Contents

1 The Real Numbers

1.1 Supremum and Infimum

Definition 1.1 (**Upper and Lower Bound**)**.** A set $A \subseteq \mathbb{R}$ is *bounded above* if there exists a number $b \in \mathbb{R}$ such that $a \leq b$ for all $a \in A$. Similarly, the set A is *bounded below* if there exists a lower bound $l \in \mathbb{R}$ satisfying $l \leq a$ for every $a \in A$.

In other words, we say that a set A is bounded above if there exists a real number b such that b is greater than any element in the set A. Likewise, we say that A is bounded below if there exists a real number l such that l is lesser than any element in A.

Note that A can be an interval such as $[1, 3]$ or a set such as $B = \{x \in \mathbb{R} : x \leq 2\}$. In the latter case, we see that A is not bounded above since it goes on to infinity; however, it is bounded below since B only contain elements of $\mathbb{R}$ greater than or equal to 2.

Consider the interval [1,2]. We see that the interval has infinitely many upper bounds and lower bounds. Thus, it is not so interesting to study, say the upper bound 500 or the lower bound 1000. What's interesting is the least upper bound and the greatest lower bound of the interval of a given set since both of them are unique.

Definition 1.2 (**Supremum (Least Upper Bound)**)**.** A real number u is the *least upper bound* for a set $A \subseteq \mathbb{R}$ if it meets the following two criteria

1. u is an upper bound for A;
2. if b is any upper bound for A, then $u \leq b$.

This means that the least upper bound is an upper bound of a set A such that it is lesser than or equal to any upper bound in the set. The infimum of the greatest lower bound is defined similarly.

Definition 1.3 (**Infimum (Greatest Lower Bound)**)**.** A real number l is the*greatest lower bound* for a set $A \subseteq \mathbb{R}$ if it meets the following two criteria:

1. l is a lower bound for A;
2. if p is any lower bound for A, then $p \leq l$.

That is, p is the greatest upper bound for A if p is a lower bound of A and given any lower bound for A, p is greater than any of them.

Definition 1.4 (Unbounded). description

Remark 1.1. A set, which is bounded above, has a unique supremum. Similarly, if a set is bounded below, then it has a unique infimum.

Proof. □

Theorem 1.1 (Characterization of $\sup S$ and $\inf S$)**.**

1. An upper bound u of a nonempty set S in $\mathbb{R}$ is the supremum of S if and only if for every $\epsilon > 0$, there exists an $s_\epsilon \in S$ such that $u - \epsilon < s_\epsilon$.

2. A lower bound l of a nonempty subset S in $\mathbb{R}$ is the infimum of S if and only if for every $\epsilon > 0$, there exists a $t_\epsilon \in S$ such that $l + \epsilon > t_\epsilon$.

The above theorem answers the question: "How do we know that an upper or lower bound is the supremum or infimum in a given set S?" Intuitively, this theorem tells us that if $u = \sup S$, then any number less than u is not an upper bound. Similarly, if $I = \inf S$, then any number greater than l is not a lower bound.

Proof. We first prove the forward direction.
($\implies$). We want to show that given $u = \sup S$, for every $\epsilon > 0$ there exists an $s_\epsilon \in S$ such that $u - \epsilon < s_\epsilon$. Let $u = \sup S$. See that for every $\epsilon > 0$, $u - \epsilon$ is not an upper bound of S. This means that there is some $s_\epsilon \in S$ such that $s - \epsilon > s_\epsilon$ as desired.

($\impliedby$). Assume, for the sake of contradiction, that $u \neq \sup S$. Then there is a least upper bound v such that $v < u$ (note that u is still an upper bound). Let $\epsilon = u - v > 0$. Then there exists some $s_\epsilon \in S$ such that $u - \epsilon < s_\epsilon$. Thus, we have

$$u - \epsilon = u - u + v = v < \epsilon$$

which contradicts the claim that v is an upper bound. Therefore, it must be the case that $u = \sup S$. □

Notes.

1. We can just let $\epsilon = u - v$ since we are assuming that ϵ can be any real number greater than 0. This also works since $v < u$ which means that the difference between u and v is greater than or equal to 0.

2. "Where did we get $\epsilon = u - v$?" We got this by playing around the inequality. Since we can substitute any values greater than 0 to ϵ, we can play around with the given inequality.

Example 1.1. Consider $S_1 = [a, b]$, where $a, b \in \mathbb{R}$ with $a < b$. Then

$$\sup S_1 = b \text{ and } \inf S_1 = a.$$

Proof. We first show that $\sup S_1 = b$. By definition of S_1, we know that b is an upper bound of S_1. Let $\epsilon > 0$. We want to show that there exists some $s_e \in S_1$ such that $b - \epsilon < s_\epsilon$. Notice that we can let $s_\epsilon = b$ which is in S_1. Thus, we have

$$b - \epsilon < b \iff b - \epsilon < s_\epsilon$$

as desired. Therefore, b is the least upper bound of S_1.

We now show that a is the infimum of S_1. By definition of S_1, we can see that a is a lower bound of S_1. Let $\epsilon > 0$ We want to find $t_\epsilon \in S$ such that $a + \epsilon > t_\epsilon$. Similar, to the above situation, we can let $t_\epsilon = a$. Hence, we have

$$a + \epsilon > t_\epsilon \iff a + \epsilon > a$$

as desired. Therefore, the greatest lower bound of S_1 is b. □

Theorem 1.2 (Properties)**.** Let S be a nonempty subset of $\mathbb{R}$ and $a \in \mathbb{R}$. Define the sets

$$a + S = \{a + s : s \in S\}, \quad \text{and} \quad -S = \{s : s \in S\}$$

1. If S is bounded above, then $\sup(a + S) = a + \sup S$.
2. if S is bounded below, then $\inf(a + S) = a + \inf S$.
3. if S is bounded, then

$$\inf(-S) = -\sup S \text{ and } \sup(-S) = -\inf S$$

Proof. (1) Let $u = \sup S$. Then u is an upper bound of S. That is, for all $s \in S$, we have

$$s \leq u.$$

Adding both sides by $a \in \mathbb{R}$ gives us

$$a + s \leq a + u.$$

Thus, $a + u$ is an upper bound of $a + S$. Let $\epsilon > 0$. We want to find some $s_\epsilon \in S$ such that $a + u - \epsilon < s_\epsilon$. By the characterization of $\sup S$, we have

$$u - \epsilon < p_e$$

for some $p_e \in S$. Adding both sides of the inequality by $a \in \mathbb{R}$ yields

$$a + u - \epsilon < a + p_\epsilon.$$

Thus, we can let $s_e = a + p_\epsilon$. Hence, we have

$$a + u - \epsilon < s_e$$

as desired. Therefore, $\sup(a + S) = a + \sup S$.

(2) Let $l = \inf S$. Then l is a lower bound of S. That is, for all $s \in S$

$$l \leq s.$$

Adding both sides by $a \in \mathbb{R}$ gives us

$$a + l \leq a + s.$$

Thus, $a + l$ is a lower bound of $a + S$. Let $\epsilon > 0$. We want to find a $t_\epsilon \in S$ such that $a + l + \epsilon > t_\epsilon$. By the characterization of $\inf S$, we have

$$l + \epsilon > s_\epsilon$$

for some $s_\epsilon \in S$. Adding both sides by a gives us

$$a + l + \epsilon > s_\epsilon + a,$$

Hence, we can let $t_\epsilon = s_e + a$. Thus, we have

$$a + l + \epsilon > t_e$$

as desired. Therefore, $\inf(a + S) = a + \inf S$.

(3) Suppose S is bounded. Let $u = \sup S$. We want to show that $\inf(-S) = -u$. Since $u = \sup S$ then u is an upper bound. Thus, for all $s \in S$, we have $s \leq u$. See that

$$s \leq u \iff -s \geq -u$$

□

Definition 1.5 (Maximum and Minimum)**.** A real number s_{max} is a **maximum** of a set S if $s_{max} \geq s$, for any $s \in S$ and $s_{max} \in S$. Similarly, s_{min} is a **minimum** of a set S if $s_{min} \leq s$, for all $s \in S$ and $s_{min} \in S$.

Example 1.2. Consider the open interval $I = (0, 1)$. Then $\sup I = 1$ and $\inf I = 0$.

1.2 Completeness Property of $\mathbb{R}$

Definition 1.6 (Completeness Property of $\mathbb{R}$)**.** Every nonempty set of real numbers that has an upper bound also has a supremum (AXIOM).

Example 1.3. Consider the set

$$S = \{x \in \mathbb{R} : x \geq 0; x^2 \leq 2\}.$$

The completeness property of $\mathbb{R}$ proves the existence of $\sqrt{2}$. The set is nonempty ($0 \in S$) and it has an upper bound (e.g., 2, 3/2,). By the completeness property, since S is bounded above, the set S has a supremum b. In fact, this supremum b satisfies $b^2 = 2$. (As an exercise, prove it).

This prove that a number $b > 0$ exists such that $b^2 = 2$. We usually denote this number b by $\sqrt{2}$.

The theorem below is one of the consequences of the completeness property.

Theorem 1.3 (Archimedean Property)**.** For every $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $x < n$ (or equivalently, the set of natural numbers is not bounded above).

Proof. Let $x \in \mathbb{R}$. We prove by contradiction. Assume, for the sake of contradiction, that there is no $n \in \mathbb{N}$ such that $x < n$. This means that for all natural number n, $x \geq n$. This means that x is an upper bound of $\mathbb{N}$. Thus, by the completeness property, $\mathbb{N}$ has a supremum (least upper bound), say $u = \sup \mathbb{N}$. Note that $u - 1 < u$. By the characterization of $\sup \mathbb{N}$ (where $\epsilon = 1$), there exists $m \in \mathbb{N}$ such that

$$u - 1 < m.$$

Adding 1 to both sides gives us

$$u < m + 1.$$

Since m is a natural number then $m + 1$ is also a natural number. This is a contradiction since u is supposed to be an upper bound of $\mathbb{N}$. Therefore, for every real number x, there is some natural number n such that $x < n$. $\square$

Corollary 1.1. Let $y, z \in \mathbb{R}^+ = \{r \in \mathbb{R} : r > 0\}$. Then

1. $\exists n \in \mathbb{N}$ such that $z < ny$.
2. $\exists n \in \mathbb{N}$ such that $0 < \frac{1}{n} < y$ (labeled as: **Corollary 2**).
3. $\exists n \in \mathbb{N}$ such that $n - 1 \leq z < n$.

Proof. (1). Let $y, z \in \mathbb{R}^+$, such that $y > 0$ and $z > 0$. Suppose $x = \frac{z}{y}$. Since $y, z \in \mathbb{R}$ then $x > 0$. By the Archimedean property, there is some natural number n such that

$$\frac{z}{y} = x < n.$$

Multiplying both sides of the inequality by y gives us

$$z < ny.$$

as desired. (Note that the inequality sign will not change since $y > 0$). Therefore, there is some natural number n such that $z < ny$.

(2). From the previous result, we can let $z = 1$. Then there is some natural number n such that

$$1 < ny \implies \frac{1}{n} < y \implies 0 < \frac{1}{n} < y$$

since $n > 0$.

(3). Consider the set $S = \{m \in \mathbb{N} : z < m\} \subseteq \mathbb{N}$. By the Archimedean Property, S is nonempty. From the Well-Ordering Property of $\mathbb{N}$ (every nonempty subset of $\mathbb{N}$ has a least element), S has a least element, say n. This means that $n - 1 \notin S$. Thus, $z < n$. Since $n - 1 \notin S$, we have, $n - 1 \leq z$. Therefore, we have

$$n - 1 \leq z < n$$

as desired. $\square$

Theorem 1.4 (Density Theorem)**.** If $a, b \in \mathbb{R}$ such that $a < b$, then there exist $r \in \mathbb{Q}$ and $y' \in \mathbb{Q}'$ such that $a < r < b$ and $a < r' < b$.

Note that $\mathbb{Q}$ is the set of all rational numbers while $\mathbb{Q}'$ is the set of all irrational numbers.

Proof. (Rational). Let $a, b \in \mathbb{R}$ such that $a < b$. We want to show that there is some rational number r such that

$$a < r < b.$$

Notice that $b - a > 0$. Then by Corollary 2, there is some natural number n such that

$$0 < \frac{1}{n} < b - a.$$

Multiplying both sides by n, we have

$$1 < n(b-a) \iff 1 < nb - na. \tag{1}$$

Consider the interval (nb, na). The length of the interval is more than 1 which follows from inequality (1). Thus, there is some integer k such that $k \in (na, nb)$. Therefore, we have

$$na < k < nb \iff a < \frac{k}{n} < b \iff a < r < b$$

as desired (where $r = \frac{k}{n} \in \mathbb{Q}$).

(Irrational). We show that there is some irrational number r' such that $a < r' < b$. Let $r' = \sqrt{2}$. Consider the interval $(a\sqrt{2}, b\sqrt{2})$. From the first part of the proof, there is some rational number r such that

$$a\sqrt{2} < r < b\sqrt{2}. \tag{2}$$

Dividing both sides by 2 gives us

$$a < \frac{r}{\sqrt{2}} < b.$$

Case 1: If $r \neq 0$, then $r' = \frac{r}{\sqrt{2}}$ is an irrational number.
Case 2: If $r = 0$, then from inequality (2)

$$a\sqrt{2} < 0 < b\sqrt{2}.$$

So, we can consider the interval $(0, b\sqrt{2})$. From the first result, there is some rational number q such that

$$a\sqrt{2} < 0 < q < b\sqrt{2} \iff a\sqrt{2} < q < b\sqrt{2}.$$

Dividing both sides by $\sqrt{2}$ gives us

$$a < \frac{q}{\sqrt{2}} < b.$$

(Note that $q \in \mathbb{Q} - \{0\}$). So taking $r' = \frac{q}{\sqrt{z}}$, we have the result. $\square$

Theorem 1.5 (Nested Interval Property)**.** For each $n \in \mathbb{N}$, assume that we have a closed interval $I_n = [a_n, b_n]$. Assume further that each I_n contains I_{n+1}, for any $n \in \mathbb{N}$. Then the resulting nested sequence of closed intervals

$$\cdots I_n \subset I_{n-1} \subset \cdots \subset I_3 \subset I_2 \subset I_1.$$

has a nonempty intersection, that is

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset$$

Moreover, if $\inf\{b_n - a_n : n \in \mathbb{N}\} = 0$, then there exists a unique x such that $x \in I_n$, for any $n \in \mathbb{N}$. That is,

$$\bigcap_{n=1}^{\infty} I_n = x$$

Proof. We need to show that there is some real number x such that

$$x \in \bigcap_{n=1}^{\infty} I_n.$$

Consider $A = \{a_n : n \in \mathbb{N}\}$, which is the set of all left endpoints. (Since the intervals are nested, for all natural number n, b_n is an upper bound for A). By the completeness property of $\mathbb{R}$, A has a supremum. Let $x = \sup A$. Then

$$a_n \leq x \tag{1}$$

where $a_n \in A$. Also since b_n is an upper bound of A, we have

$$x \leq b_n. \tag{2}$$

Combining (1) and (2) yields

$$a_n \leq x \leq b_n.$$

Thus $x \in [a_n, b_n] = I_n$. Therefore,

$$\bigcap_{n=1}^{\infty} \neq \emptyset.$$

as desired.

For the second part of the proof, assume that $\inf\{b_n - a_n : n \in \mathbb{N}\} = 0$. We want to show that

$$\bigcap_{n=1}^{\infty} I_n = \{x\}.$$

Let us consider $B = \{b_n : n \in \mathbb{N}\}$, the set of all right endpoints. Since the intervals are nested, then a_n is a lower bound of B for all natural number $\mathbb{N}$. By the consequence of the completeness property (infimum version), B has an infimum. Let $y = \inf B$.

(MISSING???)

Assume, for the sake of contradiction, that $z \notin [x, y]$. Then $z < x$ or $z > y$.

(**Case 1**). If $z < x$, then z is not an upper bound of A. This means that there is some $a_k \in A$ such that $a_k > z$. Hence, $z \notin [a_k, b_k] = I_k$. This is a contradiction since $z \in \bigcap_{n=1}^{\infty} I_n$.

(**Case 2**). If $z > y$, then z is not a lower bound of A. This implies that there is some $b_n \in B$ such that $b_n < z$. Consider the interval $[a_n, b_n]$. Then $z \in [a_n, b_n] = I_n$ which is a contradiction since $z \in \bigcap_{n=1}^{\infty} I_n$.

Therefore, $z \in [x, y]$ and hence

$$\bigcap_{n=1}^{\infty} I_n \subseteq [x, y].$$

Now, we only need to show that $x = y$. Suppose $x \neq y$. Then $y > x$. Note that

$$a_n \leq x \quad \text{and} \quad b_n \geq y, \quad \text{for all } n \in \mathbb{N}$$

Negating the left inequality gives us $-a_n \geq -x$. Thus, we have

$$0 < y - x \leq b_n - a_n$$

This means that $y - x$ is a lower bound of $\{b_n - a_n : n \in \mathbb{N}\}$. Thus,

$$0 < y - x \leq \inf\{b_n - a_n : n \in \mathbb{N}\} = 0.$$

Therefore, $x = y$ (NOT COMPLETED!!!!) $\square$

1.3 Topology of $\mathbb{R}$

1. What is the motivation of open and closed sets?
2. Check out illustrated definitions of ϵ-neighborhood and open and closed sets.
3. What is the motivation behind cluster and interior points?
4. Check out illustrated versions of the definition of both cluster and interior points.

Definition 1.7 (ϵ-neighborhood)**.** Given $a \in \mathbb{R}$ and $\epsilon > 0$, the $\epsilon-$*neighborhood* of a is the set

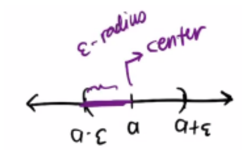
$$V_\epsilon(a) = \{x \in \mathbb{R} : |x - a| < \epsilon\},$$

where a is called the center of the neighborhood.

That is if $x \in V_\epsilon(a)$, then

$$|x - a| < \epsilon \iff -\epsilon < x - a < \epsilon \iff a - \epsilon < x < a + \epsilon.$$

Thus, x is in the interval $(a - \epsilon, a + \epsilon)$.



In other words, $V_\epsilon(a) = (a - \epsilon, a + \epsilon)$.

Definition 1.8 (Open Set)**.** A set $O \subseteq \mathbb{R}$ is *open* if for all $a \in O$, there exists an $\epsilon > 0$ such that the ϵ-neighborhood of a is a proper subset of o. That is, $V_\epsilon(a) \subset O$.

- If we want to show that a set O is open, then need to show that for any $a \in A$, there exists $\epsilon > 0$ such that

$$V_\epsilon(a) = (a - \epsilon, a + \epsilon) \subset A.$$

- The negation of the above definition is

$$\text{A set O is NOT open} \iff \exists a \in O, \forall \epsilon > 0, V_\epsilon(a) \not\subset O.$$

Example 1.4. The set of real numbers $\mathbb{R}$ and the empty set are open

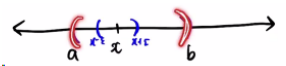
Proof. Let $a \in \mathbb{R}$. Then, we can choose any $\epsilon > 0$ so that

$$V_\epsilon(a) = (a - \epsilon, a + \epsilon) \subset \mathbb{R}$$

Thus, $\mathbb{R}$ is open. For the null set, the sufficient condition is false since there are no elements in the null set. Thus, the definition of an open set is "vacuously true". Hence, the null set is open. $\square$

Example 1.5. For any $a, b \in \mathbb{R}$ such that $a < b$, the open interval (a, b) is open.

Scratch. Let $x \in (a, b)$. We want to find $\epsilon > 0$ such that $V_\epsilon(x) \subset (a, b)$. Graphically,



In other words, we want to make sure that $x - \epsilon > a$ and $x + \epsilon < b$. Observe that

$$x - \epsilon > a \iff \epsilon < x - a$$

and

$$x + \epsilon < b \iff \epsilon < b - x.$$

Hence, we can take

$$\epsilon = \frac{\min\{x - a, b - x\}}{2}.$$

This will give us that $V_\epsilon(x) \subset (a, b)$. Therefore, the open interval (a, b) is open. $\square$

Another proof. We want to find and appropriate ϵ such that given any point $a \in S$, its ϵ-neighborhood is a subset of S. In the given set, notice that if we are close enough to one of the endpoints and ϵ is big enough then some of the points of the ϵ-neighborhood are outside of the interval. Hence, we must ensure that the neighborhood of a given point is within the interval. That is, given any point $x \in (a, b)$, we have

$$x - \epsilon > a \quad \text{and} \quad x + \epsilon < b$$

Equivalently, we have

$$\epsilon < x - a \quad \text{and} \quad \epsilon < b - x.$$

Let $\epsilon = \min\{x - a, b - x\}$. We show that $V_\epsilon(x) \subset (a, b)$. See that

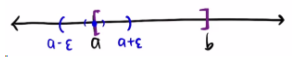
$$V_\epsilon(x) \iff -\epsilon < c - x < \epsilon \iff x - \epsilon < c < \epsilon + x$$

where $c \in \mathbb{R}$. Thus, we have

$$a = x - (x - a) < x - \epsilon < c < \epsilon + x < (b - x) + x = b.$$

Hence, $c \in (a, b)$. Since any point of an arbitrary ϵ-neighborhood is in the interval, this follows that $V_\epsilon(x) \subset (a, b)$. $\square$

Example 1.6. Any closed, half-open, or half-closed interval are not open.



Proof. Consider $[a, b]$, where $a < b$. Taking $x = 1$, we have that $\forall \epsilon > 0$,

$$V_\epsilon(a) = (a - \epsilon, a + \epsilon) \not\subset [a, b].$$

Therefore, the closed interval $[a, b]$ is not open.

The proof of other intervals are the similar. $\square$

> **Definition 1.9** (Closed)**.** A set $F \subseteq \mathbb{R}$ is **closed** if the complement of $\mathbb{F}$, $F' = \mathbb{R} - F$ is open.

- To show that a set B is closed, we need to show that its complement B' is open.
- A set which is not open is **NOT** automatically a closed set.

Example 1.7. The sets $\mathbb{R}$ and $\emptyset$ are closed.

Proof. Note that $\mathbb{R}' = \emptyset$. Since $\emptyset$ is open then $\mathbb{R}$ is closed. Similarly, $\emptyset' = \mathbb{R}$ is also open. Hence, $\emptyset$ is closed. $\square$

Note. These two sets are the only sets in the topology which are both open and closed.

Example 1.8. The complement of any open interval (a, b), which is $(-\infty, a] \cup (b, \infty)$ is closed.

Example 1.9. The closed interval $[a, b]$ is closed.

Proof. Observe that

$$[a, b]' = (\infty, a) \cup (b, +\infty)$$

EXERCISE: Show that $(-\infty, a) \cup (b, +\infty)$ is open. $\square$

Example 1.10. The interval $(a, b]$ is neither open nor closed.

Proof. By example 1.6, we have already shown that $(a, b]$ is not open. Consider the complement of

$$(a, b]' = (-\infty, a] \cup (b, +\infty)$$

EXERCISE. To show that $(-\infty, a] \cup (b, +\infty)$ is not open, choose $a \in (-\infty, a] \cup (b, +\infty)$ and argue that $\forall \epsilon > 0$, $V_\epsilon(a) \notin (-\infty, a] \cup (b, +\infty)$. $\square$

Theorem 1.6 (Union and Intersection of Open Sets)**.**

1. The union of an arbitrary collection of open subsets in $\mathbb{R}$ is open
2. The intersection of any finite collection of open sets is open.

Proof. (1). Let $\{G_\alpha : \gamma \in \Gamma\}$ be an arbitrary collection of open sets. We want to show that

$$G = \bigcup_{\gamma \in \Gamma} G_\gamma$$

is open. Let $x \in G$ Then $x \in G_\gamma$, for some $\gamma \in \Gamma$. Note that G_γ is open, so there is some $\epsilon > 0$ such that

$$V_\epsilon(x) \subset G_\gamma \subseteq \bigcup_{\gamma \in \Gamma} G_\gamma = G.$$

Since x is arbitrary, we can conclude that G is open.

(2) Let $\{G_1, G_2, G_3, \ldots, G_n\}$ be a collection of open sets. We want to show that $\bigcap_i^n G_i$ is open, that is, for any $x \in \bigcap_{i=1}^n G_i$, there exists some $\epsilon > 0$ such that $V_\epsilon(x) \subset \bigcap_{i=1}^n G_i$

Let $x_i \in \bigcap_{i=1}^n G_i$. Then $x_i \in G_i$ for all $i = 1, \ldots, n$. By hypothesis, each G_i is open. Thus, $V_{e_i}(x_i) \subset G_i$ for all $i = 1, \ldots, n$. We want an $\epsilon > 0$ such that

$$V_e(x) \subset V_{e_i}(x_i) \subset G_i, \quad \forall i = 1, \ldots n.$$

Thus, we need to find an $\epsilon > 0$ such that $\epsilon < \epsilon_i$ for all $i = 1, \ldots, n$. Choose

$$\epsilon = \frac{\min\{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\}}{2}.$$

It follows that $\epsilon < \epsilon_i$ for all $i = 1, \ldots, n$. Thus, we have

$$V_e(x) \subset V_{e_i}(x_i) \subset G_i \, \forall i = 1, \ldots, n$$
$$\implies V_e(x) \subset \bigcap_{i=1}^n G_i$$

as desired. Therefore, $\bigcap_{i=1}^n G_i$ is open. □

Theorem 1.7 (Union and Intersection of Closed Sets)**.**

1. The intersection of an arbitrary collection of closed sets is closed

2. The union of any finite collection of closed sets is closed.

Example 1.11. Let $O_n = \big(0, 1 + \frac{1}{}n\big)$ for $n \in \mathbb{N}$. The intersection

$$O = \bigcap_{n=1}^{\infty} O_n = (0, 1]$$

Example 1.12. Let $F_n = \left[\frac{1}{n}, 1\right]$, for any $n \in \mathbb{N}$. The union

$$F = \bigcup_{n=1}^{\infty} = (0, 1]$$

is not closed.

Definition 1.10 (Cluster and Interior Points)**.**

1. A point $x \in \mathbb{R}$ is a *cluster point* of X if for every $\epsilon > 0$, $V_e(x)$ contains a point of X different from x. That is, for any $\epsilon > 0$,

$$V_\epsilon(x) \cap X \neq \emptyset \quad \text{and} \quad V_\epsilon(x) \cap X \neq \{x\}$$

Note: A cluster point is sometimes called a *limit point*

2. A point $x \in \mathbb{R}$ is an *interior point* of X if there exists $\epsilon > 0$ such that $V_\epsilon(x) \subset X$.

Example 1.13. Let $a, b \in \mathbb{R}$ such that $a < b$.

1. Every $x \in (a, b)$ is an interior point of the closed interval $[a, b]$.
2. Every $y \in [a, b]$ is a cluster point of hte open interval (a,b).

Theorem 1.8 (**Characterization of Open and Closed Sets**)**.** 1.
A subset F of $\mathbb{R}$ is **closed** if and only if F contains all of its **cluster points**.

2. A subset O of $\mathbb{R}$ is **open** if and only if every point of O is an **interior point** of O.
3. A subset of $\mathbb{R}$ is **open** if and only if it is the countable union of disjoint open intervals in $\mathbb{R}$. (Proof in Bartle & Sherbert 329-330).

Definition 1.11 (Closure of a Set). Let $A \subset \mathbb{R}$ and let C_a be the set of cluster points of A. The *closure* of A, denoted by $\bar{A}$, is the set

$$\bar{A} = A \cup C_A$$

Theorem 1.9. Let $A \in \mathbb{R}$

1. The closure $\bar{A}$ is closed.
2. The closure $\bar{A}$ is the smallest closed set containing A.

Example 1.14.

1. $\overline{(a, b)} = [a, b]$
2. For any closed set F, $\bar{F} = F$.

For (2), from the characterization of closed sets, we know that $C_F \subseteq F$ where C_F is the set of all cluster points of F. So, $\bar{F} = F$.

1.3.1 Compact Sets

Definition 1.12 (Open Cover). An open cover of A is a collection of $\mathcal{O} = \{O_\alpha : \alpha \in \Gamma\}$ of open sets in $\mathbb{R}$ whose union contains A, that is,

$$A \subseteq \bigcup_{\alpha \in \Gamma} O_\alpha$$

Definition 1.13 (Subcover). A *subcover* of $\mathcal{O}$ is a subcollection of sets $\mathcal{O}'$ such that $\mathcal{O}' \subset \mathcal{O}$ and $\mathcal{O}'$ is also an open cover of A.

Definition 1.14 (Finite Subcover). If a finite subcover $\mathcal{O}'$ consists of finitely many sets, we say that $\mathcal{O}'$ is a *finite subcover*.

Example 1.15. Consider $A = [1, \infty)$. You can verify that the following collection of sets are open covers of A:

1. $\mathcal{O}_0 = \{(0, \infty)\}$
2. $\mathcal{O}_1 = \{(r - 1, r + 1) : r \in \mathbb{Q}, r > 0\}$
3. $\mathcal{O}_2 = \{(n - 1, n + 2) : n \in \mathbb{N}\}$
4. $\mathcal{O}_3 = \{(0, n) : n \in \mathbb{N}\}$

Definition 1.15 (Compact)**.** A subset K of $\mathbb{R}$ is said to be *compact* if every open cover of K has a finite subcover.

Remark 1.2. To show that a set K is compact by definition, we must consider an arbitrary open cover of K and show that we can find a finite subcover (which can be very complicated at times). However, we can still simplify this by using a later result.

Remark 1.3. To show that a set A is not compact, we just have to find an open cover of A that does not have a finite subcover.

Example 1.16.

1. Let $A = \{a_1, a_2, \dots a_n\}$ be a finite subset of $\mathbb{R}$. Then A is compact.

   Proof. Let $\mathcal{O} = \{O_\alpha\}_\alpha$ be an open cover of A. Then, $A \subseteq \bigcup_\alpha O_\alpha$. Since A is a subset of the union, then $a_i \in \bigcup_\alpha O_\alpha$ for all $i = 1, \dots, n$. It follows that there is some $O_{\alpha_i} \in \mathcal{O}$ such that $a_i O_{\alpha_i}$ for all $i = 1, \dots, n$. Thus,
   $$A = \{a_1, \dots a_n\} \subseteq \bigcup_{i=1}^{n} O_{\alpha_i}.$$
   This implies that $\{\emptyset_{\alpha_i}\}_{i=1}^n$ is a finite subcover of $\mathcal{O}$. Therefore, A is compact. $\square$

2. The closed interval $H = [0, \infty)$ is not compact.

   Proof. Consider $\mathcal{O} = \{(-1, n) : n \in \mathbb{N}\}$. (We first show that $\mathcal{O}$ is an open cover of H).

   Let $x \in H$. Then by the Archimedean property, we have
   $$-1 < 0 \leq x < n$$
   for some $n \in \mathbb{N}$. Thus, we have $x \in (-1, n)$. Hence,
   $$x \in \bigcup_{n=1}^{\infty} (-1, n)$$
   which implies that $H \subseteq \bigcup_{n=1}^{\infty} (-1, n)$ as desired.

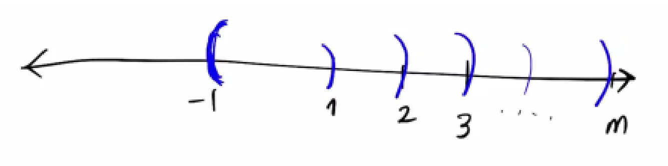
   We now show that $\mathcal{O}$ does not have a finite subcover. We prove by contradiction.

   Suppose otherwise, that there is some $m \in \mathbb{N}$ such that
   $$\mathcal{O} = \{(-1, n) : n = 1, \dots, m\}$$

is a finite subcover of $\mathcal{O}$. Then

$$H \subseteq \bigcup_{n=1}^{\infty} (-1, n) = (-1, m)$$

which means $H \not\subseteq (-1, m)$ since $m + 1 \in H$ but $m + 1 \in (-1, m)$.



Therefore, $\mathcal{O}$ does not have a finite subcover and it follows that H is not compact. $\square$

3. **EXERCISE:** The open interval $I = (0, 1)$ is not compact.

Scratch. One open cover that we can consider is

$$\left\{ \left(0, 1 - \frac{1}{n}\right) : n \in \mathbb{N} \right\}$$

$\square$

The theorem below is an easier way to show that a subset K of $\mathbb{R}$ is compact.

Theorem 1.10 (Heine-Borel Theorem)**.** A subset K of $\mathbb{R}$ is compact if and only if it is closed and bounded.

To show that a subset K of $\mathbb{R}$ is not compact we need to show that K is not closed or not bounded.

Remark 1.4. In our previous examples, the set $[0, \infty)$ is closed but not bounded while the set $(0, 1)$ is bounded but not closed. On both cases, we showed that they are not compact.

2 Sequences in $\mathbb{R}$

2.1 Limit of a Sequence

Definition 2.1 (Sequence)**.** A sequence of real numbers (or a sequence in $\mathbb{R}$) is a function defined on the set $\mathbb{N} = \{1, 2, 3, \ldots\}$ of natural numbers whose range is contained in the set $\mathbb{R}$ of real numbers

Notations.

1. If A is a sequence, then it assigned a natural number n to a uniquely determined real number, which we will denote by a_n rather than the usual notation $A(n)$ for function value.

2. The values a_n **elements** or **terms** of the sequence. We usually say that a_n is the nth term of the sequence.

3. We will denote sequences as follows

$$A = \{a_n\} \quad \text{and} \quad X = \{x_n\}$$

   We usually drop the A and X and leave the

$$\{a_n\} \quad \text{and} \quad \{x_n\}$$

Example 2.1. Below are some examples of sequences.

1. $a_n = \frac{1}{n} = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$
2. $a_n = (-1)^{n+1} = (-1)^{n-1} = \{1, -1, 1, -1, \ldots\}$
3. $a_n = \{1, 1, 2, 3, 5, \ldots\}$. Let $a_1 = a_2 = 1$.
   Then $a_n = a_{n-2} + a_{n-1}$, $n \geq 3$.

Definition 2.2 (Convergence of a Sequence)**.** A sequence $\{a_n\}$ *converges* to a, or a is said to be a limit of $\{a_n\}$ if for every $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that for every $n \geq N_\epsilon$,

$$|a_n - a| < \epsilon.$$

We denote the limit a by

$$a = \lim_{n \to \infty} a_n.$$

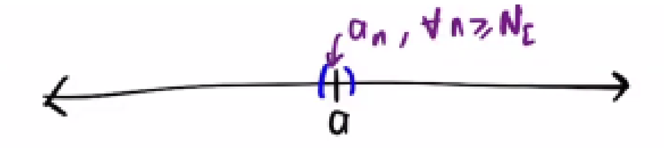
If a sequence has a limit, we say that $\{a_n\}$ *convergent*. Otherwise, we say that the sequence $\{a_n\}$ is *divergent*.

Recall: Let $\epsilon > 0$. The ϵ-neighborhood of a real number x is the set

$$V_\epsilon(x) = \{y \in \mathbb{R} : |y - x| < \epsilon\} = (x - \epsilon, x + \epsilon).$$

Thus, if a sequence $\{a_n\}$ converges to a, then we can find $\mathbb{N}_\epsilon \in \mathbb{N}$ such that for all $n \geq N_\epsilon$

$$|a_n - a| < \epsilon \implies a_n \in V_\epsilon(a), \quad \forall n \in N_\epsilon.$$



That is, any ϵ-neighborhood of a limit a contains infinitely many terms **MISSING**.

Theorem 2.1. If a sequence converges, then its limit is unique.

Proof. Let $\{a_n\}$ be a convergent sequence. Suppose that the sequence converges to a' and a''. We show that $a' = a''$ Let $\epsilon > 0$. From the definition of convergence, there is some $N_{\epsilon'} \in \mathbb{N}$ such that

$$|a_n - a| < \frac{\epsilon}{2},$$

and some $N_{\epsilon''}$ such that

$$|a_n - a''| < \frac{\epsilon}{2}, \quad n \geq N_{\frac{\epsilon}{2}}$$

Let $N_\epsilon = \max\{N'_{\frac{\epsilon}{2}}, N''_{\frac{\epsilon}{2}}\}$. Then if

STOPPED THIS PART. □

3 List of Theorems and Definitions

Definition 3.1 (**Upper and Lower Bound**)**.** A set $A \subseteq \mathbb{R}$ is *bounded above* if there exists a number $b \in \mathbb{R}$ such that $a \leq b$ for all $a \in A$. Similarly, the set A is *bounded below* if there exists a lower bound $l \in \mathbb{R}$ satisfying $l \leq a$ for every $a \in A$.

Definition 3.2 (**Supremum (Least Upper Bound)**)**.** A real number u is the *least upper bound* for a set $A \subseteq \mathbb{R}$ if it meets the following two criteria

1. u is an upper bound for A;
2. if b is any upper bound for A, then $u \leq b$.

Definition 3.3 (**Infimum (Greatest Lower Bound)**)**.** A real number l is the *greatest lower bound* for a set $A \subseteq \mathbb{R}$ if it meets the following two criteria:

1. l is a lower bound for A;
2. if p is any lower bound for A, then $p \leq l$.

Theorem 3.1 (Characterization of $\sup S$ and $\inf S$)**.**

1. An upper bound u of a nonempty set S in $\mathbb{R}$ is the supremum of S if and only if for every $\epsilon > 0$, there exists an $s_\epsilon \in S$ such that $u - \epsilon < s_\epsilon$.
2. A lower bound l of a nonempty subset S in $\mathbb{R}$ is the infimum of S if and only if for every $\epsilon > 0$, there exists a $t_\epsilon \in S$ such that $l + \epsilon > t_\epsilon$.

Theorem 3.2 (Properties)**.** Let S be a nonempty subset of $\mathbb{R}$ and $a \in \mathbb{R}$. Define the sets

$$a + S = \{a + s : s \in S\}, \quad \text{and} \quad -S = \{s : s \in S\}$$

1. If S is bounded above, then $\sup(a + S) = a + \sup S$.
2. if S is bounded below, then $\inf(a + S) = a + \inf S$.

3. if S is bounded, then

$$\inf(-S) = -\sup S \text{ and } \sup(-S) = -\inf S$$

Definition 3.4 (Maximum and Minimum)**.** A real number s_{max} is a **maximum** of a set S if $s_{max} \geq s$, for any $s \in S$ and $s_{max} \in S$. Similarly, s_{min} is a **minimum** of a set S if $s_{min} \leq s$, for all $s \in S$ and $s_{min} \in S$.

Definition 3.5 (Completeness Property of $\mathbb{R}$)**.** Every nonempty set of real numbers that has an upper bound also has a supremum (AXIOM).

Theorem 3.3 (Archimedean Property)**.** For every $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $x < n$ (or equivalently, the set of natural numbers is not bounded above).

Corollary 3.1. Let $y, z \in \mathbb{R}^+ = \{r \in \mathbb{R} : r > 0\}$. Then

1. $\exists n \in \mathbb{N}$ such that $z < ny$.
2. $\exists n \in \mathbb{N}$ such that $0 < \frac{1}{n} < y$ (labeled as: **Corollary 2**).
3. $\exists n \in \mathbb{N}$ such that $n - 1 \leq z < n$.

Theorem 3.4 (Density Theorem)**.** If $a, b \in \mathbb{R}$ such that $a < b$, then there exist $r \in \mathbb{Q}$ and $y' \in \mathbb{Q}'$ such that $a < r < b$ and $a < r' < b$.

Theorem 3.5 (Nested Interval Property)**.** For each $n \in \mathbb{N}$, assume that we have a closed interval $I_n = [a_n, b_n]$. Assume further that each I_n contains I_{n+1}, for any $n \in \mathbb{N}$. Then the resulting nested sequence of closed intervals

$$\cdots I_n \subset I_{n-1} \subset \cdots \subset I_3 \subset I_2 \subset I_1.$$

has a nonempty intersection, that is

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset$$

Moreover, if $\inf\{b_n - a_n : n \in \mathbb{N}\} = 0$, then there exists a unique x such that $x \in I_n$, for any $n \in \mathbb{N}$. That is,

$$\bigcap_{n=1}^{\infty} I_n = x$$

Definition 3.6 (ϵ-neighborhood)**.** Given $a \in \mathbb{R}$ and $\epsilon > 0$, the $\epsilon-$*neighborhood* of a is the set

$$V_\epsilon(a) = \{x \in \mathbb{R} : |x - a| < \epsilon\},$$

where a is called the center of the neighborhood.

Definition 3.7 (Open Set)**.** A set $O \subseteq \mathbb{R}$ is *open* if for all $a \in O$, there exists an $\epsilon > 0$ such that the ϵ-neighborhood of a is a proper subset of O. That is, $V_\epsilon(a) \subset O$.

Definition 3.8 (Closed)**.** A set $F \subseteq \mathbb{R}$ is **closed** if the complement of F, $F' = \mathbb{R} - F$ is open.

Theorem 3.6 (Union and Intersection of Open Sets)**.**

1. The union of an arbitrary collection of open subsets in $\mathbb{R}$ is open
2. The intersection of any finite collection of open sets is open.

Theorem 3.7 (Union and Intersection of Closed Sets)**.**

1. The intersection of an arbitrary collection of closed sets is closed
2. The union of any finite collection of closed sets is closed.

Definition 3.9 (Cluster and Interior Points)**.**

1. A point $x \in \mathbb{R}$ is a *cluster point* of X if for every $\epsilon > 0$, $V_e(x)$ contains a point of X different from x. That is, for any $\epsilon > 0$,

$$V_\epsilon(x) \cap X \neq \emptyset \quad \text{and} \quad V_\epsilon(x) \cap X \neq \{x\}$$

Note: A cluster point is sometimes called a *limit point*

2. A point $x \in \mathbb{R}$ is an *interior point* of X if there exists $\epsilon > 0$ such that $V_\epsilon(x) \subset X$.

Theorem 3.8 (**Characterization of Open and Closed Sets**). 1.

A subset F of $\mathbb{R}$ is **closed** if and only if F contains all of its **cluster points**.

2. A subset O of $\mathbb{R}$ is **open** if and only if every point of O is an **interior point** of O.

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Definition 3.13 (Compact)**.** A subset K of $\mathbb{R}$ is said to be *compact* if every open cover of K has a finite subcover.

Theorem 3.10 (Heine-Borel Theorem)**.** A subset K of $\mathbb{R}$ is compact if and only if it is closed and bounded.