1 First time of reach for a sum of a Markov sum

1.1 Defining the problem

Let $(X_n)_{n \geq 0}$ be a markov chain such as :

- $\forall n \geq 0 \quad X_n \in \{-1, 1\}$

- $\forall i \geq 0 \quad \mathbb{P}(X_{i+1} = 1 | X_i = 1) = \mathbb{P}(X_{i+1} = -1 | X_i = -1) = p = 1 - q$

Let $(S_n)_{n \geq 0}$ be a sequence such as :

$$S_n = \sum_{k=1}^{n} X_k$$

Let T be the first time when S_n reaches -1, i.e :

$$T = \inf\{n \geq 0, S_n = -1\}$$

Let us define the probability-generating function of T such as :

$$g_s^x(z) = \mathbb{E}(z^T | S_0 = s, X_0 = x) \quad \forall x \in \{-1, 1\} \ , \ s \in \mathbb{N} \ , \ z \in [0, 1]$$

1.2 The probability-generating function dynamic

Given the transition matrix of $(X_n)_{n \geq 0}$, we can infer that for all $x \in \{-1, 1\}$, $s \in \mathbb{N}$ and $z \in [0, 1]$ [1] :

$$\begin{cases} g_s^1 = z \left(p \, g_{s+1}^1 + q \, g_{s-1}^{-1} \right) & (1) \\ g_s^{-1} = z \left(q \, g_{s+1}^1 + p \, g_{s-1}^{-1} \right) & (2) \end{cases}$$

Thus :

$$g_{s+1}^1 = \frac{1}{zp} \, g_s^1 - \frac{q}{p} \, g_{s-1}^{-1} \tag{3}$$

$$g_{s+1}^1 = \frac{1}{zq} \, g_s^{-1} - \frac{p}{q} \, g_{s-1}^{-1} \tag{4}$$

Which implies that :

$$g_s^{-1} = \frac{q}{p} \, g_s^1 + \frac{z \, (p^2 - q^2)}{p} \, g_{s-1}^{-1} \tag{5}$$

Leading to :

$$g_{s+1}^{-1} = \left(\frac{1}{z \, p} + z \left(2 - \frac{1}{p} \right) \right) g_s^{-1} - g_{s-1}^{-1} \tag{6}$$

Consequently :

$$G_{s+1} = \begin{pmatrix} g_{s+1}^1 \\ g_{s+1}^{-1} \\ g_s^1 \\ g_s^{-1} \end{pmatrix} = \begin{pmatrix} \frac{1}{zp} & 0 & 0 & -\frac{q}{p} \\ 0 & \frac{1}{zp} + z(2 - \frac{1}{p}) & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} g_s^1 \\ g_s^{-1} \\ g_{s-1}^1 \\ g_{s-1}^{-1} \end{pmatrix}$$

Noting :

$$A(z) = \begin{pmatrix} \frac{1}{zp} & 0 & 0 & -\frac{q}{p} \\ 0 & \frac{1}{zp} + z(2 - \frac{1}{p}) & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

and by imposing a Dirichlet boundary condition we finally get the following second order system :

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[1] For clarity, we omit to note the dependence of G in z

$$\begin{cases} G_{s+1}(z) = A(z).G_s(z) & (7) \\ g_{-1}^1 = g_{-1}^{-1} = 0 & (8) \\ \lim_{s \to +\infty} g_s^{-1} = \lim_{s \to +\infty} g_s^1 = 0 & (9) \end{cases}$$

Autrement:

$$\begin{cases} G_{s+1}(z) = A(z).G_s(z) & (10) \\ G_0(z) = \begin{pmatrix} x(z) & y(z) & 1 & 1 \end{pmatrix}^T & (11) \\ \lim_{s \to +\infty} A(z)^s.G_0(z) = 0 & (12) \end{cases}$$

$A(z)$ is diagnolisable, therefore there exist a matrix $Q(z)$, and $\lambda_i \in R, i = 1..4$ so that:

$$A(z) = Q(z)diag(\lambda_i, i = 1..4)Q(z)^{-1}$$

A quick analysis using Maple shows that exactly two eigen values have theire absolute value greater than 1, hence we can write that: $|_i| < 1$ for $i = 1, 2$ and $|\lambda_i| \geq 1$ for $i = 3, 4$.
Let $U(z) = Q(z)^-1G_0(z)$, dirichlet condition in infinity are equivalent to

$$\lambda_i^s.(U(z))_i \overset{\infty}{\to} 0$$

ie

$$\begin{cases} U(z)_3 = 0 & (13) \\ U(z)_4 = 0 & (14) \end{cases}$$

these 2x2 system gives $x(z)$ and $y(z)$