## 第 12 章 d: 函数展开成幂级数

数学系 梁卓滨

2016-2017 **学年** II



## Outline

$$f(x) \stackrel{?}{=} a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

$$f(x) \neq a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots + a_n(x-x_0)^n + \cdots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

$$f(x) \stackrel{?}{=} a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

$$f^{(k)}(x) = \left[\sum_{n=0}^{\infty} a_n (x - x_0)^n\right]^{(k)}$$

$$f(x) \neq a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

$$f^{(k)}(x) = \left[\sum_{n=0}^{\infty} a_n (x - x_0)^n\right]^{(k)} = \sum_{n=0}^{\infty} \left[a_n (x - x_0)^n\right]^{(k)}$$

$$f(x) \neq a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

$$f^{(k)}(x) = \left[\sum_{n=0}^{\infty} a_n (x - x_0)^n\right]^{(k)} = \sum_{n=0}^{\infty} \left[a_n (x - x_0)^n\right]^{(k)}$$

$$n \cdot (n-1) \cdots (n-k+1) \cdot (x-x_0)^{n-k}$$

$$f(x) \neq a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

$$f^{(k)}(x) = \left[\sum_{n=0}^{\infty} a_n (x - x_0)^n\right]^{(k)} = \sum_{n=0}^{\infty} \left[a_n (x - x_0)^n\right]^{(k)}$$
$$= \sum_{n=0}^{\infty} a_n \cdot n \cdot (n-1) \cdots (n-k+1) \cdot (x - x_0)^{n-k}$$

$$f(x) \neq a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

$$f^{(k)}(x) = \left[\sum_{n=0}^{\infty} a_n (x - x_0)^n\right]^{(k)} = \sum_{n=0}^{\infty} \left[a_n (x - x_0)^n\right]^{(k)}$$
$$= \sum_{n=k}^{\infty} a_n \cdot n \cdot (n-1) \cdots (n-k+1) \cdot (x - x_0)^{n-k}$$
$$= a_k \cdot k!$$

$$f(x) \stackrel{?}{=} a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

$$f^{(k)}(x) = \left[\sum_{n=0}^{\infty} a_n (x - x_0)^n\right]^{(k)} = \sum_{n=0}^{\infty} \left[a_n (x - x_0)^n\right]^{(k)}$$
$$= \sum_{n=k}^{\infty} a_n \cdot n \cdot (n-1) \cdots (n-k+1) \cdot (x - x_0)^{n-k}$$
$$= a_k \cdot k! + a_{k+1} \cdot (k+1) \cdots 2 \cdot (x - x_0)$$

$$f(x) \stackrel{?}{=} a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

$$f^{(k)}(x) = \left[\sum_{n=0}^{\infty} a_n (x - x_0)^n\right]^{(k)} = \sum_{n=0}^{\infty} \left[a_n (x - x_0)^n\right]^{(k)}$$

$$= \sum_{n=k}^{\infty} a_n \cdot n \cdot (n-1) \cdots (n-k+1) \cdot (x - x_0)^{n-k}$$

$$= a_k \cdot k! + a_{k+1} \cdot (k+1) \cdots 2 \cdot (x - x_0)$$

$$+ a_{k+2} \cdot (k+2) \cdots 3 \cdot (x - x_0)^2 + \cdots$$

$$f(x) \neq a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

证明 逐项求 k 次导得:

$$f^{(k)}(x) = \left[\sum_{n=0}^{\infty} a_n (x - x_0)^n\right]^{(k)} = \sum_{n=0}^{\infty} \left[a_n (x - x_0)^n\right]^{(k)}$$
$$= \sum_{n=k}^{\infty} a_n \cdot n \cdot (n-1) \cdots (n-k+1) \cdot (x - x_0)^{n-k}$$
$$= a_k \cdot k! + a_{k+1} \cdot (k+1) \cdots 2 \cdot (x - x_0)$$

 $+ a_{k+2} \cdot (k+2) \cdots 3 \cdot (x-x_0)^2 + \cdots$ 

取  $x = x_0$  得  $a_k = \frac{1}{k!} f^{(k)}(x_0)$ 



$$f(x) \stackrel{?}{=} a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$



$$f(x) \neq a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots + a_n(x-x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

注 1

 $f(x_0)$ 

$$f(x) \neq a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

$$f(x_0)$$
  $f'(x_0)$ 

$$f(x) \neq a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

$$f(x_0)$$
  $f'(x_0)$   $\frac{1}{2!}f''(x_0)$ 

$$f(x) \neq a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

$$f(x_0)$$
  $f'(x_0)$   $\frac{1}{2!}f''(x_0)$   $\cdots$   $\frac{1}{n!}f^{(n)}(x_0)$ 

$$f(x) \neq a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

注 1 也就是, f(x) 至多能展成如下形式的幂级数:

$$f(x_0)+f'(x_0)(x-x_0)+\frac{1}{2!}f''(x_0)(x-x_0)^2+\cdots+\frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n+\cdots$$

$$f(x) \neq a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

注 1 也就是,f(x) 至多能展成如下形式的幂级数:

$$f(x_0)+f'(x_0)(x-x_0)+\frac{1}{2!}f''(x_0)(x-x_0)^2+\cdots+\frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n+\cdots$$

- 此级数称为 f(x) 在  $x_0$  处的 泰勒级数。

$$f(x) \stackrel{?}{=} a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots + a_n(x-x_0)^n + \cdots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

注 1 也就是, f(x) 至多能展成如下形式的幂级数:

$$f(x_0)+f'(x_0)(x-x_0)+\frac{1}{2!}f''(x_0)(x-x_0)^2+\cdots+\frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n+\cdots$$

- 此级数称为 *f*(x) 在 x<sub>0</sub> 处的 泰勒级数。
  - 此级数称为 J(X) 在  $X_0$  处的 泰剌级数。
  - 级数前 n+1 项的部分和记为  $p_n$ ,称为 n 次泰勒多项式

$$f(x) \stackrel{?}{=} a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

注 1 也就是, f(x) 至多能展成如下形式的幂级数:

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

 $f(x_0)+f'(x_0)(x-x_0)+\frac{1}{2!}f''(x_0)(x-x_0)^2+\cdots+\frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n+\cdots$ 

- 此级数称为 
$$f(x)$$
 在  $x_0$  处的 泰勒级数。

- 级数前 n+1 项的部分和记为  $p_n$ ,称为 n 次泰勒多项式

注 2
$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0) (x - x_0)^n$$



$$f(x) \stackrel{?}{=} a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

 $a_n = \frac{1}{n!} f^{(n)}(x_0).$ 

性质 若 f(x) 能展成上述幂级数,则

$$a_n = -\frac{1}{n!}f^{(n)}(x)$$

注 1 也就是,f(x) 至多能展成如下形式的幂级数:

- 此级数称为 
$$f(x)$$
 在  $x_0$  处的 泰勒级数。

 $f(x_0)+f'(x_0)(x-x_0)+\frac{1}{2!}f''(x_0)(x-x_0)^2+\cdots+\frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n+\cdots$ 

- 级数前 n+1 项的部分和记为  $p_n$ ,称为 n 次泰勒多项式

注 2
$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0) (x - x_0)^n \quad \Leftrightarrow \quad f(x) = \lim_{n \to \infty} p_n(x)$$

解 取 
$$x_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

解 取 
$$x_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 
$$f(x) = e^x$$
 时,

$$f(x) = f'(x) = f''(x) = f'''(x) = \dots = f^{(n)}(x) = e^x$$

解 取 
$$x_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 
$$f(x) = e^x$$
 时,

$$f(x) = f'(x) = f''(x) = f'''(x) = \dots = f^{(n)}(x) = e^x$$

$$\Rightarrow f(0) = f'(0) = f''(0) = f'''(0) = \dots = f^{(n)}(0) = 1$$

解 取 
$$x_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 
$$f(x) = e^x$$
 时,

$$f(x) = f'(x) = f''(x) = f'''(x) = \cdots = f^{(n)}(x) = e^x$$

$$\Rightarrow f(0) = f'(0) = f''(0) = f'''(0) = \dots = f^{(n)}(0) = 1$$

⇒ 
$$\bar{x}$$
 \$\text{\$\pi\$}\$ \$\pi\$\$ \$\pi\$ \$\p

解 取  $x_0 = 0$  时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当  $f(x) = e^x$  时,

$$f(x) = f'(x) = f''(x) = f'''(x) = \dots = f^{(n)}(x) = e^x$$

$$\Rightarrow f(0) = f'(0) = f''(0) = f'''(0) = \dots = f^{(n)}(0) = 1$$

$$\Rightarrow f(0) = f(0) = f(0) = f(0) = \cdots = f(0) = 1$$

$$= \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{3}$$

⇒ 
$$\bar{x}$$
 \$\text{\$\pi\$}\$ \$\tag{\pi}\$ \$\ta

注 n 次泰勒多项式是:

$$p_n(x) =$$

解 取  $x_0 = 0$  时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 
$$f(x) = e^x$$
 时,

$$f(x) = f'(x) = f''(x) = f'''(x) = \dots = f^{(n)}(x) = e^{x}$$
  

$$\Rightarrow f(0) = f'(0) = f''(0) = f'''(0) = \dots = f^{(n)}(0) = 1$$

⇒ 
$$\bar{x}$$
 \$\text{\$\pi\$}\$ \$\pi\$\$ \$\pi\$ \$\pi\$\$ \$\pi\$ \$\

## 注 n 次泰勒多项式是:

$$p_n(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots + \frac{1}{n!}x^n$$



解 取 
$$x_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 
$$f(x) = \sin x$$
 时,

解 取  $x_0 = 0$  时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当  $f(x) = \sin x$  时,

	$f^{(n)}(x)$	$f^{(n)}(0)$
n = 0, 4, 8	sin x	0
<i>n</i> = 1, 5, 9	cosx	1
n = 2, 6, 10	— sin <i>x</i>	0
n = 3, 7, 11	— cos x	-1

解 取  $x_0 = 0$  时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当  $f(x) = \sin x$  时,

	$f^{(n)}(x) = \sin(x + \frac{n}{2}\pi)$	$f^{(n)}(0)$
n = 0, 4, 8	sin x	0
n = 1, 5, 9	cosx	1
n = 2, 6, 10	— sin <i>x</i>	0
n = 3, 7, 11	— cos x	-1

解 取  $x_0 = 0$  时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当  $f(x) = \sin x$  时,

	$f^{(n)}(x) = \sin(x + \frac{n}{2}\pi)$	$f^{(n)}(0) = \sin(\frac{n}{2}\pi)$
n = 0, 4, 8	sin x	0
<i>n</i> = 1, 5, 9	cosx	1
n = 2, 6, 10	— sin <i>x</i>	0
n = 3, 7, 11	— cos x	-1



解 取  $x_0 = 0$  时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当  $f(x) = \sin x$  时,

	$f^{(n)}(x) = \sin(x + \frac{n}{2}\pi)$	$f^{(n)}(0) = \sin(\frac{n}{2}\pi)$
n = 0, 4, 8	sin x	0
<i>n</i> = 1, 5, 9	cosx	1
<i>n</i> = 2, 6, 10	— sin <i>x</i>	0
<i>n</i> = 3, 7, 11	— cos x	-1

所以泰勒级数是

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \cdots$$

 $\mathbf{m} \mathbf{n} \mathbf{x}_0 = \mathbf{0} \mathbf{n}$  时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当  $f(x) = \sin x$  时,

	$f^{(n)}(x) = \sin(x + \frac{n}{2}\pi)$	$f^{(n)}(0) = \sin(\frac{n}{2}\pi)$
n = 0, 4, 8	sin x	0
<i>n</i> = 1, 5, 9	cosx	1
<i>n</i> = 2, 6, 10	— sin <i>x</i>	0
<i>n</i> = 3, 7, 11	— cos <i>x</i>	-1

所以泰勒级数是

所以泰朝级致是
$$x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\frac{1}{7!}x^7+\frac{1}{9!}x^9-\frac{1}{11!}x^{11}+\cdots+(-1)^m\frac{1}{(2m+1)!}x^{2m+1}+\cdots$$

展开成幂级数

$$x - \frac{1}{3!}x^{3} + \frac{1}{5!}x^{5} - \frac{1}{7!}x^{7} + \frac{1}{9!}x^{9} - \frac{1}{11!}x^{11} + \dots + (-1)^{m} \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

$$x - \frac{1}{3!}x^{3} + \frac{1}{5!}x^{5} - \frac{1}{7!}x^{7} + \frac{1}{9!}x^{9} - \frac{1}{11!}x^{11} + \dots + (-1)^{m} \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

$$p_1 = x$$
;

$$x - \frac{1}{3!}x^{3} + \frac{1}{5!}x^{5} - \frac{1}{7!}x^{7} + \frac{1}{9!}x^{9} - \frac{1}{11!}x^{11} + \dots + (-1)^{m} \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

$$p_1 = p_2 = x$$
;

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

$$p_1 = p_2 = x;$$
  
 $p_3 = x - \frac{1}{3!}x^3;$ 

$$p_3 = x - \frac{1}{3!}x^3$$

 $p_1 = p_2 = x$ ;

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

$$p_3 = p_4 = x - \frac{1}{3!}x^3;$$

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

sinx的n次泰勒多项式是:

$$p_1 = p_2 = x;$$

$$p_3 = p_4 = x - \frac{1}{3!}x^3;$$

$$p_5 = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5;$$

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

sinx的n次泰勒多项式是:

$$p_1 = p_2 = x;$$

$$p_3 = p_4 = x - \frac{1}{3!}x^3;$$

$$p_5 = p_6 = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5;$$

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

$$n_1 = n_2 = Y$$
:

 $p_1 = p_2 = x$ ;

$$p_3 = p_4 = x - \frac{1}{3!}x^3;$$

$$p_5 = p_6 = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5;$$

 $p_{2m+1}$ 



$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

 $p_1 = p_2 = x$ ;

 $p_{2m+1}$ 

sinx的n次泰勒多项式是:

 $p_3 = p_4 = x - \frac{1}{3!}x^3;$ 

 $p_5 = p_6 = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5;$ 

 $= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1}$ 

- sinx 的泰勒级数是:

$$\frac{1}{2}$$
  $\frac{1}{2}$   $\frac{1}{2}$   $\frac{1}{2}$   $\frac{1}{2}$   $\frac{1}{2}$ 

 $p_1 = p_2 = x$ ;

 $x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$ 

$$\frac{1}{3}$$
  $\frac{1}{5}$   $\frac{1}{5}$ 

sinx的n次泰勒多项式是:

 $p_3 = p_4 = x - \frac{1}{3!}x^3;$ 



 $p_5 = p_6 = x - \frac{1}{31}x^3 + \frac{1}{51}x^5;$ 

 $p_{2m+1} = p_{2m+2} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1}$ 

- sinx 的泰勒级数是:

解 取 
$$x_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 
$$f(x) = \cos x$$
 时,

解 取  $x_0 = 0$  时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当  $f(x) = \cos x$  时,

	$f^{(n)}(x)$	$f^{(n)}(0)$
n = 0, 4, 8	cosx	1
n = 1, 5, 9	— sin <i>x</i>	0
n = 2, 6, 10	— cos x	-1
n = 3, 7, 11	sin x	0

解 取  $x_0 = 0$  时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当  $f(x) = \cos x$  时,

	$f^{(n)}(x) = \cos(x + \frac{n}{2}\pi)$	$f^{(n)}(0)$
n = 0, 4, 8	cosx	1
n = 1, 5, 9	— sin <i>x</i>	0
n = 2, 6, 10	— cos x	-1
n = 3, 7, 11	sin x	0

解 取  $x_0 = 0$  时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当  $f(x) = \cos x$  时,

	$f^{(n)}(x) = \cos(x + \frac{n}{2}\pi)$	$f^{(n)}(0) = \cos(\frac{n}{2}\pi)$
n = 0, 4, 8	cosx	1
$n = 1, 5, 9 \dots$	— sin <i>x</i>	0
n = 2, 6, 10	— cos x	-1
<i>n</i> = 3, 7, 11	sin x	0

解 取  $x_0 = 0$  时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当  $f(x) = \cos x$  时,

	$f^{(n)}(x) = \cos(x + \frac{n}{2}\pi)$	$f^{(n)}(0) = \cos(\frac{n}{2}\pi)$
n = 0, 4, 8	cosx	1
<i>n</i> = 1, 5, 9	— sin <i>x</i>	0
n = 2, 6, 10	— cos x	-1
n = 3, 7, 11	sin x	0

所以泰勒级数是

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \cdots$$



解 取  $x_0 = 0$  时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当  $f(x) = \cos x$  时,

	$f^{(n)}(x) = \cos(x + \frac{n}{2}\pi)$	$f^{(n)}(0) = \cos(\frac{n}{2}\pi)$
n = 0, 4, 8	cosx	1
n = 1, 5, 9	— sin <i>x</i>	0
<i>n</i> = 2, 6, 10	— cos x	-1
<i>n</i> = 3, 7, 11	sin x	0

所以泰勒级数是

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$



$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

$$p_0 = 1;$$

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

$$p_0 = p_1 = 1$$
;

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

$$p_0 = p_1 = 1;$$
  
 $p_2 = 1 - \frac{1}{2!}x^2;$ 

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

$$p_0 = p_1 = 1;$$
  
 $p_2 = p_3 = 1 - \frac{1}{2!}x^2;$ 

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

$$p_0 = p_1 = 1;$$

$$p_2 = p_3 = 1 - \frac{1}{2!}x^2;$$

$$p_4 = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4;$$



$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

$$p_0 = p_1 = 1;$$

$$p_2 = p_3 = 1 - \frac{1}{2!}x^2;$$

$$p_4 = p_5 = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4;$$

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

• cos x 的 n 次泰勒多项式是:

$$p_0 = p_1 = 1;$$

$$p_2 = p_3 = 1 - \frac{1}{2!}x^2;$$

$$p_4 = p_5 = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4;$$
:

 $p_{2m}(x)$ 



$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

• 
$$\cos x$$
 的  $n$  次泰勒多项式是: 
$$p_0 = p_1 = 1;$$

 $p_2 = p_3 = 1 - \frac{1}{2!}x^2;$  $p_4 = p_5 = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4;$ 

$$p_4 = p_5 = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4;$$

 $=1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\cdots+(-1)^m\frac{1}{(2m)!}x^{2m}$ 

 $p_{2m}(x)$ 

COS X 的泰勒级数是:

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

cos x 的 n 次泰勒多项式是:

 $p_0 = p_1 = 1$ :

$$p_2 = p_3 = 1 - \frac{1}{2!}x^2;$$

$$p_4 = p_5 = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4;$$

 $p_{2m}(x) = p_{2m+1}(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots + (-1)^m \frac{1}{(2m)!}x^{2m}$ 

解 取  $x_0 = 0$  时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当  $f(x) = \ln(1+x)$  时,

解 取 
$$x_0 = 0$$
 时,泰勒级数是  $f''(0)$   $f'''(0)$ 

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 
$$f(x) = \ln(1+x)$$
 时,  
 $f = \ln(1+x)$ ,  $f' = \frac{1}{1+x}$ ,

$$f = \ln(1+x), \quad f' = \frac{1}{1+x}$$

$$\mathbf{m} \mathbf{n} \mathbf{x}_0 = \mathbf{0} \mathbf{n}$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 
$$f(x) = \ln(1+x)$$
 时,  
 $f = \ln(1+x)$ ,  $f' = \frac{1}{1+x}$ ,  $f'' = \frac{(-1)}{(1+x)^2}$ ,



解 取 
$$x_0 = 0$$
 时,泰勒级数是 
$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 
$$f(x) = \ln(1+x)$$
 时,  
 $f = \ln(1+x)$ ,  $f' = \frac{1}{1+x}$ ,  $f'' = \frac{(-1)}{(1+x)^2}$ ,  $f''' = \frac{2}{(1+x)^3}$ ,



解 取 
$$x_0 = 0$$
 时,泰勒级数是 
$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

$$f(0) + f'(0)x + \frac{f'(0)}{2!}x^2 + \frac{f'(0)}{3!}x^3 + \dots + \frac{f'(0)}{n!}x^n + \dots$$

$$\stackrel{\text{def}}{=} f(x) = \ln(1+x) \text{ pt},$$

$$f = \ln(1+x), \quad f' = \frac{1}{1+x}, \quad f'' = \frac{(-1)}{(1+x)^2}, \quad f''' = \frac{2}{(1+x)^3},$$

$$f^{(4)} = \frac{-2 \cdot 3}{(1+x)^4},$$

解 取 
$$x_0 = 0$$
 时,泰勒级数是 
$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

$$f(0) + f'(0)x + \frac{f'(0)}{2!}x^2 + \frac{f'(0)}{3!}x^3 + \dots + \frac{f'(0)}{n!}x^n + \dots$$

$$\stackrel{\text{def}}{=} f(x) = \ln(1+x) \text{ pt},$$

$$f = \ln(1+x), \quad f' = \frac{1}{1+x}, \quad f'' = \frac{(-1)}{(1+x)^2}, \quad f''' = \frac{2}{(1+x)^3},$$

$$f^{(4)} = \frac{-2 \cdot 3}{(1+x)^4}, \quad f^{(5)} = \frac{2 \cdot 3 \cdot 4}{(1+x)^5}, \dots,$$

$$\mathbf{H}$$
 取  $\mathbf{X}_0 = \mathbf{0}$  时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 
$$f(x) = \ln(1+x)$$
 时,

$$f = \ln(1+x), \quad f' = \frac{1}{1+x}, \quad f'' = \frac{(-1)}{(1+x)^2}, \quad f''' = \frac{2}{(1+x)^3},$$

$$f = \ln(1+x), \quad f' = \frac{1}{1+x}, \quad f'' = \frac{1}{(1+x)^2}, \quad f''' = \frac{1}{(1+x)^3},$$

$$f^{(4)} = \frac{-2 \cdot 3}{(1+x)^4}, \quad f^{(5)} = \frac{2 \cdot 3 \cdot 4}{(1+x)^5}, \dots, f^{(n)} = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}, \dots$$

解 取 
$$x_0 = 0$$
 时,泰勒级数是 
$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

$$f(0) + f'(0)x + \frac{f'(0)}{2!}x^2 + \frac{f'(0)}{3!}x^3 + \dots + \frac{f''(0)}{n!}x^n + \dots$$

当 
$$f(x) = \ln(1+x)$$
 时,  
 $f = \ln(1+x)$ ,  $f' = \frac{1}{1+x}$ ,  $f'' = \frac{(-1)}{(1+x)^2}$ ,  $f''' = \frac{2}{(1+x)^3}$ ,

$$f^{(4)} = \frac{-2 \cdot 3}{(1+x)^4}, \quad f^{(5)} = \frac{2 \cdot 3 \cdot 4}{(1+x)^5}, \dots, f^{(n)} = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}, \dots$$

所以 
$$\frac{1}{n!}f^{(n)}(0) = \frac{(-1)^{n-1}}{n}$$
,

例 求  $f(x) = \ln(1+x)$  在 x = 0 处泰勒级数。

解 取 
$$x_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 
$$f(x) = \ln(1+x)$$
 时,

$$f = \ln(1+x), \quad f' = \frac{1}{1+x}, \quad f'' = \frac{(-1)}{(1+x)^2}, \quad f''' = \frac{2}{(1+x)^3},$$

$$f^{(4)} = \frac{-2 \cdot 3}{(1+x)^4}, \quad f^{(5)} = \frac{2 \cdot 3 \cdot 4}{(1+x)^5}, \dots, f^{(n)} = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}, \dots$$

所以 
$$\frac{1}{n!}f^{(n)}(0) = \frac{(-1)^{n-1}}{n}$$
,泰勒级数是 
$$x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{1}{4}x^4 + \cdots$$



例 求  $f(x) = \ln(1+x)$  在 x = 0 处泰勒级数。

解 取 
$$x_0 = 0$$
 时,泰勒级数是 
$$f''(0) = f'''(0) = f^{(n)}(0)$$

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 
$$f(x) = \ln(1+x)$$
 时,  
 $f = \ln(1+x)$ ,  $f' = \frac{1}{1+x}$ ,  $f'' = \frac{(-1)}{(1+x)^2}$ ,  $f''' = \frac{2}{(1+x)^3}$ ,

$$f^{(4)} = \frac{-2 \cdot 3}{(1+x)^4}, \quad f^{(5)} = \frac{2 \cdot 3 \cdot 4}{(1+x)^5}, \dots, f^{(n)} = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}, \dots$$
所以  $\frac{1}{n!} f^{(n)}(0) = \frac{(-1)^{n-1}}{n}, \,\,$ 泰勒级数是

$$x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots$$



解 取  $x_0 = 0$  时,泰勒级数是

$$\mathbf{H}$$
 以  $X_0 = \mathbf{U}$  的, 泰剌级数  $f''(\mathbf{0})$ 

$$f(0) + f'(0)x + \frac{f'(0)}{2!}$$

$$f(0) + f'(0)x + \frac{f'(0)}{2!}x$$

 $f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{2!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$ 

所以  $\frac{1}{n!}f^{(n)}(0) = \frac{(-1)^{n-1}}{n}$ ,泰勒级数是

$$f(0) + f'(0)x + \frac{f'(0)}{2!}x$$

当  $f(x) = \ln(1+x)$  时,

例 求  $f(x) = \ln(1+x)$  在 x = 0 处泰勒级数。

 $f = \ln(1+x), \quad f' = \frac{1}{1+x}, \quad f'' = \frac{(-1)}{(1+x)^2}, \quad f''' = \frac{2}{(1+x)^3},$ 

 $f^{(4)} = \frac{-2 \cdot 3}{(1+x)^4}, \quad f^{(5)} = \frac{2 \cdot 3 \cdot 4}{(1+x)^5}, \dots, f^{(n)} = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}, \dots$ 

 $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots$ 

解 取  $x_0 = 0$  时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 
$$f(x) = (1+x)^{\alpha}$$
 时,

 $\mathbf{H}$  取  $\mathbf{x}_0 = \mathbf{0}$  时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 
$$f(x) = (1+x)^{\alpha}$$
 时,  
 $f = (1+x)^{\alpha}$ ,  $f' = \alpha(1+x)^{\alpha-1}$ ,

解 取 
$$x_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 
$$f(x) = (1+x)^{\alpha}$$
 时,

$$f = (1+x)^{\alpha}, \quad f' = \alpha(1+x)^{\alpha-1}, \quad f'' = \alpha(\alpha-1)(1+x)^{\alpha-2},$$

 $\mathbf{H}$  取  $\mathbf{x}_0 = \mathbf{0}$  时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 
$$f(x) = (1+x)^{\alpha}$$
 时,

$$f = (1+x)^{\alpha}, \quad f' = \alpha(1+x)^{\alpha-1}, \quad f'' = \alpha(\alpha-1)(1+x)^{\alpha-2},$$

$$\ldots, f^{(n)} = \alpha(\alpha - 1) \cdots (\alpha - n + 1)(1 + x)^{\alpha - n}, \cdots$$

解 取 
$$x_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 
$$f(x) = (1+x)^{\alpha}$$
 时,

$$f = (1+x)^{\alpha}, \quad f' = \alpha(1+x)^{\alpha-1}, \quad f'' = \alpha(\alpha-1)(1+x)^{\alpha-2},$$

$$\ldots, f^{(n)} = \alpha(\alpha - 1) \cdots (\alpha - n + 1)(1 + x)^{\alpha - n}, \cdots$$

所以 
$$\frac{1}{n!}f^{(n)}(0) = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$$
,

解 取 
$$x_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 
$$f(x) = (1+x)^{\alpha}$$
 时,

$$f = (1+x)^{\alpha}$$
,  $f' = \alpha(1+x)^{\alpha-1}$ ,  $f'' = \alpha(\alpha-1)(1+x)^{\alpha-2}$ ,

$$\ldots, f^{(n)} = \alpha(\alpha - 1) \cdots (\alpha - n + 1)(1 + x)^{\alpha - n}, \cdots$$

所以 
$$\frac{1}{n!}f^{(n)}(0) = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$$
,泰勒级数是 
$$1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots$$

解 取 
$$x_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 
$$f(x) = (1+x)^{\alpha}$$
 时,

$$f = (1+x)^{\alpha}, \quad f' = \alpha(1+x)^{\alpha-1}, \quad f'' = \alpha(\alpha-1)(1+x)^{\alpha-2},$$
  
...,  $f^{(n)} = \alpha(\alpha-1)\cdots(\alpha-n+1)(1+x)^{\alpha-n}, \cdots$ 

所以 
$$\frac{1}{n!}f^{(n)}(0) = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$$
,泰勒级数是

$$1 + \alpha x + \frac{\alpha(\alpha - 1)}{2!} x^2 + \dots + \frac{\alpha(\alpha - 1) \cdots (\alpha - n + 1)}{n!} x^n + \dots$$



 $\mathbf{m}$  取  $x_0 = 0$  时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{2!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当  $f(x) = (1+x)^{\alpha}$  时,

$$f = (1+x)^{\alpha}, \quad f' = \alpha(1+x)^{\alpha-1}, \quad f'' = \alpha(\alpha-1)(1+x)^{\alpha-2},$$

$$\ldots, f^{(n)} = \alpha(\alpha - 1) \cdots (\alpha - n + 1)(1 + x)^{\alpha - n}, \cdots$$

所以 
$$\frac{1}{n!}f^{(n)}(0) = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$$
,泰勒级数是 
$$1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n + \cdots$$

注 n 次泰勒多项式是:

$$p_n(x) =$$



 $f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{2!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$ 

 $\mathbf{H}$  取  $x_0 = 0$  时,泰勒级数是

例 求  $f(x) = (1+x)^{\alpha}$  在 x = 0 处的 n 次泰勒多项式  $p_n(x)$ 

 $1 + \alpha x + \frac{\alpha(\alpha - 1)}{2!} x^2 + \dots + \frac{\alpha(\alpha - 1) \cdots (\alpha - n + 1)}{2!} x^n + \dots$ 

所以  $\frac{1}{n!}f^{(n)}(0) = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$ , 泰勒级数是

 $p_n(x) = 1 + \alpha x + \frac{\alpha(\alpha - 1)}{2!}x^2 + \dots + \frac{\alpha(\alpha - 1)\cdots(\alpha - n + 1)}{n!}x^n$ 

 $\dots, f^{(n)} = \alpha(\alpha - 1) \cdots (\alpha - n + 1)(1 + x)^{\alpha - n}, \dots$ 

**注** *n* 次泰勒多项式是:

当  $f(x) = (1+x)^{\alpha}$  时,  $f = (1+x)^{\alpha}$ ,  $f' = \alpha(1+x)^{\alpha-1}$ ,  $f'' = \alpha(\alpha-1)(1+x)^{\alpha-2}$ 

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0) (x - x_0)^n$$

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0) (x - x_0)^n \iff f(x) = \lim_{n \to \infty} p_n(x)$$

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0) (x - x_0)^n \iff f(x) = \lim_{n \to \infty} p_n(x)$$

$$\Leftrightarrow \lim_{n\to\infty} [f(x) - p_n(x)] = 0$$

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0) (x - x_0)^n \iff f(x) = \lim_{n \to \infty} p_n(x)$$

$$\Leftrightarrow \lim_{n \to \infty} [f(x) - p_n(x)] = 0$$

$$( \Leftrightarrow R_n(x) = f(x) - p_n(x) )$$

$$\Leftrightarrow \lim_{n \to \infty} R_n(x) = 0$$

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0) (x - x_0)^n \iff f(x) = \lim_{n \to \infty} p_n(x)$$

$$\Leftrightarrow \lim_{n \to \infty} [f(x) - p_n(x)] = 0$$

$$( R_n(x) = f(x) - p_n(x) )$$

$$\Leftrightarrow \lim_{n \to \infty} R_n(x) = 0$$

注  $R_n(x) = f(x) - p_n(x)$ ,或者  $f(x) = p_n(x) + R_n(x)$ ,刻画了原函数 f(x) 与其泰勒多项式  $p_n(x)$  的差异。



回忆 泰勒中值定理 1 若 f 具有 n 阶导数,则

$$R_n(x) = o((x-x_0)^n).$$

回忆 泰勒中值定理 1 若 f 具有 n 阶导数,则

$$R_n(x) = o((x-x_0)^n).$$

特别地,

$$f(x) = p_n(x) + R_n(x)$$

回忆 泰勒中值定理 1 若 f 具有 n 阶导数,则

$$R_n(x) = o((x-x_0)^n).$$

特别地,

$$f(x) = p_n(x) + R_n(x)$$

$$= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + o((x - x_0)^n)$$

 $e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + o(x^{n})$ 

$$(1+x)^{\alpha} = 1 + \alpha x$$

 $(1+x)^{\alpha} = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \dots + \frac{\alpha(\alpha-1) \cdots (\alpha-n+1)}{n!} x^n + o(x^n)$ 

 $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots + (-1)^{n-1}\frac{1}{n}x^n + o(x^n)$ 

 $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + o(x^{2m+2})$ 

 $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + o(x^{2m+1})$ 

例求  $\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$ ,  $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1-x)]}$ 

例求 
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
,  $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1-x)]}$ 

$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$

例求 
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
,  $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1-x)]}$ 

$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$$

例求 
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
,  $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1-x)]}$ 

$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$$

$$= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3}$$

$$= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3}$$



例求 
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
,  $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1-x)]}$ 

$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$$

$$= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3} = \frac{1}{3}$$

例 求 
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
,  $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1-x)]}$ 

$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$$

$$\frac{1}{2}x^3 + o(x^4) = 1$$

$$\sin^{3} x \qquad x \to 0 
= \lim_{x \to 0} \frac{\frac{1}{3}x^{3} + o(x^{4})}{x^{3}} = \frac{1}{3}$$

$$= \lim_{X \to 0} \frac{\frac{1}{3}X^3 + o(X^4)}{X^3} =$$

$$\cos X - e^{-\frac{X^2}{2}}$$

$$\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2 [x + \ln(1 - x)]}$$

$$\lim_{x\to 0} \frac{1}{x^2 [x + \ln(1-x)]}$$

$$= \lim_{x \to 0} \frac{\left[ \frac{1}{x + \ln(1 - x)} \right]}{x^2 \left[ x + \left( \frac{1}{x + \ln(1 - x)} \right) \right]}$$



例 求 
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
,  $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1-x)]}$ 

例 求 
$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
,  $\lim_{x \to 0} \frac{\cos x - e^{-2}}{x^2 [x + \ln(1 - x)]}$ 

$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$$

$$= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3} = \frac{1}{3}$$

$$\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2 [x + \ln(1 - x)]}$$

$$x \to 0 \quad x^{2} \left[ x + \ln(1 - x) \right]$$

$$= \lim_{x \to 0} \frac{\left[ 1 - \frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[ \frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[ \frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[ \frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[ \frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[ \frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[ \frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[ \frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[ \frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[ \frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[ \frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[ \frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[ \frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[ \frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[ \frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[ \frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[ \frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[ \frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[ \frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[ \frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[ \frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[ \frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[ \frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[ \frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[ \frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[ \frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[ \frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[ \frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[ \frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[ \frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[ \frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[ \frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[ \frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[ \frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[ \frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[ \frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[ \frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[ \frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[ \frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[$$

例 求 
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
,  $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1-x)]}$ 

$$\lim_{x \to 0} \frac{\sin^3 x}{\sin^3 x} + \lim_{x \to 0} \frac{x^2 [x + \ln(1 - x)]}{x^2 [x + \ln(1 - x)]}$$

$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$$

$$\begin{array}{ll}
\Rightarrow 0 & \sin^3 x & x \to 0 \\
& = \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3} = \frac{1}{3}
\end{array}$$

$$\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2 [x + \ln(1 - x)]}$$

$$\frac{1}{x^2} \frac{1}{x^2} \frac{1}$$

$$\frac{1}{2} x^2 + \frac{1}{2} x^4$$

 $= \lim_{x \to 0} \frac{\left[1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + o(x^5)\right] - \left[1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + o(x^4)\right]}{x^2 \left[x + \left(\frac{1}{2}x^2 + \frac{1}{8}x^4 + o(x^4)\right]\right]}$ 

例 求 
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
,  $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1-x)]}$ 

$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$$
$$= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3} = \frac{1}{3}$$

$$= \lim_{x \to 0} \frac{3}{x^3}$$

$$\cos x - e^{-\frac{x^2}{2}}$$

$$\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2 \left[ x + \ln(1 - x) \right]}$$

$$= \lim_{x \to 0} \frac{\left[ 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 + o(x^5) \right] - \left[ 1 - \frac{1}{2} x^2 + \frac{1}{8} x^4 + o(x^4) \right]}{x^2 \left[ x + \left( -x - \frac{1}{2} x^2 + o(x^2) \right) \right]}$$



$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} =$$

$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$$
$$= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3} = \frac{1}{3}$$

$$\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2 [x + \ln(1 - x)]}$$

$$[x + ln($$

 $= \lim_{x \to 0} \frac{-\frac{1}{12}x^4 + o(x^4)}{-\frac{1}{2}x^4 + o(x^4)}$ 

 $= \lim_{x \to 0} \frac{\left[1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + o(x^5)\right] - \left[1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + o(x^4)\right]}{x^2 \left[x + \left(-x - \frac{1}{2}x^2 + o(x^2)\right)\right]}$ 

$$x^2[x]$$

例 求  $\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$ ,  $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1-x)]}$ 解

 $\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$ 

 $\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2 [x + \ln(1 - x)]}$ 

 $= \lim_{x \to 0} \frac{\left[1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + o(x^5)\right] - \left[1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + o(x^4)\right]}{x^2 \left[x + \left(-x - \frac{1}{2}x^2 + o(x^2)\right)\right]}$ 

函数展开成幂级数

 $= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3} = \frac{1}{2}$ 

 $= \lim_{x \to 0} \frac{-\frac{1}{12}x^4 + o(x^4)}{-\frac{1}{2}x^4 + o(x^4)} = \lim_{x \to 0} \frac{-\frac{1}{12} + o(x^4)/x^4}{-\frac{1}{2} + o(x^4)/x^4}$ 

例 求 
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
,  $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x}{2}}}{x^2[x + \ln(1 - x)]}$ 

$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x\to 0} \frac{\left[x - \frac{1}{3!}x^3 + c^3\right]}{x^2[x + \ln(1 - x)]}$$

$$\lim_{x \to \infty} \frac{\sin x - x \cos x}{\sin x - x \cos x}$$

 $\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$  $= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3} = \frac{1}{2}$ 

$$\lim_{n \to \infty} \frac{\cos x - e^{n}}{n}$$

$$\lim_{x \to 0} \frac{\cos x - \cos x}{x^2 [x + \ln(x)]}$$

$$x \to 0 \quad X^{2} \left[ X + \ln(1 - \frac{1}{2!} X^{2} + \frac{1}{2!} X^{2} + \frac{1}{2!} X^{2} + \frac{1}{2!} X^{2} \right]$$

$$= \lim_{x \to 0} \frac{\left[ 1 - \frac{1}{2!} X^{2} + \frac{1}{2!} X^{2} + \frac{1}{2!} X^{2} + \frac{1}{2!} X^{2} + \frac{1}{2!} X^{2} \right]}{x^{2} \left[ 1 - \frac{1}{2!} X^{2} + \frac{1}{2$$

函数展开成幂级数

 $\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2 [x + \ln(1 - x)]}$ 

 $= \lim_{x \to 0} \frac{\left[1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + o(x^5)\right] - \left[1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + o(x^4)\right]}{x^2 \left[x + \left(-x - \frac{1}{2}x^2 + o(x^2)\right)\right]}$ 

 $= \lim_{x \to 0} \frac{-\frac{1}{12}x^4 + o(x^4)}{-\frac{1}{2}x^4 + o(x^4)} = \lim_{x \to 0} \frac{-\frac{1}{12} + o(x^4)/x^4}{-\frac{1}{2} + o(x^4)/x^4} = \frac{1}{6}$ 

泰勒中值定理 2 若 f 具有 n+1 阶导数,则

$$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) (x - x_0)^{n+1}$$

其中 ξ 是  $x_0$  与 x 之间的某个值

$$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) (x - x_0)^{n+1}$$

$$(1-\theta)x_0+\theta x$$

其中  $\xi$  是  $x_0$  与 x 之间的某个值, $0 < \theta < 1$ 。

$$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) (x - x_0)^{n+1}$$

$$\stackrel{or}{=} \frac{1}{(n+1)!} f^{(n+1)}((1-\theta)x_0 + \theta x) (x - x_0)^{n+1},$$

其中  $\xi$  是  $x_0$  与 x 之间的某个值, $0 < \theta < 1$ 。

$$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) (x - x_0)^{n+1}$$

$$\stackrel{or}{=} \frac{1}{(n+1)!} f^{(n+1)}((1-\theta)x_0 + \theta x) (x - x_0)^{n+1},$$

其中  $\xi$  是  $x_0$  与 x 之间的某个值, $0 < \theta < 1$ 。

## 注

1. ξ (以及 θ) 不是固定不变的,而是随 x 和 n 的改变而变化。

$$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) (x - x_0)^{n+1}$$

$$\stackrel{or}{=} \frac{1}{(n+1)!} f^{(n+1)} ((1-\theta)x_0 + \theta x) (x - x_0)^{n+1},$$

其中  $\xi$  是  $x_0$  与 x 之间的某个值, $0 < \theta < 1$ 。

## 注

- 1. ξ (以及 θ) 不是固定不变的,而是随 x 和 n 的改变而变化。
- 2. 当  $x_0 = 0$  时,则余项可写成

$$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1}, \qquad (0 < \theta < 1)$$



$$e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty).$$

$$e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty).$$

证明

1. 只需证明对任意 x, 成立  $\lim_{n\to\infty} R_n(x) = 0$ 。

$$e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty).$$

- 1. 只需证明对任意 x,成立  $\lim_{n\to\infty} R_n(x) = 0$ 。
- 2. 由泰勒中值定理 2.

$$|R_n(x)| = \left| \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1} \right|$$

$$e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty).$$

- 1. 只需证明对任意 x,成立  $\lim_{n\to\infty} R_n(x) = 0$ 。
- 2. 由泰勒中值定理 2.

$$|R_n(x)| = \left| \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1} \right| = \left| \frac{e^{\theta x} x^{n+1}}{(n+1)!} \right|$$



$$e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty).$$

- 1. 只需证明对任意 x,成立  $\lim_{n\to\infty} R_n(x) = 0$ 。
- 由泰勒中值定理 2.

$$|R_n(x)| = \left| \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1} \right| = \left| \frac{e^{\theta x} x^{n+1}}{(n+1)!} \right| \le \frac{|x|^{n+1}}{(n+1)!} e^{|x|}$$

$$e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty).$$

- 1. 只需证明对任意 x, 成立  $\lim_{n \to \infty} R_n(x) = 0$ 。
- 2. 由泰勒中值定理 2.

$$|R_n(x)| = \left| \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1} \right| = \left| \frac{e^{\theta x} x^{n+1}}{(n+1)!} \right| \le \frac{|x|^{n+1}}{(n+1)!} e^{|x|} \to 0$$

$$e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty).$$

证明

- 1. 只需证明对任意 x, 成立  $\lim_{n \to \infty} R_n(x) = 0$ 。
- 由泰勒中值定理 2.

$$|R_n(x)| = \left| \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1} \right| = \left| \frac{e^{\theta x} x^{n+1}}{(n+1)!} \right| \le \frac{|x|^{n+1}}{(n+1)!} e^{|x|} \to 0$$
(已知级数  $\sum \frac{|x|^{n+1}}{n}$  收敛,所以一般项  $\frac{|x|^{n+1}}{n} \to 0$ )

(已知级数  $\sum \frac{|x|^{n+1}}{(n+1)!}$  收敛,所以一般项  $\frac{|x|^{n+1}}{(n+1)!} \to 0$ )

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

其中 
$$x \in (-\infty, \infty)$$
。

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

其中  $x \in (-\infty, \infty)$ 。

证明

1. 只需证明对任意 x,成立  $\lim_{n\to\infty} R_n(x) = 0$ 。

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

其中  $x \in (-\infty, \infty)$ 。

- 1. 只需证明对任意 x, 成立  $\lim_{n\to\infty} R_n(x) = 0$ 。
- 2. 由泰勒中值定理 2,

$$|R_n(x)| = \left| \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1} \right|$$



$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

其中  $x \in (-\infty, \infty)$ 。

- 1. 只需证明对任意 x, 成立  $\lim_{n \to \infty} R_n(x) = 0$ 。
- 2. 由泰勒中值定理 2,

$$|R_n(x)| = \left| \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1} \right| = \left| \frac{\sin\left(\theta x + \frac{n+1}{2}\pi\right) x^{n+1}}{(n+1)!} \right|$$



$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

其中  $x \in (-\infty, \infty)$ 。

- 1. 只需证明对任意 x, 成立  $\lim_{n \to \infty} R_n(x) = 0$ 。
- 2. 由泰勒中值定理 2.

$$|R_n(x)| = \left| \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1} \right| = \left| \frac{\sin\left(\theta x + \frac{n+1}{2}\pi\right) x^{n+1}}{(n+1)!} \right|$$

$$\leq \frac{|x|^{n+1}}{(n+1)!}$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

其中  $x \in (-\infty, \infty)$ 。

- 1. 只需证明对任意 x, 成立  $\lim_{n \to \infty} R_n(x) = 0$ 。
- 2. 由泰勒中值定理 2.

$$|R_n(x)| = \left| \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1} \right| = \left| \frac{\sin\left(\theta x + \frac{n+1}{2}\pi\right) x^{n+1}}{(n+1)!} \right|$$

$$\leq \frac{|x|^{n+1}}{(n+1)!} \to 0$$



$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

其中  $x \in (-\infty, \infty)$ 。

证明

- 1. 只需证明对任意 x, 成立  $\lim_{n \to \infty} R_n(x) = 0$ 。
- $n \to \infty$

2. 由泰勒中值定理 2,
$$|R_n(x)| = \left| \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1} \right| = \left| \frac{\sin\left(\theta x + \frac{n+1}{2}\pi\right) x^{n+1}}{(n+1)!} \right|$$

 $\leq \frac{|\mathsf{X}|^{n+1}}{(n+1)!} \to 0$ (已知级数  $\sum_{|\mathsf{X}|^{n+1}}^{|\mathsf{X}|^{n+1}}$  收敛,所以一般项  $\frac{|\mathsf{X}|^{n+1}}{(n+1)!} \to 0$ )

— (n+1): (n+1):



$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

其中  $x \in (-\infty, \infty)$ 。

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

其中  $x \in (-\infty, \infty)$ 。

证明

1. 只需证明对任意 x,成立  $\lim_{n\to\infty} R_n(x) = 0$ 。

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

其中  $x \in (-\infty, \infty)$ 。

- 1. 只需证明对任意 x,成立  $\lim_{n\to\infty} R_n(x) = 0$ 。
- 2. 由泰勒中值定理 2,

$$|R_n(x)| = \left| \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1} \right|$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

其中  $x \in (-\infty, \infty)$ 。

- 1. 只需证明对任意 x, 成立  $\lim_{n \to \infty} R_n(x) = 0$ 。
- 2. 由泰勒中值定理 2.

$$|R_n(x)| = \left| \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1} \right| = \left| \frac{\cos\left(\theta x + \frac{n+1}{2}\pi\right) x^{n+1}}{(n+1)!} \right|$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

其中  $x \in (-\infty, \infty)$ 。

- 1. 只需证明对任意 x, 成立  $\lim_{n\to\infty} R_n(x) = 0$ 。
- 2. 由泰勒中值定理 2.

$$|R_n(x)| = \left| \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1} \right| = \left| \frac{\cos\left(\theta x + \frac{n+1}{2}\pi\right) x^{n+1}}{(n+1)!} \right|$$

$$\leq \frac{|x|^{n+1}}{(n+1)!}$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

其中  $x \in (-\infty, \infty)$ 。

- 1. 只需证明对任意 x, 成立  $\lim_{n\to\infty} R_n(x) = 0$ 。
- 2. 由泰勒中值定理 2.

$$|R_n(x)| = \left| \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1} \right| = \left| \frac{\cos\left(\theta x + \frac{n+1}{2}\pi\right) x^{n+1}}{(n+1)!} \right|$$

$$\leq \frac{|x|^{n+1}}{(n+1)!} \to 0$$

性质 对任意  $x \in (-\infty, \infty)$ ,  $\sin x$  等于其泰勒级数。即

1 1 1 1 1 1 1

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

证明

- 1. 只需证明对任意 x, 成立  $\lim_{n \to \infty} R_n(x) = 0$ 。
- 2. 由泰勒中值定理 2.

其中  $x \in (-\infty, \infty)$ 。

$$|R_n(x)| = \left| \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1} \right| = \left| \frac{\cos\left(\theta x + \frac{n+1}{2}\pi\right) x^{n+1}}{(n+1)!} \right|$$

$$\leq \frac{|x|^{n+1}}{(n+1)!} \to 0$$

(已知级数  $\sum_{(n+1)}^{|x|^{n+1}}$  收敛,所以一般项  $\frac{|x|^{n+1}}{(n+1)} \to 0$ )



• 至此,我们知道  $e^x$ ,  $\sin x$ ,  $\cos x$  以及  $\frac{1}{1+x}$  是等于其泰勒级数,即

$$e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty)$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots + (-1)^n \frac{1}{(2n+1)!}x^{2n+1} + \dots, x \in (-\infty, \infty)$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{2!}x^4 - \frac{1}{2!}x^6 + \dots + (-1)^n \frac{1}{2!}x^{2n} + \dots \qquad x \in (-\infty, \infty)$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots + (-1)^n \frac{1}{(2n)!}x^{2n} + \dots, \quad x \in (-\infty, \infty)$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots + (-1)^n x^n + \dots, x \in (-1, 1)$$



• 至此,我们知道  $e^x$ , $\sin x$ , $\cos x$  以及  $\frac{1}{1+x}$  是等于其泰勒级数,即 1 1 1 1

$$e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty)$$

 $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots + (-1)^n \frac{1}{(2n+1)!}x^{2n+1} + \dots, x \in (-\infty, \infty)$   $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots + (-1)^n \frac{1}{(2n)!}x^{2n} + \dots, \quad x \in (-\infty, \infty)$ 

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots + (-1)^n \frac{1}{(2n)!}x^{2n} + \dots, \quad x \in (-\infty, \infty)$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots + (-1)^n x^n + \dots, x \in (-1, 1)$$
• 利用上述结果,及逐项积分公式,可进一步求出

• 利用工处结果,及逐项积分公式,可进一步来出 $\ln(1+x)$ ,  $\arctan x$ 

的幂级数展开。



性质 成立 
$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$$

 $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$ 

 $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$ 

$$\ln(1+x) = \int_0^x \frac{1}{1+t} dt$$



2.

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$$

$$\ln(1+x) = \int_0^x \frac{1}{1+t} dt = \int_0^x \sum_{n=0}^\infty (-1)^n t^n dt$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$$

2. 
$$\ln(1+x) = \int_0^x \frac{1}{1+t} dt = \int_0^x \sum_{n=0}^\infty (-1)^n t^n dt = \sum_{n=0}^\infty (-1)^n \int_0^x t^n dt$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$$

$$\ln(1+x) = \int_0^x \frac{1}{1+t} dt = \int_0^x \sum_{n=0}^\infty (-1)^n t^n dt = \sum_{n=0}^\infty (-1)^n \int_0^x t^n dt$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1} x^{n+1}$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$$

$$\ln(1+x) = \int_0^x \frac{1}{1+t} dt = \int_0^x \sum_{n=0}^\infty (-1)^n t^n dt = \sum_{n=0}^\infty (-1)^n \int_0^x t^n dt$$

$$\ln(1+x) = \int_0^{\infty} \frac{1}{1+t} dt = \int_0^{\infty} \sum_{n=0}^{\infty} (-1)^n t^n dt = \sum_{n=0}^{\infty} (-1)^n \int_0^{\infty} t^n dt$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1} x^{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1} \frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$$

证明 1. 幂级数的收敛域是 (-1, 1], 故上式至多对  $x \in (-1, 1]$  成立。

2. 当  $x \in (-1, 1)$  时,利用逐项积分可得

$$\ln(1+x) = \int_0^x \frac{1}{1+t} dt = \int_0^x \sum_{n=0}^\infty (-1)^n t^n dt = \sum_{n=0}^\infty (-1)^n \int_0^x t^n dt$$
$$= \sum_{n=0}^\infty (-1)^n \frac{1}{1+t} x^{n+1} = \sum_{n=0}^\infty \frac{(-1)^{n-1}}{1+t} x^n$$

$$=\sum_{n=0}^{\infty}(-1)^n\frac{1}{n+1}x^{n+1}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n$$



$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$$

证明 1. 幂级数的收敛域是 (-1, 1],故上式至多对  $x \in (-1, 1]$  成立。

2. 当 
$$x \in (-1, 1)$$
 时,利用逐项积分可得
$$\ln(1+x) = \int_0^x \frac{1}{1+t} dt = \int_0^x \sum_{n=0}^\infty (-1)^n t^n dt = \sum_{n=0}^\infty (-1)^n \int_0^x t^n dt$$
$$= \sum_{n=0}^\infty (-1)^n \frac{1}{n+1} x^{n+1} = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n} x^n$$

3. 注意到  $\sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n} x^n$  收敛域是 (-1, 1], 由连续性, 当 x=1 时也

成立 
$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n} x^n.$$



性质 成立  $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$ 

证明 1. 幂级数的收敛域是 (-1, 1],故上式至多对  $x \in (-1, 1]$  成立。

2. 当 
$$x \in (-1, 1)$$
 时,利用逐项积分可得

$$\ln(1+x) = \int_0^x \frac{1}{1+t} dt = \int_0^x \sum_{n=0}^\infty (-1)^n t^n dt = \sum_{n=0}^\infty (-1)^n \int_0^x t^n dt$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1} x^{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$$

3. 注意到 
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$$
 收敛域是  $(-1, 1]$ , 由连续性, 当  $x = 1$  时也

成立  $ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n} x^n.$ 

成立 
$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{x^n}.$$

(这是f(1) == S(1)

 $21/27 \triangleleft \triangleright \triangle \nabla$ 

性质 成立  $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$ 

证明 1. 幂级数的收敛域是 (-1, 1],故上式至多对  $x \in (-1, 1]$  成立。

2. 当  $x \in (-1, 1)$  时,利用逐项积分可得

$$\ln(1+x) = \int_0^x \frac{1}{1+t} dt = \int_0^x \sum_{n=0}^\infty (-1)^n t^n dt = \sum_{n=0}^\infty (-1)^n \int_0^x t^n dt$$

 $=\sum_{n=0}^{\infty}(-1)^n\frac{1}{n+1}x^{n+1}=\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{n}x^n$ 

 $\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n} x^{n}.$ 成立

3. 注意到 
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$$
 收敛域是 (-1, 1], 由连续性, 当  $x = 1$  时也

第 12 章 d: 函数展开成幂级数

(这是 $f(1) = \lim_{x \to 1^-} \ln(1+x)$ = S(1)

性质 成立 
$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1} \frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$$

证明 1. 幂级数的收敛域是 (-1, 1],故上式至多对  $x \in (-1, 1]$  成立。

2. 当  $x \in (-1, 1)$  时,利用逐项积分可得

$$\ln(1+x) = \int_0^x \frac{1}{1+t} dt = \int_0^x \sum_{n=0}^\infty (-1)^n t^n dt = \sum_{n=0}^\infty (-1)^n \int_0^x t^n dt$$

$$=\sum_{n=0}^{\infty}(-1)^n\frac{1}{n+1}x^{n+1}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n$$

3. 注意到 
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$$
 收敛域是 (-1, 1], 由连续性, 当  $x=1$  时也

 $\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n} x^{n}.$ 成立

3. 注意到 
$$\sum_{n=1}^{\infty} \frac{(-1, 1)}{n}$$
 收敛域是  $(-1, 1)$ ,由连续性,当  $x = 1$  时也  $\infty$  ( 1) $n = 1$ 

 $\lim_{x\to 1^-} S(x) = S(1)$ (这是 $f(1) = \lim_{x \to 1^-} \ln(1+x)$ 

21/27 ▷ ▷ ▷ ♡

性质 成立 
$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1} \frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$$

证明 1. 幂级数的收敛域是 (-1, 1],故上式至多对  $x \in (-1, 1]$  成立。

2. 当  $x \in (-1, 1)$  时,利用逐项积分可得

$$\ln(1+x) = \int_0^x \frac{1}{1+t} dt = \int_0^x \sum_{n=0}^\infty (-1)^n t^n dt = \sum_{n=0}^\infty (-1)^n \int_0^x t^n dt$$

$$=\sum_{n=0}^{\infty}(-1)^n\frac{1}{n+1}x^{n+1}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n$$

3. 注意到  $\sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n} x^n$  收敛域是 (-1, 1], 由连续性, 当 x=1 时也

成立  $\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n} x^n.$ 

(这是 $f(1) = \lim_{x \to 1^{-}} \ln(1+x)$   $\lim_{x \to 1^{-}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n = \lim_{x \to 1^{-}} S(x) = S(1)$ )

12 章 d: 函数展开成幂级数

性质 成立 
$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1} \frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$$

证明 1. 幂级数的收敛域是 (-1, 1],故上式至多对  $x \in (-1, 1]$  成立。

2. 当  $x \in (-1, 1)$  时,利用逐项积分可得

$$\ln(1+x) = \int_0^x \frac{1}{1+t} dt = \int_0^x \sum_{n=0}^\infty (-1)^n t^n dt = \sum_{n=0}^\infty (-1)^n \int_0^x t^n dt$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1} x^{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$$

3. 注意到  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$  收敛域是 (-1, 1], 由连续性, 当 x=1 时也

成立  $\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n} x^n.$ 

(这是 $f(1) = \lim_{x \to 1^{-}} \ln(1+x) = \lim_{x \to 1^{-}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^{n} = \lim_{x \to 1^{-}} S(x) = S(1)$ )

性质 成立  $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots + (-1)^n \frac{1}{2n+1}x^{2n+1} + \dots, x \in [-1,1].$ 

性质 成立  $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots + (-1)^n \frac{1}{2n+1}x^{2n+1} + \dots, x \in [-1,1].$ 证明 1. 幂级数的收敛域是 [-1, 1], 故上式至多对  $x \in [-1, 1]$  成立。

$$\arctan x = \int_{0}^{x} \frac{1}{1+t^2} dt$$

证明 1. 幂级数的收敛域是 [-1, 1], 故上式至多对  $x \in [-1, 1]$  成立。 2.

性质 成立  $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots + (-1)^n \frac{1}{2n+1}x^{2n+1} + \dots, x \in [-1,1].$ 



 $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots + (-1)^n \frac{1}{2n+1}x^{2n+1} + \dots, x \in [-1,1].$ 

性质 成立

证明 1. 幂级数的收敛域是 
$$[-1, 1]$$
,故上式至多对  $x \in [-1, 1]$  成立。

2.

$$\arctan x = \int_0^{x} \frac{1}{1+t^2} dt = \int_0^{x} \sum_{n=0}^{\infty} (-1)^n t^{2n} dt$$



证明 1. 幂级数的收敛域是 [-1, 1], 故上式至多对  $x \in [-1, 1]$  成立。

 $\arctan x = \int_0^x \frac{1}{1+t^2} dt = \int_0^x \sum_{n=0}^\infty (-1)^n t^{2n} dt = \sum_{n=0}^\infty (-1)^n \int_0^x t^{2n} dt$ 

性质 成立  $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots + (-1)^n \frac{1}{2n+1}x^{2n+1} + \dots, x \in [-1,1].$ 

第 12 章 d: 函数展开成幂级数

性质 成立  $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots + (-1)^n \frac{1}{2n+1}x^{2n+1} + \dots, x \in [-1,1].$ 

证明 1. 幂级数的收敛域是 [-1, 1], 故上式至多对  $x \in [-1, 1]$  成立。

$$\arctan x = \int_0^x \frac{1}{1+t^2} dt = \int_0^x \sum_{n=0}^\infty (-1)^n t^{2n} dt = \sum_{n=0}^\infty (-1)^n \int_0^x t^{2n} dt$$
$$= \sum_{n=0}^\infty (-1)^n \frac{1}{2n+1} x^{2n+1}$$

性质 成立  $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots + (-1)^n \frac{1}{2n+1}x^{2n+1} + \dots, x \in [-1,1].$ 

证明 1. 幂级数的收敛域是 [-1, 1], 故上式至多对  $x \in [-1, 1]$  成立。

2. 当  $x \in (-1, 1)$  时,利用逐项积分可得

 $\arctan x = \int_0^x \frac{1}{1+t^2} dt = \int_0^x \sum_{n=0}^\infty (-1)^n t^{2n} dt = \sum_{n=0}^\infty (-1)^n \int_0^x t^{2n} dt$ 

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1}$$

性质 成立  $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots + (-1)^n \frac{1}{2n+1}x^{2n+1} + \dots, x \in [-1,1].$ 

证明 1. 幂级数的收敛域是 
$$[-1, 1]$$
,故上式至多对  $x \in [-1, 1]$  成立。

证明 1. 希级数的收敛或是 [-1,1],战上式至多为  $X \in [-1,1]$  成立。

2. 当 x ∈ (-1, 1) 时, 利用逐项积分可得

$$\arctan x = \int_0^x \frac{1}{1+t^2} dt = \int_0^x \sum_{n=0}^\infty (-1)^n t^{2n} dt = \sum_{n=0}^\infty (-1)^n \int_0^x t^{2n} dt$$
$$= \sum_{n=0}^\infty (-1)^n \frac{1}{2n+1} x^{2n+1}$$

$$\sum_{n=0}^{\infty} (-1)^n x^{2n+1}$$
 收敛域是 [-1 1] 中连续性 当  $x$ 

3. 注意到  $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$  收敛域是 [-1, 1], 由连续性, 当  $x = \pm 1$  时地有

也有  $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1}$ .



证明 1. 幂级数的收敛域是 [-1, 1], 故上式至多对  $x \in [-1, 1]$  成立。

 $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots + (-1)^n \frac{1}{2n+1}x^{2n+1} + \dots, x \in [-1,1].$ 

2. 当 
$$x \in (-1, 1)$$
 时,利用逐项积分可得  $C^{X}$  0  $C^{X}$ 

$$\arctan x = \int_0^x \frac{1}{1+t^2} dt = \int_0^x \sum_{n=0}^\infty (-1)^n t^{2n} dt = \sum_{n=0}^\infty (-1)^n \int_0^x t^{2n} dt$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1}$$

 $\arctan x = \sum_{n=1}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1}.$ 也有

3. 注意到  $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$  收敛域是 [-1, 1], 由连续性, 当  $x = \pm 1$  时

(如f(1) == S(1)

3. 注意到  $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$  收敛域是 [-1, 1], 由连续性, 当  $x = \pm 1$  时

性质 成立

也有

 $(\inf(1) = \lim_{x \to 1^{-}} \operatorname{arctan} x$ 

 $\arctan x = \int_0^{\infty} \frac{1}{1+t^2} dt = \int_0^{\infty} \sum_{n=0}^{\infty} (-1)^n t^{2n} dt = \sum_{n=0}^{\infty} (-1)^n \int_0^{\infty} t^{2n} dt$ 

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1}$$

$$=\sum_{n=0}^{\infty}(-1)^{n}\frac{1}{x^{2n+1}}$$

 $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots + (-1)^n \frac{1}{2n+1}x^{2n+1} + \dots, x \in [-1,1].$ 

证明 1. 幂级数的收敛域是 [-1, 1], 故上式至多对  $x \in [-1, 1]$  成立。

$$\int_{0}^{\infty} \frac{1+t^{2}}{1+t^{2}} dt = \int_{0}^{\infty} \int_{n=0}^{\infty} (-1)^{n} t dt = \sum_{n=0}^{\infty} (-1)^{n} \int_{0}^{\infty} t dt$$

2. 当  $x \in (-1, 1)$  时,利用逐项积分可得

 $\arctan x = \sum_{n=1}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1}.$ 

第 12 章 d: 函数展开成幂级数

 $22/27 \triangleleft \triangleright \triangle \nabla$ 

= S(1)

第 12 章 d: 函数展开成幂级数

也有

 $\arctan x = \sum_{n=1}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1}.$ 

 $(\inf(1) = \lim_{x \to 1^{-}} \operatorname{arctan} x$ 

 $\lim_{x\to 1^-} S(x) = S(1)$ 

3. 注意到  $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$  收敛域是 [-1, 1], 由连续性, 当  $x = \pm 1$  时

 $\arctan x = x - \frac{1}{2}x^3 + \frac{1}{5}x^5 + \dots + (-1)^n \frac{1}{2n+1}x^{2n+1} + \dots, x \in [-1,1].$ 

 $=\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1}$ 

2. 当  $x \in (-1, 1)$  时,利用逐项积分可得  $\arctan x = \int_0^{\infty} \frac{1}{1+t^2} dt = \int_0^{\infty} \sum_{n=0}^{\infty} (-1)^n t^{2n} dt = \sum_{n=0}^{\infty} (-1)^n \int_0^{\infty} t^{2n} dt$ 

证明 1. 幂级数的收敛域是 [-1, 1],故上式至多对  $x \in [-1, 1]$  成立。

也有

性质 成立

证明 1. 幂级数的收敛域是 [-1, 1], 故上式至多对  $x \in [-1, 1]$  成立。 2. 当  $x \in (-1, 1)$  时,利用逐项积分可得

 $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots + (-1)^n \frac{1}{2n+1}x^{2n+1} + \dots, x \in [-1,1].$ 

 $\arctan x = \int_0^x \frac{1}{1+t^2} dt = \int_0^x \sum_{n=0}^\infty (-1)^n t^{2n} dt = \sum_{n=0}^\infty (-1)^n \int_0^x t^{2n} dt$ 

 $=\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{2n+1}x^{2n+1}$ 

3. 注意到  $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$  收敛域是 [-1, 1], 由连续性, 当  $x = \pm 1$  时

 $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1}.$ 

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1}.$$

 $(\inf(1) = \lim_{x \to 1^{-}} \arctan x \quad \lim_{x \to 1^{-}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = \lim_{x \to 1^{-}} S(x) = S(1))$ 

2. 当  $x \in (-1, 1)$  时,利用逐项积分可得

 $\arctan x = \int_0^x \frac{1}{1+t^2} dt = \int_0^x \sum_{n=0}^\infty (-1)^n t^{2n} dt = \sum_{n=0}^\infty (-1)^n \int_0^x t^{2n} dt$ 

 $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots + (-1)^n \frac{1}{2n+1}x^{2n+1} + \dots, x \in [-1,1].$ 

证明 1. 幂级数的收敛域是 [-1, 1], 故上式至多对  $x \in [-1, 1]$  成立。

$$=\sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}x^{2n+1}$$

3. 注意到  $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$  收敛域是 [-1, 1], 由连续性, 当  $x = \pm 1$  时

 $\arctan x = \sum_{n=1}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1}.$ 也有

 $(\inf(1) = \lim_{x \to 1^{-}} \arctan x = \lim_{x \to 1^{-}} \sum_{n=0}^{\infty} \frac{(-1)^{n}}{2n+1} x^{2n+1} = \lim_{x \to 1^{-}} S(x) = S(1))$ 

$$\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots + (-1)^n \frac{1}{2n+1}x^{2n+1} + \dots, \quad x \in [-1, 1]$$



注 取 x=1,则得到

 $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots + (-1)^n \frac{1}{2n+1}x^{2n+1} + \dots, \quad x \in [-1, 1]$ 

$$\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots + (-1)^n \frac{1}{2n+1}x^{2n+1} + \dots, \quad x \in [-1, 1]$$

注 取 
$$x=1$$
,则得到

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$$



 $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots + (-1)^n x^n + \dots, x \in (-1, 1)$ 

 $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1]$ 

• 至此, 得出如下常用函数的幂级数展开式:

 $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots + (-1)^n \frac{1}{2n+1}x^{2n+1} + \dots, x \in [-1,1]$ 

 $e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty)$ 

 $\sin x = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 + \dots + (-1)^n \frac{1}{(2n+1)!} x^{2n+1} + \dots, \ x \in (-\infty, \infty)$  $\cos x = 1 - \tfrac{1}{2!} x^2 + \tfrac{1}{4!} x^4 - \tfrac{1}{6!} x^6 + \dots + (-1)^n \tfrac{1}{(2n)!} x^{2n} + \dots \,, \ x \in (-\infty, \infty)$ 

 $\sin x = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 + \dots + (-1)^n \frac{1}{(2n+1)!} x^{2n+1} + \dots, \ x \in (-\infty, \infty)$ 

• 至此, 得出如下常用函数的幂级数展开式:

 $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1]$ 

 $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots + (-1)^n \frac{1}{2n+1}x^{2n+1} + \dots, x \in [-1,1]$ 

 $e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty)$ 

 $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots + (-1)^n x^n + \dots, x \in (-1, 1)$ 

 $\cos x = 1 - \tfrac{1}{2!} x^2 + \tfrac{1}{4!} x^4 - \tfrac{1}{6!} x^6 + \dots + (-1)^n \tfrac{1}{(2n)!} x^{2n} + \dots \,, \; x \in (-\infty, \infty)$ 

• 用上述结果, 及逐项求导、积分公式, 可求更多函数的泰勒级数展开

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$$

所以当 
$$x \in (-1, 1]$$
 时,  

$$(1-x)\ln(1+x) = (1-x)\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$$

所以当 
$$x \in (-1, 1]$$
 时,
$$(1-x)\ln(1+x) = (1-x)\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n$$

$$=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}x^{n}-\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}x^{n+1}$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$$

所以当 
$$x \in (-1, 1]$$
 时,  

$$(1-x)\ln(1+x) = (1-x)\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n - \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^{n+1}$$
$$\sum_{n=1}^{\infty} (-1)^{n-2} \frac{1}{n-1} x^n$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$$

所以当 
$$x \in (-1, 1]$$
 时,  

$$(1-x)\ln(1+x) = (1-x)\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n - \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^{n+1}$$
$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n - \sum_{n=2}^{\infty} (-1)^{n-2} \frac{1}{n-1} x^n$$

解 利用

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$$

所以当  $x \in (-1, 1]$  时,  $(1-x)\ln(1+x) = (1-x)\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n$ 

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n - \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^{n+1}$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n - \sum_{n=2}^{\infty} (-1)^{n-2} \frac{1}{n-1} x^n$$

$$= x + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{1}{n} x^n$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$$

所以当 
$$x \in (-1, 1]$$
 时,  
 $(1-x)\ln(1+x) = (1-x)\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n$ 

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n - \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^{n+1}$$
$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n - \sum_{n=2}^{\infty} (-1)^{n-2} \frac{1}{n-1} x^n$$

$$-\sum_{n=1}^{\infty} (-1)^{n} \frac{1}{n} x^{n} - \sum_{n=2}^{\infty} (-1)^{n} \frac{1}{n-1} x^{n}$$

$$= x + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^{n} - \sum_{n=2}^{\infty} (-1)^{n-2} \frac{1}{n-1} x^{n}$$

解利用

所以当 
$$x \in (-1, 1]$$
 时, $(1-x)\ln(1+x) = (-1, 1)$ 

$$(1-x)\ln(1+x) = (1-x)\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n - \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^{n+1}$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n - \sum_{n=1}^{\infty} (-1)^{n-2} \frac{1}{n} x^{n+1}$$

例 把函数  $f(x) = (1-x) \ln(1+x)$  展开成 x 的幂级数。

 $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$ 

 $=x+\sum_{n=0}^{\infty}\left(\frac{(-1)^{n-1}}{n}-\frac{(-1)^n}{n-1}\right)x^n$ 

$$=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}x^{n}-\sum_{n=2}^{\infty}(-1)^{n-2}\frac{1}{n-1}x^{n}$$

$$0 - \sum_{n=0}^{\infty} (-1)^{n-2} \frac{1}{n}$$

$$= x + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{1}{n} x^n - \sum_{n=2}^{\infty} (-1)^{n-2} \frac{1}{n-1} x^n$$

解 利用

$$\cos t = 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6 + \dots + (-1)^n \frac{1}{(2n)!}t^{2n} + \dots, \ t \in (-\infty, \infty)$$

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$$

解 利用

$$\cos t = 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6 + \dots + (-1)^n \frac{1}{(2n)!}t^{2n} + \dots, \ t \in (-\infty, \infty)$$

所以当 
$$x \in (-\infty, \infty)$$
 时,

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$$
$$= \frac{1}{2} + \frac{1}{2}\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} (2x)^{2n}$$

解 利用

$$\cos t = 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6 + \dots + (-1)^n \frac{1}{(2n)!}t^{2n} + \dots, \ t \in (-\infty, \infty)$$

所以当 
$$x \in (-\infty, \infty)$$
 时,

$$\cos^{2} x = \frac{1}{2} + \frac{1}{2} \cos 2x$$

$$= \frac{1}{2} + \frac{1}{2} \sum_{n=0}^{\infty} (-1)^{n} \frac{1}{(2n)!} (2x)^{2n}$$

$$= \frac{1}{2} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^{n} 2^{2n}}{(2n)!} x^{2n}$$

例 把函数  $f(x) = \cos^2 x$  展开成 x 的幂级数。

解 利用

 $\cos t = 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6 + \dots + (-1)^n \frac{1}{(2n)!}t^{2n} + \dots, \ t \in (-\infty, \infty)$ 

所以当  $x \in (-\infty, \infty)$  时,

第 12 章 d: 函数展开成幂级数

 $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$  $= \frac{1}{2} + \frac{1}{2} \sum_{n=1}^{\infty} (-1)^n \frac{1}{(2n)!} (2x)^{2n}$  $= \frac{1}{2} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!} x^{2n}$ 

 $=1+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n}2^{2n}}{(2n)!}x^{2n}$ 

解 1. 注意到 
$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$
.

解 1. 注意到 
$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$
.

将 
$$\frac{1}{x+1}$$
,  $\frac{1}{x+2}$  分别展开成  $(x+4)$  的幂级数:

解 1. 注意到 
$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$
.

2. 利用 
$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \dots + t^n + \dots$$
,  $t \in (-1, 1)$ 

将 
$$\frac{1}{x+1}$$
,  $\frac{1}{x+2}$  分别展开成  $(x+4)$  的幂级数: 令  $t=x+4$ , 则

$$* \quad \frac{1}{x+1} = \frac{1}{t-3}$$

解 1. 注意到 
$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$
.

2. 利用 
$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots + t^n + \cdots$$
,  $t \in (-1, 1)$ 

将 
$$\frac{1}{x+1}$$
,  $\frac{1}{x+2}$  分别展开成  $(x+4)$  的幂级数: 令  $t=x+4$ , 则

\* 
$$\frac{1}{x+1} = \frac{1}{t-3} = \frac{1}{-3} \cdot \frac{1}{1-\frac{t}{3}}$$

解 1. 注意到 
$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$
.

2. 利用 
$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots + t^n + \cdots$$
,  $t \in (-1, 1)$ 

将 
$$\frac{1}{x+1}$$
,  $\frac{1}{x+2}$  分别展开成  $(x+4)$  的幂级数: 令  $t=x+4$ , 则

\* 
$$\frac{1}{x+1} = \frac{1}{t-3} = \frac{1}{-3} \cdot \frac{1}{1-\frac{t}{3}} = -\frac{1}{3} \sum_{n=0}^{\infty} \frac{t^n}{3^n}$$

解 1. 注意到 
$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$
.

2. 利用 
$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots + t^n + \cdots, t \in (-1, 1)$$

将 
$$\frac{1}{x+1}$$
,  $\frac{1}{x+2}$  分别展开成  $(x+4)$  的幂级数: 令  $t=x+4$ , 则

\*  $\frac{1}{x+1} = \frac{1}{t-3} = \frac{1}{-3} \cdot \frac{1}{1-\frac{t}{3}} = -\frac{1}{3} \sum_{n=0}^{\infty} \frac{t^n}{3^n} = -\sum_{n=0}^{\infty} \frac{(x+4)^n}{3^{n+1}}$ 

解 1. 注意到 
$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$
.

2. 利用 
$$\frac{1}{1+t} = 1 + t + t^2 + t^3 + \cdots + t^n + \cdots, t \in (-1, 1)$$

将 
$$\frac{1}{x+1}$$
,  $\frac{1}{x+2}$  分别展开成  $(x+4)$  的幂级数: 令  $t=x+4$ , 则 1 1 1 1  $\frac{\infty}{x+4}$ 

\* 
$$\frac{1}{x+1} = \frac{1}{t-3} = \frac{1}{-3} \cdot \frac{1}{1-\frac{t}{3}} = -\frac{1}{3} \sum_{n=0}^{\infty} \frac{t^n}{3^n} = -\sum_{n=0}^{\infty} \frac{(x+4)^n}{3^{n+1}}$$

其中 
$$|\frac{t}{2}| < 1$$



解 1. 注意到 
$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$
.

将 
$$\frac{1}{x+1}$$
,  $\frac{1}{x+2}$  分别展开成  $(x+4)$  的幂级数: 令  $t=x+4$ , 则

\* 
$$\frac{1}{x+1} = \frac{1}{t-3} = \frac{1}{-3} \cdot \frac{1}{1-\frac{t}{3}} = -\frac{1}{3} \sum_{n=0}^{\infty} \frac{t^n}{3^n} = -\sum_{n=0}^{\infty} \frac{(x+4)^n}{3^{n+1}}$$

其中 
$$\left|\frac{x+4}{3}\right| = \left|\frac{t}{3}\right| < 1$$



解 1. 注意到 
$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$
.

2. 利用 
$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots + t^n + \cdots, t \in (-1, 1)$$

将 
$$\frac{1}{x+1}$$
,  $\frac{1}{x+2}$  分别展开成  $(x+4)$  的幂级数: 令  $t=x+4$ , 则

\* 
$$\frac{1}{x+1} = \frac{1}{t-3} = \frac{1}{-3} \cdot \frac{1}{1-\frac{t}{3}} = -\frac{1}{3} \sum_{n=0}^{\infty} \frac{t^n}{3^n} = -\sum_{n=0}^{\infty} \frac{(x+4)^n}{3^{n+1}}$$

其中 
$$\left|\frac{x+4}{3}\right| = \left|\frac{t}{3}\right| < 1$$
,即  $-7 < x < -1$ 。



解 1. 注意到 
$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$
.

2. 利用  $\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots + t^n + \cdots$ ,  $t \in (-1, 1)$ 

将 
$$\frac{1}{x+1}$$
,  $\frac{1}{x+2}$  分别展开成  $(x+4)$  的幂级数: 令  $t=x+4$ , 则

\* 
$$\frac{1}{x+1} = \frac{1}{t-3} = \frac{1}{-3} \cdot \frac{1}{1-\frac{t}{3}} = -\frac{1}{3} \sum_{n=0}^{\infty} \frac{t^n}{3^n} = -\sum_{n=0}^{\infty} \frac{(x+4)^n}{3^{n+1}}$$

其中  $\left|\frac{x+4}{3}\right| = \left|\frac{t}{3}\right| < 1$ ,即 -7 < x < -1。

\* 
$$\frac{1}{x+2} = \frac{1}{t-2}$$

解 1. 注意到 
$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$
.

2. 利用 
$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots + t^n + \cdots, t \in (-1, 1)$$

将 
$$\frac{1}{x+1}$$
,  $\frac{1}{x+2}$  分别展开成  $(x+4)$  的幂级数: 令  $t=x+4$ , 则

\* 
$$\frac{1}{x+1} = \frac{1}{t-3} = \frac{1}{-3} \cdot \frac{1}{1-\frac{t}{3}} = -\frac{1}{3} \sum_{n=0}^{\infty} \frac{t^n}{3^n} = -\sum_{n=0}^{\infty} \frac{(x+4)^n}{3^{n+1}}$$

其中 
$$\left|\frac{x+4}{3}\right| = \left|\frac{t}{3}\right| < 1$$
,即  $-7 < x < -1$ 。

\* 
$$\frac{1}{x+2} = \frac{1}{t-2} = \frac{1}{-2} \cdot \frac{1}{1-\frac{t}{2}}$$

解 1. 注意到 
$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$
.

2. 利用 
$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \dots + t^n + \dots$$
,  $t \in (-1, 1)$ 

将 
$$\frac{1}{x+1}$$
,  $\frac{1}{x+2}$  分别展开成  $(x+4)$  的幂级数: 令  $t=x+4$ , 则

\* 
$$\frac{1}{x+1} = \frac{1}{t-3} = \frac{1}{-3} \cdot \frac{1}{1-\frac{t}{3}} = -\frac{1}{3} \sum_{n=0}^{\infty} \frac{t^n}{3^n} = -\sum_{n=0}^{\infty} \frac{(x+4)^n}{3^{n+1}}$$

其中 
$$\left|\frac{x+4}{3}\right| = \left|\frac{t}{3}\right| < 1$$
,即  $-7 < x < -1$ 。

\* 
$$\frac{1}{x+2} = \frac{1}{t-2} = \frac{1}{-2} \cdot \frac{1}{1-\frac{t}{2}} = -\frac{1}{2} \sum_{n=0}^{\infty} \frac{t^n}{2^n}$$



解 1. 注意到 
$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$
.

2. 利用 
$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots + t^n + \cdots, t \in (-1, 1)$$

将 
$$\frac{1}{x+1}$$
,  $\frac{1}{x+2}$  分别展开成  $(x+4)$  的幂级数: 令  $t=x+4$ , 则 1 1 1 1  $\stackrel{\frown}{=}$   $t^n$   $\stackrel{\frown}{=}$   $(x+4)^n$ 

\* 
$$\frac{1}{x+1} = \frac{1}{t-3} = \frac{1}{-3} \cdot \frac{1}{1-\frac{t}{3}} = -\frac{1}{3} \sum_{n=0}^{\infty} \frac{t^n}{3^n} = -\sum_{n=0}^{\infty} \frac{(x+4)^n}{3^{n+1}}$$

其中 
$$\left|\frac{x+4}{3}\right| = \left|\frac{t}{3}\right| < 1$$
,即  $-7 < x < -1$ 。

\* 
$$\frac{1}{x+2} = \frac{1}{t-2} = \frac{1}{-2} \cdot \frac{1}{1-\frac{t}{2}} = -\frac{1}{2} \sum_{n=0}^{\infty} \frac{t^n}{2^n} = -\sum_{n=0}^{\infty} \frac{(x+4)^n}{2^{n+1}}$$



解 1. 注意到 
$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$
.

\* 
$$\frac{1}{x+1} = \frac{1}{t-3} = \frac{1}{-3} \cdot \frac{1}{1-\frac{t}{3}} = -\frac{1}{3} \sum_{n=0}^{\infty} \frac{t^n}{3^n} = -\sum_{n=0}^{\infty} \frac{(x+4)^n}{3^{n+1}}$$

其中 
$$\left|\frac{x+4}{3}\right| = \left|\frac{t}{3}\right| < 1$$
,即  $-7 < x < -1$ 。

\* 
$$\frac{1}{x+2} = \frac{1}{t-2} = \frac{1}{-2} \cdot \frac{1}{1-\frac{t}{2}} = -\frac{1}{2} \sum_{n=0}^{\infty} \frac{t^n}{2^n} = -\sum_{n=0}^{\infty} \frac{(x+4)^n}{2^{n+1}}$$

其中 
$$|\frac{t}{2}| < 1$$

解 1. 注意到 
$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$
.

\* 
$$\frac{1}{x+1} = \frac{1}{t-3} = \frac{1}{-3} \cdot \frac{1}{1-\frac{t}{3}} = -\frac{1}{3} \sum_{n=0}^{\infty} \frac{t^n}{3^n} = -\sum_{n=0}^{\infty} \frac{(x+4)^n}{3^{n+1}}$$

其中 
$$\left|\frac{x+4}{3}\right| = \left|\frac{t}{3}\right| < 1$$
,即  $-7 < x < -1$ 。

\* 
$$\frac{1}{x+2} = \frac{1}{t-2} = \frac{1}{-2} \cdot \frac{1}{1-\frac{t}{2}} = -\frac{1}{2} \sum_{n=0}^{\infty} \frac{t^n}{2^n} = -\sum_{n=0}^{\infty} \frac{(x+4)^n}{2^{n+1}}$$

其中
$$\left|\frac{x+4}{2}\right| = \left|\frac{t}{2}\right| < 1$$



解 1. 注意到 
$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$
.

将 
$$\frac{1}{x+1}$$
,  $\frac{1}{x+2}$  分别展开成  $(x+4)$  的幂级数: 令  $t=x+4$ , 则

\*  $\frac{1}{x+1} = \frac{1}{t-3} = \frac{1}{-3} \cdot \frac{1}{1-\frac{t}{3}} = -\frac{1}{3} \sum_{n=0}^{\infty} \frac{t^n}{3^n} = -\sum_{n=0}^{\infty} \frac{(x+4)^n}{3^{n+1}}$ 

其中 
$$\left|\frac{x+4}{3}\right| = \left|\frac{t}{3}\right| < 1$$
,即  $-7 < x < -1$ 。

\* 
$$\frac{1}{x+2} = \frac{1}{t-2} = \frac{1}{-2} \cdot \frac{1}{1-\frac{t}{2}} = -\frac{1}{2} \sum_{n=0}^{\infty} \frac{t^n}{2^n} = -\sum_{n=0}^{\infty} \frac{(x+4)^n}{2^{n+1}}$$

其中
$$\left|\frac{x+4}{2}\right| = \left|\frac{t}{2}\right| < 1$$
,即  $-6 < x < -2$ 。

解 1. 注意到 
$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$
.

2. 利用  $\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots + t^n + \cdots$ ,  $t \in (-1, 1)$ 

将 
$$\frac{1}{x+1}$$
,  $\frac{1}{x+2}$  分别展开成  $(x+4)$  的幂级数: 令  $t=x+4$ , 则

\*  $\frac{1}{x+1} = \frac{1}{t-3} = \frac{1}{-3} \cdot \frac{1}{1-\frac{t}{3}} = -\frac{1}{3} \sum_{n=0}^{\infty} \frac{t^n}{3^n} = -\sum_{n=0}^{\infty} \frac{(x+4)^n}{3^{n+1}}$ 

其中 
$$\left|\frac{x+4}{3}\right| = \left|\frac{t}{3}\right| < 1$$
,即  $-7 < x < -1$ 。

\* 
$$\frac{1}{x+2} = \frac{1}{t-2} = \frac{1}{-2} \cdot \frac{1}{1-\frac{t}{2}} = -\frac{1}{2} \sum_{n=0}^{\infty} \frac{t^n}{2^n} = -\sum_{n=0}^{\infty} \frac{(x+4)^n}{2^{n+1}}$$

其中
$$\left|\frac{x+4}{2}\right| = \left|\frac{t}{2}\right| < 1$$
,即  $-6 < x < -2$ 。

3. 所以-6 < x < -2时

解 1. 注意到  $\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$ .

例 把函数  $f(x) = \frac{1}{x^2 + 3x + 2}$  展开成 (x + 4) 的幂级数。

2. 利用 
$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \dots + t^n + \dots$$
,  $t \in (-1, 1)$ 

将  $\frac{1}{x+1}$ ,  $\frac{1}{x+2}$  分别展开成 (x+4) 的幂级数: 令 t=x+4, 则

\*  $\frac{1}{x+2} = \frac{1}{t-2} = \frac{1}{-2} \cdot \frac{1}{1-\frac{t}{2}} = -\frac{1}{2} \sum_{n=0}^{\infty} \frac{t^n}{2^n} = -\sum_{n=0}^{\infty} \frac{(x+4)^n}{2^{n+1}}$ 其中 $\left|\frac{x+4}{2}\right| = \left|\frac{t}{2}\right| < 1$ ,即 -6 < x < -2。

$$\mathcal{H}^{\top}$$
  $| \frac{1}{2} | - | \frac{1}{2} | < 1$ ,  $| \mathcal{H}$ 

$$\frac{1}{x^2 + 3x + 2} = \frac{1}{x + 1} - \frac{1}{x + 2} = \sum_{n=0}^{\infty} \left( \frac{1}{2^{n+1}} - \frac{1}{3^{n+1}} \right) (x + 4)_{0}^{n}$$