第 08 周作业解答

练习 1. 画出积分区域，并计算二重积分：

1. $\iint_D x\sqrt{y}d\sigma$，其中 D 是由两条抛物线 $y=\sqrt{x}$ 和 $y=x^2$ 所围成的闭区域；
2. $\iint_D (x^2+y^2-x)d\sigma$，其中 D 是由直线 $y=2$，$y=x$ 和 $y=2x$ 所围成的闭区域；

解 1. 将 D 视为 X 型区域：$D=\{(x,y)|\, x^2\le y\le\sqrt{x}, 0\le x\le 1\}$。所以

$$\begin{aligned}\iint_D x\sqrt{y}d\sigma &= \int_0^1\left[\int_{x^2}^{\sqrt{x}} x\sqrt{y}dy\right]dx = \int_0^1\left[\frac{2}{3}xy^{\frac{3}{2}}\Big|_{x^2}^{\sqrt{x}}\right]dx = \int_0^1\left[\frac{2}{3}x^{\frac{7}{4}}-\frac{2}{3}x^4\right]dx\\ &= \left(\frac{8}{33}x^{\frac{11}{4}}-\frac{2}{15}x^5\right)\Big|_0^1 = \frac{8}{33}-\frac{2}{15}=\frac{6}{55}.\end{aligned}$$

也可以将 D 视为 Y 型区域：$D=\{(x,y)|\, y^2\le x\le\sqrt{y}, 0\le y\le 1\}$。所以

$$\begin{aligned}\iint_D x\sqrt{y}d\sigma &= \int_0^1\left[\int_{y^2}^{\sqrt{y}} x\sqrt{y}dx\right]dy = \int_0^1\left[\frac{1}{2}x^2y^{\frac{1}{2}}\Big|_{y^2}^{\sqrt{y}}\right]dy = \int_0^1\left[\frac{1}{2}y^{\frac{3}{2}}-\frac{1}{2}x^{\frac{9}{2}}\right]dy\\ &= \left(\frac{1}{5}x^{\frac{5}{2}}-\frac{1}{11}x^{\frac{11}{2}}\right)\Big|_0^1 = \frac{1}{5}-\frac{1}{11}=\frac{6}{55}.\end{aligned}$$

2. 将 D 视为 Y 型区域：$D=\{(x,y)|\, \frac{1}{2}y\le x\le y, 0\le y\le 2\}$。所以

$$\begin{aligned}\iint_D (x^2+y^2-x)d\sigma &= \int_0^2\left[\int_{\frac{1}{2}y}^{y}(x^2+y^2-x)dx\right]dy = \int_0^2\left[\frac{1}{3}x^3+xy^2-\frac{1}{2}x^2\Big|_{\frac{1}{2}y}^{y}\right]dy = \int_0^1\left[\frac{19}{24}y^3-\frac{3}{8}y^2\right]dy\\ &= \left(\frac{19}{96}y^4-\frac{1}{8}y^3\right)\Big|_0^2 = \frac{19}{6}-1=\frac{13}{6}.\end{aligned}$$

练习 2. 画出积分区域，并计算二重积分：

1. $\iint_D x\cos(x+y)d\sigma$，其中 D 是顶点分别为 $(0,0)$，$(\pi,0)$ 和 (π,π) 的三角区闭区域；
2. $\iint_D e^{x+y}d\sigma$，其中 $D=\{(x,y)|\,|x|\le 1, |y|\le 1\}$.

解 1. 将 D 视为 Y 型区域：$D=\{(x,\,y)|\,y\le x\le\pi,\,0\le y\le\pi\}$。所以

$$\begin{aligned}\iint_D x\cos(x+y)d\sigma &= \int_0^\pi\left[\int_y^\pi x\cos(x+y)dx\right]dy=\int_0^\pi\left[\int_y^\pi xd\sin(x+y)\right]dy\\ &=\int_0^\pi\left[x\sin(x+y)\big|_y^\pi-\int_y^\pi\sin(x+y)dx\right]dy\\ &=\int_0^\pi[\pi\sin(\pi+y)-y\sin(2y)+\cos(\pi+y)-\cos(2y)]\,dy\\ &=\int_0^\pi[-\pi\sin y-y\sin(2y)-\cos y-\cos(2y)]\,dy\\ &=\pi\cos y\big|_0^\pi-\sin y\big|_0^\pi-\frac{1}{2}\sin 2y\big|_0^\pi+\frac{1}{2}\int_0^\pi yd\cos(2y)\\ &=-2\pi+\frac{1}{2}\left(y\cos(2y)\big|_0^\pi-\int_0^\pi\cos(2y)dy\right)\\ &=-2\pi+\frac{1}{2}\left(\pi-\frac{1}{2}\sin(2y)\big|_0^\pi\right)=-\frac{3}{2}\pi.\end{aligned}$$

也可以将 D 视为 X 型区域：$D=\{(x,\,y)|\,0\le y\le x,\,0\le x\le\pi\}$。所以

$$\begin{aligned}\iint_D x\cos(x+y)d\sigma &= \int_0^\pi\left[\int_0^x x\cos(x+y)dy\right]dx=\int_0^\pi\left[x\sin(x+y)\Big|_0^x\right]dx=\int_0^\pi[x\sin(2x)-x\sin x]\,dx\\ &=\int_0^\pi xd\left(-\frac{1}{2}\cos(2x)+\cos x\right)=x\left(-\frac{1}{2}\cos(2x)+\cos x\right)\Big|_0^\pi-\int_0^\pi\left(-\frac{1}{2}\cos(2x)+\cos x\right)dx\\ &=-\frac{3}{2}\pi-\left[-\frac{1}{4}\sin(2x)+\sin x\right]\Big|_0^\pi=-\frac{3}{2}\pi.\end{aligned}$$

2.

$$\begin{aligned}\iint_D e^{x+y}d\sigma &= \int_{-1}^1\left(\int_{-1}^1 e^{x+y}dx\right)dy=\int_{-1}^1\left(e^{x+y}\big|_{-1}^1\right)dy=\int_0^1 e^{1+y}-e^{-1+y}dy\\ &=e^{1+y}-e^{-1+y}\big|_{-1}^1=e^2+e^{-2}-2.\end{aligned}$$

练习 3. 计算二重积分 $\iint_D e^{x+y}d\sigma$，其中 $D=\{(x,\,y)|\,|x|+|y|\le 1\}$.

解将 D 视为两个 X 型区域之并：

$$D=D_1\cup D_2=\{(x,\,y)|\,-x-1\le y\le x+1,\,-1\le x\le 0\}\cup\{(x,\,y)|\,-1+x\le y\le 1-x,\,0\le x\le 1\}.$$

所以

$$\begin{aligned}
\iint_D e^{x+y}d\sigma &= \iint_{D_1} e^{x+y}d\sigma + \iint_{D_2} e^{x+y}d\sigma \\
&= \int_{-1}^{0}\left[\int_{-x-1}^{x+1} e^{x+y}dy\right]dx + \int_{0}^{1}\left[\int_{-1+x}^{1-x} e^{x+y}dy\right]dx \\
&= \int_{-1}^{0}\left[e^{x+y}\Big|_{-x-1}^{x+1}\right]dx + \int_{0}^{1}\left[e^{x+y}\Big|_{-1+x}^{1-x}\right]dx \\
&= \int_{-1}^{0}\left[e^{2x+1}-e^{-1}\right]dx + \int_{0}^{1}\left[e-e^{2x-1}\right]dx \\
&= (\frac{1}{2}e^{2x+1}-e^{-1}x)\big|_{-1}^{0} + (ex-\frac{1}{2}e^{2x-1})\big|_{0}^{1} \\
&= \frac{1}{2}e - 0 - \frac{1}{2}e^{-1} - e^{-1} + e - \frac{1}{2}e + \frac{1}{2}e^{-1} \\
&= e - e^{-1}.
\end{aligned}$$

也可以将 D 视为两个 Y 型区域之并：

$$D = D_1 \cup D_2 = \{(x,\, y)|\, -y-1 \le x \le y+1,\, -1 \le y \le 0\} \cup \{(x,\, y)|\, y-1 \le x \le -y+1,\, 0 \le y \le 1\}.$$

所以

$$\begin{aligned}
\iint_D e^{x+y}d\sigma &= \iint_{D_1} e^{x+y}d\sigma + \iint_{D_2} e^{x+y}d\sigma \\
&= \int_{-1}^{0}\left[\int_{-y-1}^{y+1} e^{x+y}dx\right]dy + \int_{0}^{1}\left[\int_{y-1}^{-y+1} e^{x+y}dx\right]dy \\
&= \int_{-1}^{0}\left[e^{x+y}\Big|_{-y-1}^{y+1}\right]dy + \int_{0}^{1}\left[e^{x+y}\Big|_{-1+y}^{1-y}\right]dy \\
&= \int_{-1}^{0}\left[e^{2y+1}-e^{-1}\right]dy + \int_{0}^{1}\left[e-e^{2y-1}\right]dy \\
&= (\frac{1}{2}e^{2y+1}-e^{-1}y)\big|_{-1}^{0} + (ey-\frac{1}{2}e^{2y-1})\big|_{0}^{1} \\
&= \frac{1}{2}e - 0 - \frac{1}{2}e^{-1} - e^{-1} + e - \frac{1}{2}e + \frac{1}{2}e^{-1} \\
&= e - e^{-1}.
\end{aligned}$$

练习 4. 交换二次积分 $\int_1^2\left[\int_{2-x}^{\sqrt{2x-x^2}} f(x,\, y)dy\right]dx$ 的积分次序。

解 1. $\int_1^2\left[\int_{2-x}^{\sqrt{2x-x^2}} f(x,\, y)dy\right]dx = \iint_D f(x,\, y)d\sigma$，其中 $D = \{(x,\, y)|\, 2-x \le y \le \sqrt{2x-x^2},\, 1 \le x \le 2\}$ 视为 X 型区域。

2. D 也可视为 Y 型区域：$D = \{(x,\, y)|\, 2-y \le x \le 1+\sqrt{1-y^2},\, 0 \le y \le 1\}$。所以

$$\int_1^2\left[\int_{2-x}^{\sqrt{2x-x^2}} f(x,\, y)dy\right]dx = \iint_D f(x,\, y)d\sigma = \int_0^1\left[\int_{2-y}^{1+\sqrt{1-y^2}} f(x,\, y)dx\right]dy.$$

练习 5. 通过交换积分次序计算二次积分 $\int_0^2 dx\int_x^2 e^{-y^2}dy$。

解 1. $\int_0^2 \left[\int_x^2 e^{-y^2}dy\right]dx = \iint_D f(x,\,y)d\sigma$，其中 $D = \{(x,\,y)|\, x \le y \le 2,\, 0 \le x \le 2\}$ 视为 X 型区域。
2. D 也可视为 Y 型区域：$D = \{(x,\,y)|\, 0 \le x \le y,\, 0 \le y \le 2\}$。所以

$$
\begin{aligned}
\int_0^2 \left[\int_x^2 e^{-y^2}dy\right]dx &= \iint_D f(x,\,y)d\sigma \\
&= \int_0^2 \left[\int_0^y e^{-y^2}dx\right]dy \\
&= \int_0^2 e^{-y^2}ydy \\
&= \frac{1}{2}\int_0^2 e^{-y^2}dy^2 \\
&= \frac{1}{2}\int_0^4 e^{-u}du \\
&= -\frac{1}{2}e^{-u}\big|_0^4 = \frac{1}{2}(1-e^{-4}).
\end{aligned}
$$

练习 6. 计算 $\iint_D |x^2+y^2-4|d\sigma$，其中 D 为圆盘 $x^2+y^2 \le 16$。

解在极坐标下 $D = \{(\rho,\,\theta)|\, 0 \le \rho \le 4,\, 0 \le \theta \le 2\pi\}$，所以

$$
\begin{aligned}
\iint_D |x^2+y^2-4|d\sigma &= \iint_D |\rho^2-4|\rho d\rho d\theta \\
&= \int_0^{2\pi}\left[\int_0^4 |\rho^2-4|\rho d\rho\right]d\theta = 2\pi\left[\int_0^4 |\rho^2-4|\rho d\rho\right] \\
&= 2\pi\left[\int_0^2 |\rho^2-4|\rho d\rho + \int_2^4 |\rho^2-4|\rho d\rho\right] = 2\pi\left[\int_0^2 (4-\rho^2)\rho d\rho + \int_2^4 (\rho^2-4)\rho d\rho\right] \\
&= 2\pi\left[\left(2\rho^2-\frac{1}{4}\rho^4\right)\big|_0^2 + \left(\frac{1}{4}\rho^4-2\rho^2\right)\big|_2^4\right] = 80\pi.
\end{aligned}
$$

练习 7. 计算 $D = \iint_D \arctan\frac{y}{x}d\sigma$，其中 D 是由圆周 $x^2+y^2=4$，$x^2+y^2=1$ 及直线 $y=0$，$y=x$ 所围成的在第一象限内的闭区域。

解在极坐标下 $D = \{(\rho,\,\theta)|\, 1 \le \rho \le 2,\, 0 \le \theta \le \frac{\pi}{4}\}$，$\arctan\frac{y}{x} = \theta$，所以

$$
\begin{aligned}
\iint_D \arctan\frac{y}{x}d\sigma &= \iint_D \theta\rho d\rho d\theta \\
&= \int_0^{\frac{1}{4}\pi}\left[\int_1^2 \rho\theta d\rho\right]d\theta = \int_0^{\frac{1}{4}\pi}\left(\frac{1}{2}\theta\rho^2\right)\big|_1^2 d\theta \\
&= \int_0^{\frac{1}{4}\pi}\frac{3}{2}\theta d\theta = \frac{3}{4}\theta^2\big|_0^{\frac{1}{4}\pi} = \frac{3}{64}\pi^2.
\end{aligned}
$$