第 3 章 b：向量与向量组的线性组合

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20-2021 学年 I

向量

- n 维**行向量**

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- 行向量、列向量统称**向量**。
  根据上下文判断“向量”是行向量，或列向量
- **零向量** $O = (0, 0, \cdots, 0)$

向量的线性运算

- 设 $\alpha = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$，$\beta = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$，$k \in \mathbb{R}$，则

$$\alpha + \beta = \qquad\qquad\qquad, \quad \alpha - \beta = \qquad\qquad\qquad, \quad k\alpha =$$

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- 行向量类似

线性组合问题

- 给定向量组

$$\alpha_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \alpha_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \ldots, \alpha_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

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设 $k_1, k_2, \cdots, k_n$ 为任意数，则称

$$k_1\alpha_1 + k_2\alpha_2 + \cdots + k_n\alpha_n$$

为向量组 $\alpha_1, \alpha_2, \cdots, \alpha_n$ 的**线性组合**。

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- **问题** 给定向量 $\beta = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$，问 β 能否由 $\alpha_1, \alpha_2, \cdots, \alpha_n$ **线性表示**？

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即：是否存在数 $k_1, k_2, \ldots, k_n$ 使得：

$$\beta = k_1\alpha_1 + k_2\alpha_2 + \cdots + k_n\alpha_n\,?$$

线性组合问题

例 判断 β 能否由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式 $\beta = k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3$。

(1) 问

$$\begin{array}{cccc} \beta & \alpha_1 & \alpha_2 & \alpha_3 \\ \begin{pmatrix} 2 \\ -7 \\ 5 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \end{array}$$

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所以 $\beta = 2\alpha_1 - 7\alpha_2 + \frac{5}{2}\alpha_3$；

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$$\begin{array}{cccc} \beta & \alpha_1 & \alpha_2 & \alpha_3 \\ \begin{pmatrix} 2 \\ -7 \\ 5 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \end{array}$$

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所以 β 不能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示！

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$$\begin{array}{ccccccc} \beta & & \alpha_1 & & \alpha_2 & & \alpha_3 \\ \begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix} & = \underline{\quad} & \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix} & + \underline{\quad} & \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix} & + \underline{\quad} & \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix} \end{array}$$

即：β 能否由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示？如果能，线性表达式是什么？

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- 如果能线性表出，如何求出 $k_1, k_2, \ldots, k_n$ 使

$$k_1\alpha_1 + k_2\alpha_2 + \cdots + k_n\alpha_n = \beta\,?$$

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$$\overset{\beta}{\begin{pmatrix}2\\3\\0\\5\end{pmatrix}} = \_\_\overset{\alpha_1}{\begin{pmatrix}1\\0\\1\\2\end{pmatrix}} + \_\_\overset{\alpha_2}{\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}} + \_\_\overset{\alpha_3}{\begin{pmatrix}3\\2\\0\\1\end{pmatrix}}$$

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$$k_1\alpha_1 + k_2\alpha_2 + \cdots + k_n\alpha_n = \beta\ ?$$

不难看出，$k_1, \cdots, k_n$ 的求解可归结为线性方程组的求解。

$$
\begin{array}{ccccc}
\alpha_1 & \alpha_2 & & \alpha_n & \beta \\
\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} & \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} & \cdots & \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} & \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}
\end{array}
$$

β可由$\alpha_1, \alpha_2, \cdots, \alpha_n$线性表示

$$
\begin{array}{ccccc}
\alpha_1 & \alpha_2 & & \alpha_n & \beta \\
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$$

β可由$\alpha_1, \alpha_2, \cdots, \alpha_n$线性表示

$$\Longleftrightarrow \quad \begin{array}{cccccccc} & \alpha_1 & & \alpha_2 & & \alpha_n & & \beta \\ k_1 & \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} & + k_2 & \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} & + \cdots + k_n & \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} & = & \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \end{array}$$

β可由$\alpha_1, \alpha_2, \cdots, \alpha_n$线性表示

$$\Leftrightarrow \quad k_1 \begin{array}{c} \alpha_1 \\ \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} \end{array} + k_2 \begin{array}{c} \alpha_2 \\ \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} \end{array} + \cdots + k_n \begin{array}{c} \alpha_n \\ \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \end{array} = \begin{array}{c} \beta \\ \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \end{array}$$

$$\begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & & \alpha_n \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix}$$

β可由$\alpha_1, \alpha_2, \cdots, \alpha_n$线性表示

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$$\Leftrightarrow \quad \begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & & \alpha_n \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} = \begin{array}{c} \beta \\ \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \end{array}$$

β可由$\alpha_1, \alpha_2, \cdots, \alpha_n$线性表示

$$\Leftrightarrow \quad k_1 \overset{\alpha_1}{\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}} + k_2 \overset{\alpha_2}{\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}} + \cdots + k_n \overset{\alpha_n}{\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}} = \overset{\beta}{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}$$

$$\Leftrightarrow \quad \underbrace{\begin{array}{cccc} \alpha_1 & \alpha_2 & & \alpha_n \\ \multicolumn{4}{c}{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}} \end{array}}_{A} \underbrace{\begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix}}_{x} = \overset{\beta}{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}$$

β可由$\alpha_1, \alpha_2, \cdots, \alpha_n$线性表示

$$\Leftrightarrow \quad k_1 \begin{array}{c} \alpha_1 \\ \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} \end{array} + k_2 \begin{array}{c} \alpha_2 \\ \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} \end{array} + \cdots + k_n \begin{array}{c} \alpha_n \\ \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \end{array} = \begin{array}{c} \beta \\ \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \end{array}$$

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$$\Leftrightarrow \quad Ax = \beta \text{有解}$$

β可由$\alpha_1, \alpha_2, \cdots, \alpha_n$线性表示

$$\Leftrightarrow \quad k_1 \begin{array}{c} \alpha_1 \\ \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} \end{array} + k_2 \begin{array}{c} \alpha_2 \\ \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} \end{array} + \cdots + k_n \begin{array}{c} \alpha_n \\ \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \end{array} = \begin{array}{c} \beta \\ \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \end{array}$$

$$\Leftrightarrow \quad \underbrace{\begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & & \alpha_n \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array}}_{A} \underbrace{\begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix}}_{x} = \begin{array}{c} \beta \\ \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \end{array}$$

$\Leftrightarrow \quad Ax = \beta$有解 （$k_1, \cdots, k_n$是方程的解）

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$$\Leftrightarrow \quad k_1 \begin{array}{c} \alpha_1 \\ \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} \end{array} + k_2 \begin{array}{c} \alpha_2 \\ \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} \end{array} + \cdots + k_n \begin{array}{c} \alpha_n \\ \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \end{array} = \begin{array}{c} \beta \\ \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \end{array}$$

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$\Leftrightarrow \quad Ax = \beta$有解　　（$k_1, \cdots, k_n$是方程的解）

$\Leftrightarrow \quad r(A) = r(A \vdots \beta)$

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$$\Leftrightarrow \quad k_1 \begin{array}{c} \alpha_1 \\ \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} \end{array} + k_2 \begin{array}{c} \alpha_2 \\ \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} \end{array} + \cdots + k_n \begin{array}{c} \alpha_n \\ \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \end{array} = \begin{array}{c} \beta \\ \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \end{array}$$

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$\Leftrightarrow \quad Ax = \beta$有解 （$k_1, \cdots, k_n$是方程的解）

$\Leftrightarrow \quad r(A) = r(A \vdots \beta) \quad \Leftrightarrow \quad (\alpha_1 \alpha_2 \cdots \alpha_n) \quad (\alpha_1 \alpha_2 \cdots \alpha_n \beta)$

β可由$\alpha_1, \alpha_2, \cdots, \alpha_n$线性表示

$$\Leftrightarrow \quad k_1 \begin{array}{c} \alpha_1 \\ \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} \end{array} + k_2 \begin{array}{c} \alpha_2 \\ \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} \end{array} + \cdots + k_n \begin{array}{c} \alpha_n \\ \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \end{array} = \begin{array}{c} \beta \\ \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \end{array}$$

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$\Leftrightarrow \quad Ax = \beta$有解 （$k_1, \cdots, k_n$是方程的解）

$$\Leftrightarrow \quad r(A) = r(A \vdots \beta) \quad \Leftrightarrow \quad r(\alpha_1 \alpha_2 \cdots \alpha_n) = r(\alpha_1 \alpha_2 \cdots \alpha_n \beta)$$

β可由$\alpha_1, \alpha_2, \cdots, \alpha_n$线性表示

$$\Leftrightarrow \quad k_1 \begin{array}{c} \alpha_1 \\ \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} \end{array} + k_2 \begin{array}{c} \alpha_2 \\ \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} \end{array} + \cdots + k_n \begin{array}{c} \alpha_n \\ \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \end{array} = \begin{array}{c} \beta \\ \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \end{array}$$

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$\Leftrightarrow \quad Ax = \beta$有解　　（$k_1, \cdots, k_n$是方程的解）

$$\Leftrightarrow \quad r(A) = r(A \vdots \beta) \quad \Leftrightarrow \quad r(\alpha_1\alpha_2\cdots\alpha_n) = r(\alpha_1\alpha_2\cdots\alpha_n\beta)$$

定理 β 可由 $\alpha_1, \alpha_2, \cdots, \alpha_n$ **线性表示** $\Leftrightarrow r(\alpha_1\alpha_2\cdots\alpha_n) = r(\alpha_1\alpha_2\cdots\alpha_n\beta)$

β可由$\alpha_1,\alpha_2,\cdots,\alpha_n$线性表示

$$\Leftrightarrow \quad k_1 \begin{array}{c} \alpha_1 \\ \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} \end{array} + k_2 \begin{array}{c} \alpha_2 \\ \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} \end{array} + \cdots + k_n \begin{array}{c} \alpha_n \\ \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \end{array} = \begin{array}{c} \beta \\ \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \end{array}$$

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$\Leftrightarrow \quad Ax=\beta$有解 （$k_1,\cdots,k_n$是方程的解）

$\Leftrightarrow \quad r(A)=r(A\vdots\beta) \quad \Leftrightarrow \quad r(\alpha_1\alpha_2\cdots\alpha_n)=r(\alpha_1\alpha_2\cdots\alpha_n\beta)$

定理 β 可由 $\alpha_1,\alpha_2,\cdots,\alpha_n$ **线性表示** $\Leftrightarrow r(\alpha_1\alpha_2\cdots\alpha_n)=r(\alpha_1\alpha_2\cdots\alpha_n\beta)$

注 实际中，$k_1,\cdots,k_n$ 的求解不需要特意解方程 $Ax=\beta$，方法见下例

初等行变换求线性表示问题——例 1

例 判断 β 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

(1)

$$(\,\alpha_1\ \ \alpha_2\ \ \alpha_3 \,|\, \beta\,) = \left(\begin{array}{rrr|r} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{array}\right)$$

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- $r(\alpha_1\alpha_2\alpha_3) = \quad, \ r(\alpha_1\alpha_2\alpha_3\beta) = \quad,$

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- $r(\alpha_1\alpha_2\alpha_3) = \quad$，$r(\alpha_1\alpha_2\alpha_3\beta) = \quad$，

初等行变换求线性表示问题——例 1

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- $r(\alpha_1\alpha_2\alpha_3) =$ ，$r(\alpha_1\alpha_2\alpha_3\beta) = 3$，

初等行变换求线性表示问题——例 1

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- 所以 $r(\alpha_1\alpha_2\alpha_3) = 3$，$r(\alpha_1\alpha_2\alpha_3\beta) = 3$，

初等行变换求线性表示问题——例 1

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- 所以 $r(\alpha_1\alpha_2\alpha_3) = 3$，$r(\alpha_1\alpha_2\alpha_3\beta) = 3$，成立

$$r(\alpha_1\alpha_2\alpha_3) = r(\alpha_1\alpha_2\alpha_3\beta)$$

  β 可由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示。

- 显然 $\beta' = \alpha_1' - \alpha_2' + \alpha_3'$，

初等行变换求线性表示问题——例 1

例 判断 β 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

(1)

$$(\alpha_1\ \ \alpha_2\ \ \alpha_3 \mid \beta) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{array}\right) \xrightarrow{\text{初等行变换}} \begin{array}{cccc} \alpha_1' & \alpha_2' & \alpha_3' & \beta' \\ \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

- 所以 $r(\alpha_1\alpha_2\alpha_3) = 3$，$r(\alpha_1\alpha_2\alpha_3\beta) = 3$，成立

$$r(\alpha_1\alpha_2\alpha_3) = r(\alpha_1\alpha_2\alpha_3\beta)$$

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初等行变换求线性表示问题——例 1

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- 所以 $r(\alpha_1\alpha_2\alpha_3) = 3$，$r(\alpha_1\alpha_2\alpha_3\beta) = 3$，成立

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初等行变换求线性表示问题——例 1

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- 所以 $r(\alpha_1\alpha_2\alpha_3) = 3$，$r(\alpha_1\alpha_2\alpha_3\beta) = 3$，成立

$$r(\alpha_1\alpha_2\alpha_3) = r(\alpha_1\alpha_2\alpha_3\beta)$$

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注 可证明：作初等行变换不改变列与列之间的“线性关系”。

初等行变换求线性表示问题——例 2

例 判断 β 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

(2)

$$(\alpha_1 \ \ \alpha_2 \ \ \alpha_3 \mid \beta) = \left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 2 & -1 & 3 & 3 \\ -1 & 1 & -2 & 0 \\ 5 & 1 & 4 & 11 \end{array}\right)$$

初等行变换求线性表示问题——例 2

例 判断 β 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

(2)

$$(\,\alpha_1\ \ \alpha_2\ \ \alpha_3 \,|\, \beta\,) = \left(\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 2 & -1 & 3 & 3 \\ -1 & 1 & -2 & 0 \\ 5 & 1 & 4 & 11 \end{array}\right)$$

- $r(\alpha_1\alpha_2\alpha_3) = \quad$，$r(\alpha_1\alpha_2\alpha_3\beta) = \quad$，

初等行变换求线性表示问题——例 2

例 判断 β 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

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$$(\,\alpha_1 \;\; \alpha_2 \;\; \alpha_3 \,|\, \beta\,) = \left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 2 & -1 & 3 & 3 \\ -1 & 1 & -2 & 0 \\ 5 & 1 & 4 & 11 \end{array}\right) \xrightarrow{\text{初等行变换}} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

- $r(\alpha_1\alpha_2\alpha_3) = \quad$ ， $r(\alpha_1\alpha_2\alpha_3\beta) = \quad$ ，

初等行变换求线性表示问题——例 2

例 判断 β 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

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- $r(\alpha_1\alpha_2\alpha_3) =$ ， $r(\alpha_1\alpha_2\alpha_3\beta) =$ ，

初等行变换求线性表示问题——例 2

例 判断 β 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

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- $r(\alpha_1\alpha_2\alpha_3) = \quad$ ，$r(\alpha_1\alpha_2\alpha_3\beta) = 3$，

初等行变换求线性表示问题——例 2

例 判断 β 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

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$$(\,\alpha_1 \;\; \alpha_2 \;\; \alpha_3 \,|\, \beta\,) = \left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 2 & -1 & 3 & 3 \\ -1 & 1 & -2 & 0 \\ 5 & 1 & 4 & 11 \end{array}\right) \xrightarrow{\text{初等行变换}} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

- $r(\alpha_1\alpha_2\alpha_3) = 2$，$r(\alpha_1\alpha_2\alpha_3\beta) = 3$，

初等行变换求线性表示问题——例 2

例 判断 β 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

(2)

$$(\alpha_1 \ \ \alpha_2 \ \ \alpha_3 \,|\, \beta) = \left(\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 2 & -1 & 3 & 3 \\ -1 & 1 & -2 & 0 \\ 5 & 1 & 4 & 11 \end{array}\right) \xrightarrow{\text{初等行变换}} \left(\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

- 所以 $r(\alpha_1\alpha_2\alpha_3) = 2$，$r(\alpha_1\alpha_2\alpha_3\beta) = 3$，成立

$$r(\alpha_1\alpha_2\alpha_3) \neq r(\alpha_1\alpha_2\alpha_3\beta)$$

初等行变换求线性表示问题——例 2

例 判断 β 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

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$$(\,\alpha_1\ \ \alpha_2\ \ \alpha_3 \,|\, \beta\,) = \left(\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 2 & -1 & 3 & 3 \\ -1 & 1 & -2 & 0 \\ 5 & 1 & 4 & 11 \end{array}\right) \xrightarrow{\text{初等行变换}} \left(\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

- 所以 $r(\alpha_1\alpha_2\alpha_3) = 2$，$r(\alpha_1\alpha_2\alpha_3\beta) = 3$，成立

$$r(\alpha_1\alpha_2\alpha_3) \neq r(\alpha_1\alpha_2\alpha_3\beta)$$

β 不能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示。

初等行变换求线性表示问题——总结

问题 β 能否由 $\alpha_1, \alpha_2, \ldots, \alpha_n$ 线性表示？若能，写出线性表示等式。

步骤

初等行变换求线性表示问题——总结

问题 β 能否由 $\alpha_1, \alpha_2, ..., \alpha_n$ 线性表示？若能，写出线性表示等式。

步骤

$$\left(\begin{array}{cccc|c} \alpha_1 & \alpha_2 & \cdots & \alpha_n & \beta \end{array}\right)$$

初等行变换求线性表示问题——总结

问题 β 能否由 $\alpha_1, \alpha_2, \ldots, \alpha_n$ 线性表示？若能，写出线性表示等式。

步骤

$$\left(\alpha_1 \ \ \alpha_2 \ \ \cdots \ \ \alpha_n \,\middle|\, \beta \right) \xrightarrow{\text{初等行变换}}$$

初等行变换求线性表示问题——总结

问题 β 能否由 $\alpha_1, \alpha_2, \ldots, \alpha_n$ 线性表示？若能，写出线性表示等式。

步骤

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \mid \beta \end{pmatrix} \xrightarrow{\text{初等行变换}} \begin{pmatrix} \alpha_1' & \alpha_2' & \cdots & \alpha_n' \mid \beta' \end{pmatrix}_{\substack{\text{(简化)} \\ \text{阶梯型矩阵}}}$$

初等行变换求线性表示问题——总结

问题 β 能否由 $\alpha_1, \alpha_2, \ldots, \alpha_n$ 线性表示？若能，写出线性表示等式。

步骤

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n & | & \beta \end{pmatrix} \xrightarrow{\text{初等行变换}} \begin{pmatrix} \alpha_1' & \alpha_2' & \cdots & \alpha_n' & | & \beta' \end{pmatrix}_{\text{阶梯型矩阵}}^{\text{(简化)}}$$

1. β 由 $\alpha_1, \alpha_2, \ldots, \alpha_n$ 线性表示 $\Leftrightarrow r(\alpha_1 \cdots \alpha_n) = r(\alpha_1 \cdots \alpha_n\, \beta)$.

初等行变换求线性表示问题——总结

问题 β 能否由 $\alpha_1, \alpha_2, \dots, \alpha_n$ 线性表示？若能，写出线性表示等式。

步骤

$$\left(\alpha_1 \ \ \alpha_2 \ \ \cdots \ \ \alpha_n \,\middle|\, \beta\right) \xrightarrow{\text{初等行变换}} \left(\alpha_1' \ \ \alpha_2' \ \ \cdots \ \ \alpha_n' \,\middle|\, \beta'\right)_{\text{阶梯型矩阵}}^{\text{(简化)}}$$

1. β 由 $\alpha_1, \alpha_2, \dots, \alpha_n$ 线性表示 $\Leftrightarrow r(\alpha_1 \cdots \alpha_n) = r(\alpha_1 \cdots \alpha_n\, \beta)$.

   也就是，在阶梯型矩阵中

$$(\alpha_1'\alpha_2'\cdots\alpha_n') \text{ 非零行数} = (\alpha_1'\alpha_2'\cdots\alpha_n' \,|\, \beta) \text{ 非零行数}$$

初等行变换求线性表示问题——总结

问题 β 能否由 $\alpha_1, \alpha_2, \ldots, \alpha_n$ 线性表示？若能，写出线性表示等式。

步骤

$$(\,\alpha_1\ \ \alpha_2\ \ \cdots\ \ \alpha_n \,|\, \beta\,) \xrightarrow{\text{初等行变换}} (\,\alpha_1'\ \ \alpha_2'\ \ \cdots\ \ \alpha_n' \,|\, \beta'\,)_{\substack{\text{(简化)}\\ \text{阶梯型矩阵}}}$$

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$$(\alpha_1'\alpha_2'\cdots\alpha_n')\ \text{非零行数} = (\alpha_1'\alpha_2'\cdots\alpha_n' \,|\, \beta)\ \text{非零行数}$$

2. 行变换前后列与列的线性关系不变，即：

初等行变换求线性表示问题——总结

问题 β 能否由 $\alpha_1, \alpha_2, \ldots, \alpha_n$ 线性表示？若能，写出线性表示等式。

步骤

$$\left(\,\alpha_1 \;\; \alpha_2 \;\; \cdots \;\; \alpha_n \,\middle|\, \beta\,\right) \xrightarrow{\text{初等行变换}} \left(\,\alpha_1' \;\; \alpha_2' \;\; \cdots \;\; \alpha_n' \,\middle|\, \beta'\,\right)_{\substack{\text{(简化)}\\ \text{阶梯型矩阵}}}$$

1. β 由 $\alpha_1, \alpha_2, \ldots, \alpha_n$ 线性表示 $\Leftrightarrow r(\alpha_1 \cdots \alpha_n) = r(\alpha_1 \cdots \alpha_n\, \beta)$.

   也就是，在阶梯型矩阵中

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2. 行变换前后列与列的线性关系不变，即：

   $$\beta' = k_1 \alpha_1' + \cdots + k_n \alpha_n' \quad \Rightarrow$$

初等行变换求线性表示问题——总结

问题 β 能否由 $\alpha_1, \alpha_2, \ldots, \alpha_n$ 线性表示？若能，写出线性表示等式。

步骤

$$\left(\ \alpha_1\ \ \alpha_2\ \ \cdots\ \ \alpha_n \,\middle|\, \beta\ \right) \xrightarrow{\text{初等行变换}} \left(\ \alpha_1'\ \ \alpha_2'\ \ \cdots\ \ \alpha_n' \,\middle|\, \beta'\ \right)_{\substack{\text{(简化)}\\\text{阶梯型矩阵}}}$$

1. β 由 $\alpha_1, \alpha_2, \ldots, \alpha_n$ 线性表示 $\Leftrightarrow r(\alpha_1 \cdots \alpha_n) = r(\alpha_1 \cdots \alpha_n\, \beta)$.

   也就是，在阶梯型矩阵中

$$(\alpha_1'\alpha_2'\cdots\alpha_n')\ \text{非零行数} = (\alpha_1'\alpha_2'\cdots\alpha_n' \,|\, \beta)\ \text{非零行数}$$

2. 行变换前后列与列的线性关系不变，即：

$$\beta' = k_1\alpha_1' + \cdots + k_n\alpha_n' \quad \Rightarrow \quad \beta = k_1\alpha_1 + \cdots + k_n\alpha_n$$

例 1 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示?

解

例 1 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示?

解

$$\begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \beta \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{array}\right)$$

例 1 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示?

解

$$\begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \beta \end{array} \\ \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{array}\right) \end{array} \xrightarrow[r_4-2r_1]{r_3-r_1}$$

例 1 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示?

解

$$\begin{array}{c}\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array}\\ \left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\ 0 & -1 & 2 & 3\\ 1 & 1 & 0 & 0\\ 2 & -2 & 1 & 5\end{array}\right)\end{array}\xrightarrow[r_4-2r_1]{r_3-r_1}\left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\ 0 & -1 & 2 & 3\\ & & & \\ & & & \end{array}\right)$$

例 1 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示?

解

$$\begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \beta \end{array} \\ \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{array}\right) \end{array} \xrightarrow[r_4-2r_1]{r_3-r_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 0 & -1 & -3 & -2 \\ & & & \end{array}\right)$$

例 1 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示?

解

$$\begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \beta \end{array} \\ \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{array}\right) \end{array} \xrightarrow[r_4-2r_1]{r_3-r_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 0 & -1 & -3 & -2 \\ 0 & -6 & -5 & 1 \end{array}\right)$$

例 1 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示?

解

$$\begin{array}{c} \quad\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \beta \\ \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{array}\right) \end{array} \xrightarrow[r_4-2r_1]{r_3-r_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 0 & -1 & -3 & -2 \\ 0 & -6 & -5 & 1 \end{array}\right) \xrightarrow{(-1)\times r_2}$$

例 1 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示?

解

$$\begin{array}{c} \begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array} \\ \left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\0 & -1 & 2 & 3\\1 & 1 & 0 & 0\\2 & -2 & 1 & 5\end{array}\right)\end{array} \xrightarrow[r_4-2r_1]{r_3-r_1} \left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\0 & -1 & 2 & 3\\0 & -1 & -3 & -2\\0 & -6 & -5 & 1\end{array}\right) \xrightarrow{(-1)\times r_2} \left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\0 & 1 & -2 & -3\\0 & -1 & -3 & -2\\0 & -6 & -5 & 1\end{array}\right)$$

例 1 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示?

解

$$\begin{array}{c}\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array}\\ \left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\1&1&0&0\\2&-2&1&5\end{array}\right)\end{array}\xrightarrow[r_4-2r_1]{r_3-r_1}\left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)\xrightarrow{(-1)\times r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)$$

$$\xrightarrow[r_4+6r_2]{r_3+r_2}$$

例 1 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示?

解

$$\begin{array}{c}\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array}\\ \left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\1&1&0&0\\2&-2&1&5\end{array}\right)\end{array}\xrightarrow[r_4-2r_1]{r_3-r_1}\left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)\xrightarrow{(-1)\times r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)$$

$$\xrightarrow[r_4+6r_2]{r_3+r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\&&&\\&&&\end{array}\right)$$

例 1 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示?

解 $\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \beta$

$$\left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\1&1&0&0\\2&-2&1&5\end{array}\right)\xrightarrow[r_4-2r_1]{r_3-r_1}\left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)\xrightarrow{(-1)\times r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)$$

$$\xrightarrow[r_4+6r_2]{r_3+r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&-5&-5\\&&&\end{array}\right)$$

例 1 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示?

解

$$\begin{array}{c}\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array}\\ \left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\0 & -1 & 2 & 3\\1 & 1 & 0 & 0\\2 & -2 & 1 & 5\end{array}\right)\end{array}\xrightarrow[r_4-2r_1]{r_3-r_1}\left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\0 & -1 & 2 & 3\\0 & -1 & -3 & -2\\0 & -6 & -5 & 1\end{array}\right)\xrightarrow{(-1)\times r_2}\left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\0 & 1 & -2 & -3\\0 & -1 & -3 & -2\\0 & -6 & -5 & 1\end{array}\right)$$

$$\xrightarrow[r_4+6r_2]{r_3+r_2}\left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\0 & 1 & -2 & -3\\0 & 0 & -5 & -5\\0 & 0 & -17 & -17\end{array}\right)$$

例 1 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示?

解

$$\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array}$$

$$\left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\1&1&0&0\\2&-2&1&5\end{array}\right)\xrightarrow[r_4-2r_1]{r_3-r_1}\left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)\xrightarrow{(-1)\times r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)$$

$$\xrightarrow[r_4+6r_2]{r_3+r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&-5&-5\\0&0&-17&-17\end{array}\right)\longrightarrow\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&1&1\\0&0&1&1\end{array}\right)$$

例 1 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示?

解

$$\begin{array}{c}\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array}\\ \left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\1&1&0&0\\2&-2&1&5\end{array}\right)\end{array}\xrightarrow[r_4-2r_1]{r_3-r_1}\left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)\xrightarrow{(-1)\times r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)$$

$$\xrightarrow[r_4+6r_2]{r_3+r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&-5&-5\\0&0&-17&-17\end{array}\right)\longrightarrow\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&1&1\\0&0&1&1\end{array}\right)\xrightarrow{r_4-r_3}$$

例 1 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示?

解

$$\begin{array}{c}\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array}\\ \left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\ 0 & -1 & 2 & 3\\ 1 & 1 & 0 & 0\\ 2 & -2 & 1 & 5\end{array}\right)\end{array}\xrightarrow[r_4-2r_1]{r_3-r_1}\left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\ 0 & -1 & 2 & 3\\ 0 & -1 & -3 & -2\\ 0 & -6 & -5 & 1\end{array}\right)\xrightarrow{(-1)\times r_2}\left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\ 0 & 1 & -2 & -3\\ 0 & -1 & -3 & -2\\ 0 & -6 & -5 & 1\end{array}\right)$$

$$\xrightarrow[r_4+6r_2]{r_3+r_2}\left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\ 0 & 1 & -2 & -3\\ 0 & 0 & -5 & -5\\ 0 & 0 & -17 & -17\end{array}\right)\longrightarrow\left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\ 0 & 1 & -2 & -3\\ 0 & 0 & 1 & 1\\ 0 & 0 & 1 & 1\end{array}\right)\xrightarrow{r_4-r_3}\left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\ 0 & 1 & -2 & -3\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0\end{array}\right)$$

例 1 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示?

解

$$\begin{array}{c}\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array}\\ \left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\1&1&0&0\\2&-2&1&5\end{array}\right)\end{array}\xrightarrow[r_4-2r_1]{r_3-r_1}\left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)\xrightarrow{(-1)\times r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)$$

$$\xrightarrow[r_4+6r_2]{r_3+r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&-5&-5\\0&0&-17&-17\end{array}\right)\longrightarrow\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&1&1\\0&0&1&1\end{array}\right)\xrightarrow{r_4-r_3}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&1&1\\0&0&0&0\end{array}\right)$$

$$\xrightarrow[r_1-3r_3]{r_2-2r_3}$$

例 1 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示?

解

$$\begin{array}{c}\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array}\\ \left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\1&1&0&0\\2&-2&1&5\end{array}\right)\end{array}\xrightarrow[r_4-2r_1]{r_3-r_1}\left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)\xrightarrow{(-1)\times r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)$$

$$\xrightarrow[r_4+6r_2]{r_3+r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&-5&-5\\0&0&-17&-17\end{array}\right)\longrightarrow\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&1&1\\0&0&1&1\end{array}\right)\xrightarrow{r_4-r_3}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&1&1\\0&0&0&0\end{array}\right)$$

$$\xrightarrow[r_1-3r_3]{r_2-2r_3}\left(\begin{array}{ccc|c}&&&\\&&&\\0&0&1&1\\0&0&0&0\end{array}\right)$$

例 1 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示?

解

$$\begin{array}{c}\alpha_1\quad\alpha_2\quad\alpha_3\quad\beta\\ \left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\1&1&0&0\\2&-2&1&5\end{array}\right)\end{array}\xrightarrow[r_4-2r_1]{r_3-r_1}\left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)\xrightarrow{(-1)\times r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)$$

$$\xrightarrow[r_4+6r_2]{r_3+r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&-5&-5\\0&0&-17&-17\end{array}\right)\longrightarrow\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&1&1\\0&0&1&1\end{array}\right)\xrightarrow{r_4-r_3}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&1&1\\0&0&0&0\end{array}\right)$$

$$\xrightarrow[r_1-3r_3]{r_2-2r_3}\left(\begin{array}{ccc|c}0&1&0&-1\\0&0&1&1\\0&0&0&0\end{array}\right)$$

例 1 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示?

解

$$\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array}$$

$$\left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\0 & -1 & 2 & 3\\1 & 1 & 0 & 0\\2 & -2 & 1 & 5\end{array}\right)\xrightarrow[r_4-2r_1]{r_3-r_1}\left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\0 & -1 & 2 & 3\\0 & -1 & -3 & -2\\0 & -6 & -5 & 1\end{array}\right)\xrightarrow{(-1)\times r_2}\left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\0 & 1 & -2 & -3\\0 & -1 & -3 & -2\\0 & -6 & -5 & 1\end{array}\right)$$

$$\xrightarrow[r_4+6r_2]{r_3+r_2}\left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\0 & 1 & -2 & -3\\0 & 0 & -5 & -5\\0 & 0 & -17 & -17\end{array}\right)\longrightarrow\left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\0 & 1 & -2 & -3\\0 & 0 & 1 & 1\\0 & 0 & 1 & 1\end{array}\right)\xrightarrow{r_4-r_3}\left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\0 & 1 & -2 & -3\\0 & 0 & 1 & 1\\0 & 0 & 0 & 0\end{array}\right)$$

$$\xrightarrow[r_1-3r_3]{r_2-2r_3}\left(\begin{array}{ccc|c}1 & 2 & 0 & -1\\0 & 1 & 0 & -1\\0 & 0 & 1 & 1\\0 & 0 & 0 & 0\end{array}\right)$$

例 1 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示?

解

$$\begin{array}{c} \begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array} \\ \left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\0 & -1 & 2 & 3\\1 & 1 & 0 & 0\\2 & -2 & 1 & 5\end{array}\right)\end{array} \xrightarrow[r_4-2r_1]{r_3-r_1} \left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\0 & -1 & 2 & 3\\0 & -1 & -3 & -2\\0 & -6 & -5 & 1\end{array}\right) \xrightarrow{(-1)\times r_2} \left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\0 & 1 & -2 & -3\\0 & -1 & -3 & -2\\0 & -6 & -5 & 1\end{array}\right)$$

$$\xrightarrow[r_4+6r_2]{r_3+r_2} \left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\0 & 1 & -2 & -3\\0 & 0 & -5 & -5\\0 & 0 & -17 & -17\end{array}\right) \longrightarrow \left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\0 & 1 & -2 & -3\\0 & 0 & 1 & 1\\0 & 0 & 1 & 1\end{array}\right) \xrightarrow{r_4-r_3} \left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\0 & 1 & -2 & -3\\0 & 0 & 1 & 1\\0 & 0 & 0 & 0\end{array}\right)$$

$$\xrightarrow[r_1-3r_3]{r_2-2r_3} \left(\begin{array}{ccc|c}1 & 2 & 0 & -1\\0 & 1 & 0 & -1\\0 & 0 & 1 & 1\\0 & 0 & 0 & 0\end{array}\right) \xrightarrow{r_1-2r_2}$$

例 1 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示?

解 $\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \beta$

$$\left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\1&1&0&0\\2&-2&1&5\end{array}\right)\xrightarrow[r_4-2r_1]{r_3-r_1}\left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)\xrightarrow{(-1)\times r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)$$

$$\xrightarrow[r_4+6r_2]{r_3+r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&-5&-5\\0&0&-17&-17\end{array}\right)\longrightarrow\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&1&1\\0&0&1&1\end{array}\right)\xrightarrow{r_4-r_3}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&1&1\\0&0&0&0\end{array}\right)$$

$$\xrightarrow[r_1-3r_3]{r_2-2r_3}\left(\begin{array}{ccc|c}1&2&0&-1\\0&1&0&-1\\0&0&1&1\\0&0&0&0\end{array}\right)\xrightarrow{r_1-2r_2}\left(\begin{array}{ccc|c}1&0&0&1\\0&1&0&-1\\0&0&1&1\\0&0&0&0\end{array}\right)$$

例 1 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示?

解 $\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \beta$

$$\left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\1&1&0&0\\2&-2&1&5\end{array}\right)\xrightarrow[r_4-2r_1]{r_3-r_1}\left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)\xrightarrow{(-1)\times r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)$$

$$\xrightarrow[r_4+6r_2]{r_3+r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&-5&-5\\0&0&-17&-17\end{array}\right)\longrightarrow\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&1&1\\0&0&1&1\end{array}\right)\xrightarrow{r_4-r_3}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&1&1\\0&0&0&0\end{array}\right)$$

$$\xrightarrow[r_1-3r_3]{r_2-2r_3}\left(\begin{array}{ccc|c}1&2&0&-1\\0&1&0&-1\\0&0&1&1\\0&0&0&0\end{array}\right)\xrightarrow{r_1-2r_2}\left(\begin{array}{ccc|c}1&0&0&1\\0&1&0&-1\\0&0&1&1\\0&0&0&0\end{array}\right)$$

例 1 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示?

解

$$\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array}$$

$$\left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\1&1&0&0\\2&-2&1&5\end{array}\right)\xrightarrow[r_4-2r_1]{r_3-r_1}\left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)\xrightarrow{(-1)\times r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)$$

$$\xrightarrow[r_4+6r_2]{r_3+r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&-5&-5\\0&0&-17&-17\end{array}\right)\longrightarrow\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&1&1\\0&0&1&1\end{array}\right)\xrightarrow{r_4-r_3}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&1&1\\0&0&0&0\end{array}\right)$$

$$\xrightarrow[r_1-3r_3]{r_2-2r_3}\left(\begin{array}{ccc|c}1&2&0&-1\\0&1&0&-1\\0&0&1&1\\0&0&0&0\end{array}\right)\xrightarrow{r_1-2r_2}\left(\begin{array}{ccc|c}1&0&0&1\\0&1&0&-1\\0&0&1&1\\0&0&0&0\end{array}\right)$$

所以 $r(\alpha_1\alpha_2\alpha_3)=r(\alpha_1\alpha_2\alpha_3\beta)$,

例 1 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示?

解

$$\begin{array}{c}\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array}\\ \left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\1&1&0&0\\2&-2&1&5\end{array}\right)\end{array}\xrightarrow[r_4-2r_1]{r_3-r_1}\left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)\xrightarrow{(-1)\times r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)$$

$$\xrightarrow[r_4+6r_2]{r_3+r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&-5&-5\\0&0&-17&-17\end{array}\right)\longrightarrow\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&1&1\\0&0&1&1\end{array}\right)\xrightarrow{r_4-r_3}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&1&1\\0&0&0&0\end{array}\right)$$

$$\xrightarrow[r_1-3r_3]{r_2-2r_3}\left(\begin{array}{ccc|c}1&2&0&-1\\0&1&0&-1\\0&0&1&1\\0&0&0&0\end{array}\right)\xrightarrow{r_1-2r_2}\left(\begin{array}{ccc|c}1&0&0&1\\0&1&0&-1\\0&0&1&1\\0&0&0&0\end{array}\right)$$

所以 $r(\alpha_1\alpha_2\alpha_3)=r(\alpha_1\alpha_2\alpha_3\beta)$，能线性表示

例 1 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示?

解

$$\begin{array}{c}\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array}\\ \left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\1&1&0&0\\2&-2&1&5\end{array}\right)\end{array}\xrightarrow[r_4-2r_1]{r_3-r_1}\left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)\xrightarrow{(-1)\times r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)$$

$$\xrightarrow[r_4+6r_2]{r_3+r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&-5&-5\\0&0&-17&-17\end{array}\right)\longrightarrow\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&1&1\\0&0&1&1\end{array}\right)\xrightarrow{r_4-r_3}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&1&1\\0&0&0&0\end{array}\right)$$

$$\xrightarrow[r_1-3r_3]{r_2-2r_3}\left(\begin{array}{ccc|c}1&2&0&-1\\0&1&0&-1\\0&0&1&1\\0&0&0&0\end{array}\right)\xrightarrow{r_1-2r_2}\left(\begin{array}{ccc|c}1&0&0&1\\0&1&0&-1\\0&0&1&1\\0&0&0&0\end{array}\right)$$

所以 $r(\alpha_1\alpha_2\alpha_3)=r(\alpha_1\alpha_2\alpha_3\beta)$，能线性表示，且 $\beta=\alpha_1-\alpha_2+\alpha_3$

例 2 $\beta=\begin{pmatrix}1\\2\\-1\\6\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}2\\-1\\1\\1\end{pmatrix}$, $\alpha_2=\begin{pmatrix}-1\\3\\-2\\4\end{pmatrix}$, $\alpha_3=\begin{pmatrix}4\\3\\0\\11\end{pmatrix}$线性表示?

解

例 2 $\beta=\begin{pmatrix}1\\2\\-1\\6\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}2\\-1\\1\\1\end{pmatrix}$, $\alpha_2=\begin{pmatrix}-1\\3\\-2\\4\end{pmatrix}$, $\alpha_3=\begin{pmatrix}4\\3\\0\\11\end{pmatrix}$线性表示?

解

$$\begin{array}{c}
\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array}\\
\left(\begin{array}{ccc|c}2 & -1 & 4 & 1\\ -1 & 3 & 3 & 2\\ 1 & -2 & 0 & -1\\ 1 & 4 & 11 & 6\end{array}\right)
\end{array}$$

例 2 $\beta=\begin{pmatrix}1\\2\\-1\\6\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}2\\-1\\1\\1\end{pmatrix}$, $\alpha_2=\begin{pmatrix}-1\\3\\-2\\4\end{pmatrix}$, $\alpha_3=\begin{pmatrix}4\\3\\0\\11\end{pmatrix}$线性表示?

解

$$\begin{array}{c} \begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array} \\ \left(\begin{array}{ccc|c} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{array}\right) \end{array} \xrightarrow{r_1 \leftrightarrow r_3}$$

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解

$$\begin{array}{c}\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array}\\ \left(\begin{array}{ccc|c}2 & -1 & 4 & 1\\ -1 & 3 & 3 & 2\\ 1 & -2 & 0 & -1\\ 1 & 4 & 11 & 6\end{array}\right)\end{array} \xrightarrow{r_1\leftrightarrow r_3} \left(\begin{array}{ccc|c}1 & -2 & 0 & -1\\ -1 & 3 & 3 & 2\\ 2 & -1 & 4 & 1\\ 1 & 4 & 11 & 6\end{array}\right)$$

例 2 $\beta=\begin{pmatrix}1\\2\\-1\\6\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}2\\-1\\1\\1\end{pmatrix}$, $\alpha_2=\begin{pmatrix}-1\\3\\-2\\4\end{pmatrix}$, $\alpha_3=\begin{pmatrix}4\\3\\0\\11\end{pmatrix}$线性表示?

解

$$\begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \beta \end{array} \\ \left(\begin{array}{ccc|c} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{array}\right) \end{array} \xrightarrow{r_1\leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ -1 & 3 & 3 & 2 \\ 2 & -1 & 4 & 1 \\ 1 & 4 & 11 & 6 \end{array}\right)$$

$$\xrightarrow[\substack{r_3-2r_1\\ r_4-r_1}]{r_2+r_1}$$

例 2 $\beta=\begin{pmatrix}1\\2\\-1\\6\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}2\\-1\\1\\1\end{pmatrix}$, $\alpha_2=\begin{pmatrix}-1\\3\\-2\\4\end{pmatrix}$, $\alpha_3=\begin{pmatrix}4\\3\\0\\11\end{pmatrix}$线性表示?

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$$\begin{array}{c} \begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array} \\ \left(\begin{array}{ccc|c} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{array}\right) \end{array} \xrightarrow{r_1\leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ -1 & 3 & 3 & 2 \\ 2 & -1 & 4 & 1 \\ 1 & 4 & 11 & 6 \end{array}\right)$$

$$\xrightarrow[\substack{r_3-2r_1\\ r_4-r_1}]{r_2+r_1} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ & & & \\ & & & \\ & & & \end{array}\right)$$

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$$\begin{array}{c} \begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array} \\ \left(\begin{array}{ccc|c} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{array}\right) \end{array} \xrightarrow{r_1\leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ -1 & 3 & 3 & 2 \\ 2 & -1 & 4 & 1 \\ 1 & 4 & 11 & 6 \end{array}\right)$$

$$\xrightarrow[\substack{r_3-2r_1\\ r_4-r_1}]{r_2+r_1} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ & & & \\ & & & \end{array}\right)$$

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解

$$\begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \beta \end{array} \\ \left(\begin{array}{ccc|c} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{array}\right) \end{array} \xrightarrow{r_1 \leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ -1 & 3 & 3 & 2 \\ 2 & -1 & 4 & 1 \\ 1 & 4 & 11 & 6 \end{array}\right)$$

$$\xrightarrow[\substack{r_3-2r_1 \\ r_4-r_1}]{r_2+r_1} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 3 & 4 & 3 \\ & & & \end{array}\right)$$

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解

$$\begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \beta \end{array} \\ \left(\begin{array}{ccc|c} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{array}\right) \end{array} \xrightarrow{r_1 \leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ -1 & 3 & 3 & 2 \\ 2 & -1 & 4 & 1 \\ 1 & 4 & 11 & 6 \end{array}\right)$$

$$\xrightarrow[\substack{r_3-2r_1 \\ r_4-r_1}]{r_2+r_1} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 3 & 4 & 3 \\ 0 & 6 & 11 & 7 \end{array}\right)$$

例 2 $\beta=\begin{pmatrix}1\\2\\-1\\6\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}2\\-1\\1\\1\end{pmatrix}$, $\alpha_2=\begin{pmatrix}-1\\3\\-2\\4\end{pmatrix}$, $\alpha_3=\begin{pmatrix}4\\3\\0\\11\end{pmatrix}$线性表示?

解

$$\begin{array}{c} \quad\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \beta \\ \left(\begin{array}{ccc|c} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{array}\right) \end{array} \xrightarrow{r_1 \leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ -1 & 3 & 3 & 2 \\ 2 & -1 & 4 & 1 \\ 1 & 4 & 11 & 6 \end{array}\right)$$

$$\xrightarrow[\substack{r_3-2r_1 \\ r_4-r_1}]{r_2+r_1} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 3 & 4 & 3 \\ 0 & 6 & 11 & 7 \end{array}\right) \xrightarrow[r_4-6r_2]{r_3-3r_2}$$

例 2 $\beta=\begin{pmatrix}1\\2\\-1\\6\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}2\\-1\\1\\1\end{pmatrix}$，$\alpha_2=\begin{pmatrix}-1\\3\\-2\\4\end{pmatrix}$，$\alpha_3=\begin{pmatrix}4\\3\\0\\11\end{pmatrix}$线性表示?

解

$$\begin{array}{c} \quad\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \beta \\ \left(\begin{array}{ccc|c} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{array}\right) \end{array} \xrightarrow{r_1 \leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ -1 & 3 & 3 & 2 \\ 2 & -1 & 4 & 1 \\ 1 & 4 & 11 & 6 \end{array}\right)$$

$$\xrightarrow[\substack{r_3-2r_1\\ r_4-r_1}]{r_2+r_1} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 3 & 4 & 3 \\ 0 & 6 & 11 & 7 \end{array}\right) \xrightarrow[r_4-6r_2]{r_3-3r_2} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ & & & \\ & & & \end{array}\right)$$

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解

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$$\xrightarrow[\substack{r_3-2r_1\\ r_4-r_1}]{r_2+r_1} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1\\ 0 & 1 & 3 & 1\\ 0 & 3 & 4 & 3\\ 0 & 6 & 11 & 7 \end{array}\right) \xrightarrow[r_4-6r_2]{r_3-3r_2} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1\\ 0 & 1 & 3 & 1\\ 0 & 0 & -5 & 0\\ & & & \end{array}\right)$$

例 2 $\beta=\begin{pmatrix}1\\2\\-1\\6\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}2\\-1\\1\\1\end{pmatrix}$, $\alpha_2=\begin{pmatrix}-1\\3\\-2\\4\end{pmatrix}$, $\alpha_3=\begin{pmatrix}4\\3\\0\\11\end{pmatrix}$线性表示?

解

$$\begin{array}{c} \quad\ \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \beta \\ \left(\begin{array}{ccc|c} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{array}\right) \end{array} \xrightarrow{r_1 \leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ -1 & 3 & 3 & 2 \\ 2 & -1 & 4 & 1 \\ 1 & 4 & 11 & 6 \end{array}\right)$$

$$\xrightarrow[\substack{r_3-2r_1\\ r_4-r_1}]{r_2+r_1} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 3 & 4 & 3 \\ 0 & 6 & 11 & 7 \end{array}\right) \xrightarrow[r_4-6r_2]{r_3-3r_2} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & -7 & 1 \end{array}\right)$$

例 2 $\beta=\begin{pmatrix}1\\2\\-1\\6\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}2\\-1\\1\\1\end{pmatrix}$, $\alpha_2=\begin{pmatrix}-1\\3\\-2\\4\end{pmatrix}$, $\alpha_3=\begin{pmatrix}4\\3\\0\\11\end{pmatrix}$线性表示?

解

$$\begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \beta \end{array} \\ \left(\begin{array}{ccc|c} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{array}\right) \end{array} \xrightarrow{r_1 \leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ -1 & 3 & 3 & 2 \\ 2 & -1 & 4 & 1 \\ 1 & 4 & 11 & 6 \end{array}\right)$$

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$$\xrightarrow{-\frac{1}{5}\times r_3}$$

例 2 $\beta=\begin{pmatrix}1\\2\\-1\\6\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}2\\-1\\1\\1\end{pmatrix}$, $\alpha_2=\begin{pmatrix}-1\\3\\-2\\4\end{pmatrix}$, $\alpha_3=\begin{pmatrix}4\\3\\0\\11\end{pmatrix}$线性表示?

解

$$\begin{array}{c} \begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array} \\ \left(\begin{array}{ccc|c} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{array}\right) \end{array} \xrightarrow{r_1 \leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ -1 & 3 & 3 & 2 \\ 2 & -1 & 4 & 1 \\ 1 & 4 & 11 & 6 \end{array}\right)$$

$$\xrightarrow[\substack{r_3-2r_1\\ r_4-r_1}]{r_2+r_1} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 3 & 4 & 3 \\ 0 & 6 & 11 & 7 \end{array}\right) \xrightarrow[r_4-6r_2]{r_3-3r_2} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & -7 & 1 \end{array}\right)$$

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$$\xrightarrow[\substack{r_3-2r_1 \\ r_4-r_1}]{r_2+r_1} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 3 & 4 & 3 \\ 0 & 6 & 11 & 7 \end{array}\right) \xrightarrow[r_4-6r_2]{r_3-3r_2} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & -7 & 1 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{5}\times r_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -7 & 1 \end{array}\right) \xrightarrow{r_4+7r_3}$$

例 2 $\beta=\begin{pmatrix}1\\2\\-1\\6\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}2\\-1\\1\\1\end{pmatrix}$, $\alpha_2=\begin{pmatrix}-1\\3\\-2\\4\end{pmatrix}$, $\alpha_3=\begin{pmatrix}4\\3\\0\\11\end{pmatrix}$线性表示?

解

$$\begin{array}{c} \begin{matrix}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{matrix} \\ \left(\begin{array}{ccc|c} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{array}\right) \end{array} \xrightarrow{r_1 \leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ -1 & 3 & 3 & 2 \\ 2 & -1 & 4 & 1 \\ 1 & 4 & 11 & 6 \end{array}\right)$$

$$\xrightarrow[\substack{r_3-2r_1\\ r_4-r_1}]{r_2+r_1} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 3 & 4 & 3 \\ 0 & 6 & 11 & 7 \end{array}\right) \xrightarrow[r_4-6r_2]{r_3-3r_2} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & -7 & 1 \end{array}\right)$$

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例 2 $\beta=\begin{pmatrix}1\\2\\-1\\6\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}2\\-1\\1\\1\end{pmatrix}$, $\alpha_2=\begin{pmatrix}-1\\3\\-2\\4\end{pmatrix}$, $\alpha_3=\begin{pmatrix}4\\3\\0\\11\end{pmatrix}$线性表示?

解

$$\begin{array}{c}
\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array}\\
\left(\begin{array}{ccc|c}2 & -1 & 4 & 1\\ -1 & 3 & 3 & 2\\ 1 & -2 & 0 & -1\\ 1 & 4 & 11 & 6\end{array}\right)
\end{array}
\xrightarrow{r_1\leftrightarrow r_3}
\left(\begin{array}{ccc|c}1 & -2 & 0 & -1\\ -1 & 3 & 3 & 2\\ 2 & -1 & 4 & 1\\ 1 & 4 & 11 & 6\end{array}\right)$$

$$\xrightarrow[\substack{r_3-2r_1\\ r_4-r_1}]{r_2+r_1}
\left(\begin{array}{ccc|c}1 & -2 & 0 & -1\\ 0 & 1 & 3 & 1\\ 0 & 3 & 4 & 3\\ 0 & 6 & 11 & 7\end{array}\right)
\xrightarrow[r_4-6r_2]{r_3-3r_2}
\left(\begin{array}{ccc|c}1 & -2 & 0 & -1\\ 0 & 1 & 3 & 1\\ 0 & 0 & -5 & 0\\ 0 & 0 & -7 & 1\end{array}\right)$$

$$\xrightarrow{-\frac{1}{5}\times r_3}
\left(\begin{array}{ccc|c}1 & -2 & 0 & -1\\ 0 & 1 & 3 & 1\\ 0 & 0 & 1 & 0\\ 0 & 0 & -7 & 1\end{array}\right)
\xrightarrow{r_4+7r_3}
\left(\begin{array}{ccc|c}1 & -2 & 0 & -1\\ 0 & 1 & 3 & 1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right)$$

例 2 $\beta=\begin{pmatrix}1\\2\\-1\\6\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}2\\-1\\1\\1\end{pmatrix}$, $\alpha_2=\begin{pmatrix}-1\\3\\-2\\4\end{pmatrix}$, $\alpha_3=\begin{pmatrix}4\\3\\0\\11\end{pmatrix}$线性表示?

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$$\begin{array}{c}
\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array}\\
\left(\begin{array}{ccc|c}2 & -1 & 4 & 1\\ -1 & 3 & 3 & 2\\ 1 & -2 & 0 & -1\\ 1 & 4 & 11 & 6\end{array}\right)
\end{array}
\xrightarrow{r_1\leftrightarrow r_3}
\left(\begin{array}{ccc|c}1 & -2 & 0 & -1\\ -1 & 3 & 3 & 2\\ 2 & -1 & 4 & 1\\ 1 & 4 & 11 & 6\end{array}\right)$$

$$\xrightarrow[\substack{r_3-2r_1\\ r_4-r_1}]{r_2+r_1}
\left(\begin{array}{ccc|c}1 & -2 & 0 & -1\\ 0 & 1 & 3 & 1\\ 0 & 3 & 4 & 3\\ 0 & 6 & 11 & 7\end{array}\right)
\xrightarrow[r_4-6r_2]{r_3-3r_2}
\left(\begin{array}{ccc|c}1 & -2 & 0 & -1\\ 0 & 1 & 3 & 1\\ 0 & 0 & -5 & 0\\ 0 & 0 & -7 & 1\end{array}\right)$$

$$\xrightarrow{-\frac{1}{5}\times r_3}
\left(\begin{array}{ccc|c}1 & -2 & 0 & -1\\ 0 & 1 & 3 & 1\\ 0 & 0 & 1 & 0\\ 0 & 0 & -7 & 1\end{array}\right)
\xrightarrow{r_4+7r_3}
\left(\begin{array}{ccc|c}1 & -2 & 0 & -1\\ 0 & 1 & 3 & 1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right)$$

可见 $r(\alpha_1\alpha_2\alpha_3\beta)=4>3=r(\alpha_1\alpha_2\alpha_3)$,

例 2 $\beta=\begin{pmatrix}1\\2\\-1\\6\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}2\\-1\\1\\1\end{pmatrix}$，$\alpha_2=\begin{pmatrix}-1\\3\\-2\\4\end{pmatrix}$，$\alpha_3=\begin{pmatrix}4\\3\\0\\11\end{pmatrix}$线性表示?

解

$$\begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \beta \end{array} \\ \left(\begin{array}{ccc|c} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{array}\right) \end{array} \xrightarrow{r_1 \leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ -1 & 3 & 3 & 2 \\ 2 & -1 & 4 & 1 \\ 1 & 4 & 11 & 6 \end{array}\right)$$

$$\xrightarrow[\substack{r_3-2r_1\\ r_4-r_1}]{r_2+r_1} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 3 & 4 & 3 \\ 0 & 6 & 11 & 7 \end{array}\right) \xrightarrow[r_4-6r_2]{r_3-3r_2} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & -7 & 1 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{5}\times r_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -7 & 1 \end{array}\right) \xrightarrow{r_4+7r_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

可见 $r(\alpha_1\alpha_2\alpha_3\beta)=4>3=r(\alpha_1\alpha_2\alpha_3)$，所以不能线性表示。

向量组的线性组合

定义 设有两个向量组

$$(A): \quad \alpha_1, \alpha_2, \ldots, \alpha_s$$

$$(B): \quad \beta_1, \beta_2, \ldots, \beta_t$$

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如果中 (A) 中每一向量均可由 (B) 线性表示，则称向量组 (A) 可由向量组 (B) **线性表示**。

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例

向量组 α_1, α_2 可由向量组 $\beta_1, \beta_2, \beta_3$ 线性表示

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例 向量组 α_1, α_2 可由向量组 $\beta_1, \beta_2, \beta_3$ 线性表示

$$\Rightarrow \begin{cases} \alpha_1 = \quad \beta_1 + \quad \beta_2 + \quad \beta_3 \\ \alpha_2 = \quad \beta_1 + \quad \beta_2 + \quad \beta_3 \end{cases}$$

向量组的线性组合

定义 设有两个向量组

$$(A): \quad \alpha_1, \alpha_2, \ldots, \alpha_s$$

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如果中 (A) 中每一向量均可由 (B) 线性表示，则称向量组 (A) 可由向量组 (B) **线性表示**。

例

向量组 α_1, α_2 可由向量组 $\beta_1, \beta_2, \beta_3$ 线性表示

$$\Rightarrow \begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 + a_{31}\beta_3 \\ \alpha_2 = \qquad \beta_1 + \qquad \beta_2 + \qquad \beta_3 \end{cases}$$

向量组的线性组合

定义 设有两个向量组

$$(A): \quad \alpha_1, \alpha_2, \ldots, \alpha_s$$

$$(B): \quad \beta_1, \beta_2, \ldots, \beta_t$$

如果中 (A) 中每一向量均可由 (B) 线性表示，则称向量组 (A) 可由向量组 (B) **线性表示**。

例 向量组 α_1, α_2 可由向量组 $\beta_1, \beta_2, \beta_3$ 线性表示

$$\Rightarrow \begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 + a_{31}\beta_3 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 + a_{32}\beta_3 \end{cases}$$

矩阵乘积与向量组线性组合

$$A = BP$$

矩阵乘积与向量组线性组合

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$$\Leftrightarrow \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{ms} \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1t} \\ b_{21} & b_{22} & \cdots & b_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mt} \end{pmatrix}}_{B} \underbrace{\begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{t1} & p_{t2} & \cdots & p_{tn} \end{pmatrix}}_{P}$$

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$$\Leftrightarrow (\alpha_1 \ \alpha_2 \ \cdots \ \alpha_s) = (\beta_1 \ \beta_2 \ \cdots \ \beta_t) \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{t1} & p_{t2} & \cdots & p_{tn} \end{pmatrix}$$

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$$\Leftrightarrow \begin{cases} \alpha_1 = p_{11}\beta_1 + p_{21}\beta_2 + \cdots + p_{t1}\beta_t \\ \\ \\ \end{cases}$$

矩阵乘积与向量组线性组合

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$$\Leftrightarrow (\alpha_1\ \alpha_2\ \cdots\ \alpha_s) = (\beta_1\ \beta_2\ \cdots\ \beta_t) \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{t1} & p_{t2} & \cdots & p_{tn} \end{pmatrix}$$

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$\Leftrightarrow$ $\{\alpha_1, \alpha_2, \cdots \alpha_s\}$由$\{\beta_1, \beta_2 \cdots \beta_t\}$线性表示

矩阵乘积与向量组线性组合

$$A = BP$$

$$\Leftrightarrow \underbrace{\begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & & \alpha_s \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{ms} \end{pmatrix} \end{array}}_{A} = \underbrace{\begin{array}{c} \begin{array}{cccc} \beta_1 & \beta_2 & & \beta_t \end{array} \\ \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1t} \\ b_{21} & b_{22} & \cdots & b_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mt} \end{pmatrix} \end{array}}_{B} \underbrace{\begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{t1} & p_{t2} & \cdots & p_{tn} \end{pmatrix}}_{P}$$

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$\Leftrightarrow \{\alpha_1, \alpha_2, \cdots \alpha_s\}$由$\{\beta_1, \beta_2 \cdots \beta_t\}$线性表示，$P$的列是线性表示系数

定理（向量组线性表示的传递性） 假设向量组 $(A), (B), (C)$ 满足：(A) 可由 (B) 线性表示，(B) 可由 (C) 线性表示，则 (A) 可由 (C) 线性表示。

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证明 设向量均为列向量。

设向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 可由向量组 $\beta_1, \beta_2, \ldots, \beta_t$ 线性表示：

向量组 $\beta_1, \beta_2, \ldots, \beta_t$ 可由向量组 $\gamma_1, \gamma_2, \ldots, \gamma_k$ 线性表示：

定理（向量组线性表示的传递性） 假设向量组 $(A), (B), (C)$ 满足：(A) 可由 (B) 线性表示，(B) 可由 (C) 线性表示，则 (A) 可由 (C) 线性表示。

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设向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 可由向量组 $\beta_1, \beta_2, \ldots, \beta_t$ 线性表示：

$$(\alpha_1, \alpha_2, \ldots, \alpha_s) = (\beta_1, \beta_2, \ldots, \beta_t)A_{t\times s}.$$

向量组 $\beta_1, \beta_2, \ldots, \beta_t$ 可由向量组 $\gamma_1, \gamma_2, \ldots, \gamma_k$ 线性表示：

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证明 设向量均为列向量。

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$$(\beta_1, \beta_2, \ldots, \beta_t) = (\gamma_1, \gamma_2, \ldots, \gamma_k)B_{k\times t}.$$

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将第 2 式代入第 1 式，可得

定理（向量组线性表示的传递性） 假设向量组 (A), (B), (C) 满足：(A) 可由 (B) 线性表示，(B) 可由 (C) 线性表示，则 (A) 可由 (C) 线性表示。

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向量组 $\beta_1, \beta_2, \ldots, \beta_t$ 可由向量组 $\gamma_1, \gamma_2, \ldots, \gamma_k$ 线性表示：

$$(\beta_1, \beta_2, \ldots, \beta_t) = (\gamma_1, \gamma_2, \ldots, \gamma_k)B_{k\times t}.$$

将第 2 式代入第 1 式，可得

$(\alpha_1, \alpha_2, \ldots, \alpha_s) = (\gamma_1, \gamma_2, \ldots, \gamma_k)B_{k\times t}A_{t\times s}$

定理（向量组线性表示的传递性） 假设向量组 (A), (B), (C) 满足：(A) 可由 (B) 线性表示，(B) 可由 (C) 线性表示，则 (A) 可由 (C) 线性表示。

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向量组 $\beta_1, \beta_2, \ldots, \beta_t$ 可由向量组 $\gamma_1, \gamma_2, \ldots, \gamma_k$ 线性表示：

$$(\beta_1, \beta_2, \ldots, \beta_t) = (\gamma_1, \gamma_2, \ldots, \gamma_k)B_{k\times t}.$$

将第 2 式代入第 1 式，可得

$$(\alpha_1, \alpha_2, \ldots, \alpha_s) = (\gamma_1, \gamma_2, \ldots, \gamma_k)\underbrace{B_{k\times t}A_{t\times s}}_{C_{k\times s}}$$

定理（向量组线性表示的传递性） 假设向量组 $(A), (B), (C)$ 满足：(A) 可由 (B) 线性表示，(B) 可由 (C) 线性表示，则 (A) 可由 (C) 线性表示。

证明 设向量均为列向量。

设向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 可由向量组 $\beta_1, \beta_2, \ldots, \beta_t$ 线性表示：

$$(\alpha_1, \alpha_2, \ldots, \alpha_s) = (\beta_1, \beta_2, \ldots, \beta_t)A_{t\times s}.$$

向量组 $\beta_1, \beta_2, \ldots, \beta_t$ 可由向量组 $\gamma_1, \gamma_2, \ldots, \gamma_k$ 线性表示：

$$(\beta_1, \beta_2, \ldots, \beta_t) = (\gamma_1, \gamma_2, \ldots, \gamma_k)B_{k\times t}.$$

将第 2 式代入第 1 式，可得

$$(\alpha_1, \alpha_2, \ldots, \alpha_s) = (\gamma_1, \gamma_2, \ldots, \gamma_k)\underbrace{B_{k\times t}A_{t\times s}}_{C_{k\times s}} = (\gamma_1, \gamma_2, \ldots, \gamma_k)C.$$

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所以向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 可由向量组 $\gamma_1, \gamma_2, \ldots, \gamma_k$ 线性表示。

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将第 2 式代入第 1 式，可得

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所以向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 可由向量组 $\gamma_1, \gamma_2, \ldots, \gamma_k$ 线性表示。
(并且，线性组合的系数就是矩阵 C 的列。)

例 $\left.\begin{array}{r}\alpha_1,\ \alpha_2\text{由}\beta_1,\ \beta_2\text{线性表示}\\ \beta_1,\ \beta_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示}\end{array}\right\}\Rightarrow \alpha_1,\ \alpha_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示}$

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具体地，设

$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases}$$

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则

$\alpha_1 =$

$\alpha_2 =$

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则

$$\alpha_1 = a_{11}(\qquad\qquad\qquad\qquad) + a_{21}(\qquad\qquad\qquad\qquad)$$

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则

$$\alpha_1 = a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(\qquad\qquad\qquad\qquad)$$

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$$\alpha_2 = a_{12}(\qquad\qquad\qquad\qquad) + a_{22}(\qquad\qquad\qquad\qquad)$$

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$$\begin{aligned}\alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)\\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3\end{aligned}$$

$$\begin{aligned}\alpha_2 &= a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)\\ &= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (a_{12}b_{31} + a_{22}b_{32})\gamma_3\end{aligned}$$

例 $\left.\begin{array}{r}\alpha_1,\ \alpha_2 由 \beta_1,\ \beta_2 线性表示 \\ \beta_1,\ \beta_2 由 \gamma_1,\ \gamma_2,\ \gamma_3 线性表示\end{array}\right\} \Rightarrow \alpha_1,\ \alpha_2 由 \gamma_1,\ \gamma_2,\ \gamma_3 线性表示$

具体地，设

$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases}$$

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases}$$

则

$$\begin{aligned} \alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3 \\ &= c_{11}\gamma_1 + \\ \alpha_2 &= a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (a_{12}b_{31} + a_{22}b_{32})\gamma_3 \end{aligned}$$

例 $\left.\begin{array}{r}\alpha_1,\ \alpha_2\text{由}\beta_1,\ \beta_2\text{线性表示}\\ \beta_1,\ \beta_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示}\end{array}\right\} \Rightarrow \alpha_1,\ \alpha_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示}$

具体地，设

$$\begin{cases}\alpha_1 = a_{11}\beta_1 + a_{21}\beta_2\\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2\end{cases}$$

$$\begin{cases}\beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3\\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3\end{cases}$$

则

$$\begin{aligned}\alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)\\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3\\ &= c_{11}\gamma_1 + c_{21}\gamma_2 +\\ \alpha_2 &= a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)\\ &= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (a_{12}b_{31} + a_{22}b_{32})\gamma_3\end{aligned}$$

例 $\left.\begin{array}{r} \alpha_1, \alpha_2\text{由}\beta_1, \beta_2\text{线性表示} \\ \beta_1, \beta_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示} \end{array}\right\} \Rightarrow \alpha_1, \alpha_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示}$

具体地，设

$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases}$$

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases}$$

则

$$\begin{aligned} \alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3 \\ &= c_{11}\gamma_1 + c_{21}\gamma_2 + c_{31}\gamma_3 \\ \alpha_2 &= a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (a_{12}b_{31} + a_{22}b_{32})\gamma_3 \end{aligned}$$

例 $\left.\begin{array}{r}\alpha_1, \alpha_2\text{由}\beta_1, \beta_2\text{线性表示}\\ \beta_1, \beta_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示}\end{array}\right\} \Rightarrow \alpha_1, \alpha_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示}$

具体地，设

$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases}$$

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases}$$

则

$$\begin{aligned} \alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3 \\ &= c_{11}\gamma_1 + c_{21}\gamma_2 + c_{31}\gamma_3 \\ \alpha_2 &= a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (a_{12}b_{31} + a_{22}b_{32})\gamma_3 \\ &= c_{12}\gamma_1 + \end{aligned}$$

例 $\left.\begin{array}{r} \alpha_1, \alpha_2\text{由}\beta_1, \beta_2\text{线性表示} \\ \beta_1, \beta_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示} \end{array}\right\} \Rightarrow \alpha_1, \alpha_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示}$

具体地，设

$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases}$$

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases}$$

则

$$\begin{aligned} \alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3 \\ &= c_{11}\gamma_1 + c_{21}\gamma_2 + c_{31}\gamma_3 \\ \alpha_2 &= a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (a_{12}b_{31} + a_{22}b_{32})\gamma_3 \\ &= c_{12}\gamma_1 + c_{22}\gamma_2 + \end{aligned}$$

例 $\left.\begin{array}{r}\alpha_1, \alpha_2 \text{由} \beta_1, \beta_2 \text{线性表示} \\ \beta_1, \beta_2 \text{由} \gamma_1, \gamma_2, \gamma_3 \text{线性表示}\end{array}\right\} \Rightarrow \alpha_1, \alpha_2 \text{由} \gamma_1, \gamma_2, \gamma_3 \text{线性表示}$

具体地，设

$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases}$$

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases}$$

则

$$\begin{aligned} \alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3 \\ &= c_{11}\gamma_1 + c_{21}\gamma_2 + c_{31}\gamma_3 \\ \alpha_2 &= a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (a_{12}b_{31} + a_{22}b_{32})\gamma_3 \\ &= c_{12}\gamma_1 + c_{22}\gamma_2 + c_{32}\gamma_3 \end{aligned}$$

例 $\left.\begin{array}{r} \alpha_1, \alpha_2\text{由}\beta_1, \beta_2\text{线性表示} \\ \beta_1, \beta_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示} \end{array}\right\} \Rightarrow \alpha_1, \alpha_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示}$

具体地，设

$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases}$$

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases}$$

则

$$\begin{aligned} \alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3 \\ &= c_{11}\gamma_1 + c_{21}\gamma_2 + c_{31}\gamma_3 \\ \alpha_2 &= a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (a_{12}b_{31} + a_{22}b_{32})\gamma_3 \\ &= c_{12}\gamma_1 + c_{22}\gamma_2 + c_{32}\gamma_3 \end{aligned}$$

其中

$$(c_{ij}) = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{12} & a_{12}b_{11} + a_{22}b_{12} \\ a_{11}b_{21} + a_{21}b_{22} & a_{12}b_{21} + a_{22}b_{22} \\ a_{11}b_{31} + a_{21}b_{32} & a_{12}b_{31} + a_{22}b_{32} \end{pmatrix}$$

例 $\left.\begin{array}{r}\alpha_1,\ \alpha_2\text{由}\beta_1,\ \beta_2\text{线性表示}\\ \beta_1,\ \beta_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示}\end{array}\right\}\Rightarrow \alpha_1,\ \alpha_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示}$

具体地，设

$$\begin{cases}\alpha_1 = a_{11}\beta_1 + a_{21}\beta_2\\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2\end{cases}$$

$$\begin{cases}\beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3\\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3\end{cases}$$

则

$$\begin{aligned}\alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)\\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3\\ &= c_{11}\gamma_1 + c_{21}\gamma_2 + c_{31}\gamma_3\\ \alpha_2 &= a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)\\ &= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (a_{12}b_{31} + a_{22}b_{32})\gamma_3\\ &= c_{12}\gamma_1 + c_{22}\gamma_2 + c_{32}\gamma_3\end{aligned}$$

其中

$$(c_{ij}) = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{12} & a_{12}b_{11} + a_{22}b_{12}\\ a_{11}b_{21} + a_{21}b_{22} & a_{12}b_{21} + a_{22}b_{22}\\ a_{11}b_{31} + a_{21}b_{32} & a_{12}b_{31} + a_{22}b_{32}\end{pmatrix} = \begin{pmatrix} b_{11} & b_{12}\\ b_{21} & b_{22}\\ b_{31} & b_{32}\end{pmatrix}\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$$

例 $\left.\begin{array}{r}\alpha_1,\ \alpha_2\text{由}\beta_1,\ \beta_2\text{线性表示}\\ \beta_1,\ \beta_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示}\end{array}\right\} \Rightarrow \alpha_1,\ \alpha_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示}$

具体地，设

$$\begin{cases}\alpha_1 = a_{11}\beta_1 + a_{21}\beta_2\\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2\end{cases} \Rightarrow (\alpha_1,\ \alpha_2) = (\beta_1,\ \beta_2)\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$$

$$\begin{cases}\beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3\\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3\end{cases}$$

则

$$\begin{aligned}\alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)\\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3\\ &= c_{11}\gamma_1 + c_{21}\gamma_2 + c_{31}\gamma_3\\ \alpha_2 &= a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)\\ &= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (a_{12}b_{31} + a_{22}b_{32})\gamma_3\\ &= c_{12}\gamma_1 + c_{22}\gamma_2 + c_{32}\gamma_3\end{aligned}$$

其中

$$(c_{ij}) = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{12} & a_{12}b_{11} + a_{22}b_{12}\\ a_{11}b_{21} + a_{21}b_{22} & a_{12}b_{21} + a_{22}b_{22}\\ a_{11}b_{31} + a_{21}b_{32} & a_{12}b_{31} + a_{22}b_{32}\end{pmatrix} = \begin{pmatrix} b_{11} & b_{12}\\ b_{21} & b_{22}\\ b_{31} & b_{32}\end{pmatrix}\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$$

例 $\left.\begin{array}{r}\alpha_1,\ \alpha_2 \text{由} \beta_1,\ \beta_2 \text{线性表示} \\ \beta_1,\ \beta_2 \text{由} \gamma_1,\ \gamma_2,\ \gamma_3 \text{线性表示}\end{array}\right\} \Rightarrow \alpha_1,\ \alpha_2 \text{由} \gamma_1,\ \gamma_2,\ \gamma_3 \text{线性表示}$

具体地，设

$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases} \Rightarrow (\alpha_1,\ \alpha_2) = (\beta_1,\ \beta_2)\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases} \Rightarrow (\beta_1,\ \beta_2) = (\gamma_1,\ \gamma_2,\ \gamma_3)\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}$$

则

$$\begin{aligned} \alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3 \\ &= c_{11}\gamma_1 + c_{21}\gamma_2 + c_{31}\gamma_3 \\ \alpha_2 &= a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (a_{12}b_{31} + a_{22}b_{32})\gamma_3 \\ &= c_{12}\gamma_1 + c_{22}\gamma_2 + c_{32}\gamma_3 \end{aligned}$$

其中

$$(c_{ij}) = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{12} & a_{12}b_{11} + a_{22}b_{12} \\ a_{11}b_{21} + a_{21}b_{22} & a_{12}b_{21} + a_{22}b_{22} \\ a_{11}b_{31} + a_{21}b_{32} & a_{12}b_{31} + a_{22}b_{32} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

例 $\left.\begin{array}{r}\alpha_1,\ \alpha_2\text{由}\beta_1,\ \beta_2\text{线性表示}\\ \beta_1,\ \beta_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示}\end{array}\right\}\Rightarrow \alpha_1,\ \alpha_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示}$

具体地，设

$$\begin{cases}\alpha_1 = a_{11}\beta_1 + a_{21}\beta_2\\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2\end{cases} \Rightarrow (\alpha_1,\ \alpha_2) = (\beta_1,\ \beta_2)\underbrace{\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}}_{A}$$

$$\begin{cases}\beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3\\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3\end{cases} \Rightarrow (\beta_1,\ \beta_2) = (\gamma_1,\ \gamma_2,\ \gamma_3)\begin{pmatrix} b_{11} & b_{12}\\ b_{21} & b_{22}\\ b_{31} & b_{32}\end{pmatrix}$$

则

$$\begin{aligned}\alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)\\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3\\ &= c_{11}\gamma_1 + c_{21}\gamma_2 + c_{31}\gamma_3\\ \alpha_2 &= a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)\\ &= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (a_{12}b_{31} + a_{22}b_{32})\gamma_3\\ &= c_{12}\gamma_1 + c_{22}\gamma_2 + c_{32}\gamma_3\end{aligned}$$

其中

$$(c_{ij}) = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{12} & a_{12}b_{11} + a_{22}b_{12}\\ a_{11}b_{21} + a_{21}b_{22} & a_{12}b_{21} + a_{22}b_{22}\\ a_{11}b_{31} + a_{21}b_{32} & a_{12}b_{31} + a_{22}b_{32}\end{pmatrix} = \begin{pmatrix} b_{11} & b_{12}\\ b_{21} & b_{22}\\ b_{31} & b_{32}\end{pmatrix}\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$$

例 $\left.\begin{array}{r}\alpha_1,\ \alpha_2\text{由}\beta_1,\ \beta_2\text{线性表示}\\ \beta_1,\ \beta_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示}\end{array}\right\}\Rightarrow \alpha_1,\ \alpha_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示}$

具体地，设

$$\begin{cases}\alpha_1 = a_{11}\beta_1 + a_{21}\beta_2\\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2\end{cases} \Rightarrow (\alpha_1,\ \alpha_2) = (\beta_1,\ \beta_2)\underbrace{\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}}_{A}$$

$$\begin{cases}\beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3\\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3\end{cases} \Rightarrow (\beta_1,\ \beta_2) = (\gamma_1,\ \gamma_2,\ \gamma_3)\underbrace{\begin{pmatrix} b_{11} & b_{12}\\ b_{21} & b_{22}\\ b_{31} & b_{32}\end{pmatrix}}_{B}$$

则

$$\begin{aligned}\alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)\\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3\\ &= c_{11}\gamma_1 + c_{21}\gamma_2 + c_{31}\gamma_3\\ \alpha_2 &= a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)\\ &= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (a_{12}b_{31} + a_{22}b_{32})\gamma_3\\ &= c_{12}\gamma_1 + c_{22}\gamma_2 + c_{32}\gamma_3\end{aligned}$$

其中

$$(c_{ij}) = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{12} & a_{12}b_{11} + a_{22}b_{12}\\ a_{11}b_{21} + a_{21}b_{22} & a_{12}b_{21} + a_{22}b_{22}\\ a_{11}b_{31} + a_{21}b_{32} & a_{12}b_{31} + a_{22}b_{32}\end{pmatrix} = \begin{pmatrix} b_{11} & b_{12}\\ b_{21} & b_{22}\\ b_{31} & b_{32}\end{pmatrix}\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$$

例 $\left.\begin{array}{r}\alpha_1,\ \alpha_2\text{由}\beta_1,\ \beta_2\text{线性表示}\\ \beta_1,\ \beta_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示}\end{array}\right\}\Rightarrow \alpha_1,\ \alpha_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示}$

具体地，设

$$\begin{cases}\alpha_1=a_{11}\beta_1+a_{21}\beta_2\\ \alpha_2=a_{12}\beta_1+a_{22}\beta_2\end{cases}\Rightarrow(\alpha_1,\ \alpha_2)=(\beta_1,\ \beta_2)\underbrace{\begin{pmatrix}a_{11}&a_{12}\\ a_{21}&a_{22}\end{pmatrix}}_{A}$$

$$\begin{cases}\beta_1=b_{11}\gamma_1+b_{21}\gamma_2+b_{31}\gamma_3\\ \beta_2=b_{12}\gamma_1+b_{22}\gamma_2+b_{32}\gamma_3\end{cases}\Rightarrow(\beta_1,\ \beta_2)=(\gamma_1,\ \gamma_2,\ \gamma_3)\underbrace{\begin{pmatrix}b_{11}&b_{12}\\ b_{21}&b_{22}\\ b_{31}&b_{32}\end{pmatrix}}_{B}$$

则

$$\begin{aligned}\alpha_1&=a_{11}(b_{11}\gamma_1+b_{21}\gamma_2+b_{31}\gamma_3)+a_{21}(b_{12}\gamma_1+b_{22}\gamma_2+b_{32}\gamma_3)\\ &=(a_{11}b_{11}+a_{21}b_{12})\gamma_1+(a_{11}b_{21}+a_{21}b_{22})\gamma_2+(a_{11}b_{31}+a_{21}b_{32})\gamma_3\\ &=c_{11}\gamma_1+c_{21}\gamma_2+c_{31}\gamma_3\\ \alpha_2&=a_{12}(b_{11}\gamma_1+b_{21}\gamma_2+b_{31}\gamma_3)+a_{22}(b_{12}\gamma_1+b_{22}\gamma_2+b_{32}\gamma_3)\\ &=(a_{12}b_{11}+a_{22}b_{12})\gamma_1+(a_{12}b_{21}+a_{22}b_{22})\gamma_2+(a_{12}b_{31}+a_{22}b_{32})\gamma_3\\ &=c_{12}\gamma_1+c_{22}\gamma_2+c_{32}\gamma_3\end{aligned}$$

其中

$$(c_{ij})=\begin{pmatrix}a_{11}b_{11}+a_{12}b_{12}&a_{12}b_{11}+a_{22}b_{12}\\ a_{11}b_{21}+a_{21}b_{22}&a_{12}b_{21}+a_{22}b_{22}\\ a_{11}b_{31}+a_{21}b_{32}&a_{12}b_{31}+a_{22}b_{32}\end{pmatrix}=\begin{pmatrix}b_{11}&b_{12}\\ b_{21}&b_{22}\\ b_{31}&b_{32}\end{pmatrix}\begin{pmatrix}a_{11}&a_{12}\\ a_{21}&a_{22}\end{pmatrix}=BA$$

定义 设有两个向量组

$$(A):\quad \alpha_1, \alpha_2, \ldots, \alpha_s$$

$$(B):\quad \beta_1, \beta_2, \ldots, \beta_t$$

如果 (A) 与 (B) 可相互线性表示，则称向量组 (A) 与 (B) **等价**。