### 第 11 章 α: 对弧长的曲线积分

数学系 梁卓滨

2017-2018 学年 II



### **Outline**

1. 对弧长的曲线积分: 概念与性质

2. 对弧长的曲线积分: 计算法

3. 对弧长的曲线积分: 空间曲线



### We are here now...

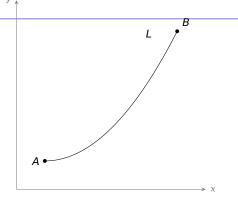
1. 对弧长的曲线积分: 概念与性质

2. 对弧长的曲线积分: 计算法

3. 对弧长的曲线积分: 空间曲线

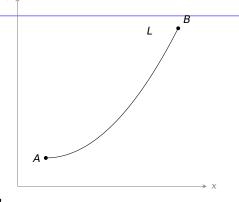
假设平面曲线 L

- 线密度为 μ(x, y)
- 质量为 m



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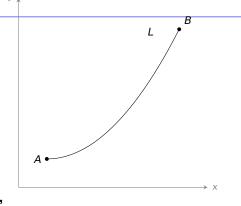


当曲线是均匀时(μ=常数),

当曲线非均匀时(μ = μ(x, y) 为 L 上函数)

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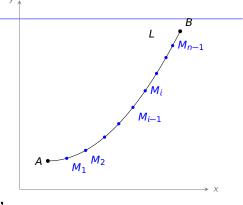
$$m = \mu \cdot \text{Length}(L)$$

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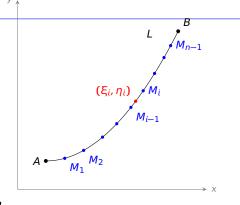
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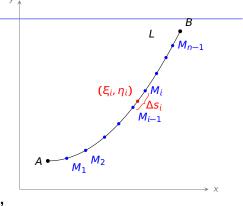
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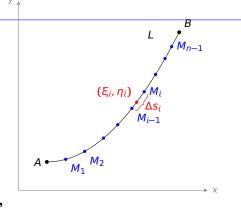
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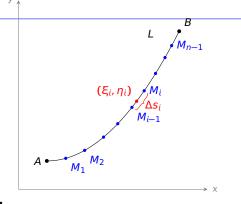
$$m = \mu \cdot \text{Length}(L)$$

$$\mu(\xi_i, \eta_i)\Delta s_i$$



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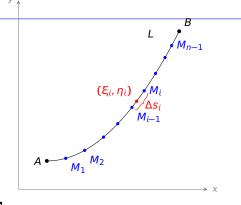
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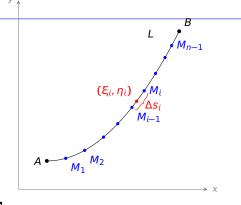
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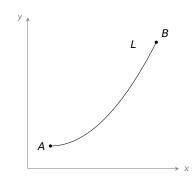
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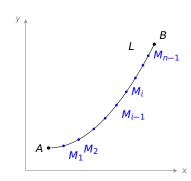
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- L 是平面上分段光滑曲线,
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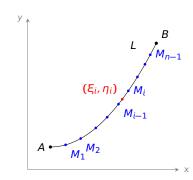
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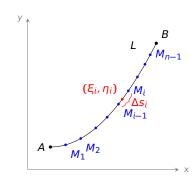
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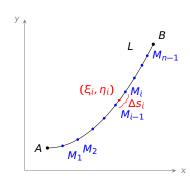
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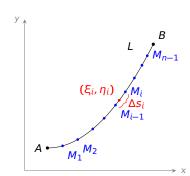
$$\sum_{i=1}^n f(\xi_i, \eta_i) \Delta s_i$$



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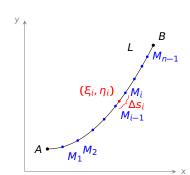


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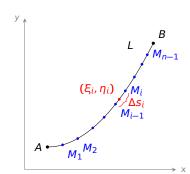
• 极限  $\lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_i, \eta_i) \Delta s_i$ 存在,



#### 对弧长的曲线积分定义 设

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- 极限  $\lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_i, \eta_i) \Delta s_i$ 存在,且极限
- 与上述 L 的划分、 $(ξ_i, η_i)$  的选取无关,

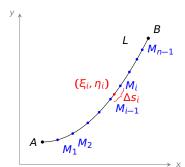


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$$\int_{L} f(x, y) ds = \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_{i}, \eta_{i}) \Delta s_{i}$$



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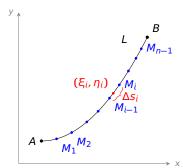
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称为 f(x, y) 在曲线 L 上的对弧长的曲线积分。



#### 对弧长的曲线积分定义 设

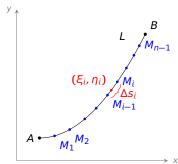
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- 与上述 L 的划分、 $(\xi_i, \eta_i)$  的选取无关,

$$\int_{I} f(x, y) ds = \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_{i}, \eta_{i}) \Delta s_{i}$$

称为 f(x, y) 在曲线 L 上的对弧长的曲线积分。 ds 称为弧长元素。



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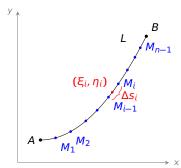
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注 对弧长的曲线积分的定义式与重积分的类似,故性质也类似



• 存在性 若 L 是分段光滑曲线, f(x, y) 在 L 上连续, 则

$$\int_{1} f(x, y) ds$$

◆ 存在性 若 L 是分段光滑曲线, f(x, y) 在 L 上连续, 则∫ , , , ,

$$\int_{L} f(x, y) ds$$

存在。

• 线性性  $\int_L (\alpha f + \beta g) ds = \alpha \int_L f ds + \beta \int_L g ds$ 



• 存在性 若 L 是分段光滑曲线,f(x, y) 在 L 上连续,则  $\int_{L} f(x, y) ds$ 

- 线性性  $\int_{I} (\alpha f + \beta g) ds = \alpha \int_{I} f ds + \beta \int_{I} g ds$
- 可加性  $\int_{L} f(x,y) ds = \int_{L_1} f(x,y) ds + \int_{L_2} f(x,y) ds$

• 存在性 若 L 是分段光滑曲线,f(x, y) 在 L 上连续,则  $\int_{\Gamma} f(x, y) ds$ 

- 线性性  $\int_L (\alpha f + \beta g) ds = \alpha \int_L f ds + \beta \int_L g ds$
- 可加性  $\int_L f(x,y)ds = \int_{L_1} f(x,y)ds + \int_{L_2} f(x,y)ds$
- $\int_L 1ds = \text{Length}(L)$

◆ 存在性 若 L 是分段光滑曲线, f(x, y) 在 L 上连续, 则

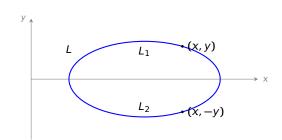
$$\int_{L} f(x, y) ds$$

- 线性性  $\int_L (\alpha f + \beta g) ds = \alpha \int_L f ds + \beta \int_L g ds$
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- $\int_L 1ds = \text{Length}(L)$
- 若 f(x,y) ≤ g(x,y), 则

$$\int_{L} f(x, y) ds \le \int_{L} g(x, y) ds$$

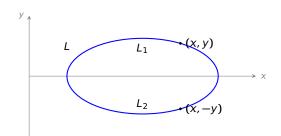


性质 设平面曲线 L 关于 x 轴对称,



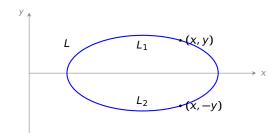
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• 若 f(x,y) 关于 y 是奇函数 (即: f(x,-y) = -f(x,y)),则  $\int_{1}^{1} f(x,y)ds = 0$ 

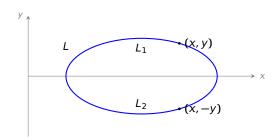


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$$\int_{L} f(x, y) ds = 0$$

• 若 f(x, y) 关于 y 是偶函数 (即: f(x, -y) = f(x, y)),则



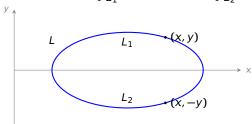
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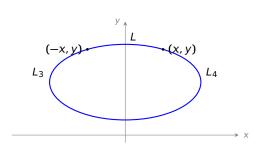
$$\int_{L} f(x, y) ds = 0$$

• 若 f(x, y) 关于 y 是偶函数 (即: f(x, -y) = f(x, y)),则

$$\int_{L} f(x, y) ds = 2 \int_{L_{1}} f(x, y) ds = 2 \int_{L_{2}} f(x, y) ds$$

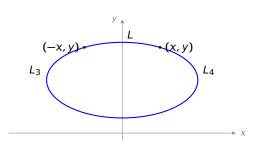


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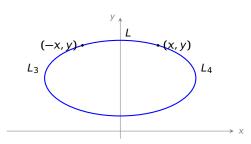
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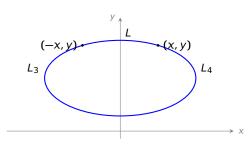


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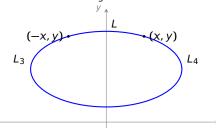
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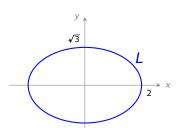
$$\int_{L} f(x, y) ds = 2 \int_{L_{3}} f(x, y) ds = 2 \int_{L_{4}} f(x, y) ds$$



计算

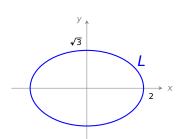
例 已知椭圆 
$$L: \frac{x^2}{4} + \frac{y^2}{3} = 1$$
 的周长是  $\alpha$ ,

$$\int_{1}^{2} 2xy + 3x^2 + 4y^2 ds$$



例 已知椭圆  $L: \frac{x^2}{4} + \frac{y^2}{3} = 1$  的周长是  $\alpha$ , 计算

$$\int_{1} 2xy + 3x^2 + 4y^2 ds$$

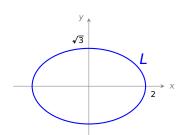


原式 = 
$$\int_{1}^{1} 2xyds + \int_{1}^{1} 12(\frac{x^{2}}{4} + \frac{y^{2}}{3})ds$$

计算

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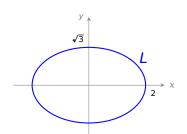
$$\int_{I} 2xy + 3x^2 + 4y^2 ds$$



原式 = 
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例 已知椭圆  $L: \frac{x^2}{4} + \frac{y^2}{3} = 1$  的周长是  $\alpha$ , 计算

$$\int_{1}^{\infty} 2xy + 3x^2 + 4y^2 ds$$



原式 = 
$$\int_{1}^{1} 2xyds + \int_{1}^{1} 12(\frac{x^2}{4} + \frac{y^2}{3})ds = 0 + \int_{1}^{1} 12ds$$

计算

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$$L: \frac{x^2}{4} + \frac{y^2}{3} = 1$$
 的周长是  $a$ ,  
计算 
$$\int_{L} 2xy + 3x^2 + 4y^2 ds$$

原式 = 
$$\int_{1}^{1} 2xyds + \int_{1}^{1} 12(\frac{x^2}{4} + \frac{y^2}{3})ds = 0 + \int_{1}^{1} 12ds = 12a$$

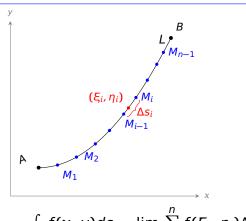


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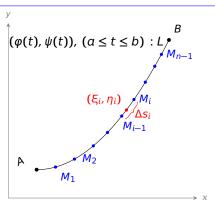
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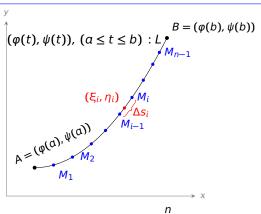


$$\int_L f(x, y) ds = \lim_{\lambda \to 0} \sum_{i=1}^n f(\xi_i, \eta_i) \Delta s_i$$

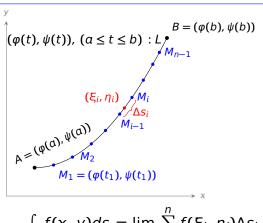


$$\int_{L} f(x, y) ds = \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_{i}, \eta_{i}) \Delta s_{i}$$

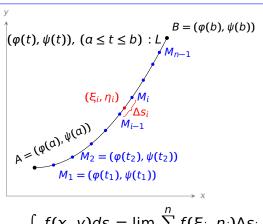




$$\int_{L} f(x, y) ds = \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_{i}, \eta_{i}) \Delta s_{i}$$

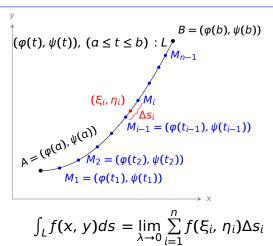


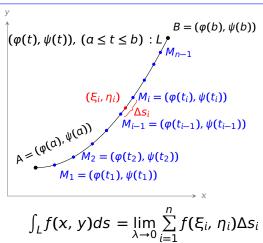
$$\int_L f(x, y) ds = \lim_{\lambda \to 0} \sum_{i=1}^n f(\xi_i, \eta_i) \Delta s_i$$



$$\int_L f(x, y) ds = \lim_{\lambda \to 0} \sum_{i=1}^n f(\xi_i, \eta_i) \Delta s_i$$







$$B = (\varphi(b), \psi(b))$$

$$(\varphi(t), \psi(t)), (a \le t \le b) : L$$

$$M_{n-1} = (\varphi(t_{n-1}), \psi(t_{n-1}))$$

$$M_{i} = (\varphi(t_{i}), \psi(t_{i}))$$

$$M_{i-1} = (\varphi(t_{i-1}), \psi(t_{i-1}))$$

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$$M_{i} = (\varphi(t_{i-1}), \psi(t_{i-1}))$$

$$\int_{L} f(x, y) ds = \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_{i}, \eta_{i}) \Delta s_{i}$$



$$B = (\varphi(b), \psi(b))$$

$$(\varphi(t), \psi(t)), (\alpha \le t \le b) : L$$

$$M_{n-1} = (\varphi(t_{n-1}), \psi(t_{n-1}))$$

$$\Delta s_{i} \qquad \Delta s_{i} \approx |M_{i-1}M_{i}|$$

$$M_{i-1} = (\varphi(t_{i-1}), \psi(t_{i-1})) = \sqrt{(\varphi(t_{i-1}) - \varphi(t_{i}))^{2} + (\psi(t_{i-1}) - \psi(t_{i}))^{2}}$$

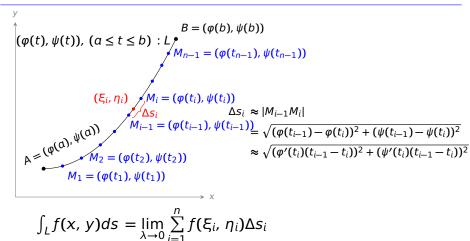
$$A = (\varphi(t_{1}), \psi(t_{1}))$$

$$M_{1} = (\varphi(t_{1}), \psi(t_{1}))$$

$$M_{1} = (\varphi(t_{1}), \psi(t_{1}))$$

$$\int_{L} f(x, y) ds = \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_{i}, \eta_{i}) \Delta s_{i}$$







$$B = (\varphi(b), \psi(b))$$

$$(\varphi(t), \psi(t)), (a \le t \le b) : L$$

$$M_{n-1} = (\varphi(t_{n-1}), \psi(t_{n-1}))$$

$$(\xi_{i}, \eta_{i}) \qquad M_{i} = (\varphi(t_{i}), \psi(t_{i}))$$

$$\Delta s_{i} \qquad \Delta s_{i} \approx |M_{i-1}M_{i}|$$

$$M_{i-1} = (\varphi(t_{i-1}), \psi(t_{i-1})) = \sqrt{(\varphi(t_{i-1}) - \varphi(t_{i}))^{2} + (\psi(t_{i-1}) - \psi(t_{i}))^{2}}$$

$$\approx \sqrt{(\varphi'(t_{i})(t_{i-1} - t_{i}))^{2} + (\psi'(t_{i})(t_{i-1} - t_{i}))^{2}}$$

$$\approx \sqrt{(\varphi'(t_{i})(t_{i-1} - t_{i}))^{2} + (\psi'(t_{i})(t_{i-1} - t_{i}))^{2}}$$

$$= \sqrt{\varphi'(t_{i})^{2} + \psi'(t_{i})^{2}}(t_{i} - t_{i-1})$$

$$\int_{L} f(x, y) ds = \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_{i}, \eta_{i}) \Delta s_{i}$$



$$B = (\varphi(b), \psi(b))$$

$$(\varphi(t), \psi(t)), (a \le t \le b) : L$$

$$M_{n-1} = (\varphi(t_{n-1}), \psi(t_{n-1}))$$

$$(\xi_{i}, \eta_{i}) \qquad M_{i} = (\varphi(t_{i}), \psi(t_{i}))$$

$$\Delta s_{i} \qquad \Delta s_{i} \approx |M_{i-1}M_{i}|$$

$$M_{i-1} = (\varphi(t_{i-1}), \psi(t_{i-1})) = \sqrt{(\varphi(t_{i-1}) - \varphi(t_{i}))^{2} + (\psi(t_{i-1}) - \psi(t_{i}))^{2}}$$

$$\approx \sqrt{(\varphi'(t_{i})(t_{i-1} - t_{i}))^{2} + (\psi'(t_{i})(t_{i-1} - t_{i}))^{2}}$$

$$\approx \sqrt{(\varphi'(t_{i})(t_{i-1} - t_{i}))^{2} + (\psi'(t_{i})(t_{i-1} - t_{i}))^{2}}$$

$$= \sqrt{\varphi'(t_{i})^{2} + \psi'(t_{i})^{2}}(t_{i} - t_{i-1})$$

$$= \sqrt{\varphi'(t_{i})^{2} + \psi'(t_{i})^{2}}\Delta t_{i}$$

$$\int_{L} f(x, y) ds = \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_{i}, \eta_{i}) \Delta s_{i}$$



$$B = (\varphi(b), \psi(b))$$

$$(\varphi(t), \psi(t)), (\alpha \leq t \leq b) : L^{\bullet}$$

$$M_{n-1} = (\varphi(t_{n-1}), \psi(t_{n-1}))$$

$$(\xi_{i}, \eta_{i}) \qquad M_{i} = (\varphi(t_{i}), \psi(t_{i}))$$

$$\Delta s_{i} \qquad \Delta s_{i} \approx |M_{i-1}M_{i}|$$

$$M_{i-1} = (\varphi(t_{i-1}), \psi(t_{i-1})) = \sqrt{(\varphi(t_{i-1}) - \varphi(t_{i}))^{2} + (\psi(t_{i-1}) - \psi(t_{i}))^{2}}$$

$$\approx \sqrt{(\varphi'(t_{i})(t_{i-1} - t_{i}))^{2} + (\psi'(t_{i})(t_{i-1} - t_{i}))^{2}}$$

$$M_{1} = (\varphi(t_{1}), \psi(t_{1})) \qquad = \sqrt{\varphi'(t_{i})^{2} + \psi'(t_{i})^{2}}(t_{i} - t_{i-1})$$

$$= \sqrt{\varphi'(t_{i})^{2} + \psi'(t_{i})^{2}}\Delta t_{i}$$

$$\int_{L} f(x, y) ds = \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_{i}, \eta_{i}) \Delta s_{i}$$

$$\sqrt{\varphi'(t_{i})^{2} + \psi'(t_{i})^{2}} \Delta t_{i}$$



$$B = (\varphi(b), \psi(b))$$

$$(\varphi(t), \psi(t)), (a \le t \le b) : L \nearrow$$

$$M_{n-1} = (\varphi(t_{n-1}), \psi(t_{n-1}))$$

$$(\xi_{i}, \eta_{i}) \qquad M_{i} = (\varphi(t_{i}), \psi(t_{i}))$$

$$\Delta s_{i} \qquad \Delta s_{i} \approx |M_{i-1}M_{i}|$$

$$M_{i-1} = (\varphi(t_{i-1}), \psi(t_{i-1})) = \sqrt{(\varphi(t_{i-1}) - \varphi(t_{i}))^{2} + (\psi(t_{i-1}) - \psi(t_{i}))^{2}}$$

$$\approx \sqrt{(\varphi'(t_{i})(t_{i-1} - t_{i}))^{2} + (\psi'(t_{i})(t_{i-1} - t_{i}))^{2}}$$

$$= \sqrt{\varphi'(t_{i})^{2} + \psi'(t_{i})^{2}}(t_{i} - t_{i-1})$$

$$= \sqrt{\varphi'(t_{i})^{2} + \psi'(t_{i})^{2}}\Delta t_{i}$$

$$f(\varphi(t_{i}), \psi(t_{i})) \sqrt{\varphi'(t_{i})^{2} + \psi'(t_{i})^{2}}\Delta t_{i}$$



$$B = (\varphi(b), \psi(b))$$

$$(\varphi(t), \psi(t)), (a \le t \le b) : L$$

$$M_{n-1} = (\varphi(t_{n-1}), \psi(t_{n-1}))$$

$$(\xi_{i}, \eta_{i}) \qquad M_{i} = (\varphi(t_{i}), \psi(t_{i}))$$

$$\Delta s_{i} \qquad \Delta s_{i} \approx |M_{i-1}M_{i}|$$

$$M_{i-1} = (\varphi(t_{i-1}), \psi(t_{i-1})) = \sqrt{(\varphi(t_{i-1}) - \varphi(t_{i}))^{2} + (\psi(t_{i-1}) - \psi(t_{i}))^{2}}$$

$$\approx \sqrt{(\varphi'(t_{i})(t_{i-1} - t_{i}))^{2} + (\psi'(t_{i})(t_{i-1} - t_{i}))^{2}}$$

$$= \sqrt{\varphi'(t_{i})^{2} + \psi'(t_{i})^{2}}(t_{i} - t_{i-1})$$

$$= \sqrt{\varphi'(t_{i})^{2} + \psi'(t_{i})^{2}}\Delta t_{i}$$

$$= \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\varphi(t_{i}), \psi(t_{i})) \sqrt{\varphi'(t_{i})^{2} + \psi'(t_{i})^{2}}\Delta t_{i}$$



$$B = (\varphi(b), \psi(b))$$

$$(\varphi(t), \psi(t)), (a \le t \le b) : L \xrightarrow{M_{n-1}} = (\varphi(t_{n-1}), \psi(t_{n-1}))$$

$$(\xi_{i}, \eta_{i}) \qquad M_{i} = (\varphi(t_{i}), \psi(t_{i}))$$

$$\Delta s_{i} \qquad \Delta s_{i} \approx |M_{i-1}M_{i}|$$

$$M_{i-1} = (\varphi(t_{i-1}), \psi(t_{i-1})) = \sqrt{(\varphi(t_{i-1}) - \varphi(t_{i}))^{2} + (\psi(t_{i-1}) - \psi(t_{i}))^{2}}$$

$$\approx \sqrt{(\varphi'(t_{i})(t_{i-1} - t_{i}))^{2} + (\psi'(t_{i})(t_{i-1} - t_{i}))^{2}}$$

$$= \sqrt{\varphi'(t_{i})^{2} + \psi'(t_{i})^{2}}(t_{i} - t_{i-1})$$

$$= \sqrt{\varphi'(t_{i})^{2} + \psi'(t_{i})^{2}}\Delta t_{i}$$

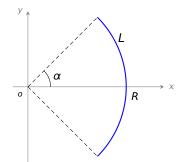
$$= \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_{i}, \eta_{i})\Delta s_{i}$$

$$= \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\varphi(t_{i}), \psi(t_{i})) \sqrt{\varphi'(t_{i})^{2} + \psi'(t_{i})^{2}}\Delta t_{i}$$

$$= \int_{a}^{b} f(\varphi(t), \psi(t)) \sqrt{\varphi'(t)^{2} + \psi'(t_{i})^{2}}dt$$

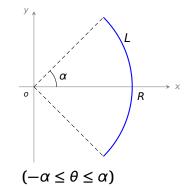
#### 从上述推导可知:

性质 设平面曲线 L 的参数方程为  $x = \varphi(t)$ ,  $y = \psi(t)$ , 则弧长元素  $ds = \sqrt{\varphi'(t)^2 + \psi'(t)^2} dt.$ 

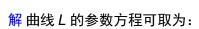


#### $\mathbf{m}$ 曲线 $\mathbf{L}$ 的参数方程可取为:

$$x = R \cos \theta$$
,  $y = R \sin \theta$   $(-\alpha \le \theta \le \alpha)$ 

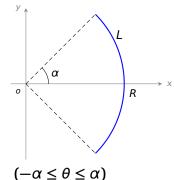


#### 例 1 计算 $\int_{L} y^{2} ds$ ,其中曲线 L 如右图所 示

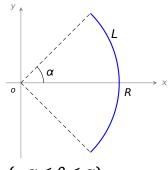


$$x = R \cos \theta$$
,  $y = R \sin \theta$   $(-\alpha \le \theta \le \alpha)$ 

$$\int_{L} y^{2} ds = \int_{-\alpha}^{\alpha} R^{2} \sin^{2} \theta \cdot$$



$$(-\alpha \le \theta \le \alpha)$$

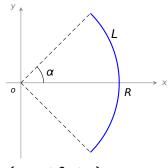


解曲线 L 的参数方程可取为:

$$x = R\cos\theta$$
,  $y = R\sin\theta$   $(-\alpha \le \theta \le \alpha)$ 

所以

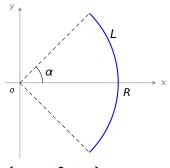
$$\int_{-\alpha}^{\alpha} y^2 ds = \int_{-\alpha}^{\alpha} R^2 \sin^2 \theta \cdot \sqrt{\left[ (R \cos \theta)' \right]^2 + \left[ (R \sin \theta)' \right]^2} d\theta$$



解曲线 L 的参数方程可取为:

$$x = R\cos\theta$$
,  $y = R\sin\theta$   $(-\alpha \le \theta \le \alpha)$ 

$$\int_{L} y^{2} ds = \int_{-\alpha}^{\alpha} R^{2} \sin^{2} \theta \cdot \sqrt{\left[ (R \cos \theta)' \right]^{2} + \left[ (R \sin \theta)' \right]^{2}} d\theta$$
$$= \int_{-\alpha}^{\alpha} R^{2} \sin^{2} \theta \cdot R d\theta$$



解曲线 L 的参数方程可取为:

$$x = R\cos\theta$$
,  $y = R\sin\theta$   $(-\alpha \le \theta \le \alpha)$ 

所以

$$\int_{L} y^{2} ds = \int_{-\alpha}^{\alpha} R^{2} \sin^{2} \theta \cdot \sqrt{\left[ (R \cos \theta)' \right]^{2} + \left[ (R \sin \theta)' \right]^{2}} d\theta$$
$$= \int_{-\alpha}^{\alpha} R^{2} \sin^{2} \theta \cdot R d\theta = R^{3} \int_{-\alpha}^{\alpha} \frac{1}{2} (1 - \cos 2\theta) d\theta$$

$$M_1$$
计算  $\int_L y^2 ds$ ,其中曲线  $L$  如右图所

 $(-\alpha < \theta < \alpha)$ 

解 曲线 L 的参数方程可取为:

$$x = R \cos \theta$$
,  $y = R \sin \theta$   $(-\alpha \le \theta \le \alpha)$ 

所以

$$\int_{L} y^{2} ds = \int_{-\alpha}^{\alpha} R^{2} \sin^{2} \theta \cdot \sqrt{\left[ (R \cos \theta)' \right]^{2} + \left[ (R \sin \theta)' \right]^{2}} d\theta$$

$$= \int_{-\alpha}^{\alpha} R^{2} \sin^{2} \theta \cdot R d\theta = R^{3} \int_{-\alpha}^{\alpha} \frac{1}{2} (1 - \cos 2\theta) d\theta$$

$$= \frac{1}{2} R^{3} (\theta - \frac{1}{2} \sin 2\theta)$$



$$0$$
 1 计算  $\int_L y^2 ds$ ,其中曲线  $L$  如右图所

 $\alpha$   $R \rightarrow X$ 

m 曲线 L 的参数方程可取为:

$$x = R\cos\theta, \quad y = R\sin\theta \quad (-\alpha \le \theta \le \alpha)$$

$$\int_{L} y^{2} ds = \int_{-\alpha}^{\alpha} R^{2} \sin^{2} \theta \cdot \sqrt{\left[ (R \cos \theta)' \right]^{2} + \left[ (R \sin \theta)' \right]^{2}} d\theta$$

$$= \int_{-\alpha}^{\alpha} R^{2} \sin^{2} \theta \cdot R d\theta = R^{3} \int_{-\alpha}^{\alpha} \frac{1}{2} (1 - \cos 2\theta) d\theta$$

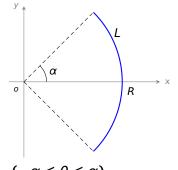
 $= \frac{1}{2}R^3(\theta - \frac{1}{2}\sin 2\theta)\Big|^{\alpha}$ 



第11章α:对弧长的曲线积分

13/19 ⊲ ⊳ ∆ ⊽

例 1 计算  $\int_L y^2 ds$ ,其中曲线 L 如右图所



解 曲线 L 的参数方程可取为:

$$x = R\cos\theta, \quad y = R\sin\theta \quad (-\alpha \le \theta \le \alpha)$$

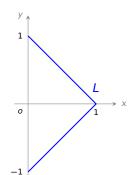
所以

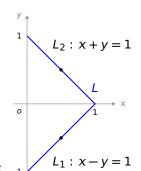
$$\int_{L} y^{2} ds = \int_{-\alpha}^{\alpha} R^{2} \sin^{2} \theta \cdot \sqrt{\left[ (R \cos \theta)' \right]^{2} + \left[ (R \sin \theta)' \right]^{2}} d\theta$$

$$= \int_{-\alpha}^{\alpha} R^{2} \sin^{2} \theta \cdot R d\theta = R^{3} \int_{-\alpha}^{\alpha} \frac{1}{2} (1 - \cos 2\theta) d\theta$$

 $= \frac{1}{2}R^3(\theta - \frac{1}{2}\sin 2\theta)\Big|_{\alpha}^{\alpha} = R^3(\alpha - \frac{1}{2}\sin(2\alpha))$ 

**(a**) <u>1</u>

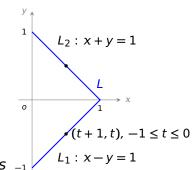




解

$$\int_{L} e^{x+y} ds = \int_{L_{1}} e^{x+y} ds + \int_{L_{2}} e^{x+y} ds \Big|_{-1} \Big|_{L_{1}: x-y=1}$$

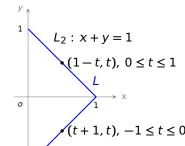




$$M_2$$
 计算  $\int_L e^{x+y} ds$ ,其中  $L$  如右图所示

$$\int_{L} e^{x+y} ds = \int_{L_{1}} e^{x+y} ds + \int_{L_{2}} e^{x+y} ds \Big|_{-1}$$
  $(t+1,t), -1$ 





$$\int_{L} e^{x+y} ds = \int_{L_{1}} e^{x+y} ds + \int_{L_{2}} e^{x+y} ds _{-1}$$
  $(t+1,t), -1$ 



$$\int_{L} e^{x+y} ds = \int_{L_{1}} e^{x+y} ds + \int_{L_{2}} e^{x+y} dt$$
$$= \int_{-1}^{0} e^{2t+1} \cdot \sqrt{\left[(t+1)'\right]^{2} + \left[t'\right]^{2}} dt$$



例 2 计算 
$$\int_{L} e^{x+y} ds$$
,其中  $L$  如右图所示

$$\begin{array}{c|c}
 & L_2: x + y = 1 \\
 & (1 - t, t), \ 0 \le t \le 1 \\
\hline
 & c & t \le 1
\end{array}$$

$$\begin{array}{c|c}
 & (t + 1, t), \ -1 \le t \le 0
\end{array}$$

$$\int_{L} e^{x+y} ds = \int_{L_{1}} e^{x+y} ds + \int_{L_{2}} e^{x+y} ds = \int_{L_{1}}^{0} e^{x+y} ds + \int_{L_{2}}^{0} e^{x+y} ds = \int_{L_{1}}^{0} e^{x+y} ds + \int_{L_{1}}^{0} e^{x+y} ds + \int_{L_{1}}^{0} e^{x+y} ds = \int_{L_{1}}^{0} e^{x+y} ds + \int_{L_{1}}^{0} e^{x+y} ds = \int_{L_{1}}^{0} e^{x+y} ds + \int_{L_{1}$$

$$= \int_{-1}^{0} e^{2t+1} \cdot \sqrt{\left[(t+1)'\right]^2 + \left[t'\right]^2} dt + \int_{0}^{1} e^{1} \cdot \sqrt{\left[(1-t)'\right]^2 + \left[t'\right]^2} dt$$



$$\begin{array}{c|c}
 & L_2: x + y = 1 \\
 & (1 - t, t), \ 0 \le t \le 1 \\
\hline
 & (t + 1, t), \ -1 \le t \le 0 \\
 & L_1: x - y = 1
\end{array}$$

$$\int_{L} e^{x+y} ds = \int_{L_{1}} e^{x+y} ds + \int_{L_{2}} e^{x+y} ds \Big|_{-1} \Big|_{L_{1} : x-y=1}$$

$$= \int_{-1}^{0} e^{2t+1} \cdot \sqrt{[(t+1)']^{2} + [t']^{2}} dt + \int_{0}^{1} e^{1} \cdot \sqrt{[(1-t)']^{2} + [t']^{2}} dt$$

$$= \sqrt{2} \int_{0}^{0} e^{2t+1} dt + \sqrt{2} \int_{0}^{1} e^{1} dt$$



第 11 章 α: 对弧长的曲线积分

 $\int_{a}^{b} e^{x+y} ds = \int_{a}^{b} e^{x+y} ds + \int_{a}^{b} e^{x+y} ds = 1$ 

 $= \int_{-1}^{0} e^{2t+1} \cdot \sqrt{\left[(t+1)'\right]^2 + \left[t'\right]^2} dt + \int_{0}^{1} e^{1} \cdot \sqrt{\left[(1-t)'\right]^2 + \left[t'\right]^2} dt$  $=\sqrt{2}\int_{0}^{0}e^{2t+1}dt+\sqrt{2}\int_{0}^{1}e^{1}dt$ 

 $=\sqrt{2}\cdot\frac{1}{2}e^{2t+1}$ 



 $\int_{a}^{b} e^{x+y} ds = \int_{a}^{b} e^{x+y} ds + \int_{a}^{b} e^{x+y} ds = 1$ 

 $= \int_{-1}^{0} e^{2t+1} \cdot \sqrt{\left[(t+1)'\right]^2 + \left[t'\right]^2} dt + \int_{0}^{1} e^{1} \cdot \sqrt{\left[(1-t)'\right]^2 + \left[t'\right]^2} dt$ 

 $=\sqrt{2}\cdot\frac{1}{2}e^{2t+1}\Big|_{0}$ 

 $=\sqrt{2}\int_{0}^{0}e^{2t+1}dt+\sqrt{2}\int_{0}^{1}e^{1}dt$ 

$$\int e^{x+y} ds = \int e^{x+y} ds + \int e^{x+y} ds$$

 $\int_{C} e^{x+y} ds = \int_{C} e^{x+y} ds + \int_{C} e^{x+y} ds \Big|_{-1} \Big|_{L_{1}: x-y=1}$ 

 $= \int_{-1}^{0} e^{2t+1} \cdot \sqrt{\left[(t+1)'\right]^2 + \left[t'\right]^2} dt + \int_{0}^{1} e^{1} \cdot \sqrt{\left[(1-t)'\right]^2 + \left[t'\right]^2} dt$  $=\sqrt{2}\int_{0}^{0}e^{2t+1}dt+\sqrt{2}\int_{0}^{1}e^{1}dt$ 

 $=\sqrt{2}\cdot\frac{1}{2}e^{2t+1}\Big|_{1}^{0}+\sqrt{2}e^{-t}$ 



$$\int_{L} e^{x+y} ds = \int_{L_1} e^{x+y} ds$$

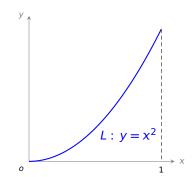
$$\int_{L} e^{x+y} ds = \int_{L_{1}} e^{x+y} ds + \int_{L_{2}} e^{x+y} ds \Big|_{-1}$$

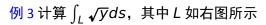
$$L_{1}: x-y=1$$

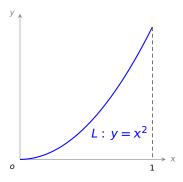
 $= \int_{-1}^{0} e^{2t+1} \cdot \sqrt{\left[(t+1)'\right]^2 + \left[t'\right]^2} dt + \int_{0}^{1} e^{1} \cdot \sqrt{\left[(1-t)'\right]^2 + \left[t'\right]^2} dt$  $=\sqrt{2}\int_{1}^{0}e^{2t+1}dt+\sqrt{2}\int_{0}^{1}e^{1}dt$ 

$$\begin{cases} s & -1 \\ \int_0^1 e^1 \end{aligned}$$

 $= \sqrt{2} \cdot \frac{1}{2} e^{2t+1} \Big|_{1}^{0} + \sqrt{2} e = \frac{\sqrt{2}}{2} (3e - e^{-1})$ 

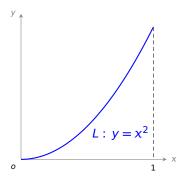






解曲线 L 的参数方程可取为:

$$x = t, \quad y = t^2 \qquad (0 \le t \le 1)$$

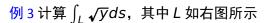


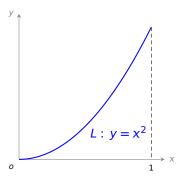
解 曲线 L 的参数方程可取为:

$$x = t, \quad y = t^2 \qquad (0 \le t \le 1)$$

$$\int_{t} \sqrt{y} ds = \int_{0}^{1} \sqrt{t^{2}} \cdot$$





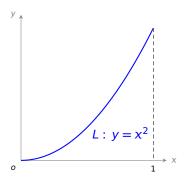


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$$x = t, \quad y = t^2 \qquad (0 \le t \le 1)$$

$$\int_{0}^{1} \sqrt{y} ds = \int_{0}^{1} \sqrt{t^{2}} \cdot \sqrt{[t']^{2} + [(t^{2})']^{2}} dt$$



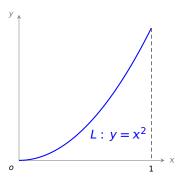


m 曲线 L 的参数方程可取为:

$$x = t, \quad y = t^2 \qquad (0 \le t \le 1)$$

$$\int_{0}^{1} \sqrt{y} ds = \int_{0}^{1} \sqrt{t^{2}} \cdot \sqrt{[t']^{2} + [(t^{2})']^{2}} dt = \int_{0}^{1} t \cdot \sqrt{1 + 4t^{2}} dt$$





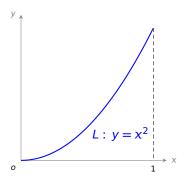
解 曲线 L 的参数方程可取为:

$$x = t, \quad y = t^2 \qquad (0 \le t \le 1)$$

$$\int_{L} \sqrt{y} ds = \int_{0}^{1} \sqrt{t^{2}} \cdot \sqrt{[t']^{2} + [(t^{2})']^{2}} dt = \int_{0}^{1} t \cdot \sqrt{1 + 4t^{2}} dt$$



第 11 章 a:对弧长的曲线积分



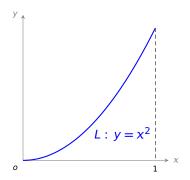
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$$= \frac{u = 1 + 4t^{2}}{u + 4t^{2}} \int_{0}^{5} \sqrt{u} \cdot dt$$





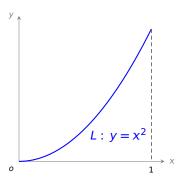
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$$= \frac{u = 1 + 4t^{2}}{2} \int_{1}^{5} \sqrt{u} \cdot \frac{1}{8} du$$



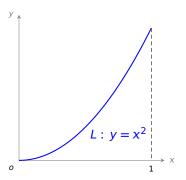


m 曲线 L 的参数方程可取为:

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$$\frac{u = 1 + 4t^{2}}{1 + 4t^{2}} \int_{1}^{5} \sqrt{u} \cdot \frac{1}{8} du = \frac{1}{8} \cdot \frac{2}{3} u^{\frac{3}{2}}$$



m 曲线 L 的参数方程可取为:

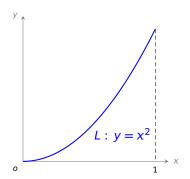
$$x = t$$
,  $y = t^2$   $(0 \le t \le 1)$ 

$$\int_{L} \sqrt{y} ds = \int_{0}^{1} \sqrt{t^{2}} \cdot \sqrt{[t']^{2} + [(t^{2})']^{2}} dt = \int_{0}^{1} t \cdot \sqrt{1 + 4t^{2}} dt$$

$$\frac{u = 1 + 4t^{2}}{1 + 4t^{2}} \int_{1}^{5} \sqrt{u} \cdot \frac{1}{8} du = \frac{1}{8} \cdot \frac{2}{3} u^{\frac{3}{2}} \Big|_{1}^{5}$$



第 11 章 a:对弧长的曲线积分



m 曲线 L 的参数方程可取为:

$$x = t, \quad y = t^2 \qquad (0 \le t \le 1)$$

$$\int_{L} \sqrt{y} ds = \int_{0}^{1} \sqrt{t^{2}} \cdot \sqrt{[t']^{2} + [(t^{2})']^{2}} dt = \int_{0}^{1} t \cdot \sqrt{1 + 4t^{2}} dt$$

$$\frac{u = 1 + 4t^{2}}{1 + 4t^{2}} \int_{1}^{5} \sqrt{u} \cdot \frac{1}{8} du = \frac{1}{8} \cdot \frac{2}{3} u^{\frac{3}{2}} \Big|_{1}^{5} = \frac{1}{12} (5\sqrt{5} - 1)$$

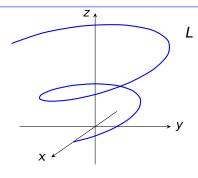


### We are here now...

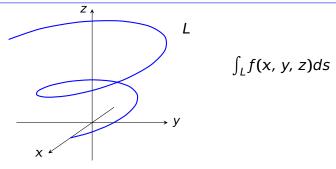
1. 对弧长的曲线积分: 概念与性质

2. 对弧长的曲线积分: 计算法

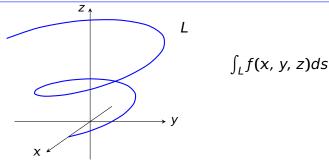
3. 对弧长的曲线积分:空间曲线



 $\int_L f(x, y, z) ds$ 

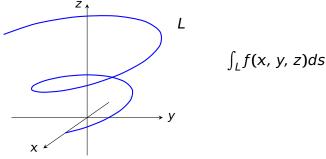


• 当 f(x, y, z) 是线密度时,  $\int_L f(x, y, z) ds$  表示曲线的质量



- 当 f(x, y, z) 是线密度时,  $\int_{\mathcal{C}} f(x, y, z) ds$  表示曲线的质量
- 若曲线 L 的参数方程是  $\begin{cases} x = \varphi(t) \\ y = \psi(t) , & (\alpha \le t \le b), \\ z = \zeta(t) \end{cases}$



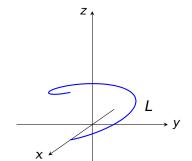


• 当 f(x, y, z) 是线密度时,  $\int_{I} f(x, y, z) ds$  表示曲线的质量

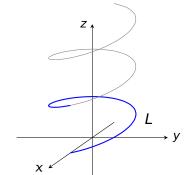
• 若曲线 L 的参数方程是  $\begin{cases} x = \varphi(t) \\ y = \psi(t) , & (\alpha \le t \le b), \\ z = \zeta(t) \end{cases}$ 

$$\int_{a}^{b} f(x, y, z) ds = \int_{a}^{b} f(\varphi(t), \psi(t), \zeta(t)) \sqrt{\varphi'(t)^{2} + \psi'(t)^{2} + \zeta'(t)^{2}} dt$$

例 1 计算  $\int_L (x^2 + y^2 + z^2) ds$ ,其中 L 为 螺旋线  $x = a \cos t$ ,  $y = a \sin t$ , z = bt  $(0 \le t \le 2\pi)$ 



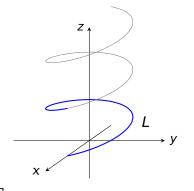
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例 1 计算 
$$\int_L (x^2 + y^2 + z^2) ds$$
,其中  $L$  为 螺旋线  $x = a \cos t$ ,  $y = a \sin t$ ,  $z = bt$   $(0 \le t \le 2\pi)$ 

$$\Re \int_{L} (x^{2} + y^{2} + z^{2}) ds$$

$$= \int_{0}^{2\pi} \left[ (a\cos t)^{2} + (a\sin t)^{2} + (bt)^{2} \right] \cdot$$



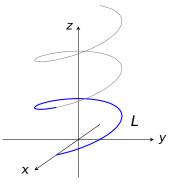
例 1 计算 
$$\int_L (x^2 + y^2 + z^2) ds$$
,其中  $L$  为 螺旋线  $x = a \cos t$ ,  $y = a \sin t$ ,  $z = bt$   $(0 < t < 2\pi)$ 

例 1 计算 
$$\int_{L} (x^{2} + y^{2} + z^{2}) ds$$
, 其中  $L$  为 螺旋线  $x = a \cos t$ ,  $y = a \sin t$ ,  $z = bt$   $(0 \le t \le 2\pi)$  解  $\int_{L} (x^{2} + y^{2} + z^{2}) ds$ 

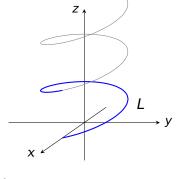
$$\int_{L}^{2\pi} (x^2 + y^2 + z^2) ds$$

$$= \int_{0}^{2\pi} \left[ (a\cos t)^{2} + (a\sin t)^{2} + (bt)^{2} \right] \cdot$$

$$\sqrt{[(a\cos t)']^2 + [(a\sin t)']^2 + [(bt)']^2}dt$$



例 1 计算 
$$\int_L (x^2 + y^2 + z^2) ds$$
,其中  $L$  为 螺旋线  $x = a \cos t$ ,  $y = a \sin t$ ,  $z = bt$   $(0 \le t \le 2\pi)$ 



$$\mathbf{H} \int_{L} (x^2 + y^2 + z^2) ds$$

$$= \int_{0}^{2\pi} \left[ (a\cos t)^{2} + (a\sin t)^{2} + (bt)^{2} \right] \cdot$$

$$\sqrt{[(a\cos t)']^2 + [(a\sin t)']^2 + [(bt)']^2}dt$$

$$= \int_{0}^{2\pi} \left[ a^{2} + b^{2}t^{2} \right] \cdot \sqrt{a^{2} + b^{2}} dt$$

例 1 计算 
$$\int_L (x^2 + y^2 + z^2) ds$$
,其中  $L$  为 螺旋线  $x = a \cos t$ ,  $y = a \sin t$ ,  $z = bt$   $(0 \le t \le 2\pi)$ 

期 1 计算 
$$\int_L (x^2 + y^2 + z^2) ds$$
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$$\prod_{i=1}^{n} (x^2 + y^2 + z^2) ds$$

$$=\int_{L}^{2\pi} (a\cos t)^{2} + (a\sin t)^{2}$$

$$= \int_0^{2\pi} \left[ (a\cos t)^2 + (a\sin t)^2 + (bt)^2 \right].$$

$$\sqrt{[(a\cos t)']^2 + [(a\sin t)']^2 + [(bt)']^2} dt$$

$$= \int_{0}^{2\pi} \left[ a^{2} + b^{2}t^{2} \right] \cdot \sqrt{a^{2} + b^{2}} dt = \sqrt{a^{2} + b^{2}} \cdot \left( a^{2}t + \frac{1}{3}b^{2}t^{3} \right) \Big|_{0}^{2\pi}$$



第 11 章 α:对弧长的曲线积分

例 1 计算 
$$\int_L (x^2 + y^2 + z^2) ds$$
,其中  $L$  为 螺旋线  $x = a \cos t$ ,  $y = a \sin t$ ,  $z = bt$   $(0 \le t \le 2\pi)$ 

 $\Re \int_{a}^{b} (x^2 + y^2 + z^2) ds$ 

$$=\int_0^{2\pi} \left[ (a\cos t)^2 + (a\sin t)^2 + (bt)^2 \right] \cdot$$

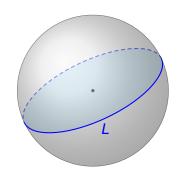
$$\int_{0}^{2\pi} \left[ (a\cos t)' + (a\sin t)' \right]^{2} + \left[ (bt)' \right]^{2} dt$$

$$= \int_{0}^{2\pi} \left[ a^{2} + b^{2}t^{2} \right] \cdot \sqrt{a^{2} + b^{2}} dt = \sqrt{a^{2} + b^{2}} \cdot \left( a^{2}t + \frac{1}{3}b^{2}t^{3} \right) \Big|_{0}^{2\pi}$$

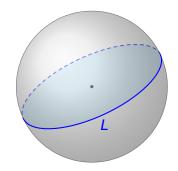
$$= \frac{2}{3}\pi\sqrt{a^2 + b^2} \cdot (3a^2 + 4b^2\pi^2)$$



例 2 计算  $\int_L x^2 ds$ ,其中 L 为球面  $x^2 + y^2 + z^2 = 1$  与平面 x + y + z = 0 的交线。



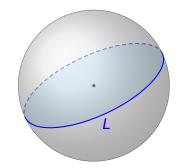
例 2 计算  $\int_L x^2 ds$ ,其中 L 为球面  $x^2 + y^2 + z^2 = 1$  与平面 x + y + z = 0 的交线。



### 解 由对称性可知:

$$\int_{L} x^2 ds = \int_{L} y^2 ds = \int_{L} z^2 ds$$

例 2 计算 
$$\int_L x^2 ds$$
,其中  $L$  为球面  $x^2 + y^2 + z^2 = 1$  与平面  $x + y + z = 0$  的交线。

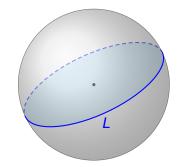


$$\int_{L} x^{2} ds = \int_{L} y^{2} ds = \int_{L} z^{2} ds$$

$$\int_{1} x^{2} ds = \frac{1}{3} \int_{1} (x^{2} + y^{2} + z^{2}) ds$$



例 2 计算 
$$\int_L x^2 ds$$
,其中  $L$  为球面  $x^2 + y^2 + z^2 = 1$  与平面  $x + y + z = 0$  的交线。

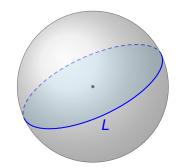


$$\int_{L} x^{2} ds = \int_{L} y^{2} ds = \int_{L} z^{2} ds$$

$$\int_{L} x^{2} ds = \frac{1}{3} \int_{L} (x^{2} + y^{2} + z^{2}) ds = \frac{1}{3} \int_{L} 1 ds$$



例 2 计算 
$$\int_L x^2 ds$$
,其中  $L$  为球面  $x^2 + y^2 + z^2 = 1$  与平面  $x + y + z = 0$  的交线。

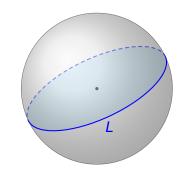


$$\int_{I} x^{2} ds = \int_{I} y^{2} ds = \int_{I} z^{2} ds$$

$$\int_{C} x^{2} ds = \frac{1}{3} \int_{C} (x^{2} + y^{2} + z^{2}) ds = \frac{1}{3} \int_{C} 1 ds = \frac{1}{3} \text{Length}(L)$$



例 2 计算 
$$\int_L x^2 ds$$
,其中  $L$  为球面  $x^2 + y^2 + z^2 = 1$  与平面  $x + y + z = 0$  的交线。



$$\int_{I} x^{2} ds = \int_{I} y^{2} ds = \int_{I} z^{2} ds$$

$$\int_{L} x^{2} ds = \frac{1}{3} \int_{L} (x^{2} + y^{2} + z^{2}) ds = \frac{1}{3} \int_{L} 1 ds = \frac{1}{3} \text{Length}(L) = \frac{1}{3} \cdot 2\pi$$

