§3.1　线性方程组的消元解法

数学系　梁卓滨

2017 - 2018 学年 I

记号

考虑 n 个未知量 m 个方程的线性方程组:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

记号

考虑 n 个未知量 m 个方程的线性方程组：

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

可以，等价地，改写成矩阵形式

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m\times n} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

记号

考虑 n 个未知量 m 个方程的线性方程组：

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

可以，等价地，改写成矩阵形式

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m\times n}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}_{b}$$

记号

考虑 n 个未知量 m 个方程的线性方程组：

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

可以，等价地，改写成矩阵形式

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m\times n}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}_{b} \quad \Rightarrow \quad Ax = b$$

记号

考虑 n 个未知量 m 个方程的线性方程组：

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

可以，等价地，改写成矩阵形式

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m\times n}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}_{b} \quad \Rightarrow \quad Ax = b$$

整个方程组的信息包含在：

$$(A \vdots b) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

记号

考虑 n 个未知量 m 个方程的线性方程组：

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

可以，等价地，改写成矩阵形式

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m\times n}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}_{b} \quad \Rightarrow \quad Ax = b$$

整个方程组的信息包含在：

增广矩阵 $$(A \vdots b) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

消元法求解线性方程组——示例

例 解方程组：
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases}$$

消元法求解线性方程组——示例

例 解方程组：$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases}$

消元法步骤示例：

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases}$$

消元法求解线性方程组——示例

例 解方程组：
$$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 4x_1+7x_2-x_3=-1 \\ -3x_1-4x_2+2x_3=-3 \end{cases}$$

消元法步骤示例：

$$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 4x_1+7x_2-x_3=-1 \\ -3x_1-4x_2+2x_3=-3 \end{cases} \qquad (2)-(1)$$

消元法求解线性方程组——示例

例 解方程组：
$$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 4x_1+7x_2-x_3=-1 \\ -3x_1-4x_2+2x_3=-3 \end{cases}$$

消元法步骤示例：

$$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 4x_1+7x_2-x_3=-1 \\ -3x_1-4x_2+2x_3=-3 \end{cases} \longrightarrow \begin{cases} x_1+x_2-x_3=2 \\ x_2+x_3=-3 \\ 4x_1+7x_2-x_3=-1 \\ -3x_1-4x_2+2x_3=-3 \end{cases} \quad (2)-(1)$$

消元法求解线性方程组——示例

例 解方程组：
$$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 4x_1+7x_2-x_3=-1 \\ -3x_1-4x_2+2x_3=-3 \end{cases}$$

消元法步骤示例：

$$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 4x_1+7x_2-x_3=-1 \\ -3x_1-4x_2+2x_3=-3 \end{cases} \longrightarrow \begin{cases} x_1+x_2-x_3=2 \\ x_2+x_3=-3 \\ 4x_1+7x_2-x_3=-1 \\ -3x_1-4x_2+2x_3=-3 \end{cases} \quad (2)-(1)$$

$$\Updownarrow$$

$$(A \vdots b)=\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right)$$

消元法求解线性方程组——示例

例 解方程组：
$$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 4x_1+7x_2-x_3=-1 \\ -3x_1-4x_2+2x_3=-3 \end{cases}$$

消元法步骤示例：

$$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 4x_1+7x_2-x_3=-1 \\ -3x_1-4x_2+2x_3=-3 \end{cases} \longrightarrow \begin{cases} x_1+x_2-x_3=2 \\ x_2+x_3=-3 \\ 4x_1+7x_2-x_3=-1 \\ -3x_1-4x_2+2x_3=-3 \end{cases} \quad (2)-(1)$$

$$\Updownarrow$$

$$(A \vdots b)=\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right) \qquad r_2-r_1$$

消元法求解线性方程组——示例

例 解方程组：
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases}$$

消元法步骤示例：

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases} \longrightarrow \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases} \quad (2)-(1)$$

$$\Updownarrow$$

$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right) \quad r_2 - r_1$$

消元法求解线性方程组——示例

例 解方程组：
$$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 4x_1+7x_2-x_3=-1 \\ -3x_1-4x_2+2x_3=-3 \end{cases}$$

消元法步骤示例：

$$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 4x_1+7x_2-x_3=-1 \\ -3x_1-4x_2+2x_3=-3 \end{cases} \longrightarrow \begin{cases} x_1+x_2-x_3=2 \\ x_2+x_3=-3 \\ 4x_1+7x_2-x_3=-1 \\ -3x_1-4x_2+2x_3=-3 \end{cases} \quad (2)-(1)$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$(A\vdots b)=\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right) \quad r_2-r_1$$

消元法求解线性方程组——示例

例 解方程组：
$$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 4x_1+7x_2-x_3=-1 \\ -3x_1-4x_2+2x_3=-3 \end{cases}$$

消元法步骤示例：

$$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 4x_1+7x_2-x_3=-1 \\ -3x_1-4x_2+2x_3=-3 \end{cases} \longrightarrow \begin{cases} x_1+x_2-x_3=2 \\ x_2+x_3=-3 \\ 4x_1+7x_2-x_3=-1 \\ -3x_1-4x_2+2x_3=-3 \end{cases} \quad (3)-4(1)$$

$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$$

$$(A\vdots b)=\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right) \quad r_2-r_1$$

消元法求解线性方程组——示例

例 解方程组：
$$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 4x_1+7x_2-x_3=-1 \\ -3x_1-4x_2+2x_3=-3 \end{cases}$$

消元法步骤示例：

$$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 4x_1+7x_2-x_3=-1 \\ -3x_1-4x_2+2x_3=-3 \end{cases} \longrightarrow \begin{cases} x_1+x_2-x_3=2 \\ x_2+x_3=-3 \\ 3x_2+3x_3=-9 \\ -3x_1-4x_2+2x_3=-3 \end{cases} \quad (3)-4(1)$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$(A\vdots b)=\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right) \quad r_2-r_1$$

消元法求解线性方程组——示例

例 解方程组：
$$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 4x_1+7x_2-x_3=-1 \\ -3x_1-4x_2+2x_3=-3 \end{cases}$$

消元法步骤示例：

$$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 4x_1+7x_2-x_3=-1 \\ -3x_1-4x_2+2x_3=-3 \end{cases} \longrightarrow \begin{cases} x_1+x_2-x_3=2 \\ x_2+x_3=-3 \\ 3x_2+3x_3=-9 \\ -3x_1-4x_2+2x_3=-3 \end{cases} \quad (3)-4(1)$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$(A \vdots b)=\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right) \quad r_3-4r_1$$

消元法求解线性方程组——示例

例 解方程组：$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases}$

消元法步骤示例：

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases} \longrightarrow \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \\ 3x_2 + 3x_3 = -9 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases} \quad (3)-4(1)$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 3 & -9 \\ -3 & -4 & 2 & -3 \end{array}\right) \quad r_3 - 4r_1$$

消元法求解线性方程组——示例

例 解方程组：
$$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 4x_1+7x_2-x_3=-1 \\ -3x_1-4x_2+2x_3=-3 \end{cases}$$

消元法步骤示例：

$$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 4x_1+7x_2-x_3=-1 \\ -3x_1-4x_2+2x_3=-3 \end{cases} \longrightarrow \begin{cases} x_1+x_2-x_3=2 \\ x_2+x_3=-3 \\ 3x_2+3x_3=-9 \\ -3x_1-4x_2+2x_3=-3 \end{cases} \quad (4)+3(1)$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$(A\vdots b)=\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 3 & -9 \\ -3 & -4 & 2 & -3 \end{array}\right) \quad r_3-4r_1$$

消元法求解线性方程组——示例

例 解方程组：
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases}$$

消元法步骤示例：

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases} \longrightarrow \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \\ 3x_2 + 3x_3 = -9 \\ -x_2 - x_3 = 3 \end{cases} \quad (4)+3(1)$$

$$\Updownarrow \qquad\qquad\qquad \Updownarrow$$

$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 3 & -9 \\ -3 & -4 & 2 & -3 \end{array}\right) \quad r_3 - 4r_1$$

消元法求解线性方程组——示例

例 解方程组:
$$\begin{cases} x_1+\ \ x_2-\ \ x_3=2 \\ x_1+2x_2 \qquad\quad =-1 \\ 4x_1+7x_2-\ \ x_3=-1 \\ -3x_1-4x_2+2x_3=-3 \end{cases}$$

消元法步骤示例:

$$\begin{cases} x_1+\ \ x_2-\ \ x_3=2 \\ x_1+2x_2 \qquad\quad =-1 \\ 4x_1+7x_2-\ \ x_3=-1 \\ -3x_1-4x_2+2x_3=-3 \end{cases} \longrightarrow \begin{cases} x_1+\ \ x_2-\ \ x_3=2 \\ \qquad x_2+\ \ x_3=-3 \\ \quad 3x_2+3x_3=-9 \\ -\ \ x_2-\ \ x_3=3 \end{cases} \quad (4)+3(1)$$

$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$$

$$(A \vdots b)=\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 3 & -9 \\ -3 & -4 & 2 & -3 \end{array}\right) \quad r_4+3r_1$$

消元法求解线性方程组——示例

例 解方程组：
$$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 4x_1+7x_2-x_3=-1 \\ -3x_1-4x_2+2x_3=-3 \end{cases}$$

消元法步骤示例：

$$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 4x_1+7x_2-x_3=-1 \\ -3x_1-4x_2+2x_3=-3 \end{cases} \longrightarrow \begin{cases} x_1+x_2-x_3=2 \\ x_2+x_3=-3 \\ 3x_2+3x_3=-9 \\ -x_2-x_3=3 \end{cases} \quad (4)+3(1)$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$(A\vdots b)=\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 3 & -9 \\ 0 & -1 & -1 & 3 \end{array}\right) \quad r_4+3r_1$$

消元法求解线性方程组——示例

例 解方程组：
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases}$$

消元法步骤示例：

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases} \longrightarrow \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \\ 3x_2 + 3x_3 = -9 \\ -x_2 - x_3 = 3 \end{cases} \quad (3) - 3(2)$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 3 & -9 \\ 0 & -1 & -1 & 3 \end{array}\right) \qquad r_4 + 3r_1$$

消元法求解线性方程组——示例

例 解方程组:
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases}$$

消元法步骤示例:

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases} \longrightarrow \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \\ 0 = 0 \\ -x_2 - x_3 = 3 \end{cases} \quad (3) - 3(2)$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 3 & -9 \\ 0 & -1 & -1 & 3 \end{array}\right) \quad r_4 + 3r_1$$

消元法求解线性方程组——示例

例 解方程组:
$$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 4x_1+7x_2-x_3=-1 \\ -3x_1-4x_2+2x_3=-3 \end{cases}$$

消元法步骤示例:

$$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 4x_1+7x_2-x_3=-1 \\ -3x_1-4x_2+2x_3=-3 \end{cases} \longrightarrow \begin{cases} x_1+x_2-x_3=2 \\ x_2+x_3=-3 \\ 0=0 \\ -x_2-x_3=3 \end{cases} \quad (3)-3(2)$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$(A\vdots b)=\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 3 & -9 \\ 0 & -1 & -1 & 3 \end{array}\right) \quad r_3-2r_2$$

消元法求解线性方程组——示例

例 解方程组：
$$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 4x_1+7x_2-x_3=-1 \\ -3x_1-4x_2+2x_3=-3 \end{cases}$$

消元法步骤示例：

$$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 4x_1+7x_2-x_3=-1 \\ -3x_1-4x_2+2x_3=-3 \end{cases} \longrightarrow \begin{cases} x_1+x_2-x_3=2 \\ x_2+x_3=-3 \\ 0=0 \\ -x_2-x_3=3 \end{cases} \quad (3)-3(2)$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$(A\vdots b)=\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 3 \end{array}\right) \quad r_3-2r_2$$

消元法求解线性方程组——示例

例 解方程组：
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases}$$

消元法步骤示例：

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases} \longrightarrow \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \\ 0 = 0 \\ -x_2 - x_3 = 3 \end{cases} \quad (4)+(2)$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 3 \end{array}\right) \quad r_3 - 3r_2$$

消元法求解线性方程组——示例

例 解方程组：
$$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 4x_1+7x_2-x_3=-1 \\ -3x_1-4x_2+2x_3=-3 \end{cases}$$

消元法步骤示例：

$$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 4x_1+7x_2-x_3=-1 \\ -3x_1-4x_2+2x_3=-3 \end{cases} \longrightarrow \begin{cases} x_1+x_2-x_3=2 \\ x_2+x_3=-3 \\ 0=0 \\ 0=0 \end{cases} \quad (4)+(2)$$

$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$$

$$(A\vdots b)=\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 3 \end{array}\right) \quad r_3-3r_2$$

消元法求解线性方程组——示例

例 解方程组：
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases}$$

消元法步骤示例：

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases} \longrightarrow \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \\ 0 = 0 \\ 0 = 0 \end{cases} \quad (4) + (2)$$

$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$$

$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 3 \end{array}\right) \quad r_4 + r_2$$

暨南大学 JINAN UNIVERSITY

消元法求解线性方程组——示例

例 解方程组：
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases}$$

消元法步骤示例：

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases} \longrightarrow \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \\ 0 = 0 \\ 0 = 0 \end{cases} \quad (4) + (2)$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad r_4 + r_2$$

消元法求解线性方程组——示例

例 解方程组：
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases}$$

消元法步骤示例：

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases} \longrightarrow \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \\ 0 = 0 \\ 0 = 0 \end{cases} \quad (1)-(2)$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad r_4 + r_2$$

消元法求解线性方程组——示例

例 解方程组：$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases}$

消元法步骤示例：

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases} \longrightarrow \begin{cases} x_1 \quad - 2x_3 = 5 \\ x_2 + x_3 = -3 \\ 0 = 0 \\ 0 = 0 \end{cases} \quad (1)-(2)$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad r_4 + r_2$$

消元法求解线性方程组——示例

例 解方程组：
$$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 4x_1+7x_2-x_3=-1 \\ -3x_1-4x_2+2x_3=-3 \end{cases}$$

消元法步骤示例：

$$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 4x_1+7x_2-x_3=-1 \\ -3x_1-4x_2+2x_3=-3 \end{cases} \longrightarrow \begin{cases} x_1-2x_3=5 \\ x_2+x_3=-3 \\ 0=0 \\ 0=0 \end{cases} \quad (1)-(2)$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$(A \vdots b)=\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad r_1-r_2$$

消元法求解线性方程组——示例

例 解方程组：$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases}$

消元法步骤示例：

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases} \longrightarrow \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \\ 0 = 0 \\ 0 = 0 \end{cases} \quad (1)-(2)$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad r_1 - r_2$$

消元法求解线性方程组——示例

例 解方程组：
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases}$$

消元法步骤示例：

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases} \longrightarrow \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \\ 0 = 0 \\ 0 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 5 + 2x_3 \\ x_2 = -3 - x_3 \end{cases}$$

$$\Updownarrow \qquad\qquad\qquad \Updownarrow$$

$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

消元法求解线性方程组——示例

例 解方程组：
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases}$$

消元法步骤示例：

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases} \longrightarrow \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \\ 0 = 0 \\ 0 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 5 + 2x_3 \\ x_2 = -3 - x_3 \end{cases}$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{cases} x_1 = 5 + 2x_3 \\ x_2 = -3 - x_3 \end{cases}$$

消元法求解线性方程组——示例

例 解方程组：
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases}$$

消元法步骤示例：

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases} \longrightarrow \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \\ 0 = 0 \\ 0 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 5 + 2x_3 \\ x_2 = -3 - x_3 \end{cases}$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{cases} x_1 = 5 + 2x_3 \\ x_2 = -3 - x_3 \end{cases}$$

初等行变换求解线性方程组

$$Ax = b \quad \Longrightarrow \quad (A \vdots b) \xrightarrow{\text{初等行变换}} \text{简化的阶梯型矩阵}$$

回忆：阶梯形矩阵

形如：

$$
A=\begin{pmatrix}
0 & \cdots & 0 & b_1 & \cdots & * & * & \cdots & * & * & \cdots & \cdots & * & * & \cdots & * \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & b_2 & \cdots & * & * & \cdots & \cdots & * & * & \cdots & * \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & b_3 & \cdots & \cdots & * & * & \cdots & * \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & b_r & \cdots & * \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0
\end{pmatrix}
$$

的矩阵，其中 $b_1, b_2, \ldots, b_r \neq 0$，称为阶梯型矩阵。

简化的阶梯型矩阵

形如：

$$
A=\begin{pmatrix}
0 & \cdots & 0 & 1 & \cdots & * & * & \cdots & * & * & \cdots & \cdots & * & * & \cdots & * \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \cdots & * & * & \cdots & \cdots & * & * & \cdots & * \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \cdots & \cdots & * & * & \cdots & * \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \cdots & * \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0
\end{pmatrix}
$$

简化的阶梯型矩阵

形如：

$$A=\begin{pmatrix}
0 & \cdots & 0 & 1 & \cdots & * & 0 & \cdots & * & 0 & \cdots & \cdots & * & 0 & \cdots & * \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \cdots & * & 0 & \cdots & \cdots & * & 0 & \cdots & * \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \cdots & \cdots & * & 0 & \cdots & * \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \cdots & * \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0
\end{pmatrix}$$

简化的阶梯型矩阵

形如：

$$
A = \begin{pmatrix}
0 & \cdots & 0 & 1 & \cdots & * & 0 & \cdots & * & 0 & \cdots & \cdots & * & 0 & \cdots & * \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \cdots & * & 0 & \cdots & \cdots & * & 0 & \cdots & * \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \cdots & \cdots & * & 0 & \cdots & * \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \cdots & * \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0
\end{pmatrix}
$$

称为简化的阶梯型矩阵。

例 1 解方程组：
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$

例 1 解方程组：
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$

解
$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{array}\right)$$

例 1 解方程组：$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$

解 $(A \vdots b) = \left(\begin{array}{rrr|r} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{array}\right) \xrightarrow[\substack{r_3 - 2r_1 \\ r_4 + 2r_1}]{r_2 - r_1}$

例 1 解方程组：
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$

解
$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{array}\right) \xrightarrow[\substack{r_3 - 2r_1 \\ r_4 + 2r_1}]{r_2 - r_1} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ & & & \\ & & & \\ & & & \end{array}\right)$$

例 1 解方程组：
$$\begin{cases} x_1+x_2-x_3=2\\ x_1+2x_2=-1\\ 2x_1+5x_2+x_3=-5\\ -2x_1-3x_2+x_3=-1\end{cases}$$

解 $$(A\vdots b)=\left(\begin{array}{ccc|c}1&1&-1&2\\1&2&0&-1\\2&5&1&-5\\-2&-3&1&-1\end{array}\right)\xrightarrow[\substack{r_3-2r_1\\r_4+2r_1}]{r_2-r_1}\left(\begin{array}{ccc|c}1&1&-1&2\\0&1&1&-3\\&&&\\&&&\end{array}\right)$$

例 1 解方程组：$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 2x_1+5x_2+x_3=-5 \\ -2x_1-3x_2+x_3=-1 \end{cases}$

解 $(A\vdots b)=\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{array}\right) \xrightarrow[\substack{r_3-2r_1 \\ r_4+2r_1}]{r_2-r_1} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 3 & -9 \end{array}\right)$

例 1 解方程组：$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$

解 $(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{array}\right) \xrightarrow[\substack{r_3-2r_1 \\ r_4+2r_1}]{r_2-r_1} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 3 & -9 \\ 0 & -1 & -1 & 3 \end{array}\right)$

例 1 解方程组：$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$

解 $(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{array}\right) \xrightarrow[\substack{r_3-2r_1 \\ r_4+2r_1}]{r_2-r_1} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 3 & -9 \\ 0 & -1 & -1 & 3 \end{array}\right)$

$\xrightarrow[r_4+r_2]{r_3-3r_2}$

例 1 解方程组：
$$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 2x_1+5x_2+x_3=-5 \\ -2x_1-3x_2+x_3=-1 \end{cases}$$

解
$$(A\vdots b)=\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{array}\right) \xrightarrow[\substack{r_3-2r_1 \\ r_4+2r_1}]{r_2-r_1} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 3 & -9 \\ 0 & -1 & -1 & 3 \end{array}\right)$$

$$\xrightarrow[r_4+r_2]{r_3-3r_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ & & & \\ & & & \end{array}\right)$$

例 1 解方程组：$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 2x_1+5x_2+x_3=-5 \\ -2x_1-3x_2+x_3=-1 \end{cases}$

解

$$(A\vdots b)=\left(\begin{array}{rrr|r} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{array}\right) \xrightarrow[\substack{r_3-2r_1 \\ r_4+2r_1}]{r_2-r_1} \left(\begin{array}{rrr|r} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 3 & -9 \\ 0 & -1 & -1 & 3 \end{array}\right)$$

$$\xrightarrow[r_4+r_2]{r_3-3r_2} \left(\begin{array}{rrr|r} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

例 1 解方程组：
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$

解
$$(A\vdots b) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{array}\right) \xrightarrow[\substack{r_3-2r_1 \\ r_4+2r_1}]{r_2-r_1} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 3 & -9 \\ 0 & -1 & -1 & 3 \end{array}\right)$$

$$\xrightarrow[r_4+r_2]{r_3-3r_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

例 1 解方程组：
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$

解
$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{array}\right) \xrightarrow[\substack{r_3-2r_1 \\ r_4+2r_1}]{r_2-r_1} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 3 & -9 \\ 0 & -1 & -1 & 3 \end{array}\right)$$

$$\xrightarrow[r_4+r_2]{r_3-3r_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{r_1-r_2}$$

例 1 解方程组：
$$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 2x_1+5x_2+x_3=-5 \\ -2x_1-3x_2+x_3=-1 \end{cases}$$

解
$$(A\vdots b)=\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{array}\right) \xrightarrow[\substack{r_3-2r_1 \\ r_4+2r_1}]{r_2-r_1} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 3 & -9 \\ 0 & -1 & -1 & 3 \end{array}\right)$$

$$\xrightarrow[r_4+r_2]{r_3-3r_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{r_1-r_2} \left(\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

例 1 解方程组：$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 2x_1+5x_2+x_3=-5 \\ -2x_1-3x_2+x_3=-1 \end{cases}$

解

$$(A\vdots b)=\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{array}\right) \xrightarrow[\substack{r_3-2r_1 \\ r_4+2r_1}]{r_2-r_1} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 3 & -9 \\ 0 & -1 & -1 & 3 \end{array}\right)$$

$$\xrightarrow[r_4+r_2]{r_3-3r_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{r_1-r_2} \left(\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

所以原方程组等价于

例 1 解方程组：
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$

解
$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{array}\right) \xrightarrow[\substack{r_3-2r_1 \\ r_4+2r_1}]{r_2-r_1} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 3 & -9 \\ 0 & -1 & -1 & 3 \end{array}\right)$$

$$\xrightarrow[r_4+r_2]{r_3-3r_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{r_1-r_2} \left(\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

所以原方程组等价于
$$\begin{cases} x_1 + \quad - 2x_3 = 5 \\ \qquad x_2 + x_3 = -3 \end{cases}$$

例 1 解方程组：
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$

解
$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{array}\right) \xrightarrow[\substack{r_3-2r_1 \\ r_4+2r_1}]{r_2-r_1} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 3 & -9 \\ 0 & -1 & -1 & 3 \end{array}\right)$$

$$\xrightarrow[r_4+r_2]{r_3-3r_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{r_1-r_2} \left(\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

所以原方程组等价于
$$\begin{cases} x_1 + \quad - 2x_3 = 5 \\ x_2 + x_3 = -3 \end{cases} \Leftrightarrow \begin{cases} x_1 + \quad = 5 + 2x_3 \\ x_2 = -3 - x_3 \end{cases}$$

例 1 解方程组：
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$

解
$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{array}\right) \xrightarrow[\substack{r_3-2r_1 \\ r_4+2r_1}]{r_2-r_1} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 3 & -9 \\ 0 & -1 & -1 & 3 \end{array}\right)$$

$$\xrightarrow[r_4+r_2]{r_3-3r_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{r_1-r_2} \left(\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

所以原方程组等价于
$$\begin{cases} x_1 + \quad - 2x_3 = 5 \\ x_2 + x_3 = -3 \end{cases} \Leftrightarrow \begin{cases} x_1 + \quad = 5 + 2x_3 \\ x_2 = -3 - x_3 \end{cases}$$

所以通解是：
$$\begin{cases} \\ \\ x_3 = c_1 \end{cases} \qquad (c_1\text{为任意常数})$$

例 1 解方程组:
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$

解
$$(A\vdots b) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{array}\right) \xrightarrow[\substack{r_3-2r_1 \\ r_4+2r_1}]{r_2-r_1} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 3 & -9 \\ 0 & -1 & -1 & 3 \end{array}\right)$$

$$\xrightarrow[r_4+r_2]{r_3-3r_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{r_1-r_2} \left(\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

所以原方程组等价于
$$\begin{cases} x_1 + \quad - 2x_3 = 5 \\ x_2 + x_3 = -3 \end{cases} \Leftrightarrow \begin{cases} x_1 + \quad = 5 + 2x_3 \\ x_2 = -3 - x_3 \end{cases}$$

所以通解是:
$$\begin{cases} x_1 = 5 + 2c_1 \\ \\ x_3 = c_1 \end{cases} \quad (c_1\text{为任意常数})$$

例 1 解方程组: $\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$

解 $(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{array}\right) \xrightarrow[\substack{r_3-2r_1 \\ r_4+2r_1}]{r_2-r_1} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 3 & -9 \\ 0 & -1 & -1 & 3 \end{array}\right)$

$$\xrightarrow[r_4+r_2]{r_3-3r_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{r_1-r_2} \left(\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

所以原方程组等价于

$$\begin{cases} x_1 + \quad - 2x_3 = 5 \\ \quad x_2 + x_3 = -3 \end{cases} \Leftrightarrow \begin{cases} x_1 + \quad = 5 + 2x_3 \\ \quad x_2 = -3 - x_3 \end{cases}$$

所以通解是: $\begin{cases} x_1 = 5 + 2c_1 \\ x_2 = -3 - c_1 \\ x_3 = c_1 \end{cases}$ (c_1为任意常数)

例 1 解方程组：
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$

解
$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{array}\right) \xrightarrow[\substack{r_3-2r_1 \\ r_4+2r_1}]{r_2-r_1} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 3 & -9 \\ 0 & -1 & -1 & 3 \end{array}\right)$$

$$\xrightarrow[r_4+r_2]{r_3-3r_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{r_1-r_2} \left(\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

所以原方程组等价于
$$\begin{cases} x_1 + \quad - 2x_3 = 5 \\ x_2 + x_3 = -3 \end{cases} \Leftrightarrow \begin{cases} x_1 + \quad = 5 + 2x_3 \\ x_2 = -3 - x_3 \end{cases}$$

所以通解是：
$$\begin{cases} x_1 = 5 + 2c_1 \\ x_2 = -3 - c_1 \quad (c_1\text{为任意常数}) \\ x_3 = c_1 \end{cases}$$

例 2 解方程组：
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

例 2 解方程组：
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

解
$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{array}\right)$$

例 2 解方程组：
$$\begin{cases} x_1+2x_2+4x_3=28 \\ -2x_1-3x_2-9x_3=-53 \\ 3x_1+6x_2+13x_3=88 \\ 5x_1+9x_2+22x_3=141 \end{cases}$$

解 $(A \vdots b)=\left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{array}\right) \xrightarrow[\substack{r_3-3r_1 \\ r_4-5r_1}]{r_2+2r_1}$

例 2 解方程组：
$$\begin{cases} x_1+2x_2+4x_3=28 \\ -2x_1-3x_2-9x_3=-53 \\ 3x_1+6x_2+13x_3=88 \\ 5x_1+9x_2+22x_3=141 \end{cases}$$

解 $$(A \vdots b)=\left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{array}\right) \xrightarrow[\substack{r_3-3r_1 \\ r_4-5r_1}]{r_2+2r_1} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ & & & \\ & & & \\ & & & \end{array}\right)$$

例 2 解方程组：$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

解 $$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{array}\right) \xrightarrow[\substack{r_3-3r_1 \\ r_4-5r_1}]{r_2+2r_1} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ & & & \\ & & & \end{array}\right)$$

例 2 解方程组：
$$\begin{cases} x_1+2x_2+4x_3=28 \\ -2x_1-3x_2-9x_3=-53 \\ 3x_1+6x_2+13x_3=88 \\ 5x_1+9x_2+22x_3=141 \end{cases}$$

解
$$(A\vdots b)=\left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{array}\right) \xrightarrow[\substack{r_3-3r_1 \\ r_4-5r_1}]{r_2+2r_1} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ & & & \end{array}\right)$$

例 2 解方程组：$\begin{cases} x_1+2x_2+4x_3=28 \\ -2x_1-3x_2-9x_3=-53 \\ 3x_1+6x_2+13x_3=88 \\ 5x_1+9x_2+22x_3=141 \end{cases}$

解 $(A \vdots b)=\left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{array}\right) \xrightarrow[\substack{r_3-3r_1 \\ r_4-5r_1}]{r_2+2r_1} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{array}\right)$

例 2 解方程组：$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$

解 $$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{array}\right) \xrightarrow[\substack{r_3-3r_1 \\ r_4-5r_1}]{r_2+2r_1} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{array}\right)$$

$$\xrightarrow{r_4+r_2}$$

例 2 解方程组：
$$\begin{cases} x_1+2x_2+4x_3=28 \\ -2x_1-3x_2-9x_3=-53 \\ 3x_1+6x_2+13x_3=88 \\ 5x_1+9x_2+22x_3=141 \end{cases}$$

解
$$(A\vdots b)=\left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{array}\right) \xrightarrow[\substack{r_3-3r_1 \\ r_4-5r_1}]{r_2+2r_1} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{array}\right)$$

$$\xrightarrow{r_4+r_2} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{array}\right)$$

例 2 解方程组：
$$\begin{cases} x_1+2x_2+4x_3=28 \\ -2x_1-3x_2-9x_3=-53 \\ 3x_1+6x_2+13x_3=88 \\ 5x_1+9x_2+22x_3=141 \end{cases}$$

解
$$(A \vdots b)=\left(\begin{array}{rrr|r} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{array}\right) \xrightarrow[\substack{r_3-3r_1 \\ r_4-5r_1}]{r_2+2r_1} \left(\begin{array}{rrr|r} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{array}\right)$$

$$\xrightarrow{r_4+r_2} \left(\begin{array}{rrr|r} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{array}\right) \xrightarrow{r_4-r_3}$$

例 2 解方程组：$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$

解

$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{array}\right) \xrightarrow[\substack{r_3-3r_1 \\ r_4-5r_1}]{r_2+2r_1} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{array}\right)$$

$$\xrightarrow{r_4+r_2} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{array}\right) \xrightarrow{r_4-r_3} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

例 2 解方程组：$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$

解

$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{array}\right) \xrightarrow[\substack{r_3-3r_1 \\ r_4-5r_1}]{r_2+2r_1} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{array}\right)$$

$$\xrightarrow{r_4+r_2} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{array}\right) \xrightarrow{r_4-r_3} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[r_1-4r_3]{r_2+r_3}$$

$$
\text{例 2 解方程组：}\left\{\begin{array}{rrrr}
x_1+ & 2x_2+ & 4x_3 = & 28 \\
-2x_1- & 3x_2- & 9x_3 = & -53 \\
3x_1+ & 6x_2+ & 13x_3 = & 88 \\
5x_1+ & 9x_2+ & 22x_3 = & 141
\end{array}\right.
$$

解

$$
(A\vdots b)=\left(\begin{array}{rrr|r}
1 & 2 & 4 & 28 \\
-2 & -3 & -9 & -53 \\
3 & 6 & 13 & 88 \\
5 & 9 & 22 & 141
\end{array}\right)
\xrightarrow[\substack{r_3-3r_1 \\ r_4-5r_1}]{r_2+2r_1}
\left(\begin{array}{rrr|r}
1 & 2 & 4 & 28 \\
0 & 1 & -1 & 3 \\
0 & 0 & 1 & 4 \\
0 & -1 & 2 & 1
\end{array}\right)
$$

$$
\xrightarrow{r_4+r_2}
\left(\begin{array}{rrr|r}
1 & 2 & 4 & 28 \\
0 & 1 & -1 & 3 \\
0 & 0 & 1 & 4 \\
0 & 0 & 1 & 4
\end{array}\right)
\xrightarrow{r_4-r_3}
\left(\begin{array}{rrr|r}
1 & 2 & 4 & 28 \\
0 & 1 & -1 & 3 \\
0 & 0 & 1 & 4 \\
0 & 0 & 0 & 0
\end{array}\right)
$$

$$
\xrightarrow[r_1-4r_3]{r_2+r_3}
\left(\begin{array}{rrr|r}
 & & & \\
 & & & \\
0 & 0 & 1 & 4 \\
0 & 0 & 0 & 0
\end{array}\right)
$$

例 2 解方程组：$\begin{cases} x_1+2x_2+4x_3=28 \\ -2x_1-3x_2-9x_3=-53 \\ 3x_1+6x_2+13x_3=88 \\ 5x_1+9x_2+22x_3=141 \end{cases}$

解

$$(A\vdots b)=\left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{array}\right) \xrightarrow[\substack{r_3-3r_1 \\ r_4-5r_1}]{r_2+2r_1} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{array}\right)$$

$$\xrightarrow{r_4+r_2} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{array}\right) \xrightarrow{r_4-r_3} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[r_1-4r_3]{r_2+r_3} \left(\begin{array}{ccc|c} 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

例 2 解方程组：
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

解
$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{array}\right) \xrightarrow[\substack{r_3-3r_1 \\ r_4-5r_1}]{r_2+2r_1} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{array}\right)$$

$$\xrightarrow{r_4+r_2} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{array}\right) \xrightarrow{r_4-r_3} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[r_1-4r_3]{r_2+r_3} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 12 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

例 2 解方程组：$\begin{cases} x_1+2x_2+4x_3=28 \\ -2x_1-3x_2-9x_3=-53 \\ 3x_1+6x_2+13x_3=88 \\ 5x_1+9x_2+22x_3=141 \end{cases}$

解

$$(A \vdots b)=\left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{array}\right) \xrightarrow[\substack{r_3-3r_1 \\ r_4-5r_1}]{r_2+2r_1} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{array}\right)$$

$$\xrightarrow{r_4+r_2} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{array}\right) \xrightarrow{r_4-r_3} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[r_1-4r_3]{r_2+r_3} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 12 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{r_1-2r_2}$$

例 2 解方程组：$\begin{cases} x_1+2x_2+4x_3=28 \\ -2x_1-3x_2-9x_3=-53 \\ 3x_1+6x_2+13x_3=88 \\ 5x_1+9x_2+22x_3=141 \end{cases}$

解

$$(A\vdots b)=\left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{array}\right) \xrightarrow[\substack{r_3-3r_1 \\ r_4-5r_1}]{r_2+2r_1} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{array}\right)$$

$$\xrightarrow{r_4+r_2} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{array}\right) \xrightarrow{r_4-r_3} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[r_1-4r_3]{r_2+r_3} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 12 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{r_1-2r_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

例 2 解方程组：
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

解
$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{array}\right) \xrightarrow[\substack{r_3-3r_1 \\ r_4-5r_1}]{r_2+2r_1} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{array}\right)$$

$$\xrightarrow{r_4+r_2} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{array}\right) \xrightarrow{r_4-r_3} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[r_1-4r_3]{r_2+r_3} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 12 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{r_1-2r_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

所以原方程组等价于
$$\begin{cases} x_1 = -2 \\ x_2 = 7 \\ x_3 = 4 \end{cases}$$

例 2 解方程组：
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

解
$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{array}\right) \xrightarrow[\substack{r_3-3r_1 \\ r_4-5r_1}]{r_2+2r_1} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{array}\right)$$

$$\xrightarrow{r_4+r_2} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{array}\right) \xrightarrow{r_4-r_3} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[r_1-4r_3]{r_2+r_3} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 12 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{r_1-2r_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

所以原方程组等价于
$$\begin{cases} x_1 = -2 \\ x_2 = 7 \\ x_3 = 4 \end{cases}$$

例 3 解方程组：$$\begin{cases} 4x_1 + 2x_2 - 7x_3 = -3 \\ 2x_1 + x_2 - 4x_3 = -1 \\ 5x_1 + 3x_2 - 11x_3 = 2 \\ x_1 + x_2 - 4x_3 = 2 \end{cases}$$

例 3 解方程组：$\begin{cases} 4x_1+2x_2-7x_3=-3 \\ 2x_1+x_2-4x_3=-1 \\ 5x_1+3x_2-11x_3=2 \\ x_1+x_2-4x_3=2 \end{cases}$

解

$$(A \vdots b)=\left(\begin{array}{ccc|c} 4 & 2 & -7 & -3 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 1 & 1 & -4 & 2 \end{array}\right)$$

例 3 解方程组：
$$\begin{cases} 4x_1 + 2x_2 - 7x_3 = -3 \\ 2x_1 + x_2 - 4x_3 = -1 \\ 5x_1 + 3x_2 - 11x_3 = 2 \\ x_1 + x_2 - 4x_3 = 2 \end{cases}$$

解

$$(A \vdots b) = \left(\begin{array}{ccc|c} 4 & 2 & -7 & -3 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 1 & 1 & -4 & 2 \end{array}\right) \xrightarrow{r_1 \leftrightarrow r_4}$$

例 3 解方程组:
$$\begin{cases} 4x_1 + 2x_2 - 7x_3 = -3 \\ 2x_1 + x_2 - 4x_3 = -1 \\ 5x_1 + 3x_2 - 11x_3 = 2 \\ x_1 + x_2 - 4x_3 = 2 \end{cases}$$

解

$$(A \vdots b) = \left(\begin{array}{ccc|c} 4 & 2 & -7 & -3 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 1 & 1 & -4 & 2 \end{array}\right) \xrightarrow{r_1 \leftrightarrow r_4} \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 4 & 2 & -7 & -3 \end{array}\right)$$

例 3 解方程组：$\begin{cases} 4x_1+2x_2-7x_3=-3 \\ 2x_1+x_2-4x_3=-1 \\ 5x_1+3x_2-11x_3=2 \\ x_1+x_2-4x_3=2 \end{cases}$

解

$$(A\vdots b)=\left(\begin{array}{ccc|c} 4 & 2 & -7 & -3 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 1 & 1 & -4 & 2 \end{array}\right) \xrightarrow{r_1 \leftrightarrow r_4} \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 4 & 2 & -7 & -3 \end{array}\right) \xrightarrow[\substack{r_3-5r_1 \\ r_4-4r_1}]{r_2-2r_1}$$

例 3 解方程组：
$$\begin{cases} 4x_1+2x_2-7x_3=-3 \\ 2x_1+x_2-4x_3=-1 \\ 5x_1+3x_2-11x_3=2 \\ x_1+x_2-4x_3=2 \end{cases}$$

解

$$(A \vdots b)=\left(\begin{array}{ccc|c} 4 & 2 & -7 & -3 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 1 & 1 & -4 & 2 \end{array}\right) \xrightarrow{r_1 \leftrightarrow r_4} \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 4 & 2 & -7 & -3 \end{array}\right) \xrightarrow[\substack{r_3-5r_1 \\ r_4-4r_1}]{r_2-2r_1}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ & & & \\ & & & \end{array}\right)$$

例 3 解方程组：
$$\begin{cases} 4x_1 + 2x_2 - 7x_3 = -3 \\ 2x_1 + x_2 - 4x_3 = -1 \\ 5x_1 + 3x_2 - 11x_3 = 2 \\ x_1 + x_2 - 4x_3 = 2 \end{cases}$$

解

$$(A \vdots b) = \left(\begin{array}{cccc|c} 4 & 2 & -7 & -3 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 1 & 1 & -4 & 2 \end{array}\right) \xrightarrow{r_1 \leftrightarrow r_4} \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 4 & 2 & -7 & -3 \end{array}\right) \xrightarrow[\substack{r_3 - 5r_1 \\ r_4 - 4r_1}]{r_2 - 2r_1}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ & & & \\ & & & \end{array}\right)$$

例 3 解方程组：$\begin{cases} 4x_1 + 2x_2 - 7x_3 = -3 \\ 2x_1 + x_2 - 4x_3 = -1 \\ 5x_1 + 3x_2 - 11x_3 = 2 \\ x_1 + x_2 - 4x_3 = 2 \end{cases}$

解

$$(A \vdots b) = \left(\begin{array}{ccc|c} 4 & 2 & -7 & -3 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 1 & 1 & -4 & 2 \end{array}\right) \xrightarrow{r_1 \leftrightarrow r_4} \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 4 & 2 & -7 & -3 \end{array}\right) \xrightarrow[\substack{r_3 - 5r_1 \\ r_4 - 4r_1}]{r_2 - 2r_1}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & -2 & 9 & -8 \\ & & & \end{array}\right)$$

例 3 解方程组：
$$\begin{cases} 4x_1 + 2x_2 - 7x_3 = -3 \\ 2x_1 + x_2 - 4x_3 = -1 \\ 5x_1 + 3x_2 - 11x_3 = 2 \\ x_1 + x_2 - 4x_3 = 2 \end{cases}$$

解

$$(A \vdots b) = \left(\begin{array}{ccc|c} 4 & 2 & -7 & -3 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 1 & 1 & -4 & 2 \end{array}\right) \xrightarrow{r_1 \leftrightarrow r_4} \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 4 & 2 & -7 & -3 \end{array}\right) \xrightarrow[\substack{r_3 - 5r_1 \\ r_4 - 4r_1}]{r_2 - 2r_1}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & -2 & 9 & -8 \\ 0 & -2 & 9 & -11 \end{array}\right)$$

例 3 解方程组：$\begin{cases} 4x_1 + 2x_2 - 7x_3 = -3 \\ 2x_1 + x_2 - 4x_3 = -1 \\ 5x_1 + 3x_2 - 11x_3 = 2 \\ x_1 + x_2 - 4x_3 = 2 \end{cases}$

解

$$(A \vdots b) = \left(\begin{array}{ccc|c} 4 & 2 & -7 & -3 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 1 & 1 & -4 & 2 \end{array}\right) \xrightarrow{r_1 \leftrightarrow r_4} \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 4 & 2 & -7 & -3 \end{array}\right) \xrightarrow[\substack{r_3 - 5r_1 \\ r_4 - 4r_1}]{r_2 - 2r_1}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & -2 & 9 & -8 \\ 0 & -2 & 9 & -11 \end{array}\right) \xrightarrow[r_4 - 2r_2]{r_3 - 2r_2}$$

例 3 解方程组：
$$\begin{cases} 4x_1 + 2x_2 - 7x_3 = -3 \\ 2x_1 + x_2 - 4x_3 = -1 \\ 5x_1 + 3x_2 - 11x_3 = 2 \\ x_1 + x_2 - 4x_3 = 2 \end{cases}$$

解

$$(A \vdots b) = \left(\begin{array}{ccc|c} 4 & 2 & -7 & -3 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 1 & 1 & -4 & 2 \end{array}\right) \xrightarrow{r_1 \longleftrightarrow r_4} \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 4 & 2 & -7 & -3 \end{array}\right) \xrightarrow[\substack{r_3 - 5r_1 \\ r_4 - 4r_1}]{r_2 - 2r_1}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & -2 & 9 & -8 \\ 0 & -2 & 9 & -11 \end{array}\right) \xrightarrow[r_4 - 2r_2]{r_3 - 2r_2} \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ & & & \\ & & & \end{array}\right)$$

例 3 解方程组：
$$\begin{cases} 4x_1+2x_2-7x_3=-3 \\ 2x_1+x_2-4x_3=-1 \\ 5x_1+3x_2-11x_3=2 \\ x_1+x_2-4x_3=2 \end{cases}$$

解

$$(A \vdots b)=\left(\begin{array}{ccc|c} 4 & 2 & -7 & -3 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 1 & 1 & -4 & 2 \end{array}\right) \xrightarrow{r_1 \longleftrightarrow r_4} \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 4 & 2 & -7 & -3 \end{array}\right) \xrightarrow[\substack{r_3-5r_1 \\ r_4-4r_1}]{r_2-2r_1}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & -2 & 9 & -8 \\ 0 & -2 & 9 & -11 \end{array}\right) \xrightarrow[r_4-2r_2]{r_3-2r_2} \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ & & & \end{array}\right)$$

例 3 解方程组：
$$\begin{cases} 4x_1+2x_2-7x_3=-3 \\ 2x_1+x_2-4x_3=-1 \\ 5x_1+3x_2-11x_3=2 \\ x_1+x_2-4x_3=2 \end{cases}$$

解

$$(A\vdots b)=\left(\begin{array}{ccc|c} 4 & 2 & -7 & -3 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 1 & 1 & -4 & 2 \end{array}\right) \xrightarrow{r_1\leftrightarrow r_4} \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 4 & 2 & -7 & -3 \end{array}\right) \xrightarrow[\substack{r_3-5r_1\\ r_4-4r_1}]{r_2-2r_1}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & -2 & 9 & -8 \\ 0 & -2 & 9 & -11 \end{array}\right) \xrightarrow[r_4-2r_2]{r_3-2r_2} \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

例 3 解方程组：
$$\begin{cases} 4x_1+2x_2-7x_3=-3 \\ 2x_1+x_2-4x_3=-1 \\ 5x_1+3x_2-11x_3=2 \\ x_1+x_2-4x_3=2 \end{cases}$$

解

$$(A \vdots b)=\left(\begin{array}{ccc|c} 4 & 2 & -7 & -3 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 1 & 1 & -4 & 2 \end{array}\right) \xrightarrow{r_1 \longleftrightarrow r_4} \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 4 & 2 & -7 & -3 \end{array}\right) \xrightarrow[\substack{r_3-5r_1 \\ r_4-4r_1}]{r_2-2r_1}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & -2 & 9 & -8 \\ 0 & -2 & 9 & -11 \end{array}\right) \xrightarrow[r_4-2r_2]{r_3-2r_2} \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right) \xrightarrow{r_4-r_3}$$

例 3 解方程组：$\begin{cases} 4x_1+2x_2-7x_3=-3 \\ 2x_1+x_2-4x_3=-1 \\ 5x_1+3x_2-11x_3=2 \\ x_1+x_2-4x_3=2 \end{cases}$

解

$$(A\vdots b)=\left(\begin{array}{ccc|c} 4 & 2 & -7 & -3 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 1 & 1 & -4 & 2 \end{array}\right) \xrightarrow{r_1\leftrightarrow r_4} \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 4 & 2 & -7 & -3 \end{array}\right) \xrightarrow[\substack{r_3-5r_1 \\ r_4-4r_1}]{r_2-2r_1}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & -2 & 9 & -8 \\ 0 & -2 & 9 & -11 \end{array}\right) \xrightarrow[r_4-2r_2]{r_3-2r_2} \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right) \xrightarrow{r_4-r_3} \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -3 \end{array}\right)$$

例 3 解方程组：$\begin{cases} 4x_1+2x_2-7x_3=-3 \\ 2x_1+x_2-4x_3=-1 \\ 5x_1+3x_2-11x_3=2 \\ x_1+x_2-4x_3=2 \end{cases}$

解

$$(A\,\vdots\,b)=\left(\begin{array}{ccc|c} 4 & 2 & -7 & -3 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 1 & 1 & -4 & 2 \end{array}\right) \xrightarrow{r_1\leftrightarrow r_4} \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 4 & 2 & -7 & -3 \end{array}\right) \xrightarrow[\substack{r_3-5r_1 \\ r_4-4r_1}]{r_2-2r_1}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & -2 & 9 & -8 \\ 0 & -2 & 9 & -11 \end{array}\right) \xrightarrow[r_4-2r_2]{r_3-2r_2} \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right) \xrightarrow{r_4-r_3} \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -3 \end{array}\right)$$

所以原方程组等价于

例 3 解方程组:
$$\begin{cases} 4x_1+2x_2-7x_3=-3 \\ 2x_1+x_2-4x_3=-1 \\ 5x_1+3x_2-11x_3=2 \\ x_1+x_2-4x_3=2 \end{cases}$$

解

$$(A\vdots b)=\left(\begin{array}{ccc|c} 4 & 2 & -7 & -3 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 1 & 1 & -4 & 2 \end{array}\right) \xrightarrow{r_1\leftrightarrow r_4} \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 4 & 2 & -7 & -3 \end{array}\right) \xrightarrow[\substack{r_3-5r_1 \\ r_4-4r_1}]{r_2-2r_1}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & -2 & 9 & -8 \\ 0 & -2 & 9 & -11 \end{array}\right) \xrightarrow[r_4-2r_2]{r_3-2r_2} \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right) \xrightarrow{r_4-r_3} \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -3 \end{array}\right)$$

所以原方程组等价于
$$\begin{cases} x_1+x_2-4x_3=2 \\ -x_2+4x_3=-5 \\ x_3=2 \\ 0=-3 \end{cases}$$

例 3 解方程组：
$$\begin{cases} 4x_1 + 2x_2 - 7x_3 = -3 \\ 2x_1 + x_2 - 4x_3 = -1 \\ 5x_1 + 3x_2 - 11x_3 = 2 \\ x_1 + x_2 - 4x_3 = 2 \end{cases}$$

解

$$(A \vdots b) = \left(\begin{array}{ccc|c} 4 & 2 & -7 & -3 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 1 & 1 & -4 & 2 \end{array}\right) \xrightarrow{r_1 \leftrightarrow r_4} \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 4 & 2 & -7 & -3 \end{array}\right) \xrightarrow[\substack{r_3 - 5r_1 \\ r_4 - 4r_1}]{r_2 - 2r_1}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & -2 & 9 & -8 \\ 0 & -2 & 9 & -11 \end{array}\right) \xrightarrow[r_4 - 2r_2]{r_3 - 2r_2} \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right) \xrightarrow{r_4 - r_3} \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -3 \end{array}\right)$$

所以原方程组等价于
$$\begin{cases} x_1 + x_2 - 4x_3 = 2 \\ -x_2 + 4x_3 = -5 \\ x_3 = 2 \\ 0 = -3 \end{cases} \Rightarrow \quad \text{无解!}$$

例 3 解方程组：
$$\begin{cases} 4x_1 + 2x_2 - 7x_3 = -3 \\ 2x_1 + x_2 - 4x_3 = -1 \\ 5x_1 + 3x_2 - 11x_3 = 2 \\ x_1 + x_2 - 4x_3 = 2 \end{cases}$$

解

$$(A \vdots b) = \left(\begin{array}{ccc|c} 4 & 2 & -7 & -3 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 1 & 1 & -4 & 2 \end{array}\right) \xrightarrow{r_1 \leftrightarrow r_4} \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 4 & 2 & -7 & -3 \end{array}\right) \xrightarrow[\substack{r_3-5r_1 \\ r_4-4r_1}]{r_2-2r_1}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & -2 & 9 & -8 \\ 0 & -2 & 9 & -11 \end{array}\right) \xrightarrow[r_4-2r_2]{r_3-2r_2} \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right) \xrightarrow{r_4-r_3} \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -3 \end{array}\right)$$

所以原方程组等价于
$$\begin{cases} x_1 + x_2 - 4x_3 = 2 \\ -x_2 + 4x_3 = -5 \\ x_3 = 2 \\ 0 = -3 \end{cases} \Rightarrow \quad \text{无解！}$$

总结

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases} \qquad \begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases} \qquad \begin{cases} 4x_1 + 2x_2 - 7x_3 = -3 \\ 2x_1 + x_2 - 4x_3 = -1 \\ 5x_1 + 3x_2 - 11x_3 = 2 \\ x_1 + x_2 - 4x_3 = 2 \end{cases}$$

总结

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases} \quad \begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases} \quad \begin{cases} 4x_1 + 2x_2 - 7x_3 = -3 \\ 2x_1 + x_2 - 4x_3 = -1 \\ 5x_1 + 3x_2 - 11x_3 = 2 \\ x_1 + x_2 - 4x_3 = 2 \end{cases}$$

$$\Downarrow \qquad\qquad \Downarrow \qquad\qquad \Downarrow$$

$$(A \vdots b) \qquad\qquad (A \vdots b) \qquad\qquad (A \vdots b)$$

总结

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases} \quad \begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases} \quad \begin{cases} 4x_1 + 2x_2 - 7x_3 = -3 \\ 2x_1 + x_2 - 4x_3 = -1 \\ 5x_1 + 3x_2 - 11x_3 = 2 \\ x_1 + x_2 - 4x_3 = 2 \end{cases}$$

$$\Downarrow \qquad\qquad \Downarrow \qquad\qquad \Downarrow$$

$$(A \vdots b) \qquad\qquad (A \vdots b) \qquad\qquad (A \vdots b)$$

初等 $\Downarrow$ 行变换　　初等 $\Downarrow$ 行变换　　初等 $\Downarrow$ 行变换

总结

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$

$$\Downarrow$$

$$(A \vdots b)$$

初等 $\Downarrow$ 行变换

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$\Downarrow$$

$$(A \vdots b)$$

初等 $\Downarrow$ 行变换

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} 4x_1 + 2x_2 - 7x_3 = -3 \\ 2x_1 + x_2 - 4x_3 = -1 \\ 5x_1 + 3x_2 - 11x_3 = 2 \\ x_1 + x_2 - 4x_3 = 2 \end{cases}$$

$$\Downarrow$$

$$(A \vdots b)$$

初等 $\Downarrow$ 行变换

$$\left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -3 \end{array}\right)$$

总结

$$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 2x_1+5x_2+x_3=-5 \\ -2x_1-3x_2+x_3=-1 \end{cases} \qquad \begin{cases} x_1+2x_2+4x_3=28 \\ -2x_1-3x_2-9x_3=-53 \\ 3x_1+6x_2+13x_3=88 \\ 5x_1+9x_2+22x_3=141 \end{cases} \qquad \begin{cases} 4x_1+2x_2-7x_3=-3 \\ 2x_1+x_2-4x_3=-1 \\ 5x_1+3x_2-11x_3=2 \\ x_1+x_2-4x_3=2 \end{cases}$$

$$\Downarrow \qquad\qquad \Downarrow \qquad\qquad \Downarrow$$

$$(A\vdots b) \qquad\qquad (A\vdots b) \qquad\qquad (A\vdots b)$$

初等⇓行变换　　初等⇓行变换　　初等⇓行变换

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -3 \end{array}\right)$$

无穷多解　　唯一解　　无解

总结

$$\begin{cases} x_1+x_2-x_3=2\\ x_1+2x_2=-1\\ 2x_1+5x_2+x_3=-5\\ -2x_1-3x_2+x_3=-1 \end{cases}$$

$$\Downarrow$$

$$(A \vdots b)$$

初等 $\Downarrow$ 行变换

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 5\\ 0 & 1 & 1 & -3\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{array}\right)$$

无穷多解

$$\begin{cases} x_1+2x_2+4x_3=28\\ -2x_1-3x_2-9x_3=-53\\ 3x_1+6x_2+13x_3=88\\ 5x_1+9x_2+22x_3=141 \end{cases}$$

$$\Downarrow$$

$$(A \vdots b)$$

初等 $\Downarrow$ 行变换

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -2\\ 0 & 1 & 0 & 7\\ 0 & 0 & 1 & 4\\ 0 & 0 & 0 & 0 \end{array}\right)$$

唯一解

$$\begin{cases} 4x_1+2x_2-7x_3=-3\\ 2x_1+x_2-4x_3=-1\\ 5x_1+3x_2-11x_3=2\\ x_1+x_2-4x_3=2 \end{cases}$$

$$\Downarrow$$

$$(A \vdots b)$$

初等 $\Downarrow$ 行变换

$$\left(\begin{array}{ccc|c} 1 & 1 & -4 & 2\\ 0 & -1 & 4 & -5\\ 0 & 0 & 1 & 2\\ 0 & 0 & 0 & -3 \end{array}\right)$$

无解

总结

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases} \quad \begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases} \quad \begin{cases} 4x_1 + 2x_2 - 7x_3 = -3 \\ 2x_1 + x_2 - 4x_3 = -1 \\ 5x_1 + 3x_2 - 11x_3 = 2 \\ x_1 + x_2 - 4x_3 = 2 \end{cases}$$

$$\Downarrow \qquad\qquad \Downarrow \qquad\qquad \Downarrow$$

$$(A \vdots b) \qquad\qquad (A \vdots b) \qquad\qquad (A \vdots b)$$

初等 $\Downarrow$ 行变换　　　初等 $\Downarrow$ 行变换　　　初等 $\Downarrow$ 行变换

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -3 \end{array}\right)$$

无穷多解　　　唯一解　　　无解

$r(A) \quad r(A \vdots b)$　　　$r(A) \quad r(A \vdots b)$　　　$r(A) \quad r(A \vdots b)$

总结

$$\begin{cases} x_1+ \ x_2- \ x_3 = 2 \\ x_1+ 2x_2 = -1 \\ 2x_1+ 5x_2+ x_3 = -5 \\ -2x_1- 3x_2+ x_3 = -1 \end{cases}$$

$$\Downarrow$$

$$(A \vdots b)$$

初等 $\Downarrow$ 行变换

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

无穷多解

$$r(A) \quad r(A \vdots b)$$

$$\begin{cases} x_1+ 2x_2+ 4x_3 = 28 \\ -2x_1- 3x_2- 9x_3 = -53 \\ 3x_1+ 6x_2+ 13x_3 = 88 \\ 5x_1+ 9x_2+ 22x_3 = 141 \end{cases}$$

$$\Downarrow$$

$$(A \vdots b)$$

初等 $\Downarrow$ 行变换

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

唯一解

$$r(A) \quad r(A \vdots b)$$

$$\begin{cases} 4x_1+ 2x_2- 7x_3 = -3 \\ 2x_1+ x_2- 4x_3 = -1 \\ 5x_1+ 3x_2- 11x_3 = 2 \\ x_1+ x_2- 4x_3 = 2 \end{cases}$$

$$\Downarrow$$

$$(A \vdots b)$$

初等 $\Downarrow$ 行变换

$$\left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -3 \end{array}\right)$$

无解

$$r(A) \neq r(A \vdots b)$$

总结

$$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 2x_1+5x_2+x_3=-5 \\ -2x_1-3x_2+x_3=-1 \end{cases} \qquad \begin{cases} x_1+2x_2+4x_3=28 \\ -2x_1-3x_2-9x_3=-53 \\ 3x_1+6x_2+13x_3=88 \\ 5x_1+9x_2+22x_3=141 \end{cases} \qquad \begin{cases} 4x_1+2x_2-7x_3=-3 \\ 2x_1+x_2-4x_3=-1 \\ 5x_1+3x_2-11x_3=2 \\ x_1+x_2-4x_3=2 \end{cases}$$

$$\Downarrow \qquad\qquad \Downarrow \qquad\qquad \Downarrow$$

$$(A \vdots b) \qquad\qquad (A \vdots b) \qquad\qquad (A \vdots b)$$

初等 $\Downarrow$ 行变换　　初等 $\Downarrow$ 行变换　　初等 $\Downarrow$ 行变换

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -3 \end{array}\right)$$

无穷多解　　唯一解　　无解

$$r(A)=r(A \vdots b) \qquad r(A)=r(A \vdots b) \qquad r(A)\neq r(A \vdots b)$$

总结

$$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 2x_1+5x_2+x_3=-5 \\ -2x_1-3x_2+x_3=-1 \end{cases}$$

$$\Downarrow$$

$$(A \vdots b)$$

初等 $\Downarrow$ 行变换

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

无穷多解

$$r(A)=r(A \vdots b)$$

$$\begin{cases} x_1+2x_2+4x_3=28 \\ -2x_1-3x_2-9x_3=-53 \\ 3x_1+6x_2+13x_3=88 \\ 5x_1+9x_2+22x_3=141 \end{cases}$$

$$\Downarrow$$

$$(A \vdots b)$$

初等 $\Downarrow$ 行变换

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

唯一解

$$r(A)=r(A \vdots b)$$

$$\begin{cases} 4x_1+2x_2-7x_3=-3 \\ 2x_1+x_2-4x_3=-1 \\ 5x_1+3x_2-11x_3=2 \\ x_1+x_2-4x_3=2 \end{cases}$$

$$\Downarrow$$

$$(A \vdots b)$$

初等 $\Downarrow$ 行变换

$$\left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -3 \end{array}\right)$$

无解

$$r(A)<r(A \vdots b)$$

总结

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$

$$\Downarrow$$

$$(A \vdots b)$$

初等 $\Downarrow$ 行变换

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

无穷多解

$$r(A) = r(A \vdots b) < n$$

$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$\Downarrow$$

$$(A \vdots b)$$

初等 $\Downarrow$ 行变换

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

唯一解

$$r(A) = r(A \vdots b)$$

$$\begin{cases} 4x_1 + 2x_2 - 7x_3 = -3 \\ 2x_1 + x_2 - 4x_3 = -1 \\ 5x_1 + 3x_2 - 11x_3 = 2 \\ x_1 + x_2 - 4x_3 = 2 \end{cases}$$

$$\Downarrow$$

$$(A \vdots b)$$

初等 $\Downarrow$ 行变换

$$\left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -3 \end{array}\right)$$

无解

$$r(A) < r(A \vdots b)$$

总结

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases} \qquad \begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases} \qquad \begin{cases} 4x_1 + 2x_2 - 7x_3 = -3 \\ 2x_1 + x_2 - 4x_3 = -1 \\ 5x_1 + 3x_2 - 11x_3 = 2 \\ x_1 + x_2 - 4x_3 = 2 \end{cases}$$

$$\Downarrow \qquad\qquad \Downarrow \qquad\qquad \Downarrow$$

$$(A \vdots b) \qquad\qquad (A \vdots b) \qquad\qquad (A \vdots b)$$

初等 $\Downarrow$ 行变换 　　 初等 $\Downarrow$ 行变换 　　 初等 $\Downarrow$ 行变换

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -3 \end{array}\right)$$

无穷多解 　　 唯一解 　　 无解

$$r(A) = r(A \vdots b) < n \qquad r(A) = r(A \vdots b) = n \qquad r(A) < r(A \vdots b)$$

总结

定理 方程组 $\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$ $\Leftrightarrow Ax = b$ 的

解有如下情形：

总结

定理 方程组 $\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$ $\Leftrightarrow Ax = b$ 的

解有如下情形：

1. $r(A \vdots b) = r(A)$

2. $r(A) \neq r(A \vdots b)$

总结

定理 方程组 $\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \Leftrightarrow Ax = b$ 的

解有如下情形：

1. $r(A \vdots b) = r(A)$

2. $r(A) \neq r(A \vdots b) \Leftrightarrow r(A) < r(A \vdots b)$

总结

定理 方程组 $\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$ $\Leftrightarrow Ax = b$ 的

解有如下情形：

1. $r(A \vdots b) = r(A)$

$$r(A) = r(A \vdots b) < n$$

$$r(A) = r(A \vdots b) = n$$

2. $r(A) \neq r(A \vdots b) \Leftrightarrow r(A) < r(A \vdots b)$

总结

定理 方程组
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \Leftrightarrow Ax = b \text{ 的}$$

解有如下情形：

1. $$r(A \vdots b) = r(A)$$

$$r(A) = r(A \vdots b) < n$$

- 只有唯一解 $\Leftrightarrow r(A) = r(A \vdots b) = n$

2. $$r(A) \neq r(A \vdots b) \Leftrightarrow r(A) < r(A \vdots b)$$

总结

定理 方程组 $\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$ $\Leftrightarrow Ax = b$ 的

解有如下情形：

1. $$r(A \vdots b) = r(A)$$

- 有无穷多解 $\Leftrightarrow r(A) = r(A \vdots b) < n$
- 只有唯一解 $\Leftrightarrow r(A) = r(A \vdots b) = n$

2. $$r(A) \neq r(A \vdots b) \Leftrightarrow r(A) < r(A \vdots b)$$

总结

定理 方程组
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \Leftrightarrow Ax = b \text{ 的}$$
解有如下情形：

1. 有解 $\Leftrightarrow r(A \vdots b) = r(A)$
   - 有无穷多解 $\Leftrightarrow r(A) = r(A \vdots b) < n$
   - 只有唯一解 $\Leftrightarrow r(A) = r(A \vdots b) = n$
2. $r(A) \neq r(A \vdots b) \Leftrightarrow r(A) < r(A \vdots b)$

总结

定理 方程组 $\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$ $\Leftrightarrow Ax = b$ 的

解有如下情形：

1. 有解 $\Leftrightarrow r(A \vdots b) = r(A)$
   - 有无穷多解 $\Leftrightarrow r(A) = r(A \vdots b) < n$
   - 只有唯一解 $\Leftrightarrow r(A) = r(A \vdots b) = n$
2. 无解 $\Leftrightarrow r(A) \neq r(A \vdots b) \Leftrightarrow r(A) < r(A \vdots b)$

总结

定理 方程组 $\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$ $\Leftrightarrow Ax = b$ 的

解有如下情形：

1. 有解 $\Leftrightarrow r(A \vdots b) = r(A)$

   - 有无穷多解 $\Leftrightarrow r(A) = r(A \vdots b) < n$
   - 只有唯一解 $\Leftrightarrow r(A) = r(A \vdots b) = n$

2. 无解 $\Leftrightarrow$ $r(A) \neq r(A \vdots b)$ $\Leftrightarrow r(A) < r(A \vdots b)$

注

- $r(A \vdots b) = r(A)$ 的值，相当于方程组中“独立”方程个数；此时

总结

定理 方程组 $\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$ $\Leftrightarrow Ax = b$ 的

解有如下情形：

1. 有解 $\Leftrightarrow r(A \vdots b) = r(A)$
   - 有无穷多解 $\Leftrightarrow r(A) = r(A \vdots b) < n$
   - 只有唯一解 $\Leftrightarrow r(A) = r(A \vdots b) = n$
2. 无解 $\Leftrightarrow$ $r(A) \neq r(A \vdots b)$ $\Leftrightarrow r(A) < r(A \vdots b)$

注

- $r(A \vdots b) = r(A)$ 的值，相当于方程组中“独立”方程个数；此时
- $n - r(A)$ 为自由变量的个数

练习 求解 $\begin{cases} x_1+2x_2+x_3+x_4+x_5=1 \\ 2x_1+4x_2+3x_3+x_4+x_5=3 \\ -x_1-2x_2+x_3+3x_4-3x_5=7 \\ 2x_3+5x_4-2x_5=9 \end{cases}$

练习 求解 $\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1 \\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3 \\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7 \\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$

解

$$(A \vdots b) = \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{array}\right)$$

练习 求解 $\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1 \\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3 \\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7 \\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$

解

$$(A \vdots b) = \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{array}\right) \xrightarrow[r_3+r_1]{r_2-2r_1}$$

练习 求解 $\begin{cases} x_1+2x_2+x_3+x_4+x_5=1 \\ 2x_1+4x_2+3x_3+x_4+x_5=3 \\ -x_1-2x_2+x_3+3x_4-3x_5=7 \\ 2x_3+5x_4-2x_5=9 \end{cases}$

解

$$(A \vdots b)=\left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{array}\right) \xrightarrow[r_3+r_1]{r_2-2r_1} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{array}\right)$$

练习 求解 $\begin{cases} x_1+2x_2+x_3+x_4+x_5=1 \\ 2x_1+4x_2+3x_3+x_4+x_5=3 \\ -x_1-2x_2+x_3+3x_4-3x_5=7 \\ 2x_3+5x_4-2x_5=9 \end{cases}$

解

$$(A \vdots b)=\left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{array}\right) \xrightarrow[r_3+r_1]{r_2-2r_1} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{array}\right)$$

$$\xrightarrow[r_4-2r_2]{r_3-2r_2}$$

练习 求解 $\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1 \\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3 \\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7 \\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$

解

$$(A \vdots b) = \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{array}\right) \xrightarrow[r_3+r_1]{r_2-2r_1} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{array}\right)$$

$$\xrightarrow[r_4-2r_2]{r_3-2r_2} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 6 & 0 & 6 \\ 0 & 0 & 0 & 7 & 0 & 7 \end{array}\right)$$

练习 求解 $\begin{cases} x_1+2x_2+x_3+x_4+x_5=1 \\ 2x_1+4x_2+3x_3+x_4+x_5=3 \\ -x_1-2x_2+x_3+3x_4-3x_5=7 \\ 2x_3+5x_4-2x_5=9 \end{cases}$

解

$$(A \vdots b)=\left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{array}\right) \xrightarrow[r_3+r_1]{r_2-2r_1} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{array}\right)$$

$$\xrightarrow[r_4-2r_2]{r_3-2r_2} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 6 & 0 & 6 \\ 0 & 0 & 0 & 7 & 0 & 7 \end{array}\right) \xrightarrow[\frac{1}{7}\times r_4]{\frac{1}{6}\times r_3}$$

练习 求解 $\begin{cases} x_1+2x_2+x_3+x_4+x_5=1 \\ 2x_1+4x_2+3x_3+x_4+x_5=3 \\ -x_1-2x_2+x_3+3x_4-3x_5=7 \\ 2x_3+5x_4-2x_5=9 \end{cases}$

解

$$(A\vdots b)=\left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{array}\right) \xrightarrow[r_3+r_1]{r_2-2r_1} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{array}\right)$$

$$\xrightarrow[r_4-2r_2]{r_3-2r_2} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 6 & 0 & 6 \\ 0 & 0 & 0 & 7 & 0 & 7 \end{array}\right) \xrightarrow[\frac{1}{7}\times r_4]{\frac{1}{6}\times r_3} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right)$$

练习 求解 $\begin{cases} x_1+ & 2x_2+ & x_3+ & x_4+ & x_5= & 1 \\ 2x_1+ & 4x_2+ & 3x_3+ & x_4+ & x_5= & 3 \\ -x_1- & 2x_2+ & x_3+ & 3x_4- & 3x_5= & 7 \\ & & 2x_3+ & 5x_4- & 2x_5= & 9 \end{cases}$

解

$$(A \vdots b)=\left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{array}\right) \xrightarrow[r_3+r_1]{r_2-2r_1} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{array}\right)$$

$$\xrightarrow[r_4-2r_2]{r_3-2r_2} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 6 & 0 & 6 \\ 0 & 0 & 0 & 7 & 0 & 7 \end{array}\right) \xrightarrow[\frac{1}{7}\times r_4]{\frac{1}{6}\times r_3} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{r_4-r_3}$$

练习 求解 $\begin{cases} x_1+2x_2+x_3+x_4+x_5=1 \\ 2x_1+4x_2+3x_3+x_4+x_5=3 \\ -x_1-2x_2+x_3+3x_4-3x_5=7 \\ 2x_3+5x_4-2x_5=9 \end{cases}$

解

$$(A\,\vdots\,b)=\left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{array}\right) \xrightarrow[r_3+r_1]{r_2-2r_1} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{array}\right)$$

$$\xrightarrow[r_4-2r_2]{r_3-2r_2} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 6 & 0 & 6 \\ 0 & 0 & 0 & 7 & 0 & 7 \end{array}\right) \xrightarrow[\frac{1}{7}\times r_4]{\frac{1}{6}\times r_3} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{r_4-r_3} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

练习 求解 $\begin{cases} x_1+2x_2+x_3+x_4+x_5=1 \\ 2x_1+4x_2+3x_3+x_4+x_5=3 \\ -x_1-2x_2+x_3+3x_4-3x_5=7 \\ 2x_3+5x_4-2x_5=9 \end{cases}$

解

$$(A \vdots b)=\left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{array}\right) \xrightarrow[r_3+r_1]{r_2-2r_1} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{array}\right)$$

$$\xrightarrow[r_4-2r_2]{r_3-2r_2} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 6 & 0 & 6 \\ 0 & 0 & 0 & 7 & 0 & 7 \end{array}\right) \xrightarrow[\frac{1}{7}\times r_4]{\frac{1}{6}\times r_3} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{r_4-r_3} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[r_1-r_3]{r_2+r_3}$$

练习 求解
$$\begin{cases} x_1+2x_2+x_3+x_4+x_5=1 \\ 2x_1+4x_2+3x_3+x_4+x_5=3 \\ -x_1-2x_2+x_3+3x_4-3x_5=7 \\ 2x_3+5x_4-2x_5=9 \end{cases}$$

解

$$(A\,\vdots\,b)=\left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{array}\right) \xrightarrow[r_3+r_1]{r_2-2r_1} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{array}\right)$$

$$\xrightarrow[r_4-2r_2]{r_3-2r_2} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 6 & 0 & 6 \\ 0 & 0 & 0 & 7 & 0 & 7 \end{array}\right) \xrightarrow[\frac{1}{7}\times r_4]{\frac{1}{6}\times r_3} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{r_4-r_3} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[r_1-r_3]{r_2+r_3} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

练习 求解 $\begin{cases} x_1+2x_2+x_3+x_4+x_5=1 \\ 2x_1+4x_2+3x_3+x_4+x_5=3 \\ -x_1-2x_2+x_3+3x_4-3x_5=7 \\ 2x_3+5x_4-2x_5=9 \end{cases}$

解

$$(A \vdots b) = \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{array}\right) \xrightarrow[r_3+r_1]{r_2-2r_1} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{array}\right)$$

$$\xrightarrow[r_4-2r_2]{r_3-2r_2} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 6 & 0 & 6 \\ 0 & 0 & 0 & 7 & 0 & 7 \end{array}\right) \xrightarrow[\frac{1}{7}\times r_4]{\frac{1}{6}\times r_3} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{r_4-r_3} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[r_1-r_3]{r_2+r_3} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{r_1-r_2}$$

练习 求解 $\begin{cases} x_1+2x_2+x_3+x_4+x_5=1 \\ 2x_1+4x_2+3x_3+x_4+x_5=3 \\ -x_1-2x_2+x_3+3x_4-3x_5=7 \\ 2x_3+5x_4-2x_5=9 \end{cases}$

解

$$(A\vdots b)=\left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{array}\right) \xrightarrow[r_3+r_1]{r_2-2r_1} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{array}\right)$$

$$\xrightarrow[r_4-2r_2]{r_3-2r_2} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 6 & 0 & 6 \\ 0 & 0 & 0 & 7 & 0 & 7 \end{array}\right) \xrightarrow[\frac{1}{7}\times r_4]{\frac{1}{6}\times r_3} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{r_4-r_3} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[r_1-r_3]{r_2+r_3} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{r_1-r_2} \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

练习 求解 $\begin{cases} x_1+2x_2+x_3+x_4+x_5=1 \\ 2x_1+4x_2+3x_3+x_4+x_5=3 \\ -x_1-2x_2+x_3+3x_4-3x_5=7 \\ 2x_3+5x_4-2x_5=9 \end{cases}$

解

$$(A\vdots b)=\left(\begin{array}{ccccc|c} 1&2&1&1&1&1\\ 2&4&3&1&1&3\\ -1&-2&1&3&-3&7\\ 0&0&2&5&-2&9 \end{array}\right) \xrightarrow[r_3+r_1]{r_2-2r_1} \left(\begin{array}{ccccc|c} 1&2&1&1&1&1\\ 0&0&1&-1&-1&1\\ 0&0&2&4&-2&8\\ 0&0&2&5&-2&9 \end{array}\right)$$

$$\xrightarrow[r_4-2r_2]{r_3-2r_2} \left(\begin{array}{ccccc|c} 1&2&1&1&1&1\\ 0&0&1&-1&-1&1\\ 0&0&0&6&0&6\\ 0&0&0&7&0&7 \end{array}\right) \xrightarrow[\frac{1}{7}\times r_4]{\frac{1}{6}\times r_3} \left(\begin{array}{ccccc|c} 1&2&1&1&1&1\\ 0&0&1&-1&-1&1\\ 0&0&0&1&0&1\\ 0&0&0&1&0&1 \end{array}\right)$$

$$\xrightarrow{r_4-r_3} \left(\begin{array}{ccccc|c} 1&2&1&1&1&1\\ 0&0&1&-1&-1&1\\ 0&0&0&1&0&1\\ 0&0&0&0&0&0 \end{array}\right) \xrightarrow[r_1-r_3]{r_2+r_3} \left(\begin{array}{ccccc|c} 1&2&1&0&1&0\\ 0&0&1&0&-1&2\\ 0&0&0&1&0&1\\ 0&0&0&0&0&0 \end{array}\right)$$

$$\xrightarrow{r_1-r_2} \left(\begin{array}{ccccc|c} 1&2&0&0&2&-2\\ 0&0&1&0&-1&2\\ 0&0&0&1&0&1\\ 0&0&0&0&0&0 \end{array}\right)$$

练习 求解 $\begin{cases} x_1+2x_2+x_3+x_4+x_5=1 \\ 2x_1+4x_2+3x_3+x_4+x_5=3 \\ -x_1-2x_2+x_3+3x_4-3x_5=7 \\ 2x_3+5x_4-2x_5=9 \end{cases}$

解

$$(A \vdots b) \longrightarrow \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

练习 求解 $\begin{cases} x_1+ & 2x_2+ & x_3+ & x_4+ & x_5 = & 1 \\ 2x_1+ & 4x_2+ & 3x_3+ & x_4+ & x_5 = & 3 \\ -x_1- & 2x_2+ & x_3+ & 3x_4- & 3x_5 = & 7 \\ & & 2x_3+ & 5x_4- & 2x_5 = & 9 \end{cases}$

解

$$(A \vdots b) \longrightarrow \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

- 可见 $r(A) = r(A \vdots b) = 3 < 5$，有无穷多的解，

练习 求解 $\begin{cases} x_1+2x_2+x_3+x_4+x_5=1 \\ 2x_1+4x_2+3x_3+x_4+x_5=3 \\ -x_1-2x_2+x_3+3x_4-3x_5=7 \\ 2x_3+5x_4-2x_5=9 \end{cases}$

解

$$(A \vdots b) \longrightarrow \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

- 可见 $r(A)=r(A \vdots b)=3<5$，有无穷多的解，含 $5-3=2$ 个自由变量

练习 求解 $\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1 \\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3 \\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7 \\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$

解

$$(A \vdots b) \longrightarrow \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

- 可见 $r(A) = r(A \vdots b) = 3 < 5$，有无穷多的解，含 $5 - 3 = 2$ 个自由变量
- 由既约阶梯形矩阵可知，原方程组等价于

$$\begin{cases} x_1 + 2x_2 + 2x_5 = -2 \\ x_3 - x_5 = 2 \\ x_4 = 1 \end{cases}$$

练习 求解 $\begin{cases} x_1+2x_2+x_3+x_4+x_5=1 \\ 2x_1+4x_2+3x_3+x_4+x_5=3 \\ -x_1-2x_2+x_3+3x_4-3x_5=7 \\ 2x_3+5x_4-2x_5=9 \end{cases}$

解

$$(A \vdots b) \longrightarrow \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

- 可见 $r(A) = r(A \vdots b) = 3 < 5$，有无穷多的解，含 $5-3=2$ 个自由变量
- 由既约阶梯形矩阵可知，原方程组等价于

$$\begin{cases} x_1+2x_2 \qquad\qquad + 2x_5 = -2 \\ \qquad\qquad x_3 \qquad - x_5 = 2 \\ \qquad\qquad\qquad x_4 \qquad = 1 \end{cases} \Rightarrow \begin{cases} x_1 = -2-2x_2-2x_5 \\ x_3 = 2+x_5 \\ x_4 = 1 \end{cases}$$

练习 求解 $\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1 \\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3 \\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7 \\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$

解

$$(A \vdots b) \longrightarrow \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

- 可见 $r(A) = r(A \vdots b) = 3 < 5$，有无穷多的解，含 $5 - 3 = 2$ 个自由变量
- 由既约阶梯形矩阵可知，原方程组等价于

$$\begin{cases} x_1 + 2x_2 + 2x_5 = -2 \\ x_3 - x_5 = 2 \\ x_4 = 1 \end{cases} \Rightarrow \begin{cases} x_1 = -2 - 2x_2 - 2x_5 \\ x_3 = 2 + x_5 \\ x_4 = 1 \end{cases}$$

所以通解是

$$\begin{cases} x_1 = \\ x_2 = \\ x_3 = \\ x_4 = \\ x_5 = \end{cases}$$

练习 求解
$$\begin{cases} x_1+ & 2x_2+ & x_3+ & x_4+ & x_5= & 1\\ 2x_1+ & 4x_2+ & 3x_3+ & x_4+ & x_5= & 3\\ -x_1- & 2x_2+ & x_3+ & 3x_4- & 3x_5= & 7\\ & & 2x_3+ & 5x_4- & 2x_5= & 9 \end{cases}$$

解

$$(A\vdots b)\longrightarrow\left(\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 2 & -2\\ 0 & 0 & 1 & 0 & -1 & 2\\ 0 & 0 & 0 & 1 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

- 可见 $r(A)=r(A\vdots b)=3<5$，有无穷多的解，含 $5-3=2$ 个自由变量
- 由既约阶梯形矩阵可知，原方程组等价于

$$\begin{cases} x_1+ \quad 2x_2 \qquad\qquad + \; 2x_5 = -2\\ \qquad\qquad x_3 \qquad - \; x_5 = 2\\ \qquad\qquad\qquad x_4 \qquad = 1 \end{cases} \Rightarrow \begin{cases} x_1=-2-2x_2-2x_5\\ x_3=2+x_5\\ x_4=1 \end{cases}$$

所以通解是

$$\begin{cases} x_1=\\ x_2=c_1\\ x_3=\\ x_4=\\ x_5=c_2 \end{cases} \qquad (c_1,\ c_2\text{为任意常数})$$

练习 求解 $\begin{cases} x_1+2x_2+x_3+x_4+x_5=1 \\ 2x_1+4x_2+3x_3+x_4+x_5=3 \\ -x_1-2x_2+x_3+3x_4-3x_5=7 \\ 2x_3+5x_4-2x_5=9 \end{cases}$

解

$$(A\vdots b)\longrightarrow\left(\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

- 可见 $r(A)=r(A\vdots b)=3<5$，有无穷多的解，含 $5-3=2$ 个自由变量
- 由既约阶梯形矩阵可知，原方程组等价于

$$\begin{cases} x_1+2x_2+2x_5=-2 \\ x_3-x_5=2 \\ x_4=1 \end{cases}\Rightarrow\begin{cases} x_1=-2-2x_2-2x_5 \\ x_3=2+x_5 \\ x_4=1 \end{cases}$$

所以通解是

$$\begin{cases} x_1=-2-2c_1-2c_2 \\ x_2=c_1 \\ x_3= \\ x_4= \\ x_5=c_2 \end{cases}\qquad (c_1,c_2\text{为任意常数})$$

练习 求解 $\begin{cases} x_1+2x_2+x_3+x_4+x_5=1 \\ 2x_1+4x_2+3x_3+x_4+x_5=3 \\ -x_1-2x_2+x_3+3x_4-3x_5=7 \\ 2x_3+5x_4-2x_5=9 \end{cases}$

解

$$(A\vdots b)\longrightarrow\left(\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

- 可见 $r(A)=r(A\vdots b)=3<5$，有无穷多的解，含 $5-3=2$ 个自由变量
- 由既约阶梯形矩阵可知，原方程组等价于

$$\begin{cases} x_1+2x_2+2x_5=-2 \\ x_3-x_5=2 \\ x_4=1 \end{cases} \Rightarrow \begin{cases} x_1=-2-2x_2-2x_5 \\ x_3=2+x_5 \\ x_4=1 \end{cases}$$

所以通解是

$$\begin{cases} x_1=-2-2c_1-2c_2 \\ x_2=c_1 \\ x_3=2+c_2 \\ x_4= \\ x_5=c_2 \end{cases} \qquad (c_1,c_2\text{为任意常数})$$

练习 求解 $\begin{cases} x_1+2x_2+x_3+x_4+x_5=1 \\ 2x_1+4x_2+3x_3+x_4+x_5=3 \\ -x_1-2x_2+x_3+3x_4-3x_5=7 \\ 2x_3+5x_4-2x_5=9 \end{cases}$

解

$$(A\vdots b)\longrightarrow\left(\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

- 可见 $r(A)=r(A\vdots b)=3<5$，有无穷多的解，含 $5-3=2$ 个自由变量
- 由既约阶梯形矩阵可知，原方程组等价于

$$\begin{cases} x_1+2x_2+2x_5=-2 \\ x_3-x_5=2 \\ x_4=1 \end{cases} \Rightarrow \begin{cases} x_1=-2-2x_2-2x_5 \\ x_3=2+x_5 \\ x_4=1 \end{cases}$$

所以通解是

$$\begin{cases} x_1=-2-2c_1-2c_2 \\ x_2=c_1 \\ x_3=2+c_2 \\ x_4=1 \\ x_5=c_2 \end{cases} \qquad (c_1,\ c_2\text{为任意常数})$$

例 讨论 a, b 取何值时，方程组

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 2x_4 = 1 \\ -x_2 + (a-3)x_3 - 2x_4 = b \\ 3x_1 + 2x_2 + x_3 + ax_4 = -1 \end{cases}$$

有无穷解、唯一解，及无解？

例 讨论 a, b 取何值时，方程组

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 2x_4 = 1 \\ -x_2 + (a-3)x_3 - 2x_4 = b \\ 3x_1 + 2x_2 + x_3 + ax_4 = -1 \end{cases}$$

有无穷解、唯一解，及无解？

解

$$(A \vdots b) = \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & -1 & a-3 & -2 & b \\ 3 & 2 & 1 & a & -1 \end{array}\right)$$

例 讨论 a, b 取何值时，方程组

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 2x_4 = 1 \\ -x_2 + (a-3)x_3 - 2x_4 = b \\ 3x_1 + 2x_2 + x_3 + ax_4 = -1 \end{cases}$$

有无穷解、唯一解，及无解？

解

$$(A \vdots b) = \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & -1 & a-3 & -2 & b \\ 3 & 2 & 1 & a & -1 \end{array}\right) \xrightarrow{r_4 - 3r_1}$$

例 讨论 a, b 取何值时，方程组

$$\begin{cases} x_1+x_2+x_3+x_4=0 \\ x_2+2x_3+2x_4=1 \\ -x_2+(a-3)x_3-2x_4=b \\ 3x_1+2x_2+x_3+ax_4=-1 \end{cases}$$

有无穷解、唯一解，及无解？

解

$$(A \vdots b) = \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & -1 & a-3 & -2 & b \\ 3 & 2 & 1 & a & -1 \end{array}\right) \xrightarrow{r_4-3r_1} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & -1 & a-3 & -2 & b \\ 0 & -1 & -2 & a-3 & -1 \end{array}\right)$$

例 讨论 a, b 取何值时，方程组

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 2x_4 = 1 \\ -x_2 + (a-3)x_3 - 2x_4 = b \\ 3x_1 + 2x_2 + x_3 + ax_4 = -1 \end{cases}$$

有无穷解、唯一解，及无解？

解

$$(A \vdots b) = \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & -1 & a-3 & -2 & b \\ 3 & 2 & 1 & a & -1 \end{array}\right) \xrightarrow{r_4 - 3r_1} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & -1 & a-3 & -2 & b \\ 0 & -1 & -2 & a-3 & -1 \end{array}\right)$$

$$\xrightarrow[r_4 + r_2]{r_3 + r_2}$$

例 讨论 a, b 取何值时，方程组

$$\begin{cases} x_1+x_2+x_3+x_4=0 \\ x_2+2x_3+2x_4=1 \\ -x_2+(a-3)x_3-2x_4=b \\ 3x_1+2x_2+x_3+ax_4=-1 \end{cases}$$

有无穷解、唯一解，及无解？

解

$$(A\vdots b)=\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & -1 & a-3 & -2 & b \\ 3 & 2 & 1 & a & -1 \end{array}\right) \xrightarrow{r_4-3r_1} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & -1 & a-3 & -2 & b \\ 0 & -1 & -2 & a-3 & -1 \end{array}\right)$$

$$\xrightarrow[r_4+r_2]{r_3+r_2} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & a-1 & 0 & b+1 \\ 0 & 0 & 0 & a-1 & 0 \end{array}\right)$$

例 讨论 a, b 取何值时，方程组

$$\begin{cases} x_1+ & x_2+ & x_3+ & x_4= & 0 \\ & x_2+ & 2x_3+ & 2x_4= & 1 \\ & -x_2+ & (a-3)x_3- & 2x_4= & b \\ 3x_1+ & 2x_2+ & x_3+ & ax_4= & -1 \end{cases}$$

有无穷解、唯一解，及无解？

解

$$(A\vdots b)\longrightarrow\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & a-1 & 0 & b+1 \\ 0 & 0 & 0 & a-1 & 0 \end{array}\right)$$

例 讨论 a, b 取何值时，方程组

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 2x_4 = 1 \\ -x_2 + (a-3)x_3 - 2x_4 = b \\ 3x_1 + 2x_2 + x_3 + ax_4 = -1 \end{cases}$$

有无穷解、唯一解，及无解？

解

$$(A \vdots b) \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & a-1 & 0 & b+1 \\ 0 & 0 & 0 & a-1 & 0 \end{array}\right)$$

- 当 $a \neq 1$ 时
- 当 $a = 1$ 时

例 讨论 a, b 取何值时，方程组

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 2x_4 = 1 \\ -x_2 + (a-3)x_3 - 2x_4 = b \\ 3x_1 + 2x_2 + x_3 + ax_4 = -1 \end{cases}$$

有无穷解、唯一解，及无解？

解

$$(A \vdots b) \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & a-1 & 0 & b+1 \\ 0 & 0 & 0 & a-1 & 0 \end{array}\right)$$

- 当 $a \neq 1$ 时
- 当 $a = 1$ 时

例 讨论 a, b 取何值时，方程组

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 2x_4 = 1 \\ -x_2 + (a-3)x_3 - 2x_4 = b \\ 3x_1 + 2x_2 + x_3 + ax_4 = -1 \end{cases}$$

有无穷解、唯一解，及无解？

解

$$(A \vdots b) \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & a-1 & 0 & b+1 \\ 0 & 0 & 0 & a-1 & 0 \end{array}\right)$$

- 当 $a \neq 1$ 时（b 为任意数），$r(A) = r(A \vdots b) = 4$，
- 当 $a = 1$ 时

例 讨论 a, b 取何值时，方程组

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 2x_4 = 1 \\ -x_2 + (a-3)x_3 - 2x_4 = b \\ 3x_1 + 2x_2 + x_3 + ax_4 = -1 \end{cases}$$

有无穷解、唯一解，及无解？

解

$$(A \vdots b) \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & a-1 & 0 & b+1 \\ 0 & 0 & 0 & a-1 & 0 \end{array}\right)$$

- 当 $a \neq 1$ 时（b 为任意数），$r(A) = r(A \vdots b) = 4$，有唯一解；
- 当 $a = 1$ 时

例 讨论 a, b 取何值时，方程组

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 2x_4 = 1 \\ -x_2 + (a-3)x_3 - 2x_4 = b \\ 3x_1 + 2x_2 + x_3 + ax_4 = -1 \end{cases}$$

有无穷解、唯一解，及无解？

解

$$(A \vdots b) \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & a-1 & 0 & b+1 \\ 0 & 0 & 0 & a-1 & 0 \end{array}\right)$$

- 当 $a \neq 1$ 时（b 为任意数），$r(A) = r(A \vdots b) = 4$，有唯一解；
- 当 $a = 1$ 时

例 讨论 a, b 取何值时，方程组

$$\begin{cases} x_1+ & x_2+ & x_3+ & x_4 = & 0 \\ & x_2+ & 2x_3+ & 2x_4 = & 1 \\ & -x_2+ & (a-3)x_3- & 2x_4 = & b \\ 3x_1+ & 2x_2+ & x_3+ & ax_4 = & -1 \end{cases}$$

有无穷解、唯一解，及无解？

解

$$(A \vdots b) \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & a-1 & 0 & b+1 \\ 0 & 0 & 0 & a-1 & 0 \end{array}\right)$$

- 当 $a \neq 1$ 时（b 为任意数），$r(A) = r(A \vdots b) = 4$，有唯一解；
- 当 $a = 1$ 时

$$(A \vdots b) \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & b+1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

例 讨论 a, b 取何值时，方程组

$$\begin{cases} x_1+ x_2+ x_3+ x_4 = 0 \\ x_2+ 2x_3+ 2x_4 = 1 \\ -x_2+ (a-3)x_3- 2x_4 = b \\ 3x_1+ 2x_2+ x_3+ ax_4 = -1 \end{cases}$$

有无穷解、唯一解，及无解？

解

$$(A \vdots b) \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & a-1 & 0 & b+1 \\ 0 & 0 & 0 & a-1 & 0 \end{array}\right)$$

- 当 $a \neq 1$ 时（b 为任意数），$r(A) = r(A \vdots b) = 4$，有唯一解；
- 当 $a = 1$ 时

$$(A \vdots b) \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & b+1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

  - $a = 1, b = -1$ 时
  - $a = 1, b \neq -1$ 时

例 讨论 a, b 取何值时，方程组

$$\begin{cases} x_1+ & x_2+ & x_3+ & x_4 = & 0 \\ & x_2+ & 2x_3+ & 2x_4 = & 1 \\ & -x_2+ & (a-3)x_3- & 2x_4 = & b \\ 3x_1+ & 2x_2+ & x_3+ & ax_4 = & -1 \end{cases}$$

有无穷解、唯一解，及无解？

解

$$(A\vdots b) \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & a-1 & 0 & b+1 \\ 0 & 0 & 0 & a-1 & 0 \end{array}\right)$$

- 当 $a \neq 1$ 时（b 为任意数），$r(A) = r(A\vdots b) = 4$，有唯一解；
- 当 $a = 1$ 时

$$(A\vdots b) \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & b+1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

  - $a = 1, b = -1$ 时
  - $a = 1, b \neq -1$ 时

例 讨论 a, b 取何值时，方程组

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 2x_4 = 1 \\ -x_2 + (a-3)x_3 - 2x_4 = b \\ 3x_1 + 2x_2 + x_3 + ax_4 = -1 \end{cases}$$

有无穷解、唯一解，及无解？

解

$$(A \vdots b) \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & a-1 & 0 & b+1 \\ 0 & 0 & 0 & a-1 & 0 \end{array}\right)$$

- 当 $a \neq 1$ 时（b 为任意数），$r(A) = r(A \vdots b) = 4$，有唯一解；
- 当 $a = 1$ 时

$$(A \vdots b) \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & b+1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

  - $a = 1, b = -1$ 时，$r(A) = r(A \vdots b) = 2 < 4$，
  - $a = 1, b \neq -1$ 时

例 讨论 a, b 取何值时，方程组

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 2x_4 = 1 \\ -x_2 + (a-3)x_3 - 2x_4 = b \\ 3x_1 + 2x_2 + x_3 + ax_4 = -1 \end{cases}$$

有无穷解、唯一解，及无解？

解

$$(A \vdots b) \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & a-1 & 0 & b+1 \\ 0 & 0 & 0 & a-1 & 0 \end{array}\right)$$

- 当 $a \neq 1$ 时（b 为任意数），$r(A) = r(A \vdots b) = 4$，有唯一解；
- 当 $a = 1$ 时

$$(A \vdots b) \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & b+1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

  - $a = 1, b = -1$ 时，$r(A) = r(A \vdots b) = 2 < 4$，有无穷多解
  - $a = 1, b \neq -1$ 时

例 讨论 a, b 取何值时，方程组

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 2x_4 = 1 \\ -x_2 + (a-3)x_3 - 2x_4 = b \\ 3x_1 + 2x_2 + x_3 + ax_4 = -1 \end{cases}$$

有无穷解、唯一解，及无解？

解

$$(A \vdots b) \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & a-1 & 0 & b+1 \\ 0 & 0 & 0 & a-1 & 0 \end{array}\right)$$

- 当 $a \neq 1$ 时（b 为任意数），$r(A) = r(A \vdots b) = 4$，有唯一解；
- 当 $a = 1$ 时

$$(A \vdots b) \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & b+1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

  - $a = 1, b = -1$ 时，$r(A) = r(A \vdots b) = 2 < 4$，有无穷多解
  - $a = 1, b \neq -1$ 时

例 讨论 a, b 取何值时，方程组

$$\begin{cases} x_1+ x_2+ x_3+ x_4 = 0 \\ x_2+ 2x_3+ 2x_4 = 1 \\ -x_2+ (a-3)x_3- 2x_4 = b \\ 3x_1+ 2x_2+ x_3+ ax_4 = -1 \end{cases}$$

有无穷解、唯一解，及无解？

解

$$(A\vdots b)\longrightarrow\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & a-1 & 0 & b+1 \\ 0 & 0 & 0 & a-1 & 0 \end{array}\right)$$

- 当 $a \neq 1$ 时（b 为任意数），$r(A) = r(A\vdots b) = 4$，有唯一解；
- 当 $a = 1$ 时

$$(A\vdots b)\longrightarrow\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & b+1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

  - $a = 1, b = -1$ 时，$r(A) = r(A\vdots b) = 2 < 4$，有无穷多解
  - $a = 1, b \neq -1$ 时，$r(A) = 2 < 3 = r(A\vdots b)$，

例 讨论 a, b 取何值时，方程组

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 2x_4 = 1 \\ -x_2 + (a-3)x_3 - 2x_4 = b \\ 3x_1 + 2x_2 + x_3 + ax_4 = -1 \end{cases}$$

有无穷解、唯一解，及无解？

解

$$(A \vdots b) \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & a-1 & 0 & b+1 \\ 0 & 0 & 0 & a-1 & 0 \end{array}\right)$$

- 当 $a \neq 1$ 时（b 为任意数），$r(A) = r(A \vdots b) = 4$，有唯一解；
- 当 $a = 1$ 时

$$(A \vdots b) \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & b+1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

  - $a = 1, b = -1$ 时，$r(A) = r(A \vdots b) = 2 < 4$，有无穷多解
  - $a = 1, b \neq -1$ 时，$r(A) = 2 < 3 = r(A \vdots b)$，无解

例 讨论 a, b 取何值时，方程组 $\begin{cases} x_1+2x_2+3x_3=1 \\ 2x_1+3x_2+5x_3=-1 \\ 3x_1+4x_2+ax_3=b \end{cases}$ 有无穷解、唯一解，及无解？

例 讨论 a, b 取何值时，方程组 $\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \\ 3x_1 + 4x_2 + ax_3 = b \end{cases}$ 有无穷解、唯一解，及无解？

解

$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 3 & 5 & -1 \\ 3 & 4 & a & b \end{array}\right)$$

例 讨论 a, b 取何值时，方程组 $\begin{cases} x_1+ & 2x_2+ & 3x_3 = & 1 \\ 2x_1+ & 3x_2+ & 5x_3 = & -1 \\ 3x_1+ & 4x_2+ & ax_3 = & b \end{cases}$ 有无穷解、唯一解，及无解？

解

$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 3 & 5 & -1 \\ 3 & 4 & a & b \end{array}\right) \xrightarrow[r_3-3r_1]{r_2-2r_1}$$

例 讨论 a, b 取何值时，方程组 $\begin{cases} x_1+ \ 2x_2+ \ 3x_3 = 1 \\ 2x_1+ \ 3x_2+ \ 5x_3 = -1 \\ 3x_1+ \ 4x_2+ \ ax_3 = b \end{cases}$ 有无穷解、唯一解，及无解？

解

$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 3 & 5 & -1 \\ 3 & 4 & a & b \end{array}\right) \xrightarrow[r_3-3r_1]{r_2-2r_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ & & & \\ & & & \end{array}\right)$$

例 讨论 a, b 取何值时，方程组 $\begin{cases} x_1+ & 2x_2+ & 3x_3 = & 1 \\ 2x_1+ & 3x_2+ & 5x_3 = & -1 \\ 3x_1+ & 4x_2+ & ax_3 = & b \end{cases}$ 有无穷解、唯一解，及无解？

解

$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 3 & 5 & -1 \\ 3 & 4 & a & b \end{array}\right) \xrightarrow[r_3-3r_1]{r_2-2r_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ & & & \end{array}\right)$$

例 讨论 a, b 取何值时，方程组 $\begin{cases} x_1+2x_2+3x_3=1 \\ 2x_1+3x_2+5x_3=-1 \\ 3x_1+4x_2+ax_3=b \end{cases}$ 有无穷解、唯一解，及无解？

解

$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 3 & 5 & -1 \\ 3 & 4 & a & b \end{array}\right) \xrightarrow[r_3-3r_1]{r_2-2r_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & -2 & a-9 & b-3 \end{array}\right)$$

例 讨论 a, b 取何值时，方程组 $\begin{cases} x_1+2x_2+3x_3=1 \\ 2x_1+3x_2+5x_3=-1 \\ 3x_1+4x_2+ax_3=b \end{cases}$ 有无穷解、唯一解，及无解？

解

$$(A\vdots b)=\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 3 & 5 & -1 \\ 3 & 4 & a & b \end{array}\right) \xrightarrow[r_3-3r_1]{r_2-2r_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & -2 & a-9 & b-3 \end{array}\right)$$

$$\xrightarrow{r_3-2r_2}$$

例 讨论 a, b 取何值时，方程组 $\begin{cases} x_1+2x_2+3x_3=1 \\ 2x_1+3x_2+5x_3=-1 \\ 3x_1+4x_2+ax_3=b \end{cases}$ 有无穷解、唯一解，及无解？

解

$$(A\vdots b)=\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 3 & 5 & -1 \\ 3 & 4 & a & b \end{array}\right) \xrightarrow[r_3-3r_1]{r_2-2r_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & -2 & a-9 & b-3 \end{array}\right)$$

$$\xrightarrow{r_3-2r_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & a-7 & b+3 \end{array}\right)$$

例 讨论 a, b 取何值时，方程组 $\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \\ 3x_1 + 4x_2 + ax_3 = b \end{cases}$ 有无穷解、唯一解，及无解？

解

$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 3 & 5 & -1 \\ 3 & 4 & a & b \end{array}\right) \xrightarrow[r_3-3r_1]{r_2-2r_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & -2 & a-9 & b-3 \end{array}\right)$$

$$\xrightarrow{r_3-2r_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & a-7 & b+3 \end{array}\right)$$

- 当 $a \neq 7$ 时
- 当 $a = 7$ 时

例 讨论 a, b 取何值时，方程组 $\begin{cases} x_1+2x_2+3x_3=1 \\ 2x_1+3x_2+5x_3=-1 \\ 3x_1+4x_2+ax_3=b \end{cases}$ 有无穷解、唯一解，及无解？

解

$$(A\,\vdots\,b)=\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 3 & 5 & -1 \\ 3 & 4 & a & b \end{array}\right) \xrightarrow[r_3-3r_1]{r_2-2r_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & -2 & a-9 & b-3 \end{array}\right)$$

$$\xrightarrow{r_3-2r_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & a-7 & b+3 \end{array}\right)$$

- 当 $a \neq 7$ 时
- 当 $a = 7$ 时

例 讨论 a, b 取何值时，方程组 $\begin{cases} x_1+ & 2x_2+ & 3x_3 = & 1 \\ 2x_1+ & 3x_2+ & 5x_3 = & -1 \\ 3x_1+ & 4x_2+ & ax_3 = & b \end{cases}$ 有无穷解、唯一解，及无解？

解

$$(A\vdots b) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 3 & 5 & -1 \\ 3 & 4 & a & b \end{array}\right) \xrightarrow[r_3-3r_1]{r_2-2r_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & -2 & a-9 & b-3 \end{array}\right)$$

$$\xrightarrow{r_3-2r_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & a-7 & b+3 \end{array}\right)$$

- 当 $a \neq 7$ 时（b 为任意数），$r(A\vdots b) = r(A) = 3$，
- 当 $a = 7$ 时

例 讨论 a, b 取何值时，方程组 $\begin{cases} x_1+ & 2x_2+ & 3x_3 = & 1 \\ 2x_1+ & 3x_2+ & 5x_3 = & -1 \\ 3x_1+ & 4x_2+ & ax_3 = & b \end{cases}$ 有无穷解、唯一解，及无解？

解

$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 3 & 5 & -1 \\ 3 & 4 & a & b \end{array}\right) \xrightarrow[r_3-3r_1]{r_2-2r_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & -2 & a-9 & b-3 \end{array}\right)$$

$$\xrightarrow{r_3-2r_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & a-7 & b+3 \end{array}\right)$$

- 当 $a \neq 7$ 时（b 为任意数），$r(A \vdots b) = r(A) = 3$，有唯一解；
- 当 $a = 7$ 时

例 讨论 a, b 取何值时，方程组 $\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \\ 3x_1 + 4x_2 + ax_3 = b \end{cases}$ 有无穷解、唯一解，及无解？

解

$$(A\vdots b) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 3 & 5 & -1 \\ 3 & 4 & a & b \end{array}\right) \xrightarrow[r_3-3r_1]{r_2-2r_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & -2 & a-9 & b-3 \end{array}\right)$$

$$\xrightarrow{r_3-2r_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & a-7 & b+3 \end{array}\right)$$

- 当 $a \neq 7$ 时（b 为任意数），$r(A\vdots b) = r(A) = 3$，有唯一解；
- 当 $a = 7$ 时

例 讨论 a, b 取何值时，方程组 $\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \\ 3x_1 + 4x_2 + ax_3 = b \end{cases}$ 有无穷解、唯一解，及无解？

解

$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 3 & 5 & -1 \\ 3 & 4 & a & b \end{array}\right) \xrightarrow[r_3-3r_1]{r_2-2r_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & -2 & a-9 & b-3 \end{array}\right)$$

$$\xrightarrow{r_3-2r_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & a-7 & b+3 \end{array}\right)$$

- 当 $a \neq 7$ 时（b 为任意数），$r(A \vdots b) = r(A) = 3$，有唯一解；
- 当 $a = 7$ 时

$$(A \vdots b) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & b+3 \end{array}\right)$$

例 讨论 a, b 取何值时，方程组 $\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \\ 3x_1 + 4x_2 + ax_3 = b \end{cases}$ 有无穷解、唯一解，及无解？

解

$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 3 & 5 & -1 \\ 3 & 4 & a & b \end{array}\right) \xrightarrow[r_3-3r_1]{r_2-2r_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & -2 & a-9 & b-3 \end{array}\right)$$

$$\xrightarrow{r_3-2r_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & a-7 & b+3 \end{array}\right)$$

- 当 $a \neq 7$ 时（b 为任意数），$r(A \vdots b) = r(A) = 3$，有唯一解；
- 当 $a = 7$ 时

$$(A \vdots b) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & b+3 \end{array}\right)$$

  - $a = 7, b = -3$ 时
  - $a = 7, b \neq -3$ 时

例 讨论 a, b 取何值时，方程组 $\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \\ 3x_1 + 4x_2 + ax_3 = b \end{cases}$ 有无穷解、唯一解，及无解？

解

$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 3 & 5 & -1 \\ 3 & 4 & a & b \end{array}\right) \xrightarrow[r_3 - 3r_1]{r_2 - 2r_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & -2 & a-9 & b-3 \end{array}\right)$$

$$\xrightarrow{r_3 - 2r_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & a-7 & b+3 \end{array}\right)$$

- 当 $a \neq 7$ 时（b 为任意数），$r(A \vdots b) = r(A) = 3$，有唯一解；
- 当 $a = 7$ 时 $$(A \vdots b) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & b+3 \end{array}\right)$$
  - $a = 7, b = -3$ 时
  - $a = 7, b \neq -3$ 时

例 讨论 a, b 取何值时，方程组 $\begin{cases} x_1+2x_2+3x_3=1 \\ 2x_1+3x_2+5x_3=-1 \\ 3x_1+4x_2+ax_3=b \end{cases}$ 有无穷解、唯一解，及无解？

解

$$(A\vdots b)=\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 3 & 5 & -1 \\ 3 & 4 & a & b \end{array}\right) \xrightarrow[r_3-3r_1]{r_2-2r_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & -2 & a-9 & b-3 \end{array}\right)$$

$$\xrightarrow{r_3-2r_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & a-7 & b+3 \end{array}\right)$$

- 当 $a \neq 7$ 时（b 为任意数），$r(A\vdots b)=r(A)=3$，有唯一解；
- 当 $a=7$ 时 $$(A\vdots b) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & b+3 \end{array}\right)$$
  - $a=7, b=-3$ 时，$r(A\vdots b)=r(A)=2<3$，
  - $a=7, b\neq -3$ 时

例 讨论 a, b 取何值时，方程组 $\begin{cases} x_1+\ 2x_2+\ 3x_3 = \ \ 1 \\ 2x_1+\ 3x_2+\ 5x_3 = -1 \\ 3x_1+\ 4x_2+\ ax_3 = \ \ b \end{cases}$ 有无穷解、唯一解，及无解？

解

$$(A\vdots b) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 3 & 5 & -1 \\ 3 & 4 & a & b \end{array}\right) \xrightarrow[r_3-3r_1]{r_2-2r_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & -2 & a-9 & b-3 \end{array}\right)$$

$$\xrightarrow{r_3-2r_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & a-7 & b+3 \end{array}\right)$$

- 当 $a \neq 7$ 时（b 为任意数），$r(A\vdots b) = r(A) = 3$，有唯一解；
- 当 $a = 7$ 时

$$(A\vdots b) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & b+3 \end{array}\right)$$

  - $a = 7, b = -3$ 时，$r(A\vdots b) = r(A) = 2 < 3$，有无穷多解
  - $a = 7, b \neq -3$ 时

例 讨论 a, b 取何值时，方程组 $\begin{cases} x_1+2x_2+3x_3=1 \\ 2x_1+3x_2+5x_3=-1 \\ 3x_1+4x_2+ax_3=b \end{cases}$ 有无穷解、唯一解，及无解？

解

$$(A\vdots b)=\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 3 & 5 & -1 \\ 3 & 4 & a & b \end{array}\right) \xrightarrow[r_3-3r_1]{r_2-2r_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & -2 & a-9 & b-3 \end{array}\right)$$

$$\xrightarrow{r_3-2r_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & a-7 & b+3 \end{array}\right)$$

- 当 $a \neq 7$ 时（b 为任意数），$r(A\vdots b)=r(A)=3$，有唯一解；
- 当 $a=7$ 时

$$(A\vdots b) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & b+3 \end{array}\right)$$

  - $a=7, b=-3$ 时，$r(A\vdots b)=r(A)=2<3$，有无穷多解
  - $a=7, b\neq-3$ 时

例 讨论 a, b 取何值时，方程组 $\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \\ 3x_1 + 4x_2 + ax_3 = b \end{cases}$ 有无穷解、唯一解，及无解？

解

$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 3 & 5 & -1 \\ 3 & 4 & a & b \end{array}\right) \xrightarrow[r_3 - 3r_1]{r_2 - 2r_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & -2 & a-9 & b-3 \end{array}\right)$$

$$\xrightarrow{r_3 - 2r_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & a-7 & b+3 \end{array}\right)$$

- 当 $a \neq 7$ 时（b 为任意数），$r(A \vdots b) = r(A) = 3$，有唯一解；
- 当 $a = 7$ 时 $$(A \vdots b) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & b+3 \end{array}\right)$$
  - $a = 7, b = -3$ 时，$r(A \vdots b) = r(A) = 2 < 3$，有无穷多解
  - $a = 7, b \neq -3$ 时

例 讨论 a, b 取何值时，方程组 $\begin{cases} x_1+2x_2+3x_3=1 \\ 2x_1+3x_2+5x_3=-1 \\ 3x_1+4x_2+ax_3=b \end{cases}$ 有无穷解、唯一解，及无解？

解

$$(A\vdots b)=\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 3 & 5 & -1 \\ 3 & 4 & a & b \end{array}\right) \xrightarrow[r_3-3r_1]{r_2-2r_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & -2 & a-9 & b-3 \end{array}\right)$$

$$\xrightarrow{r_3-2r_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & a-7 & b+3 \end{array}\right)$$

- 当 $a \neq 7$ 时（b 为任意数），$r(A\vdots b)=r(A)=3$，有唯一解；
- 当 $a=7$ 时 $$(A\vdots b) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & b+3 \end{array}\right)$$
  - $a=7, b=-3$ 时，$r(A\vdots b)=r(A)=2<3$，有无穷多解
  - $a=7, b\neq -3$ 时，$r(A\vdots b)=3\neq 2=r(A)$，

例 讨论 a, b 取何值时，方程组 $\begin{cases} x_1+2x_2+3x_3=1 \\ 2x_1+3x_2+5x_3=-1 \\ 3x_1+4x_2+ax_3=b \end{cases}$ 有无穷解、唯一解，及无解？

解

$$(A\vdots b)=\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 3 & 5 & -1 \\ 3 & 4 & a & b \end{array}\right) \xrightarrow[r_3-3r_1]{r_2-2r_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & -2 & a-9 & b-3 \end{array}\right)$$

$$\xrightarrow{r_3-2r_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & a-7 & b+3 \end{array}\right)$$

- 当 $a\neq 7$ 时（b 为任意数），$r(A\vdots b)=r(A)=3$，有唯一解；
- 当 $a=7$ 时 $$(A\vdots b)\longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & b+3 \end{array}\right)$$
  - $a=7, b=-3$ 时，$r(A\vdots b)=r(A)=2<3$，有无穷多解
  - $a=7, b\neq -3$ 时，$r(A\vdots b)=3\neq 2=r(A)$，无解

线性方程组解的分类

- 一般线性方程组 $A_{m\times n}x=b$（m 个方程，n 个未知量）

$Ax=b$	有无穷解	有唯一解	无解
	$r(A)=r(A\vdots b)<n$	$r(A)=r(A\vdots b)=n$	$r(A)<r(A\vdots b)$

线性方程组解的分类

- 一般线性方程组 $A_{m\times n}x = b$（m 个方程，n 个未知量）

$Ax = b$	有无穷解	有唯一解	无解
	$r(A) = r(A \vdots b) < n$	$r(A) = r(A \vdots b) = n$	$r(A) < r(A \vdots b)$

- 齐次线性方程组 $A_{m\times n}x = 0$，一定有 $r(A) = r(A \vdots 0)$，所以一定有解（至少有零解）

线性方程组解的分类

- 一般线性方程组 $A_{m\times n}x = b$（m 个方程，n 个未知量）

$Ax = b$	有无穷解	有唯一解	无解
	$r(A) = r(A \vdots b) < n$	$r(A) = r(A \vdots b) = n$	$r(A) < r(A \vdots b)$

- 齐次线性方程组 $A_{m\times n}x = 0$，一定有 $r(A) = r(A \vdots 0)$，所以一定有解（至少有零解）

$Ax = 0$	有无穷解	有唯一解（零解）

线性方程组解的分类

- 一般线性方程组 $A_{m\times n}x=b$（m 个方程，n 个未知量）

$Ax=b$	有无穷解	有唯一解	无解
	$r(A)=r(A\vdots b)<n$	$r(A)=r(A\vdots b)=n$	$r(A)<r(A\vdots b)$

- 齐次线性方程组 $A_{m\times n}x=0$，一定有 $r(A)=r(A\vdots 0)$，所以一定有解（至少有零解）

$Ax=0$	有无穷解	有唯一解（零解）
	$r(A)<n$	

线性方程组解的分类

- 一般线性方程组 $A_{m\times n}x=b$（m 个方程，n 个未知量）

$Ax=b$	有无穷解	有唯一解	无解
	$r(A)=r(A\vdots b)<n$	$r(A)=r(A\vdots b)=n$	$r(A)<r(A\vdots b)$

- 齐次线性方程组 $A_{m\times n}x=0$，一定有 $r(A)=r(A\vdots 0)$，所以一定有解（至少有零解）

$Ax=0$	有无穷解	有唯一解（零解）
	$r(A)<n$	$r(A)=n$

例 解齐次线性方程组 $\begin{cases} x_1 - x_2 + 5x_3 - x_4 = 0 \\ x_1 + x_2 - 2x_3 + 3x_4 = 0 \\ 3x_1 - x_2 + 8x_3 + x_4 = 0 \\ x_1 + 3x_2 - 9x_3 + 7x_4 = 0 \end{cases}$

例 解齐次线性方程组 $\begin{cases} x_1 - x_2 + 5x_3 - x_4 = 0 \\ x_1 + x_2 - 2x_3 + 3x_4 = 0 \\ 3x_1 - x_2 + 8x_3 + x_4 = 0 \\ x_1 + 3x_2 - 9x_3 + 7x_4 = 0 \end{cases}$

解

$$(A \vdots b) = \left(\begin{array}{cccc|c} 1 & -1 & 5 & -1 & 0 \\ 1 & 1 & -2 & 3 & 0 \\ 3 & -1 & 8 & 1 & 0 \\ 1 & 3 & -9 & 7 & 0 \end{array}\right)$$

例 解齐次线性方程组
$$\begin{cases} x_1 - x_2 + 5x_3 - x_4 = 0 \\ x_1 + x_2 - 2x_3 + 3x_4 = 0 \\ 3x_1 - x_2 + 8x_3 + x_4 = 0 \\ x_1 + 3x_2 - 9x_3 + 7x_4 = 0 \end{cases}$$

解

$$(A \vdots b) = \left(\begin{array}{cccc|c} 1 & -1 & 5 & -1 & 0 \\ 1 & 1 & -2 & 3 & 0 \\ 3 & -1 & 8 & 1 & 0 \\ 1 & 3 & -9 & 7 & 0 \end{array}\right) \xrightarrow[\substack{r_3 - 3r_1 \\ r_4 - r_1}]{r_2 - r_1}$$

例 解齐次线性方程组 $\begin{cases} x_1 - x_2 + 5x_3 - x_4 = 0 \\ x_1 + x_2 - 2x_3 + 3x_4 = 0 \\ 3x_1 - x_2 + 8x_3 + x_4 = 0 \\ x_1 + 3x_2 - 9x_3 + 7x_4 = 0 \end{cases}$

解

$$(A \vdots b) = \left(\begin{array}{cccc|c} 1 & -1 & 5 & -1 & 0 \\ 1 & 1 & -2 & 3 & 0 \\ 3 & -1 & 8 & 1 & 0 \\ 1 & 3 & -9 & 7 & 0 \end{array}\right) \xrightarrow[\substack{r_3 - 3r_1 \\ r_4 - r_1}]{r_2 - r_1} \left(\begin{array}{cccc|c} 1 & -1 & 5 & -1 & 0 \\ & & & & \\ & & & & \\ & & & & \end{array}\right)$$

例 解齐次线性方程组
$$\begin{cases} x_1 - x_2 + 5x_3 - x_4 = 0 \\ x_1 + x_2 - 2x_3 + 3x_4 = 0 \\ 3x_1 - x_2 + 8x_3 + x_4 = 0 \\ x_1 + 3x_2 - 9x_3 + 7x_4 = 0 \end{cases}$$

解

$$(A \vdots b) = \left(\begin{array}{cccc|c} 1 & -1 & 5 & -1 & 0 \\ 1 & 1 & -2 & 3 & 0 \\ 3 & -1 & 8 & 1 & 0 \\ 1 & 3 & -9 & 7 & 0 \end{array}\right) \xrightarrow[\substack{r_3-3r_1 \\ r_4-r_1}]{r_2-r_1} \left(\begin{array}{cccc|c} 1 & -1 & 5 & -1 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ & & & & \\ & & & & \end{array}\right)$$

例 解齐次线性方程组 $\begin{cases} x_1 - x_2 + 5x_3 - x_4 = 0 \\ x_1 + x_2 - 2x_3 + 3x_4 = 0 \\ 3x_1 - x_2 + 8x_3 + x_4 = 0 \\ x_1 + 3x_2 - 9x_3 + 7x_4 = 0 \end{cases}$

解

$$(A \vdots b) = \left(\begin{array}{cccc|c} 1 & -1 & 5 & -1 & 0 \\ 1 & 1 & -2 & 3 & 0 \\ 3 & -1 & 8 & 1 & 0 \\ 1 & 3 & -9 & 7 & 0 \end{array}\right) \xrightarrow[\substack{r_3 - 3r_1 \\ r_4 - r_1}]{r_2 - r_1} \left(\begin{array}{cccc|c} 1 & -1 & 5 & -1 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ & & & & \end{array}\right)$$

例 解齐次线性方程组 $\begin{cases} x_1 - x_2 + 5x_3 - x_4 = 0 \\ x_1 + x_2 - 2x_3 + 3x_4 = 0 \\ 3x_1 - x_2 + 8x_3 + x_4 = 0 \\ x_1 + 3x_2 - 9x_3 + 7x_4 = 0 \end{cases}$

解

$$(A \vdots b) = \left(\begin{array}{cccc|c} 1 & -1 & 5 & -1 & 0 \\ 1 & 1 & -2 & 3 & 0 \\ 3 & -1 & 8 & 1 & 0 \\ 1 & 3 & -9 & 7 & 0 \end{array}\right) \xrightarrow[\substack{r_3-3r_1 \\ r_4-r_1}]{r_2-r_1} \left(\begin{array}{cccc|c} 1 & -1 & 5 & -1 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 4 & -14 & 8 & 0 \end{array}\right)$$

例 解齐次线性方程组 $\begin{cases} x_1 - x_2 + 5x_3 - x_4 = 0 \\ x_1 + x_2 - 2x_3 + 3x_4 = 0 \\ 3x_1 - x_2 + 8x_3 + x_4 = 0 \\ x_1 + 3x_2 - 9x_3 + 7x_4 = 0 \end{cases}$

解

$$(A \vdots b) = \left(\begin{array}{cccc|c} 1 & -1 & 5 & -1 & 0 \\ 1 & 1 & -2 & 3 & 0 \\ 3 & -1 & 8 & 1 & 0 \\ 1 & 3 & -9 & 7 & 0 \end{array}\right) \xrightarrow[\substack{r_3-3r_1 \\ r_4-r_1}]{r_2-r_1} \left(\begin{array}{cccc|c} 1 & -1 & 5 & -1 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 4 & -14 & 8 & 0 \end{array}\right)$$

$$\xrightarrow[r_4-2r_2]{r_3-r_2}$$

例 解齐次线性方程组 $\begin{cases} x_1 - x_2 + 5x_3 - x_4 = 0 \\ x_1 + x_2 - 2x_3 + 3x_4 = 0 \\ 3x_1 - x_2 + 8x_3 + x_4 = 0 \\ x_1 + 3x_2 - 9x_3 + 7x_4 = 0 \end{cases}$

解

$$(A \vdots b) = \left(\begin{array}{cccc|c} 1 & -1 & 5 & -1 & 0 \\ 1 & 1 & -2 & 3 & 0 \\ 3 & -1 & 8 & 1 & 0 \\ 1 & 3 & -9 & 7 & 0 \end{array}\right) \xrightarrow[\substack{r_3 - 3r_1 \\ r_4 - r_1}]{r_2 - r_1} \left(\begin{array}{cccc|c} 1 & -1 & 5 & -1 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 4 & -14 & 8 & 0 \end{array}\right)$$

$$\xrightarrow[r_4 - 2r_2]{r_3 - r_2} \left(\begin{array}{cccc|c} 1 & -1 & 5 & -1 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

例 解齐次线性方程组 $\begin{cases} x_1 - x_2 + 5x_3 - x_4 = 0 \\ x_1 + x_2 - 2x_3 + 3x_4 = 0 \\ 3x_1 - x_2 + 8x_3 + x_4 = 0 \\ x_1 + 3x_2 - 9x_3 + 7x_4 = 0 \end{cases}$

解

$$(A \vdots b) = \left(\begin{array}{cccc|c} 1 & -1 & 5 & -1 & 0 \\ 1 & 1 & -2 & 3 & 0 \\ 3 & -1 & 8 & 1 & 0 \\ 1 & 3 & -9 & 7 & 0 \end{array}\right) \xrightarrow[\substack{r_3-3r_1 \\ r_4-r_1}]{r_2-r_1} \left(\begin{array}{cccc|c} 1 & -1 & 5 & -1 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 4 & -14 & 8 & 0 \end{array}\right)$$

$$\xrightarrow[r_4-2r_2]{r_3-r_2} \left(\begin{array}{cccc|c} 1 & -1 & 5 & -1 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{1}{2}\times r_2}$$

例 解齐次线性方程组 $\begin{cases} x_1 - x_2 + 5x_3 - x_4 = 0 \\ x_1 + x_2 - 2x_3 + 3x_4 = 0 \\ 3x_1 - x_2 + 8x_3 + x_4 = 0 \\ x_1 + 3x_2 - 9x_3 + 7x_4 = 0 \end{cases}$

解

$$(A \vdots b) = \left(\begin{array}{cccc|c} 1 & -1 & 5 & -1 & 0 \\ 1 & 1 & -2 & 3 & 0 \\ 3 & -1 & 8 & 1 & 0 \\ 1 & 3 & -9 & 7 & 0 \end{array}\right) \xrightarrow[\substack{r_3-3r_1 \\ r_4-r_1}]{r_2-r_1} \left(\begin{array}{cccc|c} 1 & -1 & 5 & -1 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 4 & -14 & 8 & 0 \end{array}\right)$$

$$\xrightarrow[r_4-2r_2]{r_3-r_2} \left(\begin{array}{cccc|c} 1 & -1 & 5 & -1 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{1}{2}\times r_2} \left(\begin{array}{cccc|c} 1 & -1 & 5 & -1 & 0 \\ 0 & 1 & -7/2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

例 解齐次线性方程组 $\begin{cases} x_1 - x_2 + 5x_3 - x_4 = 0 \\ x_1 + x_2 - 2x_3 + 3x_4 = 0 \\ 3x_1 - x_2 + 8x_3 + x_4 = 0 \\ x_1 + 3x_2 - 9x_3 + 7x_4 = 0 \end{cases}$

解

$$(A \vdots b) = \left(\begin{array}{cccc|c} 1 & -1 & 5 & -1 & 0 \\ 1 & 1 & -2 & 3 & 0 \\ 3 & -1 & 8 & 1 & 0 \\ 1 & 3 & -9 & 7 & 0 \end{array}\right) \xrightarrow[\substack{r_3 - 3r_1 \\ r_4 - r_1}]{r_2 - r_1} \left(\begin{array}{cccc|c} 1 & -1 & 5 & -1 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 4 & -14 & 8 & 0 \end{array}\right)$$

$$\xrightarrow[r_4 - 2r_2]{r_3 - r_2} \left(\begin{array}{cccc|c} 1 & -1 & 5 & -1 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{1}{2} \times r_2} \left(\begin{array}{cccc|c} 1 & -1 & 5 & -1 & 0 \\ 0 & 1 & -7/2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{r_1 + r_2}$$

例 解齐次线性方程组 $\begin{cases} x_1 - x_2 + 5x_3 - x_4 = 0 \\ x_1 + x_2 - 2x_3 + 3x_4 = 0 \\ 3x_1 - x_2 + 8x_3 + x_4 = 0 \\ x_1 + 3x_2 - 9x_3 + 7x_4 = 0 \end{cases}$

解

$$(A \vdots b) = \left(\begin{array}{cccc|c} 1 & -1 & 5 & -1 & 0 \\ 1 & 1 & -2 & 3 & 0 \\ 3 & -1 & 8 & 1 & 0 \\ 1 & 3 & -9 & 7 & 0 \end{array}\right) \xrightarrow[\substack{r_3-3r_1 \\ r_4-r_1}]{r_2-r_1} \left(\begin{array}{cccc|c} 1 & -1 & 5 & -1 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 4 & -14 & 8 & 0 \end{array}\right)$$

$$\xrightarrow[r_4-2r_2]{r_3-r_2} \left(\begin{array}{cccc|c} 1 & -1 & 5 & -1 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{1}{2}\times r_2} \left(\begin{array}{cccc|c} 1 & -1 & 5 & -1 & 0 \\ 0 & 1 & -7/2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{r_1+r_2} \left(\begin{array}{cccc|c} 1 & 0 & 3/2 & 1 & 0 \\ 0 & 1 & -7/2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

例 解齐次线性方程组 $\begin{cases} x_1 - x_2 + 5x_3 - x_4 = 0 \\ x_1 + x_2 - 2x_3 + 3x_4 = 0 \\ 3x_1 - x_2 + 8x_3 + x_4 = 0 \\ x_1 + 3x_2 - 9x_3 + 7x_4 = 0 \end{cases}$

解

$$(A \vdots b) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 3/2 & 1 & 0 \\ 0 & 1 & -7/2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

例 解齐次线性方程组 $\begin{cases} x_1 - x_2 + 5x_3 - x_4 = 0 \\ x_1 + x_2 - 2x_3 + 3x_4 = 0 \\ 3x_1 - x_2 + 8x_3 + x_4 = 0 \\ x_1 + 3x_2 - 9x_3 + 7x_4 = 0 \end{cases}$

解

$$(A \vdots b) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 3/2 & 1 & 0 \\ 0 & 1 & -7/2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

所以原方程组等价于

例 解齐次线性方程组 $\begin{cases} x_1 - x_2 + 5x_3 - x_4 = 0 \\ x_1 + x_2 - 2x_3 + 3x_4 = 0 \\ 3x_1 - x_2 + 8x_3 + x_4 = 0 \\ x_1 + 3x_2 - 9x_3 + 7x_4 = 0 \end{cases}$

解

$$(A \vdots b) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 3/2 & 1 & 0 \\ 0 & 1 & -7/2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

所以原方程组等价于

$$\begin{cases} x_1 + \frac{3}{2}x_3 + x_4 = 0 \\ x_2 - \frac{7}{2}x_3 + 2x_4 = 0 \end{cases}$$

例 解齐次线性方程组 $\begin{cases} x_1 - x_2 + 5x_3 - x_4 = 0 \\ x_1 + x_2 - 2x_3 + 3x_4 = 0 \\ 3x_1 - x_2 + 8x_3 + x_4 = 0 \\ x_1 + 3x_2 - 9x_3 + 7x_4 = 0 \end{cases}$

解

$$(A \vdots b) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 3/2 & 1 & 0 \\ 0 & 1 & -7/2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

所以原方程组等价于

$$\begin{cases} x_1 + \frac{3}{2}x_3 + x_4 = 0 \\ x_2 - \frac{7}{2}x_3 + 2x_4 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = -\frac{3}{2}x_3 - x_4 \\ x_2 = \frac{7}{2}x_3 - 2x_4 \end{cases}$$

例 解齐次线性方程组 $\begin{cases} x_1 - x_2 + 5x_3 - x_4 = 0 \\ x_1 + x_2 - 2x_3 + 3x_4 = 0 \\ 3x_1 - x_2 + 8x_3 + x_4 = 0 \\ x_1 + 3x_2 - 9x_3 + 7x_4 = 0 \end{cases}$

解

$$(A \vdots b) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 3/2 & 1 & 0 \\ 0 & 1 & -7/2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

所以原方程组等价于

$$\begin{cases} x_1 + \frac{3}{2}x_3 + x_4 = 0 \\ x_2 - \frac{7}{2}x_3 + 2x_4 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = -\frac{3}{2}x_3 - x_4 \\ x_2 = \frac{7}{2}x_3 - 2x_4 \end{cases}$$

所以 $\begin{cases} \\ \\ x_3 = c_1 \\ x_4 = c_2 \end{cases}$

例 解齐次线性方程组 $\begin{cases} x_1 - x_2 + 5x_3 - x_4 = 0 \\ x_1 + x_2 - 2x_3 + 3x_4 = 0 \\ 3x_1 - x_2 + 8x_3 + x_4 = 0 \\ x_1 + 3x_2 - 9x_3 + 7x_4 = 0 \end{cases}$

解

$$(A \vdots b) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 3/2 & 1 & 0 \\ 0 & 1 & -7/2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

所以原方程组等价于

$$\begin{cases} x_1 + \frac{3}{2}x_3 + x_4 = 0 \\ x_2 - \frac{7}{2}x_3 + 2x_4 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = -\frac{3}{2}x_3 - x_4 \\ x_2 = \frac{7}{2}x_3 - 2x_4 \end{cases}$$

所以 $\begin{cases} x_1 = -\frac{3}{2}c_1 - c_2 \\ \\ x_3 = c_1 \\ x_4 = c_2 \end{cases}$

例 解齐次线性方程组 $\begin{cases} x_1 - x_2 + 5x_3 - x_4 = 0 \\ x_1 + x_2 - 2x_3 + 3x_4 = 0 \\ 3x_1 - x_2 + 8x_3 + x_4 = 0 \\ x_1 + 3x_2 - 9x_3 + 7x_4 = 0 \end{cases}$

解

$$(A \vdots b) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 3/2 & 1 & 0 \\ 0 & 1 & -7/2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

所以原方程组等价于

$$\begin{cases} x_1 + \frac{3}{2}x_3 + x_4 = 0 \\ x_2 - \frac{7}{2}x_3 + 2x_4 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 + = -\frac{3}{2}x_3 - x_4 \\ x_2 = \frac{7}{2}x_3 - 2x_4 \end{cases}$$

所以 $\begin{cases} x_1 = -\frac{3}{2}c_1 - c_2 \\ x_2 = \frac{7}{2}c_1 - 2c_2 \\ x_3 = c_1 \\ x_4 = c_2 \end{cases}$

例 解齐次线性方程组 $\begin{cases} x_1 - x_2 + 5x_3 - x_4 = 0 \\ x_1 + x_2 - 2x_3 + 3x_4 = 0 \\ 3x_1 - x_2 + 8x_3 + x_4 = 0 \\ x_1 + 3x_2 - 9x_3 + 7x_4 = 0 \end{cases}$

解

$$(A\vdots b) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 3/2 & 1 & 0 \\ 0 & 1 & -7/2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

所以原方程组等价于

$$\begin{cases} x_1 + \frac{3}{2}x_3 + x_4 = 0 \\ x_2 - \frac{7}{2}x_3 + 2x_4 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = -\frac{3}{2}x_3 - x_4 \\ x_2 = \frac{7}{2}x_3 - 2x_4 \end{cases}$$

所以 $\begin{cases} x_1 = -\frac{3}{2}c_1 - c_2 \\ x_2 = \frac{7}{2}c_1 - 2c_2 \\ x_3 = c_1 \\ x_4 = c_2 \end{cases}$ （注自由变量个数 $= 2 = 4 - r(A)$）