1 First time of reach for a sum of a Markov sum

1.1 Defining the problem

Let $(X_n)_{n\geq 0}$ be a markov chain such as :

- $\forall\, n \geq 0 \quad X_n \in \{-1, 1\}$
- $\forall\, i \geq 0 \quad \mathbb{P}(X_{i+1} = 1 | X_i = 1) = \mathbb{P}(X_{i+1} = -1 | X_i = -1) = p = 1 - q$

Let $(S_n)_{n\geq 0}$ be a sequence such as :

$$S_n = \sum_{k=1}^{n} X_k$$

Let T be the first time when S_n reaches -1, i.e :

$$T = \inf\{n \geq 0, S_n = -1\}$$

Let us define the probability-generating function of T such as :

$$g_s^x(z) = \mathbb{E}(z^T | S_0 = s, X_0 = x) \quad \forall\, x \in \{-1, 1\}\ ,\ s \in \mathbb{N}\ ,\ z \in [0, 1]$$

1.2 The probability-generating function dynamic

Given the transition matrix of $(X_n)_{n\geq 0}$, we can infer that for all $x \in \{-1, 1\}$, $s \in \mathbb{N}$ and $z \in [0, 1]$ [1] :

$$\begin{cases} g_s^1 = z\,(p\, g_{s+1}^1 + q\, g_{s-1}^{-1}) & (1) \\ g_s^{-1} = z\,(q\, g_{s+1}^1 + p\, g_{s-1}^{-1}) & (2) \end{cases}$$

Thus :

$$g_{s+1}^1 = \frac{1}{zp}\, g_s^1 - \frac{q}{p}\, g_{s-1}^{-1} \tag{3}$$

$$g_{s+1}^1 = \frac{1}{zq}\, g_s^{-1} - \frac{p}{q}\, g_{s-1}^{-1} \tag{4}$$

Which implies that :

$$g_s^{-1} = \frac{q}{p}\, g_s^1 + \frac{z\,(p^2 - q^2)}{p}\, g_{s-1}^{-1} \tag{5}$$

Leading to :

$$g_{s+1}^{-1} = (\frac{1}{z\,p} + z\,(2 - \frac{1}{p}))\, g_s^{-1} - g_{s-1}^{-1} \tag{6}$$

Consequently :

$$G_{s+1} = \begin{pmatrix} g_{s+1}^1 \\ g_{s+1}^{-1} \\ g_s^1 \\ g_s^{-1} \end{pmatrix} = \begin{pmatrix} \frac{1}{zp} & 0 & 0 & -\frac{q}{p} \\ 0 & \frac{1}{zp} + z(2 - \frac{1}{p}) & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} g_s^1 \\ g_s^{-1} \\ g_{s-1}^1 \\ g_{s-1}^{-1} \end{pmatrix}$$

Noting :

$$A(z) = \begin{pmatrix} \frac{1}{zp} & 0 & 0 & -\frac{q}{p} \\ 0 & \frac{1}{zp} + z(2 - \frac{1}{p}) & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

and by imposing a Dirichlet boundary condition we finally get the following second order system :

[1] For clarity, we omit to note the dependence of G in z

$$\begin{cases} G_{s+1}(z) = A(z).G_s(z) & (7) \\ g^1_{-1} = g^{-1}_{-1} = 0 & (8) \\ \lim\limits_{s\to+\infty} g^{-1}_s = \lim\limits_{s\to+\infty} g^1_s = 0 & (9) \end{cases}$$

Autrement:

$$\begin{cases} G_{s+1}(z) = A(z).G_s(z) & (10) \\ G_0(z) = \begin{pmatrix} x(z) & y(z) & 1 & 1 \end{pmatrix}^T & (11) \\ \lim\limits_{s\to+\infty} A(z)^s.G_0(z) = 0 & (12) \end{cases}$$

$A(z)$is diagnolisable, therefore there exist a matrix $Q(z)$, and $\lambda_i \in \mathbb{R}, i = 1..4$ so that:

$$A(z) = Q(z)diag(\lambda_i, i = 1..4)Q(z)^{-1}$$

A quick analysis using Maple shows that exactly two eigen values have theire absolute value greater than 1, hence we can write that: $|\lambda_i| < 1$ for $i = 1, 2$ and $|\lambda_i| \geq 1$ for $i = 3, 4$.

Let $U(z) = Q(z)^-1G_0(z)$, dirichlet condition in infinity are equivalent to

$$\lambda_i^s.(U(z))_i \overset{\infty}{\rightarrow} 0$$

ie

$$\begin{cases} U(z)_3 = 0 & (13) \\ U(z)_4 = 0 & (14) \end{cases}$$

these 2x2 system gives $x(z)$ and $y(z)$

1.3 First results

$$\mathbb{E}[T|S_0 = 0, x_0 = i] = g_0^{i\prime}(1) = +\infty$$

$$\mathbb{P}(T = k|S_0 = 0, x_0 = i) = \frac{(g_0^i)^k(0)}{k!}$$