第 03 周作业解答

练习 1. 设行列式 $D = \begin{vmatrix} 2 & -1 & 3 \\ 0 & 1 & 1 \\ -1 & -2 & 0 \end{vmatrix}$，求出其所有代数余子式 A_{ij}。

解

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 1 & 1 \\ -2 & 0 \end{vmatrix} = 2, \qquad A_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} = -1, \qquad A_{13} = (-1)^{1+3}\begin{vmatrix} 0 & 1 \\ -1 & -2 \end{vmatrix} = 1$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} -1 & 3 \\ -2 & 0 \end{vmatrix} = -6, \qquad A_{22} = (-1)^{2+2}\begin{vmatrix} 2 & 3 \\ -1 & 0 \end{vmatrix} = 3, \qquad A_{23} = (-1)^{2+3}\begin{vmatrix} 2 & -1 \\ -1 & -2 \end{vmatrix} = 5$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} -1 & 3 \\ 1 & 1 \end{vmatrix} = -4, \qquad A_{32} = (-1)^{3+2}\begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix} = -2, \qquad A_{33} = (-1)^{3+3}\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} = 2$$

练习 2. 利用降阶法求解行列式 $D_1 = \begin{vmatrix} 1 & 2 & -1 & 0 \\ -2 & 4 & 5 & -1 \\ 2 & 3 & 1 & 3 \\ 3 & 1 & -2 & 0 \end{vmatrix}$ 和 $D_2 = \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{vmatrix}$

解

$$D_1 = \begin{vmatrix} 1 & 2 & -1 & 0 \\ -2 & 4 & 5 & -1 \\ 2 & 3 & 1 & 3 \\ 3 & 1 & -2 & 0 \end{vmatrix} \xlongequal{r_3+3r_2} \begin{vmatrix} 1 & 2 & -1 & 0 \\ -2 & 4 & 5 & -1 \\ -4 & 15 & 16 & 0 \\ 3 & 1 & -2 & 0 \end{vmatrix} \xlongequal{\text{按第四列展开}} (-1)\cdot(-1)^{2+4}\cdot\begin{vmatrix} 1 & 2 & -1 \\ -4 & 15 & 16 \\ 3 & 1 & -2 \end{vmatrix}$$

$$\xlongequal[c_3+c_1]{c_2-2c_1} -\begin{vmatrix} 1 & 0 & 0 \\ -4 & 23 & 12 \\ 3 & -5 & 1 \end{vmatrix} = -\begin{vmatrix} 23 & 12 \\ -5 & 1 \end{vmatrix} = -83$$

$$D_2 = \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{vmatrix} \xlongequal{\text{依次将 2, 3, 4 列加到第 1 列}} \begin{vmatrix} 6 & 1 & 2 & 3 \\ 6 & 2 & 3 & 0 \\ 6 & 3 & 0 & 1 \\ 6 & 0 & 1 & 2 \end{vmatrix} = 6\begin{vmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \\ 1 & 3 & 0 & 1 \\ 1 & 0 & 1 & 2 \end{vmatrix}$$

$$\xlongequal[\substack{r_3-r_1 \\ r_4-r_1}]{r_2-r_1} 6\begin{vmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & -3 \\ 0 & 2 & -2 & -2 \\ 0 & -1 & -1 & -1 \end{vmatrix} = 6\begin{vmatrix} 1 & 1 & -3 \\ 2 & -2 & -2 \\ -1 & -1 & -1 \end{vmatrix} = 12\begin{vmatrix} 1 & 1 & -3 \\ 1 & -1 & -1 \\ -1 & -1 & -1 \end{vmatrix}$$

$$\xlongequal[r_3+r_1]{r_2-r_1} 12\begin{vmatrix} 1 & 1 & -3 \\ 0 & -2 & 2 \\ 0 & 0 & -4 \end{vmatrix} = 96$$

练习 3. 设 $D = \begin{vmatrix} 1 & 0 & 4 & 0 \\ 2 & -1 & -1 & 2 \\ 0 & -6 & 0 & 0 \\ 2 & 4 & -1 & 2 \end{vmatrix}$，求第四列各元素的余子式之和，即 $M_{14} + M_{24} + M_{34} + M_{44}$

解

$$
\begin{aligned}
M_{14}+M_{24}+M_{34}+M_{44} &= (-1)\cdot A_{14}+1\cdot A_{24}+(-1)\cdot A_{34}+1\cdot A_{44}\\
&= \begin{vmatrix} 1 & 0 & 4 & -1\\ 2 & -1 & -1 & 1\\ 0 & -6 & 0 & -1\\ 2 & 4 & -1 & 1 \end{vmatrix} \xlongequal[\substack{r_3+r_2\\ r_4-r_2}]{r_1+r_2} \begin{vmatrix} 3 & -1 & 3 & 0\\ 2 & -1 & -1 & 1\\ 2 & -7 & -1 & 0\\ 0 & 5 & 0 & 0 \end{vmatrix}\\
&\xlongequal{\text{按第四列展开}} 1\cdot(-1)^{2+4}\cdot \begin{vmatrix} 3 & -1 & 3\\ 2 & -7 & -1\\ 0 & 5 & 0 \end{vmatrix}\\
&\xlongequal{\text{按第三行展开}} 5\cdot(-1)^{3+2}\cdot \begin{vmatrix} 3 & 3\\ 2 & -1 \end{vmatrix} = 45
\end{aligned}
$$

练习 4. 如果齐次线性方程组 $\begin{cases} kx & +y & +z & =0\\ x & +ky & -z & =0\\ 2x & -y & +z & =0 \end{cases}$ 有非零解，k 应取什么值?

解系数行列式为

$$
D=\begin{vmatrix} k & 1 & 1\\ 1 & k & -1\\ 2 & -1 & 1 \end{vmatrix} \xlongequal[r_3-r_1]{r_2+r_1} \begin{vmatrix} k & 1 & 1\\ k+1 & k+1 & 0\\ 2-k & -2 & 0 \end{vmatrix} = (k+1)\begin{vmatrix} k & 1 & 1\\ 1 & 1 & 0\\ 2-k & -2 & 0 \end{vmatrix} = (k+1)\begin{vmatrix} 1 & 1\\ 2-k & -2 \end{vmatrix} = (k+1)(k-4)
$$

齐次线性方程组有非零解当且仅当 $D=0$，所以 $k=-1$ 或 $k=4$。