3 List of Theorems and Definitions

Definition 3.1 (**Upper and Lower Bound**)**.** A set $A \subseteq \mathbb{R}$ is *bounded above* if there exists a number $b \in \mathbb{R}$ such that $a \leq b$ for all $a \in A$. Similarly, the set A is *bounded below* if there exists a lower bound $l \in \mathbb{R}$ satisfying $l \leq a$ for every $a \in A$.

Definition 3.2 (**Supremum (Least Upper Bound)**)**.** A real number u is the *least upper bound* for a set $A \subseteq \mathbb{R}$ if it meets the following two criteria

1. u is an upper bound for A;
2. if b is any upper bound for A, then $u \leq b$.

Definition 3.3 (**Infimum (Greatest Lower Bound)**)**.** A real number l is the *greatest lower bound* for a set $A \subseteq \mathbb{R}$ if it meets the following two criteria:

1. l is a lower bound for A;
2. if p is any lower bound for A, then $p \leq l$.

Theorem 3.1 (Characterization of $\sup S$ and $\inf S$)**.**

1. An upper bound u of a nonempty set S in $\mathbb{R}$ is the supremum of S if and only if for every $\epsilon > 0$, there exists an $s_\epsilon \in S$ such that $u - \epsilon < s_\epsilon$.
2. A lower bound l of a nonempty subset S in $\mathbb{R}$ is the infimum of S if and only if for every $\epsilon > 0$, there exists a $t_\epsilon \in S$ such that $l + \epsilon > t_\epsilon$.

Theorem 3.2 (Properties)**.** Let S be a nonempty subset of $\mathbb{R}$ and $a \in \mathbb{R}$. Define the sets

$$a + S = \{a + s : s \in S\}, \quad \text{and} \quad -S = \{s : s \in S\}$$

1. If S is bounded above, then $\sup(a + S) = a + \sup S$.
2. if S is bounded below, then $\inf(a + S) = a + \inf S$.

3. if S is bounded, then

$$\inf(-S) = -\sup S \text{ and } \sup(-S) = -\inf S$$

Definition 3.4 (Maximum and Minimum)**.** A real number s_{max} is a **maximum** of a set S if $s_{max} \geq s$, for any $s \in S$ and $s_{max} \in S$. Similarly, s_{min} is a **minimum** of a set S if $s_{min} \leq s$, for all $s \in S$ and $s_{min} \in S$.

Definition 3.5 (Completeness Property of $\mathbb{R}$)**.** Every nonempty set of real numbers that has an upper bound also has a supremum (AXIOM).

Theorem 3.3 (Archimedean Property)**.** For every $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $x < n$ (or equivalently, the set of natural numbers is not bounded above).

Corollary 3.1. Let $y, z \in \mathbb{R}^+ = \{r \in \mathbb{R} : r > 0\}$. Then

1. $\exists n \in \mathbb{N}$ such that $z < ny$.
2. $\exists n \in \mathbb{N}$ such that $0 < \frac{1}{n} < y$ (labeled as: **Corollary 2**).
3. $\exists n \in \mathbb{N}$ such that $n - 1 \leq z < n$.

Theorem 3.4 (Density Theorem)**.** If $a, b \in \mathbb{R}$ such that $a < b$, then there exist $r \in \mathbb{Q}$ and $y' \in \mathbb{Q}'$ such that $a < r < b$ and $a < r' < b$.

Theorem 3.5 (Nested Interval Property)**.** For each $n \in \mathbb{N}$, assume that we have a closed interval $I_n = [a_n, b_n]$. Assume further that each I_n contains I_{n+1}, for any $n \in \mathbb{N}$. Then the resulting nested sequence of closed intervals

$$\cdots I_n \subset I_{n-1} \subset \cdots \subset I_3 \subset I_2 \subset I_1.$$

has a nonempty intersection, that is

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset$$

Moreover, if $\inf\{b_n - a_n : n \in \mathbb{N}\} = 0$, then there exists a unique x such that $x \in I_n$, for any $n \in \mathbb{N}$. That is,

$$\bigcap_{n=1}^{\infty} I_n = x$$

Definition 3.6 (ϵ-neighborhood)**.** Given $a \in \mathbb{R}$ and $\epsilon > 0$, the $\epsilon-$*neighborhood* of a is the set

$$V_\epsilon(a) = \{x \in \mathbb{R} : |x - a| < \epsilon\},$$

where a is called the center of the neighborhood.

Definition 3.7 (Open Set)**.** A set $O \subseteq \mathbb{R}$ is *open* if for all $a \in O$, there exists an $\epsilon > 0$ such that the ϵ-neighborhood of a is a proper subset of O. That is, $V_\epsilon(a) \subset O$.

Definition 3.8 (Closed)**.** A set $F \subseteq \mathbb{R}$ is **closed** if the complement of F, $F' = \mathbb{R} - F$ is open.

Theorem 3.6 (Union and Intersection of Open Sets)**.**

1. The union of an arbitrary collection of open subsets in $\mathbb{R}$ is open

2. The intersection of any finite collection of open sets is open.

Theorem 3.7 (Union and Intersection of Closed Sets)**.**

1. The intersection of an arbitrary collection of closed sets is closed

2. The union of any finite collection of closed sets is closed.

Definition 3.9 (Cluster and Interior Points)**.**

1. A point $x \in \mathbb{R}$ is a *cluster point* of X if for every $\epsilon > 0$, $V_e(x)$ contains a point of X different from x. That is, for any $\epsilon > 0$,

$$V_\epsilon(x) \cap X \neq \emptyset \quad \text{and} \quad V_\epsilon(x) \cap X \neq \{x\}$$

Note: A cluster point is sometimes called a *limit point*

2. A point $x \in \mathbb{R}$ is an *interior point* of X if there exists $\epsilon > 0$ such that $V_\epsilon(x) \subset X$.

Theorem 3.8 (Characterization of Open and Closed Sets). 1.

1. A subset F of $\mathbb{R}$ is **closed** if and only if F contains all of its **cluster points**.

2. A subset O of $\mathbb{R}$ is **open** if and only if every point of O is an **interior point** of O.

3. A subset of $\mathbb{R}$ is **open** if and only if it is the countable union of disjoint open intervals in $\mathbb{R}$. (Proof in Bartle & Sherbert 329-330).

Definition 3.10 (Closure of a Set)**.** Let $A \subset \mathbb{R}$ and let C_a be the set of cluster points of A. The *closure* of A, denoted by $\bar{A}$, is the set

$$\bar{A} = A \cup C_A$$

Theorem 3.9. Let $A \subseteq \mathbb{R}$.

1. The closure $\bar{A}$ is closed

2. The closure $\bar{A}$ is the smallest closed set containing A.

Definition 3.11 (Open Cover)**.** An open cover of A is a collection of $\mathcal{O} = \{O_\alpha\}$ of open sets in $\mathbb{R}$ whose union contains A, that is,

$$A \subseteq \bigcup_\alpha O_\alpha$$

Definition 3.12 (Subcover)**.** A *subcover* of $\mathcal{O}$ is a subcollection of sets $\mathcal{O}'$ such that $\mathcal{O}' \subset \mathcal{O}$ and $\mathcal{O}'$ is also an open cover of A.