第12章 d: 函数展开成幂级数

数学系 梁卓滨

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Outline



$$f(x) \neq a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

$$f(x) \stackrel{?}{=} a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

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$$n \cdot (n-1) \cdot \cdot \cdot (n-k+1) \cdot (x-x_0)^{n-k}$$

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$$= \sum_{n=0}^{\infty} a_n \cdot n \cdot (n-1) \cdots (n-k+1) \cdot (x - x_0)^{n-k}$$

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$$= \sum_{n=k}^{\infty} a_n \cdot n \cdot (n-1) \cdots (n-k+1) \cdot (x - x_0)^{n-k}$$
$$= a_k \cdot k!$$

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$$= \sum_{n=k}^{\infty} a_n \cdot n \cdot (n-1) \cdots (n-k+1) \cdot (x - x_0)^{n-k}$$
$$= a_k \cdot k! + a_{k+1} \cdot (k+1) \cdots 2 \cdot (x - x_0)$$

$$f(x) \stackrel{?}{=} a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

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$$= \sum_{n=k}^{\infty} a_n \cdot n \cdot (n-1) \cdots (n-k+1) \cdot (x - x_0)^{n-k}$$

$$= a_k \cdot k! + a_{k+1} \cdot (k+1) \cdots 2 \cdot (x - x_0)$$

$$+ a_{k+2} \cdot (k+2) \cdots 3 \cdot (x - x_0)^2 + \cdots$$

$$f(x) \stackrel{?}{=} a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots + a_n(x-x_0)^n + \cdots$$

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证明 逐项求 k 次导得:

$$f^{(k)}(x) = \left[\sum_{n=0}^{\infty} a_n (x - x_0)^n\right]^{(k)} = \sum_{n=0}^{\infty} \left[a_n (x - x_0)^n\right]^{(k)}$$

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$$+ a_{k+2} \cdot (k+2) \cdots 3 \cdot (x - x_0)^2 + \cdots$$

取 $x = x_0$ 得 $a_k = \frac{1}{k!} f^{(k)}(x_0)$



$$f(x) \stackrel{?}{=} a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

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注1

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$$f(x_0) f'(x_0)$$

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$$f(x_0)+f'(x_0)(x-x_0)+\frac{1}{2!}f''(x_0)(x-x_0)^2+\cdots+\frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n+\cdots$$

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- 此级数称为 f(x) 在 x_0 处的 泰勒级数。

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- 此级数称为 f(x) 在 x_0 处的 泰勒级数。
- 级数前 n+1 项的部分和记为 p_n , 称为 n 次泰勒多项式

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注 2 $f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0) (x - x_0)^n$

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 $f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0) (x - x_0)^n \quad \Longleftrightarrow \quad f(x) = \lim_{n \to \infty} p_n(x)$

解 取
$$x_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

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当
$$f(x) = e^x$$
时,

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⇒
$$\bar{x}$$
 \$\text{\$\pi\$}\$ \$\pi\$\$ \$\pi\$ \$\p

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 \$\text{\$\pi\$}\$ \$\pi\$\$ \$\pi\$ \$\pi\$ \$\pi\$\$ \$\pi\$ \$\pi\$\$ \$\pi\$\$ \$\pi\$ \$\pi

 $\frac{1}{2!}$ $\frac{1}{3!}$ $\frac{1}{n!}$

注 n 次泰勒多项式是:

$$p_n(x) =$$

解 取 $x_0 = 0$ 时,泰勒级数是

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⇒
$$\bar{x}$$
 \$\text{\$\pi\$}\$ \$\tag{\pi}\$ \$\ta

$$\Rightarrow \quad \text{$\frac{3}{4}$ } \text{$\frac{3!}{2!}$ } + \frac{-1}{3!} \text{$\frac{1}{4}$ } + \cdots + \frac{-1}{n!} \text{$\frac{1}{4}$ } + \cdots$$

注 n 次泰勒多项式是:

$$p_n(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots + \frac{1}{n!}x^n$$

例 2 求 $f(x) = \sin x$ 在 x = 0 处的泰勒级数。

$$\mathbf{H}$$
 取 $x_0 = 0$ 时,泰勒级数是

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当 $f(x) = \sin x$ 时,

	$f^{(n)}(x)$	$f^{(n)}(0)$
n = 0, 4, 8	sin x	0
<i>n</i> = 1, 5, 9	cosx	1
n = 2, 6, 10	— sin <i>x</i>	0
n = 3, 7, 11	— cos x	-1

解 取 $x_0 = 0$ 时,泰勒级数是

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	$f^{(n)}(x) = \sin(x + \frac{n}{2}\pi)$	$f^{(n)}(0)$
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所以泰勒级数是

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \cdots$$

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$$x - \frac{1}{3!}x^{3} + \frac{1}{5!}x^{5} - \frac{1}{7!}x^{7} + \frac{1}{9!}x^{9} - \frac{1}{11!}x^{11} + \dots + (-1)^{m} \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

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$$p_1 = x$$
;

$$x - \frac{1}{3!}x^{3} + \frac{1}{5!}x^{5} - \frac{1}{7!}x^{7} + \frac{1}{9!}x^{9} - \frac{1}{11!}x^{11} + \dots + (-1)^{m} \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

$$p_1 = p_2 = x$$
;

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

$$p_1 = p_2 = x;$$

 $p_3 = x - \frac{1}{3!}x^3;$

$$=x-\frac{1}{3!}x^3$$

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

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$$p_1 = p_2 = x;$$

$$p_3 = p_4 = x - \frac{1}{3!}x^3;$$

$$p_5 = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5;$$

$$-\frac{1}{2!}x^3 + \frac{1}{5!}x^5;$$

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

$$p_1 = p_2 = x;$$

$$p_3 = p_4 = x - \frac{1}{3!}x^3;$$

$$p_5 = p_6 = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5;$$

$$\frac{1}{3!}x^3 + \frac{1}{5!}x^5$$
;

 $p_1 = p_2 = x$;

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• sin x 的 n 次泰勒多项式是:

$$p_3 = p_4 = x - \frac{1}{3!}x^3;$$

$$p_5 = p_6 = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5;$$

$$\vdots$$

 p_{2m+1}



$$\frac{1}{-}x^3 + \frac{1}{-}x^5 - \frac{1}{-}x^7 +$$

 $p_1 = p_2 = x$;

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

sin x 的 n 次泰勒多项式是:

 $p_3 = p_4 = x - \frac{1}{3!}x^3;$

 $p_5 = p_6 = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5;$

 $= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1}$

- sin x 的泰勒级数是:

 p_{2m+1}

$$\frac{1}{-}$$
 $x^3 + \frac{1}{-}$ $x^5 - \frac{1}{-}$ $x^7 + \frac{1}{-}$

 $p_1 = p_2 = x$;

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

sin x 的 n 次泰勒多项式是:

 $p_3 = p_4 = x - \frac{1}{3!}x^3;$

 $p_5 = p_6 = x - \frac{1}{2!}x^3 + \frac{1}{5!}x^5;$

 $p_{2m+1} = p_{2m+2} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1}$

解 取
$$x_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = \cos x$$
时,

解 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = \cos x$ 时,

	$f^{(n)}(x)$	$f^{(n)}(0)$
n = 0, 4, 8	cosx	1
<i>n</i> = 1, 5, 9	— sin <i>x</i>	0
n = 2, 6, 10	— cos x	-1
n = 3, 7, 11	sin x	0

解 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = \cos x$ 时,

	$f^{(n)}(x) = \cos(x + \frac{n}{2}\pi)$	$f^{(n)}(0)$
n = 0, 4, 8	cosx	1
<i>n</i> = 1, 5, 9	— sin <i>x</i>	0
n = 2, 6, 10	— cos x	-1
n = 3, 7, 11	sin x	0

解 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = \cos x$ 时,

	$f^{(n)}(x) = \cos(x + \frac{n}{2}\pi)$	$f^{(n)}(0) = \cos(\frac{n}{2}\pi)$
n = 0, 4, 8	cosx	1
$n = 1, 5, 9 \dots$	— sin <i>x</i>	0
n = 2, 6, 10	— cos <i>x</i>	-1
n = 3, 7, 11	sin x	0

例 $3 \, \bar{x} f(x) = \cos x \, \bar{x} = 0$ 处的泰勒级数。

 \mathbf{H} 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = \cos x$ 时,

	$f^{(n)}(x) = \cos(x + \frac{n}{2}\pi)$	$f^{(n)}(0) = \cos(\frac{n}{2}\pi)$
n = 0, 4, 8	cosx	1
$n = 1, 5, 9 \dots$	— sin <i>x</i>	0
n = 2, 6, 10	— cos x	-1
n = 3, 7, 11	sin x	0

所以泰勒级数是

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \cdots$$



解 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = \cos x$ 时,

	$f^{(n)}(x) = \cos(x + \frac{n}{2}\pi)$	$f^{(n)}(0) = \cos(\frac{n}{2}\pi)$
n = 0, 4, 8	cosx	1
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$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

$$p_0 = 1;$$

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

$$p_0 = p_1 = 1$$
;

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

$$p_0 = p_1 = 1;$$
 $p_2 = 1 - \frac{1}{2!}x^2;$

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

• $\cos x$ 的 n 次泰勒多项式是: $p_0 = p_1 = 1$;

$$p_2 = p_3 = 1 - \frac{1}{2!}x^2;$$

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

cos x 的 n 次泰勒多项式是:

 $p_0 = p_1 = 1$;

$$p_2 = p_3 = 1 - \frac{1}{2!}x^2;$$

$$p_4 = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4;$$



$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

$$p_0 = p_1 = 1;$$

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$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

• cos x 的 n 次泰勒多项式是:

$$p_0 = p_1 = 1;$$

$$p_2 = p_3 = 1 - \frac{1}{2!}x^2;$$

$$p_4 = p_5 = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4;$$
:

 $p_{2m}(x)$



$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

 $p_0 = p_1 = 1$;

$$p_2 = p_3 = 1 - \frac{1}{2!}x^2;$$

$$p_4 = p_5 = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4;$$

 $p_{2m}(x)$

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

2! 4! 6! 8! 10! (2*m*)!

•
$$\cos x$$
 的 n 次泰勒多项式是:

 $p_0 = p_1 = 1$;

 $p_2 = p_3 = 1 - \frac{1}{2!}x^2;$ $p_4 = p_5 = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4$;

$$0_5 = 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^2;$$

 $p_{2m}(x) = p_{2m+1}(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots + (-1)^m \frac{1}{(2m)!}x^{2m}$

 $\mathbf{H} \mathbf{H} \mathbf{X}_0 = \mathbf{0} \mathbf{H}$,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = \ln(1+x)$$
时,

解 取
$$x_0 = 0$$
 时,泰勒级数是
$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

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 $f = \ln(1+x)$, $f' = \frac{1}{1+x}$,

$$f = \ln(1+x), \quad f' = \frac{1}{1+x}$$

解 取
$$x_0 = 0$$
 时,泰勒级数是
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当
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 $f = \ln(1+x)$, $f' = \frac{1}{1+x}$, $f'' = \frac{(-1)}{(1+x)^2}$,

解 取
$$x_0 = 0$$
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解 取
$$x_0 = 0$$
 时,泰勒级数是
$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

$$f = \ln(1+x), \quad f' = \frac{1}{1+x}, \quad f'' = \frac{(-1)}{(1+x)^2}, \quad f''' = \frac{2}{(1+x)^3},$$

$$f^{(4)} = \frac{-2 \cdot 3}{(1+x)^4},$$

解 取
$$x_0 = 0$$
 时,泰勒级数是
$$f(0) + f'(0)x + \frac{f''(0)}{x^2} + \frac{f'''(0)}{x^3} + \dots + \frac{f^{(n)}(0)}{x^n} + \dots$$

$$f^{(4)} = \frac{-2 \cdot 3}{(1+x)^4}, \quad f^{(5)} = \frac{2 \cdot 3 \cdot 4}{(1+x)^5}, \dots,$$

$$f^{(4)} = \frac{-2 \cdot 3}{(1+x)^4}, \quad f^{(5)} = \frac{2 \cdot 3 \cdot 4}{(1+x)^5}, \dots,$$

 \mathbf{H} 取 $\mathbf{x}_0 = \mathbf{0}$ 时,泰勒级数是

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$$f^{(4)} = \frac{-2 \cdot 3}{(1+x)^4}, \quad f^{(5)} = \frac{2 \cdot 3 \cdot 4}{(1+x)^5}, \dots, f^{(n)} = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}, \dots$$

解 取
$$x_0 = 0$$
 时,泰勒级数是
$$f''(0) = f'''(0) = f'''(0) = f^{(n)}(0) = f^{(n)}(0)$$

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = \ln(1+x)$$
 时,
 $f = \ln(1+x)$, $f' = \frac{1}{1+x}$, $f'' = \frac{(-1)}{(1+x)^2}$, $f''' = \frac{2}{(1+x)^3}$,

$$f^{(4)} = \frac{-2 \cdot 3}{(1+x)^4}, \quad f^{(5)} = \frac{2 \cdot 3 \cdot 4}{(1+x)^5}, \dots, f^{(n)} = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}, \dots$$

所以
$$\frac{1}{n!}f^{(n)}(0) = \frac{(-1)^{n-1}}{n}$$
,

例 $4 \, \bar{x} \, f(x) = \ln(1+x) \, \bar{x} \, x = 0 \,$ 处泰勒级数。

m 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = \ln(1+x)$$
时,

$$f = \ln(1+x), \quad f' = \frac{1}{1+x}, \quad f'' = \frac{(-1)}{(1+x)^2}, \quad f''' = \frac{2}{(1+x)^3},$$

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所以
$$\frac{1}{n!}f^{(n)}(0) = \frac{(-1)^{n-1}}{n}$$
,泰勒级数是
$$x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{1}{4}x^4 + \cdots$$

例 $4 \, \bar{x} f(x) = \ln(1+x) \, \bar{x} = 0 \,$ 处泰勒级数。

解 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = \ln(1+x)$ 时,

$$f = \ln(1+x), \quad f' = \frac{1}{1+x}, \quad f'' = \frac{(-1)}{(1+x)^2}, \quad f''' = \frac{2}{(1+x)^3},$$

$$f^{(4)} = \frac{-2 \cdot 3}{(1+x)^4}, \quad f^{(5)} = \frac{2 \cdot 3 \cdot 4}{(1+x)^5}, \dots, f^{(n)} = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}, \dots$$

所以
$$\frac{1}{n!}f^{(n)}(0) = \frac{(-1)^{n-1}}{n}$$
,泰勒级数是
$$x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{1}{4}x^4 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots$$



$$\mathbf{H}$$
 取 $\mathbf{x}_0 = \mathbf{0}$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{}$$

$$f(0) + f'(0)x + \frac{f''(0)}{2}$$

$$f(0) + f'(0)x + \frac{1}{2!}x^{2}$$

例 4 求 $f(x) = \ln(1 + x)$ 在 x = 0 处泰勒级数。

$$f(0) + f'(0)x + \frac{f'(0)}{2!}$$

$$f(0) + f'(0)x + \frac{f''(0)}{2}$$

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = \ln(1+x)$$
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 $f = \ln(1+x)$, $f' = \frac{1}{1+x}$, $f'' = \frac{(-1)}{(1+x)^2}$, $f''' = \frac{2}{(1+x)^3}$,

$$f^{(4)} = \frac{-2 \cdot 3}{(1+x)^4}, \quad f^{(5)} = \frac{2 \cdot 3 \cdot 4}{(1+x)^5}, \dots, f^{(n)} = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}, \dots$$

$$x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots$$

$$p_n(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots + (-1)^{n-1}\frac{1}{n}x^n$$

解 取
$$x_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = (1+x)^{\alpha}$$
 时,

$$\mathbf{H}$$
 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = (1+x)^{\alpha}$$
 时,
 $f = (1+x)^{\alpha}$, $f' = \alpha(1+x)^{\alpha-1}$,

$$\mathbf{H}$$
 取 $\mathbf{x}_0 = \mathbf{0}$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{2!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = (1+x)^{\alpha}$$
时,

$$f = (1+x)^{\alpha}, \quad f' = \alpha(1+x)^{\alpha-1}, \quad f'' = \alpha(\alpha-1)(1+x)^{\alpha-2},$$

解 取
$$x_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

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$$\ldots, f^{(n)} = \alpha(\alpha - 1) \cdots (\alpha - n + 1)(1 + x)^{\alpha - n}, \cdots$$

 \mathbf{H} 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

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所以
$$\frac{1}{n!}f^{(n)}(0) = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$$
,

 $\mathbf{H} \mathbf{H} \mathbf{X}_0 = \mathbf{0} \mathbf{H}$,泰勒级数是

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$$f(x) = (1+x)^{\alpha}$$
 时,

$$f = (1+x)^{\alpha}$$
, $f' = \alpha(1+x)^{\alpha-1}$, $f'' = \alpha(\alpha-1)(1+x)^{\alpha-2}$,

$$\ldots, f^{(n)} = \alpha(\alpha - 1) \cdots (\alpha - n + 1)(1 + x)^{\alpha - n}, \cdots$$

所以
$$\frac{1}{n!}f^{(n)}(0) = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$$
,泰勒级数是
$$1 + \alpha x + \frac{\alpha(\alpha-1)}{m!}x^2 + \cdots$$

解
$$\mathbf{x}_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = (1+x)^{\alpha}$$
 时,

$$f = (1+x)^{\alpha}, \quad f' = \alpha(1+x)^{\alpha-1}, \quad f'' = \alpha(\alpha-1)(1+x)^{\alpha-2},$$

$$\ldots, f^{(n)} = \alpha(\alpha - 1) \cdots (\alpha - n + 1)(1 + x)^{\alpha - n}, \cdots$$

所以
$$\frac{1}{n!}f^{(n)}(0) = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$$
,泰勒级数是
$$1 + \alpha x + \frac{\alpha(\alpha-1)}{n!}x^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n + \cdots$$

$$1 + \alpha x + \frac{\alpha(\alpha - 1)}{2!}x^2 + \dots + \frac{\alpha(\alpha - 1)\cdots(\alpha - n + 1)}{n!}x^n + \dots$$

 $\mathbf{H} \mathbf{H} \mathbf{X}_0 = \mathbf{0} \mathbf{H}$,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{2!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = (1+x)^{\alpha}$ 时,

$$f = (1+x)^{\alpha}, \quad f' = \alpha(1+x)^{\alpha-1}, \quad f'' = \alpha(\alpha-1)(1+x)^{\alpha-2},$$

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所以
$$\frac{1}{n!}f^{(n)}(0) = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$$
,泰勒级数是
$$\alpha(\alpha-1) \qquad \alpha(\alpha-1)\cdots(\alpha-n+1)$$

$$1 + \alpha x + \frac{\alpha(\alpha - 1)}{2!}x^2 + \dots + \frac{\alpha(\alpha - 1)\cdots(\alpha - n + 1)}{n!}x^n + \dots$$

注 n 次泰勒多项式是:

$$p_n(x) =$$



 $\mathbf{H} \mathbf{H} \mathbf{X}_0 = \mathbf{0} \mathbf{H}$,泰勒级数是

当 $f(x) = (1+x)^{\alpha}$ 时,

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

例 5 求 $f(x) = (1 + x)^{\alpha}$ 在 x = 0 处的 n 次泰勒多项式 $p_n(x)$

 $f = (1+x)^{\alpha}$, $f' = \alpha(1+x)^{\alpha-1}$, $f'' = \alpha(\alpha-1)(1+x)^{\alpha-2}$ $f^{(n)} = \alpha(\alpha - 1) \cdots (\alpha - n + 1)(1 + x)^{\alpha - n}, \cdots$

$$f = (1+x)^{\alpha}, \quad f' = \alpha(1+x)^{\alpha-1}, \quad f'' = \alpha(\alpha-1)(1+x)^{\alpha-2},$$

$$\dots, f^{(n)} = \alpha(\alpha-1)\cdots(\alpha-n+1)(1+x)^{\alpha-n}, \cdots$$
所以 $\frac{1}{n!}f^{(n)}(0) = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}, \quad$ 泰勒级数是
$$1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n + \cdots$$

注 n 次泰勒多项式是:

 $p_n(x) = 1 + \alpha x + \frac{\alpha(\alpha - 1)}{2!}x^2 + \dots + \frac{\alpha(\alpha - 1)\cdots(\alpha - n + 1)}{n!}x^n$

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0) (x - x_0)^n$$

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0) (x - x_0)^n \quad \Longleftrightarrow \quad f(x) = \lim_{n \to \infty} p_n(x)$$

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0) (x - x_0)^n \quad \Leftrightarrow \quad f(x) = \lim_{n \to \infty} p_n(x)$$

$$\Leftrightarrow \lim_{n\to\infty} [f(x) - p_n(x)] = 0$$

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0) (x - x_0)^n \quad \Leftrightarrow \quad f(x) = \lim_{n \to \infty} p_n(x)$$

$$\Leftrightarrow \quad \lim_{n \to \infty} [f(x) - p_n(x)] = 0$$

$$(\Leftrightarrow R_n(x) = f(x) - p_n(x))$$

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0) (x - x_0)^n \iff f(x) = \lim_{n \to \infty} p_n(x)$$

$$\Leftrightarrow \lim_{n \to \infty} [f(x) - p_n(x)] = 0$$

$$(R_n(x) = f(x) - p_n(x))$$

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$$(R_n(x) = f(x) - p_n(x))$$

$$\Leftrightarrow \lim_{n \to \infty} R_n(x) = 0$$

注 $R_n(x) = f(x) - p_n(x)$,或者 $f(x) = p_n(x) + R_n(x)$,刻画了原函数 f(x) 与其泰勒多项式 $p_n(x)$ 的差异。



回忆 泰勒中值定理 1 若 f 具有 n 阶导数,则

$$R_n(x) = o((x-x_0)^n).$$

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特别地,

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回忆 泰勒中值定理 1 若 f 具有 n 阶导数,则

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特别地,

$$f(x) = p_n(x) + R_n(x)$$

$$= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + o((x - x_0)^n)$$

 $e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + o(x^{n})$

 $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + o(x^{2m+1})$ $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots + (-1)^{n-1}\frac{1}{n}x^n + o(x^n)$

 $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + o(x^{2m+2})$

 $(1+x)^{\alpha} = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \dots + \frac{\alpha(\alpha-1) \cdots (\alpha-n+1)}{n!} x^n + o(x^n)$

例求 $\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$, $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1-x)]}$

例求
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
, $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1-x)]}$

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例求
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
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$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$$



例求
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
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$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$$
$$= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3}$$

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$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
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$$= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3} = \frac{1}{3}$$

$$\lim_{x \to 0} \frac{1}{\sin^3 x} = \lim_{x \to 0} \frac{1}{\sin^3 x}$$

例求
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
, $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1-x)]}$

例求
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
, $\lim_{x\to 0} \frac{\cos x - e^{-2x}}{x^2 [x + \ln(1 - x)]}$

$$\lim_{x \to 0} \frac{1}{\sin^3 x}, \lim_{x \to 0} \frac{1}{x^2 [x + \ln(1 - x)]}$$

$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$$

$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right]}{\frac{1}{2}x^3 + o(x^4)}$$

$$= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^3 + o$$

$$= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3} = \frac{1}{3}$$

$$\lim_{x \to 0} x$$

$$\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{\cos x + \cos x}$$

$$\lim_{x \to 0} \frac{\cos x - e^{-2}}{x^2 [x + \ln(1 - x)]}$$

$$\lim_{x\to 0} \frac{1}{x^2[x+\ln(1-x)]}$$

$$= \lim_{x \to 0} \frac{\left[\right] - \left[\right]}{x^2 \left[x + \left(\right] \right]}$$

第 12 章 d: 函数展开成幂级数

例求
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
, $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1-x)]}$

$$\frac{1}{1} \times \lim_{x \to 0} \frac{1}{\sin^3 x}, \lim_{x \to 0} \frac{1}{x^2 [x + \ln(1 - x)]}$$

$$\frac{1}{1} \sin x - x \cos x \qquad \left[x - \frac{1}{3!} x^3 + o(x) \right]$$

$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$$

$$= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3} = \frac{1}{3}$$

$$\cos x - e^{-\frac{x^2}{2}}$$

$$\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2 [x + \ln(1 - x)]}$$

$$\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2 [x + \ln(1 - x)]}$$

$$= \lim_{x \to 0} \frac{\left[1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + o(x^5)\right] - \left[\frac{1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + o(x^5)\right]}{x^2 [x + \frac{1}{4!}x^4 + o(x^5)]}$$

$$\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2 [x + \ln(1 - x)]}$$

$$= \lim_{x \to 0} \frac{\left[1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + o(x^5)\right] - \left[\right]}{x^2 [x + ()]}$$

$$\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2 \left[x + \ln(1 - x) \right]}$$

$$\lim_{x \to 0} \left[1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 + o(x^5) \right] - \left[\frac{1}{2!} x^2 + \frac{1}{4!} x^4 + o(x^5) \right]$$

例求
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
, $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1-x)]}$

$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$$

$$= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3} = \frac{1}{3}$$

$$\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2 [x + \ln(1 - x)]}$$

$$\lim_{x \to 0} \frac{1}{x^{2} [x + \ln(1 - x)]}$$

$$= \lim_{x \to 0} \frac{\left[1 - \frac{1}{2!}x^{2} + \frac{1}{4!}x^{4} + o(x^{5})\right] - \left[1 - \frac{1}{2}x^{2} + \frac{1}{8}x^{4} + o(x^{4})\right]}{x^{2} [x + (y^{2})]}$$



例求
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
, $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1-x)]}$

例求
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
, $\lim_{x\to 0} \frac{\cos x - e^{-2}}{x^2[x + \ln(1 - x)]}$

$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x\to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$$

$$= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3} = \frac{1}{3}$$

$$\cos x - e^{-\frac{x^2}{2}}$$

$$\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2 [x + \ln(1 - x)]}$$

$$\lim_{x \to 0} \frac{\cos x - e^{-\frac{x}{2}}}{x^2 [x + \ln(1 - x)]}$$

$$\lim_{x \to 0} \frac{\cos x - e^{-x}}{x^2 [x + \ln(1 - x)]}$$

$$\left[1 - \frac{1}{2}x^2 + \frac{1}{2}x^4 + o(x^5)\right] - \left[1 - \frac{1}{2}x^4 + o(x^5)\right]$$

$$= \lim_{x \to 0} \frac{\left[1 - \frac{1}{2!}x^2 + \ln(1 - x)\right]}{x^2 \left[x + \ln(1 - x)\right]}$$

$$= \lim_{x \to 0} \frac{\left[1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + o(x^5)\right] - \left[1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + o(x^4)\right]}{x^2 \left[x + \left(-x - \frac{1}{2}x^2 + o(x^2)\right)\right]}$$

例求 $\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$, $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2 [x + \ln(1-x)]}$

$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$$

$$= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3} = \frac{1}{3}$$

 $\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2 [x + \ln(1 - x)]}$

$$\lim_{x \to 0} \frac{1}{x^{2} [x + \ln(1 - x)]}$$

$$= \lim_{x \to 0} \frac{\left[1 - \frac{1}{2!}x^{2} + \frac{1}{4!}x^{4} + o(x^{5})\right] - \left[1 - \frac{1}{2}x^{2} + \frac{1}{8}x^{4} + o(x^{4})\right]}{x^{2} \left[x + \left(-x - \frac{1}{2}x^{2} + o(x^{2})\right)\right]}$$

 $= \lim_{x \to 0} \frac{-\frac{1}{12}x^4 + o(x^4)}{-\frac{1}{2}x^4 + o(x^4)}$

例求 $\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$, $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2 [x + \ln(1-x)]}$

 $\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$

 $= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3} = \frac{1}{3}$

 $\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2 [x + \ln(1 - x)]}$

 $= \lim_{x \to 0} \frac{\left[1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + o(x^5)\right] - \left[1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + o(x^4)\right]}{x^2 \left[x + \left(-x - \frac{1}{2}x^2 + o(x^2)\right)\right]}$

$$= \lim_{X \to 0} \frac{-\frac{1}{12}X^4 + o(X^4)}{-\frac{1}{2}X^4 + o(X^4)} = \lim_{X \to 0} \frac{-\frac{1}{12} + o(X^4)/X^4}{-\frac{1}{2} + o(X^4)/X^4}$$

$$2 \stackrel{?}{=} d: \text{ adj}_{R, T, R}$$

$$\lim_{x\to 0} \frac{1}{x^2}$$

 $\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$

例求
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
, $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1 - x)]}$

$$\frac{-x\cos x}{\ln^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{x}\right]}{1 + \frac{1}{x}}$$

 $= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3} = \frac{1}{3}$

- $\lim_{x \to 0} \frac{\cos x e^{-\frac{x^2}{2}}}{x^2 [x + \ln(1 x)]}$
 - $= \lim_{x \to 0} \frac{\left[1 \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + o(x^5)\right] \left[1 \frac{1}{2}x^2 + \frac{1}{8}x^4 + o(x^4)\right]}{x^2 \left[x + \left(-x \frac{1}{2}x^2 + o(x^2)\right)\right]}$
 - $= \lim_{x \to 0} \frac{-\frac{1}{12}x^4 + o(x^4)}{-\frac{1}{2}x^4 + o(x^4)} = \lim_{x \to 0} \frac{-\frac{1}{12} + o(x^4)/x^4}{-\frac{1}{2} + o(x^4)/x^4} = \frac{1}{6}$

泰勒中值定理 2 若 f 具有 n+1 阶导数,则

$$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) (x - x_0)^{n+1}$$

其中 ξ 是 x_0 与 x 之间的某个值

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$$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) (x - x_0)^{n+1}$$

$$(1-\theta)x_0+\theta x$$

其中 ξ 是 x_0 与 x 之间的某个值, $0 < \theta < 1$ 。

泰勒中值定理 2 若 f 具有 n+1 阶导数,则

$$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) (x - x_0)^{n+1}$$

$$\stackrel{or}{=} \frac{1}{(n+1)!} f^{(n+1)}((1-\theta)x_0 + \theta x) (x - x_0)^{n+1},$$

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注

1. ξ (以及 θ) 不是固定不变的,而是随 x 和 n 的改变而变化。

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注

- 1. ξ (以及 θ) 不是固定不变的,而是随 x 和 n 的改变而变化。
- 2. 当 $x_0 = 0$ 时,则余项可写成

$$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1}, \qquad (0 < \theta < 1)$$

$$e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty).$$

$$e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty).$$

证明

1. 只需证明对任意 x,成立 $\lim_{n\to\infty} R_n(x) = 0$ 。

$$e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty).$$

- 1. 只需证明对任意 x,成立 $\lim_{n\to\infty} R_n(x) = 0$ 。
- 2. 由泰勒中值定理 2.

$$|R_n(x)| = \left| \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1} \right|$$

$$e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty).$$

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- 2. 由泰勒中值定理 2.

$$|R_n(x)| = \left| \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1} \right| = \left| \frac{e^{\theta x} x^{n+1}}{(n+1)!} \right|$$

$$e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty).$$

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- 2. 由泰勒中值定理 2.

$$|R_n(x)| = \left| \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1} \right| = \left| \frac{e^{\theta x} x^{n+1}}{(n+1)!} \right| \le \frac{|x|^{n+1}}{(n+1)!} e^{|x|}$$

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• 至此,我们知道 e^x , $\sin x$, $\cos x$ 以及 $\frac{1}{1+x}$ 是等于其泰勒级数,即

$$e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty)$$

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• 利用上还结果,及逐坝积分公式,可进一步水出 $\ln(1+x)$, $\operatorname{arctan} x$

的幂级数展开。



性质 成立
$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots$$
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3. 注意到 $\sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n} x^n$ 收敛域是 (-1, 1], 由连续性, 当 x=1 时也成

$$\frac{1}{2}$$
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(这是*f*(1)=

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第 12 章 d: 函数展开成幂级数

= S(1)

(这是 $f(1) = \lim_{x \to 1^-} \ln(1+x)$

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$$\int_{0}^{\infty} 1+t \qquad \int_{0}^{\infty} \sum_{n=0}^{\infty} (-1)^{n} \frac{1}{n+1} x^{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^{n}$$

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(这是 $f(1) = \lim_{x \to 1^{-}} \ln(1+x)$ $\lim_{x \to 1^{-}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n = \lim_{x \to 1^{-}} S(x) = S(1)$)

$$n=0$$
 $n=1$ $n=1$

第 12 章 d:函数展开成幂级数

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1} \frac{1}{n}x^n + \dots$$
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2. 当 $x \in (-1, 1)$ 时,利用逐项积分可得 $\ln(1+x) = \int_0^x \frac{1}{1+t} dt = \int_0^x \sum_{n=0}^\infty (-1)^n t^n dt = \sum_{n=0}^\infty (-1)^n \int_0^x t^n dt$

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$$= \sum_{n=0}^\infty (-1)^n \frac{1}{n+1} x^{n+1} = \sum_{n=0}^\infty \frac{(-1)^{n-1}}{n} x^n$$

3. 注意到 $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$ 收敛域是 (-1, 1],由连续性,当 x = 1 时也成

立
$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n.$$

性质 成立 $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots + (-1)^n \frac{1}{2n+1}x^{2n+1} + \dots, x \in [-1,1].$

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$$\begin{array}{cccc} C^{x} & 1 & C^{x} \xrightarrow{\infty} \end{array}$$

$$\arctan x = \int_0^x \frac{1}{1+t^2} dt = \int_0^x \sum_{n=0}^{\infty} (-1)^n t^{2n} dt$$

2.

$$rctan v = v$$

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是引工带级效的代数域是[1, 1],成工汽至乡村入C[1, 1] 成立。

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3. 注意到 $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$ 收敛域是 [-1, 1], 由连续性, 当 $x = \pm 1$ 时也

有 $\operatorname{arctan} x = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1}.$



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(如f(1) == S(1)

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(如
$$f(1) = \lim_{x \to 1^{-}} \arctan x$$

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 $\lim_{x\to 1^-} S(x) = S(1)$

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 $(\inf(1) = \lim_{x \to 1^{-}} \arctan x \quad \lim_{x \to 1^{-}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = \lim_{x \to 1^{-}} S(x) = S(1))$

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注 $\mathbf{x} = 1$,则得到

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$$\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots + (-1)^n \frac{1}{2n+1}x^{2n+1} + \dots, \quad x \in [-1, 1]$$

注 $\mathbf{x} = 1$,则得到

 $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$

$$\cos x = 1 - \frac{1}{2!}x$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots + (-1)^n \frac{1}{(2n+1)!}x^{2n+1} + \dots, \ x \in (-\infty, \infty)$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots + (-1)^n \frac{1}{(2n)!}x^{2n} + \dots, \ x \in (-\infty, \infty)$$

 $e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty)$

• 至此, 得出如下常用函数的幂级数展开式:

 $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots + (-1)^n x^n + \dots, x \in (-1, 1)$

 $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1]$

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• 用上述结果, 及逐项求导、积分公式, 可求更多函数的泰勒级数展开

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$$

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所以当
$$x \in (-1, 1]$$
时,
$$(1-x)\ln(1+x) = (1-x)\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n$$

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$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n - \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^{n+1}$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1} \frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$$

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$$\sum_{n=2}^{\infty} (-1)^{n-2} \frac{1}{n-1} x^n$$

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$$= x + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{1}{n} x^n$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$$

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 $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1} \frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$

例 1 把函数 $f(x) = (1-x)\ln(1+x)$ 展开成 x 的幂级数。

所以当 $x \in (-1, 1]$ 时, $(1-x)\ln(1+x) = (1-x)\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n$



 $=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}x^n-\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}x^{n+1}$ $=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}x^n-\sum_{n=2}^{\infty}(-1)^{n-2}\frac{1}{n-1}x^n$

 $= x + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{1}{n} x^n - \sum_{n=1}^{\infty} (-1)^{n-2} \frac{1}{n-1} x^n$ $= x + \sum_{n=2}^{\infty} \left(\frac{(-1)^{n-1}}{n} - \frac{(-1)^n}{n-1} \right) x^n$

$$\cos t = 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6 + \dots + (-1)^n \frac{1}{(2n)!}t^{2n} + \dots, \ t \in (-\infty, \infty)$$

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$$

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$$x \in (-\infty, \infty)$$
 时,

$$\cos^{2} x = \frac{1}{2} + \frac{1}{2} \cos 2x$$

$$= \frac{1}{2} + \frac{1}{2} \sum_{n=0}^{\infty} (-1)^{n} \frac{1}{(2n)!} (2x)^{2n}$$

$$= \frac{1}{2} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^{n} 2^{2n}}{(2n)!} x^{2n}$$

例 2 把函数 $f(x) = \cos^2 x$ 展开成 x 的幂级数。

 $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$

解利用

- $\cos t = 1 \frac{1}{2!}t^2 + \frac{1}{4!}t^4 \frac{1}{6!}t^6 + \dots + (-1)^n \frac{1}{(2n)!}t^{2n} + \dots, \ t \in (-\infty, \infty)$
- 所以当 $x \in (-\infty, \infty)$ 时,

 $= \frac{1}{2} + \frac{1}{2} \sum_{n=1}^{\infty} (-1)^n \frac{1}{(2n)!} (2x)^{2n}$

 $= \frac{1}{2} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!} x^{2n}$

 $=1+\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n2^{2n}}{(2n)!}x^{2n}$

解 1. 注意到
$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$
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2. 利用 $\frac{1}{1-t} = 1 + t + t^2 + t^3 + \dots + t^n + \dots$, $t \in (-1, 1)$ 将 $\frac{1}{v+1}$, $\frac{1}{v+2}$ 分别展开成 (x+4) 的幂级数:

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$$\frac{1}{x+1}$$
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其中 $\left|\frac{x+4}{2}\right| = \left|\frac{t}{2}\right| < 1$,即-6 < x < -2。

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3. 所以 -6 < x < -2 时

解 1. 注意到 $\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$.

例 3 把函数 $f(x) = \frac{1}{x^2 + 3x + 2}$ 展开成 (x + 4) 的幂级数。

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$$x < -2$$

 $\frac{1}{x^2 + 3x + 2} = \frac{1}{x + 1} - \frac{1}{x + 2} = \sum_{n=0}^{\infty} \left(\frac{1}{2^{n+1}} - \frac{1}{3^{n+1}} \right) (x + 4)_{0}^{n}$