

§3.4 向量组的秩

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向量组的极大无关组

$$\alpha_1, \alpha_2, \dots, \alpha_s$$

向量组的极大无关组

$\alpha_1, \alpha_2, \dots, \alpha_s$ $\xrightarrow[\text{能被其余向量线性表示的向量}]{\text{逐个剔除}}$

向量组的极大无关组

$\alpha_1, \alpha_2, \dots, \alpha_s$ $\xrightarrow[\text{直到不能再剔除为止}]{\begin{array}{c} \text{逐个剔除} \\ \text{能被其余向量线性表示的向量} \end{array}}$

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例 求 $\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ 的一个极大无关组。

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解

$\alpha_1, \alpha_2, \alpha_3, \alpha_4$

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$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_4 = 2\alpha_1 + 0\alpha_2 + 0\alpha_3}$$

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向量组的极大无关组

$\alpha_1, \alpha_2, \dots, \alpha_s$ $\xrightarrow[\text{直到不能再剔除为止}]{\substack{\text{逐个剔除} \\ \text{能被其余向量线性表示的向量}}} \alpha_{j_1}, \alpha_{j_2}, \dots, \alpha_{j_r}$ (极大无关组)

例 求 $\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ 的一个极大无关组。

解

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow[\text{剔除}\alpha_4]{\alpha_4 = 2\alpha_1 + 0\alpha_2 + 0\alpha_3} \alpha_1, \alpha_2, \alpha_3 \xrightarrow[\text{剔除}\alpha_3]{\alpha_3 = \alpha_1 + \alpha_2} \alpha_1, \alpha_2 \quad \text{极大无关组}$$

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow[\text{剔除}\alpha_1]{\alpha_1 = -\alpha_2 + \alpha_3 + 0\alpha_4} \alpha_2, \alpha_3, \alpha_4 \xrightarrow[\text{剔除}\alpha_2]{\alpha_2 = \alpha_3 - \frac{1}{2}\alpha_4} \alpha_3, \alpha_4 \quad \text{极大无关组}$$

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还有其他极大无关组吗？

向量组的极大无关组

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还有其他极大无关组吗？

注 极大无关组不唯一！

极大无关组的性质

定理 $\alpha_{j_1}, \alpha_{j_2}, \dots, \alpha_{j_r}$ 是 $\alpha_1, \alpha_2, \dots, \alpha_s$ 的极大无关组, 当且仅当

- $\alpha_1, \alpha_2, \dots, \alpha_s$ 中每个向量都可由 $\alpha_{j_1}, \alpha_{j_2}, \dots, \alpha_{j_r}$ 线性表示
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$$\alpha_{j_1}, \alpha_{j_2}, \dots, \alpha_{j_r}; \quad \beta_{k_1}, \beta_{k_2}, \dots, \beta_{k_t}$$

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 $\alpha_1, \alpha_2; \quad \alpha_1, \alpha_3; \quad \alpha_2, \alpha_3; \quad \alpha_2, \alpha_4; \quad \alpha_3, \alpha_4$

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可见, 每个极大无关组都由 2 个向量构成。

向量组的秩

定义 向量组 $\alpha_1, \alpha_2, \dots, \alpha_s$ 的极大无关组所包含向量的个数，称向量组的秩，记为：

$$r(\alpha_1, \alpha_2, \dots, \alpha_s)$$

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注 $r(\alpha_1, \alpha_2, \dots, \alpha_s) \leq s$ 且 $\leq m$ (维数)。

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$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 3 & 5 \\ 0 & -2 & -4 \end{pmatrix}$$

• $r(\alpha_1, \alpha_2, \alpha_3) = 2$:

$$\alpha_1, \alpha_2, \alpha_3 \xrightarrow[\text{剔除}\alpha_3]{\alpha_3 = -\alpha_1 + 2\alpha_2} \alpha_1, \alpha_2 \xrightarrow{\alpha_1, \alpha_2 \text{ 线性无关}} \alpha_1, \alpha_2 \text{ 为极大无关组}$$

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可见，以上三个秩均相等，即 $r(A) = r(\alpha_1, \alpha_2, \alpha_3) = r(\beta_1, \beta_2)$ 。

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这不是巧合，而是恒成立！

秩

设

$$A_{m \times n} = \begin{pmatrix} & \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

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定义

- $r(\alpha_1, \alpha_2, \dots, \alpha_n)$ 称为 A 的列秩;
- $r(\beta_1, \beta_2, \dots, \beta_m)$ 称为 A 的行秩;

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应用 计算向量组的秩可转化为计算矩阵的秩。

初等变换求极大无关组

问题 给出 m 维的向量组 $\alpha_1, \alpha_2, \dots, \alpha_n$, 如何求出其一组极大无关组?

步骤

初等变换求极大无关组

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2. 通过简化的阶梯型矩阵, 求出 $r(A)$ 。

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利用 $r(\alpha_1, \alpha_2, \dots, \alpha_n) = r(A)$, 得出极大无关组所包含向量的个数

初等变换求极大无关组

问题 给出 m 维的向量组 $\alpha_1, \alpha_2, \dots, \alpha_n$, 如何求出其一组极大无关组?

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3. 通过简化的阶梯型矩阵, 容易看出线性无关的 $r(A)$ 列, 这就找到一组极大无关组

初等变换求极大无关组

问题 给出 m 维的向量组 $\alpha_1, \alpha_2, \dots, \alpha_n$, 如何求出其一组极大无关组?

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4. 通过简化的阶梯型矩阵, 容易看出其余列如何用该选定极大无关组线性表示

例 1 求向量组 $\alpha_1 = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

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解

	α_1	α_2	α_3	α_4
$\begin{pmatrix} 2 & 1 & 2 & 3 \\ 4 & 1 & 3 & 5 \\ 2 & 0 & 1 & 2 \end{pmatrix}$				

例 1 求向量组 $\alpha_1 = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$ 的一个极大无关组; 并把其余向量用该极大无关组线性表示。

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$$\begin{array}{cccc} & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \left(\begin{array}{cccc} 2 & 1 & 2 & 3 \\ 4 & 1 & 3 & 5 \\ 2 & 0 & 1 & 2 \end{array} \right) & \xrightarrow[r_3-r_1]{r_2-2r_1} & & & \end{array}$$

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$$\xrightarrow[r_3-r_1]{r_2-2r_1} \begin{pmatrix} 2 & 1 & 2 & 3 \\ 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 \end{pmatrix} \rightarrow$$

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解 $\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4$

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$$\xrightarrow{r_1-r_2}$$

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$$\xrightarrow{r_1-r_2} \begin{pmatrix} 2 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\frac{1}{2} \times r_1}$$

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所以

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所以

- $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2;$

例 1 求向量组 $\alpha_1 = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$ 的一个极大无关组; 并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \left(\begin{array}{cccc} 2 & 1 & 2 & 3 \\ 4 & 1 & 3 & 5 \\ 2 & 0 & 1 & 2 \end{array} \right) & \xrightarrow[r_3-r_1]{r_2-2r_1} & \left(\begin{array}{cccc} 2 & 1 & 2 & 3 \\ 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 \end{array} \right) & \rightarrow & \left(\begin{array}{cccc} 2 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right) \\ & & \xrightarrow{r_1-r_2} & \left(\begin{array}{cccc} 2 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right) & \xrightarrow{\frac{1}{2} \times r_1} & \left(\begin{array}{cccc} 1 & 0 & \frac{1}{2} & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right) \end{array}$$

所以

- $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2$;
- α_1, α_2 是极大无关组;

例 1 求向量组 $\alpha_1 = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$ 的一个极大无关组; 并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \left(\begin{array}{cccc} 2 & 1 & 2 & 3 \\ 4 & 1 & 3 & 5 \\ 2 & 0 & 1 & 2 \end{array} \right) & \xrightarrow[r_3-r_1]{r_2-2r_1} & \left(\begin{array}{cccc} 2 & 1 & 2 & 3 \\ 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 \end{array} \right) & \rightarrow & \left(\begin{array}{cccc} 2 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right) \\ & & \xrightarrow{r_1-r_2} & \left(\begin{array}{cccc} 2 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right) & \xrightarrow{\frac{1}{2} \times r_1} & \left(\begin{array}{cccc} 1 & 0 & \frac{1}{2} & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right) \end{array}$$

所以

- $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2$;
- α_1, α_2 是极大无关组;
- $\alpha_3 = \frac{1}{2}\alpha_1 + \alpha_2, \quad \alpha_4 = \alpha_1 + \alpha_2$

例 2 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一

个极大无关组；并把其余向量用该极大无关组线性表示。

例 2 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一个

极大无关组；并把其余向量用该极大无关组线性表示。

解

	α_1	α_2	α_3	α_4
$\begin{pmatrix}$	1	0	1	2
$\begin{pmatrix}$	2	1	1	4
$\begin{pmatrix}$	1	1	0	3
$\begin{pmatrix}$	0	2	-2	3
$\end{pmatrix}$				

例 2 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一个

极大无关组；并把其余向量用该极大无关组线性表示。

解

α_1	α_2	α_3	α_4
1	0	1	2
2	1	1	4
1	1	0	3
0	2	-2	3

$$\xrightarrow[r_3-r_1]{r_2-2r_1}$$

例 2 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一个

极大无关组；并把其余向量用该极大无关组线性表示。

解

α_1	α_2	α_3	α_4
1	0	1	2
2	1	1	4
1	1	0	3
0	2	-2	3

$$\xrightarrow[r_3-r_1]{r_2-2r_1} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 3 \end{pmatrix}$$

例2 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一个

极大无关组；并把其余向量用该极大无关组线性表示。

解

α_1	α_2	α_3	α_4
1	0	1	2
2	1	1	4
1	1	0	3
0	2	-2	3

$$\xrightarrow[r_3-r_1]{r_2-2r_1} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow[r_4-2r_2]{r_3-r_2}$$

例2 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一个

极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \left(\begin{array}{cccc} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 0 & 2 & -2 & 3 \end{array} \right) & \xrightarrow[r_3-r_1]{r_2-2r_1} & \left(\begin{array}{cccc} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 3 \end{array} \right) & \xrightarrow[r_4-2r_2]{r_3-r_2} & \left(\begin{array}{cccc} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{array} \right) \end{array}$$

例2 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一个

极大无关组；并把其余向量用该极大无关组线性表示。

解 $\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4$

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow[r_3-r_1]{r_2-2r_1} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow[r_4-2r_2]{r_3-r_2} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$\xrightarrow[r_1-2r_3]{r_4-3r_3}$$

例2 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一个

极大无关组；并把其余向量用该极大无关组线性表示。

解 $\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4$

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow[r_3-r_1]{r_2-2r_1} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow[r_4-2r_2]{r_3-r_2} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$
$$\xrightarrow[r_1-2r_3]{r_4-3r_3} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

例 2 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解 $\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4$

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow[r_3-r_1]{r_2-2r_1} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow[r_4-2r_2]{r_3-r_2} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$\xrightarrow[r_1-2r_3]{r_4-3r_3} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

例 2 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{cccc}
 \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\
 \left(\begin{array}{cccc} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 0 & 2 & -2 & 3 \end{array} \right) & \xrightarrow[r_3-r_1]{r_2-2r_1} & \left(\begin{array}{cccc} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 3 \end{array} \right) & \xrightarrow[r_4-2r_2]{r_3-r_2} \left(\begin{array}{cccc} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{array} \right) \\
 & & \xrightarrow[r_1-2r_3]{r_4-3r_3} & \left(\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right)
 \end{array}$$

所以

例2 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一个极大无关组; 并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \left(\begin{array}{cccc} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 0 & 2 & -2 & 3 \end{array} \right) & \xrightarrow[r_3-r_1]{r_2-2r_1} & \left(\begin{array}{cccc} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 3 \end{array} \right) & \xrightarrow[r_4-2r_2]{r_3-r_2} \left(\begin{array}{cccc} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{array} \right) \\ & & \xrightarrow[r_1-2r_3]{r_4-3r_3} & \left(\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right) \end{array}$$

所以

• $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 3;$

例2 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一个

极大无关组；并把其余向量用该极大无关组线性表示。

解 $\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4$

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow[r_3-r_1]{r_2-2r_1} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow[r_4-2r_2]{r_3-r_2} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$
$$\xrightarrow[r_1-2r_3]{r_4-3r_3} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

所以

• $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 3;$

例2 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一个

极大无关组；并把其余向量用该极大无关组线性表示。

解 $\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4$

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow[r_3-r_1]{r_2-2r_1} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow[r_4-2r_2]{r_3-r_2} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$
$$\xrightarrow[r_1-2r_3]{r_4-3r_3} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

所以

- $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 3;$

- $\alpha_1, \alpha_2, \alpha_4$ 是极大无关组;

例2 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一个

极大无关组；并把其余向量用该极大无关组线性表示。

解 $\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4$

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow[r_3-r_1]{r_2-2r_1} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow[r_4-2r_2]{r_3-r_2} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$
$$\xrightarrow[r_1-2r_3]{r_4-3r_3} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

所以

- $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 3$;
- $\alpha_1, \alpha_2, \alpha_4$ 是极大无关组;
- $\alpha_3 = \alpha_1 - \alpha_2$

例 3 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

例 3 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个

极大无关组；并把其余向量用该极大无关组线性表示。

解

α_1	α_2	α_3	α_4
1	2	3	4
2	3	4	5
3	4	5	6
4	5	6	7

例3 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个

极大无关组；并把其余向量用该极大无关组线性表示。

解 $\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \xrightarrow[r_4-4r_1]{\begin{matrix} r_2-2r_1 \\ r_3-3r_1 \end{matrix}}$$

例3 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个

极大无关组；并把其余向量用该极大无关组线性表示。

解 $\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \xrightarrow[r_4-4r_1]{\begin{matrix} r_2-2r_1 \\ r_3-3r_1 \end{matrix}} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{pmatrix}$$

例3 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个

极大无关组；并把其余向量用该极大无关组线性表示。

解 $\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \xrightarrow[r_4-4r_1]{\begin{matrix} r_2-2r_1 \\ r_3-3r_1 \end{matrix}} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{pmatrix}$$

$$\xrightarrow[r_4-3r_2]{r_3-2r_2}$$

例3 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个

极大无关组；并把其余向量用该极大无关组线性表示。

解 $\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \xrightarrow[r_4-4r_1]{\begin{matrix} r_2-2r_1 \\ r_3-3r_1 \end{matrix}} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{pmatrix}$$
$$\xrightarrow[r_4-3r_2]{r_3-2r_2} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

例3 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个

极大无关组；并把其余向量用该极大无关组线性表示。

解 $\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \xrightarrow[r_4-4r_1]{\begin{matrix} r_2-2r_1 \\ r_3-3r_1 \end{matrix}} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{pmatrix}$$
$$\xrightarrow[r_4-3r_2]{r_3-2r_2} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

例3 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个

极大无关组；并把其余向量用该极大无关组线性表示。

解 $\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \xrightarrow[r_4-4r_1]{\begin{matrix} r_2-2r_1 \\ r_3-3r_1 \end{matrix}} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{pmatrix}$$

$$\xrightarrow[r_4-3r_2]{r_3-2r_2} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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极大无关组；并把其余向量用该极大无关组线性表示。

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例 假设向量组 $\alpha_1, \alpha_2, \dots, \alpha_s$ 可由 $\beta_1, \beta_2, \dots, \beta_t$ 线性表示, 则

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证明 设

$$r_1 = r(\alpha_1, \alpha_2, \dots, \alpha_s), \quad \alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_{r_1}} \text{ 是极大无关组}$$

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即

$$(\gamma_1 \ \gamma_2 \ \cdots \ \gamma_s) = (\alpha_1 \ \alpha_2 \ \cdots \ \alpha_n) \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}$$

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$$(\gamma_1 \ \gamma_2 \ \cdots \ \gamma_s) = (\alpha_1 \ \alpha_2 \ \cdots \ \alpha_n) \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}$$

$$\Rightarrow \gamma_1 = b_{11}\alpha_1 + b_{21}\alpha_2 + \cdots + b_{n1}\alpha_n$$

例 设 $A_{m \times n}$, $B_{n \times s}$ 为矩阵, 则 $r(AB) \leq \min\{r(A), r(B)\}$ 。

证明 设 $AB = C_{m \times s}$

$$\underbrace{\begin{pmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_s \\ c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_C = \underbrace{\begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_A \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_B$$

即

$$(\gamma_1 \ \gamma_2 \ \cdots \ \gamma_s) = (\alpha_1 \ \alpha_2 \ \cdots \ \alpha_n) \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}$$

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即

$$(\gamma_1 \ \gamma_2 \ \cdots \ \gamma_s) = (\alpha_1 \ \alpha_2 \ \cdots \ \alpha_n) \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}$$

$$\Rightarrow \gamma_1 = b_{11}\alpha_1 + b_{21}\alpha_2 + \cdots + b_{n1}\alpha_n \quad \text{等等}$$

可见 $\gamma_1, \dots, \gamma_s$ 由 $\alpha_1, \dots, \alpha_n$ 线性表示,

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即

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证明 设 $AB = C_{m \times s}$

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可见 $\gamma_1, \dots, \gamma_s$ 由 $\alpha_1, \dots, \alpha_n$ 线性表示, 所以

$$r(\gamma_1, \dots, \gamma_s) \leq r(\alpha_1, \dots, \alpha_n) = r(A)$$

例 设 $A_{m \times n}, B_{n \times s}$ 为矩阵, 则 $r(AB) \leq \min\{r(A), r(B)\}$ 。

证明 设 $AB = C_{m \times s}$

$$\underbrace{\begin{pmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_s \\ C_{11} & C_{12} & \cdots & C_{1s} \\ C_{21} & C_{22} & \cdots & C_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ C_{m1} & C_{m2} & \cdots & C_{ms} \end{pmatrix}}_C = \underbrace{\begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_A \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_B$$

即

$$(\gamma_1 \ \gamma_2 \ \cdots \ \gamma_s) = (\alpha_1 \ \alpha_2 \ \cdots \ \alpha_n) \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}$$

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$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_C = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_A \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_B \beta_1$$

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证明 设 $AB = C_{m \times s}$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_C = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_A \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_B \begin{matrix} \beta_1 \\ \beta_2 \end{matrix}$$

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证明 设 $AB = C_{m \times s}$

$$\delta_1 \underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_C = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_A \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_B \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$$

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证明 设 $AB = C_{m \times s}$

$$\begin{array}{l} \delta_1 \\ \delta_2 \end{array} \underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_C = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_A \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_B \begin{array}{l} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{array}$$

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$$\begin{array}{c} \delta_1 \\ \delta_2 \\ \vdots \end{array} \underbrace{\begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1s} \\ C_{21} & C_{22} & \cdots & C_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ C_{m1} & C_{m2} & \cdots & C_{ms} \end{pmatrix}}_C = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_A \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_B \begin{array}{c} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{array}$$

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$$\begin{matrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{matrix} \underbrace{\begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1s} \\ C_{21} & C_{22} & \cdots & C_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ C_{m1} & C_{m2} & \cdots & C_{ms} \end{pmatrix}}_C = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_A \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_B \begin{matrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{matrix}$$

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即

$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$$

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$$\begin{matrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{matrix} \underbrace{\begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1s} \\ C_{21} & C_{22} & \cdots & C_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ C_{m1} & C_{m2} & \cdots & C_{ms} \end{pmatrix}}_C = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_A \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_B \begin{matrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{matrix}$$

即

$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$$

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即

$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$$

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证明 设 $AB = C_{m \times s}$

$$\begin{matrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{matrix} \underbrace{\begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1s} \\ C_{21} & C_{22} & \cdots & C_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ C_{m1} & C_{m2} & \cdots & C_{ms} \end{pmatrix}}_C = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_A \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_B \begin{matrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{matrix}$$

即

$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$$

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可见 $\delta_1, \dots, \delta_m$ 由 β_1, \dots, β_n 线性表示,

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证明 设 $AB = C_{m \times s}$

$$\begin{matrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{matrix} \underbrace{\begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1s} \\ C_{21} & C_{22} & \cdots & C_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ C_{m1} & C_{m2} & \cdots & C_{ms} \end{pmatrix}}_C = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_A \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_B \begin{matrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{matrix}$$

即

$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$$

$$\Rightarrow \delta_1 = a_{11}\beta_1 + a_{12}\beta_2 + \cdots + a_{1n}\beta_n \quad \text{等等}$$

可见 $\delta_1, \dots, \delta_m$ 由 β_1, \dots, β_n 线性表示, 所以

$$r(\delta_1, \dots, \delta_m) \leq r(\beta_1, \dots, \beta_n)$$

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