# 第3章 d: 向量组的秩

数学系 梁卓滨

2019-2020 学年 I

 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 

逐个剔除 能被其余向量线性表示的向量<sub>.</sub>  $\alpha_1, \alpha_2, \ldots, \alpha_s$ 

逐个剔除

 $\alpha_1,\alpha_2,\ldots,\alpha_s$  能被其余向量线性表示的向量

直到不能再剔除为止

逐个剔除

能被其余向量线性表示的向量  $\alpha_1, \alpha_2, \ldots, \alpha_s$ 

直到不能再剔除为止

 $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_r}$ 

逐个剔除

能被其余向量线性表示的向量  $\alpha_1, \alpha_2, \ldots, \alpha_s$ 

直到不能再剔除为止

 $\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}$  极大无关组

逐个剔除

 $\alpha_1, \alpha_2, \ldots, \alpha_s \xrightarrow{\text{fixtyschill}} \alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}$  极大无关组

**例** 求 
$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
,  $\alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ 的一个极大无关组。

逐个剔除

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$$\alpha_1, \alpha_2, \alpha_3, \alpha_4$$

逐个剔除

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$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_4 = 2\alpha_1 + 0\alpha_2 + 0\alpha_3}$$

逐个剔除

 $\alpha_1, \alpha_2, \ldots, \alpha_s \xrightarrow{\text{fixtyschill}} \alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}$  极大无关组

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$$\xrightarrow{\text{SIR}\alpha_4}$$

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$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_4 = 2\alpha_1 + 0\alpha_2 + 0\alpha_3} \alpha_1, \alpha_2, \alpha_3$$

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$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_4 = 2\alpha_1 + 0\alpha_2 + 0\alpha_3} \alpha_1, \alpha_2, \alpha_3 \xrightarrow{\alpha_3 = \alpha_1 + \alpha_2}$$

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$$lpha_1, lpha_2, lpha_3, lpha_4 \xrightarrow{lpha_4 = 2lpha_1 + 0lpha_2 + 0lpha_3} lpha_1, lpha_2, lpha_3 \xrightarrow{lpha_3 = lpha_1 + lpha_2} \ rac{lpha_3 = lpha_1 + lpha_2}{
ell 
m shk lpha_3}$$

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 $lpha_1,lpha_2,\ldots,lpha_s$  能被其余向量线性表示的向量  $lpha_{j_1},lpha_{j_2},\ldots,lpha_{j_r}$  极大无关组  $lpha_{j_1},lpha_{j_2},\ldots,lpha_{j_r}$ 

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$$\alpha_1, \alpha_2, \alpha_3$$

$$\xrightarrow{\alpha_3=\alpha_1+\alpha_2}$$

$$lpha_1,lpha_2$$
 极大

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4$$

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$$lpha_1,lpha_2,lpha_3,lpha_4 \xrightarrow{lpha_4=2lpha_1+0lpha_2+0lpha_3} lpha_1,lpha_2,lpha_3 \xrightarrow{lpha_3=lpha_1+lpha_2} lpha_1,lpha_2 \overset{ ext{RX}}{ ext{TX}}$$

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_1 = -\alpha_2 + \alpha_3 + 0\alpha_4}$$

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$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
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$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_1 = -\alpha_2 + \alpha_3 + 0\alpha_4}$$
 $\Rightarrow$ 

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$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_1 = -\alpha_2 + \alpha_3 + 0\alpha_4} \alpha_2, \alpha_3, \alpha_4$$

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$$lpha_1,lpha_2,lpha_3,lpha_4 \xrightarrow{lpha_4=2lpha_1+0lpha_2+0lpha_3} lpha_1,lpha_2,lpha_3 \xrightarrow{lpha_3=lpha_1+lpha_2} lpha_1,lpha_2 \xrightarrow{ ext{RX}} lpha_1,lpha_2 \xrightarrow{ ext{RX}}$$

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_1 = -\alpha_2 + \alpha_3 + 0\alpha_4} \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2 = \alpha_3 - \frac{1}{2}\alpha_4}$$

例 求 
$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
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$$lpha_1,lpha_2,lpha_3,lpha_4 \xrightarrow{lpha_4=2lpha_1+0lpha_2+0lpha_3} lpha_1,lpha_2,lpha_3 \xrightarrow{lpha_3=lpha_1+lpha_2} lpha_1,lpha_2 \xrightarrow{ ext{RX}} lpha_1,lpha_2 \xrightarrow{ ext{RX}}$$

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_1 = -\alpha_2 + \alpha_3 + 0\alpha_4} \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2 = \alpha_3 - \frac{1}{2}\alpha_4}$$
 $\xrightarrow{\text{slik} \alpha_1}$ 

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$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_1 = -\alpha_2 + \alpha_3 + 0\alpha_4} \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2 = \alpha_3 - \frac{1}{2}\alpha_4} \alpha_3, \alpha_4$$

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$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_1 = -\alpha_2 + \alpha_3 + 0\alpha_4} \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2 = \alpha_3 - \frac{1}{2}\alpha_4} \alpha_3, \alpha_4 \xrightarrow{\text{RX}} \alpha_5$$

 $lpha_1,lpha_2,\ldots,lpha_s$  能被其余向量线性表示的向量  $lpha_{j_1},lpha_{j_2},\ldots,lpha_{j_r}$  极大无关组  $lpha_{j_1},lpha_{j_2},\ldots,lpha_{j_r}$ 

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$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_1 = -\alpha_2 + \alpha_3 + 0\alpha_4} \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2 = \alpha_3 - \frac{1}{2}\alpha_4} \alpha_3, \alpha_4 \xrightarrow{\text{RF}}$$

 $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ 

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$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_1 = -\alpha_2 + \alpha_3 + 0\alpha_4} \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2 = \alpha_3 - \frac{1}{2}\alpha_4} \alpha_3, \alpha_4 \xrightarrow{\text{RF}}$$

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2 = 0 \alpha_1 + \alpha_3 - \frac{1}{2} \alpha_4}$$

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$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2 = 0\alpha_1 + \alpha_3 - \frac{1}{2}\alpha_4}$$
 $\Rightarrow$ 

 $lpha_1,lpha_2,\ldots,lpha_s$  能被其余向量线性表示的向量  $lpha_{j_1},lpha_{j_2},\ldots,lpha_{j_r}$  极大无关组  $lpha_{j_1},lpha_{j_2},\ldots,lpha_{j_r}$ 

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$$lpha_1,lpha_2,lpha_3,lpha_4 \xrightarrow{lpha_4=2lpha_1+0lpha_2+0lpha_3} lpha_1,lpha_2,lpha_3 \xrightarrow{lpha_3=lpha_1+lpha_2} lpha_1,lpha_2 \xrightarrow{\mathrm{k} \chi_4} lpha_2 + lpha_2 \xrightarrow{\mathrm{k} \chi_4} lpha_1,lpha_2 \xrightarrow{\mathrm{k} \chi_4} lpha_2 + lpha_2 \xrightarrow{\mathrm{k} \chi_4} lpha_2 + lpha_3 \xrightarrow{\mathrm{k} \chi_4} lpha_3 + lpha_2 \xrightarrow{\mathrm{k} \chi_4} lpha_3 + lpha_3 \xrightarrow{\mathrm{k} \chi_4} lpha_3 + lpha_3 \xrightarrow{\mathrm{k} \chi_4} lpha_4 \xrightarrow{\mathrm{k}$$

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2 = 0\alpha_1 + \alpha_3 - \frac{1}{2}\alpha_4} \alpha_1, \alpha_3, \alpha_4$$

 $lpha_1,lpha_2,\ldots,lpha_s$  能被其余向量线性表示的向量  $lpha_{j_1},lpha_{j_2},\ldots,lpha_{j_r}$  极大无关组  $lpha_{j_1},lpha_{j_2},\ldots,lpha_{j_r}$ 

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$$lpha_1,lpha_2,lpha_3,lpha_4 \xrightarrow{lpha_4=2lpha_1+0lpha_2+0lpha_3} lpha_1,lpha_2,lpha_3 \xrightarrow{lpha_3=lpha_1+lpha_2} lpha_1,lpha_2 \overset{\mathrm{dk}}{\mathrm{T}}_{\mathrm{T}}$$

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2 = 0\alpha_1 + \alpha_3 - \frac{1}{2}\alpha_4} \alpha_1, \alpha_3, \alpha_4 \xrightarrow{\alpha_4 = 2\alpha_1 + 0\alpha_3}$$

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$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_1 = -\alpha_2 + \alpha_3 + 0\alpha_4} \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2 = \alpha_3 - \frac{1}{2}\alpha_4} \alpha_3, \alpha_4 \xrightarrow{\text{RF}}$$

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$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_1 = -\alpha_2 + \alpha_3 + 0\alpha_4} \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2 = \alpha_3 - \frac{1}{2}\alpha_4} \alpha_3, \alpha_4 \xrightarrow{\text{KT}}$$

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2 = 0\alpha_1 + \alpha_3 - \frac{1}{2}\alpha_4} \alpha_1, \alpha_3, \alpha_4 \xrightarrow{\alpha_4 = 2\alpha_1 + 0\alpha_3} \alpha_1, \alpha_3$$

 $\alpha_1, \alpha_2, \ldots, \alpha_s$  能被其余向量线性表示的向量  $\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}$  极大无关组

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$$lpha_1,lpha_2,lpha_3,lpha_4 \xrightarrow{lpha_4=2lpha_1+0lpha_2+0lpha_3} lpha_1,lpha_2,lpha_3 \xrightarrow{lpha_3=lpha_1+lpha_2} lpha_1,lpha_2 \overset{ ext{kb}}{ ext{T}}_{ ext{E}}$$

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_1 = -\alpha_2 + \alpha_3 + 0\alpha_4} \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2 = \alpha_3 - \frac{1}{2}\alpha_4} \alpha_3, \alpha_4 \xrightarrow{\text{Mb}}$$

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\text{Min}} \alpha_1 \xrightarrow{\text{Min}} \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\text{Min}} \alpha_2 \xrightarrow{\text{Min}} \alpha_3, \alpha_4 \xrightarrow{\text{Min}} \alpha_2$$

$$lpha_1,lpha_2,lpha_3,lpha_4 \xrightarrow{lpha_2=0lpha_1+lpha_3-rac{1}{2}lpha_4} lpha_1,lpha_3,lpha_4 \xrightarrow{lpha_4=2lpha_1+0lpha_3} lpha_1,lpha_3 \xrightarrow{ ext{k} ext{K} ext{L}} lpha_4$$

还有其他极大无关组吗?

 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 能被其余向量线性表示的向量
直到不能再剔除为止  $\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}$ 极大无关组

例 求 
$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
,  $\alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ 的一个极大无关组。

$$lpha_1,lpha_2,lpha_3,lpha_4 \xrightarrow{lpha_4=2lpha_1+0lpha_2+0lpha_3} lpha_1,lpha_2,lpha_3 \xrightarrow{lpha_3=lpha_1+lpha_2} lpha_1,lpha_2 \xrightarrow{ ext{RK4}}$$

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_1 = -\alpha_2 + \alpha_3 + 0\alpha_4} \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2 = \alpha_3 - \frac{1}{2}\alpha_4} \alpha_3, \alpha_4 \xrightarrow{\text{RE}} \alpha_1$$

$$lpha_1, lpha_2, lpha_3, lpha_4 \xrightarrow{lpha_2 = 0lpha_1 + lpha_3 - rac{1}{2}lpha_4} lpha_1, lpha_3, lpha_4 \xrightarrow{lpha_4 = 2lpha_1 + 0lpha_3} lpha_1, lpha_3 \xrightarrow{ ext{RF}} lpha_1, lpha_3 \xrightarrow{ ext{RF}}$$

还有其他极大无关组吗?

注 极大无关组不唯一!

**定理**  $\alpha_{j_1}$ ,  $\alpha_{j_2}$ , . . . ,  $\alpha_{j_r}$  是  $\alpha_1$ ,  $\alpha_2$ , . . . ,  $\alpha_s$  的极大无关组,当且仅当

- $\alpha_1$ ,  $\alpha_2$ , ···· ,  $\alpha_s$  中每个向量都可由  $\alpha_{i_1}$ ,  $\alpha_{i_2}$ , ... ,  $\alpha_{i_r}$  线性表示
- α<sub>j1</sub>, α<sub>j2</sub>, . . . , α<sub>jr</sub> 线性无关

秩

**定理**  $\alpha_{j_1}$ ,  $\alpha_{j_2}$ , . . . ,  $\alpha_{j_r}$  是  $\alpha_1$ ,  $\alpha_2$ , . . . ,  $\alpha_s$  的极大无关组,当且仅当

- $\alpha_1$ ,  $\alpha_2$ ,  $\cdots$ ,  $\alpha_s$  中每个向量都可由  $\alpha_{j_1}$ ,  $\alpha_{j_2}$ ,  $\ldots$ ,  $\alpha_{j_r}$  线性表示
- α<sub>i1</sub>, α<sub>i2</sub>, . . . , α<sub>ir</sub> 线性无关

定理 极大无关组所包含向量的个数是唯一确定的。

**定理**  $\alpha_{j_1}$ ,  $\alpha_{j_2}$ , . . . ,  $\alpha_{j_r}$  是  $\alpha_1$ ,  $\alpha_2$ , . . . ,  $\alpha_s$  的极大无关组,当且仅当

- $\alpha_1$ ,  $\alpha_2$ , ···· ,  $\alpha_s$  中每个向量都可由  $\alpha_{i_1}$ ,  $\alpha_{i_2}$ , ···· ,  $\alpha_{i_r}$  线性表示
- α<sub>j1</sub>, α<sub>j2</sub>, ..., α<sub>jr</sub> 线性无关

定理 极大无关组所包含向量的个数是唯一确定的。即:若

$$\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}; \qquad \beta_{k_1}, \beta_{k_2}, \ldots, \beta_{k_t}$$

都是  $\alpha_1$ ,  $\alpha_2$ , ...,  $\alpha_s$  的极大无关组,则

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# 极大无关组的性质

**定理**  $\alpha_{j_1}$ ,  $\alpha_{j_2}$ , . . . ,  $\alpha_{j_r}$  是  $\alpha_1$ ,  $\alpha_2$ , . . . ,  $\alpha_s$  的极大无关组,当且仅当

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都是  $\alpha_1$ ,  $\alpha_2$ , ...,  $\alpha_s$  的极大无关组,则 r = t。

例 设 
$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
,  $\alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ , 则极大无关组是:  $\alpha_1, \alpha_2; \quad \alpha_1, \alpha_3; \quad \alpha_2, \alpha_3; \quad \alpha_2, \alpha_4; \quad \alpha_3, \alpha_4$ 

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# 极大无关组的性质

**定理**  $\alpha_{j_1}$ ,  $\alpha_{j_2}$ , . . . ,  $\alpha_{j_r}$  是  $\alpha_1$ ,  $\alpha_2$ , . . . ,  $\alpha_s$  的极大无关组,当且仅当

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$$\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}; \qquad \beta_{k_1}, \beta_{k_2}, \ldots, \beta_{k_t}$$

都是  $\alpha_1$ ,  $\alpha_2$ , ...,  $\alpha_s$  的极大无关组,则 r = t。

例 设 
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$$\alpha_1, \alpha_2; \quad \alpha_1, \alpha_3; \quad \alpha_2, \alpha_3; \quad \alpha_2, \alpha_4; \quad \alpha_3, \alpha_4$$

可见,每个极大无关组都由2个向量构成。

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**定义** 向量组  $\alpha_1$ ,  $\alpha_2$ , ...,  $\alpha_s$  的极大无关组所包含向量的个数,称向量组的 **秩**,记为:

 $r(\alpha_1, \alpha_2, \ldots, \alpha_s)$ 

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例 设 
$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$

r(A) = :

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$$A = \left(\begin{array}{ccc} 1 & 3 & 5 \\ 2 & 4 & 6 \end{array}\right)$$

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$$r(A) = :$$

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} \xrightarrow{r_2 - 2r_1}$$

**例** 设 
$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$

$$r(A) = :$$

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 3 & 5 \\ 0 & -2 & -4 \end{pmatrix}$$

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 $r(\alpha_1,\alpha_2,\alpha_3) = :$ 

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 $r(\alpha_1, \alpha_2, \alpha_3) = :$ 

$$\alpha_1, \alpha_2, \alpha_3$$

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•  $r(\alpha_1, \alpha_2, \alpha_3) = :$ 

$$\alpha_1, \alpha_2, \alpha_3 \xrightarrow{\alpha_3 = -\alpha_1 + 2\alpha_2}$$

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 $\Rightarrow$ 

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 $\bullet \ r(\alpha_1,\alpha_2,\alpha_3) = :$ 

$$\alpha_1, \alpha_2, \alpha_3 \xrightarrow{\alpha_3 = -\alpha_1 + 2\alpha_2} \alpha_1, \alpha_2$$

例设
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 $r(\alpha_1, \alpha_2, \alpha_3) = :$ 

$$\alpha_1, \alpha_2, \alpha_3 \xrightarrow{\alpha_3 = -\alpha_1 + 2\alpha_2} \alpha_1, \alpha_2 \xrightarrow{\alpha_1, \alpha_2 \text{ det } \mathbb{X} \times \mathbb{X}}$$

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$$\alpha_1, \alpha_2, \alpha_3 \xrightarrow{\alpha_3 = -\alpha_1 + 2\alpha_2} \alpha_1, \alpha_2 \xrightarrow{\alpha_1, \alpha_2 \text{线性无关}} \alpha_1, \alpha_2$$
为极大无关组

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为极大无关组

例设
$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}_{\beta_2}^{\beta_1}$$

 $\alpha_1$   $\alpha_2$ 

• r(A) = 2:

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 3 & 5 \\ 0 & -2 & -4 \end{pmatrix}$$

•  $r(\alpha_1, \alpha_2, \alpha_3) = 2$ :

$$\alpha_1, \alpha_2, \alpha_3 \xrightarrow[]{\alpha_3 = -\alpha_1 + 2\alpha_2} \alpha_1, \alpha_2 \xrightarrow[]{\alpha_1, \alpha_2 \mbox{ det } \mathbb{R}} \alpha_1, \alpha_2$$
为极大无关组

 $r(\beta_1,\beta_2) = :$ 

例设
$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}_{\beta_2}^{\beta_1}$$

 $\alpha_1$   $\alpha_2$ 

• r(A) = 2:

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为极大无关组

 $\bullet$   $r(\beta_1, \beta_2) = :$ 

$$\beta_1, \beta_2$$

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$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}_{\beta_2}^{\beta_1}$$

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为极大无关组

 $r(\beta_1,\beta_2) = :$ 

$$\beta_1, \beta_2 \xrightarrow{\beta_1, \beta_2 \text{ det } \mathcal{E} \times}$$

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为极大无关组

•  $r(\beta_1, \beta_2) = :$ 

$$\beta_1, \beta_2 \xrightarrow{\beta_1, \beta_2 \stackrel{\text{gt}. \text{KE}}{\longrightarrow}} \beta_1, \beta_2$$
为极大无关组

例设
$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$
 $\beta_2$ 

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 3 & 5 \\ 0 & -2 & -4 \end{pmatrix}$$

•  $r(\alpha_1, \alpha_2, \alpha_3) = 2$ :

$$\alpha_1, \alpha_2, \alpha_3 \xrightarrow[]{\alpha_3 = -\alpha_1 + 2\alpha_2} \alpha_1, \alpha_2 \xrightarrow[]{\alpha_1, \alpha_2 \mbox{ det } \mathbb{R}} \alpha_1, \alpha_2$$
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•  $r(\beta_1, \beta_2) = 2$ :

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为极大无关组

可见,以上上个秩均相等,即  $r(A) = r(\alpha_1, \alpha_2, \alpha_3) = r(\beta_1, \beta_2)$ 。

例 设 
$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}_{\beta_2}^{\beta_1}$$

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 3 & 5 \\ 0 & -2 & -4 \end{pmatrix}$$

•  $r(\alpha_1, \alpha_2, \alpha_3) = 2$ :

$$lpha_1,lpha_2,lpha_3 \xrightarrow{lpha_3=-lpha_1+2lpha_2} lpha_1,lpha_2 \xrightarrow{lpha_1,lpha_2$$
线性无关 $lpha_1,lpha_2$ 为极大无关组

•  $r(\beta_1, \beta_2) = 2$ :

$$eta_1,eta_2 \stackrel{eta_1,eta_2 ext{\$tensh}}{\longrightarrow} eta_1,eta_2$$
为极大无关组

可见,以上上个秩均相等,即  $r(A) = r(\alpha_1, \alpha_2, \alpha_3) = r(\beta_1, \beta_2)$ 。

这不是巧合,而是恒成立!

设

$$A_{m \times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

设

$$A_{m \times n} = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_n \\ \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{pmatrix}$$

# 秩

设

$$A_{m \times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \beta_{1} \\ \beta_{2} \\ \vdots \\ \beta_{m} \end{pmatrix}$$

#### 定义

- r(α<sub>1</sub>, α<sub>2</sub>, ..., α<sub>n</sub>) 称为 A 的列秩;
- r(β<sub>1</sub>, β<sub>2</sub>, ..., β<sub>m</sub>) 称为 A 的 行秩;

# 秩

设

$$A_{m \times n} = \begin{pmatrix} \alpha_{1} & \alpha_{2} & \alpha_{n} \\ \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix}_{\beta_{m}}^{\beta_{1}}$$

### 定义

- r(α<sub>1</sub>, α<sub>2</sub>, ..., α<sub>n</sub>) 称为 A 的 列秩;
- $r(\beta_1, \beta_2, ..., \beta_m)$  称为 A 的 **行秩**;

定理 
$$r(A) = r(\alpha_1, \alpha_2, \ldots, \alpha_n) = r(\beta_1, \beta_2, \ldots, \beta_m)$$

# 秩

设

$$A_{m \times n} = \begin{pmatrix} a_{11} & a_{2} & & a_{n} \\ a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \beta_{1} \\ \beta_{2} \\ \vdots \\ \beta_{m} \end{pmatrix}$$

#### 定义

- $r(\alpha_1, \alpha_2, \ldots, \alpha_n)$  称为 A 的列秩;
- r(β<sub>1</sub>, β<sub>2</sub>, ..., β<sub>m</sub>) 称为 A 的 行秩;

定理 
$$r(A) = r(\alpha_1, \alpha_2, \ldots, \alpha_n) = r(\beta_1, \beta_2, \ldots, \beta_m)$$

应用 计算向量组的秩可转化为计算矩阵的秩。

问题 给出 m 维的向量组  $\alpha_1$ ,  $\alpha_2$ ,  $\cdots$ ,  $\alpha_n$ , 如何求出其一组极大无关组?

步骤

<mark>问题</mark> 给出 m 维的向量组  $lpha_1$  ,  $lpha_2$  ,  $\cdots$  ,  $lpha_n$  ,如何求出其一组极大无关组?

### 步骤

1. 
$$A_{m \times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

 $\overline{oldsymbol{
ho}}$ 题 给出 m 维的向量组  $lpha_1$  ,  $lpha_2$  ,  $\cdots$  ,  $lpha_n$  , 如何求出其一组极大无关组?

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简化的阶梯型矩阵

问题 给出 m 维的向量组  $lpha_1$  ,  $lpha_2$  ,  $\cdots$  ,  $lpha_n$  ,如何求出其一组极大无关组?

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简化的阶梯型矩阵

2. 通过简化的阶梯型矩阵,求出 r(A)。

## 初等变换求极大无关组

问题 给出 m 维的向量组  $lpha_1$  ,  $lpha_2$  ,  $\cdots$  ,  $lpha_n$  , 如何求出其一组极大无关组?

#### 步骤

$$A_{m \times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \xrightarrow{\text{初等行变换}}$$
简化的阶梯型矩阵

2. 通过简化的阶梯型矩阵,求出 r(A)。 利用  $r(\alpha_1, \alpha_2, ..., \alpha_n) = r(A)$ ,得出极大无关组所包含向量的个数

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## 初等变换求极大无关组

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- 3. 通过简化的阶梯型矩阵,容易看出线性无关的 *r(A)* 列,这就找到一 组极大无关组
- 4. 通过简化的阶梯型矩阵,容易看出其余列如何用该选定极大无关组 线性表示

**例 1** 求向量组  $\alpha_1 = \begin{pmatrix} 2\\4\\2 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 1\\1\\0 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 2\\3\\1 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 3\\5\\2 \end{pmatrix}$ 的一个极

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$$\left(\begin{array}{cccc}
2 & 1 & 2 & 3 \\
4 & 1 & 3 & 5 \\
2 & 0 & 1 & 2
\end{array}\right)$$

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$$\mathbf{H}$$
  $\alpha_1$   $\alpha_2$   $\alpha_3$   $\alpha_4$ 

$$\begin{pmatrix} 2 & 1 & 2 & 3 \\ 4 & 1 & 3 & 5 \\ 2 & 0 & 1 & 2 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 - r_1}$$

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$$\begin{pmatrix} 2 & 1 & 2 & 3 \\ 4 & 1 & 3 & 5 \\ 2 & 0 & 1 & 2 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 2 & 1 & 2 & 3 \\ 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 \end{pmatrix} \longrightarrow$$

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$$r_1-r_2$$

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$$\alpha_1 = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$$
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$$\mathbf{H}$$
  $\alpha_1$   $\alpha_2$   $\alpha_3$   $\alpha_4$ 

$$\left( \begin{array}{cccc} 2 & 1 & 2 & 3 \\ 4 & 1 & 3 & 5 \\ 2 & 0 & 1 & 2 \end{array} \right) \xrightarrow[r_3 - r_1]{r_2 - 2r_1} \left( \begin{array}{cccc} 2 & 1 & 2 & 3 \\ 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 \end{array} \right) \longrightarrow \left( \begin{array}{cccc} 2 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$\xrightarrow{r_1-r_2} \left(\begin{array}{cccc} 2 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

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$$\begin{pmatrix} 2 & 1 & 2 & 3 \\ 4 & 1 & 3 & 5 \\ 2 & 0 & 1 & 2 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 2 & 1 & 2 & 3 \\ 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{r_1-r_2} \left( \begin{array}{ccc} 2 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right) \xrightarrow{\frac{1}{2} \times r_1}$$

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所以

• 
$$r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2$$
;

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所以

- $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2$ ;
- α<sub>1</sub>, α<sub>2</sub> 是极大无关组;

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 $\mathbf{H}$   $\alpha_1$   $\alpha_2$   $\alpha_3$   $\alpha_4$ 

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$$\xrightarrow{r_1-r_2} \left(\begin{array}{cccc} 2 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\frac{1}{2} \times r_1} \left(\begin{array}{cccc} 1 & 0 & \frac{1}{2} & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

所以

• 
$$r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2$$
;

• 
$$\alpha_3 = \frac{1}{2}\alpha_1 + \alpha_2$$
,  $\alpha_4 = \alpha_1 + \alpha_2$ 

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**例 2** 求向量组 
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$
,  $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一

例 2 求向量组 
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$$\mathbf{H}$$
  $\alpha_1$   $\alpha_2$   $\alpha_3$   $\alpha_4$ 

$$\left(\begin{array}{cccc}
1 & 0 & 1 & 2 \\
2 & 1 & 1 & 4 \\
1 & 1 & 0 & 3 \\
0 & 2 & -2 & 3
\end{array}\right)$$

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$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$
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$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 - r_1}$$

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解 
$$\alpha_1$$
  $\alpha_2$   $\alpha_3$   $\alpha_4$ 

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 3 \end{pmatrix}$$

**例 2** 求向量组 
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$$\mathbf{H}$$
  $\alpha_1$   $\alpha_2$   $\alpha_3$   $\alpha_4$ 

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow{r_3 - r_2} \xrightarrow{r_4 - 2r_2}$$

**例 2** 求向量组 
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$
,  $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一

$$\mathbf{m}$$
  $\alpha_1$   $\alpha_2$   $\alpha_3$   $\alpha_4$ 

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow{r_3 - r_2} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

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$$r_4-3r_3$$
 $r_1-2r_2$ 

**例 2** 求向量组 
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$
,  $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一

$$\mathbf{H}$$
  $\alpha_1$   $\alpha_2$   $\alpha_3$   $\alpha_4$ 

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow{r_3 - r_2} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$\frac{r_4 - 3r_3}{r_1 - 2r_3} \begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

**例 2** 求向量组 
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$
,  $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一

$$\mathbf{H}$$
  $\alpha_1$   $\alpha_2$   $\alpha_3$   $\alpha_4$ 

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow{r_3 - r_2} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$\begin{array}{c}
r_4 - 3r_3 \\
r_1 - 2r_3
\end{array}
\left(\begin{array}{cccc}
1 & 0 & 1 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{array}\right)$$

**例 2** 求向量组 
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$
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  $\alpha_1$   $\alpha_2$   $\alpha_3$   $\alpha_4$ 

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow{r_3 - r_2} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$\frac{r_4 - 3r_3}{r_1 - 2r_3} 
\begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

所以

**例 2** 求向量组 
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$
,  $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一

$$\mathbf{H}$$
  $\alpha_1$   $\alpha_2$   $\alpha_3$   $\alpha_4$ 

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow{r_3 - r_2} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$\begin{array}{c}
r_{4}-3r_{3} \\
r_{1}-2r_{3}
\end{array}
\left(\begin{array}{cccc}
1 & 0 & 1 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{array}\right)$$

所以

• 
$$r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 3$$
;

**例 2** 求向量组 
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$
,  $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一

$$\mathbf{H}$$
  $\alpha_1$   $\alpha_2$   $\alpha_3$   $\alpha_4$ 

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow{r_3 - r_2} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$\frac{r_4 - 3r_3}{r_1 - 2r_3} \left( \begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

所以

• 
$$r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 3$$
;

例 2 求向量组 
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$
,  $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一

$$\mathbf{p}$$
  $\alpha_1$   $\alpha_2$   $\alpha_3$   $\alpha_4$ 

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow{r_3 - r_2} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$\xrightarrow[r_1-2r_3]{r_1-2r_3} \left( \begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

所以

- $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 3$ ;
- α<sub>1</sub>, α<sub>2</sub>, α<sub>4</sub> 是极大无关组;

**例 2** 求向量组 
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$
,  $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一

$$\mathbf{H}$$
  $\alpha_1$   $\alpha_2$   $\alpha_3$   $\alpha_4$ 

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow{r_3 - r_2} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$\xrightarrow[r_1-2r_3]{r_1-2r_3} \left( \begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

所以

• 
$$r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 3$$
;

**例 3** 求向量组 
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$
,  $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个

例 3 求向量组 
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$
,  $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个

$$\mathbf{\widetilde{H}}$$
  $\alpha_1$   $\alpha_2$   $\alpha_3$   $\alpha_4$ 

$$\begin{pmatrix}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 5 \\
3 & 4 & 5 & 6 \\
4 & 5 & 6 & 7
\end{pmatrix}$$

**例 3** 求向量组 
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$
,  $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个

$$\mathbf{H}$$
  $\alpha_1$   $\alpha_2$   $\alpha_3$   $\alpha_4$ 

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \xrightarrow[r_4-4r_1]{r_2-2r_1} \frac{1}{r_4-4r_1}$$

例 3 求向量组 
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$
,  $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个

$$\mathbf{H}$$
  $\alpha_1$   $\alpha_2$   $\alpha_3$   $\alpha_4$ 

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \xrightarrow[r_4-4r_1]{r_2-2r_1} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{pmatrix}$$

**例 3** 求向量组 
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$
,  $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个

 $\mathbf{H}$   $\alpha_1$   $\alpha_2$   $\alpha_3$   $\alpha_4$ 

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \xrightarrow[r_4-4r_1]{r_2-2r_1} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{pmatrix}$$

$$r_3-2r_2$$
 $r_4-3r_2$ 

例 3 求向量组 
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$
,  $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个

$$\mathbf{H}$$
  $\alpha_1$   $\alpha_2$   $\alpha_3$   $\alpha_4$ 

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \xrightarrow[r_4-4r_1]{r_2-2r_1} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{pmatrix}$$

$$\begin{array}{c}
r_{3}-2r_{2} \\
r_{4}-3r_{2}
\end{array}
\left(\begin{array}{cccc}
1 & 2 & 3 & 4 \\
0 & -1 & -2 & -3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right)$$

**例 3** 求向量组 
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$
,  $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个

$$\mathbf{H}$$
  $\alpha_1$   $\alpha_2$   $\alpha_3$   $\alpha_4$ 

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \xrightarrow[r_4-4r_1]{r_2-2r_1} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{pmatrix}$$

$$\xrightarrow[r_4-3r_2]{r_4-3r_2} \left( \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \longrightarrow \left( \begin{array}{cccc} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

**例 3** 求向量组 
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$
,  $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个

$$\mathbf{H}$$
  $\alpha_1$   $\alpha_2$   $\alpha_3$   $\alpha_4$ 

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \xrightarrow[r_4-4r_1]{r_2-2r_1} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{pmatrix}$$

$$\xrightarrow[r_4-3r_2]{\begin{array}{c}1&2&3&4\\0&-1&-2&-3\\0&0&0&0\\0&0&0&0\end{array}}\longrightarrow \left(\begin{array}{cccc}1&0&-1&-2\\0&1&2&3\\0&0&0&0\\0&0&0&0\end{array}\right)$$

例 3 求向量组 
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$
,  $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个

 $\mathbf{H}$   $\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4$ 

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \xrightarrow[r_4-4r_1]{r_2-2r_1} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{pmatrix}$$

$$\xrightarrow[r_4-3r_2]{r_4-3r_2} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

所以

• 
$$r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2$$
;

例 3 求向量组 
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$
,  $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个

 $\mathbf{H}$   $\alpha_1$   $\alpha_2$   $\alpha_3$   $\alpha_4$ 

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \xrightarrow[r_4-4r_1]{r_2-2r_1} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{pmatrix}$$

$$\xrightarrow[r_4-3r_2]{r_4-3r_2} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

所以

• 
$$r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2$$
;

**例 3** 求向量组 
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$
,  $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个

 $\mathbf{H}$   $\alpha_1$   $\alpha_2$   $\alpha_3$   $\alpha_4$ 

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \xrightarrow[r_4-4r_1]{r_2-2r_1} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{pmatrix}$$

$$\xrightarrow[r_4-3r_2]{r_4-3r_2} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

所以

• 
$$r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2$$
;

α<sub>1</sub>, α<sub>2</sub> 是极大无关组;

例 3 求向量组 
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$
,  $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个

 $\mathbf{H}$   $\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4$ 

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \xrightarrow[r_4-4r_1]{r_2-2r_1} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{pmatrix}$$

$$\xrightarrow[r_4-3r_2]{r_4-3r_2} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

所以

• 
$$r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2$$
;

- α<sub>1</sub>, α<sub>2</sub> 是极大无关组;
- $\alpha_3 = -\alpha_1 + 2\alpha_2$

**例 3** 求向量组 
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$
,  $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个

 $\mathbf{p}$   $\alpha_1$   $\alpha_2$   $\alpha_3$   $\alpha_4$ 

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \xrightarrow[r_4-4r_1]{r_2-2r_1} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{pmatrix}$$

$$\frac{r_3 - 2r_2}{r_4 - 3r_2} \begin{pmatrix}
1 & 2 & 3 & 4 \\
0 & -1 & -2 & -3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\longrightarrow
\begin{pmatrix}
1 & 0 & -1 & -2 \\
0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

所以

• 
$$r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2$$
;

- α<sub>1</sub>, α<sub>2</sub> 是极大无关组;
- $\alpha_3 = -\alpha_1 + 2\alpha_2$ ,  $\alpha_4 = -2\alpha_1 + 3\alpha_2$

**例** 假设向量组  $\alpha_1$ ,  $\alpha_2$ , ...,  $\alpha_s$  可由  $\beta_1$ ,  $\beta_2$ , ...,  $\beta_t$  线性表示,则

$$r(\alpha_1, \alpha_2, \ldots, \alpha_s) \leq r(\beta_1, \beta_2, \ldots, \beta_t).$$

$$r_1 = r(\alpha_1, \alpha_2, \dots, \alpha_s),$$
  

$$r_2 = r(\beta_1, \beta_2, \dots, \beta_t),$$

$$r_1 = r(\alpha_1, \alpha_2, \ldots, \alpha_s), \quad \alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_{r_1}}$$
 是极大无关组  $r_2 = r(\beta_1, \beta_2, \ldots, \beta_t),$ 

$$r_1 = r(\alpha_1, \alpha_2, ..., \alpha_s), \quad \alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_{r_1}}$$
 是极大无关组  $r_2 = r(\beta_1, \beta_2, ..., \beta_t), \quad \beta_{j_1}, \beta_{j_2}, ..., \beta_{j_{r_2}}$  是极大无关组

$$r_1 = r(\alpha_1, \alpha_2, \ldots, \alpha_s), \quad \alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_{r_1}}$$
 是极大无关组  $r_2 = r(\beta_1, \beta_2, \ldots, \beta_t), \quad \beta_{j_1}, \beta_{j_2}, \ldots, \beta_{j_{r_2}}$  是极大无关组 注意到  $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_{r_1}}$  能由  $\beta_{j_1}, \beta_{j_2}, \ldots, \beta_{j_{r_2}}$  线性表示,

# 证明设

$$r_1 = r(\alpha_1, \alpha_2, ..., \alpha_s), \quad \alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_{r_1}}$$
 是极大无关组  $r_2 = r(\beta_1, \beta_2, ..., \beta_t), \quad \beta_{j_1}, \beta_{j_2}, ..., \beta_{j_{r_2}}$  是极大无关组

注意到  $\alpha_{i_1}$ ,  $\alpha_{i_2}$ , . . . ,  $\alpha_{i_{r_1}}$  能由  $\beta_{j_1}$ ,  $\beta_{j_2}$ , . . . ,  $\beta_{j_{r_2}}$  线性表示,所以  $r_1 \leq r_2$ 。

**例** 假设向量组  $\alpha_1, \alpha_2, \ldots, \alpha_s$  可由  $\beta_1, \beta_2, \ldots, \beta_t$  线性表示,则  $r(\alpha_1, \alpha_2, \ldots, \alpha_s) \leq r(\beta_1, \beta_2, \ldots, \beta_t)$ .

# 证明设

$$r_1 = r(\alpha_1, \alpha_2, ..., \alpha_s), \quad \alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_{r_1}}$$
 是极大无关组  $r_2 = r(\beta_1, \beta_2, ..., \beta_t), \quad \beta_{j_1}, \beta_{j_2}, ..., \beta_{j_{r_2}}$  是极大无关组

注意到  $\alpha_{i_1}$ ,  $\alpha_{i_2}$ , . . . ,  $\alpha_{i_{r_1}}$  能由  $\beta_{j_1}$ ,  $\beta_{j_2}$ , . . . ,  $\beta_{j_{r_2}}$  线性表示,所以  $r_1 \leq r_2$ 。

定理 设有向量组 
$$(A): \alpha_1, \alpha_2, \ldots, \alpha_s$$

(B): 
$$\beta_1, \beta_2, \ldots, \beta_t$$

若它们等价,

# 证明设

$$r_1 = r(\alpha_1, \alpha_2, ..., \alpha_s), \quad \alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_{r_1}}$$
 是极大无关组  $r_2 = r(\beta_1, \beta_2, ..., \beta_t), \quad \beta_{j_1}, \beta_{j_2}, ..., \beta_{j_{r_2}}$  是极大无关组

注意到  $\alpha_{i_1}$ ,  $\alpha_{i_2}$ , ...,  $\alpha_{i_{r_1}}$  能由  $\beta_{j_1}$ ,  $\beta_{j_2}$ , ...,  $\beta_{j_{r_2}}$  线性表示,所以  $r_1 \leq r_2$ 。

**定理** 设有向量组  $(A): \alpha_1, \alpha_2, \ldots, \alpha_s$ 

(B):  $\beta_1, \beta_2, \ldots, \beta_t$ 

若它们等价,则  $r(\alpha_1, \alpha_2, \ldots, \alpha_s) = r(\beta_1, \beta_2, \ldots, \beta_t)$ 。

证明 设 $AB = C_{mxs}$ 

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}$$

证明 设 
$$AB = C_{m \times s}$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{\mathbf{c}_{m1}} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{\mathbf{c}_{m1}} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{\mathbf{c}_{m1}}$$

证明 设 
$$AB = C_{m \times s}$$

$$\begin{pmatrix}
c_{11} & c_{12} & \cdots & c_{1s} \\
c_{21} & c_{22} & \cdots & c_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
c_{m1} & c_{m2} & \cdots & c_{ms}
\end{pmatrix} = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix} \begin{pmatrix}
b_{11} & b_{12} & \cdots & b_{1s} \\
b_{21} & b_{22} & \cdots & b_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
b_{n1} & b_{n2} & \cdots & b_{ns}
\end{pmatrix}$$

证明 设 
$$AB = C_{m \times s}$$

$$\begin{pmatrix}
c_{11} & c_{12} & \cdots & c_{1s} \\
c_{21} & c_{22} & \cdots & c_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
c_{m1} & c_{m2} & \cdots & c_{ms}
\end{pmatrix} = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\begin{pmatrix}
b_{11} & b_{12} & \cdots & b_{1s} \\
b_{21} & b_{22} & \cdots & b_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
b_{n1} & b_{n2} & \cdots & b_{ns}
\end{pmatrix}$$

证明 设 
$$AB = C_{m \times s}$$

$$\underbrace{\begin{pmatrix}
c_{11} & c_{12} & \cdots & c_{1s} \\
c_{21} & c_{22} & \cdots & c_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
c_{m1} & c_{m2} & \cdots & c_{ms}
\end{pmatrix}}_{c} = \underbrace{\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}}_{c} \underbrace{\begin{pmatrix}
b_{11} & b_{12} & \cdots & b_{1s} \\
b_{21} & b_{22} & \cdots & b_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
b_{n1} & b_{n2} & \cdots & b_{ns}
\end{pmatrix}}_{c}$$

证明 设 
$$AB = C_{m \times s}$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{c_{m1}} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{c_{m1}} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{c_{m1}}$$

证明 设 
$$AB = C_{m \times s}$$

$$\underbrace{\begin{pmatrix}
c_{11} & c_{12} & \cdots & c_{1s} \\
c_{21} & c_{22} & \cdots & c_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
c_{m1} & c_{m2} & \cdots & c_{ms}
\end{pmatrix}}_{C} = \underbrace{\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}}_{A} \underbrace{\begin{pmatrix}
b_{11} & b_{12} & \cdots & b_{1s} \\
b_{21} & b_{22} & \cdots & b_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
b_{n1} & b_{n2} & \cdots & b_{ns}
\end{pmatrix}}_{B}$$

证明 设 
$$AB = C_{m \times s}$$

$$\underbrace{\begin{pmatrix}
c_{11} & c_{12} & \cdots & c_{1s} \\
c_{21} & c_{22} & \cdots & c_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
c_{m1} & c_{m2} & \cdots & c_{ms}
\end{pmatrix}}_{C} = \underbrace{\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}}_{A} \begin{pmatrix}
b_{11} & b_{12} & \cdots & b_{1s} \\
b_{21} & b_{22} & \cdots & b_{ns}
\end{pmatrix}}_{B}$$
即

即

$$(\gamma_1 \ \gamma_2 \cdots \gamma_s) = (\alpha_1 \ \alpha_2 \cdots \alpha_n) \begin{pmatrix} b_{11} \ b_{12} \cdots b_{1s} \\ b_{21} \ b_{22} \cdots b_{2s} \\ \vdots \ \vdots \ \ddots \ \vdots \\ b_{n1} \ b_{n2} \cdots b_{ns} \end{pmatrix}$$

证明 设 
$$AB = C_{m \times s}$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{ns} \end{pmatrix}}_{B}$$
即

$$(\gamma_1 \ \gamma_2 \cdots \gamma_s) = (\alpha_1 \ \alpha_2 \cdots \alpha_n) \begin{pmatrix} b_{11} \ b_{12} \cdots b_{1s} \\ b_{21} \ b_{22} \cdots b_{2s} \\ \vdots \ \vdots \ \ddots \ \vdots \\ b_{n1} \ b_{n2} \cdots b_{ns} \end{pmatrix}$$

$$\Rightarrow \gamma_1 = b_{11}\alpha_1 + b_{21}\alpha_2 + \cdots + b_{n1}\alpha_n$$

证明 设 
$$AB = C_{m \times s}$$

$$\underbrace{\begin{pmatrix}
c_{11} & c_{12} & \cdots & c_{1s} \\
c_{21} & c_{22} & \cdots & c_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
c_{m1} & c_{m2} & \cdots & c_{ms}
\end{pmatrix}}_{C} = \underbrace{\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}}_{A} \begin{pmatrix}
b_{11} & b_{12} & \cdots & b_{1s} \\
b_{21} & b_{22} & \cdots & b_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
b_{n1} & b_{n2} & \cdots & b_{ns}
\end{pmatrix}}_{B}$$

$$(\gamma_1 \ \gamma_2 \cdots \gamma_s) = (\alpha_1 \ \alpha_2 \cdots \alpha_n) \begin{pmatrix} b_{11} \ b_{12} \cdots b_{1s} \\ b_{21} \ b_{22} \cdots b_{2s} \\ \vdots \ \vdots \ \ddots \ \vdots \\ b_{n1} \ b_{n2} \cdots b_{ns} \end{pmatrix}$$

⇒ 
$$\gamma_1 = b_{11}\alpha_1 + b_{21}\alpha_2 + \cdots + b_{n1}\alpha_n$$
 等等

证明 设 
$$AB = C_{m \times s}$$

$$\begin{pmatrix}
c_{11} & c_{12} & \cdots & c_{1s} \\
c_{21} & c_{22} & \cdots & c_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
c_{m1} & c_{m2} & \cdots & c_{ms}
\end{pmatrix} = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix} \begin{pmatrix}
b_{11} & b_{12} & \cdots & b_{1s} \\
b_{21} & b_{22} & \cdots & b_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
b_{n1} & b_{n2} & \cdots & b_{ns}
\end{pmatrix}$$

即

$$(\gamma_1 \ \gamma_2 \cdots \gamma_s) = (\alpha_1 \ \alpha_2 \cdots \alpha_n) \begin{pmatrix} b_{11} \ b_{12} \cdots b_{1s} \\ b_{21} \ b_{22} \cdots b_{2s} \\ \vdots \ \vdots \ \ddots \ \vdots \\ b_{n1} \ b_{n2} \cdots b_{ns} \end{pmatrix}$$

$$\Rightarrow$$
  $\gamma_1 = b_{11}\alpha_1 + b_{21}\alpha_2 + \cdots + b_{n1}\alpha_n$  等等

可见  $\gamma_1, \ldots, \gamma_s$  由  $\alpha_1, \ldots, \alpha_n$  线性表示,

证明 设 
$$AB = C_{m \times s}$$

$$\underbrace{\begin{pmatrix}
c_{11} & c_{12} & \cdots & c_{1s} \\
c_{21} & c_{22} & \cdots & c_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
c_{m1} & c_{m2} & \cdots & c_{ms}
\end{pmatrix}}_{C} = \underbrace{\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}}_{A} \underbrace{\begin{pmatrix}
b_{11} & b_{12} & \cdots & b_{1s} \\
b_{21} & b_{22} & \cdots & b_{2s}
\end{pmatrix}}_{B}$$

$$\underbrace{\begin{pmatrix}
b_{11} & b_{12} & \cdots & b_{1s} \\
b_{n1} & b_{n2} & \cdots & b_{ns}
\end{pmatrix}}_{B}$$

即

$$(\gamma_1 \ \gamma_2 \cdots \gamma_s) = (\alpha_1 \ \alpha_2 \cdots \alpha_n) \begin{pmatrix} b_{11} \ b_{12} \cdots b_{1s} \\ b_{21} \ b_{22} \cdots b_{2s} \\ \vdots \ \vdots \ \ddots \ \vdots \\ b_{n1} \ b_{n2} \cdots b_{ns} \end{pmatrix}$$

$$\Rightarrow \gamma_1 = b_{11}\alpha_1 + b_{21}\alpha_2 + \cdots + b_{n1}\alpha_n$$
 等等

可见  $\gamma_1, \ldots, \gamma_s$  由  $\alpha_1, \ldots, \alpha_n$  线性表示,所以

$$r(\gamma_1, \ldots, \gamma_s) \leq r(\alpha_1, \ldots, \alpha_n)$$

证明 设 
$$AB = C_{m \times s}$$

$$\underbrace{\begin{pmatrix}
c_{11} & c_{12} & \cdots & c_{1s} \\
c_{21} & c_{22} & \cdots & c_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
c_{m1} & c_{m2} & \cdots & c_{ms}
\end{pmatrix}}_{C} = \underbrace{\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}}_{A} \underbrace{\begin{pmatrix}
b_{11} & b_{12} & \cdots & b_{1s} \\
b_{21} & b_{22} & \cdots & b_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
b_{n1} & b_{n2} & \cdots & b_{ns}
\end{pmatrix}}_{B}$$

BD

$$\begin{pmatrix}
b_{11} & b_{12} & \cdots & b_{1s} \\
b_{21} & b_{22} & \cdots & b_{2s}
\end{pmatrix}$$

即

$$(\gamma_1 \ \gamma_2 \cdots \gamma_s) = (\alpha_1 \ \alpha_2 \cdots \alpha_n) \begin{pmatrix} b_{11} \ b_{12} \cdots b_{1s} \\ b_{21} \ b_{22} \cdots b_{2s} \\ \vdots \ \vdots \ \ddots \ \vdots \\ b_{n1} \ b_{n2} \cdots b_{ns} \end{pmatrix}$$

$$\Rightarrow \gamma_1 = b_{11}\alpha_1 + b_{21}\alpha_2 + \cdots + b_{n1}\alpha_n$$
 等等

可见  $\gamma_1, \ldots, \gamma_s$  由  $\alpha_1, \ldots, \alpha_n$  线性表示,所以

$$r(\gamma_1, \ldots, \gamma_s) \le r(\alpha_1, \ldots, \alpha_n) = r(A)$$

证明 设 
$$AB = C_{m \times s}$$

$$\underbrace{\begin{pmatrix}
c_{11} & c_{12} & \cdots & c_{1s} \\
c_{21} & c_{22} & \cdots & c_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
c_{m1} & c_{m2} & \cdots & c_{ms}
\end{pmatrix}}_{C} = \underbrace{\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}}_{A} \underbrace{\begin{pmatrix}
b_{11} & b_{12} & \cdots & b_{1s} \\
b_{21} & b_{22} & \cdots & b_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
b_{n1} & b_{n2} & \cdots & b_{ns}
\end{pmatrix}}_{B}$$

Here,  $b_{11}$ ,  $b_{12}$ ,  $b_{23}$ ,  $b_{24}$ ,  $b_{25}$ 

即

$$(\gamma_1 \ \gamma_2 \cdots \gamma_s) = (\alpha_1 \ \alpha_2 \cdots \alpha_n) \begin{pmatrix} b_{11} \ b_{12} \cdots b_{1s} \\ b_{21} \ b_{22} \cdots b_{2s} \\ \vdots \ \vdots \ \ddots \ \vdots \\ b_{n1} \ b_{n2} \cdots b_{ns} \end{pmatrix}$$

$$\Rightarrow \gamma_1 = b_{11}\alpha_1 + b_{21}\alpha_2 + \cdots + b_{n1}\alpha_n$$
 等等

可见  $\gamma_1, \ldots, \gamma_s$  由  $\alpha_1, \ldots, \alpha_n$  线性表示,所以

$$r(AB) = r(\gamma_1, \ldots, \gamma_s) \le r(\alpha_1, \ldots, \alpha_n) = r(A)$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{G} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}^{\beta_{1}}$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}^{\beta_{1}}$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{\beta_{n}}^{\beta_{1}}$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{\beta_{n}}^{\beta_{1}}$$

$$\frac{\delta_{1}}{\delta_{2}} \underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{G} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{\beta_{n}}_{\beta_{n}}$$

$$\underbrace{\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{bmatrix}}_{C} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{bmatrix}}_{\beta_{n}}^{\beta_{1}}$$

$$\underbrace{\begin{bmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{bmatrix} \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1s} \\ C_{21} & C_{22} & \cdots & C_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ C_{m1} & C_{m2} & \cdots & C_{ms} \end{pmatrix}}_{G} = \underbrace{\begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{\beta_n} \beta_n$$

## 证明 设 $AB = C_{m \times s}$

$$\frac{\delta_{1}}{\delta_{2}} \begin{pmatrix} c_{11} & c_{12} \cdots c_{1s} \\ c_{21} & c_{22} \cdots c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} \cdots c_{ms} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} \cdots a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \cdots b_{1s} \\ b_{21} & b_{22} \cdots b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} \cdots b_{ns} \end{pmatrix} \begin{pmatrix} \beta_{1} \\ \beta_{2} \\ \vdots \\ \beta_{n} \end{pmatrix}$$

$$\frac{\delta_{1}}{\delta_{2}} \begin{pmatrix} \delta_{1} \\ \delta_{2} \\ \vdots \\ \delta_{n1} & a_{n2} \cdots a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} \cdots a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{nn} & a_{nn} & a_{nn} \end{pmatrix} \begin{pmatrix} \beta_{1} \\ \beta_{2} \\ \vdots \\ \beta_{nn} \end{pmatrix}$$

## 证明 设 $AB = C_{m \times s}$

$$\frac{\delta_{1}}{\delta_{2}} \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix} \begin{pmatrix} \beta_{1} \\ \beta_{2} \\ \vdots \\ \beta_{n} \end{pmatrix}$$

$$\Rightarrow \delta_{1} = a_{11}\beta_{1} + a_{12}\beta_{2} + \cdots + a_{1n}\beta_{n}$$

## 证明 设 $AB = C_{m \times s}$

$$\frac{\delta_{1}}{\delta_{2}} \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix} \begin{pmatrix} \beta_{1} \\ \beta_{2} \\ \vdots \\ \beta_{n} \end{pmatrix}$$

$$\frac{\delta_{1}}{\delta_{2}} \begin{pmatrix} \delta_{1} \\ \delta_{2} \\ \vdots \\ \delta_{m} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \beta_{1} \\ \beta_{2} \\ \vdots \\ \beta_{n} \end{pmatrix}$$

$$\Rightarrow \delta_{1} = a_{11}\beta_{1} + a_{12}\beta_{2} + \cdots + a_{1n}\beta_{n} \quad \text{\textweather}$$

## 证明 设 $AB = C_{m \times s}$

$$\frac{\delta_{1}}{\delta_{2}} \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix} \beta_{1} \\ \beta_{2} \\ \vdots \\ \beta_{n} \end{pmatrix}$$

$$\frac{\delta_{1}}{\delta_{2}} \begin{pmatrix} \delta_{1} \\ \delta_{2} \\ \vdots \\ \delta_{m} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \beta_{1} \\ \beta_{2} \\ \vdots \\ \beta_{n} \end{pmatrix}$$

$$\Rightarrow \delta_{1} = a_{11}\beta_{1} + a_{12}\beta_{2} + \cdots + a_{1n}\beta_{n} \quad \text{\textwith}$$

可见  $\delta_1, \ldots, \delta_m$  由  $\beta_1, \ldots, \beta_n$  线性表示,

## 证明 设 $AB = C_{m \times s}$

$$\frac{\delta_{1}}{\delta_{2}} \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix} \beta_{1} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix} \beta_{n}$$

$$\frac{\delta_{1}}{\delta_{2}} \begin{pmatrix} \delta_{1} \\ \delta_{2} \\ \vdots \\ \delta_{m} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \beta_{1} \\ \beta_{2} \\ \vdots \\ \beta_{n} \end{pmatrix}$$

$$\Rightarrow \delta_{1} = a_{11}\beta_{1} + a_{12}\beta_{2} + \cdots + a_{1n}\beta_{n} \quad \text{\textwith}$$

可见  $\delta_1, \ldots, \delta_m$  由  $\beta_1, \ldots, \beta_n$  线性表示,所以

$$r(\delta_1,\ldots,\delta_m) \leq r(\beta_1,\ldots,\beta_n)$$

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## 证明 设 $AB = C_{m \times s}$

$$\frac{\delta_{1}}{\delta_{2}} \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix} \beta_{1} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix} \beta_{n}$$

$$\frac{\delta_{1}}{\delta_{2}} \begin{pmatrix} \delta_{1} \\ \delta_{2} \\ \vdots \\ \delta_{m} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \beta_{1} \\ \beta_{2} \\ \vdots \\ \beta_{n} \end{pmatrix}$$

$$\Rightarrow \delta_{1} = a_{11}\beta_{1} + a_{12}\beta_{2} + \cdots + a_{1n}\beta_{n} \quad \text{\textwith}$$

可见  $\delta_1, \ldots, \delta_m$  由  $\beta_1, \ldots, \beta_n$  线性表示,所以

$$r(\delta_1, \ldots, \delta_m) \le r(\beta_1, \ldots, \beta_n) = r(B)$$

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## 证明 设 $AB = C_{m \times s}$

$$\frac{\delta_{1}}{\delta_{2}} \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix} \beta_{1} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix} \beta_{n}$$

$$\frac{\delta_{1}}{\delta_{2}} \begin{pmatrix} \delta_{1} \\ \delta_{2} \\ \vdots \\ \delta_{m} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \beta_{1} \\ \beta_{2} \\ \vdots \\ \beta_{n} \end{pmatrix}$$

$$\Rightarrow \delta_{1} = a_{11}\beta_{1} + a_{12}\beta_{2} + \cdots + a_{1n}\beta_{n} \quad \text{\textwith}$$

可见  $\delta_1, \ldots, \delta_m$  由  $\beta_1, \ldots, \beta_n$  线性表示,所以

$$r(AB) = r(\delta_1, \ldots, \delta_m) \le r(\beta_1, \ldots, \beta_n) = r(B)$$

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