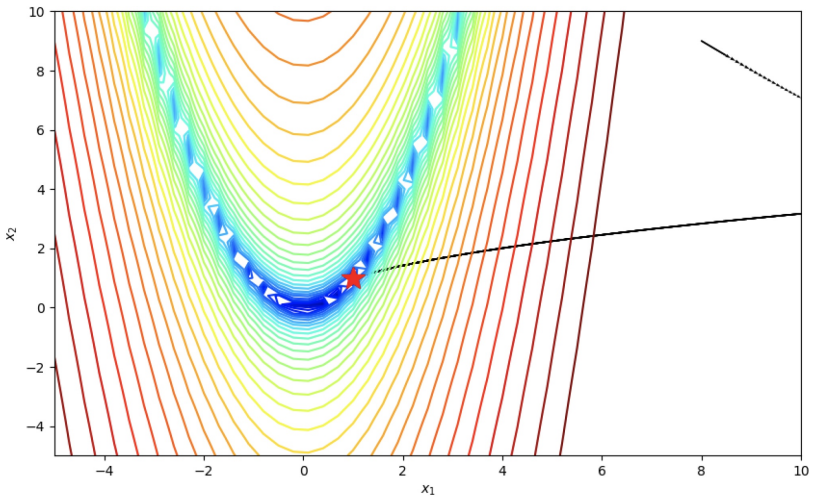


Figure 12: The optimal convergence pathway of the Rosenbrock function

5.3 Rosenbrock Function

For the Rosenbrock function the optimal model was achieved by the Adam method, see Appendix A. This final model was able to achieve an ϵ_x of 9.835347909592552e-15 and an ϵ_f of 1.6093797846446743e-26. On average the model converged in 1.0645 seconds. Both the Adam and Adagrad methods presented similar behavior in minimizing the Rosenbrock function although Adam converged slightly quicker. The other three algorithms took much lower learning rates to converge and thus took much longer to converge.

5.4 Ackley Function

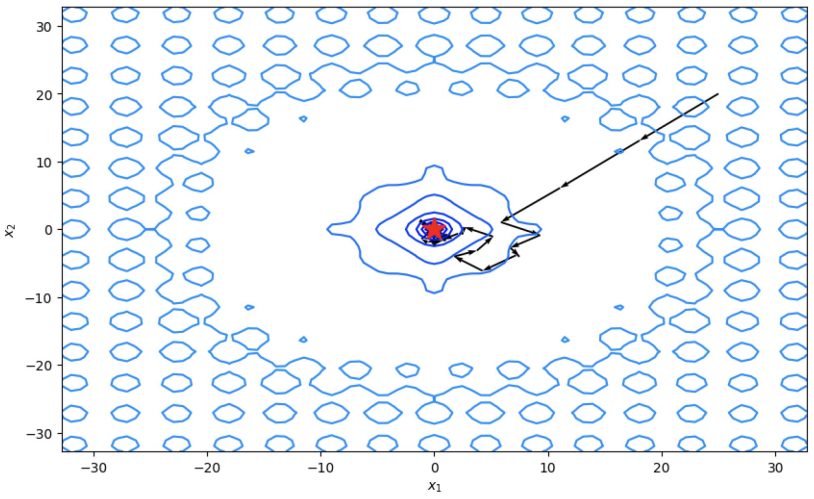


Figure 13: The optimal convergence pathway of the Ackley function

For the Ackley function the optimal model was achieved by the Adagrad method, see Appendix A. This final model was able to achieve an ϵ_x of 6.356646485595074e-06 and an ϵ_f of 1.7980387315930102e-05. On average the model converged in 6.896 seconds. The Ackley function was by far the most difficult function to optimize. My implemented hyper-parameter tuning function was unable to find hyper-parameters that lead to convergence. Thus it took immense manually hyper-parameter tuning to find convergence. I implemented a deep multi-level learning rate function that reduced the learning rate drastically as t increased. It took an initial high learning rate for the algorithms to not get stuck in a local minimum. But then it took a lower learning rate for the algorithms to not over shoot the global minimum. Thus a learning rate step down function was implemented and proved successful.

6 Conclusion

This paper presented an investigation into the performance of different optimization algorithms for gradient descent on challenging objective functions. The objective functions tested were the Booth, Beale,

Rosenbrock, and Ackley functions. The optimization algorithms tested were vanilla gradient descent, momentum descent, Nesterov Accelerated Gradient, Adagrad, and Adam. The results showed that each algorithm performed differently on each objective function, with some algorithms performing better on certain functions than others. The Ackley function was found to be the most challenging to optimize.

7 Literature Review

Introduction: I will be performing a literature review on "A Universal Catalyst for First-Order Optimization" [3]. In this paper a generic scheme for accelerating first-order optimization methods which builds upon a new analysis of the accelerated proximal point algorithm.

Method: Their approach consists of minimizing a convex objective by approximately solving a sequence of well-chosen auxiliary problems, leading to faster convergence. It consists of replacing, at iteration k, the original objective function $F(x)$ by an auxiliary objective $G_k(x)$ close to $F(x)$ up to a quadratic term:

$$G_k(x) \triangle = F(x) + \frac{k}{2} \|x - y_{k-1}\|^2$$

where k will be specified later and y_k is obtained by an extrapolation step. Then at iteration k, the accelerated algorithm A minimizes $G_k(x)$ up to accuracy ϵ_k. Minimizing this substituted equation is only an approximation of the original equation unless k = 0; but this equation has a better condition number than the original objective thus making it easier to minimize. This accelerated approach applies to a large class of algorithms and thus problems. Such as gradient descent, block coordinate descent, SAG, SAGA, SDCA, SVRG, Finito/MISO, and their proximal variants. By analogy this generic approach could be thought of as a chemical "catalyst". The approach provides a universal positive answer regardless of the strong convexity of the objective.

References

[1] S. Ruder, "An overview of gradient descent optimization algorithms," Mar 2020.

[2] S. Surjanovic and D. Bingham, "Virtual library of simulation experiments:."

[3] H. Lin, J. Mairal, and Z. Harchaoui, "A universal catalyst for first-order optimization," 2015.

8 Appendix A:

In [31]:
```
%matplotlib inline
import matplotlib.pyplot as plt
import numpy as np
import time

from mpl_toolkits.mplot3d import Axes3D
from matplotlib.colors import LogNorm
from matplotlib import animation

from scipy.optimize import minimize, OptimizeResult
from collections import defaultdict
from itertools import zip_longest
from functools import partial

from answer import Answer
```

In [32]:
```
import warnings
warnings.filterwarnings('ignore')
```

Implementation (Students do)

Methods

You will implement five optimization algorithms (descriptions available here).

- Gradient descent (`gd`)
- Momentum gradient method (`momentum`)
- Nesterov's accelerated gradient method (`nag`)
- Adaptive gradient method (`adagrad`)
- Adaptive moment estimation (`adam`)

The last is a very common optimizer used in practical applications -- possibly the most common in the world.

Make note of the function headers: `def gd(func, x, lr, num_iters, jac, tol, callback, *args, **kwargs):` . Each method will satisfy this header format in accordance with the specification of custom minimizers used with `scipy.optimize.minimize` . This function is well-documented, but the highlights of the arguments are below.

- `func` : [type: function] The loss function. Takes in a point of type np.ndarray (2,) and returns a float representing the value of the function at that point.
- `x` : [type: np.ndarray (2,)] The starting point of the optimization.
- `lr` : [type: function] Learning rate schedule. Takes in an argument of type int representing the iteration number, and returns the learning rate to be used for that iteration.

- `num_iters` : [type: int] The number of iterations of the optimization method to run.
- `jac` : [type: function] The gradient of the loss function. "Jac" stands for Jacobian, which is out of scope for this class, but for scalar-valued functions, it is the transpose of the gradient. Takes in a point of type np.ndarray (2,) and returns an np.ndarray (2,) representing the gradient of the function at that point.
- `tol` : [type: float] The tolerance within which the optimization method is deemed to have converged.
- `callback` : [type: function] A function to be called on each iterate over the course of the optimization.
- `*args` and `**kwargs` : You will not need to use these, but they are present for compatibility with the `scipy.optimize.minimize` API.

Each function will need to return a two-tuple containing

- An instance of `scipy.optimize.OptimizeResult`, described here.
- A `np.ndarray` containing the function value at the initial point and each iterate over the course of the optimization.

In [33]:
```
def result_init():
    result = OptimizeResult()
    result.x = None
    result.success = False
    result.status = 1
    result.message = "Optimization in Progress"
    result.fun = []
    result.jac = []
    result.nfev = 0
    result.nhev = 0
    result.nit = 0
    result.njev = 0
    return result

def result_update(result, func_x, jac_x):
    result.fun = np.append(result.fun, func_x)
    result.nfev += 1
    result.jac = np.append(result.jac, jac_x)
    result.njev += 1
    result.nit += 1

def result_post(result, curr_x, success, status, message):
    result.x = curr_x
    result.success = success
    result.status = status
    result.message = message


def gd(func, x, lr, num_iters, jac, tol, callback, *args, **kwargs):
    curr_x = x

    result = result_init()
```

```
    while result.nit < num_iters:
        prev_x = curr_x
        curr_x = curr_x - lr(result.nit) * jac(prev_x)
        delta_x = abs(curr_x - prev_x)

        # Update Result
        result_update(result, func(curr_x), jac(curr_x))

        #callback("Iteration: ", result.nit, "Current X Value: ", curr_x)
        callback(curr_x)
        if np.linalg.norm(delta_x) <= tol:
            # Optimization Converged
            result_post(result, curr_x, True, 0, "Optimization Successfully Converg
            return result, np.array(result.fun)

    # Optimization Failed to Converge
    result_post(result, curr_x, False, -1, "Optimization Failed to Converge")
    return result, np.array(result.fun)




def momentum(func, x, lr, num_iters, jac, tol, callback, gamma=0.9, *args, **kwargs
    curr_x = x
    velocity = 0

    result = result_init()

    while result.nit < num_iters:
        prev_x = curr_x

        velocity = gamma * velocity + lr(result.nit) * jac(curr_x)
        curr_x = curr_x - velocity

        delta_x = abs(curr_x - prev_x)

        # Update Result
        result_update(result, func(curr_x), jac(curr_x))


        #callback("Iteration: ", result.nit, "Current X Value: ", curr_x)
        callback(curr_x)
        if np.linalg.norm(delta_x) <= tol:
            # Optimization Converged
            result_post(result, curr_x, True, 0, "Optimization Successfully Converg
            return result, np.array(result.fun)

    # Optimization Failed to Converge
    result_post(result, curr_x, False, -1, "Optimization Failed to Converge")
    return result, np.array(result.fun)

def nag(func, x, lr, num_iters, jac, tol, callback, gamma=0.9, *args, **kwargs):
    curr_x = x
    velocity = 0
```

```
    result = result_init()

    while result.nit < num_iters:
        prev_x = curr_x

        velocity = gamma * velocity + lr(result.nit) * jac(curr_x - (gamma*velocity
        curr_x = curr_x - velocity

        delta_x = abs(curr_x - prev_x)

        # Update Result
        result_update(result, func(curr_x), jac(curr_x))


        #callback("Iteration: ", result.nit, "Current X Value: ", curr_x)
        callback(curr_x)
        if np.linalg.norm(delta_x) <= tol:
            # Optimization Converged
            result_post(result, curr_x, True, 0, "Optimization Successfully Converg
            return result, np.array(result.fun)

    # Optimization Failed to Converge
    result_post(result, curr_x, False, -1, "Optimization Failed to Converge")
    return result, np.array(result.fun)

def adagrad(func, x, lr, num_iters, jac, tol, callback, eps=1e-5, *args, **kwargs):
    curr_x = x
    G = 0

    result = result_init()

    while result.nit < num_iters:
        prev_x = curr_x

        G += (jac(curr_x))**2
        curr_x = curr_x - (lr(result.nit) / np.sqrt( G + eps)) * jac(curr_x)

        delta_x = abs(curr_x - prev_x)

        # Update Result
        result_update(result, func(curr_x), jac(curr_x))


        #print("Iteration: ", result.nit, "Current X Value: ", curr_x)
        callback(curr_x)
        if np.linalg.norm(delta_x) <= tol:
            # Optimization Converged
            result_post(result, curr_x, True, 0, "Optimization Successfully Converg
            return result, np.array(result.fun)

    # Optimization Failed to Converge
    result_post(result, curr_x, False, -1, "Optimization Failed to Converge")
    return result, np.array(result.fun)

def adam(func, x, lr, num_iters, jac, tol, callback, beta1=0.9, beta2=0.999, eps=1e
    curr_x = x
```

```
    momentum = 0
    velocity = 0

    result = result_init()

    while result.nit < num_iters:
        prev_x = curr_x

        momentum = (beta1 * momentum) + (1 - beta1)*jac(curr_x)
        velocity = (beta2 * velocity) + (1 - beta2)*(jac(curr_x)**2)

        momentum_hat = momentum / (1 - (beta1**(result.nit + 1)))
        velocity_hat = velocity / (1 - (beta2**(result.nit + 1)))

        curr_x = curr_x - (lr(result.nit) / (np.sqrt(velocity_hat) + eps) )*momentu

        delta_x = abs(curr_x - prev_x)

        # Update Result
        result_update(result, func(curr_x), jac(curr_x))

        #print("Iteration: ", result.nit, "Current X Value: ", curr_x)
        callback(curr_x)
        if np.linalg.norm(delta_x) <= tol:
            # Optimization Converged
            result_post(result, curr_x, True, 0, "Optimization Successfully Converg
            return result, np.array(result.fun)

    # Optimization Failed to Converge
    result_post(result, curr_x, False, -1, "Optimization Failed to Converge")
    return result, np.array(result.fun)
```

Functions and gradients

You have been given the implementation of four functions ($\mathbb{R}^2 \to \mathbb{R}$), given below. You will need to implement `grad`, which returns their gradients as `np.ndarray` (2,). There is a field below for you to submit the gradient in $\frac{\partial f}{\partial x}$.

- Booth function: $f_1(x) = (x_1 + 2x_2 - 7)^2 + (2x_1 + x_2 - 5)^2$
- Beale function:
  $f_2(x) = (1.5 - x_1 + x_1x_2)^2 + (2.25 - x_1 + x_1x_2^2)^2 + (2.625 - x_1 + x_1x_2^3)^2$
- Rosenbrock function: $f_3(x) = 100 \cdot (x_2 - x_1^2)^2 + (x_1 - 1)^2$
- Ackley function:
  $f_4(x) = -20 \cdot \exp\left(-\frac{1}{5}\sqrt{\frac{x_1^2 + x_2^2}{2}}\right) - \exp\frac{(\cos 2\pi x_1 + \cos 2\pi x_2)}{2} + 20 + \exp(1)$

In [34]:
```
def func(fn, x_1, x_2):
    if fn == 'booth':
        return (x_1 + 2*x_2 - 7)**2 + (2*x_1 + x_2 - 5)**2
    elif fn == 'beale':
```

```
        return (1.5 - x_1 + x_1*x_2)**2 + (2.25 - x_1 + x_1*(x_2**2))**2 + (2.625 -
    elif fn == 'rosen2d':
        return 100 * (x_2 - (x_1**2))**2 + (x_1 - 1)**2
    elif fn == 'ackley2d':
        return -20 * np.exp((-1/5)*np.sqrt((x_1**2 + x_2**2)/2)) - np.exp((np.cos(2
    else:
        raise ValueError('Function %s not supported.' % fn)

def grad(fn, x_1, x_2):
    gradient = np.zeros_like(x_1)
    if fn == 'booth':
        g1 = 10*x_1 + 8*x_2 - 34
        g2 = 8*x_1 + 10*x_2 - 38
    elif fn == 'beale':
        g1 = (2*x_1*(x_2**6) + 2*x_1*(x_2**4)) + 5.25*(x_2**3) - 4*x_1*(x_2**3) + 4
        g2 = 6*(x_1**2)*(x_2**5) + 4*(x_1**2)*(x_2**3) - 6*(x_1**2)*(x_2**2) - 2*(x
    elif fn == 'rosen2d':
        g1 = 202*x_1 - 200*(x_2**2) - 2
        g2 = 400*(x_2**3) - 400*x_1*x_2
    elif fn == 'ackley2d':
        g1 = ( (2**1.5) * np.exp(-1 * (x_1**2 + x_2**2)**0.5 / (5* (2**0.5))) * x_1
        g2 = ( (2**1.5) * np.exp(-1 * (x_1**2 + x_2**2)**0.5 / (5* (2**0.5))) * x_2
    else:
        raise ValueError('Function %s not supported.' % fn)
    return np.stack((g1, g2), axis=-1)
```

FIX

Submission: Gradient values $LaTeX$

Enter the gradients you calculated below.

- Booth: $\nabla_x f_1(x) = \begin{bmatrix} 10x_1+8x_2-34 \\ 8x_1+10x_2-38 \end{bmatrix}$
- Beale: $\nabla_x f_2(x) = \begin{bmatrix} 6x_1x_2^2+6x_1-12x_1x_2+12.75x_2-12.75 \\ 6x_1^2x_2-6x_1^2+12.75x_1 \end{bmatrix}$
- Rosenbrock: $\nabla_x f_3(x) = \begin{bmatrix} 202x_1-200x_2-2 \\ 200x_2-200x_1 \end{bmatrix}$
- Ackley: $\nabla_x f_4(x) = \begin{bmatrix} \frac{\sqrt{2}e^{-\frac{\sqrt{(x_1+x_2)}}{5\cdot\sqrt{2}}}}{\sqrt{x_1+x_2}}+\pi e^{\frac{\cos(2\pi x_1)+\cos(2\pi x_2)}{2}}\sin(2\pi x_1) \\ \frac{\sqrt{2}e^{-\frac{\sqrt{(x_1+x_2)}}{5\cdot\sqrt{2}}}}{\sqrt{x_1+x_2}}+\pi e^{\frac{\cos(2\pi x_1)+\cos(2\pi x_2)}{2}}\sin(2\pi x_2) \end{bmatrix}$

Student-facing `Answer` class (provided)

You have been provided a class called `Answer` which will be helpful for the remainder of the project. It can be found in the `answer.py` file. You are welcome to read and modify it, but this is not required. All information you need about this class is documented here, and examples of usage are given below.

Documentation

- `__init__(self, methods, func, grad)`
  - Instantiates the `Answer` class with the functions you have implemented. `methods` is a dictionary mapping algorithm names to the functions that implement them, and `func` and `grad` are the functions of the same name that you have implemented.
- `set_fn_settings(self, fn_name)`
  - Sets the instance variables needed for visualizing `fn_name` with `plot2d` and `plot3d`. Needs to be called before calling these functions.
- `set_settings(self, fn_name, method, x0, **kwargs)`
  - Sets the instance variables needed for visualizing `method` optimizing `fn_name` starting at `x0` with `path2d`, `path3d`, `video2d`, and `video3d`. Any additional `kwargs` (likely `lr` and `num_iters`) will be passed on to `method`. Needs to be called before calling these functions or `compare`.
- `get_settings(self)`
  - Returns the arguments passed into `set_settings`: `fn_name`, `method`, `x0`, and `kwargs`.
- `compare(self, method, start_iter=0, **kwargs)`
  - Generates training loss graph comparing `method` with the previously set method on the previously set loss function and starting point, starting at iteration `start_iter`. Additional `kwargs` (likely `lr` and `num_iters`) will be passed on to `method`.
- `get_xs_losses(self)`
  - Returns a tuple containing
    - [type: `np.ndarray` (1 + `n_iters`, 2)] All iterates (including the initial point).
    - [type: `np.ndarray` (1 + `n_iters`,)] The loss at each iterate.
- `get_min_errs(self)`
  - Returns a tuple containing
    - `float` representing the closest (in L2 norm) the optimization procedure got to the global minimizer.
    - `float` representing the closest the optimization procedure got to the global minimum function value.
- `func_val(self, x)`
  - Returns `float` value of the previously set loss function evaluated at `x`. Convenience tool for debugging.
- `grad_val(self, x)`

    - Returns `np.ndarray` (2,) gradient of the previously set loss function evaluated at `x`. Convenience tool for debugging.
- `plot2d(self)`
    - Plots contours of the previously set loss function.
- `plot3d(self)`
    - Plots the previously set loss function.
- `path2d(self)`
    - Plots the sequence of iterates produced by the set method on the set loss function on a 2D contour.
- `path3d(self)`
    - Plots the sequence of iterates produced by the set method on the set loss function on a 3D graph. **NOTE:** This one does not work very well.
- `video2d(self, filename=None)`
    - Creates and saves an MP4 video of the path taken in `path2d` at `filename`. File name defaults to "{function}_{method}_2d.mp4"
- `video3d(self, filename=None)`
    - Creates and saves an MP4 video of the path taken in `path3d` at `filename`. File name defaults to "{function}_{method}_3d.mp4". **NOTE:** This works better than `path3d`.

In [35]:
```
# instantiate the Answer class with the methods you have implemented! (You can impl
ans = Answer(
    {  # a mapping of algorithm names to functions implementing them
        'gd': gd,
        'momentum': momentum,
        'nag': nag,
        'adagrad': adagrad,
        'adam': adam
    },
    func,
    grad
)
```

Testing your code

We are not providing much structure here, but now is a good time to make sure your optimization methods are working well. The cell below tests your gradient descent method on the function $f(x) = x^2$. We have included the output of our solution as a comment. Note that the function you feed it needs to take in a point as its sole argument and return the function as well as the gradient evaluated at that point.

In [36]:
```
# Maybe a useful starting example for testing gradient descent on a simple function
x_squared = lambda x: (x**2, 2*x)  # returns both the function value and the gradie
opt_res, losses = minimize(x_squared, 3, jac=True, method=gd, callback=print,
                           options=dict(lr=lambda t: 0.25, x0=3, num_iters=15, tol=
print('Final iterate: %.6f. Number of iterations: %d. Final loss: %.8f.' % (opt_res
```

```
# ----------------
# Expected output (GD):
# ----------------
# [1.5]
# [0.75]
# [0.375]
# [0.1875]
# [0.09375]
# [0.046875]
# [0.0234375]
# [0.01171875]
# [0.00585938]
# [0.00292969]
# [0.00146484]
# [0.00073242]
# Final iterate: 0.000732. Number of iterations: 12. Final loss: 0.00000054.
```

```
[1.5]
[0.75]
[0.375]
[0.1875]
[0.09375]
[0.046875]
[0.0234375]
[0.01171875]
[0.00585938]
[0.00292969]
[0.00146484]
[0.00073242]
Final iterate: 0.000732. Number of iterations: 12. Final loss: 0.00000054.
```

In [37]:

```
x_squared = lambda x: (x**2, 2*x)  # returns both the function value and the gradie
opt_res, losses = minimize(x_squared, 3, jac=True, method=momentum, callback=print,
                           options=dict(lr=lambda t: 0.12, x0=3, num_iters=15, tol=
print('Final iterate: %.6f. Number of iterations: %d. Final loss: %.8f.' % (opt_res
# ----------------
# Expected output (Momentum):
# ----------------
# [2.28]
# [1.0848]
# [-0.251232]
# [-1.39336512]
# [-2.0868773]
# [-2.21018771]
# [-1.79072203]
# [-0.98342963]
# [-0.02084336]
# [0.85048669]
# [1.43056693]
# [1.60930308]
# [1.38393288]
# [0.8489558]
# [0.16372704]
# Final iterate: 0.163727. Number of iterations: 15. Final loss: 0.02680655.
```

```
[2.28]
[1.0848]
[-0.251232]
[-1.39336512]
[-2.0868773]
[-2.21018771]
[-1.79072203]
[-0.98342963]
[-0.02084336]
[0.85048669]
[1.43056693]
[1.60930308]
[1.38393288]
[0.8489558]
[0.16372704]
Final iterate: 0.163727. Number of iterations: 15. Final loss: 0.02680655.
```

In [38]:
```
x_squared = lambda x: (x**2, 2*x)  # returns both the function value and the gradie
opt_res, losses = minimize(x_squared, 3, jac=True, method=nag, callback=print,
                           options=dict(lr=lambda t: 0.4, x0=3, num_iters=15, tol=1
print('Final iterate: %.6f. Number of iterations: %d. Final loss: %.8f.' % (opt_res
# ----------------
# Expected output (NAG):
# ----------------
# [0.6]
# [-0.312]
# [-0.22656]
# [-0.0299328]
# [0.02940634]
# [0.01656231]
# [0.00100054]
# [-0.00260101]
# [-0.00116848]
# [2.41592194e-05]
# [0.00021951]
# Final iterate: 0.000220. Number of iterations: 11. Final loss: 0.00000005.
```

```
[0.6]
[-0.312]
[-0.22656]
[-0.0299328]
[0.02940634]
[0.01656231]
[0.00100054]
[-0.00260101]
[-0.00116848]
[2.41592194e-05]
[0.00021951]
Final iterate: 0.000220. Number of iterations: 11. Final loss: 0.00000005.
```

In [39]:
```
x_squared = lambda x: (x**2, 2*x)  # returns both the function value and the gradie
opt_res, losses = minimize(x_squared, 3, jac=True, method=adagrad, callback=print,
                           options=dict(lr=lambda t: 2, x0=3, num_iters=15, tol=1e-
print('Final iterate: %.6f. Number of iterations: %d. Final loss: %.8f.' % (opt_res
# ----------------
# Expected output (Adagrad):
```

```
# ----------------
# [1.00000028]
# [0.36754467]
# [0.13664342]
# [0.05087939]
# [0.01894909]
# [0.00705745]
# [0.00262851]
# [0.00097897]
# [0.00036461]
# Final iterate: 0.000365. Number of iterations: 9. Final loss: 0.00000013.
```

```
[1.00000028]
[0.36754467]
[0.13664342]
[0.05087939]
[0.01894909]
[0.00705745]
[0.00262851]
[0.00097897]
[0.00036461]
Final iterate: 0.000365. Number of iterations: 9. Final loss: 0.00000013.
```

In [40]:

```
x_squared = lambda x: (x**2, 2*x)  # returns both the function value and the gradie
opt_res, losses = minimize(x_squared, 3, jac=True, method=adam, callback=print,
                           options=dict(lr=lambda t: 2, x0=3, num_iters=15, tol=1e-
print('Final iterate: %.6f. Number of iterations: %d. Final loss: %.8f.' % (opt_res
# ----------------
# Expected output (Adam):
# ----------------
# [1.00000333]
# [-0.74212166]
# [-1.76094027]
# [-1.93985392]
# [-1.565932]
# [-0.89684359]
# [-0.12388967]
# [0.58433522]
# [1.08297399]
# [1.292603]
# [1.21773996]
# [0.91871601]
# [0.47982495]
# [-0.00561154]
# [-0.44385421]
# Final iterate: -0.443854. Number of iterations: 15. Final loss: 0.19700656.
```

```
[1.00000333]
[-0.74212166]
[-1.76094027]
[-1.93985392]
[-1.565932]
[-0.89684359]
[-0.12388967]
[0.58433522]
[1.08297399]
[1.292603]
[1.21773996]
[0.91871601]
[0.47982495]
[-0.00561154]
[-0.44385421]
Final iterate: -0.443854. Number of iterations: 15. Final loss: 0.19700656.
```

Playground and Exploration

You are free to use the functions described above to explore the behavior of the optimization algorithms you have implemented. Pick different starting points, learning rate schedules, and even tolerances to explore! Example usage of the `Answer` class is below.

Exploration

For each of the functions, start at the given initial points (x_0) and use any choice of optimization algorithm and associated hyperparameters to get within the specified distance of the global minimizer and minimum (ϵ_x, ϵ_f). *Hint: The `get_min_errs` function will be helpful.* There is a spot below for you to submit your results for each challenge.

- Booth function
  - $x_0 = [8, 9], \epsilon_x = 10^{-7}, \epsilon_f = 10^{-14}$
- Beale function
  - $x_0 = [3, 4], \epsilon_x = 0.5, \epsilon_f = 0.07$
- Rosenbrock function
  - $x_0 = [8, 9], \epsilon_x = 10^{-7}, \epsilon_f = 10^{-14}$
- Ackley function
  - $x_0 = [25, 20], \epsilon_x = 2 \cdot 10^{-4}, \epsilon_f = 5 \cdot 10^{-4}$. This function is hard. Tell us what you tried and how far you got.

In [41]:
```python
def lr_1(t):
    return 20

def lr_2(t):
    if t < 50:
        return 1e-2
    elif t < 100:
        return 1e-1
    else:
```

```
        return 0.5

def lr_3(t):
    if t < 100:
        return 1e-4
    elif t < 500:
        return 1e-2
    else:
        return 0.1

def lr_4(t):
    if t < 1000:
        return 1e-8
    elif t < 10000:
        return 1e-4
    else:
        return 1e-2

def lr_5(t):
    return 1e-5
```

Submission: Challenge

Place code in the below cells that demonstrates your results for each challenge. Each cell should end with `get_min_errs()` displaying the achieved error.

In [42]:
```
func_arr = ['gd', 'momentum', 'nag', 'adagrad', 'adam']
lr_arr = [lr_1, lr_2, lr_3, lr_4, lr_5]
num_iter_arr = [10, 20, 50, 100, 300, 500, 1000, 10000, 15000, 50000, 100000]
gamma_arr = [0.00001, 0.001, 0.01, 0.1, 0.3, 0.4, 0.5, 0.6, 0.8, 0.9, 0.999]
eps_arr = [1, 0.5, 0.1, 1e-2, 1e-3, 1e-4, 1e-5, 1e-6, 1e-7, 1e-8, 1e-9, 1e-10]

def cv_calc(func, x_0, params):
    final_params = dict()
    best_error = 10000
    if params['method'] == 'gd':
        # Vanilla GD Opti
        # Params lr and num_iter
        final_params['method'] = 'gd'
        for curr_lr in lr_arr:
            for curr_num_iter in num_iter_arr:
                params['lr'] = curr_lr
                params['num_iters'] = curr_num_iter
                ans.set_settings(fn_name=func, x0=x_0, **params)
                err = ans.get_min_errs()
                #print("X Error: ", err[0], "Loss Error: ", err[1])
                if np.linalg.norm(err) < best_error:
                    #print(curr_lr, curr_num_iter)
                    best_error = np.linalg.norm(err)
                    final_params['lr'] = curr_lr
                    final_params['num_iters'] = curr_num_iter
        return final_params
    elif params['method'] == 'momentum':
        # Momentum Opti
```

```
        # Params lr, num_iter and gamma
        final_params['method'] = 'momentum'
        for curr_lr in lr_arr:
            for curr_num_iter in num_iter_arr:
                for curr_gamma in gamma_arr:
                    params['lr'] = curr_lr
                    params['num_iters'] = curr_num_iter
                    params['gamma'] = curr_gamma
                    ans.set_settings(fn_name=func, x0=x_0, **params)
                    err = ans.get_min_errs()
                    #print("X Error: ", err[0], "Loss Error: ", err[1])
                    if np.linalg.norm(err) < best_error:
                        #print(curr_lr, curr_num_iter)
                        best_error = np.linalg.norm(err)
                        final_params['lr'] = curr_lr
                        final_params['num_iters'] = curr_num_iter
                        final_params['gamma'] = curr_gamma
        return final_params
    elif params['method'] == 'nag':
        # NAG Opti
        # Params lr, num_iter and gamma
        final_params['method'] = 'nag'
        for curr_lr in lr_arr:
            for curr_num_iter in num_iter_arr:
                for curr_gamma in gamma_arr:
                    params['lr'] = curr_lr
                    params['num_iters'] = curr_num_iter
                    params['gamma'] = curr_gamma
                    ans.set_settings(fn_name=func, x0=x_0, **params)
                    err = ans.get_min_errs()
                    #print("X Error: ", err[0], "Loss Error: ", err[1])
                    if np.linalg.norm(err) < best_error:
                        #print(curr_lr, curr_num_iter)
                        best_error = np.linalg.norm(err)
                        final_params['lr'] = curr_lr
                        final_params['num_iters'] = curr_num_iter
                        final_params['gamma'] = curr_gamma
        return final_params
    elif params['method'] == 'adagrad':
        # adagrad Opti
        # Params lr, num_iter and eps
        final_params['method'] = 'adagrad'
        for curr_lr in lr_arr:
            for curr_num_iter in num_iter_arr:
                for curr_eps in eps_arr:
                    params['lr'] = curr_lr
                    params['num_iters'] = curr_num_iter
                    params['eps'] = curr_eps
                    ans.set_settings(fn_name=func, x0=x_0, **params)
                    err = ans.get_min_errs()
                    #print("X Error: ", err[0], "Loss Error: ", err[1])
                    if np.linalg.norm(err) < best_error:
                        #print(curr_lr, curr_num_iter)
                        best_error = np.linalg.norm(err)
                        final_params['lr'] = curr_lr
                        final_params['num_iters'] = curr_num_iter
```

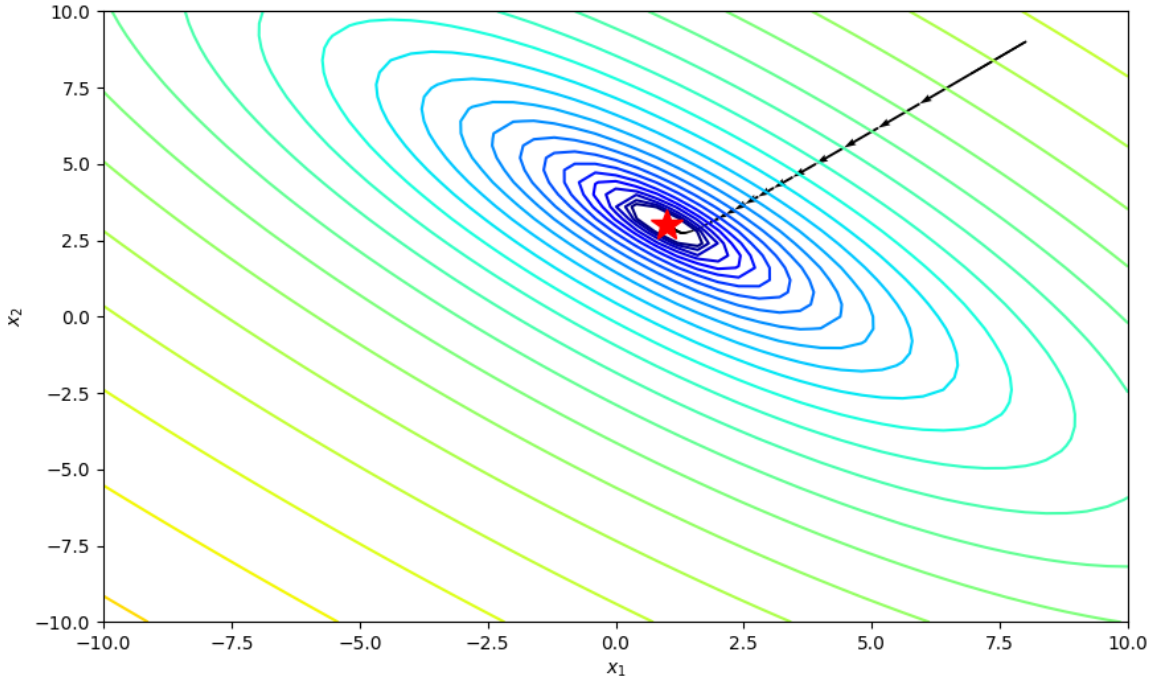
```
                        final_params['eps'] = curr_eps
        return final_params
    elif params['method'] == 'adam':
        # Adam Optimization
        # Params lr, num_iter, beta1, beta2= and eps
        final_params['method'] = 'adam'
        for curr_lr in lr_arr:
            for curr_num_iter in num_iter_arr:
                for curr_eps in eps_arr:
                    params['lr'] = curr_lr
                    params['num_iters'] = curr_num_iter
                    params['eps'] = curr_eps
                    ans.set_settings(fn_name=func, x0=x_0, **params)
                    err = ans.get_min_errs()
                    #print("X Error: ", err[0], "Loss Error: ", err[1])
                    if np.linalg.norm(err) < best_error:
                        #print(curr_lr, curr_num_iter)
                        best_error = np.linalg.norm(err)
                        final_params['lr'] = curr_lr
                        final_params['num_iters'] = curr_num_iter
                        final_params['eps'] = curr_eps
        return final_params

    else:
        print('Error: Use Implemented Optimization Method')
```

Exploratory Analysis

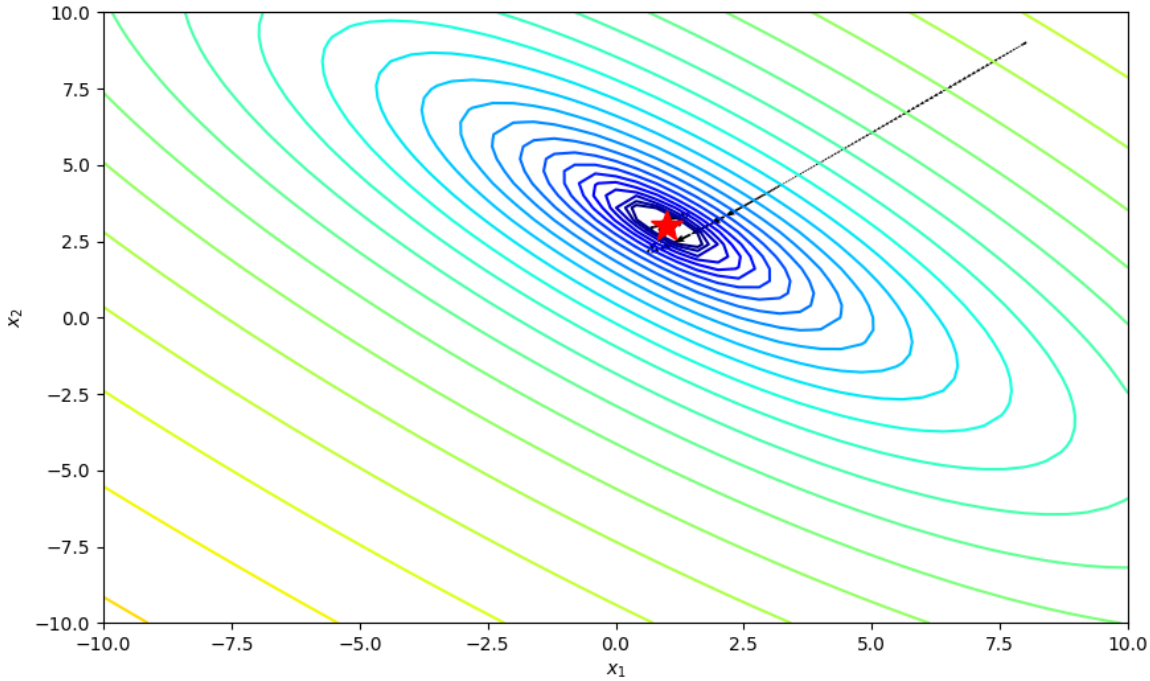
Booth function

In [43]:
```
for curr_func in func_arr:
    params = dict(
        method=curr_func,
    )
    tic = time.perf_counter()
    final_params_booth = cv_calc('booth', np.array([8, 9]), params)
    toc = time.perf_counter()
    print(f"Optimization Time: {toc - tic:0.4f} seconds")
    print(final_params_booth)
    ans.set_settings(fn_name='booth', x0=np.array([8, 9]), **final_params_booth)
    err = ans.get_min_errs()
    print("X Error: ", err[0], "Loss Error: ", err[1])
    ans.path2d()
```

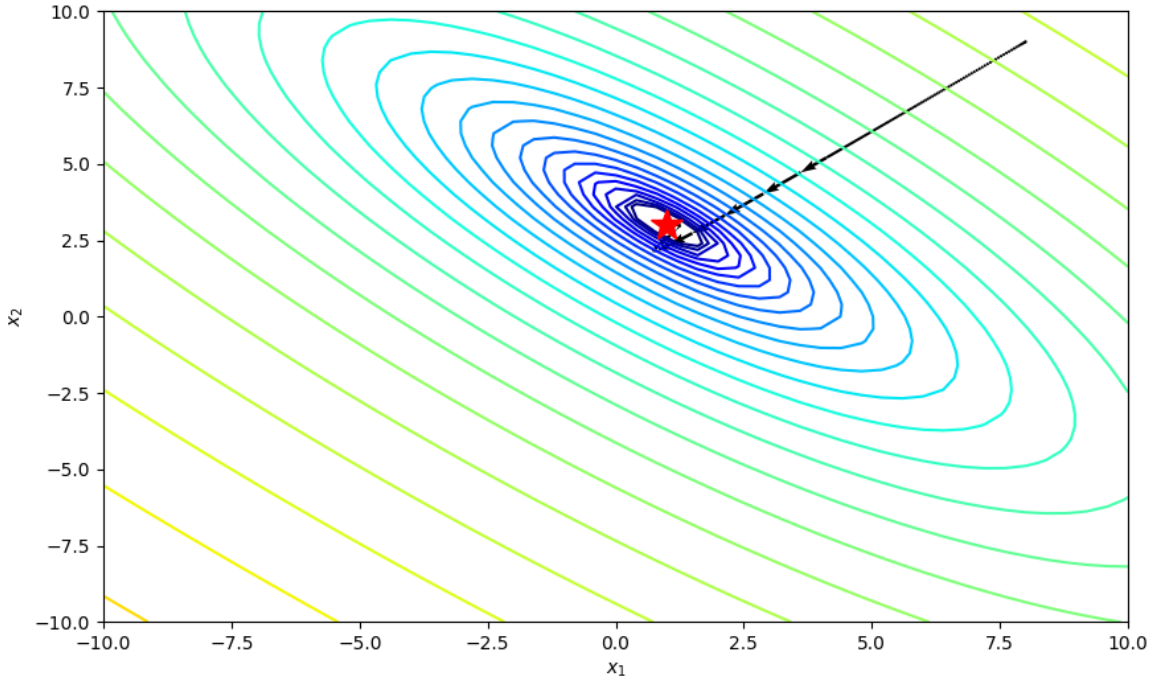
```
Optimization Time: 114.9928 seconds
{'method': 'gd', 'lr': <function lr_3 at 0x000001D54E8536D0>, 'num_iters': 1000}
X Error:  9.485749680535094e-16 Loss Error:  3.1554436208840472e-30
```



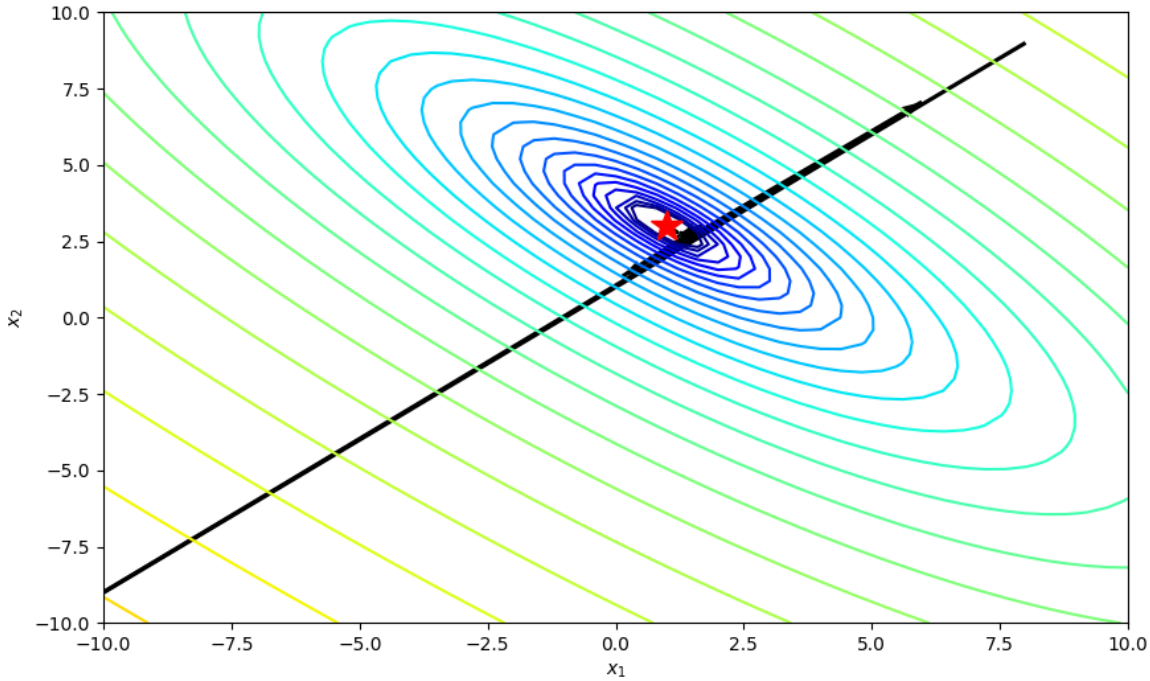

```
Optimization Time: 1163.3962 seconds
{'method': 'momentum', 'lr': <function lr_3 at 0x000001D54E8536D0>, 'num_iters': 1
000, 'gamma': 0.9}
X Error:  0.0 Loss Error:  0.0
```



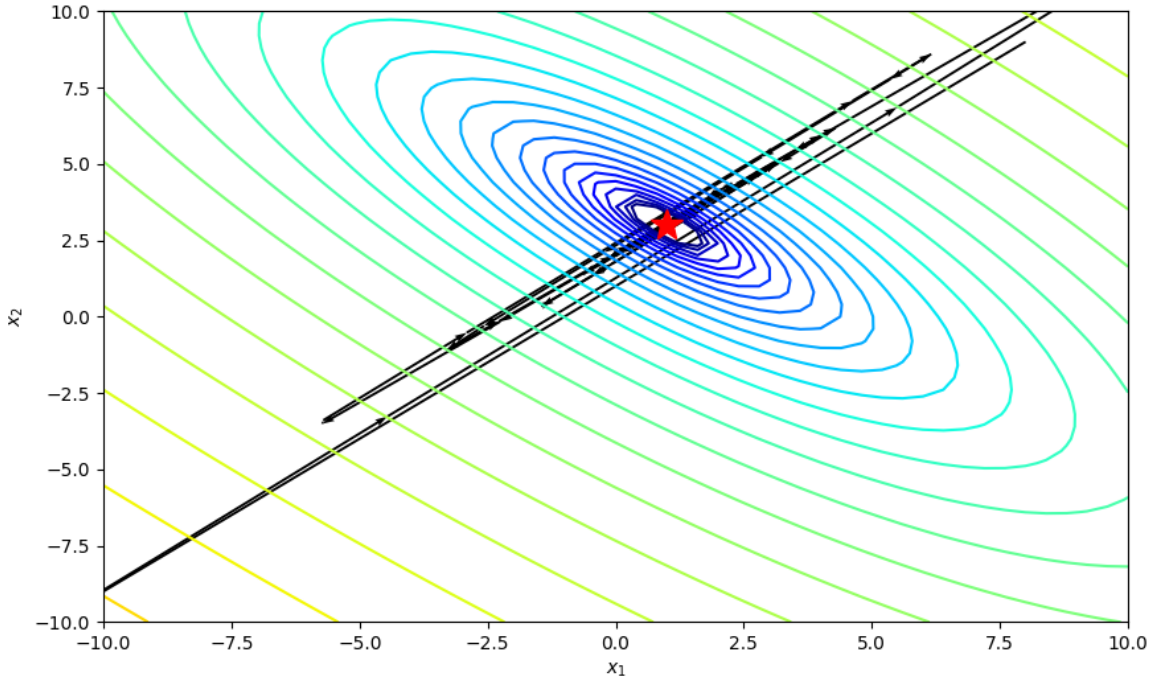

```
Optimization Time: 1332.6111 seconds
{'method': 'nag', 'lr': <function lr_3 at 0x000001D54E8536D0>, 'num_iters': 500,
'gamma': 0.8}
X Error:  0.0 Loss Error:  0.0
```




```
Optimization Time: 1093.4214 seconds
{'method': 'adagrad', 'lr': <function lr_1 at 0x000001D5583E8C10>, 'num_iters': 30
0, 'eps': 0.1}
X Error:  1.4217791915866692e-15 Loss Error:  3.944304526105059e-30
```



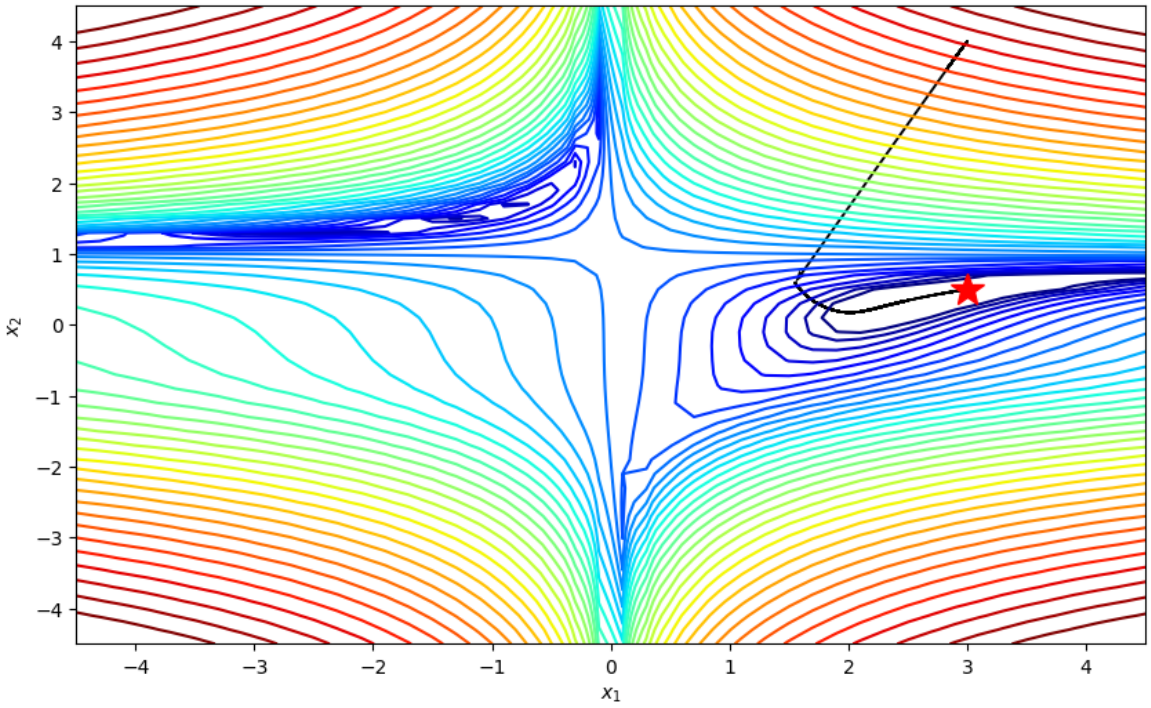

```
Optimization Time: 556.6826 seconds
{'method': 'adam', 'lr': <function lr_1 at 0x000001D5583E8C10>, 'num_iters': 1000,
'eps': 1}
X Error:  0.0 Loss Error:  0.0
```



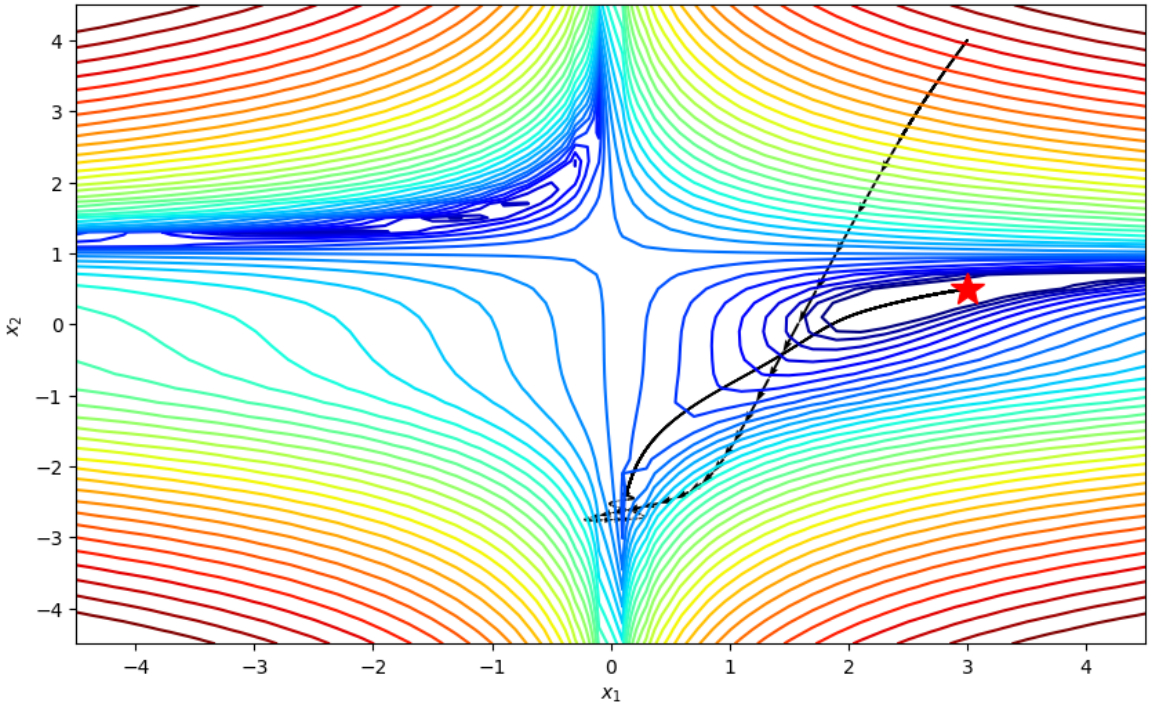
Beale function

In [44]:
```
for curr_func in func_arr:
    params = dict(
        method=curr_func,
    )
    tic = time.perf_counter()
    final_params_beale = cv_calc('beale', np.array([3, 4]), params)
    toc = time.perf_counter()
    print(f"Optimization Time: {toc - tic:0.4f} seconds")
    print(final_params_beale)
    ans.set_settings(fn_name='beale', x0=np.array([3, 4]), **final_params_beale)
    err = ans.get_min_errs()
    print("X Error: ", err[0], "Loss Error: ", err[1])
    ans.path2d()
```

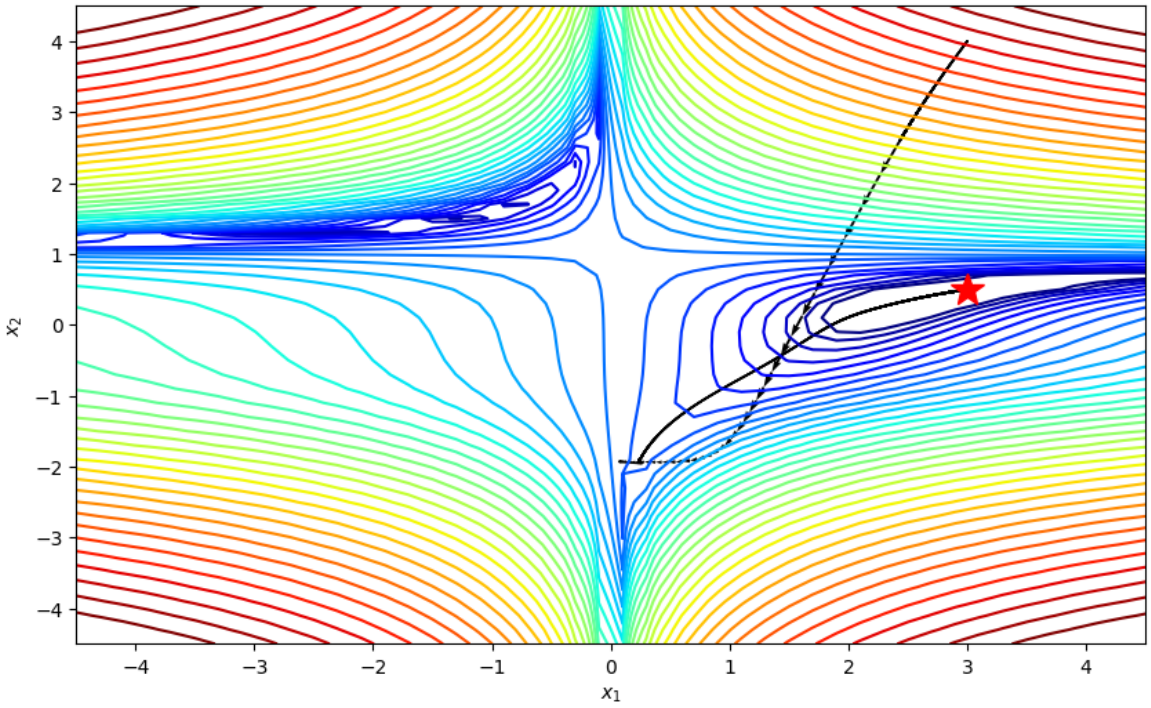
```
Optimization Time: 130.2733 seconds
{'method': 'gd', 'lr': <function lr_4 at 0x000001D54E853250>, 'num_iters': 50000}
X Error:  7.596075835309691e-14 Loss Error:  8.671676760158413e-28
```



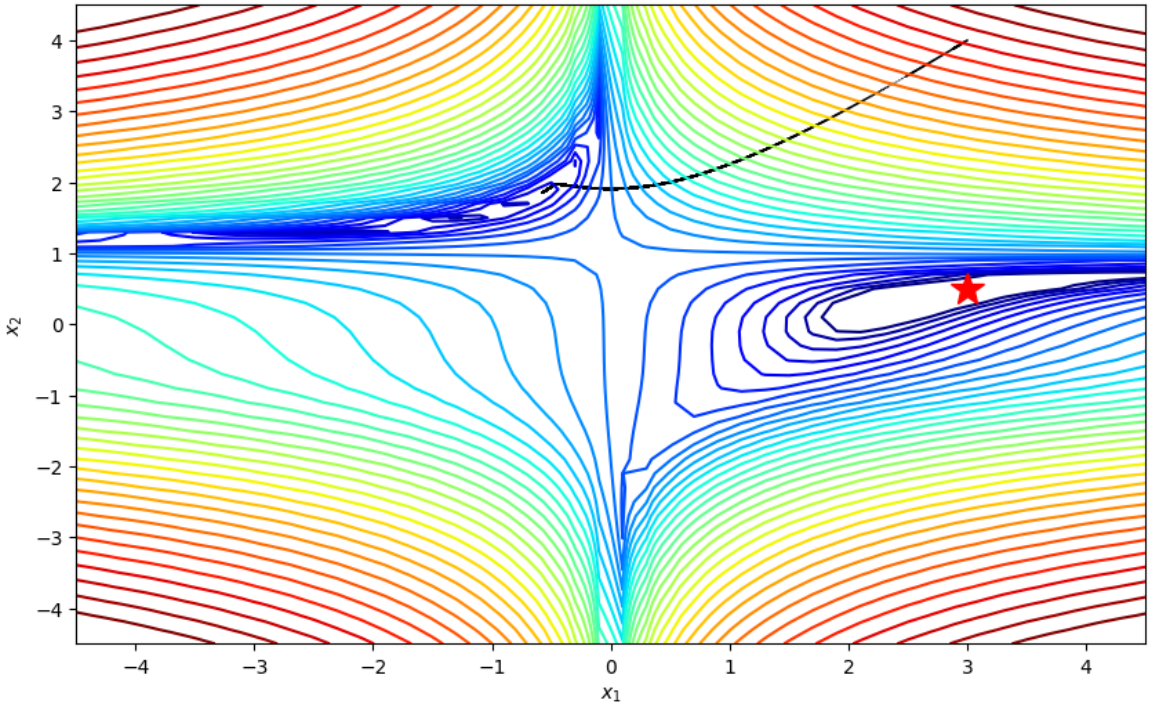

```
Optimization Time: 1457.5136 seconds
{'method': 'momentum', 'lr': <function lr_4 at 0x000001D54E853250>, 'num_iters': 1
5000, 'gamma': 0.9}
X Error:  3.233018248352212e-15 Loss Error:  1.8643001861668443e-30
```



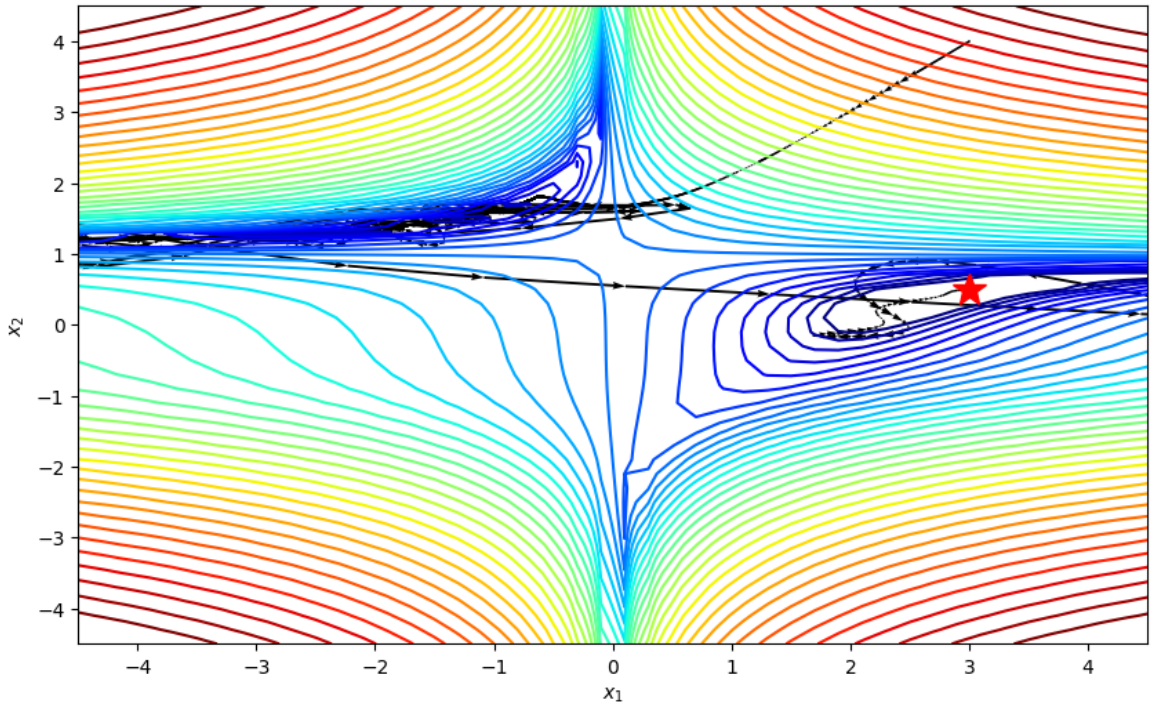

```
Optimization Time: 1534.4367 seconds
{'method': 'nag', 'lr': <function lr_4 at 0x000001D54E853250>, 'num_iters': 15000,
'gamma': 0.9}
X Error:  0.0 Loss Error:  0.0
```




```
Optimization Time: 1673.0172 seconds
{'method': 'adagrad', 'lr': <function lr_2 at 0x000001D54E8503A0>, 'num_iters': 10
0000, 'eps': 1}
X Error:  2.590231365999282 Loss Error:  1.960655375119053
```



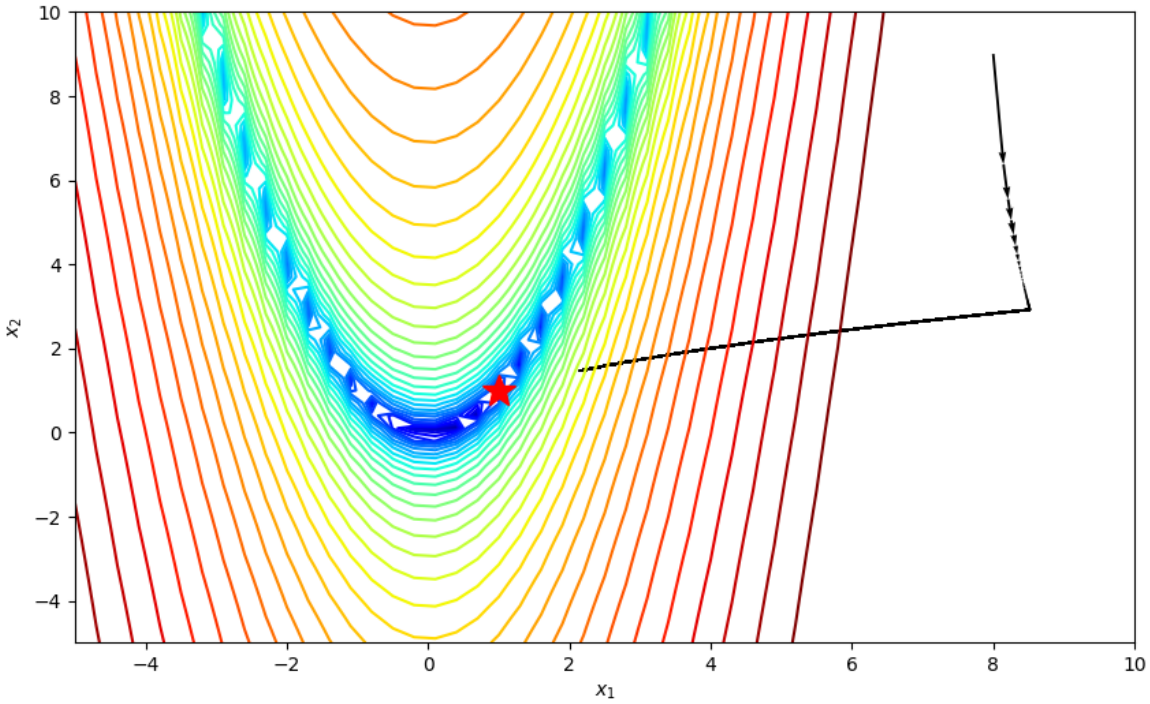

```
Optimization Time: 1287.1821 seconds
{'method': 'adam', 'lr': <function lr_2 at 0x000001D54E8503A0>, 'num_iters': 5000
0, 'eps': 1e-08}
X Error:  0.0 Loss Error:  0.0
```



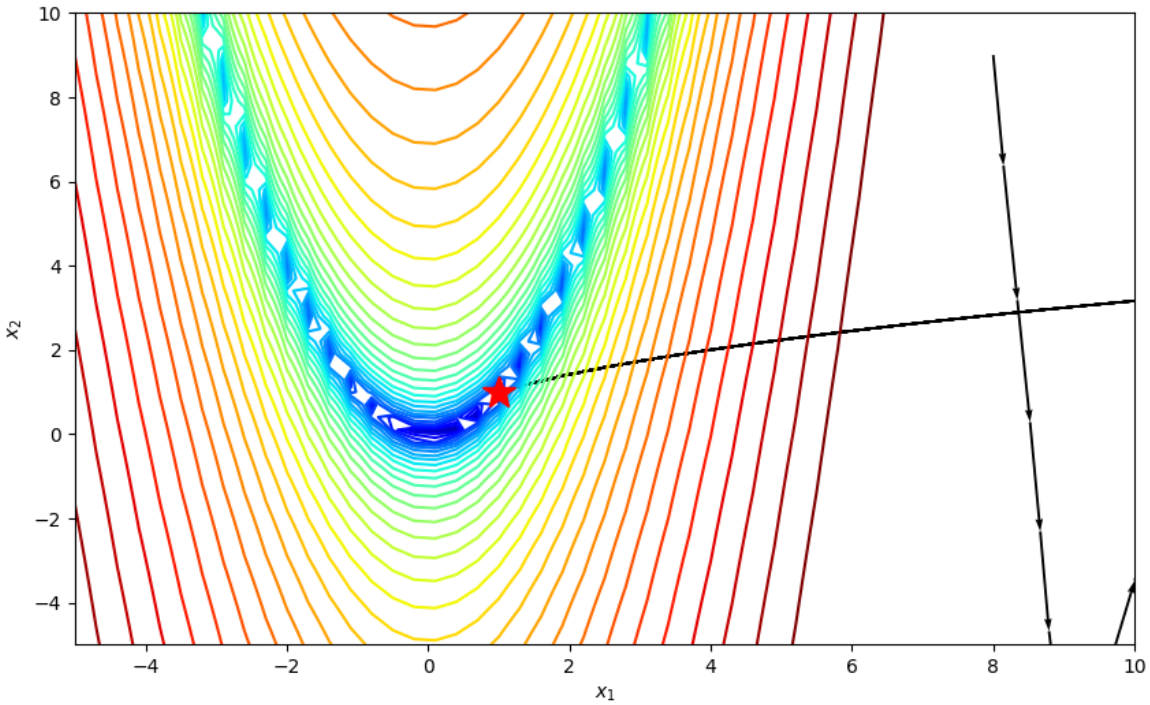
Rosenbrock function

In [45]:
```
for curr_func in func_arr:
    params = dict(
        method=curr_func,
    )
    tic = time.perf_counter()
    final_params_Rosenbrock = cv_calc('rosen2d', np.array([8, 9]), params)
    toc = time.perf_counter()
    print(f"Optimization Time: {toc - tic:0.4f} seconds")
    print(final_params_Rosenbrock)
    ans.set_settings(fn_name='rosen2d', x0=np.array([8, 9]), **final_params_Rosenbr
    err = ans.get_min_errs()
    print("X Error: ", err[0], "Loss Error: ", err[1])
    ans.path2d()
```

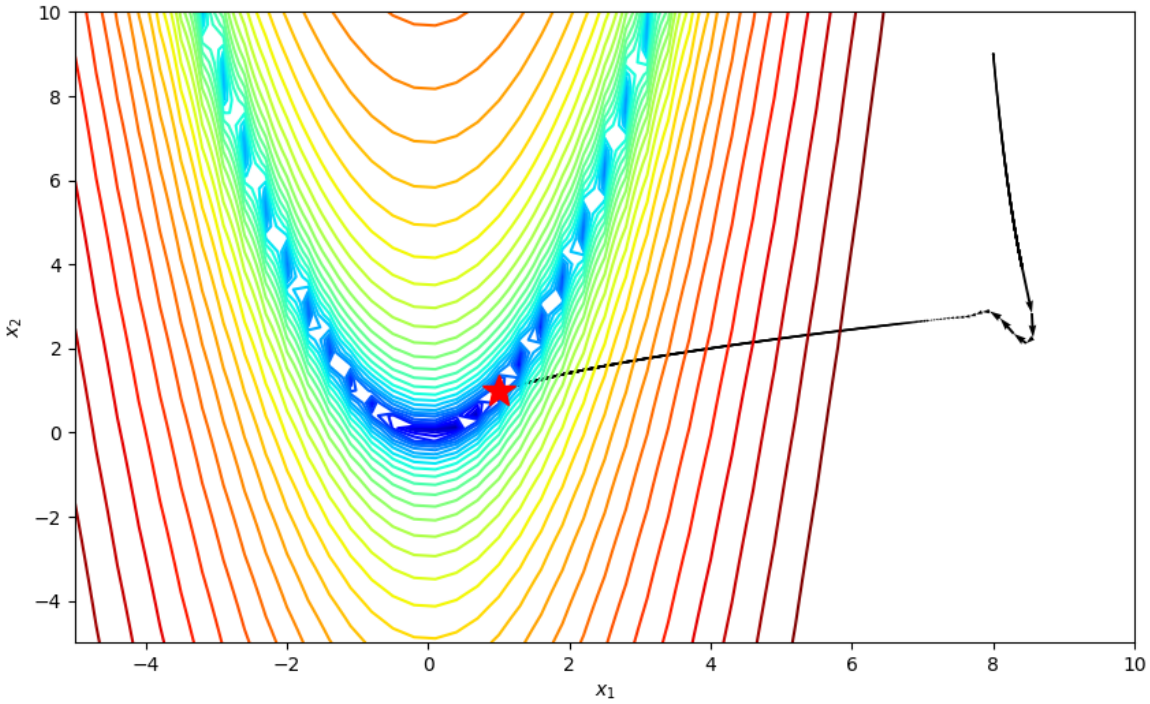
```
Optimization Time: 146.1141 seconds
{'method': 'gd', 'lr': <function lr_5 at 0x000001D54E8505E0>, 'num_iters': 100000}
X Error:  1.2451018872188626 Loss Error:  1006.77584067163
```



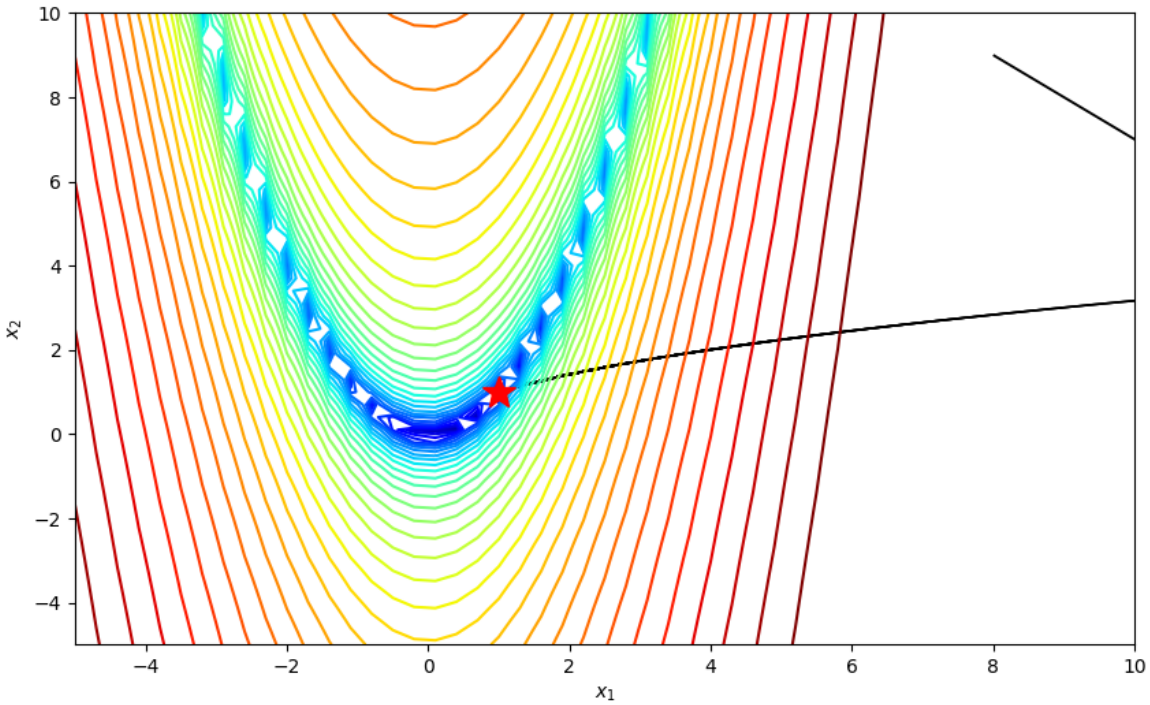

```
Optimization Time: 1646.7622 seconds
{'method': 'momentum', 'lr': <function lr_5 at 0x000001D54E8505E0>, 'num_iters': 1
00000, 'gamma': 0.9}
X Error:  7.963943453409772e-07 Loss Error:  1.1442724528376227e-10
```



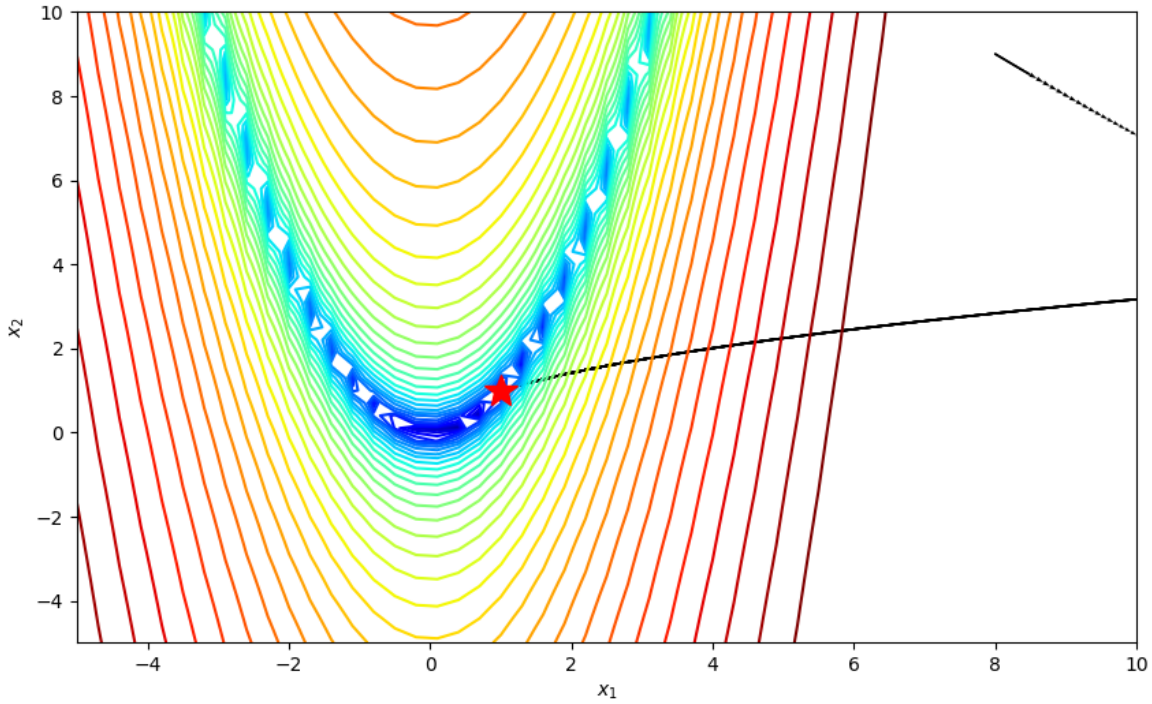

```
Optimization Time: 1707.7073 seconds
{'method': 'nag', 'lr': <function lr_4 at 0x000001D54E853250>, 'num_iters': 10000,
'gamma': 0.9}
X Error:  2.081267627276514e-06 Loss Error:  7.815014034522178e-10
```




```
Optimization Time: 1642.4683 seconds
{'method': 'adagrad', 'lr': <function lr_1 at 0x000001D5583E8C10>, 'num_iters': 10
0000, 'eps': 1}
X Error:  2.723652860591828e-12 Loss Error:  1.329403869456687e-21
```



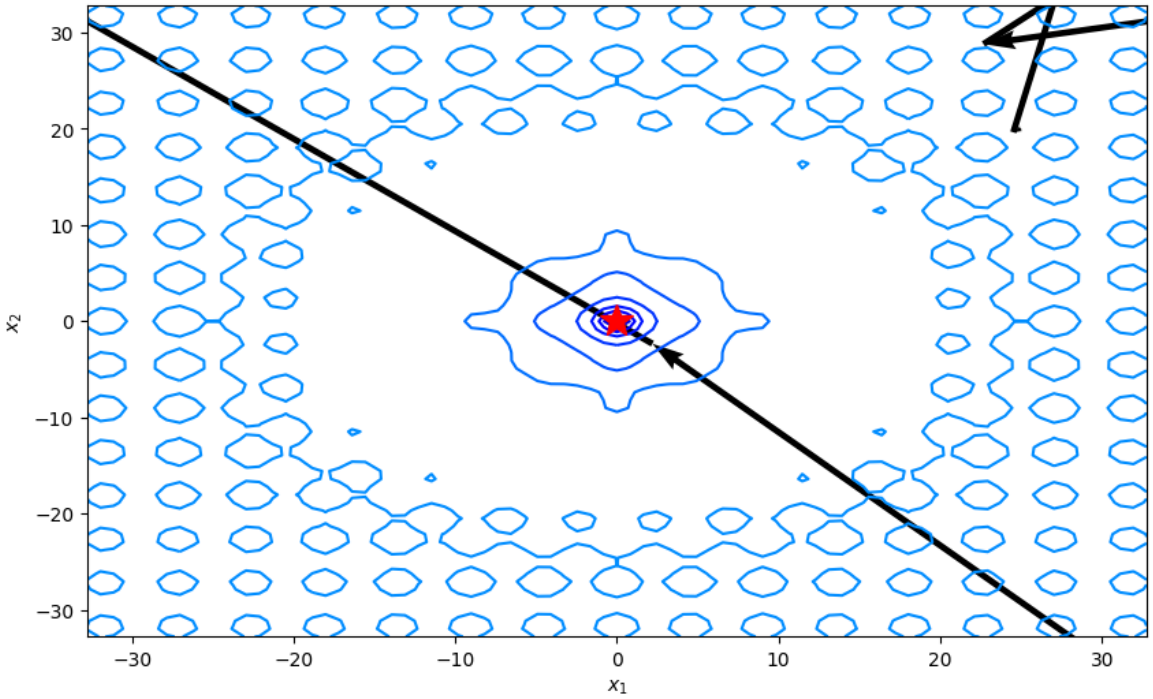

```
Optimization Time: 573.0838 seconds
{'method': 'adam', 'lr': <function lr_2 at 0x000001D54E8503A0>, 'num_iters': 1500
0, 'eps': 0.01}
X Error:  9.835347909592552e-15 Loss Error:  1.6093797846446743e-26
```



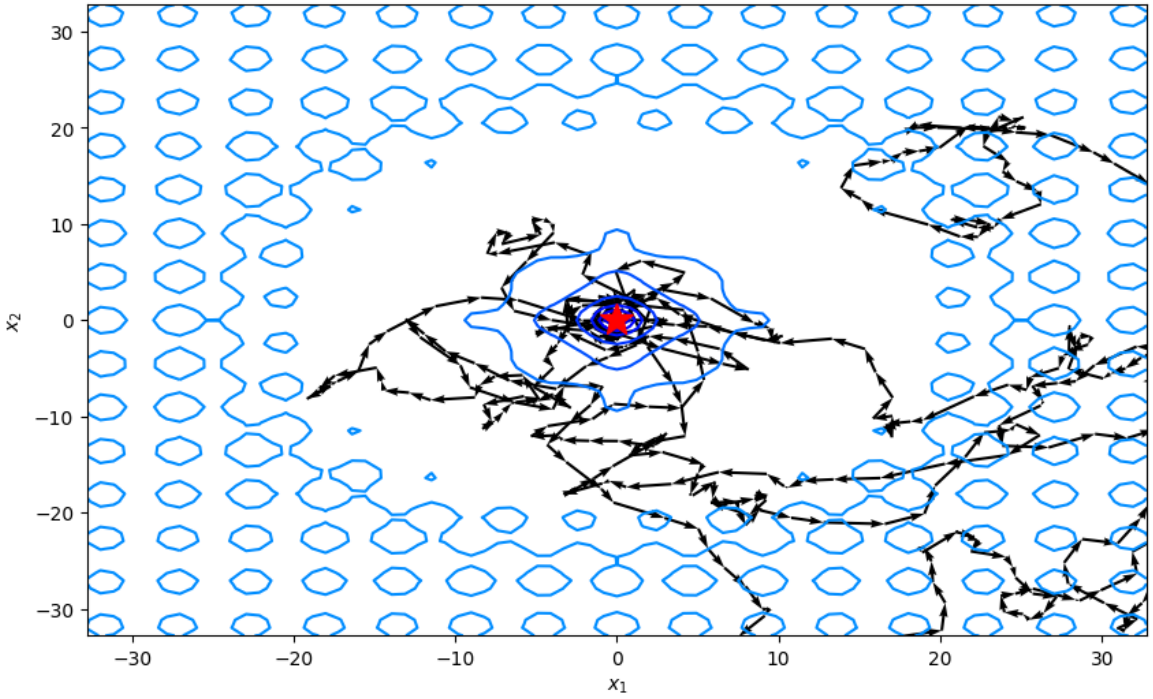
Ackley function

In [46]:
```
for curr_func in func_arr:
    params = dict(
        method=curr_func,
    )
    tic = time.perf_counter()
    final_params_ackley = cv_calc('ackley2d', np.array([25, 20]), params)
    toc = time.perf_counter()
    print(f"Optimization Time: {toc - tic:0.4f} seconds")
    print(final_params_ackley)
    ans.set_settings(fn_name='ackley2d', x0=np.array([25, 20]), **final_params_ackl
    err = ans.get_min_errs()
    print("X Error: ", err[0], "Loss Error: ", err[1])
    ans.path2d()
```

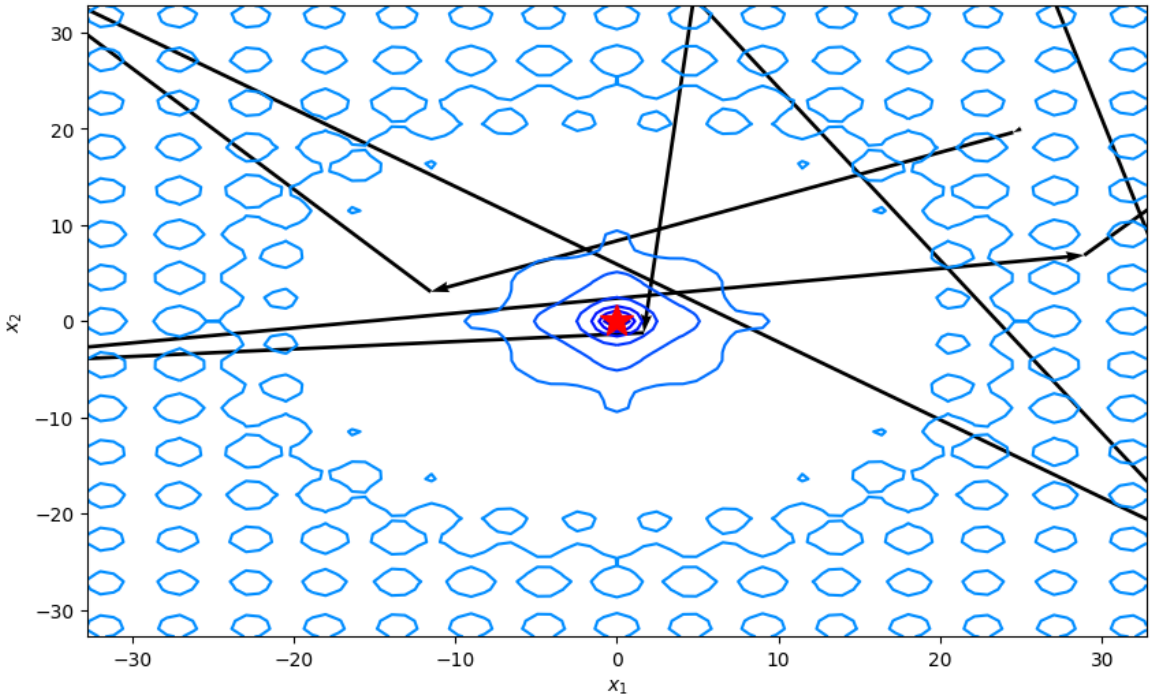
```
Optimization Time: 40.2279 seconds
{'method': 'gd', 'lr': <function lr_1 at 0x000001D5583E8C10>, 'num_iters': 100}
X Error:  3.2058428492217965 Loss Error:  9.094669523293208
```



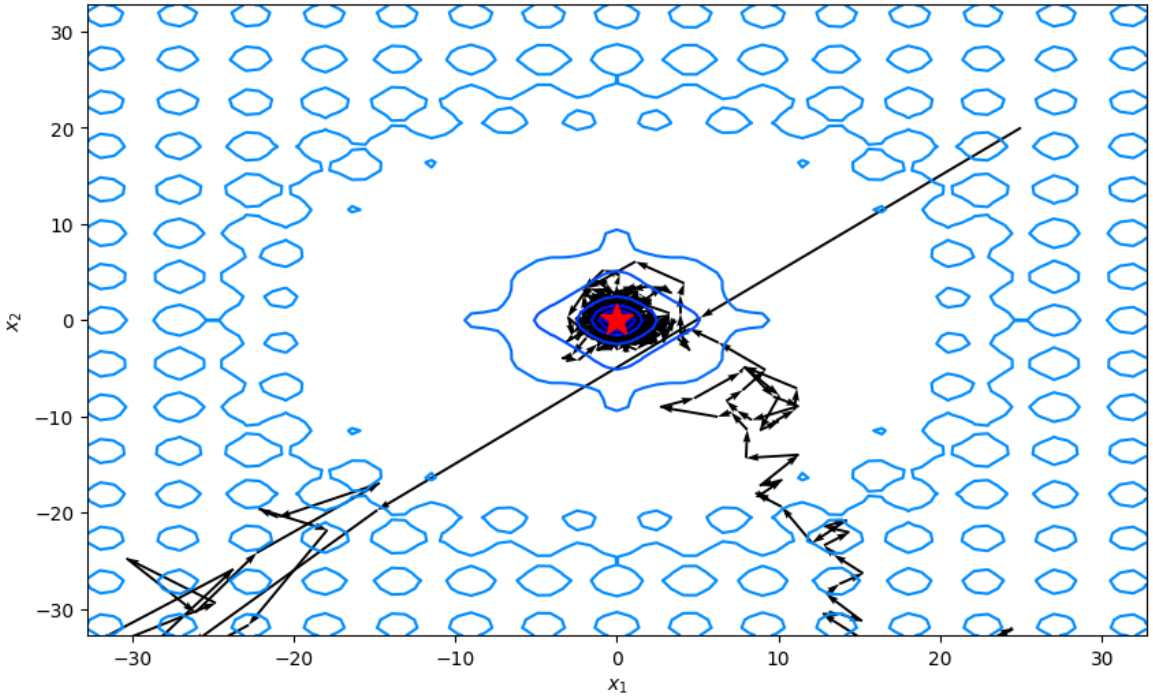

```
Optimization Time: 628.7404 seconds
{'method': 'momentum', 'lr': <function lr_2 at 0x000001D54E8503A0>, 'num_iters': 5
00, 'gamma': 0.5}
X Error:  0.13200026422887762 Loss Error:  0.7879952161785724
```



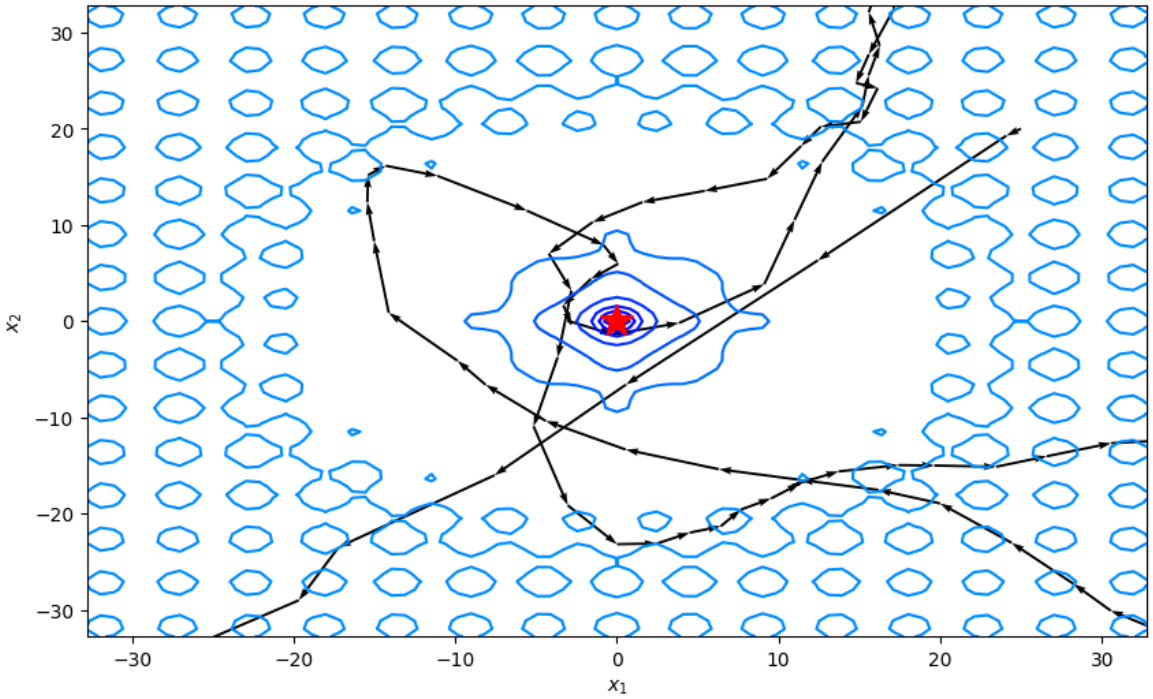

```
Optimization Time: 591.0608 seconds
{'method': 'nag', 'lr': <function lr_1 at 0x000001D5583E8C10>, 'num_iters': 300,
'gamma': 0.5}
X Error:  2.051782408494783 Loss Error:  6.875711043886893
```




```
Optimization Time: 1117.3351 seconds
{'method': 'adagrad', 'lr': <function lr_1 at 0x000001D5583E8C10>, 'num_iters': 10
000, 'eps': 1e-05}
X Error:  0.0007990017208636694 Loss Error:  0.002276917704651993
```




```
Optimization Time: 819.8973 seconds
{'method': 'adam', 'lr': <function lr_1 at 0x000001D5583E8C10>, 'num_iters': 1000
0, 'eps': 0.5}
X Error:  1.394502909646037 Loss Error:  5.624196359704598
```

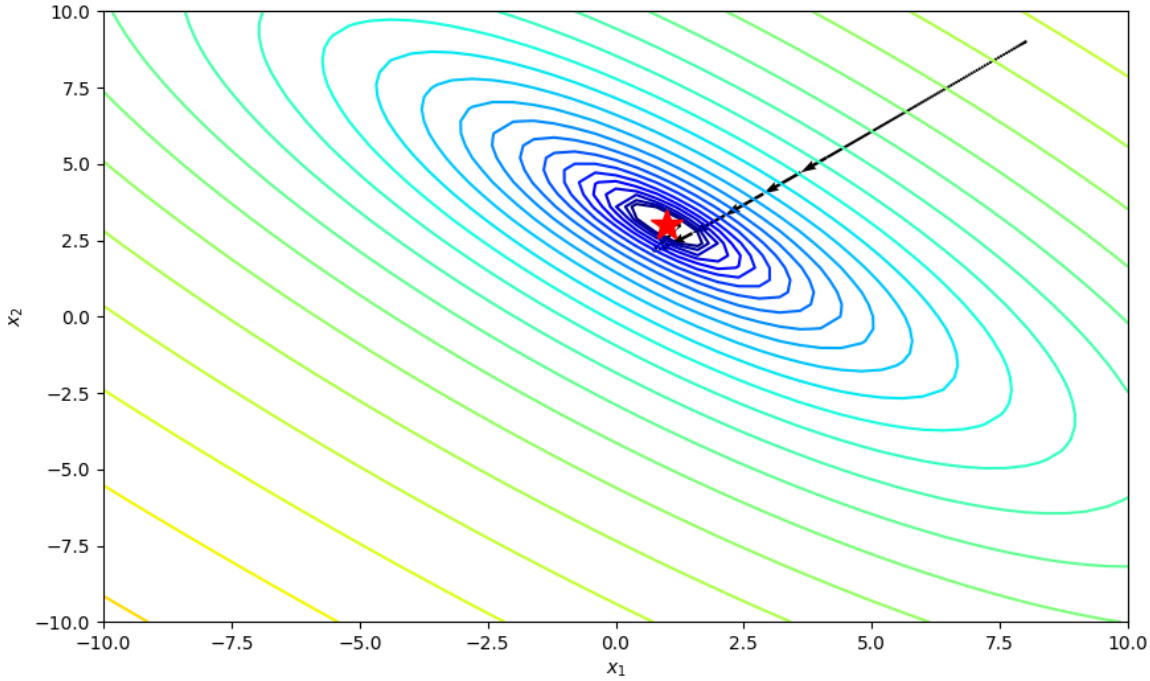


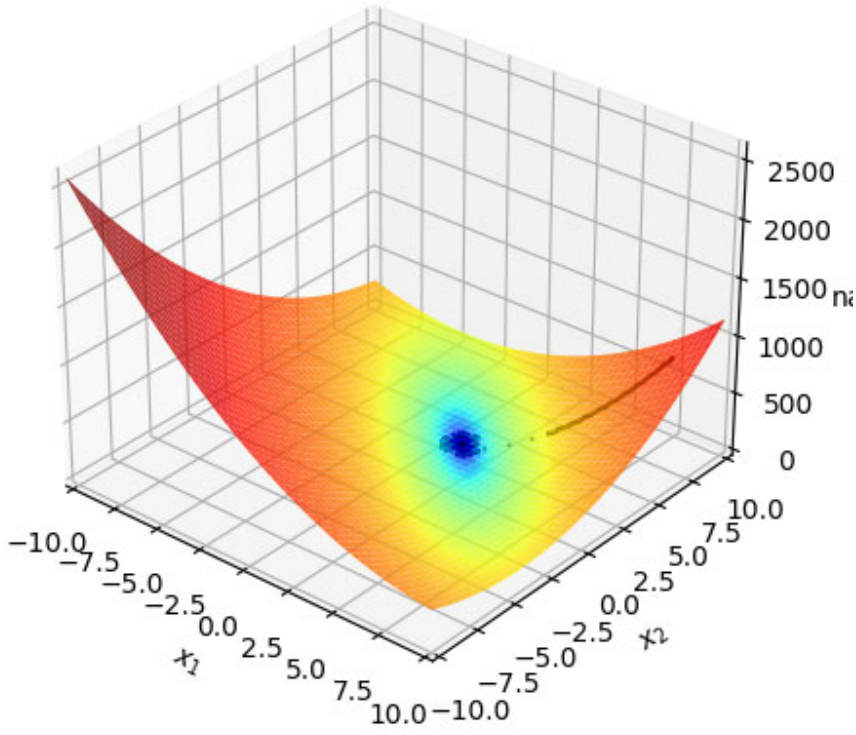
Final Models

Booth Function

In [51]:
```
# Final model due to reasonable simplicity, optimization time and directness of pat
params = dict(
    method='nag',
    lr=lr_3,
    num_iters=500,
    gamma=0.8,
)
tic = time.perf_counter()
ans.set_settings(fn_name='booth', x0=np.array([8, 9]), **params)
toc = time.perf_counter()
print(f"Optimization Time: {toc - tic:0.4f} seconds")
print(params)
err = ans.get_min_errs()
print("X Error: ", err[0], "Loss Error: ", err[1])
ans.path2d()
ans.path3d()
#ans.video3d("booth_final")
```

```
Optimization Time: 0.0364 seconds
{'method': 'nag', 'lr': <function lr_3 at 0x000001D54E8536D0>, 'num_iters': 500,
'gamma': 0.8}
X Error:  0.0 Loss Error:  0.0
```



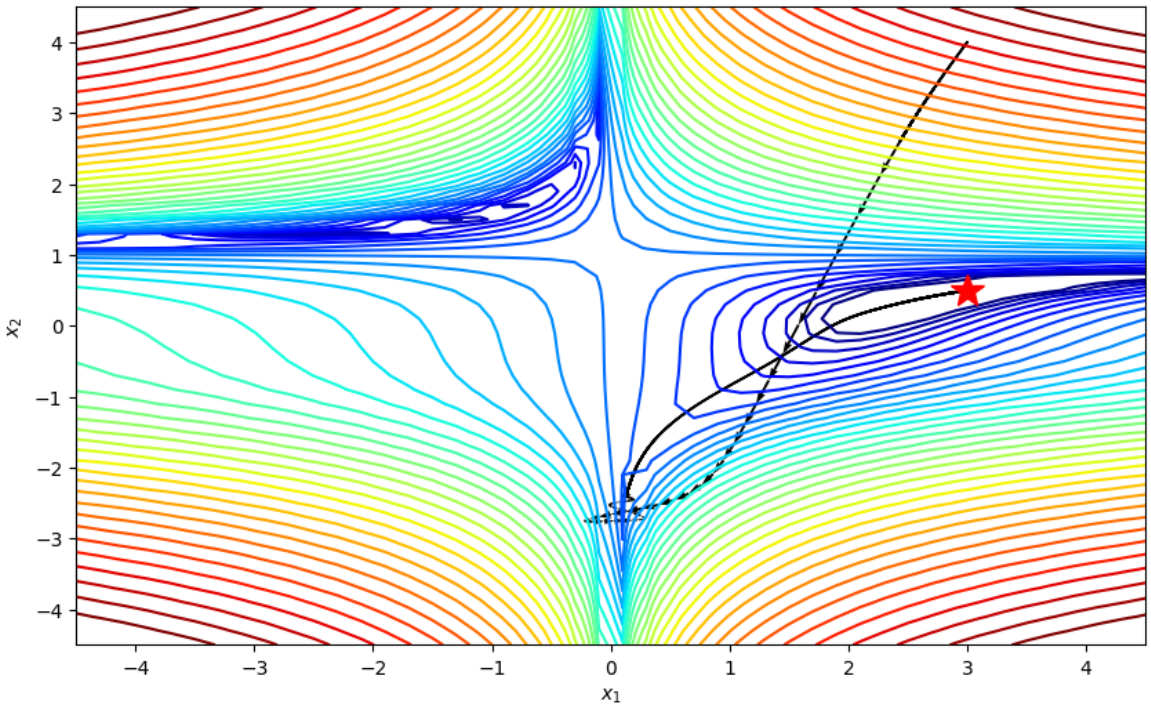


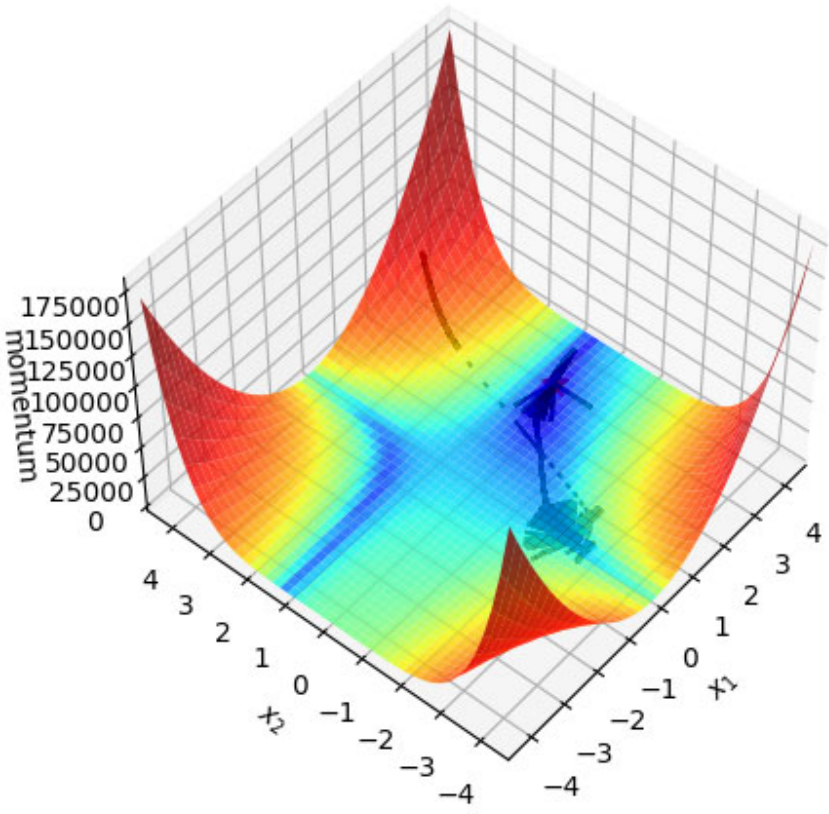
Beale Function

In [61]:
```
params = dict(
    method='momentum',
    lr=lr_4,
    num_iters=15000,
    gamma=0.9,
)
```

```
tic = time.perf_counter()
ans.set_settings(fn_name='beale', x0=np.array([3, 4]), **params)
toc = time.perf_counter()
print(f"Optimization Time: {toc - tic:0.4f} seconds")
print(params)
err = ans.get_min_errs()
print("X Error: ", err[0], "Loss Error: ", err[1])
ans.path2d()
ans.path3d()
```

```
Optimization Time: 0.7654 seconds
{'method': 'momentum', 'lr': <function lr_4 at 0x000001D54E853250>, 'num_iters': 1
5000, 'gamma': 0.9}
X Error:  3.233018248352212e-15 Loss Error:  1.8643001861668443e-30
```

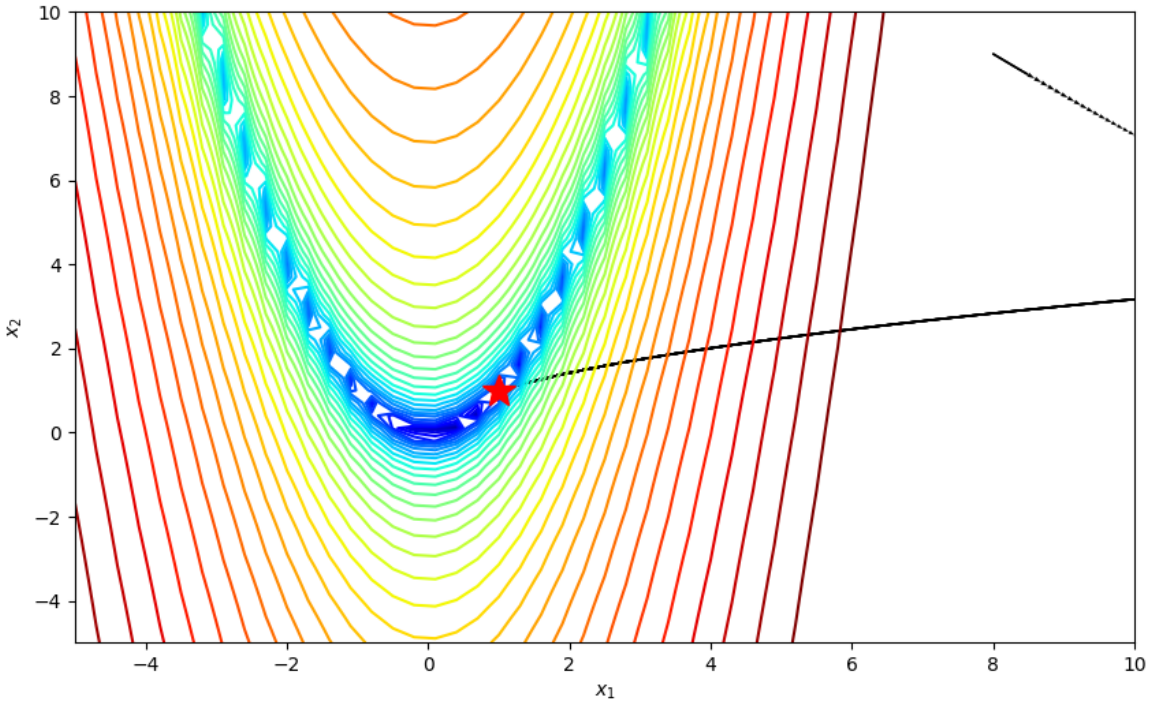


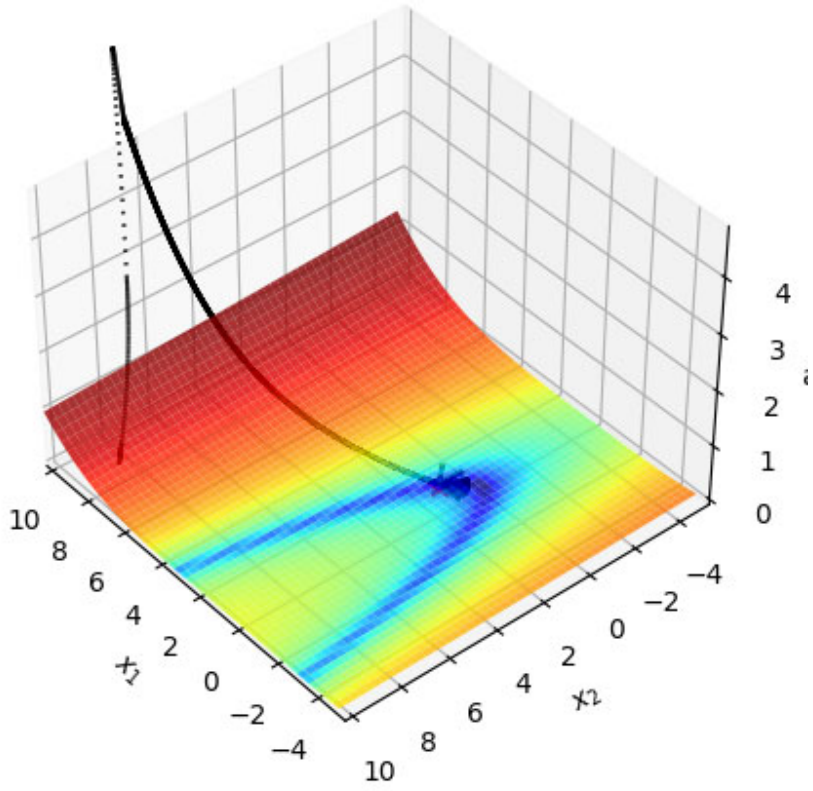


Rosenbrock Function

In [63]:
```
params = dict(
    method='adam',
    lr=lr_2,
    num_iters=15000,
    eps=0.01,
)
tic = time.perf_counter()
ans.set_settings(fn_name='rosen2d', x0=np.array([8, 9]), **params)
toc = time.perf_counter()
print(f"Optimization Time: {toc - tic:0.4f} seconds")
print(params)
err = ans.get_min_errs()
print("X Error: ", err[0], "Loss Error: ", err[1])
ans.path2d()
ans.path3d()
```

```
Optimization Time: 1.0645 seconds
{'method': 'adam', 'lr': <function lr_2 at 0x000001D54E8503A0>, 'num_iters': 1500
0, 'eps': 0.01}
X Error:  9.835347909592552e-15 Loss Error:  1.6093797846446743e-26
```





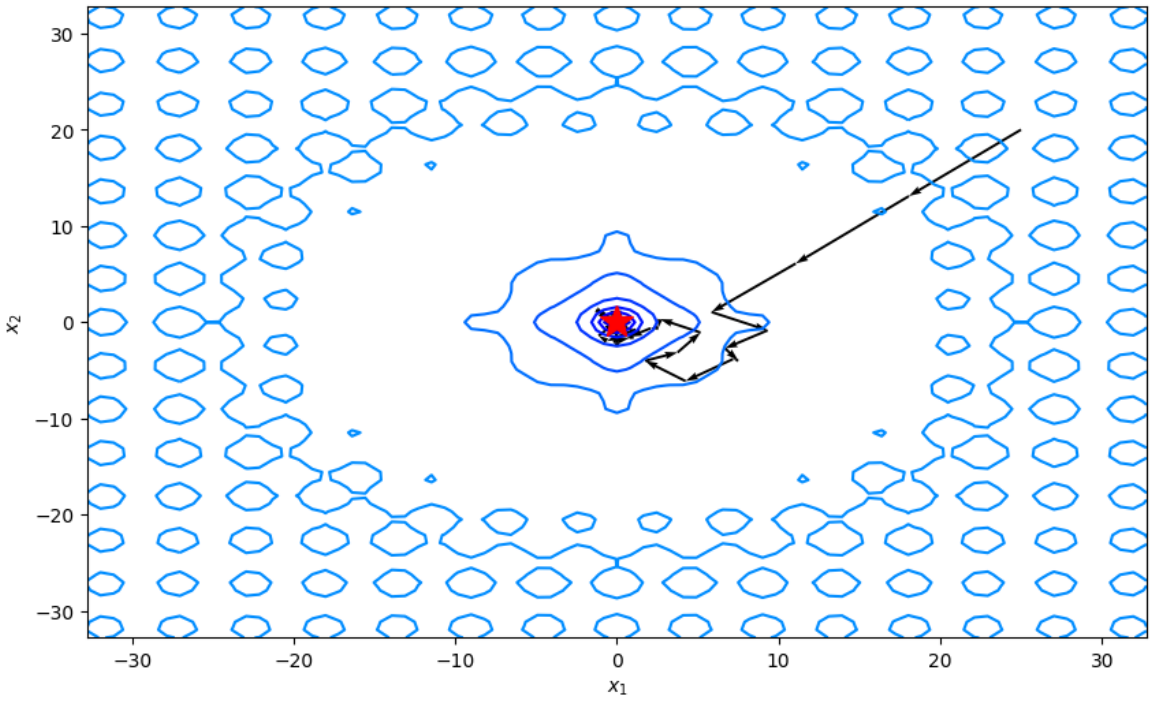
Ackley Function

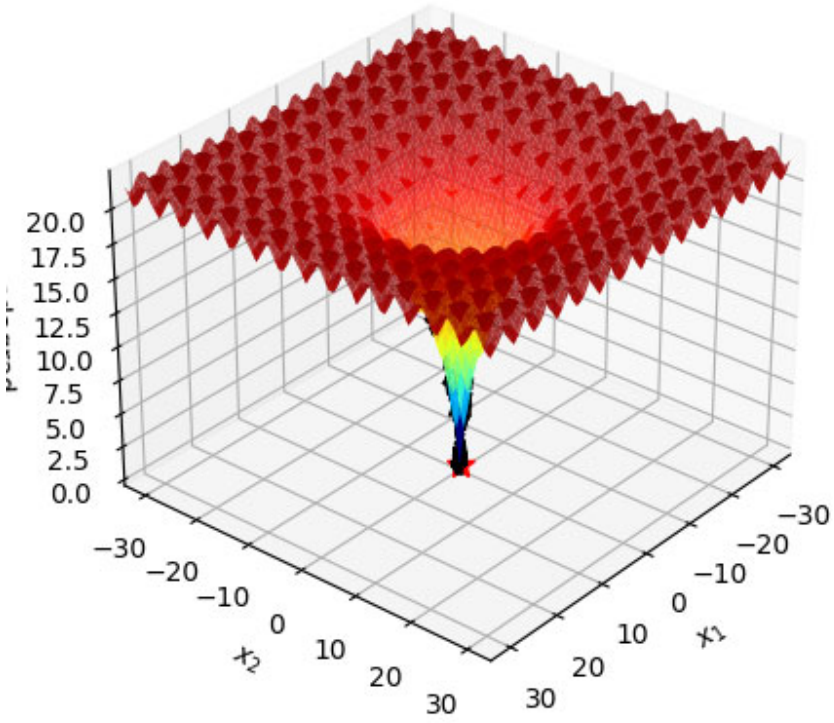
In [265…

```
def lr_7(t):
    if t < 3:
        return 7
    elif t < 20:
        return 5
```

```
    elif t < 500:
        return 3
    elif t < 1000:
        return 2
    else:
        return 1
params = dict(
    method='adagrad',
    lr=lr_7,
    num_iters=50000,
)
tic = time.perf_counter()
ans.set_settings(fn_name='ackley2d', x0=np.array([25, 20]), **params)
toc = time.perf_counter()
print(f"Optimization Time: {toc - tic:0.4f} seconds")
print(params)
err = ans.get_min_errs()
print("X Error: ", err[0], "Loss Error: ", err[1])
ans.path2d()
ans.path3d()
```

```
Optimization Time: 6.8696 seconds
{'method': 'adagrad', 'lr': <function lr_7 at 0x000001D558381360>, 'num_iters': 50
000}
X Error:  6.356646485595074e-06 Loss Error:  1.7980387315930102e-05
```





Submission: Project Report

Create a report explaining the avenues you explored after the implementation phase of the project, the process you used to select the function values for each combination of functions and initial points, and what you found or learned. You are encouraged to include explanatory images or links to videos generated in the process, showcasing the process you describe or any interesting or unusual phenomena you observe over the course of your investigation!

Please append a PDF print-out of this Jupyter notebook, including any code for extensions you choose to do (see below), to your project report.

Rubric

- A C level project would successfully implement 4 out of 5 of the optimization algorithms and 3 out of 4 of the benchmark functions (and their derivatives).
- A B level project would successfully implement all the optimization algorithms and benchmark functions (and their derivatives), as well as complete a project report.
- An A level project would do all of the above, plus one or more extensions (we suggest some below, but you can pick anything of sufficient interest and complexity).

Note that exceptional projects that go above and beyond may receive extra credit beyond at our discretion.

Extensions

Some extensions to this project you could do and include in your project report are:

- Correctly implement more classes of optimization algorithms -- do at least one algorithm for credit, and benchmark its performance. Examples include proximal gradient descent, and gradient descent with line search.
- A "literature review" of optimization algorithms. What this means is to take a research paper or two from the study of optimization algorithms and summarize it (or them) in a way that your peers can understand. A good literature review should contain: an introduction to the proposed method, a formal description of the proposed method, and discussion about why the method is useful or needed. You can use any papers from this field that you please, but here are a few in case you are stuck: "Nesterov's Accelerated Gradient and Momentum as approximations to Regularised Update Descent", "A Universal Catalyst for First-Order Optimization". You could also discuss one of the papers your TA Tarun wrote: "A Potential Reduction Inspired Algorithm for Exact Max Flow in Almost O(m^(4/3)) Time", which has interesting optimization ideas inside an algorithmic framework.
- Describe how the algorithms discussed compare to higher-order algorithms, such as Newton's method and especially interior point methods. For interior point methods, implementation will be hard due to numerical stability issues, so you can do a more theoretical review; but other comparisons should involve implementations of the comparison methods.
- Quantitatively compare how the dimensionality of the problem can affect the algorithms. How do the different algorithms fare in higher dimensions? What benchmarks or visualizations can you use for higher-dimensional optimization?

In [ ]: