§3.2 向量与向量组的线性组合

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• n 维行向量

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- 零向量 O = (0, 0, ···, 0)



• 设
$$\alpha = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \beta = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}, k \in \mathbb{R}, 则$$

$$\alpha + \beta =$$
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• 行向量类似



给定向量组

$$\alpha_1 = \begin{pmatrix} \alpha_{11} \\ \alpha_{21} \\ \vdots \\ \alpha_{m1} \end{pmatrix}, \ \alpha_2 = \begin{pmatrix} \alpha_{12} \\ \alpha_{22} \\ \vdots \\ \alpha_{m2} \end{pmatrix}, \dots, \ \alpha_n = \begin{pmatrix} \alpha_{1n} \\ \alpha_{2n} \\ \vdots \\ \alpha_{mn} \end{pmatrix}$$

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问 是否存在数 k_1 , k_2 , ..., k_n 使得:

$$\beta = k_1 \alpha_1 + k_2 \alpha_2 + \dots + k_n \alpha_n?$$

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如果能,则称 β 是向量组 α_1 , α_2 ,..., α_n 的线性组合。



• (1)
$$\[\bigcap$$
 β α_1 α_2 α_3 $\begin{pmatrix} 2 \\ -7 \\ 5 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$

• (1) i
$$\beta$$
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例 判断 β 能否由 α_1 , α_2 , α_3 线性表示,若能,写出线性表示等式 $\beta = k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3$ 。

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$$\begin{pmatrix} 2 \\ -7 \\ 5 \end{pmatrix} = \frac{2}{2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \frac{7}{2} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \frac{5}{2} \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

所以 $\beta = 2\alpha_1 - 7\alpha_2 + \frac{5}{2}\alpha_3$; β 能由 α_1 , α_2 , α_3 线性表出



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• (2)
$$\stackrel{\triangleright}{\square}$$

$$\begin{pmatrix} \beta & \alpha_1 & \alpha_2 & \alpha_3 \\ -7 & -7 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

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问题

- 一般地,如何判断β能否由α₁,α₂,...,α_n线性表出?
- 如果能线性表出,如何求出 k₁, k₂,...,kn 使

$$k_1\alpha_1 + k_2\alpha_2 + \cdots + k_n\alpha_n = \beta$$
?



问题是否存在数 k_1 , k_2 , ..., k_n 使

$$\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} = k_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + k_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \dots + k_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

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k_2 \\
\vdots \\
k_n
\end{pmatrix} = \begin{pmatrix}
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\iff Ax = \beta$$

方程有解等价于



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方程有解等价于 r(A) = r(A : β)



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方程有解等价于

 $r(A) = r(A : \beta) \iff r(\alpha_1 \alpha_2 \cdots \alpha_n) = r(\alpha_1 \alpha_2 \cdots \alpha_n \beta) \underset{\underline{\bullet} \text{ is finite}}{\underline{\bullet}}$

例 判断 β 是否能由 α_1 , α_2 , α_3 线性表示,若能,写出线性表示等式。

$$(\alpha_1 \ \alpha_2 \ \alpha_3 | \beta) = \begin{pmatrix} 1 & 2 & 3 | 2 \\ 0 & -1 & 2 | 3 \\ 1 & 1 & 0 | 0 \\ 2 & -2 & 1 | 5 \end{pmatrix}$$

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例 判断 β 是否能由 α_1 , α_2 , α_3 线性表示, 若能, 写出线性表示等式。

$$(\alpha_1 \ \alpha_2 \ \alpha_3 \,|\, \beta \) = \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{pmatrix} \xrightarrow{\text{\textit{M}$\%$frightarrow}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

• 所以
$$r(\alpha_1\alpha_2\alpha_3) = 3$$
, $r(\alpha_1\alpha_2\alpha_3\beta) =$,

例 判断 β 是否能由 α_1 , α_2 , α_3 线性表示, 若能, 写出线性表示等式。

(1)

$$(\ \alpha_1 \ \alpha_2 \ \alpha_3 \ |\ \beta\) = \left(\begin{array}{cc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{array} \right) \xrightarrow{\begin{subarray}{c} \end{subarray}} \left(\begin{array}{cc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

• 所以 $r(\alpha_1\alpha_2\alpha_3) = 3$, $r(\alpha_1\alpha_2\alpha_3\beta) = 3$,



例 判断 β 是否能由 α_1 , α_2 , α_3 线性表示,若能,写出线性表示等式。

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• 所以
$$r(\alpha_1\alpha_2\alpha_3) = 3$$
, $r(\alpha_1\alpha_2\alpha_3\beta) = 3$, 成立
$$r(\alpha_1\alpha_2\alpha_3) = r(\alpha_1\alpha_2\alpha_3\beta)$$

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(1)

$$(\ \alpha_1 \ \alpha_2 \ \alpha_3 \ |\ \beta\) = \left(\begin{array}{cc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{array} \right) \xrightarrow{\text{AMSFT} \oplus \#} \left(\begin{array}{cc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

• 所以 $r(\alpha_1\alpha_2\alpha_3) = 3$, $r(\alpha_1\alpha_2\alpha_3\beta) = 3$, 成立 $r(\alpha_1\alpha_2\alpha_3) = r(\alpha_1\alpha_2\alpha_3\beta)$

例 判断 β 是否能由 α_1 , α_2 , α_3 线性表示,若能,写出线性表示等式。

(1)

$$(\alpha_1 \ \alpha_2 \ \alpha_3 \ | \ \beta \) = \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{pmatrix} \xrightarrow{\text{\textit{diff:peh}}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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(1)
$$(\alpha_1 \ \alpha_2 \ \alpha_3 \ | \ \beta \) = \begin{pmatrix} 1 & 2 & 3 \ 2 & 3 & 3 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{pmatrix} \xrightarrow{\overline{m} \not = 0} \begin{pmatrix} \alpha_1' & \alpha_2' & 3 & 3 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

• 所以
$$r(\alpha_1\alpha_2\alpha_3) = 3$$
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$$(\alpha_1 \ \alpha_2 \ \alpha_3 \ | \ \beta \) = \begin{pmatrix} 1 & 2 & 3 \ 2 & 0 & -1 & 2 \ 3 & 1 & 1 & 0 \ 2 & -2 & 1 & 5 \end{pmatrix} \xrightarrow{\text{初等行变换}} \begin{pmatrix} \alpha_1' & \alpha_2' & \alpha_3' \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

• 所以
$$r(\alpha_1\alpha_2\alpha_3) = 3$$
, $r(\alpha_1\alpha_2\alpha_3\beta) = 3$, 成立
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例 判断 β 是否能由 α_1 , α_2 , α_3 线性表示, 若能, 写出线性表示等式。

(1)
$$(\alpha_1 \ \alpha_2 \ \alpha_3 \ | \ \beta \) = \begin{pmatrix} 1 & 2 & 3 \ 2 & 3 & 3 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{pmatrix} \xrightarrow{\text{\overline{MSTophy}}} \begin{pmatrix} \alpha_1' & \alpha_2' & \alpha_3' & \beta' \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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$$(\alpha_1 \ \alpha_2 \ \alpha_3 \ | \ \beta \) = \begin{pmatrix} 1 & 2 & 3 \ 2 & 3 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{pmatrix} \xrightarrow{\overline{ay}} \begin{pmatrix} \alpha_1' \ \alpha_2' \ \alpha_3' \ \beta' \\ \hline 1 & 0 & 0 \ 1 \\ 0 & 1 & 0 \ -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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$$r(\alpha_1\alpha_2\alpha_3) = 3$$
, $r(\alpha_1\alpha_2\alpha_3\beta) = 3$, 成立
$$r(\alpha_1\alpha_2\alpha_3) = r(\alpha_1\alpha_2\alpha_3\beta)$$
 β 可由 α_1 , α_2 , α_3 线性表示。

• 显然
$$\beta' = \alpha'_1 - \alpha'_2 + \alpha'_3$$
,

例 判断 β 是否能由 α_1 , α_2 , α_3 线性表示,若能,写出线性表示等式。

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$$(\alpha_1 \ \alpha_2 \ \alpha_3 \ | \ \beta \) = \begin{pmatrix} 1 & 2 & 3 \ 2 & 3 & 3 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{pmatrix} \xrightarrow{\overline{ay}} \begin{pmatrix} \alpha_1' \ \alpha_2' \ \alpha_3' \ \beta' \\ 1 & 0 & 0 \ 1 \\ 0 & 1 & 0 \ -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

• 所以
$$r(\alpha_1\alpha_2\alpha_3) = 3$$
, $r(\alpha_1\alpha_2\alpha_3\beta) = 3$, 成立
$$r(\alpha_1\alpha_2\alpha_3) = r(\alpha_1\alpha_2\alpha_3\beta)$$

 β 可由 α_1 , α_2 , α_3 线性表示。

• 显然 $\beta' = \alpha'_1 - \alpha'_2 + \alpha'_3$,是否也有 $\beta = \alpha_1 - \alpha_2 + \alpha_3$?



例 判断 β 是否能由 $α_1$, $α_2$, $α_3$ 线性表示,若能,写出线性表示等式。

(1)
$$(\alpha_1 \ \alpha_2 \ \alpha_3 \ | \ \beta \) = \begin{pmatrix} 1 & 2 & 3 \ 2 & 3 & 3 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{pmatrix} \xrightarrow{\text{\overline{MSToph}}} \begin{pmatrix} \alpha_1' & \alpha_2' & \alpha_3' & \beta' \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

• 所以
$$r(\alpha_1\alpha_2\alpha_3) = 3$$
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$$\beta' = \alpha'_1 - \alpha'_2 + \alpha'_3$$
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例 判断 β 是否能由 α_1 , α_2 , α_3 线性表示,若能,写出线性表示等式。

(1)
$$(\alpha_1 \ \alpha_2 \ \alpha_3 \ | \ \beta \) = \begin{pmatrix} 1 & 2 & 3 \ 0 & -1 & 2 \ 1 & 1 & 0 \ 2 & -2 & 1 \ \end{pmatrix} \xrightarrow{\overline{ag}} \begin{array}{c} \alpha_1' \ \alpha_2' \ \alpha_3' \ \beta' \\ \overline{ag} \ \overline$$

• 所以
$$r(\alpha_1\alpha_2\alpha_3) = 3$$
, $r(\alpha_1\alpha_2\alpha_3\beta) = 3$, 成立
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 β 可由 α_1 , α_2 , α_3 线性表示。

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$$\beta' = \alpha'_1 - \alpha'_2 + \alpha'_3$$
,是否也有 $\beta = \alpha_1 - \alpha_2 + \alpha_3$?

是的

注 可证明: 作初等行变换不改变列与列之间的"线性关系"。



例 判断 β 是否能由 α_1 , α_2 , α_3 线性表示, 若能, 写出线性表示等式。

$$(\alpha_1 \ \alpha_2 \ \alpha_3 | \beta) = \begin{pmatrix} 1 & 2 & -1 & | & 4 \\ 2 & -1 & 3 & | & 3 \\ -1 & 1 & -2 & | & 0 \\ 5 & 1 & 4 & | & 11 \end{pmatrix}$$

例 判断 β 是否能由 α_1 , α_2 , α_3 线性表示,若能,写出线性表示等式。

$$(\alpha_1 \ \alpha_2 \ \alpha_3 \ | \ \beta \) = \begin{pmatrix} 1 & 2 & -1 & | & 4 \\ 2 & -1 & 3 & | & 3 \\ -1 & 1 & -2 & | & 0 \\ 5 & 1 & 4 & | & 11 \end{pmatrix} \xrightarrow{\text{初等行变换}} \begin{pmatrix} 1 & 2 & -1 & | & 4 \\ 0 & 1 & -1 & | & 1 \\ 0 & 0 & 0 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

例 判断 β 是否能由 α_1 , α_2 , α_3 线性表示, 若能, 写出线性表示等式。

• 所以
$$r(\alpha_1\alpha_2\alpha_3) = r(\alpha_1\alpha_2\alpha_3\beta) = r$$

例 判断 β 是否能由 α_1 , α_2 , α_3 线性表示,若能,写出线性表示等式。

$$(\alpha_1 \ \alpha_2 \ \alpha_3 \ | \ \beta \) = \begin{pmatrix} 1 & 2 & -1 & | & 4 \\ 2 & -1 & 3 & | & 3 \\ -1 & 1 & -2 & | & 0 \\ 5 & 1 & 4 & | & 1 \end{pmatrix} \xrightarrow{\text{\overline{MSToph}}} \begin{pmatrix} 1 & 2 & -1 & | & 4 \\ 0 & 1 & -1 & | & 1 \\ 0 & 0 & 0 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

• 所以
$$r(\alpha_1\alpha_2\alpha_3) = r(\alpha_1\alpha_2\alpha_3\beta) = r$$

例 判断 β 是否能由 α_1 , α_2 , α_3 线性表示,若能,写出线性表示等式。

$$(\alpha_1 \ \alpha_2 \ \alpha_3 \ | \ \beta \) = \begin{pmatrix} 1 & 2 & -1 & | & 4 \\ 2 & -1 & 3 & | & 3 \\ -1 & 1 & -2 & | & 0 \\ 5 & 1 & 4 & | & 1 \end{pmatrix} \xrightarrow{\text{\overline{MSToph}}} \begin{pmatrix} 1 & 2 & -1 & | & 4 \\ 0 & 1 & -1 & | & 1 \\ 0 & 0 & 0 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

• 所以
$$r(\alpha_1\alpha_2\alpha_3) = 2$$
, $r(\alpha_1\alpha_2\alpha_3\beta) =$,

例 判断 β 是否能由 α_1 , α_2 , α_3 线性表示,若能,写出线性表示等式。

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• 所以
$$r(\alpha_1\alpha_2\alpha_3) = 2$$
, $r(\alpha_1\alpha_2\alpha_3\beta) = 3$,

例 判断 β 是否能由 α_1 , α_2 , α_3 线性表示,若能,写出线性表示等式。

• 所以
$$r(\alpha_1\alpha_2\alpha_3) = 2$$
, $r(\alpha_1\alpha_2\alpha_3\beta) = 3$, 成立
$$r(\alpha_1\alpha_2\alpha_3) \neq r(\alpha_1\alpha_2\alpha_3\beta)$$



例 判断 β 是否能由 α_1 , α_2 , α_3 线性表示,若能,写出线性表示等式。

(2)

$$(\alpha_1 \ \alpha_2 \ \alpha_3 \ | \ \beta \) = \begin{pmatrix} 1 & 2 & -1 & | & 4 \\ 2 & -1 & 3 & | & 3 \\ -1 & 1 & -2 & | & 0 \\ 5 & 1 & 4 & | & 1 \end{pmatrix} \xrightarrow{\text{\overline{MSToph}}} \begin{pmatrix} 1 & 2 & -1 & | & 4 \\ 0 & 1 & -1 & | & 1 \\ 0 & 0 & 0 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

• 所以
$$r(\alpha_1\alpha_2\alpha_3) = 2$$
, $r(\alpha_1\alpha_2\alpha_3\beta) = 3$, 成立
$$r(\alpha_1\alpha_2\alpha_3) \neq r(\alpha_1\alpha_2\alpha_3\beta)$$

问题 β 能否由 α_1 , α_2 , ..., α_n 线性表示? 若能, 写出线性表示等式。

步骤

问题 β 能否由 α_1 , α_2 , ..., α_n 线性表示? 若能, 写出线性表示等式。

步骤作初等行变换:

$$(\alpha_1 \ \alpha_2 \ \cdots \ \alpha_n | \beta)$$

问题 β 能否由 α_1 , α_2 , ..., α_n 线性表示? 若能, 写出线性表示等式。

步骤 作初等 行 变换:

$$(\alpha_1 \ \alpha_2 \ \cdots \ \alpha_n | \beta) \xrightarrow{\overline{\text{MS}_{7}^{\text{e}}}}$$

问题 β 能否由 $\alpha_1, \alpha_2, \ldots, \alpha_n$ 线性表示? 若能, 写出线性表示等式。

步骤 作初等 行 变换:

$$(\alpha_1 \ \alpha_2 \ \cdots \ \alpha_n | \beta) \xrightarrow{\overline{NS} + \overline{C} \circ \Phi} (\alpha'_1 \ \alpha'_2 \ \cdots \ \alpha'_n | \beta')$$
 (简化)

问题 β 能否由 $\alpha_1, \alpha_2, \ldots, \alpha_n$ 线性表示? 若能, 写出线性表示等式。

步骤 作初等 行 变换:

$$(\alpha_1 \ \alpha_2 \ \cdots \ \alpha_n | eta) \xrightarrow{ ext{ in } \beta
otag } (\alpha_1' \ \alpha_2' \ \cdots \ \alpha_n' | eta')$$
 (简化)

1.

$$\beta$$
由 $\alpha_1, \alpha_2, \ldots, \alpha_n$ 线性表示 \Leftrightarrow $r(\alpha_1 \cdots \alpha_n) = r(\alpha_1 \cdots \alpha_n \beta)$

问题 β 能否由 α_1 , α_2 , ..., α_n 线性表示? 若能, 写出线性表示等式。

步骤作初等行变换:

$$(\alpha_1 \ \alpha_2 \ \cdots \ \alpha_n | \beta) \xrightarrow{\eta \in \text{Toph}} (\alpha'_1 \ \alpha'_2 \ \cdots \ \alpha'_n | \beta')$$
 (简化)

1.

$$\beta$$
由 $\alpha_1, \alpha_2, \ldots, \alpha_n$ 线性表示 $\Leftrightarrow r(\alpha_1 \cdots \alpha_n) = r(\alpha_1 \cdots \alpha_n \beta)$

$$\uparrow r(\alpha'_1 \cdots \alpha'_n) = r(\alpha'_1 \cdots \alpha'_n \beta')$$

问题 β 能否由 $\alpha_1, \alpha_2, \ldots, \alpha_n$ 线性表示? 若能, 写出线性表示等式。

步骤 作初等 行 变换:

$$(\alpha_1 \ \alpha_2 \ \cdots \ \alpha_n | eta) \xrightarrow{\overline{NS}_{7} \oplus \overline{N}} (\alpha'_1 \ \alpha'_2 \ \cdots \ \alpha'_n | eta')$$
 (简化)

1.

$$\beta$$
由 $\alpha_1, \alpha_2, \ldots, \alpha_n$ 线性表示 $\Leftrightarrow r(\alpha_1 \cdots \alpha_n) = r(\alpha_1 \cdots \alpha_n \beta)$ \updownarrow

$$r(\alpha'_1 \cdots \alpha'_n) = r(\alpha'_1 \cdots \alpha'_n \beta')$$

2. 行变换前后列与列的线性关系不变, 即:

问题 β 能否由 α_1 , α_2 , ..., α_n 线性表示? 若能, 写出线性表示等式。

步骤作初等行变换:

$$(\alpha_1 \ \alpha_2 \ \cdots \ \alpha_n | eta) \xrightarrow{\overline{NS}^{+} \oplus \overline{NS}} (\alpha'_1 \ \alpha'_2 \ \cdots \ \alpha'_n | eta')$$
 (简化)

1

$$\beta$$
由 $\alpha_1, \alpha_2, \ldots, \alpha_n$ 线性表示 \Leftrightarrow $r(\alpha_1 \cdots \alpha_n) = r(\alpha_1 \cdots \alpha_n \beta)$

 $r(\alpha'_1 \cdots \alpha'_n) = r(\alpha'_1 \cdots \alpha'_n \beta')$

$$\beta' = k_1 \alpha'_1 + \dots + k_n \alpha'_n \Rightarrow$$



问题 β 能否由 $\alpha_1, \alpha_2, \ldots, \alpha_n$ 线性表示? 若能, 写出线性表示等式。

步骤作初等行变换:

$$(\alpha_1 \ \alpha_2 \ \cdots \ \alpha_n | eta) \xrightarrow{\overline{\eta + \gamma_{\overline{\Sigma}}}} (\alpha_1' \ \alpha_2' \ \cdots \ \alpha_n' | eta')$$
 (简化)

1.

$$\beta$$
由 $\alpha_1, \alpha_2, \ldots, \alpha_n$ 线性表示 \iff $r(\alpha_1 \cdots \alpha_n) = r(\alpha_1 \cdots \alpha_n \beta)$ \updownarrow

$$r(\alpha'_1 \cdots \alpha'_n) = r(\alpha'_1 \cdots \alpha'_n \beta')$$

2. 行变换前后列与列的线性关系不变,即:

$$\beta' = k_1 \alpha_1' + \dots + k_n \alpha_n' \Rightarrow \beta = k_1 \alpha_1 + \dots + k_n \alpha_n$$



例
$$\beta = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix}$$
能否由 $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ 线性表示?

解

例
$$\beta = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix}$$
能否由 $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ 线性表示?

$$\begin{array}{c|cccc}
\mathbf{R} & \alpha_1 & \alpha_2 & \alpha_3 & \beta \\
\begin{pmatrix}
1 & 2 & 3 & | 2 \\
0 & -1 & 2 & | 3 \\
1 & 1 & 0 & | 0 \\
2 & -2 & 1 & | 5
\end{pmatrix}$$

例
$$\beta = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix}$$
能否由 $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ 线性表示?



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$$\begin{pmatrix}
1 & 2 & 3 & | & 2 \\
0 & -1 & 2 & | & 3 \\
1 & 1 & 0 & | & 0 \\
2 & -2 & 1 & | & 5
\end{pmatrix}
\xrightarrow[r_4-2r_1]{r_3-r_1}
\begin{pmatrix}
1 & 2 & 3 & | & 2 \\
0 & -1 & 2 & | & 3 \\
0 & -1 & 2 & | & 3
\end{pmatrix}$$

例
$$\beta = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix}$$
能否由 $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ 线性表示?

$$\begin{pmatrix}
1 & 2 & 3 & 2 \\
0 & -1 & 2 & 3 \\
1 & 1 & 0 & 0 \\
2 & -2 & 1 & 5
\end{pmatrix}
\xrightarrow[r_4-2r_1]{r_3-r_1}
\begin{pmatrix}
1 & 2 & 3 & 2 \\
0 & -1 & 2 & 3 \\
0 & -1 & -3 & -2
\end{pmatrix}$$

例
$$\beta = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix}$$
能否由 $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ 线性表示?

$$\begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{pmatrix} \xrightarrow{r_3 - r_1} \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 0 & -1 & -3 & -2 \\ 0 & -6 & -5 & 1 \end{pmatrix}$$



例
$$\beta = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix}$$
能否由 $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ 线性表示?

$$\begin{pmatrix}
1 & 2 & 3 & 3 \\
0 & -1 & 2 & 3 \\
1 & 1 & 0 & 0 \\
2 & -2 & 1 & 5
\end{pmatrix}
\xrightarrow[r_4-2r_1]{r_3-r_1}
\begin{pmatrix}
1 & 2 & 3 & 2 \\
0 & -1 & 2 & 3 \\
0 & -1 & -3 & -2 \\
0 & -6 & -5 & 1
\end{pmatrix}
\xrightarrow[r_4-2r_1]{(-1)\times r_2}$$



例
$$\beta = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix}$$
能否由 $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ 线性表示?

$$\begin{pmatrix}
1 & 2 & 3 & 2 \\
0 & -1 & 2 & 3 \\
1 & 1 & 0 & 0 \\
2 & -2 & 1 & 5
\end{pmatrix}
\xrightarrow[r_4-2r_1]{r_3-r_1}
\begin{pmatrix}
1 & 2 & 3 & 2 \\
0 & -1 & 2 & 3 \\
0 & -1 & -3 & -2 \\
0 & -6 & -5 & 1
\end{pmatrix}
\xrightarrow[r_4-2r_1]{(-1)\times r_2}
\begin{pmatrix}
1 & 2 & 3 & 2 \\
0 & 1 & -2 & -3 \\
0 & -6 & -5 & 1
\end{pmatrix}$$



例
$$\beta = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix}$$
能否由 $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ 线性表示?

$$\begin{pmatrix}
1 & 2 & 3 & | & 2 \\
0 & -1 & 2 & | & 3 \\
1 & 1 & 0 & | & 0 \\
2 & -2 & 1 & | & 5
\end{pmatrix}
\xrightarrow[r_4-2r_1]{r_3-r_1}
\begin{pmatrix}
1 & 2 & 3 & | & 2 \\
0 & -1 & 2 & | & 3 \\
0 & -1 & -3 & | & -2 \\
0 & -6 & -5 & | & 1
\end{pmatrix}
\xrightarrow[0]{(-1)\times r_2}
\begin{pmatrix}
1 & 2 & 3 & | & 2 \\
0 & 1 & -2 & | & -3 \\
0 & -1 & -3 & | & -2 \\
0 & -6 & -5 & | & 1
\end{pmatrix}$$

$$\frac{r_3 + r_2}{r_4 + 6r_2}$$



例
$$\beta = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix}$$
能否由 $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ 线性表示?

$$\mathbf{R}$$
 α_1 α_2 α_3 β

$$\begin{pmatrix}
1 & 2 & 3 & | & 2 \\
0 & -1 & 2 & | & 3 \\
1 & 1 & 0 & | & 0 \\
2 & -2 & 1 & | & 5
\end{pmatrix}
\xrightarrow{r_3 - r_1}
\begin{pmatrix}
1 & 2 & 3 & | & 2 \\
0 & -1 & 2 & | & 3 \\
0 & -1 & -3 & | & -2 \\
0 & -6 & -5 & | & 1
\end{pmatrix}
\xrightarrow{(-1) \times r_2}
\begin{pmatrix}
1 & 2 & 3 & | & 2 \\
0 & 1 & -2 & | & -3 \\
0 & -1 & -3 & | & -2 \\
0 & -6 & -5 & | & 1
\end{pmatrix}$$

$$\xrightarrow[r_4+6r_2]{\begin{array}{c|cccc} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & -3 \\ \end{array}}$$



例
$$\beta = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix}$$
能否由 $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ 线性表示?

$$\mathbf{H}$$
 $\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \beta$

$$\begin{pmatrix}
1 & 2 & 3 & | & 2 \\
0 & -1 & 2 & | & 3 \\
1 & 1 & 0 & | & 0 \\
2 & -2 & 1 & | & 5
\end{pmatrix}
\xrightarrow[r_4-2r_1]{r_3-r_1}
\begin{pmatrix}
1 & 2 & 3 & | & 2 \\
0 & -1 & 2 & | & 3 \\
0 & -1 & -3 & | & -2 \\
0 & -6 & -5 & | & 1
\end{pmatrix}
\xrightarrow[(-1)\times r_2]{(-1)\times r_2}
\begin{pmatrix}
1 & 2 & 3 & | & 2 \\
0 & 1 & -2 & | & -3 \\
0 & -1 & -3 & | & -2 \\
0 & -6 & -5 & | & 1
\end{pmatrix}$$

$$\begin{array}{c|ccccc}
r_{3}+r_{2} \\
\hline
r_{4}+6r_{2}
\end{array}
\begin{pmatrix}
1 & 2 & 3 & 2 \\
0 & 1 & -2 & -3 \\
0 & 0 & -5 & -5
\end{pmatrix}$$



例
$$\beta = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix}$$
能否由 $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ 线性表示?

$$\mathbf{H}$$
 α_1 α_2 α_3 μ

$$\begin{pmatrix}
1 & 2 & 3 & | & 2 \\
0 & -1 & 2 & | & 3 \\
1 & 1 & 0 & | & 0 \\
2 & -2 & 1 & | & 5
\end{pmatrix}
\xrightarrow[r_4-2r_1]{r_3-r_1}
\begin{pmatrix}
1 & 2 & 3 & | & 2 \\
0 & -1 & 2 & | & 3 \\
0 & -1 & -3 & | & -2 \\
0 & -6 & -5 & | & 1
\end{pmatrix}
\xrightarrow[0]{(-1)\times r_2}
\begin{pmatrix}
1 & 2 & 3 & | & 2 \\
0 & 1 & -2 & | & -3 \\
0 & -1 & -3 & | & -2 \\
0 & -6 & -5 & | & 1
\end{pmatrix}$$

$$\begin{array}{c|ccccc}
r_{3}+r_{2} \\
\hline
r_{4}+6r_{2}
\end{array}
\begin{pmatrix}
1 & 2 & 3 & 2 \\
0 & 1 & -2 & -3 \\
0 & 0 & -5 & -5 \\
0 & 0 & -17 & -17
\end{pmatrix}$$



例
$$\beta = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix}$$
能否由 $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ 线性表示?

$$\mathbf{R}$$
 α_1 α_2 α_3 μ

$$\begin{pmatrix}
1 & 2 & 3 & | & 2 \\
0 & -1 & 2 & | & 3 \\
1 & 1 & 0 & | & 0 \\
2 & -2 & 1 & | & 5
\end{pmatrix}
\xrightarrow{r_3 - r_1}
\begin{pmatrix}
1 & 2 & 3 & | & 2 \\
0 & -1 & 2 & | & 3 \\
0 & -1 & -3 & | & -2 \\
0 & -6 & -5 & | & 1
\end{pmatrix}
\xrightarrow{(-1) \times r_2}
\begin{pmatrix}
1 & 2 & 3 & | & 2 \\
0 & 1 & -2 & | & -3 \\
0 & -1 & -3 & | & -2 \\
0 & -6 & -5 & | & 1
\end{pmatrix}$$

$$\xrightarrow[r_4+6r_2]{\begin{array}{c|cccc} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & -5 & -5 \\ 0 & 0 & -17 & -17 \end{array}} \rightarrow \left(\begin{array}{ccccc} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$



例
$$\beta = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix}$$
能否由 $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ 线性表示?

$$\mathbf{R}$$
 α_1 α_2 α_3 μ

$$\begin{pmatrix}
1 & 2 & 3 & | & 2 \\
0 & -1 & 2 & | & 3 \\
1 & 1 & 0 & | & 0 \\
2 & -2 & 1 & | & 5
\end{pmatrix}
\xrightarrow{r_3 - r_1}
\begin{pmatrix}
1 & 2 & 3 & | & 2 \\
0 & -1 & 2 & | & 3 \\
0 & -1 & -3 & | & -2 \\
0 & -6 & -5 & | & 1
\end{pmatrix}
\xrightarrow{(-1) \times r_2}
\begin{pmatrix}
1 & 2 & 3 & | & 2 \\
0 & 1 & -2 & | & -3 \\
0 & -1 & -3 & | & -2 \\
0 & -6 & -5 & | & 1
\end{pmatrix}$$

$$\xrightarrow[r_4+6r_2]{\begin{array}{c|cccc}
1 & 2 & 3 & 2 \\
0 & 1 & -2 & -3 \\
0 & 0 & -5 & -5 \\
0 & 0 & -17 & -17
\end{array}} \rightarrow \left(\begin{array}{ccccc}
1 & 2 & 3 & 2 \\
0 & 1 & -2 & -3 \\
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1
\end{array}\right) \xrightarrow[r_4-r_3]{r_4-r_3}$$



例
$$\beta = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix}$$
能否由 $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ 线性表示?

$$\mathbf{R}$$
 α_1 α_2 α_3

$$\begin{pmatrix}
1 & 2 & 3 & | & 2 \\
0 & -1 & 2 & | & 3 \\
1 & 1 & 0 & | & 0 \\
2 & -2 & 1 & | & 5
\end{pmatrix}
\xrightarrow{r_3 - r_1}
\begin{pmatrix}
1 & 2 & 3 & | & 2 \\
0 & -1 & 2 & | & 3 \\
0 & -1 & -3 & | & -2 \\
0 & -6 & -5 & | & 1
\end{pmatrix}
\xrightarrow{(-1) \times r_2}
\begin{pmatrix}
1 & 2 & 3 & | & 2 \\
0 & 1 & -2 & | & -3 \\
0 & -1 & -3 & | & -2 \\
0 & -6 & -5 & | & 1
\end{pmatrix}$$

$$\xrightarrow[r_4+6r_2]{\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&-5&-5\\0&0&-17&-17\end{array}} \longrightarrow \begin{pmatrix}1&2&3&2\\0&1&-2&-3\\0&0&1&1\\0&0&1&1\end{pmatrix}\xrightarrow[r_4-r_3]{\begin{pmatrix}1&2&3&2\\0&1&-2&-3\\0&0&1&1\\0&0&0&0\end{array}}$$



例
$$\beta = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix}$$
能否由 $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ 线性表示?

$$\mathbf{H}$$
 $\alpha_1 \quad \alpha_2 \quad \alpha_3$

$$\begin{pmatrix}
1 & 2 & 3 & | & 2 \\
0 & -1 & 2 & | & 3 \\
1 & 1 & 0 & | & 0 \\
2 & -2 & 1 & | & 5
\end{pmatrix}
\xrightarrow{r_3 - r_1}
\begin{pmatrix}
1 & 2 & 3 & | & 2 \\
0 & -1 & 2 & | & 3 \\
0 & -1 & -3 & | & -2 \\
0 & -6 & -5 & | & 1
\end{pmatrix}
\xrightarrow{(-1) \times r_2}
\begin{pmatrix}
1 & 2 & 3 & | & 2 \\
0 & 1 & -2 & | & -3 \\
0 & -6 & -5 & | & 1
\end{pmatrix}$$

$$\xrightarrow[r_4+6r_2]{\begin{array}{c|cccc}
1 & 2 & 3 & 2 \\
0 & 1 & -2 & -3 \\
0 & 0 & -5 & -5 \\
0 & 0 & -17 & -17
\end{array}} \longrightarrow \begin{pmatrix} 1 & 2 & 3 & 2 \\
0 & 1 & -2 & -3 \\
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1
\end{pmatrix} \xrightarrow[r_4-r_3]{\begin{array}{c|cccc}
1 & 2 & 3 & 2 \\
0 & 1 & -2 & -3 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}}$$

$$r_2-2r_3$$



例
$$\beta = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix}$$
能否由 $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ 线性表示?

$$\mathbf{R}$$
 α_1 α_2 α_3 β

$$\begin{pmatrix}
1 & 2 & 3 & 2 \\
0 & -1 & 2 & 3 \\
1 & 1 & 0 & 0 \\
2 & -2 & 1 & 5
\end{pmatrix}
\xrightarrow{r_3 - r_1}
\begin{pmatrix}
1 & 2 & 3 & 2 \\
0 & -1 & 2 & 3 \\
0 & -1 & -3 & -2 \\
0 & -6 & -5 & 1
\end{pmatrix}
\xrightarrow{(-1) \times r_2}
\begin{pmatrix}
1 & 2 & 3 & 2 \\
0 & 1 & -2 & -3 \\
0 & -6 & -5 & 1
\end{pmatrix}$$

$$\frac{r_3+r_2}{r_4+6r_2} \left(\begin{array}{ccc|c}
1 & 2 & 3 & 2 \\
0 & 1 & -2 & -3 \\
0 & 0 & -5 & -5 \\
0 & 0 & -17 & -17
\end{array}\right) \rightarrow \left(\begin{array}{ccc|c}
1 & 2 & 3 & 2 \\
0 & 1 & -2 & -3 \\
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1
\end{array}\right) \xrightarrow{r_4-r_3} \left(\begin{array}{ccc|c}
1 & 2 & 3 & 2 \\
0 & 1 & -2 & -3 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0
\end{array}\right)$$

$$\frac{r_2 - 2r_3}{r_1 - 3r_3} \left(\begin{array}{c|c} & & \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right)$$



例
$$\beta = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix}$$
能否由 $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ 线性表示?

$$\mathbf{H}$$
 α_1 α_2 α_3 μ

$$\begin{pmatrix}
1 & 2 & 3 & | & 2 \\
0 & -1 & 2 & | & 3 \\
1 & 1 & 0 & | & 0 \\
2 & -2 & 1 & | & 5
\end{pmatrix}
\xrightarrow{r_3 - r_1}
\begin{pmatrix}
1 & 2 & 3 & | & 2 \\
0 & -1 & 2 & | & 3 \\
0 & -1 & -3 & | & -2 \\
0 & -6 & -5 & | & 1
\end{pmatrix}
\xrightarrow{(-1) \times r_2}
\begin{pmatrix}
1 & 2 & 3 & | & 2 \\
0 & 1 & -2 & | & -3 \\
0 & -1 & -3 & | & -2 \\
0 & -6 & -5 & | & 1
\end{pmatrix}$$

$$\xrightarrow{r_3+r_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & -5 & -5 \\ 0 & 0 & -17 & -17 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow{r_4-r_3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[r_1-3r_3]{r_2-2r_3} \left(\begin{array}{ccc|c} 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right)$$



例
$$\beta = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix}$$
能否由 $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ 线性表示?

$$\mathbf{\mu}$$
 α_1 α_2 α_3 μ

$$\begin{pmatrix}
1 & 2 & 3 & | & 2 \\
0 & -1 & 2 & | & 3 \\
1 & 1 & 0 & | & 0 \\
2 & -2 & 1 & | & 5
\end{pmatrix}
\xrightarrow{r_3 - r_1}
\begin{pmatrix}
1 & 2 & 3 & | & 2 \\
0 & -1 & 2 & | & 3 \\
0 & -1 & -3 & | & -2 \\
0 & -6 & -5 & | & 1
\end{pmatrix}
\xrightarrow{(-1) \times r_2}
\begin{pmatrix}
1 & 2 & 3 & | & 2 \\
0 & 1 & -2 & | & -3 \\
0 & -1 & -3 & | & -2 \\
0 & -6 & -5 & | & 1
\end{pmatrix}$$

$$\xrightarrow{r_3+r_2} \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & -5 & -5 \\ 0 & 0 & -17 & -17 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \xrightarrow{r_4-r_3} \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\frac{r_2 - 2r_3}{r_1 - 3r_3} \begin{pmatrix}
1 & 2 & 0 & | & -1 \\
0 & 1 & 0 & | & -1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}$$



例
$$\beta = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix}$$
能否由 $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ 线性表示?

$$\mathbf{H}^{\alpha_1} \alpha_2 \alpha_3 \mu$$

$$\begin{pmatrix}
1 & 2 & 3 & 3 \\
0 & -1 & 2 & 3 \\
1 & 1 & 0 & 0 \\
2 & -2 & 1 & 5
\end{pmatrix}
\xrightarrow{r_3 - r_1}
\begin{pmatrix}
1 & 2 & 3 & 2 \\
0 & -1 & 2 & 3 \\
0 & -1 & -3 & -2 \\
0 & -6 & -5 & 1
\end{pmatrix}
\xrightarrow{(-1) \times r_2}
\begin{pmatrix}
1 & 2 & 3 & 2 \\
0 & 1 & -2 & -3 \\
0 & -1 & -3 & -2 \\
0 & -6 & -5 & 1
\end{pmatrix}$$

$$\xrightarrow[r_4+6r_2]{\begin{array}{cccc}
1 & 2 & 3 & 2 \\
0 & 1 & -2 & -3 \\
0 & 0 & -5 & -5 \\
0 & 0 & -17 & -17
\end{array}} \longrightarrow \begin{pmatrix} 1 & 2 & 3 & 2 \\
0 & 1 & -2 & -3 \\
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1
\end{pmatrix} \xrightarrow[r_4-r_3]{\begin{array}{ccccc}
1 & 2 & 3 & 2 \\
0 & 1 & -2 & -3 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}}$$

$$\xrightarrow[r_1-3r_3]{\begin{array}{c} 1 & 2 & 0 & | & -1 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 1 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{array}} \xrightarrow{r_1-2r_2}$$



例
$$\beta = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix}$$
能否由 $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ 线性表示?

$$\mathbf{R}$$
 α_1 α_2 α_3 μ

$$\begin{pmatrix}
1 & 2 & 3 & | & 2 \\
0 & -1 & 2 & | & 3 \\
1 & 1 & 0 & | & 0 \\
2 & -2 & 1 & | & 5
\end{pmatrix}
\xrightarrow{r_3 - r_1}
\begin{pmatrix}
1 & 2 & 3 & | & 2 \\
0 & -1 & 2 & | & 3 \\
0 & -1 & -3 & | & -2 \\
0 & -6 & -5 & | & 1
\end{pmatrix}
\xrightarrow{(-1) \times r_2}
\begin{pmatrix}
1 & 2 & 3 & | & 2 \\
0 & 1 & -2 & | & -3 \\
0 & -1 & -3 & | & -2 \\
0 & -6 & -5 & | & 1
\end{pmatrix}$$

$$\xrightarrow[r_1-3r_3]{\left(\begin{array}{cc|c}1&2&0&-1\\0&1&0&-1\\0&0&1&1\\0&0&0&0\end{array}\right)}\xrightarrow[r_1-2r_2]{\left(\begin{array}{cc|c}1&0&0&1\\0&1&0&-1\\0&0&1&1\\0&0&0&0\end{array}\right)}$$



例
$$\beta = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix}$$
能否由 $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ 线性表示?

$$\mathbf{H}$$
 α_1 α_2 α_3 μ

$$\begin{pmatrix}
1 & 2 & 3 & | & 2 \\
0 & -1 & 2 & | & 3 \\
1 & 1 & 0 & | & 0 \\
2 & -2 & 1 & | & 5
\end{pmatrix}
\xrightarrow{r_3 - r_1}
\begin{pmatrix}
1 & 2 & 3 & | & 2 \\
0 & -1 & 2 & | & 3 \\
0 & -1 & -3 & | & -2 \\
0 & -6 & -5 & | & 1
\end{pmatrix}
\xrightarrow{(-1) \times r_2}
\begin{pmatrix}
1 & 2 & 3 & | & 2 \\
0 & 1 & -2 & | & -3 \\
0 & -1 & -3 & | & -2 \\
0 & -6 & -5 & | & 1
\end{pmatrix}$$

$$\xrightarrow[r_1-3r_3]{\left(\begin{array}{cc|c}1&2&0&-1\\0&1&0&-1\\0&0&1&1\\0&0&0&0\end{array}\right)}\xrightarrow[r_1-2r_2]{\left(\begin{array}{cc|c}1&0&0&1\\0&1&0&-1\\0&0&1&1\\0&0&0&0\end{array}\right)}$$



例
$$\beta = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix}$$
能否由 $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ 线性表示?

$$\mathbf{H}$$
 α_1 α_2 α_3 μ

$$\begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{pmatrix} \xrightarrow{r_3 - r_1} \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 0 & -1 & -3 & -2 \\ 0 & -6 & -5 & 1 \end{pmatrix} \xrightarrow{(-1) \times r_2} \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & -1 & -3 & -2 \\ 0 & -6 & -5 & 1 \end{pmatrix}$$

$$\frac{r_{3}+r_{2}}{r_{4}+6r_{2}} \xrightarrow{\begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & -5 & -5 \\ 0 & 0 & -17 & -17 \end{pmatrix} \xrightarrow{\begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}} \xrightarrow{r_{4}-r_{3}} \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{r_{4}-r_{3}} \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{r_{4}-r_{3}} \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{r_{4}-r_{3}} \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{r_{4}-r_{3}} \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{r_{4}-r_{3}} \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

所以
$$r(\alpha_1\alpha_2\alpha_3) = r(\alpha_1\alpha_2\alpha_3\beta)$$
,



例
$$\beta = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix}$$
能否由 $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ 线性表示?

$$\mathbf{H}$$
 α_1 α_2 α_3 β

$$\begin{pmatrix}
1 & 2 & 3 & | & 2 \\
0 & -1 & 2 & | & 3 \\
1 & 1 & 0 & | & 0 \\
2 & -2 & 1 & | & 5
\end{pmatrix}
\xrightarrow{r_3 - r_1}
\begin{pmatrix}
1 & 2 & 3 & | & 2 \\
0 & -1 & 2 & | & 3 \\
0 & -1 & -3 & | & -2 \\
0 & -6 & -5 & | & 1
\end{pmatrix}
\xrightarrow{(-1) \times r_2}
\begin{pmatrix}
1 & 2 & 3 & | & 2 \\
0 & 1 & -2 & | & -3 \\
0 & -1 & -3 & | & -2 \\
0 & -6 & -5 & | & 1
\end{pmatrix}$$

$$\xrightarrow{r_3+r_2} \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & -5 & -5 \\ 0 & 0 & -17 & -17 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \xrightarrow{r_4-r_3} \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[r_1-3r_3]{\begin{array}{c}1&2&0&-1\\0&1&0&-1\\0&0&1&1\\0&0&0&0\end{array}}\xrightarrow[r_1-2r_2]{\begin{array}{c}1&0&0&1\\0&1&0&-1\\0&0&1&1\\0&0&0&0\end{array}}$$

所以 $r(\alpha_1\alpha_2\alpha_3) = r(\alpha_1\alpha_2\alpha_3\beta)$,能线性表示



例
$$\beta = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix}$$
能否由 $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ 线性表示?

$$\mathbf{H}$$
 $\alpha_1 \quad \alpha_2 \quad \alpha_3$

$$\begin{pmatrix}
1 & 2 & 3 & | & 2 \\
0 & -1 & 2 & | & 3 \\
1 & 1 & 0 & | & 0 \\
2 & -2 & 1 & | & 5
\end{pmatrix}
\xrightarrow{r_3 - r_1}
\begin{pmatrix}
1 & 2 & 3 & | & 2 \\
0 & -1 & 2 & | & 3 \\
0 & -1 & -3 & | & -2 \\
0 & -6 & -5 & | & 1
\end{pmatrix}
\xrightarrow{(-1) \times r_2}
\begin{pmatrix}
1 & 2 & 3 & | & 2 \\
0 & 1 & -2 & | & -3 \\
0 & -1 & -3 & | & -2 \\
0 & -6 & -5 & | & 1
\end{pmatrix}$$

$$\frac{r_{3}+r_{2}}{r_{4}+6r_{2}} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & -5 & -5 \\ 0 & 0 & -17 & -17 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow{r_{4}-r_{3}} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

$$\frac{r_{2}-2r_{3}}{r_{1}-3r_{3}} \left(\begin{array}{ccc|c} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{r_{1}-2r_{2}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

所以
$$r(\alpha_1\alpha_2\alpha_3) = r(\alpha_1\alpha_2\alpha_3\beta)$$
,能线性表示,且 $\beta = \alpha_1 - \alpha_2 + \alpha_3$

例
$$\beta = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 6 \end{pmatrix}$$
能否由 $\alpha_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} -1 \\ 3 \\ -2 \\ 4 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 4 \\ 3 \\ 0 \\ 11 \end{pmatrix}$ 线性表示?

例
$$\beta = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 6 \end{pmatrix}$$
能否由 $\alpha_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} -1 \\ 3 \\ -2 \\ 4 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 4 \\ 3 \\ 0 \\ 11 \end{pmatrix}$ 线性表示?

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$$\beta = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 6 \end{pmatrix}$$
能否由 $\alpha_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} -1 \\ 3 \\ -2 \\ 4 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 4 \\ 3 \\ 0 \\ 11 \end{pmatrix}$ 线性表示?

$$\begin{pmatrix} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_3}$$



例
$$\beta = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 6 \end{pmatrix}$$
能否由 $\alpha_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} -1 \\ 3 \\ -2 \\ 4 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 4 \\ 3 \\ 0 \\ 11 \end{pmatrix}$ 线性表示?



例
$$\beta = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 6 \end{pmatrix}$$
能否由 $\alpha_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} -1 \\ 3 \\ -2 \\ 4 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 4 \\ 3 \\ 0 \\ 11 \end{pmatrix}$ 线性表示?

$$\begin{pmatrix} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{pmatrix} 1 & -2 & 0 & | -1 \\ -1 & 3 & 3 & | & 2 \\ 2 & -1 & 4 & | & 1 \\ 1 & 4 & 11 & | & 6 \end{pmatrix}$$

$$r_2+r_1$$
 r_3-2r_1
 r_4-r_1



例
$$\beta = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 6 \end{pmatrix}$$
能否由 $\alpha_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} -1 \\ 3 \\ -2 \\ 4 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 4 \\ 3 \\ 0 \\ 11 \end{pmatrix}$ 线性表示?

$$\begin{pmatrix} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{pmatrix} 1 & -2 & 0 & | -1 \\ -1 & 3 & 3 & | & 2 \\ 2 & -1 & 4 & | & 1 \\ 1 & 4 & 11 & | & 6 \end{pmatrix}$$

$$\xrightarrow[r_{4}]{r_{2}-r_{1}} \left(\begin{array}{ccc} 1-2 & 0 & -1 \\ & & & \\ & & & \\ \end{array} \right)$$

例
$$\beta = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 6 \end{pmatrix}$$
能否由 $\alpha_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} -1 \\ 3 \\ -2 \\ 4 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 4 \\ 3 \\ 0 \\ 11 \end{pmatrix}$ 线性表示?

$$\begin{pmatrix} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{pmatrix} 1 & -2 & 0 & | -1 \\ -1 & 3 & 3 & | & 2 \\ 2 & -1 & 4 & | & 1 \\ 1 & 4 & 11 & | & 6 \end{pmatrix}$$

$$\xrightarrow[r_{4}]{r_{2}+r_{1}\atop r_{4}-r_{1}} \begin{pmatrix} 1-2 & 0 & | -1 \\ 0 & 1 & 3 & | 1 \\ & & & & | \end{pmatrix}$$

例
$$\beta = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 6 \end{pmatrix}$$
能否由 $\alpha_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} -1 \\ 3 \\ -2 \\ 4 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 4 \\ 3 \\ 0 \\ 11 \end{pmatrix}$ 线性表示?

$$\begin{pmatrix} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{pmatrix} 1 & -2 & 0 & | -1 \\ -1 & 3 & 3 & | & 2 \\ 2 & -1 & 4 & | & 1 \\ 1 & 4 & 11 & | & 6 \end{pmatrix}$$

$$\xrightarrow[r_4-r_1]{r_2+r_1} \begin{pmatrix} 1-2 & 0 & | -1 \\ 0 & 1 & 3 & | 1 \\ 0 & 3 & 4 & | 3 \end{pmatrix}$$

例
$$\beta = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 6 \end{pmatrix}$$
能否由 $\alpha_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} -1 \\ 3 \\ -2 \\ 4 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 4 \\ 3 \\ 0 \\ 11 \end{pmatrix}$ 线性表示?

$$\begin{pmatrix} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{pmatrix} 1 & -2 & 0 & | -1 \\ -1 & 3 & 3 & | & 2 \\ 2 & -1 & 4 & | & 1 \\ 1 & 4 & 11 & | & 6 \end{pmatrix}$$

$$\xrightarrow[r_4-r_1]{r_2-r_1} \begin{pmatrix} 1-2 & 0 & | -1 \\ 0 & 1 & 3 & 1 \\ 0 & 3 & 4 & 3 \\ 0 & 6 & 11 & 7 \end{pmatrix}$$

例
$$\beta = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 6 \end{pmatrix}$$
能否由 $\alpha_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} -1 \\ 3 \\ -2 \\ 4 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 4 \\ 3 \\ 0 \\ 11 \end{pmatrix}$ 线性表示?

$$\begin{pmatrix} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{pmatrix} 1 & -2 & 0 & | -1 \\ -1 & 3 & 3 & | & 2 \\ 2 & -1 & 4 & | & 1 \\ 1 & 4 & 11 & | & 6 \end{pmatrix}$$

例
$$\beta = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 6 \end{pmatrix}$$
能否由 $\alpha_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} -1 \\ 3 \\ -2 \\ 4 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 4 \\ 3 \\ 0 \\ 11 \end{pmatrix}$ 线性表示?

$$\begin{pmatrix} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{pmatrix} 1 & -2 & 0 & | -1 \\ -1 & 3 & 3 & | & 2 \\ 2 & -1 & 4 & | & 1 \\ 1 & 4 & 11 & | & 6 \end{pmatrix}$$

$$\xrightarrow[r_{4}-r_{1}]{r_{2}-2r_{1}} \begin{pmatrix}
1-2 & 0 & | -1 \\
0 & 1 & 3 & 1 \\
0 & 3 & 4 & 3 \\
0 & 6 & 11 & 7
\end{pmatrix}
\xrightarrow[r_{4}-6r_{2}]{r_{3}-3r_{2}} \begin{pmatrix}
1-2 & 0 & | -1 \\
0 & 1 & 3 & | 1 \\
0 & 1 & 3 & | 1
\end{pmatrix}$$

例
$$\beta = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 6 \end{pmatrix}$$
能否由 $\alpha_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} -1 \\ 3 \\ -2 \\ 4 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 4 \\ 3 \\ 0 \\ 11 \end{pmatrix}$ 线性表示?

$$\begin{pmatrix} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{pmatrix} 1 & -2 & 0 & | -1 \\ -1 & 3 & 3 & | & 2 \\ 2 & -1 & 4 & | & 1 \\ 1 & 4 & 11 & | & 6 \end{pmatrix}$$

$$\frac{r_{2}+r_{1}}{r_{3}-2r_{1}} \xrightarrow{\begin{pmatrix} 1-2 & 0 & | & -1 \\ 0 & 1 & 3 & | & 1 \\ 0 & 3 & 4 & | & 3 \\ 0 & 6 & 11 & 7 \end{pmatrix}} \xrightarrow{r_{3}-3r_{2}} \xrightarrow{\begin{pmatrix} 1-2 & 0 & | & -1 \\ 0 & 1 & 3 & | & 1 \\ 0 & 0 & -5 & | & 0 \end{pmatrix}$$



例
$$\beta = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 6 \end{pmatrix}$$
能否由 $\alpha_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} -1 \\ 3 \\ -2 \\ 4 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 4 \\ 3 \\ 0 \\ 11 \end{pmatrix}$ 线性表示?

$$\begin{pmatrix} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{pmatrix} 1 & -2 & 0 & | -1 \\ -1 & 3 & 3 & | & 2 \\ 2 & -1 & 4 & | & 1 \\ 1 & 4 & 11 & | & 6 \end{pmatrix}$$

$$\xrightarrow[r_{4}-r_{1}]{r_{2}-r_{1}} \begin{pmatrix}
1-2 & 0 & | -1 \\
0 & 1 & 3 & | & 1 \\
0 & 3 & 4 & | & 3 \\
0 & 6 & 11 & | & 7
\end{pmatrix} \xrightarrow[r_{4}-6r_{2}]{r_{3}-3r_{2}} \begin{pmatrix}
1-2 & 0 & | -1 \\
0 & 1 & 3 & | & 1 \\
0 & 0 & -5 & | & 0 \\
0 & 0 & -7 & | & 1
\end{pmatrix}$$

例
$$\beta = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 6 \end{pmatrix}$$
能否由 $\alpha_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} -1 \\ 3 \\ -2 \\ 4 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 4 \\ 3 \\ 0 \\ 11 \end{pmatrix}$ 线性表示?

$$\begin{pmatrix} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{pmatrix} 1 & -2 & 0 & | -1 \\ -1 & 3 & 3 & | & 2 \\ 2 & -1 & 4 & | & 1 \\ 1 & 4 & 11 & | & 6 \end{pmatrix}$$

$$\xrightarrow[r_4-r_1]{r_2+r_1} \begin{pmatrix} 1-2 & 0 & | & -1 \\ 0 & 1 & 3 & | & 1 \\ 0 & 3 & 4 & | & 3 \\ 0 & 6 & 11 & | & 7 \end{pmatrix} \xrightarrow[r_4-6r_2]{r_3-3r_2} \begin{pmatrix} 1-2 & 0 & | & -1 \\ 0 & 1 & 3 & | & 1 \\ 0 & 0 & -5 & | & 0 \\ 0 & 0 & -7 & | & 1 \end{pmatrix}$$

$$-\frac{1}{5} \times r_3$$



例
$$\beta = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 6 \end{pmatrix}$$
能否由 $\alpha_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} -1 \\ 3 \\ -2 \\ 4 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 4 \\ 3 \\ 0 \\ 11 \end{pmatrix}$ 线性表示?

$$\begin{pmatrix} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{pmatrix} 1 & -2 & 0 & | -1 \\ -1 & 3 & 3 & | & 2 \\ 2 & -1 & 4 & | & 1 \\ 1 & 4 & 11 & | & 6 \end{pmatrix}$$

$$\xrightarrow[r_4-r_1]{r_2+r_1} \begin{pmatrix} 1-2 & 0 & | & -1 \\ 0 & 1 & 3 & | & 1 \\ 0 & 3 & 4 & | & 3 \\ 0 & 6 & 11 & | & 7 \end{pmatrix} \xrightarrow[r_4-6r_2]{r_3-3r_2} \begin{pmatrix} 1-2 & 0 & | & -1 \\ 0 & 1 & 3 & | & 1 \\ 0 & 0 & -5 & | & 0 \\ 0 & 0 & -7 & | & 1 \end{pmatrix}$$

$$\xrightarrow{-\frac{1}{5} \times r_3} \begin{pmatrix} 1-2 & 0 & | -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -7 & 1 \end{pmatrix}$$



例
$$\beta = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 6 \end{pmatrix}$$
能否由 $\alpha_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} -1 \\ 3 \\ -2 \\ 4 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 4 \\ 3 \\ 0 \\ 11 \end{pmatrix}$ 线性表示?

$$\begin{pmatrix} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{pmatrix} 1 & -2 & 0 & | -1 \\ -1 & 3 & 3 & | & 2 \\ 2 & -1 & 4 & | & 1 \\ 1 & 4 & 11 & | & 6 \end{pmatrix}$$

$$\xrightarrow[r_4-r_1]{r_2-r_1} \begin{pmatrix} 1-2 & 0 & | & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 3 & 4 & 3 \\ 0 & 6 & 11 & 7 \end{pmatrix} \xrightarrow[r_4-6r_2]{r_3-3r_2} \begin{pmatrix} 1-2 & 0 & | & -1 \\ 0 & 1 & 3 & | & 1 \\ 0 & 0 & -5 & | & 0 \\ 0 & 0 & -7 & | & 1 \end{pmatrix}$$

$$\xrightarrow{-\frac{1}{5} \times r_3} \begin{pmatrix} 1 - 2 & 0 & | -1 \\ 0 & 1 & 3 & | 1 \\ 0 & 0 & 1 & | 0 \\ 0 & 0 & -7 & | 1 \end{pmatrix} \xrightarrow{r_4 + 7r_3}$$

例
$$\beta = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 6 \end{pmatrix}$$
能否由 $\alpha_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} -1 \\ 3 \\ -2 \\ 4 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 4 \\ 3 \\ 0 \\ 11 \end{pmatrix}$ 线性表示?

$$\begin{pmatrix} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{pmatrix} 1 & -2 & 0 & | -1 \\ -1 & 3 & 3 & | & 2 \\ 2 & -1 & 4 & | & 1 \\ 1 & 4 & 11 & | & 6 \end{pmatrix}$$

$$\xrightarrow[r_4-r_1]{r_2-r_1} \begin{pmatrix} 1-2 & 0 & | & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 3 & 4 & 3 \\ 0 & 6 & 11 & 7 \end{pmatrix} \xrightarrow[r_4-6r_2]{r_3-3r_2} \begin{pmatrix} 1-2 & 0 & | & -1 \\ 0 & 1 & 3 & | & 1 \\ 0 & 0 & -5 & | & 0 \\ 0 & 0 & -7 & | & 1 \end{pmatrix}$$

$$\xrightarrow{-\frac{1}{5} \times r_3} \begin{pmatrix} 1-2 & 0 & | -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -7 & 1 \end{pmatrix} \xrightarrow{r_4+7r_3} \begin{pmatrix} 1-2 & 0 | -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

例
$$\beta = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 6 \end{pmatrix}$$
能否由 $\alpha_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} -1 \\ 3 \\ -2 \\ 4 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 4 \\ 3 \\ 0 \\ 11 \end{pmatrix}$ 线性表示?

$$\begin{pmatrix} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{pmatrix} 1 & -2 & 0 & | -1 \\ -1 & 3 & 3 & | & 2 \\ 2 & -1 & 4 & | & 1 \\ 1 & 4 & 11 & | & 6 \end{pmatrix}$$

$$\xrightarrow[r_4-r_1]{r_2-r_1} \begin{pmatrix} 1-2 & 0 & | & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 3 & 4 & 3 \\ 0 & 6 & 11 & 7 \end{pmatrix} \xrightarrow[r_4-6r_2]{r_3-3r_2} \begin{pmatrix} 1-2 & 0 & | & -1 \\ 0 & 1 & 3 & | & 1 \\ 0 & 0 & -5 & | & 0 \\ 0 & 0 & -7 & | & 1 \end{pmatrix}$$

$$\xrightarrow{-\frac{1}{5} \times r_3} \begin{pmatrix} 1-2 & 0 & | & -1 \\ 0 & 1 & 3 & | & 1 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & -7 & | & 1 \end{pmatrix} \xrightarrow{r_4 + 7r_3} \begin{pmatrix} 1-2 & 0 & | & -1 \\ 0 & 1 & 3 & | & 1 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 1 \end{pmatrix}$$

例
$$\beta = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 6 \end{pmatrix}$$
能否由 $\alpha_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} -1 \\ 3 \\ -2 \\ 4 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 4 \\ 3 \\ 0 \\ 11 \end{pmatrix}$ 线性表示?

$$\xrightarrow{-\frac{1}{5} \times r_3} \begin{pmatrix} 1 - 2 & 0 & | -1 \\ 0 & 1 & 3 & | \\ 0 & 0 & 1 & | \\ 0 & 0 & -7 & | \end{pmatrix} \xrightarrow{r_4 + 7r_3} \begin{pmatrix} 1 - 2 & 0 | -1 \\ 0 & 1 & 3 & | \\ 0 & 0 & 1 & | \\ 0 & 0 & 0 & | \end{pmatrix}$$

可见 $r(\alpha_1\alpha_2\alpha_3\beta) = 4 > 3 = r(\alpha_1\alpha_2\alpha_3)$,



例
$$\beta = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 6 \end{pmatrix}$$
能否由 $\alpha_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} -1 \\ 3 \\ -2 \\ 4 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 4 \\ 3 \\ 0 \\ 11 \end{pmatrix}$ 线性表示?

可见 $r(\alpha_1\alpha_2\alpha_3\beta) = 4 > 3 = r(\alpha_1\alpha_2\alpha_3)$,所以不能线性表示。



定义 设有两个向量组

(A): $\alpha_1, \alpha_2, \ldots, \alpha_s$

(B): $\beta_1, \beta_2, \ldots, \beta_t$

定义 设有两个向量组

(A): $\alpha_1, \alpha_2, \ldots, \alpha_s$

(B): $\beta_1, \beta_2, \ldots, \beta_t$

如果中(A)中每一向量均可由(B)线性表示,则称向量组(A)可由向量组(B)线性表示。

定义 设有两个向量组

(A): $\alpha_1, \alpha_2, \ldots, \alpha_s$

(B): $\beta_1, \beta_2, \ldots, \beta_t$

如果中(A)中每一向量均可由(B)线性表示,则称向量组(A)可由向量组(B)线性表示。

例

定义 设有两个向量组

(A): $\alpha_1, \alpha_2, \ldots, \alpha_s$

(B): $\beta_1, \beta_2, \ldots, \beta_t$

如果中(A)中每一向量均可由(B)线性表示,则称向量组(A)可由向量组(B)线性表示。

例

$$\stackrel{\text{ᡯ妨设}}{\Longrightarrow} \left\{ \begin{array}{ll} \alpha_1 = & \beta_1 + & \beta_2 + & \beta_3 \\ \alpha_2 = & \beta_1 + & \beta_2 + & \beta_3 \end{array} \right.$$

定义 设有两个向量组

(A):
$$\alpha_1, \alpha_2, \ldots, \alpha_s$$

(B):
$$\beta_1, \beta_2, \ldots, \beta_t$$

如果中(A)中每一向量均可由(B)线性表示,则称向量组(A)可由向量组(B)线性表示。

例

$$\stackrel{\text{ᡯ妨设}}{\Longrightarrow} \left\{ \begin{array}{l} \alpha_1 = \alpha_{11}\beta_1 + \alpha_{21}\beta_2 + \alpha_{31}\beta_3 \\ \alpha_2 = \beta_1 + \beta_2 + \beta_3 \end{array} \right.$$

定义 设有两个向量组

(A): $\alpha_1, \alpha_2, \ldots, \alpha_s$

(B): $\beta_1, \beta_2, \ldots, \beta_t$

如果中(A)中每一向量均可由(B)线性表示,则称向量组(A)可由向量组(B)线性表示。

例

$$\stackrel{\text{不妨设}}{\Longrightarrow} \left\{ \begin{array}{l} \alpha_1 = \alpha_{11}\beta_1 + \alpha_{21}\beta_2 + \alpha_{31}\beta_3 \\ \alpha_2 = \alpha_{12}\beta_1 + \alpha_{22}\beta_2 + \alpha_{32}\beta_3 \end{array} \right.$$

定义 设有两个向量组

(A):
$$\alpha_1, \alpha_2, \ldots, \alpha_s$$

(B):
$$\beta_1, \beta_2, \ldots, \beta_t$$

如果中 (A) 中每一向量均可由 (B) 线性表示,则称向量组 (A) 可由向量组 (B)线性表示。

$$\stackrel{\text{不妨设}}{\Longrightarrow} \left\{ \begin{array}{l} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 + a_{31}\beta_3 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 + a_{32}\beta_3 \end{array} \right.$$

$$\stackrel{\text{\tiny \Delta}}{\Longrightarrow} (\alpha_1, \, \alpha_2) = (\beta_1, \, \beta_2, \, \beta_3)$$



定义 设有两个向量组

(A):
$$\alpha_1, \alpha_2, \ldots, \alpha_s$$

$$(B): \beta_1, \beta_2, \ldots, \beta_t$$

如果中 (A) 中每一向量均可由 (B) 线性表示,则称向量组 (A) 可由向量组 (B)线性表示。

 $oxed{eta}$ 向量组 $lpha_1$, $lpha_2$ 可由向量组d线性表示 eta_1 , eta_2 , eta_3



定义 设有两个向量组

(A):
$$\alpha_1, \alpha_2, \ldots, \alpha_s$$

$$(B): \beta_1, \beta_2, \ldots, \beta_t$$

如果中(A)中每一向量均可由(B)线性表示,则称向量组(A)可由向量组(B)线性表示。

例

$$\stackrel{\underline{\text{тяй}}}{\Longrightarrow} \left\{ \begin{array}{l} \alpha_1 = \alpha_{11}\beta_1 + \alpha_{21}\beta_2 + \alpha_{31}\beta_3 \\ \alpha_2 = \alpha_{12}\beta_1 + \alpha_{22}\beta_2 + \alpha_{32}\beta_3 \end{array} \right.$$

改写为

$$(\alpha_1, \alpha_2) = (\beta_1, \beta_2, \beta_3)$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

定义 设有两个向量组

(A):
$$\alpha_1, \alpha_2, \ldots, \alpha_s$$

(B):
$$\beta_1, \beta_2, \ldots, \beta_t$$

如果中(A)中每一向量均可由(B)线性表示,则称向量组(A)可由向量 组 (B)线性表示。

$$\stackrel{\text{ᡯ妨设}}{\Longrightarrow} \left\{ \begin{array}{l} \alpha_1 = \alpha_{11}\beta_1 + \alpha_{21}\beta_2 + \alpha_{31}\beta_3 \\ \alpha_2 = \alpha_{12}\beta_1 + \alpha_{22}\beta_2 + \alpha_{32}\beta_3 \end{array} \right.$$

改写为

$$(\alpha_1, \alpha_2) = (\beta_1, \beta_2, \beta_3)$$
 $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$
 $= (\beta_1, \beta_2, \beta_3)A$



注 1 一般地,向量组 α_1 , α_2 , ..., α_s 可由向量组 β_1 , β_2 , ..., β_t 线性表示,当且仅当存在矩阵 $A_{t\times s}$ 满足:

注 1 一般地,向量组 α_1 , α_2 , ..., α_s 可由向量组 β_1 , β_2 , ..., β_t 线性表示,当且仅当存在矩阵 $A_{t\times s}$ 满足:

$$(\alpha_1, \alpha_2, \ldots, \alpha_5) = (\beta_1, \beta_2, \ldots, \beta_t)A$$

注 1 一般地,向量组 α_1 , α_2 , ..., α_s 可由向量组 β_1 , β_2 , ..., β_t 线性

表示,当且仅当存在矩阵 $A_{t \times s}$ 满足:

$$(\alpha_1, \alpha_2, \dots, \alpha_s) = (\beta_1, \beta_2, \dots, \beta_t)A$$

$$= (\beta_1, \beta_2, \dots, \beta_t) \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ a_{t1} & a_{t2} & \cdots & a_{ts} \end{pmatrix}$$

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$$(\alpha_1, \alpha_2, \dots, \alpha_s) = (\beta_1, \beta_2, \dots, \beta_t)A$$

$$= (\beta_1, \beta_2, \dots, \beta_t) \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ a_{t1} & a_{t2} & \cdots & a_{ts} \end{pmatrix}$$

这时 A 的每一列表示线性组合的系数。

注 1 一般地,向量组 α_1 , α_2 ,, α_s 可由向量组 β_1 , β_2 ,, β_t 线性表示,当且仅当存在矩阵 $A_{t\times s}$ 满足:

$$(\alpha_1, \alpha_2, \dots, \alpha_s) = (\beta_1, \beta_2, \dots, \beta_t)A$$

$$= (\beta_1, \beta_2, \dots, \beta_t) \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ a_{11} & a_{12} & \cdots & a_{t} \end{pmatrix}$$

这时A的每一列表示线性组合的系数。例如,

$$\alpha_j = \beta_1 + \beta_2 + \cdots + \beta_t$$

注 1 一般地,向量组 α_1 , α_2 ,, α_s 可由向量组 β_1 , β_2 ,, β_t 线性表示,当且仅当存在矩阵 $A_{t \times s}$ 满足:

$$(\alpha_1, \alpha_2, \ldots, \alpha_s) = (\beta_1, \beta_2, \ldots, \beta_t)A$$

$$(\alpha_{11} \ \alpha_{12})$$

$$= (\beta_1, \beta_2, \dots, \beta_t) \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ a_{t1} & a_{t2} & \cdots & a_{ts} \end{pmatrix}$$

这时 A 的每一列表示线性组合的系数。例如,

$$\alpha_j = \alpha_{1j}\beta_1 + \alpha_{2j}\beta_2 + \cdots + \alpha_{tj}\beta_t$$

其中的系数就是
$$A$$
 的第 j 列 $\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{ci} \end{pmatrix}$ 。

注 1 一般地,向量组 α_1 , α_2 ,, α_s 可由向量组 β_1 , β_2 ,, β_t 线性表示,当且仅当存在矩阵 $A_{t \times s}$ 满足:

$$(\alpha_1, \alpha_2, \ldots, \alpha_s) = (\beta_1, \beta_2, \ldots, \beta_t)A$$

$$= (\beta_1, \beta_2, \dots, \beta_t) \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ a_{t1} & a_{t2} & \cdots & a_{ts} \end{pmatrix}$$

这时 A 的每一列表示线性组合的系数。例如,

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其中的系数就是
$$A$$
 的第 j 列 $\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{si} \end{pmatrix}$ 。

注2若上述向量均为列向量,



注 1 一般地,向量组 α_1 , α_2 , ..., α_s 可由向量组 β_1 , β_2 , ..., β_t 线性

表示,当且仅当存在矩阵 $A_{t imes s}$ 满足:

$$\underbrace{(\alpha_1, \alpha_2, \ldots, \alpha_s)}_{P} = (\beta_1, \beta_2, \ldots, \beta_t)A$$

$$= (\beta_1, \beta_2, \dots, \beta_t) \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ a_{t1} & a_{t2} & \cdots & a_{ts} \end{pmatrix}$$

这时A的每一列表示线性组合的系数。例如,

$$\alpha_j = \alpha_{1j}\beta_1 + \alpha_{2j}\beta_2 + \cdots + \alpha_{tj}\beta_t$$

其中的系数就是 A 的第 j 列 $\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{si} \end{pmatrix}$ 。

注2若上述向量均为列向量,



注 1 一般地,向量组 α_1 , α_2 , ..., α_s 可由向量组 β_1 , β_2 , ..., β_t 线性

表示,当且仅当存在矩阵 $A_{t\times s}$ 满足:

$$\underbrace{(\alpha_1, \alpha_2, \ldots, \alpha_s)}_{P} = \underbrace{(\beta_1, \beta_2, \ldots, \beta_t)}_{Q} A$$

$$\int_{Q} \alpha_{11} \alpha_{12} d\alpha_{13} d\alpha_{14} d\alpha_{15} d\alpha_{15$$

$$= (\beta_1, \, \beta_2, \, \dots, \, \beta_t) \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ a_{t1} & a_{t2} & \cdots & a_{ts} \end{pmatrix}$$

这时 A 的每一列表示线性组合的系数。例如,

$$\alpha_j = \alpha_{1j}\beta_1 + \alpha_{2j}\beta_2 + \cdots + \alpha_{tj}\beta_t$$

其中的系数就是 A 的第 j 列 $\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{sj} \end{pmatrix}$ 。

注 1 一般地,向量组 α_1 , α_2 , ..., α_s 可由向量组 β_1 , β_2 , ..., β_t 线性表示,当且仅当存在矩阵 $A_{t \times s}$ 满足:

$$\underbrace{(\alpha_1, \alpha_2, \ldots, \alpha_s)}_{\Omega} = \underbrace{(\beta_1, \beta_2, \ldots, \beta_t)}_{\Omega} A$$

$$= (\beta_1, \, \beta_2, \, \dots, \, \beta_t) \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ a_{t1} & a_{t2} & \cdots & a_{ts} \end{pmatrix}$$

这时A的每一列表示线性组合的系数。例如,

$$\alpha_i = \alpha_{1i}\beta_1 + \alpha_{2i}\beta_2 + \dots + \alpha_{ti}\beta_t$$

其中的系数就是 A 的第 j 列 $\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{si} \end{pmatrix}$ 。

注 2 若上述向量均为列向量,则上式正好表示矩阵乘积: P = QA

定理(向量组线性表示的传递性) 假设向量组 (A), (B), (C) 满足: (A) 可由 (B) 线性表示,(B) 可由 (C) 线性表示,则 (A) 可由 (C) 线性表示。

定理(向量组线性表示的传递性) 假设向量组 (A), (B), (C) 满足: (A) 可由 (B) 线性表示,(B) 可由 (C) 线性表示,则 (A) 可由 (C) 线性表示。证明 设向量组 α_1 , α_2 , ..., α_s 可由向量组 β_1 , β_2 , ..., β_t 线性表示:

向量组 β_1 , β_2 , ..., β_t 可由向量组 γ_1 , γ_2 , ..., γ_k 线性表示:

定理(向量组线性表示的传递性) 假设向量组 (A), (B), (C) 满足: (A) 可由 (B) 线性表示,(B) 可由 (C) 线性表示,则 (A) 可由 (C) 线性表示。

证明 设向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 可由向量组 $\beta_1, \beta_2, \ldots, \beta_t$ 线性表示:

$$(\alpha_1, \alpha_2, \ldots, \alpha_s) = (\beta_1, \beta_2, \ldots, \beta_t) A_{t \times s}.$$

向量组 β_1 , β_2 , . . . , β_t 可由向量组 γ_1 , γ_2 , . . . , γ_k 线性表示:

定理(向量组线性表示的传递性) 假设向量组 (A), (B), (C) 满足: (A) 可由 (B) 线性表示,(B) 可由 (C) 线性表示,则 (A) 可由 (C) 线性表示。

证明 设向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 可由向量组 $\beta_1, \beta_2, \ldots, \beta_t$ 线性表示:

$$(\alpha_1, \alpha_2, \ldots, \alpha_s) = (\beta_1, \beta_2, \ldots, \beta_t) A_{t \times s}.$$

向量组 $eta_1, eta_2, \ldots, eta_t$ 可由向量组 $\gamma_1, \gamma_2, \ldots, \gamma_k$ 线性表示:

$$(\beta_1, \beta_2, \ldots, \beta_t) = (\gamma_1, \gamma_2, \ldots, \gamma_k)B_{k \times t}.$$

定理(向量组线性表示的传递性) 假设向量组(A),(B),(C)满足:(A)

可由 (*B*) 线性表示,(*B*) 可由 (*C*) 线性表示,则 (*A*) 可由 (*C*) 线性表示。 证明 设向量组 α_1 , α_2 , ..., α_s 可由向量组 β_1 , β_2 , ..., β_t 线性表示:

$$(\alpha_1, \alpha_2, \ldots, \alpha_s) = (\beta_1, \beta_2, \ldots, \beta_t) A_{t \times s}.$$

向量组 $eta_1, eta_2, \ldots, eta_t$ 可由向量组 $\gamma_1, \gamma_2, \ldots, \gamma_k$ 线性表示:

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$$(\alpha_1, \alpha_2, \ldots, \alpha_s) = (\beta_1, \beta_2, \ldots, \beta_t) A_{t \times s}.$$

向量组 $eta_1, eta_2, \ldots, eta_t$ 可由向量组 $\gamma_1, \gamma_2, \ldots, \gamma_k$ 线性表示:

$$(\beta_1, \beta_2, \ldots, \beta_t) = (\gamma_1, \gamma_2, \ldots, \gamma_k)B_{k \times t}.$$

$$(\alpha_1, \alpha_2, \ldots, \alpha_s) = (\gamma_1, \gamma_2, \ldots, \gamma_k) B_{k \times t} A_{t \times s}$$

定理(向量组线性表示的传递性) 假设向量组 (A), (B), (C) 满足: (A)

可由 (B) 线性表示,(B) 可由 (C) 线性表示,则(A) 可由 (C) 线性表示。 证明 设向量组 α_1 , α_2 , ..., α_s 可由向量组 β_1 , β_2 , ..., β_t 线性表示:

$$(\alpha_1, \alpha_2, \ldots, \alpha_s) = (\beta_1, \beta_2, \ldots, \beta_t) A_{t \times s}.$$

向量组 β_1 , β_2 , ..., β_t 可由向量组 γ_1 , γ_2 , ..., γ_k 线性表示:

$$(\beta_1, \beta_2, \ldots, \beta_t) = (\gamma_1, \gamma_2, \ldots, \gamma_k) B_{k \times t}.$$

$$(\alpha_1, \alpha_2, \ldots, \alpha_s) = (\gamma_1, \gamma_2, \ldots, \gamma_k) \underbrace{B_{k \times t} A_{t \times s}}_{C_{k \times s}}$$

定理(向量组线性表示的传递性) 假设向量组 (A), (B), (C) 满足: (A)

可由 (*B*) 线性表示,(*B*) 可由 (*C*) 线性表示,则 (*A*) 可由 (*C*) 线性表示。 证明 设向量组 α_1 , α_2 , ..., α_s 可由向量组 β_1 , β_2 , ..., β_t 线性表示:

$$(\alpha_1, \alpha_2, \ldots, \alpha_s) = (\beta_1, \beta_2, \ldots, \beta_t) A_{t \times s}.$$

向量组 β_1 , β_2 , ..., β_t 可由向量组 γ_1 , γ_2 , ..., γ_k 线性表示:

$$(\beta_1, \beta_2, \ldots, \beta_t) = (\gamma_1, \gamma_2, \ldots, \gamma_k) B_{k \times t}.$$

$$(\alpha_1, \alpha_2, \ldots, \alpha_s) = (\gamma_1, \gamma_2, \ldots, \gamma_k) \underbrace{B_{k \times t} A_{t \times s}}_{C_{t \times s}} = (\gamma_1, \gamma_2, \ldots, \gamma_k) C.$$

定理(向量组线性表示的传递性) 假设向量组 (A), (B), (C) 满足: (A)

可由 (*B*) 线性表示,(*B*) 可由 (*C*) 线性表示,则 (*A*) 可由 (*C*) 线性表示。 证明 设向量组 α_1 , α_2 , ..., α_s 可由向量组 β_1 , β_2 , ..., β_t 线性表示:

$$(\alpha_1, \alpha_2, \ldots, \alpha_s) = (\beta_1, \beta_2, \ldots, \beta_t) A_{t \times s}.$$

向量组 β_1 , β_2 , ..., β_t 可由向量组 γ_1 , γ_2 , ..., γ_k 线性表示:

$$(\beta_1, \beta_2, \ldots, \beta_t) = (\gamma_1, \gamma_2, \ldots, \gamma_k) B_{k \times t}.$$

将第2式代入第1式,可得

$$(\alpha_1, \alpha_2, \ldots, \alpha_s) = (\gamma_1, \gamma_2, \ldots, \gamma_k) \underbrace{B_{k \times t} A_{t \times s}}_{C_{k \times s}} = (\gamma_1, \gamma_2, \ldots, \gamma_k) C.$$

所以向量组 α_1 , α_2 , ..., α_s 可由向量组 γ_1 , γ_2 , ..., γ_k 线性表示。

定理(向量组线性表示的传递性) 假设向量组 (A), (B), (C) 满足: (A) 可由 (B) 线性表示,(B) 可由 (C) 线性表示,则 (A) 可由 (C) 线性表示。

证明 设向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 可由向量组 $\beta_1, \beta_2, \ldots, \beta_t$ 线性表示:

$$(\alpha_1, \alpha_2, \ldots, \alpha_s) = (\beta_1, \beta_2, \ldots, \beta_t) A_{t \times s}.$$

向量组 β_1 , β_2 , ..., β_t 可由向量组 γ_1 , γ_2 , ..., γ_k 线性表示:

$$(\beta_1, \beta_2, \ldots, \beta_t) = (\gamma_1, \gamma_2, \ldots, \gamma_k) B_{k \times t}.$$

将第2式代入第1式,可得

$$(\alpha_1, \alpha_2, \ldots, \alpha_s) = (\gamma_1, \gamma_2, \ldots, \gamma_k) \underbrace{B_{k \times t} A_{t \times s}}_{C_{k \times s}} = (\gamma_1, \gamma_2, \ldots, \gamma_k) C.$$

所以向量组 α_1 , α_2 , ..., α_s 可由向量组 γ_1 , γ_2 , ..., γ_k 线性表示。 (并且、线性组合的系数就是矩阵 C 的列。)

例 $\left. \begin{array}{c} \alpha_1, \ \alpha_2 \oplus \beta_1, \ \beta_2$ 线性表示 $\beta_1, \ \beta_2 \oplus \gamma_1, \ \gamma_2, \ \gamma_3$ 线性表示 $\beta_1, \ \beta_2 \oplus \gamma_1, \ \gamma_2, \ \gamma_3$ 线性表示

例
$$\left. \begin{array}{c} \alpha_1, \ \alpha_2 \oplus \beta_1, \ \beta_2$$
 线性表示 $\beta_1, \ \beta_2 \oplus \gamma_1, \ \gamma_2, \ \gamma_3$ 线性表示 $\beta_1, \ \beta_2 \oplus \gamma_1, \ \gamma_2, \ \gamma_3$ 线性表示

$$\begin{cases} \alpha_1 = \alpha_{11}\beta_1 + \alpha_{21}\beta_2 \\ \alpha_2 = \alpha_{12}\beta_1 + \alpha_{22}\beta_2 \end{cases}$$

 $\left.\begin{array}{l} \alpha_1,\,\alpha_2 \text{由}\beta_1,\,\beta_2 \text{线性表示} \\ \beta_1,\,\beta_2 \text{由}\gamma_1,\,\gamma_2,\,\gamma_3 \text{线性表示} \end{array}\right\} \Rightarrow \alpha_1,\,\alpha_2 \text{由}\gamma_1,\,\gamma_2,\,\gamma_3 \text{线性表示}$

具体地,设

$$\begin{cases} \alpha_1 = \alpha_{11}\beta_1 + \alpha_{21}\beta_2 \\ \alpha_2 = \alpha_{12}\beta_1 + \alpha_{22}\beta_2 \end{cases}$$

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases}$$

例
$$\alpha_1, \alpha_2$$
由 β_1, β_2 线性表示 β_1, β_2 由 $\gamma_1, \gamma_2, \gamma_3$ 线性表示 β_1, β_2 由 $\gamma_1, \gamma_2, \gamma_3$ 线性表示

$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases}$$

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases}$$

$$\alpha_1 =$$

$$\alpha_2 =$$

例
$$\alpha_1, \alpha_2$$
由 β_1, β_2 线性表示 β_1, β_2 由 $\gamma_1, \gamma_2, \gamma_3$ 线性表示 β_1, β_2 由 $\gamma_1, \gamma_2, \gamma_3$ 线性表示 β_1, β_2 由 $\gamma_1, \gamma_2, \gamma_3$ 线性表示
具体地,设
$$\left\{ \begin{array}{l} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{array} \right.$$

$$\left\{ \begin{array}{l} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{array} \right.$$
 则
$$\alpha_1 = a_{11} () + a_{21} ()$$

 $\alpha_2 =$

例
$$\alpha_1$$
, α_2 由 β_1 , β_2 线性表示 β_1 , β_2 由 γ_1 , γ_2 , γ_3 线性表示 β_1 , β_2 由 γ_1 , γ_2 , γ_3 线性表示
具体地,设
$$\left\{ \begin{array}{l} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{array} \right.$$

$$\left\{ \begin{array}{l} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{array} \right.$$
 则
$$\alpha_1 = a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}($$

 $\alpha_2 =$

例
$$\left. \begin{array}{c} \alpha_1, \, \alpha_2 \oplus \beta_1, \, \beta_2 \oplus \mathbb{E} \\ \beta_1, \, \beta_2 \oplus \gamma_1, \, \gamma_2, \, \gamma_3 \oplus \mathbb{E} \\ \end{array} \right. \Rightarrow \alpha_1, \, \alpha_2 \oplus \gamma_1, \, \gamma_2, \, \gamma_3 \oplus \mathbb{E} \\ \left. \begin{array}{c} \beta_1, \, \beta_2 \oplus \gamma_1, \, \gamma_2, \, \gamma_3 \oplus \mathbb{E} \\ \end{array} \right.$$

$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases}$$

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases}$$

$$\alpha_1 = a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)$$

$$\alpha_2 =$$

例
$$\alpha_1, \alpha_2 = \beta_1, \beta_2$$
线性表示 $\beta_1, \beta_2 = \beta_1, \beta_2$ 代表示 $\beta_1, \beta_2 = \beta_1, \beta_2$ 代表示 $\beta_1, \beta_2 = \beta_1, \gamma_2, \gamma_3$ 代表示
具体地,设
$$\begin{cases} \alpha_1 = \alpha_{11}\beta_1 + \alpha_{21}\beta_2 \\ \alpha_2 = \alpha_{12}\beta_1 + \alpha_{22}\beta_2 \end{cases}$$

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases}$$
则
$$\alpha_1 = \alpha_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + \alpha_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)$$

$$= ()\gamma_1 + ()\gamma_2 + ()\gamma_3$$

$$\alpha_2 =$$

例
$$\alpha_1, \alpha_2 = \beta_1, \beta_2$$
线性表示 $\beta_1, \beta_2 = \beta_1, \beta_2$ 代表示 $\beta_1, \beta_2 = \beta_1, \beta_2$ 代表示 $\beta_1, \beta_2 = \beta_1, \beta_2 = \beta_1, \beta_2$ 代表示 $\beta_1, \beta_2 = \beta_1, \beta_1 + \alpha_2 + \beta_2$ $\beta_2 = \beta_1, \beta_1 + \beta_2 + \beta_2 + \beta_2$
$$\{ \beta_1 = \beta_1, \beta_1 + \beta_2, \beta_2 + \beta_2, \beta_2 + \beta_2, \beta_2 + \beta_2, \beta_1 + \beta_2, \beta_2 + \beta_2, \beta_2 + \beta_2, \beta_1 + \beta_2, \beta_2 + \beta_2, \beta_2 + \beta_2, \beta_2 + \beta_2, \beta_1 + \beta_2, \beta_2 + \beta_2, \beta_2 + \beta_2, \beta_1 + \beta_2, \beta_1 + \beta_2, \beta_2 + \beta_2, \beta_1 + \beta_2, \beta_1 + \beta_2, \beta_2 + \beta_2, \beta_2 + \beta_2, \beta_1 + \beta_2, \beta_1 + \beta_2, \beta_2 + \beta_2, \beta_2 + \beta_2, \beta_1 + \beta_2, \beta_2 + \beta$$

$$\alpha_2 =$$

例
$$\alpha_1$$
, α_2 由 β_1 , β_2 线性表示 β_1 , β_2 由 γ_1 , γ_2 , γ_3 线性表示 β_1 , β_2 由 γ_1 , γ_2 , γ_3 线性表示

$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases}$$

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases}$$

$$\alpha_1 = a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)$$

= $(a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + ($

$$\alpha_2 =$$

)γ3

例
$$\alpha_1$$
, α_2 由 β_1 , β_2 线性表示 β_1 , β_2 由 γ_1 , γ_2 , γ_3 线性表示 β_1 , β_2 由 γ_1 , γ_2 , γ_3 线性表示 具体地,设

$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases}$$

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases}$$

$$\begin{split} \alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3 \end{split}$$

$$\alpha_2 =$$

例
$$\alpha_1, \alpha_2 = \beta_1, \beta_2$$
线性表示 $\beta_1, \beta_2 = \beta_1, \beta_2$ 代表示 $\beta_1, \beta_2 = \beta_1, \beta_2$ 代表示 $\beta_1, \beta_2 = \beta_1, \beta_2$ 代表示 $\beta_1, \beta_2 = \beta_1, \beta_1 + \beta_2$ 代本 $\beta_1 = \beta_1, \beta_1 + \beta_2$ 代本 $\beta_2 = \beta_1, \beta_1 + \beta_2$ 代本 $\beta_1 = \beta_1, \beta_2 + \beta_2$ 作为 $\beta_2 = \beta_1, \beta_2 + \beta_2$ 的 $\beta_1 = \beta_1, \beta_2 + \beta_2$ 的 $\beta_2 = \beta_1, \beta_2 + \beta_2$ 的 $\beta_1 = \beta_1, \beta_2 + \beta_2$ 的 $\beta_2 = \beta_1, \beta_2 + \beta_2$ 的 $\beta_1 = \beta_2, \beta_2 + \beta_3$ 的 $\beta_1 = \beta_1, \beta_2 + \beta_2$ 的 $\beta_1 = \beta_2, \beta_2 + \beta_3$ 的 $\beta_1 = \beta_1, \beta_2 + \beta_2$ 的 $\beta_1 = \beta_2$ 的 $\beta_1 = \beta_1, \beta_2 + \beta_2$ 的 $\beta_1 = \beta_2, \beta_2 + \beta_3$ 的 $\beta_1 = \beta_1, \beta_2 + \beta_2$ 的 $\beta_1 = \beta_$

 $) + a_{22}($

 $\alpha_2 = a_{12}$

例
$$\alpha_1, \alpha_2 = \beta_1, \beta_2$$
线性表示 $\beta_1, \beta_2 = \beta_1, \beta_2$ 代表示 $\beta_1, \beta_2 = \beta_1, \beta_2$ 代表示 $\beta_1, \beta_2 = \beta_1, \beta_2$ 代表示 $\beta_1, \beta_2 = \beta_1, \beta_1 + \beta_2$ 代本 $\beta_1 = \beta_1, \beta_1 + \beta_2$ 代本 $\beta_2 = \beta_1, \beta_1 + \beta_2$ 代本 $\beta_1 = \beta_1, \beta_2 + \beta_2$ 作为 $\beta_2 = \beta_1, \beta_2 + \beta_2$ 的 $\beta_1 = \beta_1, \beta_2 + \beta_2$ 的 $\beta_2 = \beta_1, \beta_2 + \beta_2$ 的 $\beta_1 = \beta_1, \beta_2 + \beta_2$ 的 $\beta_2 = \beta_1, \beta_2 + \beta_2$ 的 $\beta_1 = \beta_2$ 的 $\beta_1 = \beta_1, \beta_2 = \beta_2$ 的 $\beta_1 = \beta_1, \beta_2 = \beta_2$ 的 $\beta_1 = \beta_2$ 的 $\beta_1 = \beta_1, \beta_2 = \beta_2$ 的 $\beta_1 = \beta_2, \beta_2 = \beta_2$ 的 $\beta_1 = \beta_2$ 的 $\beta_1 = \beta_2$ 的 $\beta_1 = \beta_2$ 的 $\beta_2 = \beta_2$ 的 β_2

 $\alpha_2 = a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}($

例
$$\alpha_1$$
, α_2 由 β_1 , β_2 线性表示 β_1 , β_2 由 β_1 , β_2 3线性表示 β_1 , β_2 由 β_1 , β_2 2, β_2 3线性表示 β_1 , β_2 2, β_2 3, β_2 3, β_2 4, β_2 4, β_2 5, β_2 6, β_2 7, β_2 8, β_2 8, β_2 9, β_2 9, β_2 9, β_3 9, β_3 9, β_3 9, β_3 9, β_4 9, β_4 9, β_2 9, β_3 9, β_4 9,

$$\alpha_1 = a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)$$

= $(a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3$

$$\alpha_2 = \alpha_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + \alpha_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)$$

例
$$\alpha_1, \alpha_2 \oplus \beta_1, \beta_2$$
线性表示 $\beta_1, \beta_2 \oplus \gamma_1, \gamma_2, \gamma_3$ 线性表示 $\beta_1, \beta_2 \oplus \gamma_1, \gamma_2, \gamma_3$ 线性表示

$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases}$$

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases}$$

$$\alpha_1 = a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)$$

= $(a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3$

$$\alpha_2 = a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)$$

$$= ()\gamma_1 + ()\gamma_2 + ()\gamma_3$$

例
$$\alpha_1$$
, α_2 由 β_1 , β_2 线性表示 β_1 , β_2 由 γ_1 , γ_2 , γ_3 线性表示 β_1 , β_2 由 γ_1 , γ_2 , γ_3 线性表示

$$\begin{cases} \alpha_1 = \alpha_{11}\beta_1 + \alpha_{21}\beta_2 \\ \alpha_2 = \alpha_{12}\beta_1 + \alpha_{22}\beta_2 \end{cases}$$

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases}$$

则

$$\alpha_1 = a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)$$

= $(a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3$

$$\alpha_2 = a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)$$

$$= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + ()\gamma_2 + ($$

 $)\gamma_3$

例
$$\alpha_1$$
, α_2 由 β_1 , β_2 线性表示 β_1 , β_2 由 γ_1 , γ_2 , γ_3 线性表示 β_1 , β_2 由 γ_1 , γ_2 , γ_3 线性表示

$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases}$$

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases}$$

则

$$\begin{aligned} \alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3 \end{aligned}$$

$$\alpha_2 = a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)$$

$$= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (a_{12}b_{21} + a_{22}b$$

 $)\gamma_3$

例
$$\alpha_1$$
, α_2 由 β_1 , β_2 线性表示 β_1 , β_2 由 γ_1 , γ_2 , γ_3 线性表示 β_1 , β_2 由 γ_1 , γ_2 , γ_3 线性表示

$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases}$$

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases}$$

$$\begin{aligned} \alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3 \end{aligned}$$

$$\alpha_2 = a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)$$

= $(a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (a_{12}b_{31} + a_{22}b_{32})\gamma_3$

例
$$\alpha_1$$
, α_2 由 β_1 , β_2 线性表示 β_1 , β_2 由 γ_1 , γ_2 , γ_3 线性表示 β_1 , β_2 由 γ_1 , γ_2 , γ_3 线性表示

$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases}$$

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases}$$

$$\alpha_{1} = a_{11}(b_{11}\gamma_{1} + b_{21}\gamma_{2} + b_{31}\gamma_{3}) + a_{21}(b_{12}\gamma_{1} + b_{22}\gamma_{2} + b_{32}\gamma_{3})$$

$$= (a_{11}b_{11} + a_{21}b_{12})\gamma_{1} + (a_{11}b_{21} + a_{21}b_{22})\gamma_{2} + (a_{11}b_{31} + a_{21}b_{32})\gamma_{3}$$

$$= c_{11}\gamma_{1} +$$

$$\alpha_{2} = a_{12}(b_{11}\gamma_{1} + b_{21}\gamma_{2} + b_{31}\gamma_{3}) + a_{22}(b_{12}\gamma_{1} + b_{22}\gamma_{2} + b_{32}\gamma_{3})$$

$$= (a_{12}b_{11} + a_{22}b_{12})\gamma_{1} + (a_{12}b_{21} + a_{22}b_{22})\gamma_{2} + (a_{12}b_{31} + a_{22}b_{32})\gamma_{3}$$

例
$$\alpha_1$$
, α_2 由 β_1 , β_2 线性表示 β_1 , β_2 由 γ_1 , γ_2 , γ_3 线性表示 β_1 , β_2 由 γ_1 , γ_2 , γ_3 线性表示

$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases}$$

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases}$$

$$\begin{split} \alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3 \\ &= c_{11}\gamma_1 + c_{21}\gamma_2 + \\ \alpha_2 &= a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (a_{12}b_{31} + a_{22}b_{32})\gamma_3 \end{split}$$

例
$$\alpha_1$$
, α_2 由 β_1 , β_2 线性表示 β_1 , β_2 由 γ_1 , γ_2 , γ_3 线性表示 β_1 , β_2 由 γ_1 , γ_2 , γ_3 线性表示

$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases}$$

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases}$$

$$\begin{split} \alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3 \\ &= c_{11}\gamma_1 + c_{21}\gamma_2 + c_{31}\gamma_3 \\ \alpha_2 &= a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (a_{12}b_{31} + a_{22}b_{32})\gamma_3 \end{split}$$

$$\alpha_1$$
, α_2 由 β_1 , β_2 线性表示 β_1 , β_2 由 γ_1 , γ_2 , γ_3 线性表示 β_1 , β_2 由 γ_1 , γ_2 , γ_3 线性表示

$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases}$$

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases}$$

$$\alpha_{1} = a_{11}(b_{11}\gamma_{1} + b_{21}\gamma_{2} + b_{31}\gamma_{3}) + a_{21}(b_{12}\gamma_{1} + b_{22}\gamma_{2} + b_{32}\gamma_{3})$$

$$= (a_{11}b_{11} + a_{21}b_{12})\gamma_{1} + (a_{11}b_{21} + a_{21}b_{22})\gamma_{2} + (a_{11}b_{31} + a_{21}b_{32})\gamma_{3}$$

$$= c_{11}\gamma_{1} + c_{21}\gamma_{2} + c_{31}\gamma_{3}$$

$$\alpha_{2} = a_{12}(b_{11}\gamma_{1} + b_{21}\gamma_{2} + b_{31}\gamma_{3}) + a_{22}(b_{12}\gamma_{1} + b_{22}\gamma_{2} + b_{32}\gamma_{3})$$

$$= (a_{12}b_{11} + a_{22}b_{12})\gamma_{1} + (a_{12}b_{21} + a_{22}b_{22})\gamma_{2} + (a_{12}b_{31} + a_{22}b_{32})\gamma_{3}$$

$$= c_{12}\gamma_{1} + a_{22}b_{12}\gamma_{1} + a_{22}b_{22}\gamma_{2} + a_{22}b_{22}\gamma_{2} + a_{22}b_{22}\gamma_{2}$$

$$\alpha_1$$
, α_2 由 β_1 , β_2 线性表示
 β_1 , β_2 由 γ_1 , γ_2 , γ_3 线性表示
 β_1 , β_2 由 γ_1 , γ_2 , γ_3 线性表示

$$\begin{cases} \alpha_1 = \alpha_{11}\beta_1 + \alpha_{21}\beta_2 \\ \alpha_2 = \alpha_{12}\beta_1 + \alpha_{22}\beta_2 \end{cases}$$

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases}$$

$$\alpha_{1} = a_{11}(b_{11}\gamma_{1} + b_{21}\gamma_{2} + b_{31}\gamma_{3}) + a_{21}(b_{12}\gamma_{1} + b_{22}\gamma_{2} + b_{32}\gamma_{3})$$

$$= (a_{11}b_{11} + a_{21}b_{12})\gamma_{1} + (a_{11}b_{21} + a_{21}b_{22})\gamma_{2} + (a_{11}b_{31} + a_{21}b_{32})\gamma_{3}$$

$$= c_{11}\gamma_{1} + c_{21}\gamma_{2} + c_{31}\gamma_{3}$$

$$\alpha_{2} = a_{12}(b_{11}\gamma_{1} + b_{21}\gamma_{2} + b_{31}\gamma_{3}) + a_{22}(b_{12}\gamma_{1} + b_{22}\gamma_{2} + b_{32}\gamma_{3})$$

$$= (a_{12}b_{11} + a_{22}b_{12})\gamma_{1} + (a_{12}b_{21} + a_{22}b_{22})\gamma_{2} + (a_{12}b_{31} + a_{22}b_{32})\gamma_{3}$$

$$= c_{12}\gamma_{1} + c_{22}\gamma_{2} +$$

$$\alpha_1$$
, α_2 由 β_1 , β_2 线性表示 β_1 , β_2 由 γ_1 , γ_2 , γ_3 线性表示 β_1 , β_2 由 γ_1 , γ_2 , γ_3 线性表示

$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases}$$

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$$\alpha_{1} = a_{11}(b_{11}\gamma_{1} + b_{21}\gamma_{2} + b_{31}\gamma_{3}) + a_{21}(b_{12}\gamma_{1} + b_{22}\gamma_{2} + b_{32}\gamma_{3})$$

$$= (a_{11}b_{11} + a_{21}b_{12})\gamma_{1} + (a_{11}b_{21} + a_{21}b_{22})\gamma_{2} + (a_{11}b_{31} + a_{21}b_{32})\gamma_{3}$$

$$= c_{11}\gamma_{1} + c_{21}\gamma_{2} + c_{31}\gamma_{3}$$

$$\alpha_{2} = a_{12}(b_{11}\gamma_{1} + b_{21}\gamma_{2} + b_{31}\gamma_{3}) + a_{22}(b_{12}\gamma_{1} + b_{22}\gamma_{2} + b_{32}\gamma_{3})$$

$$= (a_{12}b_{11} + a_{22}b_{12})\gamma_{1} + (a_{12}b_{21} + a_{22}b_{22})\gamma_{2} + (a_{12}b_{31} + a_{22}b_{32})\gamma_{3}$$

$$= c_{12}\gamma_{1} + c_{22}\gamma_{2} + c_{32}\gamma_{3}$$

例
$$\alpha_1$$
, α_2 由 β_1 , β_2 线性表示 β_1 , β_2 由 γ_1 , γ_2 , γ_3 线性表示 β_1 , β_2 由 γ_1 , γ_2 , γ_3 线性表示 具体地. 设

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases}$$

 $= C_{12}\gamma_1 + C_{22}\gamma_2 + C_{32}\gamma_3$

 $\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases}$

$$\alpha_{1} = a_{11}(b_{11}\gamma_{1} + b_{21}\gamma_{2} + b_{31}\gamma_{3}) + a_{21}(b_{12}\gamma_{1} + b_{22}\gamma_{2} + b_{32}\gamma_{3})$$

$$= (a_{11}b_{11} + a_{21}b_{12})\gamma_{1} + (a_{11}b_{21} + a_{21}b_{22})\gamma_{2} + (a_{11}b_{31} + a_{21}b_{32})\gamma_{3}$$

$$= c_{11}\gamma_{1} + c_{21}\gamma_{2} + c_{31}\gamma_{3}$$

$$\alpha_{2} = a_{12}(b_{11}\gamma_{1} + b_{21}\gamma_{2} + b_{31}\gamma_{3}) + a_{22}(b_{12}\gamma_{1} + b_{22}\gamma_{2} + b_{32}\gamma_{3})$$

$$= (a_{12}b_{11} + a_{22}b_{12})\gamma_{1} + (a_{12}b_{21} + a_{22}b_{22})\gamma_{2} + (a_{12}b_{31} + a_{22}b_{32})\gamma_{3}$$

其中
$$(c_{ij}) = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{12} & a_{12}b_{11} + a_{22}b_{12} \\ a_{11}b_{21} + a_{21}b_{22} & a_{12}b_{21} + a_{22}b_{22} \\ a_{11}b_{31} + a_{21}b_{32} & a_{12}b_{31} + a_{22}b_{32} \end{pmatrix}$$



例
$$\alpha_1$$
, α_2 由 β_1 , β_2 线性表示 β_1 , β_2 由 γ_1 , γ_2 , γ_3 线性表示 β_1 , β_2 由 γ_1 , γ_2 , γ_3 线性表示 具体地、设

$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases}$$

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases}$$

$$\alpha_1 = a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)$$

$$= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3$$

$$= c_{11}\gamma_1 + c_{21}\gamma_2 + c_{31}\gamma_3$$

$$\alpha_2 = a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)$$

 $=(a_{12}b_{11}+a_{22}b_{12})\gamma_1+(a_{12}b_{21}+a_{22}b_{22})\gamma_2+(a_{12}b_{31}+a_{22}b_{32})\gamma_3$

 $= C_{12}\gamma_1 + C_{22}\gamma_2 + C_{32}\gamma_3$ 其中 $(c_{ij}) = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{12} & a_{12}b_{11} + a_{22}b_{12} \\ a_{11}b_{21} + a_{21}b_{22} & a_{12}b_{21} + a_{22}b_{22} \\ a_{11}b_{31} + a_{21}b_{32} & a_{12}b_{31} + a_{22}b_{32} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

例
$$\alpha_1, \alpha_2 \oplus \beta_1, \beta_2 \otimes \mathbb{R}$$
 $\beta_1, \beta_2 \oplus \gamma_1, \gamma_2, \gamma_3 \otimes \mathbb{R}$ $\beta_1, \beta_2 \oplus \gamma_1, \gamma_2, \gamma_3 \otimes \mathbb{R}$ $\beta_1, \beta_2 \oplus \gamma_1, \gamma_2, \gamma_3 \otimes \mathbb{R}$ 具体地,设
$$\begin{cases} \alpha_1 = \alpha_{11}\beta_1 + \alpha_{21}\beta_2 \\ \alpha_2 = \alpha_{12}\beta_1 + \alpha_{22}\beta_2 \end{cases} \Rightarrow (\alpha_1, \alpha_2) = (\beta_1, \beta_2) \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}$$

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases}$$

$$\alpha_{1} = a_{11}(b_{11}\gamma_{1} + b_{21}\gamma_{2} + b_{31}\gamma_{3}) + a_{21}(b_{12}\gamma_{1} + b_{22}\gamma_{2} + b_{32}\gamma_{3})$$

$$= (a_{11}b_{11} + a_{21}b_{12})\gamma_{1} + (a_{11}b_{21} + a_{21}b_{22})\gamma_{2} + (a_{11}b_{31} + a_{21}b_{32})\gamma_{3}$$

$$= c_{11}\gamma_{1} + c_{21}\gamma_{2} + c_{31}\gamma_{3}$$

$$\alpha_{2} = a_{12}(b_{11}\gamma_{1} + b_{21}\gamma_{2} + b_{31}\gamma_{3}) + a_{22}(b_{12}\gamma_{1} + b_{22}\gamma_{2} + b_{32}\gamma_{3})$$

$$= (a_{12}b_{11} + a_{22}b_{12})\gamma_{1} + (a_{12}b_{21} + a_{22}b_{22})\gamma_{2} + (a_{12}b_{31} + a_{22}b_{32})\gamma_{3}$$

$$= c_{12}\gamma_1 + c_{22}\gamma_2 + c_{32}\gamma_3$$
其中
$$(c_{ij}) = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{12} & a_{12}b_{11} + a_{22}b_{12} \\ a_{11}b_{21} + a_{21}b_{22} & a_{12}b_{21} + a_{22}b_{22} \\ a_{11}b_{31} + a_{21}b_{32} & a_{12}b_{31} + a_{22}b_{32} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$



例
$$\alpha_1$$
, α_2 由 β_1 , β_2 线性表示 β_1 , β_2 由 β_2 由 β_1 , β_2 由 β_2 由 β_1 , β_2 由 β_2 由 β_2 由 β_1 , β_2 和 β_2 和 β_2 和 β_3 和 β_4 和 β_2 和 β_3 和 β_4 和 β_2 和 β_3 和 β_4

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases} \Rightarrow (\beta_1, \beta_2) = (\gamma_1, \gamma_2, \gamma_3) \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}$$

 $\alpha_1 = a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)$

$$= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3$$

$$= c_{11}\gamma_1 + c_{21}\gamma_2 + c_{31}\gamma_3$$

$$\alpha_2 = a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)$$

$$= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (a_{12}b_{31} + a_{22}b_{32})\gamma_3$$

$$= c_{12}\gamma_1 + c_{22}\gamma_2 + c_{32}\gamma_3$$
其中
$$(c_{ij}) = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{12} & a_{12}b_{11} + a_{22}b_{12} \\ a_{11}b_{21} + a_{21}b_{22} & a_{12}b_{21} + a_{22}b_{22} \\ a_{11}b_{31} + a_{21}b_{32} & a_{12}b_{31} + a_{22}b_{32} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$



具体地,设
$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_2 \\ \alpha_2 = a_{12}\beta_1 + a_2 \end{cases}$$

$$\begin{cases} \alpha_{1} = a_{11}\beta_{1} + a_{21}\beta_{2} \\ \alpha_{2} = a_{12}\beta_{1} + a_{22}\beta_{2} \end{cases} \Rightarrow (\alpha_{1}, \alpha_{2}) = (\beta_{1}, \beta_{2}) \underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{A}$$

$$\begin{cases} \beta_{1} = b_{11}\gamma_{1} + b_{21}\gamma_{2} + b_{31}\gamma_{3} \\ \beta_{2} = b_{12}\gamma_{1} + b_{22}\gamma_{2} + b_{32}\gamma_{3} \end{cases} \Rightarrow (\beta_{1}, \beta_{2}) = (\gamma_{1}, \gamma_{2}, \gamma_{3}) \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}$$

 $\alpha_1 = a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)$

 α_1, α_2 由 β_1, β_2 线性表示 β_1, β_2 由 $\gamma_1, \gamma_2, \gamma_3$ 线性表示 β_1, β_2 的 $\gamma_1, \gamma_2, \gamma_3$ 线性表示

$$= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3$$

$$= c_{11}\gamma_1 + c_{21}\gamma_2 + c_{31}\gamma_3$$

$$\alpha_2 = a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)$$

$$= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (a_{12}b_{31} + a_{22}b_{32})\gamma_3$$

$$= c_{12}\gamma_1 + c_{22}\gamma_2 + c_{32}\gamma_3$$
 其中
$$(c_{ij}) = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{12} & a_{12}b_{11} + a_{22}b_{12} \\ a_{11}b_{21} + a_{21}b_{22} & a_{12}b_{21} + a_{22}b_{22} \\ a_{11}b_{31} + a_{21}b_{32} & a_{12}b_{31} + a_{22}b_{32} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$



具体地,设
$$\left\{ \begin{array}{l} \alpha_1 = \alpha_{11}\beta_1 + \alpha_2 \\ \alpha_2 = \alpha_{12}\beta_1 + \alpha_2 \end{array} \right.$$

$$\begin{cases} \alpha_{1} = a_{11}\beta_{1} + a_{21}\beta_{2} \\ \alpha_{2} = a_{12}\beta_{1} + a_{22}\beta_{2} \end{cases} \Rightarrow (\alpha_{1}, \alpha_{2}) = (\beta_{1}, \beta_{2}) \underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{A}$$

$$\begin{cases} \beta_{1} = b_{11}\gamma_{1} + b_{21}\gamma_{2} + b_{31}\gamma_{3} \\ \beta_{2} = b_{12}\gamma_{1} + b_{22}\gamma_{2} + b_{32}\gamma_{3} \end{cases} \Rightarrow (\beta_{1}, \beta_{2}) = (\gamma_{1}, \gamma_{2}, \gamma_{3}) \underbrace{\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}}_{B_{3}}$$

 $\alpha_1 = a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)$

 $\left\{egin{array}{ll} & & & & & & & \\ & & & & & & & \\ & & & & & & \\ & & & & & \\ & & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ &$

$$= c_{11}\gamma_1 + c_{21}\gamma_2 + c_{31}\gamma_3$$

$$\alpha_2 = a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)$$

$$= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (a_{12}b_{31} + a_{22}b_{32})\gamma_3$$

 $=(a_{11}b_{11}+a_{21}b_{12})\gamma_1+(a_{11}b_{21}+a_{21}b_{22})\gamma_2+(a_{11}b_{31}+a_{21}b_{32})\gamma_3$

$$= c_{12}\gamma_1 + c_{22}\gamma_2 + c_{32}\gamma_3$$
其中
$$(c_{ij}) = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{12} & a_{12}b_{11} + a_{22}b_{12} \\ a_{11}b_{21} + a_{21}b_{22} & a_{12}b_{21} + a_{22}b_{22} \\ a_{11}b_{31} + a_{21}b_{32} & a_{12}b_{31} + a_{22}b_{32} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$



 $\left\{egin{array}{ll} & & & & & & & & \\ & & & & & & & & \\ & & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & \\ & & &$ 具体地,设 $\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases} \Rightarrow (\alpha_1, \alpha_2) = (\beta_1, \beta_2) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

则
$$\alpha_1 = a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)$$

 $=(a_{11}b_{11}+a_{21}b_{12})\gamma_1+(a_{11}b_{21}+a_{21}b_{22})\gamma_2+(a_{11}b_{31}+a_{21}b_{32})\gamma_3$ $= C_{11}\gamma_1 + C_{21}\gamma_2 + C_{31}\gamma_3$ $\alpha_2 = a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)$ $=(a_{12}b_{11}+a_{22}b_{12})\gamma_1+(a_{12}b_{21}+a_{22}b_{22})\gamma_2+(a_{12}b_{31}+a_{22}b_{32})\gamma_3$ $= C_{12}\gamma_1 + C_{22}\gamma_2 + C_{32}\gamma_3$

 $\begin{cases}
\beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\
\beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3
\end{cases} \Rightarrow (\beta_1, \beta_2) = (\gamma_1, \gamma_2, \gamma_3) \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}$