第4章 d: 有理函数的积分

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Outline



定义 设

$$P(x) = p_0 x^n + p_1 x^{n-1} + \dots + p_{n-1} x + p_n$$

$$Q(x) = q_0 x^m + q_1 x^{m-1} + \dots + q_{m-1} x + q_m$$

为多项式,并且没有公共零点(即P(x),Q(x)没有公因式),则

$$R(x) := \frac{P(x)}{Q(x)}$$

称为有理函数.



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例

真分式
$$\frac{4x+1}{x^2-6x+9}$$
 假分式
$$\frac{x^2+1}{x+1} =$$



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注 假分式 = 多项式 + 真分式.

例

真分式
$$\frac{x^2 - 6x + 9}{x^2 + 1} = \frac{x^2 - 1}{x + 1} + \frac{2}{x + 1}$$

4x + 1



定义 设

$$P(x) = p_0 x^n + p_1 x^{n-1} + \dots + p_{n-1} x + p_n$$

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例

例 真分式
$$\frac{4x+1}{x^2-6x+9}$$

$$x^2 + 1 \quad x^2$$

 $\frac{x^2+1}{x+1} = \frac{x^2-1}{x+1} + \frac{2}{x+1} = x-1 + \frac{2}{x+1}$ 假分式



$$\int \frac{sx+t}{x^2+px+q} dx.$$



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设 $\Delta = p^2 - 4q$ 为分母的判别式.

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设 $\Delta = p^2 - 4q$ 为分母的判别式. 则

•
$$\Delta$$
 < 0 ⇒ $x^2 + px + q$ 不可约,

$$\int \frac{sx+t}{x^2+px+q} dx.$$

设 $\Delta = p^2 - 4a$ 为分母的判别式. 则

•
$$\Delta > 0 \Rightarrow x^2 + px + q = (x - a)(x - b)$$
,此时
$$\frac{sx + t}{x^2 + px + q} = \frac{A}{x - a} + \frac{B}{x - b}$$

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$$\int \frac{sx+t}{x^2+px+q} dx.$$

设 $\Delta = p^2 - 4q$ 为分母的判别式. 则

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,此时
$$\frac{sx + t}{x^2 + px + q} = \frac{A}{x - a} + \frac{B}{x - b}$$

•
$$\Delta = 0 \Rightarrow x^2 + px + q = (x - a)^2$$
, 此时 $sx + t$ A B

$$\frac{sx+t}{x^2+px+q} = \frac{A}{x-a} + \frac{B}{(x-a)^2}$$

•
$$\Delta < 0 \Rightarrow x^2 + px + q$$
 不可约,



 $\int \frac{sx+t}{x^2+ny+a} dx.$

问题 本节讨论如下形式的有函数的积分:

$$\int x^2 + px + q$$

$$\frac{sx + t}{x^2 + px + q} = \frac{A}{x - a} + \frac{B}{x - b}$$

$$\frac{sx + t}{x^2 + px + q} = \frac{A}{x - a} + \frac{B}{(x - a)^2}$$

$$\frac{3x+c}{x^2+px+c}$$

$$x^2 + px$$

$$x + q$$
 $x - a$ $(x - a)^2$

•
$$\Delta < 0 \Rightarrow x^2 + px + q$$
 不可约,此时
$$\frac{sx + t}{x^2 + px + q} = \frac{A(x^2 + px + q)'}{x^2 + px + q} + \frac{B}{x^2 + px + q}$$

例 1 求不定积分 $\int \frac{2x-1}{x^2-5x+6} dx$.

Δ > 0 情形

例1 求不定积分 $\int \frac{2x-1}{x^2-5x+6} dx$.

$$\frac{\mathbf{x}}{x^2 - 5x + 6} = \frac{2x - 1}{(x - 2)(x - 3)}$$



例 1 求不定积分 $\int \frac{2x-1}{x^2-5x+6} dx$.

$$\frac{\mathbf{A}\mathbf{B}}{x^2 - 5x + 6} = \frac{2x - 1}{(x - 2)(x - 3)} = \frac{A}{x - 2} + \frac{B}{x - 3}$$



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$$\int \frac{2x-1}{x^2-5x+6} dx$$
. $\frac{A(x-3)+B(x-2)}{(x-2)(x-3)}$

$$\frac{A(x-3)+B(x-2)}{(x-2)(x-3)}$$

$$\mathbf{H} = 2x - 1$$

$$\frac{\mathbf{R}}{x^2 - 5x + 6} = \frac{2x - 1}{(x - 2)(x - 3)} = \frac{A}{x - 2} + \frac{B}{x - 3}$$



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列1 求不定积分
$$\int_{\frac{\sqrt{2}-5}{16}}^{2x-1} dx$$
.

例1 求不定积分
$$\int \frac{2x-1}{x^2-5x+6} dx$$
. $\frac{A(x-3)+B(x-2)}{(x-2)(x-3)} = \frac{(A+B)x-3A-2B}{(x-2)(x-3)} \Rightarrow \begin{cases} A+B=2\\ -3A-2B=-1 \end{cases}$

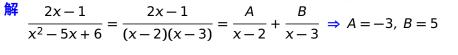


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$$\frac{2x-1}{x^2-5x+6} = \frac{2x-1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3} \Rightarrow A = -3, B = 5$$

所以
原式 =
$$-3$$
 $\int \frac{1}{x-2} dx + 5 \int \frac{1}{x-3} dx$



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$$\frac{B}{x^2 - 5x + 6} = \frac{2x - 1}{(x - 2)(x - 3)} = \frac{A}{x - 2} + \frac{B}{x - 3} \implies A = -3, B = 5$$

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原式 =
$$-3\int \frac{1}{x-2}dx + 5\int \frac{1}{x-3}dx = -3\ln|x-2| + 5\ln|x-3| + C$$

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例 2 求不定积分 $\int \frac{x+1}{\sqrt{2}} dx$.



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例 2 求不定积分
$$\int \frac{x+1}{x^2-4x+3} dx$$
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$$\frac{x+1}{x^2-4x+3} = \frac{x+1}{(x-1)(x-3)^2}$$



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例 2 求不定积分
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.

$$\frac{X+1}{x^2-4x+3} = \frac{X+1}{(x-1)(x-3)} = \frac{A(x-3)+B(x-1)}{(x-1)(x-3)} = \frac{A+B-1}{(x-1)(x-3)} \Rightarrow \begin{cases} A+B=1\\ -3A-B=1 \end{cases}$$

$$\frac{X+1}{x^2-4x+3} = \frac{X+1}{(x-1)(x-3)} = \frac{A}{x-1} + \frac{B}{x-3}$$



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$$\frac{R}{x^2 - 4x + 3} = \frac{x + 1}{(x - 1)(x - 3)} = \frac{A}{x - 1} + \frac{B}{x - 3} \implies A = -1, B = 2$$



Δ > 0 情形

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$$\frac{RR}{RR} = \frac{x+1}{x^2 - 4x + 3} = \frac{x+1}{(x-1)(x-3)} = \frac{A}{x-1} + \frac{B}{x-3} \implies A = -1, B = 2$$
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原式 = $-\int \frac{1}{x-1} dx + 2\int \frac{1}{x-3} dx$



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例1 求不定积分
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例 2 求不定积分
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例 3 求不定积分 $\int \frac{4x+1}{x^2-6x+9} dx$.



例 3 求不定积分
$$\int \frac{4x+1}{x^2-6x+9} dx$$
.

解

$$\frac{4x+1}{x^2-6x+9} = \frac{x+1}{(x-3)^2}$$



例 3 求不定积分 $\int \frac{4x+1}{x^2-6x+9} dx$.

解

$$\frac{4x+1}{x^2-6x+9} = \frac{x+1}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2}$$



例3 求不定积分
$$\int \frac{4x+1}{x^2-6x+9} dx$$
. $\frac{A(x-3)+B}{(x-3)^2}$

$$\frac{4x+1}{x^2-6x+9} = \frac{x+1}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2}$$



例3 求不定积分
$$\int \frac{4x+1}{x^2-6x+9} dx$$
.
$$\frac{A(x-3)+B}{(x-3)^2} = \frac{Ax-3A+B}{(x-3)^2}$$

$$\frac{4x+1}{x^2-6x+9} = \frac{x+1}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2}$$



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$$\frac{4x+1}{x^2-6x+9} = \frac{x+1}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2}$$



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$$\frac{4x+1}{x^2-6x+9} = \frac{x+1}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2} \implies A = 4, B = 13$$



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$$\frac{A(x-3)+B}{(x-3)^2} = \frac{Ax-3A+B}{(x-3)^2} \Rightarrow \begin{cases} A=4\\ -3A+B=1 \end{cases}$$

$$\frac{4x+1}{x^2-6x+9} = \frac{x+1}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2} \implies A = 4, B = 13$$

所以

原式 =
$$4\int \frac{1}{x-3}dx + 13\int \frac{1}{(x-3)^2}dx$$



例3 求不定积分
$$\int \frac{4x+1}{x^2-6x+9} dx$$
.
$$\frac{A(x-3)+B}{(x-3)^2} = \frac{Ax-3A+B}{(x-3)^2} \Rightarrow \begin{cases} A=4\\ -3A+B=1 \end{cases}$$

$$\frac{4x+1}{x^2-6x+9} = \frac{x+1}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2} \implies A = 4, B = 13$$

所以

原式 =
$$4\int \frac{1}{x-3}dx + 13\int \frac{1}{(x-3)^2}dx = 4\ln|x-3| - \frac{13}{x-3} + C$$



例 4 求不定积分 $\int \frac{x-3}{x^2-x+1} dx$.



例 4 求不定积分 $\int \frac{x-3}{x^2-x+1} dx$.

$$\frac{x-3}{x^2-x+1} = \frac{A(x^2-x+1)'}{x^2-x+1} + \frac{B}{x^2-x+1}$$



例 4 求不定积分
$$\int \frac{x-3}{x^2-x+1} dx$$
.

 $\frac{2Ax - A + B}{x^2 - x + 1}$

$$\frac{x-3}{x^2-x+1} = \frac{A(x^2-x+1)'}{x^2-x+1} + \frac{B}{x^2-x+1}$$



例 4 求不定积分
$$\int \frac{x-3}{x^2-x+1} dx$$
.
$$\underset{x^2-x+1}{\underbrace{2Ax-A+B}} \Rightarrow \left\{ \begin{array}{c} 2A=1\\ -A+B=-3 \end{array} \right.$$

$$\frac{x-3}{x^2-x+1} = \frac{A(x^2-x+1)'}{x^2-x+1} + \frac{B}{x^2-x+1}$$



例 4 求不定积分
$$\int \frac{x-3}{x^2-x+1} dx$$
.
$$\underset{x^2-x+1}{\underbrace{2Ax-A+B}} \Rightarrow \left\{ \begin{array}{c} 2A=1\\ -A+B=-3 \end{array} \right.$$

$$\frac{x-3}{x^2-x+1} = \frac{A(x^2-x+1)'}{x^2-x+1} + \frac{B}{x^2-x+1} \Rightarrow A = \frac{1}{2}, B = -\frac{5}{2}$$



例 4 求不定积分
$$\int \frac{x-3}{x^2-x+1} dx$$
.

$$\frac{2AX - A + B}{X^2 - X + 1} \Rightarrow \begin{cases} 2A = 1 \\ -A + B = -3 \end{cases}$$

解

$$\frac{x-3}{x^2-x+1} = \frac{A(x^2-x+1)'}{x^2-x+1} + \frac{B}{x^2-x+1} \Rightarrow A = \frac{1}{2}, B = -\frac{5}{2}$$

所以

原式 =
$$\frac{1}{2} \int \frac{(x^2 - x + 1)'}{x^2 - x + 1} dx - \frac{5}{2} \int \frac{1}{x^2 - x + 1} dx$$



例 4 求不定积分
$$\int \frac{x-3}{x^2-x+1} dx$$
.

$$\frac{2AX - A + B}{X^2 - X + 1} \Rightarrow \begin{cases} 2A = 1 \\ -A + B = -3 \end{cases}$$

解

$$\frac{x-3}{x^2-x+1} = \frac{A(x^2-x+1)'}{x^2-x+1} + \frac{B}{x^2-x+1} \Rightarrow A = \frac{1}{2}, B = -\frac{5}{2}$$

所以

原式 =
$$\frac{1}{2} \int \frac{(x^2 - x + 1)'}{x^2 - x + 1} dx - \frac{5}{2} \int \frac{1}{x^2 - x + 1} dx$$

$$\int \frac{(x^2-x+1)'}{x^2-x+1} dx$$



例 4 求不定积分
$$\int \frac{x-3}{x^2-x+1} dx$$
.

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所以

原式 =
$$\frac{1}{2} \int \frac{(x^2 - x + 1)'}{x^2 - x + 1} dx - \frac{5}{2} \int \frac{1}{x^2 - x + 1} dx$$

$$\int \frac{(x^2 - x + 1)'}{x^2 - x + 1} dx = \int \frac{d(x^2 - x + 1)}{x^2 - x + 1}$$



例 4 求不定积分
$$\int \frac{x-3}{x^2-x+1} dx$$
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所以

原式 =
$$\frac{1}{2} \int \frac{(x^2 - x + 1)'}{x^2 - x + 1} dx - \frac{5}{2} \int \frac{1}{x^2 - x + 1} dx$$

$$\int \frac{(x^2 - x + 1)^r}{x^2 - x + 1} dx = \int \frac{d(x^2 - x + 1)}{x^2 - x + 1} = \ln|x^2 - x + 1| + C$$



其中
$$\int \frac{1}{x^2 - x + 1} dx =$$



其中
$$\int \frac{1}{x^2 - x + 1} dx =$$

$$\int \frac{1}{t^2 + 1} dt$$



其中
$$\int \frac{1}{x^2 - x + 1} dx = \int \frac{1}{(x - \frac{1}{2})^2 + \frac{3}{4}} dx$$

 $\int \frac{1}{t^2 + 1} dt$



$$\int \frac{1}{x^2 - x + 1} dx = \int \frac{1}{(x - \frac{1}{2})^2 + \frac{3}{4}} dx = \frac{4}{3} \int \frac{1}{\left(\frac{2}{\sqrt{3}}x - \frac{1}{\sqrt{3}}\right)^2 + 1} dx$$

$$\int \frac{1}{t^2 + 1} dt$$

 $\int \frac{1}{x^2 - x + 1} dx = \int \frac{1}{(x - \frac{1}{2})^2 + \frac{3}{4}} dx = \frac{4}{3} \int \frac{1}{\left(\frac{2}{\sqrt{3}}x - \frac{1}{\sqrt{3}}\right)^2 + 1} dx$ $\left(t = \frac{2}{\sqrt{3}}x - \frac{1}{\sqrt{3}}\right)$

 $\int \frac{1}{x^2 - x + 1} dx = \int \frac{1}{(x - \frac{1}{2})^2 + \frac{3}{4}} dx = \frac{4}{3} \int \frac{1}{\left(\frac{2}{\sqrt{3}}x - \frac{1}{\sqrt{3}}\right)^2 + 1} dx$ $\left(t = \frac{2}{\sqrt{3}}x - \frac{1}{\sqrt{3}}\right) = \frac{4}{3} \int \frac{1}{t^2 + 1}$

 $\int \frac{1}{x^2 - x + 1} dx = \int \frac{1}{(x - \frac{1}{2})^2 + \frac{3}{4}} dx = \frac{4}{3} \int \frac{1}{\left(\frac{2}{\sqrt{3}}x - \frac{1}{\sqrt{3}}\right)^2 + 1} dx$

 $\left(t = \frac{2}{\sqrt{3}}x - \frac{1}{\sqrt{3}}\right) = \frac{4}{3} \int \frac{1}{t^2 + 1} d\left(\frac{\sqrt{3}}{2}t + \frac{1}{2}\right)$

 $\int \frac{1}{x^2 - x + 1} dx = \int \frac{1}{(x - \frac{1}{2})^2 + \frac{3}{4}} dx = \frac{4}{3} \int \frac{1}{\left(\frac{2}{\sqrt{3}}x - \frac{1}{\sqrt{3}}\right)^2 + 1} dx$

 $\left(t = \frac{2}{\sqrt{3}}x - \frac{1}{\sqrt{3}}\right) = \frac{4}{3} \int \frac{1}{t^2 + 1} d\left(\frac{\sqrt{3}}{2}t + \frac{1}{2}\right) = \frac{2}{\sqrt{3}} \int \frac{1}{t^2 + 1} dt$

 $\int \frac{1}{x^2 - x + 1} dx = \int \frac{1}{(x - \frac{1}{2})^2 + \frac{3}{4}} dx = \frac{4}{3} \int \frac{1}{\left(\frac{2}{\sqrt{3}}x - \frac{1}{\sqrt{3}}\right)^2 + 1} dx$

 $\left(t = \frac{2}{\sqrt{3}}X - \frac{1}{\sqrt{3}}\right) = \frac{4}{3} \int \frac{1}{t^2 + 1} d\left(\frac{\sqrt{3}}{2}t + \frac{1}{2}\right) = \frac{2}{\sqrt{3}} \int \frac{1}{t^2 + 1} dt$

 $=\frac{2}{\sqrt{3}} \arctan t + C$

$$\int \frac{1}{x^2 - x + 1} dx = \int \frac{1}{(x - \frac{1}{2})^2 + \frac{3}{4}} dx = \frac{4}{3} \int \frac{1}{\left(\frac{2}{\sqrt{3}}x - \frac{1}{\sqrt{3}}\right)^2 + 1} dx$$

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$$= \frac{2}{\sqrt{3}} \arctan t + C$$

 $= \frac{2}{\sqrt{3}} \arctan\left(\frac{2}{\sqrt{2}}x - \frac{1}{\sqrt{2}}\right) + C$



 $= \frac{1}{2} \ln |x^2 - x + 1| - \frac{5}{\sqrt{3}} \arctan \left(\frac{2}{\sqrt{3}} x - \frac{1}{\sqrt{3}} \right) + C$

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 $\int \frac{1}{x^2 - x + 1} dx = \int \frac{1}{(x - \frac{1}{2})^2 + \frac{3}{4}} dx = \frac{4}{3} \int \frac{1}{\left(\frac{2}{\sqrt{2}}x - \frac{1}{\sqrt{2}}\right)^2 + 1} dx$

 $\left(t = \frac{2}{\sqrt{3}}x - \frac{1}{\sqrt{3}}\right) = \frac{4}{3} \int \frac{1}{t^2 + 1} d\left(\frac{\sqrt{3}}{2}t + \frac{1}{2}\right) = \frac{2}{\sqrt{3}} \int \frac{1}{t^2 + 1} dt$



$$\frac{5x+6}{x^2+x+1} = \frac{A(x^2+x+1)'}{x^2+x+1} + \frac{B}{x^2+x+1}$$



$$\frac{5x+6}{x^2+x+1} = \frac{A(x^2+x+1)'}{x^2+x+1} + \frac{B}{x^2+x+1}$$



$$\frac{2Ax + A + B}{x^2 + x + 1} \Rightarrow \begin{cases} 2A = 5 \\ A + B = 6 \end{cases}$$

$$\frac{5x+6}{x^2+x+1} = \frac{A(x^2+x+1)'}{x^2+x+1} + \frac{B}{x^2+x+1}$$



$$\frac{2AX + A + B}{X^2 + X + 1} \Rightarrow \begin{cases} 2A = 5 \\ A + B = 6 \end{cases}$$

$$\frac{5x+6}{x^2+x+1} = \frac{A(x^2+x+1)'}{x^2+x+1} + \frac{B}{x^2+x+1} \Rightarrow A = \frac{5}{2}, B = \frac{7}{2}$$



$$\frac{2AX + A + B}{X^2 + X + 1} \Rightarrow \begin{cases} 2A = 5 \\ A + B = 6 \end{cases}$$

解

$$\frac{5x+6}{x^2+x+1} = \frac{A(x^2+x+1)'}{x^2+x+1} + \frac{B}{x^2+x+1} \implies A = \frac{5}{2}, \ B = \frac{7}{2}$$

所以

原式 =
$$\frac{5}{2} \int \frac{(x^2 + x + 1)'}{x^2 + x + 1} dx + \frac{7}{2} \int \frac{1}{x^2 + x + 1} dx$$



$$\frac{2Ax + A + B}{x^2 + x + 1} \Rightarrow \begin{cases} 2A = 5 \\ A + B = 6 \end{cases}$$

解

$$\frac{5x+6}{x^2+x+1} = \frac{A(x^2+x+1)'}{x^2+x+1} + \frac{B}{x^2+x+1} \Rightarrow A = \frac{5}{2}, B = \frac{7}{2}$$

所以

原式 =
$$\frac{5}{2} \int \frac{(x^2 + x + 1)'}{x^2 + x + 1} dx + \frac{7}{2} \int \frac{1}{x^2 + x + 1} dx$$

$$\int \frac{(x^2+x+1)'}{x^2+x+1} dx = \int \frac{d(x^2+x+1)}{x^2+x+1}$$



$$\frac{2AX + A + B}{X^2 + X + 1} \Rightarrow \begin{cases} 2A = 5 \\ A + B = 6 \end{cases}$$

解

$$\frac{5x+6}{x^2+x+1} = \frac{A(x^2+x+1)'}{x^2+x+1} + \frac{B}{x^2+x+1} \Rightarrow A = \frac{5}{2}, B = \frac{7}{2}$$

所以

原式 =
$$\frac{5}{2} \int \frac{(x^2 + x + 1)'}{x^2 + x + 1} dx + \frac{7}{2} \int \frac{1}{x^2 + x + 1} dx$$

$$\int \frac{(x^2+x+1)'}{x^2+x+1} dx = \int \frac{d(x^2+x+1)}{x^2+x+1} = \ln|x^2+x+1| + C$$



其中
$$\int \frac{1}{x^2 + x + 1} dx$$

其中
$$\int \frac{1}{x^2 + x + 1} dx$$

$$\int \frac{1}{t^2 + 1} dt$$



其中
$$\int \frac{1}{x^2 + x + 1} dx = \int \frac{1}{(x + \frac{1}{2})^2 + \frac{3}{4}} dx$$

$$\int \frac{1}{t^2 + 1} dt$$



其中 $\int \frac{1}{x^2 + x + 1} dx = \int \frac{1}{(x + \frac{1}{2})^2 + \frac{3}{4}} dx = \frac{4}{3} \int \frac{1}{\left(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right)^2 + 1} dx$ $\int \frac{1}{t^2 + 1} dt$

其中 $\int \frac{1}{x^2 + x + 1} dx = \int \frac{1}{(x + \frac{1}{2})^2 + \frac{3}{4}} dx = \frac{4}{3} \int \frac{1}{\left(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right)^2 + 1} dx$ $\left(t = \frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right)$

其中 $\int \frac{1}{x^2 + x + 1} dx = \int \frac{1}{(x + \frac{1}{2})^2 + \frac{3}{4}} dx = \frac{4}{3} \int \frac{1}{\left(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right)^2 + 1} dx$ $\left(t = \frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right) = \frac{4}{3} \int \frac{1}{t^2 + 1} d\left(\frac{\sqrt{3}}{2}t - \frac{1}{2}\right)$



$$\int \frac{1}{x^2 + x + 1} dx = \int \frac{1}{(x + \frac{1}{2})^2 + \frac{3}{4}} dx = \frac{4}{3} \int \frac{1}{\left(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right)^2 + 1} dx$$

$$\left(t = \frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right) = \frac{4}{3} \int \frac{1}{t^2 + 1} d\left(\frac{\sqrt{3}}{2}t - \frac{1}{2}\right) = \frac{2}{\sqrt{3}} \int \frac{1}{t^2 + 1} dt$$



其中
$$\int \frac{1}{x^2 + x + 1} dx = \int \frac{1}{(x + \frac{1}{2})^2 + \frac{3}{4}} dx = \frac{4}{3} \int \frac{1}{\left(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right)^2 + 1} dx$$

$$\left(t = \frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right) = \frac{4}{3} \int \frac{1}{t^2 + 1} d\left(\frac{\sqrt{3}}{2}t - \frac{1}{2}\right) = \frac{2}{\sqrt{3}} \int \frac{1}{t^2 + 1} dt$$

 $=\frac{2}{\sqrt{3}} \arctan t + C$

其中
$$\int \frac{1}{x^2 + x + 1} dx = \int \frac{1}{(x + \frac{1}{2})^2 + \frac{3}{4}} dx = \frac{4}{3} \int \frac{1}{\left(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right)^2 + 1} dx$$

$$\left(t = \frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right) = \frac{4}{3} \int \frac{1}{t^2 + 1} d\left(\frac{\sqrt{3}}{2}t - \frac{1}{2}\right) = \frac{2}{\sqrt{3}} \int \frac{1}{t^2 + 1} dt$$

 $=\frac{2}{\sqrt{3}} \arctan t + C$

$$= \frac{2}{\sqrt{3}} \arctan\left(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right) + C$$



$$\Re x = \frac{1}{2} \int \frac{(x^2 + x + 1)'}{x^2 + x + 1} dx + \frac{7}{2} \int \frac{1}{x^2 + x + 1} dx$$

$$= \frac{1}{2} \ln|x^2 + x + 1| + \frac{7}{\sqrt{3}} \arctan\left(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right) + C$$

 $=\frac{2}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}\right)+C$ 所以 原式 = $\frac{1}{2} \int \frac{(x^2 + x + 1)'}{x^2 + x + 1} dx + \frac{7}{2} \int \frac{1}{x^2 + x + 1} dx$

 $\int \frac{1}{x^2 + x + 1} dx = \int \frac{1}{(x + \frac{1}{2})^2 + \frac{3}{4}} dx = \frac{4}{3} \int \frac{1}{\left(\frac{2}{\sqrt{2}}x + \frac{1}{\sqrt{2}}\right)^2 + 1} dx$

 $\left(t = \frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right) = \frac{4}{3} \int \frac{1}{t^2 + 1} d\left(\frac{\sqrt{3}}{2}t - \frac{1}{2}\right) = \frac{2}{\sqrt{3}} \int \frac{1}{t^2 + 1} dt$

 $=\frac{2}{\sqrt{3}} \arctan t + C$

$$= \arctan t + C$$

$$= \arctan \left(\frac{2}{-x} + \frac{1}{-x} \right) + C$$