§3.4　向量组的秩

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向量组的极大无关组

m维
向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s$

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向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s \xrightarrow{\text{挑选}} \alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}$ （部分组）

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满足：

- $\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}$ 线性无关；且
- 对 $\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}$ 再加入任一 α_i 后都是线性相关，

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证明

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- $\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}$ 与 $\beta_{k_1}, \beta_{k_2}, \ldots, \beta_{k_t}$ 等价（相互线性表示）；且

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定义 设向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 的极大无关组所包含向量的个数，称向量组的秩，记为：

向量组的秩

定理 极大无关组所包含向量的个数是唯一确定的。即：若

$$\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}; \qquad \beta_{k_1}, \beta_{k_2}, \ldots, \beta_{k_t}$$

都是 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 的极大无关组，则 $r = t$

证明 注意到

- $\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}$ 与 $\beta_{k_1}, \beta_{k_2}, \ldots, \beta_{k_t}$ 等价（相互线性表示）；且
- $\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}$ 与 $\beta_{k_1}, \beta_{k_2}, \ldots, \beta_{k_t}$ 都是线性无关，

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所以 $r = t$

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注 $r(\alpha_1, \alpha_2, \ldots, \alpha_s) \leq s$ 且 $\leq m$ (维数)。

向量组的秩

例 设 $\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, 则 $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) =$ ______

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这是:

- α_1, α_2 是极大无关组,

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这是:

- α_1, α_2 是极大无关组，所以极大无关组包含个 $\underline{\quad 2 \quad}$ 向量。

向量组的秩

例 设 $\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$，则 $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \underline{\quad 2 \quad}$

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- α_1, α_2 是极大无关组，所以极大无关组包含个 $\underline{\quad 2 \quad}$ 向量。事实上，α_1, α_2 线性无关，

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  事实上，α_1, α_2 线性无关，且 α_3, α_4 均能由 α_1, α_2 线性表示

向量组的秩

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$$\alpha_3 = \alpha_1 + \alpha_2;$$

向量组的秩

例 设 $\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$，则 $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \underline{\quad 2 \quad}$

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$$\alpha_3 = \alpha_1 + \alpha_2; \quad \alpha_4 = 2\alpha_2$$

向量组的秩

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$$\alpha_3 = \alpha_1 + \alpha_2; \quad \alpha_4 = 2\alpha_2$$

或者说:

- $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \leq 2$

向量组的秩

例 设 $\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, 则 $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \underline{\quad 2 \quad}$

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或者说：

- $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \leq 2$
- 有两个线性无关向量，

向量组的秩

例 设 $\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, 则 $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \underline{\quad 2 \quad}$

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- $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \le 2$
- 有两个线性无关向量，如 α_1, α_2，

向量组的秩

例 设 $\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$，则 $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \underline{\quad 2 \quad}$

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$$\alpha_3 = \alpha_1 + \alpha_2; \quad \alpha_4 = 2\alpha_2$$

或者说:

- $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \leq 2$
- 有两个线性无关向量，如 α_1, α_2，所以 $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \geq 2$

秩

设

$$\begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & & \alpha_n \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array}$$

$$r(\alpha_1, \alpha_2, \ldots, \alpha_n)$$

秩

设

$$A_{m\times n} = \begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & & \alpha_n \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array}$$

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$$r(\alpha_1, \alpha_2, \ldots, \alpha_n)$$

秩

设

$$A_{m\times n} = \begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & & \alpha_n \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array} = (\alpha_1\,\alpha_2\,\cdots\,\alpha_n)$$

定理 $r(A) = r(\alpha_1, \alpha_2, \ldots, \alpha_n)$

秩

设

$$A_{m\times n} = \begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & & \alpha_n \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array} = (\alpha_1\, \alpha_2 \cdots \alpha_n)$$

定义

- $r(\alpha_1, \alpha_2, \ldots, \alpha_n)$ 称为 A 的列秩；

定理 $r(A) = r(\alpha_1, \alpha_2, \ldots, \alpha_n)$

秩

设

$$A_{m\times n} = \begin{array}{cc} & \begin{array}{cccc} \alpha_1 & \alpha_2 & & \alpha_n \end{array} \\ \begin{array}{c} \beta_1 \\ \\ \\ \\ \end{array} & \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array} = (\alpha_1\,\alpha_2 \cdots \alpha_n)$$

定义

- $r(\alpha_1, \alpha_2, \ldots, \alpha_n)$ 称为 A 的**列秩**；

定理 $r(A) = r(\alpha_1, \alpha_2, \ldots, \alpha_n)$

秩

设

$$A_{m\times n}=\begin{array}{cc} & \begin{array}{cccc}\alpha_1 & \alpha_2 & & \alpha_n\end{array}\\ \begin{array}{c}\beta_1\\ \beta_2\\ \\ \\ \end{array} & \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{pmatrix}\end{array}=(\alpha_1\,\alpha_2\cdots\alpha_n)$$

定义

- $r(\alpha_1,\alpha_2,\ldots,\alpha_n)$ 称为 A 的列秩；

定理 $r(A)=r(\alpha_1,\alpha_2,\ldots,\alpha_n)$

秩

设

$$A_{m\times n} = \begin{array}{c c} & \begin{array}{cccc} \alpha_1 & \alpha_2 & & \alpha_n \end{array} \\ \begin{array}{c} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{array} & \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array} = (\alpha_1\,\alpha_2\cdots\alpha_n)$$

定义

- $r(\alpha_1, \alpha_2, \ldots, \alpha_n)$ 称为 A 的列秩；

定理 $r(A) = r(\alpha_1, \alpha_2, \ldots, \alpha_n)$

秩

设

$$A_{m\times n} = \begin{array}{c} \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{array} \begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & & \alpha_n \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array} = (\alpha_1\,\alpha_2\,\cdots\,\alpha_n) = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{pmatrix}$$

定义

- $r(\alpha_1, \alpha_2, \ldots, \alpha_n)$ 称为 A 的列秩；

定理 $r(A) = r(\alpha_1, \alpha_2, \ldots, \alpha_n)$

秩

设

$$A_{m\times n} = \begin{array}{cc} & \begin{array}{cccc} \alpha_1 & \alpha_2 & & \alpha_n \end{array} \\ \begin{array}{c} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{array} & \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array} = (\alpha_1\,\alpha_2\,\cdots\,\alpha_n) = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{pmatrix}$$

定义

- $r(\alpha_1, \alpha_2, \ldots, \alpha_n)$ 称为 A 的列秩；
- $r(\beta_1, \beta_2, \ldots, \beta_m)$ 称为 A 的行秩；

定理 $r(A) = r(\alpha_1, \alpha_2, \ldots, \alpha_n)$

秩

设

$$A_{m\times n} = \begin{array}{cc} & \begin{array}{cccc} \alpha_1 & \alpha_2 & & \alpha_n \end{array} \\ \begin{array}{c} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{array} & \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array} = (\alpha_1\,\alpha_2\,\cdots\,\alpha_n) = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{pmatrix}$$

定义

- $r(\alpha_1, \alpha_2, \dots, \alpha_n)$ 称为 A 的列秩；
- $r(\beta_1, \beta_2, \dots, \beta_m)$ 称为 A 的行秩；

定理 $r(A) = r(\alpha_1, \alpha_2, \dots, \alpha_n) = r(\beta_1, \beta_2, \dots, \beta_m)$

初等变换求极大无关组

问题 给出 m 维的向量组 $\alpha_1, \alpha_2, \cdots, \alpha_n$，如何求出其一组极大无关组？

步骤

初等变换求极大无关组

问题 给出 m 维的向量组 $\alpha_1, \alpha_2, \cdots, \alpha_n$，如何求出其一组极大无关组？

步骤

1. $$A_{m\times n} = \begin{array}{c} \begin{matrix} \alpha_1 & \alpha_2 & & \alpha_n \end{matrix} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array}$$

初等变换求极大无关组

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初等变换求极大无关组

问题 给出 m 维的向量组 $\alpha_1, \alpha_2, \cdots, \alpha_n$，如何求出其一组极大无关组？

步骤

1. $$A_{m\times n} = \begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & & \alpha_n \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array} \xrightarrow{\text{初等行变换}} \text{简化的阶梯型矩阵}$$

2. 通过简化的阶梯型矩阵，求出 $r(A)$。

初等变换求极大无关组

问题 给出 m 维的向量组 $\alpha_1, \alpha_2, \cdots, \alpha_n$，如何求出其一组极大无关组？

步骤

1. $$A_{m\times n} = \begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & & \alpha_n \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array} \xrightarrow{\text{初等行变换}} \text{简化的阶梯型矩阵}$$

2. 通过简化的阶梯型矩阵，求出 $r(A)$。

   利用 $r(\alpha_1,\alpha_2,...,\alpha_n) = r(A)$，得出极大无关组所包含向量的个数

初等变换求极大无关组

问题 给出 m 维的向量组 $\alpha_1, \alpha_2, \cdots, \alpha_n$，如何求出其一组极大无关组？

步骤

1. $$A_{m\times n} = \begin{array}{c} \begin{matrix} \alpha_1 & \alpha_2 & & \alpha_n \end{matrix} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array} \xrightarrow{\text{初等行变换}} \text{简化的阶梯型矩阵}$$

2. 通过简化的阶梯型矩阵，求出 $r(A)$。

   利用 $r(\alpha_1,\alpha_2,\ldots,\alpha_n) = r(A)$，得出极大无关组所包含向量的个数

3. 通过简化的阶梯型矩阵，容易看出线性无关的 $r(A)$ 列，这就找到一组极大无关组

初等变换求极大无关组

问题 给出 m 维的向量组 $\alpha_1, \alpha_2, \cdots, \alpha_n$，如何求出其一组极大无关组？

步骤

1. $A_{m\times n} = \begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & & \alpha_n \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array} \xrightarrow{\text{初等行变换}} \text{简化的阶梯型矩阵}$
2. 通过简化的阶梯型矩阵，求出 $r(A)$。

   利用 $r(\alpha_1,\alpha_2,...,\alpha_n) = r(A)$，得出极大无关组所包含向量的个数
3. 通过简化的阶梯型矩阵，容易看出线性无关的 $r(A)$ 列，这就找到一组极大无关组
4. 通过简化的阶梯型矩阵，容易看出其余列如何用该选定极大无关组线性表示

例 1 求向量组 $\alpha_1=\begin{pmatrix}2\\4\\2\end{pmatrix}, \alpha_2=\begin{pmatrix}1\\1\\0\end{pmatrix}, \alpha_3=\begin{pmatrix}2\\3\\1\end{pmatrix}, \alpha_4=\begin{pmatrix}3\\5\\2\end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

例 1 求向量组 $\alpha_1 = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \alpha_4 = \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{array} \\ \begin{pmatrix} 2 & 1 & 2 & 3 \\ 4 & 1 & 3 & 5 \\ 2 & 0 & 1 & 2 \end{pmatrix} \end{array}$$

例 1 求向量组 $\alpha_1=\begin{pmatrix}2\\4\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}1\\1\\0\end{pmatrix}$, $\alpha_3=\begin{pmatrix}2\\3\\1\end{pmatrix}$, $\alpha_4=\begin{pmatrix}3\\5\\2\end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{c} \begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{array} \\ \left(\begin{array}{cccc} 2 & 1 & 2 & 3 \\ 4 & 1 & 3 & 5 \\ 2 & 0 & 1 & 2 \end{array}\right) \end{array} \xrightarrow[r_3-r_1]{r_2-2r_1}$$

例 1 求向量组 $\alpha_1=\begin{pmatrix}2\\4\\2\end{pmatrix}, \alpha_2=\begin{pmatrix}1\\1\\0\end{pmatrix}, \alpha_3=\begin{pmatrix}2\\3\\1\end{pmatrix}, \alpha_4=\begin{pmatrix}3\\5\\2\end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{c}\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{array}\\ \begin{pmatrix}2 & 1 & 2 & 3\\ 4 & 1 & 3 & 5\\ 2 & 0 & 1 & 2\end{pmatrix}\end{array} \xrightarrow[r_3-r_1]{r_2-2r_1} \begin{pmatrix}2 & 1 & 2 & 3\\ 0 & -1 & -1 & -1\\ 0 & -1 & -1 & -1\end{pmatrix} \longrightarrow$$

例 1 求向量组 $\alpha_1 = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \alpha_4 = \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \end{array}$$

$$\begin{pmatrix} 2 & 1 & 2 & 3 \\ 4 & 1 & 3 & 5 \\ 2 & 0 & 1 & 2 \end{pmatrix} \xrightarrow[r_3 - r_1]{r_2 - 2r_1} \begin{pmatrix} 2 & 1 & 2 & 3 \\ 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

例 1 求向量组 $\alpha_1 = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \alpha_4 = \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{array} \\ \begin{pmatrix} 2 & 1 & 2 & 3 \\ 4 & 1 & 3 & 5 \\ 2 & 0 & 1 & 2 \end{pmatrix} \end{array} \xrightarrow[r_3 - r_1]{r_2 - 2r_1} \begin{pmatrix} 2 & 1 & 2 & 3 \\ 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{r_1 - r_2}$$

例 1 求向量组 $\alpha_1=\begin{pmatrix}2\\4\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}1\\1\\0\end{pmatrix}$, $\alpha_3=\begin{pmatrix}2\\3\\1\end{pmatrix}$, $\alpha_4=\begin{pmatrix}3\\5\\2\end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{c}\begin{matrix}\alpha_1&\alpha_2&\alpha_3&\alpha_4\end{matrix}\\ \begin{pmatrix}2&1&2&3\\4&1&3&5\\2&0&1&2\end{pmatrix}\end{array}\xrightarrow[r_3-r_1]{r_2-2r_1}\begin{pmatrix}2&1&2&3\\0&-1&-1&-1\\0&-1&-1&-1\end{pmatrix}\longrightarrow\begin{pmatrix}2&1&2&3\\0&1&1&1\\0&0&0&0\end{pmatrix}$$

$$\xrightarrow{r_1-r_2}\begin{pmatrix}2&0&1&2\\0&1&1&1\\0&0&0&0\end{pmatrix}$$

例 1 求向量组 $\alpha_1=\begin{pmatrix}2\\4\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}1\\1\\0\end{pmatrix}$, $\alpha_3=\begin{pmatrix}2\\3\\1\end{pmatrix}$, $\alpha_4=\begin{pmatrix}3\\5\\2\end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{cccc}\alpha_1&\alpha_2&\alpha_3&\alpha_4\end{array}$$

$$\begin{pmatrix}2&1&2&3\\4&1&3&5\\2&0&1&2\end{pmatrix}\xrightarrow[r_3-r_1]{r_2-2r_1}\begin{pmatrix}2&1&2&3\\0&-1&-1&-1\\0&-1&-1&-1\end{pmatrix}\longrightarrow\begin{pmatrix}2&1&2&3\\0&1&1&1\\0&0&0&0\end{pmatrix}$$

$$\xrightarrow{r_1-r_2}\begin{pmatrix}2&0&1&2\\0&1&1&1\\0&0&0&0\end{pmatrix}\xrightarrow{\frac{1}{2}\times r_1}$$

例 1 求向量组 $\alpha_1=\begin{pmatrix}2\\4\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}1\\1\\0\end{pmatrix}$, $\alpha_3=\begin{pmatrix}2\\3\\1\end{pmatrix}$, $\alpha_4=\begin{pmatrix}3\\5\\2\end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{c}\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{array}\\ \begin{pmatrix}2 & 1 & 2 & 3\\ 4 & 1 & 3 & 5\\ 2 & 0 & 1 & 2\end{pmatrix}\end{array} \xrightarrow[r_3-r_1]{r_2-2r_1} \begin{pmatrix}2 & 1 & 2 & 3\\ 0 & -1 & -1 & -1\\ 0 & -1 & -1 & -1\end{pmatrix} \longrightarrow \begin{pmatrix}2 & 1 & 2 & 3\\ 0 & 1 & 1 & 1\\ 0 & 0 & 0 & 0\end{pmatrix}$$

$$\xrightarrow{r_1-r_2} \begin{pmatrix}2 & 0 & 1 & 2\\ 0 & 1 & 1 & 1\\ 0 & 0 & 0 & 0\end{pmatrix} \xrightarrow{\frac{1}{2}\times r_1} \begin{pmatrix}1 & 0 & \frac{1}{2} & 1\\ 0 & 1 & 1 & 1\\ 0 & 0 & 0 & 0\end{pmatrix}$$

例 1 求向量组 $\alpha_1=\begin{pmatrix}2\\4\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}1\\1\\0\end{pmatrix}$, $\alpha_3=\begin{pmatrix}2\\3\\1\end{pmatrix}$, $\alpha_4=\begin{pmatrix}3\\5\\2\end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{c}\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{array}\\ \begin{pmatrix}2 & 1 & 2 & 3\\ 4 & 1 & 3 & 5\\ 2 & 0 & 1 & 2\end{pmatrix}\end{array} \xrightarrow[r_3-r_1]{r_2-2r_1} \begin{pmatrix}2 & 1 & 2 & 3\\ 0 & -1 & -1 & -1\\ 0 & -1 & -1 & -1\end{pmatrix} \longrightarrow \begin{pmatrix}2 & 1 & 2 & 3\\ 0 & 1 & 1 & 1\\ 0 & 0 & 0 & 0\end{pmatrix}$$

$$\xrightarrow{r_1-r_2} \begin{pmatrix}2 & 0 & 1 & 2\\ 0 & 1 & 1 & 1\\ 0 & 0 & 0 & 0\end{pmatrix} \xrightarrow{\frac{1}{2}\times r_1} \begin{pmatrix}1 & 0 & \frac{1}{2} & 1\\ 0 & 1 & 1 & 1\\ 0 & 0 & 0 & 0\end{pmatrix}$$

例 1 求向量组 $\alpha_1=\begin{pmatrix}2\\4\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}1\\1\\0\end{pmatrix}$, $\alpha_3=\begin{pmatrix}2\\3\\1\end{pmatrix}$, $\alpha_4=\begin{pmatrix}3\\5\\2\end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{c}\begin{matrix}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{matrix}\\ \begin{pmatrix}2 & 1 & 2 & 3\\ 4 & 1 & 3 & 5\\ 2 & 0 & 1 & 2\end{pmatrix}\end{array} \xrightarrow[r_3-r_1]{r_2-2r_1} \begin{pmatrix}2 & 1 & 2 & 3\\ 0 & -1 & -1 & -1\\ 0 & -1 & -1 & -1\end{pmatrix} \longrightarrow \begin{pmatrix}2 & 1 & 2 & 3\\ 0 & 1 & 1 & 1\\ 0 & 0 & 0 & 0\end{pmatrix}$$

$$\xrightarrow{r_1-r_2} \begin{pmatrix}2 & 0 & 1 & 2\\ 0 & 1 & 1 & 1\\ 0 & 0 & 0 & 0\end{pmatrix} \xrightarrow{\frac{1}{2}\times r_1} \begin{pmatrix}1 & 0 & \frac{1}{2} & 1\\ 0 & 1 & 1 & 1\\ 0 & 0 & 0 & 0\end{pmatrix}$$

所以

- $r(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=r(A)=2$；

例 1 求向量组 $\alpha_1=\begin{pmatrix}2\\4\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}1\\1\\0\end{pmatrix}$, $\alpha_3=\begin{pmatrix}2\\3\\1\end{pmatrix}$, $\alpha_4=\begin{pmatrix}3\\5\\2\end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{c} \begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{array} \\ \begin{pmatrix}2 & 1 & 2 & 3\\ 4 & 1 & 3 & 5\\ 2 & 0 & 1 & 2\end{pmatrix}\end{array} \xrightarrow[r_3-r_1]{r_2-2r_1} \begin{pmatrix}2 & 1 & 2 & 3\\ 0 & -1 & -1 & -1\\ 0 & -1 & -1 & -1\end{pmatrix} \longrightarrow \begin{pmatrix}2 & 1 & 2 & 3\\ 0 & 1 & 1 & 1\\ 0 & 0 & 0 & 0\end{pmatrix}$$

$$\xrightarrow{r_1-r_2} \begin{pmatrix}2 & 0 & 1 & 2\\ 0 & 1 & 1 & 1\\ 0 & 0 & 0 & 0\end{pmatrix} \xrightarrow{\frac{1}{2}\times r_1} \begin{pmatrix}1 & 0 & \frac{1}{2} & 1\\ 0 & 1 & 1 & 1\\ 0 & 0 & 0 & 0\end{pmatrix}$$

所以

- $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4)=r(A)=2$；

例 1 求向量组 $\alpha_1=\begin{pmatrix}2\\4\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}1\\1\\0\end{pmatrix}$, $\alpha_3=\begin{pmatrix}2\\3\\1\end{pmatrix}$, $\alpha_4=\begin{pmatrix}3\\5\\2\end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{c} \begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{array} \\ \begin{pmatrix}2&1&2&3\\4&1&3&5\\2&0&1&2\end{pmatrix}\end{array} \xrightarrow[r_3-r_1]{r_2-2r_1} \begin{pmatrix}2&1&2&3\\0&-1&-1&-1\\0&-1&-1&-1\end{pmatrix} \longrightarrow \begin{pmatrix}2&1&2&3\\0&1&1&1\\0&0&0&0\end{pmatrix}$$

$$\xrightarrow{r_1-r_2} \begin{pmatrix}2&0&1&2\\0&1&1&1\\0&0&0&0\end{pmatrix} \xrightarrow{\frac{1}{2}\times r_1} \begin{pmatrix}1&0&\frac{1}{2}&1\\0&1&1&1\\0&0&0&0\end{pmatrix}$$

所以

- $r(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=r(A)=2$；
- α_1,α_2 是极大无关组；

例 1 求向量组 $\alpha_1 = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{array} \\ \begin{pmatrix} 2 & 1 & 2 & 3 \\ 4 & 1 & 3 & 5 \\ 2 & 0 & 1 & 2 \end{pmatrix} \end{array} \xrightarrow[r_3-r_1]{r_2-2r_1} \begin{pmatrix} 2 & 1 & 2 & 3 \\ 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{r_1-r_2} \begin{pmatrix} 2 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\frac{1}{2}\times r_1} \begin{pmatrix} 1 & 0 & \frac{1}{2} & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

所以

- $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2$；
- α_1, α_2 是极大无关组；
- $\alpha_3 = \frac{1}{2}\alpha_1 + \alpha_2, \quad \alpha_4 = \alpha_1 + \alpha_2$

例 2 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}, \alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

例 2 求向量组 $\alpha_1 = \begin{pmatrix}1\\2\\1\\0\end{pmatrix}$, $\alpha_2 = \begin{pmatrix}0\\1\\1\\2\end{pmatrix}$, $\alpha_3 = \begin{pmatrix}1\\1\\0\\-2\end{pmatrix}$, $\alpha_4 = \begin{pmatrix}2\\4\\3\\3\end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 0 & 2 & -2 & 3 \end{pmatrix} \end{array}$$

例 2 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}, \alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 0 & 2 & -2 & 3 \end{pmatrix} \end{array} \xrightarrow[r_3 - r_1]{r_2 - 2r_1}$$

例 2 求向量组 $\alpha_1=\begin{pmatrix}1\\2\\1\\0\end{pmatrix}$, $\alpha_2=\begin{pmatrix}0\\1\\1\\2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}1\\1\\0\\-2\end{pmatrix}$, $\alpha_4=\begin{pmatrix}2\\4\\3\\3\end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{array}$$

$$\begin{pmatrix}1 & 0 & 1 & 2\\2 & 1 & 1 & 4\\1 & 1 & 0 & 3\\0 & 2 & -2 & 3\end{pmatrix}\xrightarrow[r_3-r_1]{r_2-2r_1}\begin{pmatrix}1 & 0 & 1 & 2\\0 & 1 & -1 & 0\\0 & 1 & -1 & 1\\0 & 2 & -2 & 3\end{pmatrix}$$

例 2 求向量组 $\alpha_1=\begin{pmatrix}1\\2\\1\\0\end{pmatrix}, \alpha_2=\begin{pmatrix}0\\1\\1\\2\end{pmatrix}, \alpha_3=\begin{pmatrix}1\\1\\0\\-2\end{pmatrix}, \alpha_4=\begin{pmatrix}2\\4\\3\\3\end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{c}\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{array}\\ \begin{pmatrix}1 & 0 & 1 & 2\\ 2 & 1 & 1 & 4\\ 1 & 1 & 0 & 3\\ 0 & 2 & -2 & 3\end{pmatrix}\end{array} \xrightarrow[r_3-r_1]{r_2-2r_1} \begin{pmatrix}1 & 0 & 1 & 2\\ 0 & 1 & -1 & 0\\ 0 & 1 & -1 & 1\\ 0 & 2 & -2 & 3\end{pmatrix} \xrightarrow[r_4-2r_2]{r_3-r_2}$$

例 2 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}, \alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解 $\begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 0 & 2 & -2 & 3 \end{pmatrix} \end{array} \xrightarrow[r_3-r_1]{r_2-2r_1} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow[r_4-2r_2]{r_3-r_2} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

例 2 求向量组 $\alpha_1=\begin{pmatrix}1\\2\\1\\0\end{pmatrix}$, $\alpha_2=\begin{pmatrix}0\\1\\1\\2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}1\\1\\0\\-2\end{pmatrix}$, $\alpha_4=\begin{pmatrix}2\\4\\3\\3\end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 0 & 2 & -2 & 3 \end{pmatrix} \end{array} \xrightarrow[r_3-r_1]{r_2-2r_1} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow[r_4-2r_2]{r_3-r_2} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$\xrightarrow[r_1-2r_3]{r_4-3r_3}$$

例 2 求向量组 $\alpha_1=\begin{pmatrix}1\\2\\1\\0\end{pmatrix}$, $\alpha_2=\begin{pmatrix}0\\1\\1\\2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}1\\1\\0\\-2\end{pmatrix}$, $\alpha_4=\begin{pmatrix}2\\4\\3\\3\end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{c}\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{array}\\ \begin{pmatrix}1&0&1&2\\2&1&1&4\\1&1&0&3\\0&2&-2&3\end{pmatrix}\end{array} \xrightarrow[r_3-r_1]{r_2-2r_1} \begin{pmatrix}1&0&1&2\\0&1&-1&0\\0&1&-1&1\\0&2&-2&3\end{pmatrix} \xrightarrow[r_4-2r_2]{r_3-r_2} \begin{pmatrix}1&0&1&2\\0&1&-1&0\\0&0&0&1\\0&0&0&3\end{pmatrix}$$

$$\xrightarrow[r_1-2r_3]{r_4-3r_3} \begin{pmatrix}1&0&1&0\\0&1&-1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}$$

例 2 求向量组 $\alpha_1=\begin{pmatrix}1\\2\\1\\0\end{pmatrix}$, $\alpha_2=\begin{pmatrix}0\\1\\1\\2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}1\\1\\0\\-2\end{pmatrix}$, $\alpha_4=\begin{pmatrix}2\\4\\3\\3\end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{c}\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{array}\\ \begin{pmatrix}1&0&1&2\\2&1&1&4\\1&1&0&3\\0&2&-2&3\end{pmatrix}\end{array}\xrightarrow[r_3-r_1]{r_2-2r_1}\begin{pmatrix}1&0&1&2\\0&1&-1&0\\0&1&-1&1\\0&2&-2&3\end{pmatrix}\xrightarrow[r_4-2r_2]{r_3-r_2}\begin{pmatrix}1&0&1&2\\0&1&-1&0\\0&0&0&1\\0&0&0&3\end{pmatrix}$$

$$\xrightarrow[r_1-2r_3]{r_4-3r_3}\begin{pmatrix}1&0&1&0\\0&1&-1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}$$

例 2 求向量组 $\alpha_1=\begin{pmatrix}1\\2\\1\\0\end{pmatrix}$, $\alpha_2=\begin{pmatrix}0\\1\\1\\2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}1\\1\\0\\-2\end{pmatrix}$, $\alpha_4=\begin{pmatrix}2\\4\\3\\3\end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{cccc}\alpha_1&\alpha_2&\alpha_3&\alpha_4\\ \end{array}$$

$$\begin{pmatrix}1&0&1&2\\2&1&1&4\\1&1&0&3\\0&2&-2&3\end{pmatrix}\xrightarrow[r_3-r_1]{r_2-2r_1}\begin{pmatrix}1&0&1&2\\0&1&-1&0\\0&1&-1&1\\0&2&-2&3\end{pmatrix}\xrightarrow[r_4-2r_2]{r_3-r_2}\begin{pmatrix}1&0&1&2\\0&1&-1&0\\0&0&0&1\\0&0&0&3\end{pmatrix}$$

$$\xrightarrow[r_1-2r_3]{r_4-3r_3}\begin{pmatrix}1&0&1&0\\0&1&-1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}$$

所以

例 2 求向量组 $\alpha_1=\begin{pmatrix}1\\2\\1\\0\end{pmatrix}$, $\alpha_2=\begin{pmatrix}0\\1\\1\\2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}1\\1\\0\\-2\end{pmatrix}$, $\alpha_4=\begin{pmatrix}2\\4\\3\\3\end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 0 & 2 & -2 & 3 \end{pmatrix} \end{array} \xrightarrow[r_3-r_1]{r_2-2r_1} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow[r_4-2r_2]{r_3-r_2} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$\xrightarrow[r_1-2r_3]{r_4-3r_3} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

所以

- $r(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=r(A)=3$;

例 2 求向量组 $\alpha_1=\begin{pmatrix}1\\2\\1\\0\end{pmatrix}$, $\alpha_2=\begin{pmatrix}0\\1\\1\\2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}1\\1\\0\\-2\end{pmatrix}$, $\alpha_4=\begin{pmatrix}2\\4\\3\\3\end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{array}$$

$$\begin{pmatrix}1&0&1&2\\2&1&1&4\\1&1&0&3\\0&2&-2&3\end{pmatrix}\xrightarrow[r_3-r_1]{r_2-2r_1}\begin{pmatrix}1&0&1&2\\0&1&-1&0\\0&1&-1&1\\0&2&-2&3\end{pmatrix}\xrightarrow[r_4-2r_2]{r_3-r_2}\begin{pmatrix}1&0&1&2\\0&1&-1&0\\0&0&0&1\\0&0&0&3\end{pmatrix}$$

$$\xrightarrow[r_1-2r_3]{r_4-3r_3}\begin{pmatrix}1&0&1&0\\0&1&-1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}$$

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- $r(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=r(A)=3$；

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解

$$\begin{array}{c}\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{array}\\ \begin{pmatrix}1 & 0 & 1 & 2\\ 2 & 1 & 1 & 4\\ 1 & 1 & 0 & 3\\ 0 & 2 & -2 & 3\end{pmatrix}\end{array} \xrightarrow[r_3-r_1]{r_2-2r_1} \begin{pmatrix}1 & 0 & 1 & 2\\ 0 & 1 & -1 & 0\\ 0 & 1 & -1 & 1\\ 0 & 2 & -2 & 3\end{pmatrix} \xrightarrow[r_4-2r_2]{r_3-r_2} \begin{pmatrix}1 & 0 & 1 & 2\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 3\end{pmatrix}$$

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所以

- $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 3$；
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$$\xrightarrow[r_1-2r_3]{r_4-3r_3}\begin{pmatrix}1&0&1&0\\0&1&-1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}$$

所以

- $r(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=r(A)=3$；
- $\alpha_1,\alpha_2,\alpha_4$ 是极大无关组；
- $\alpha_3=\alpha_1-\alpha_2$

例 3 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}, \alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

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$$\begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{array} \\ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \end{array}$$

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解

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例 3 求向量组 $\alpha_1=\begin{pmatrix}1\\2\\3\\4\end{pmatrix}, \alpha_2=\begin{pmatrix}2\\3\\4\\5\end{pmatrix}, \alpha_3=\begin{pmatrix}3\\4\\5\\6\end{pmatrix}, \alpha_4=\begin{pmatrix}4\\5\\6\\7\end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

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解

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$$\xrightarrow[r_4-3r_2]{r_3-2r_2} \begin{pmatrix}1 & 2 & 3 & 4\\ 0 & -1 & -2 & -3\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$$

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$$\xrightarrow[r_4-3r_2]{r_3-2r_2} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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解

$$\begin{array}{c}\begin{matrix}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{matrix}\\ \begin{pmatrix}1&2&3&4\\2&3&4&5\\3&4&5&6\\4&5&6&7\end{pmatrix}\end{array} \xrightarrow[\substack{r_3-3r_1\\ r_4-4r_1}]{r_2-2r_1} \begin{pmatrix}1&2&3&4\\0&-1&-2&-3\\0&-2&-4&-6\\0&-3&-6&-9\end{pmatrix}$$

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所以

- $r(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=r(A)=2$;

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- $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2$；
- α_1, α_2 是极大无关组；

例 3 求向量组 $\alpha_1=\begin{pmatrix}1\\2\\3\\4\end{pmatrix}, \alpha_2=\begin{pmatrix}2\\3\\4\\5\end{pmatrix}, \alpha_3=\begin{pmatrix}3\\4\\5\\6\end{pmatrix}, \alpha_4=\begin{pmatrix}4\\5\\6\\7\end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\left(\begin{array}{cccc}\alpha_1&\alpha_2&\alpha_3&\alpha_4\\ 1&2&3&4\\ 2&3&4&5\\ 3&4&5&6\\ 4&5&6&7\end{array}\right) \xrightarrow[\substack{r_3-3r_1\\ r_4-4r_1}]{r_2-2r_1} \begin{pmatrix}1&2&3&4\\0&-1&-2&-3\\0&-2&-4&-6\\0&-3&-6&-9\end{pmatrix}$$

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所以

- $r(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=r(A)=2$；
- α_1,α_2 是极大无关组；
- $\alpha_3=-\alpha_1+2\alpha_2$

例 3 求向量组 $\alpha_1=\begin{pmatrix}1\\2\\3\\4\end{pmatrix}, \alpha_2=\begin{pmatrix}2\\3\\4\\5\end{pmatrix}, \alpha_3=\begin{pmatrix}3\\4\\5\\6\end{pmatrix}, \alpha_4=\begin{pmatrix}4\\5\\6\\7\end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{c} \begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{array} \\ \begin{pmatrix}1&2&3&4\\2&3&4&5\\3&4&5&6\\4&5&6&7\end{pmatrix}\end{array} \xrightarrow[\substack{r_3-3r_1\\ r_4-4r_1}]{r_2-2r_1} \begin{pmatrix}1&2&3&4\\0&-1&-2&-3\\0&-2&-4&-6\\0&-3&-6&-9\end{pmatrix}$$

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- $r(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=r(A)=2$；
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- $\alpha_3=-\alpha_1+2\alpha_2$， $\alpha_4=-2\alpha_1+3\alpha_2$

例 假设向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 可由 $\beta_1, \beta_2, \ldots, \beta_t$ 线性表示，则

$$r(\alpha_1, \alpha_2, \ldots, \alpha_s) \leq r(\beta_1, \beta_2, \ldots, \beta_t).$$

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证明 设

$$r_1 = r(\alpha_1, \alpha_2, \ldots, \alpha_s),$$

$$r_2 = r(\beta_1, \beta_2, \ldots, \beta_t),$$

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证明 设

$$r_1 = r(\alpha_1, \alpha_2, \ldots, \alpha_s), \quad \alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_{r_1}} \text{ 是极大无关组}$$

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注意到 $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_{r_1}}$ 能由 $\beta_{j_1}, \beta_{j_2}, \ldots, \beta_{j_{r_2}}$ 线性表示，

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定理 设有向量组

$$(A): \quad \alpha_1, \alpha_2, \ldots, \alpha_s$$

$$(B): \quad \beta_1, \beta_2, \ldots, \beta_t$$

若它们等价，

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$$(A): \quad \alpha_1, \alpha_2, \ldots, \alpha_s$$

$$(B): \quad \beta_1, \beta_2, \ldots, \beta_t$$

若它们等价，则 $r(\alpha_1, \alpha_2, \ldots, \alpha_s) = r(\beta_1, \beta_2, \ldots, \beta_t)$。

例 设 $A_{m\times n}, B_{n\times s}$ 为矩阵，则 $r(AB) \le \min\{r(A), r(B)\}$。

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$$\underbrace{\begin{array}{cccc} \gamma_1 & & & \\ \left(\begin{matrix} c_{11} \\ c_{21} \\ \vdots \\ c_{m1} \end{matrix}\right. & \begin{matrix} c_{12} \\ c_{22} \\ \vdots \\ c_{m2} \end{matrix} & \begin{matrix} \cdots \\ \cdots \\ \ddots \\ \cdots \end{matrix} & \left.\begin{matrix} c_{1s} \\ c_{2s} \\ \vdots \\ c_{ms} \end{matrix}\right) \end{array}}_{C} = \underbrace{\begin{array}{cccc} \alpha_1 & \alpha_2 & & \alpha_n \\ \left(\begin{matrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{matrix}\right. & \begin{matrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{matrix} & \begin{matrix} \cdots \\ \cdots \\ \ddots \\ \cdots \end{matrix} & \left.\begin{matrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{matrix}\right) \end{array}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}$$

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即

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证明 设 $AB=C_{m\times s}$

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$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B} \begin{array}{l} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{array}$$

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$$\underbrace{\begin{array}{c}\delta_1\\ \\ \\ \\ \end{array}\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s}\\ c_{21} & c_{22} & \cdots & c_{2s}\\ \vdots & \vdots & \ddots & \vdots\\ c_{m1} & c_{m2} & \cdots & c_{ms}\end{pmatrix}}_{C}=\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{pmatrix}}_{A}\underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s}\\ b_{21} & b_{22} & \cdots & b_{2s}\\ \vdots & \vdots & \ddots & \vdots\\ b_{n1} & b_{n2} & \cdots & b_{ns}\end{pmatrix}}_{B}\begin{array}{c}\beta_1\\ \beta_2\\ \vdots\\ \beta_n\end{array}$$

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$$\underbrace{\begin{array}{c}\delta_1\\\delta_2\\ \\ \\ \end{array}\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}\begin{array}{c}\beta_1\\\beta_2\\\vdots\\\beta_n\end{array}}_{B}$$

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即
$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$$

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$$\begin{pmatrix}\delta_1\\ \delta_2\\ \vdots\\ \delta_m\end{pmatrix}=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{pmatrix}\begin{pmatrix}\beta_1\\ \beta_2\\ \vdots\\ \beta_n\end{pmatrix}$$

$$\Rightarrow\quad \delta_1=a_{11}\beta_1+a_{12}\beta_2+\cdots+a_{1n}\beta_n$$

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即

$$\begin{pmatrix}\delta_1\\ \delta_2\\ \vdots\\ \delta_m\end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{pmatrix}\begin{pmatrix}\beta_1\\ \beta_2\\ \vdots\\ \beta_n\end{pmatrix}$$

$$\Rightarrow \quad \delta_1 = a_{11}\beta_1 + a_{12}\beta_2 + \cdots + a_{1n}\beta_n \quad \text{等等}$$

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可见 $\delta_1, \ldots, \delta_m$ 由 $\beta_1, \ldots, \beta_n$ 线性表示，

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证明 设 $AB = C_{m\times s}$

$$\underbrace{\begin{array}{c|cccc|} \delta_1 & c_{11} & c_{12} & \cdots & c_{1s} \\ \delta_2 & c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \delta_m & c_{m1} & c_{m2} & \cdots & c_{ms} \end{array}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{array}{|cccc|c} b_{11} & b_{12} & \cdots & b_{1s} & \beta_1 \\ b_{21} & b_{22} & \cdots & b_{2s} & \beta_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} & \beta_n \end{array}}_{B}$$

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即

$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$$

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$$r(\delta_1, \ldots, \delta_m) \leq r(\beta_1, \ldots, \beta_n) = r(B)$$

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即

$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$$

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