第 9 章 b：偏导数与全微分

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偏导数引入

- 对一元函数 $y=f(x)$：导数 $y'=f'(x) \longleftrightarrow$ 变化率
- 对二元函数 $z=f(x,y)$：导数？
  1. 固定 y，对 x 求导：$z=f(x,y)$ 关于 x 的变化率

     $$\frac{\partial z}{\partial x} \quad \text{或} \quad z'_x \quad \text{或} \quad z_x \quad \text{或} \quad f_x \qquad \text{对}x\text{偏导数}$$

  2. 固定 x，对 y 求导：$z=f(x,y)$ 关于 y 的变化率

     $$\frac{\partial z}{\partial y} \quad \text{或} \quad z'_y \quad \text{或} \quad z_y \quad \text{或} \quad f_y \qquad \text{对}y\text{偏导数}$$

例 设 $z=f(x,y)=x^2y+2x+y+1$，则

$$\frac{\partial z}{\partial x}=(x^2y+2x+y+1)'_x=2xy+2$$

$$\frac{\partial z}{\partial y}=(x^2y+2x+y+1)'_y=x^2+1$$

例 设 $z = f(x, y) = e^{xy} + 2xy^2$，求 $\frac{\partial z}{\partial x}$ 和 $\frac{\partial z}{\partial y}$

解

$$\frac{\partial z}{\partial x} = (e^{xy} + 2xy^2)'_x = (e^{xy})'_x + (2xy^2)'_x = ye^{xy} + 2y^2$$

$$\frac{\partial z}{\partial y} = (e^{xy} + 2xy^2)'_y = (e^{xy})'_y + (2xy^2)'_y = xe^{xy} + 4xy$$

例 设 $z = f(x, y) = 2y\sin(3x)$，求 $\frac{\partial z}{\partial x}$ 和 $\frac{\partial z}{\partial y}$

解

$$\frac{\partial z}{\partial x} = (2y\sin(3x))'_x = 2y(\sin(3x))'_x = 2y \cdot 3\cos(3x) = 6y\cos(3x)$$

$$\frac{\partial z}{\partial y} = (2y\sin(3x))'_y = (2y)'_y \cdot \sin(3x) = 2\sin(3x)$$

例 求三元函数 $u = xyz + \frac{z}{x}$ 的全部一阶偏导数

解

$$u_x = (xyz + \frac{z}{x})'_x = (xyz)'_x + (\frac{z}{x})'_x = yz - \frac{z}{x^2}$$

$$u_y = (xyz + \frac{z}{x})'_y = (xyz)'_y + (\frac{z}{x})'_y = xz$$

$$u_z = (xyz + \frac{z}{x})'_z = (xyz)'_z + (\frac{z}{x})'_z = xy + \frac{1}{x}$$

偏导数准确定义

- $z = f(x, y)$ 在点 (x_0, y_0) 关于 x 的偏增量：（x 方向的改变量）

$$\Delta_x z = f(x_0 + \Delta x, y_0) - f(x_0, y_0)$$

- $z = f(x, y)$ 在点 (x_0, y_0) 关于 x 的偏导数：（x 方向的导数）

$$\lim_{\Delta x \to 0} \frac{\Delta_x z}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

$$= \frac{d}{dx}[f(x, y_0)]\Big|_{x=x_0}$$

偏导数记号：

$$\frac{\partial z}{\partial x}\Big|_{\substack{x=x_0\\y=y_0}}, \qquad z'_x\Big|_{\substack{x=x_0\\y=y_0}}, \qquad z_x\Big|_{\substack{x=x_0\\y=y_0}}$$

$$\frac{\partial f}{\partial x}(x_0, y_0), \qquad f'_x(x_0, y_0), \qquad f_x(x_0, y_0)$$

偏导数准确定义

- $z=f(x,y)$ 在点 (x_0,y_0) 关于 y 的偏增量：（y 方向的改变量）

$$\Delta_y z = f(x_0, y_0+\Delta y) - f(x_0, y_0)$$

- $z=f(x,y)$ 在点 (x_0,y_0) 关于 y 的偏导数：（y 方向的导数）

$$\lim_{\Delta y\to 0}\frac{\Delta_y z}{\Delta y} = \lim_{\Delta y\to 0}\frac{f(x_0,y_0+\Delta y)-f(x_0,y_0)}{\Delta y}$$

$$= \frac{d}{dy}[f(x_0,y)]\Big|_{y=y_0}$$

偏导数记号：

$$\frac{\partial z}{\partial y}\Big|_{\substack{x=x_0\\ y=y_0}}, \qquad z'_y\Big|_{\substack{x=x_0\\ y=y_0}}, \qquad z_y\Big|_{\substack{x=x_0\\ y=y_0}}$$

$$\frac{\partial f}{\partial y}(x_0,y_0), \qquad f'_y(x_0,y_0), \qquad f_y(x_0,y_0)$$

例 设 $z = xy + \frac{x}{y}$，求 $\frac{\partial z}{\partial x}$，$\frac{\partial z}{\partial y}$ 和在点 (2, 1) 处的偏导数值

解法一

$$\frac{\partial z}{\partial x} = (xy + \frac{x}{y})'_x = (xy)'_x + (\frac{x}{y})'_x = y + \frac{1}{y}$$

$$\frac{\partial z}{\partial y} = (xy + \frac{x}{y})'_y = (xy)'_y + (\frac{x}{y})'_y = x - \frac{x}{y^2}$$

所以

$$\left.\frac{\partial z}{\partial x}\right|_{\substack{x=2\\y=1}} = \left.(y + \frac{1}{y})\right|_{\substack{x=2\\y=1}} = 1 + \frac{1}{1} = 2$$

$$\left.\frac{\partial z}{\partial y}\right|_{\substack{x=2\\y=1}} = \left.(x - \frac{x}{y^2})\right|_{\substack{x=2\\y=1}} = 2 - \frac{2}{1} = 0$$

例 设 $z = xy + \frac{x}{y}$，求 $\frac{\partial z}{\partial x}$，$\frac{\partial z}{\partial y}$ 和在点 $(2, 1)$ 处的偏导数值

解法二 利用

$$\frac{\partial z}{\partial x}(x_0, y_0) = \frac{d}{dx}[f(x, y_0)]\Big|_{x=x_0}, \qquad \frac{\partial z}{\partial y}(x_0, y_0) = \frac{d}{dy}[f(x_0, y)]\Big|_{y=y_0}$$

所以

$$f(x, 1) = 2x \quad \Rightarrow \quad \frac{d}{dx}[f(x, 1)] = 2$$

$$\Rightarrow \quad \frac{\partial z}{\partial x}\Big|_{\substack{x=2\\y=1}} = \frac{d}{dx}[f(x, 1)]\Big|_{x=2} = 2,$$

$$f(2, y) = 2y + \frac{2}{y} \quad \Rightarrow \quad \frac{d}{dy}[f(2, y)] = 2 - \frac{2}{y^2}$$

$$\Rightarrow \quad \frac{\partial z}{\partial y}\Big|_{\substack{x=2\\y=1}} = \frac{d}{dy}[f(2, y)]\Big|_{y=1} = 0.$$

例 设 $f(x, y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$，求 $f_x(0, 0), f_y(0, 0)$

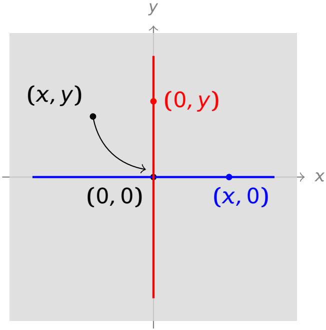
解

$$f_x(0, 0) = \frac{d}{dx}[f(x, 0)]\Big|_{x=0} = \frac{d}{dx}[0]\Big|_{x=0} = 0,$$

$$f_y(0, 0) = \frac{d}{dy}[f(0, y)]\Big|_{x=0} = \frac{d}{dy}[0]\Big|_{y=0} = 0,$$

注 偏导数存在 $\nRightarrow$ 连续

（上述 $f(x, y)$ 在 $(0, 0)$ 处存在偏导数 $f_x(0, 0)$ 和 $f_y(0,0)$，但在 $(0, 0)$ 处不连续）



二阶偏导数

$$z = f(x, y)$$

$$\frac{\partial z}{\partial x}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right): \qquad \frac{\partial^2 z}{\partial x \partial x}, \quad z_{xx}, \quad f_{xx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right): \qquad \frac{\partial^2 z}{\partial x \partial y}, \quad z_{xy}, \quad f_{xy}$$

$$\frac{\partial z}{\partial y}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right): \qquad \frac{\partial^2 z}{\partial y \partial x}, \quad z_{yx}, \quad f_{yx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right): \qquad \frac{\partial^2 z}{\partial y \partial y}, \quad z_{yy}, \quad f_{yy}$$

例 求 $z = e^{xy} + 2xy^2$ 全部二阶偏导数

解

$$z_x = (e^{xy} + 2xy^2)'_x = (e^{xy})'_x + (2xy^2)'_x = ye^{xy} + 2y^2$$

$$z_y = (e^{xy} + 2xy^2)'_y = (e^{xy})'_y + (2xy^2)'_y = xe^{xy} + 4xy$$

$$z_{xx} = (ye^{xy} + 2y^2)'_x = (ye^{xy})'_x + (2y^2)'_x = y^2e^{xy}$$

$$z_{xy} = (ye^{xy} + 2y^2)'_y = (ye^{xy})'_y + (2y^2)'_y = e^{xy} + xye^{xy} + 4y$$

$$z_{yx} = (xe^{xy} + 4xy)'_x = (xe^{xy})'_x + (4xy)'_x = e^{xy} + xye^{xy} + 4y$$

$$z_{yy} = (xe^{xy} + 4xy)'_y = (xe^{xy})'_y + (4xy)'_y = x^2e^{xy} + 4x$$

注 此例成立 $z_{xy} = z_{yx}$

三阶偏导数

$$z=f(x,y)$$

$$\frac{\partial z}{\partial x}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) \longrightarrow \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\right): \quad \frac{\partial^3 z}{\partial x\partial x\partial x},\ z_{xxx},\ f_{xxx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\right): \quad \frac{\partial^3 z}{\partial x\partial x\partial y},\ z_{xxy},\ f_{xxy}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) \longrightarrow \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\right): \quad \frac{\partial^3 z}{\partial x\partial y\partial x},\ z_{xyx},\ f_{xyx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\right): \quad \frac{\partial^3 z}{\partial x\partial y\partial y},\ z_{xyy},\ f_{xyy}$$

$$\frac{\partial z}{\partial y}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) \longrightarrow \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\right): \quad \frac{\partial^3 z}{\partial y\partial x\partial x},\ z_{yxx},\ f_{yxx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\right): \quad \frac{\partial^3 z}{\partial y\partial x\partial y},\ z_{yxy},\ f_{yxy}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) \longrightarrow \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\right): \quad \frac{\partial^3 z}{\partial y\partial y\partial x},\ z_{yyx},\ f_{yyx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\right): \quad \frac{\partial^3 z}{\partial y\partial y\partial y},\ z_{yyy},\ f_{yyy}$$

例 求 $z = x^3y^2 - 3xy^3 - xy + 1$ 全部二阶偏导数及 $\frac{\partial^3 z}{\partial x^3}$

解
$$z_x = (x^3y^2 - 3xy^3 - xy + 1)'_x = 3x^2y^2 - 3y^3 - y$$
$$z_y = (x^3y^2 - 3xy^3 - xy + 1)'_y = 2x^3y - 9xy^2 - x$$

$$z_{xx} = (3x^2y^2 - 3y^3 - y)'_x = 6xy^2$$
$$z_{xy} = (3x^2y^2 - 3y^3 - y)'_y = 6x^2y - 9y^2 - 1$$
$$z_{yx} = (2x^3y - 9xy^2 - x)'_x = 6x^2y - 9y^2 - 1$$
$$z_{yy} = (2x^3y - 9xy^2 - x)'_y = 2x^3 - 18xy$$

$$z_{xxx} = (6xy^2)'_x = 6y^2$$

注 此例成立 $z_{xy} = z_{yx}$

例 求 $z=x\sin(3y)$ 全部二阶偏导数及 z_{xyy}

解

$$z_x=(x\sin(3y))'_x=\sin(3y)$$

$$z_y=(x\sin(3y))'_y=3x\cos(3y)$$

$$z_{xx}=(\sin(3y))'_x=0$$

$$z_{xy}=(\sin(3y))'_y=3\cos(3y)$$

$$z_{yx}=(3x\cos(3y))'_x=3\cos(3y)$$

$$z_{yy}=(3x\cos(3y))'_y=-9x\sin(3y)$$

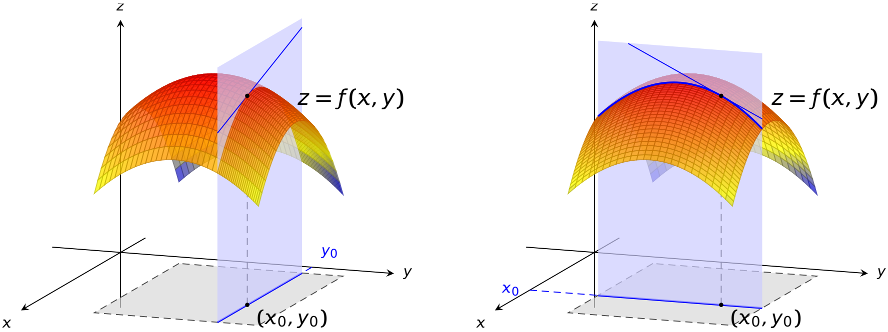
$$z_{xyy}=(3\cos(3y))'_y=-9\sin(3y)$$

注 此例成立 $z_{xy}=z_{yx}$

性质 设有二元函数 $z=f(x,y)$。若 $\frac{\partial^2 z}{\partial y\partial x}$ 和 $\frac{\partial^2 z}{\partial x\partial y}$ 均连续，则

$$\frac{\partial^2 z}{\partial y\partial x}=\frac{\partial^2 z}{\partial x\partial y}$$

偏导数的几何直观



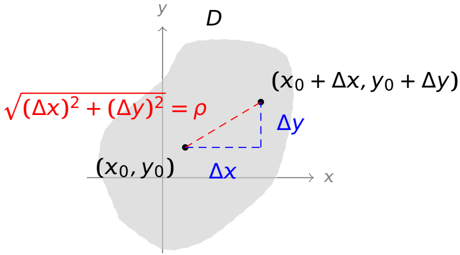
$$\frac{\partial z}{\partial x}(x_0,y_0)=\frac{d}{dx}[f(x,y_0)]\Big|_{x=x_0}$$

$$\frac{\partial z}{\partial y}(x_0,y_0)=\frac{d}{dy}[f(x_0,y)]\Big|_{y=y_0}$$

可微

- 回忆：一元函数 $z=f(x)$ 在 $x=x_0$ 处可微，指

  $\Delta z = f(x_0+\Delta x) - f(x_0) \qquad = f'(x_0)\Delta x + o($



- 二元函数 $z=f(x,y)$ 在 (x_0,y_0) 处可微，指 $\exists$ 数 A, B 使得：

  $\Delta z = f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0)$

  $= A\Delta x + B\Delta y + o(\quad) = A\Delta x + B\Delta y + o$

定理（可微必要条件） 设函数 $z = f(x, y)$ 在该点 (x_0, y_0) 处可微，则

1. 函数在该点 (x_0, y_0) 处连续；
2. 函数在该点 (x_0, y_0) 处存在偏导数 $\frac{\partial z}{\partial x}(x_0, y_0)$, $\frac{\partial z}{\partial y}(x_0, y_0)$；
3. 函数在该点 (x_0, y_0) 处的全微分为

$$dz = \frac{\partial z}{\partial x}(x_0, y_0)\Delta x + \frac{\partial z}{\partial y}(x_0, y_0)\Delta y$$

注 通常的记号：$\Delta x = dx$, $\Delta y = dy$。这样全微分（存在的话）写成：

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

定理（可微充分条件） 设函数 $z = f(x, y)$ 的偏导数 $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ 在点 (x_0, y_0) 连续，则 $z = f(x, y)$ 在该点 (x_0, y_0) 处可微，进而在该点处微分为

$$dz = \frac{\partial z}{\partial x}(x_0, y_0)dx + \frac{\partial z}{\partial y}(x_0, y_0)dy$$

例 设 $z = f(x, y) = x^2 + y^2$，证明函数可微，并计算全微分 dz

解法一（按定义）

$$\begin{aligned}
\Delta z &= f(x + \Delta x, y + \Delta y) - f(x, y) \\
&= \left[(x + \Delta x)^2 + (y + \Delta y)^2\right] - \left[x^2 + y^2\right] \\
&= 2x\Delta x + 2y\Delta y + \left[(\Delta x)^2 + (\Delta y)^2\right] \\
&= 2x\Delta x + 2y\Delta y + \rho^2 \\
&= 2x\Delta x + 2y\Delta y + o(\rho)
\end{aligned}$$

所以 $z = x^2 + y^2$ 可微，并且 $dz = 2xdx + 2ydy$

解法二（利用定理） 先计算偏导数：

$$\frac{\partial z}{\partial x} = 2x, \qquad \frac{\partial z}{\partial y} = 2y$$

可见偏导数存在，且连续。所以函数可微，并且全微分为

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = 2xdx + 2ydy$$

例 计算函数 $z = e^{\frac{y}{x}}$ 的全微分

解 先计算偏导数

$$\frac{\partial z}{\partial x} = \left(e^{\frac{y}{x}}\right)'_x = e^{\frac{y}{x}} \cdot \left(\frac{y}{x}\right)'_x = -\frac{y}{x^2} e^{\frac{y}{x}}$$

$$\frac{\partial z}{\partial x} = \left(e^{\frac{y}{x}}\right)'_y = e^{\frac{y}{x}} \cdot \left(\frac{y}{x}\right)'_y = \frac{1}{x} e^{\frac{y}{x}}$$

可见函数在其自然定义域 $D = \{(x, y) \,|\, x \neq 0\}$ 上存在偏导数且偏导数连续。所以函数可微，并且全微分为

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = -\frac{y}{x^2} e^{\frac{y}{x}} dx + \frac{1}{x} e^{\frac{y}{x}} dy$$

多元函数的全微分

- 对三元函数 $u=f(x,y,z)$，其全微分

$$du = u_x dx + u_y dy + u_z dz$$

此时

$$\begin{aligned}\Delta u &= f(x+\Delta x, y+\Delta y, z+\Delta z) - f(x,y,z) \\ &= u_x\Delta x + u_y\Delta y + u_z\Delta z + o\left(\sqrt{(\Delta x)^2+(\Delta y)^2+(\Delta z)^2}\right) \\ &\approx du\end{aligned}$$

例 设 $u = x^{yz}$，计算全微分 du

例 设 $u = x^{yz}$，计算全微分 du

解 先计算偏导数

$$u_x = (x^{yz})'_x = yz \cdot x^{yz-1}$$
$$u_y = (x^{yz})'_y = x^{yz}\ln(x) \cdot z = z \cdot x^{yz}\ln(x)$$
$$u_z = (x^{yz})'_z = x^{yz}\ln(x) \cdot y = y \cdot x^{yz}\ln(x)$$

所以

$$\begin{aligned} du &= u_x dx + u_y dy + u_z dz \\ &= yz \cdot x^{yz-1} dx + z \cdot x^{yz}\ln(x) dy + y \cdot x^{yz}\ln(x) dz \end{aligned}$$

全微分在近似计算中的应用

设 $z=f(x,y)$，则 $f(x+\Delta x, y+\Delta y)-f(x,y)=dz+o(\rho)\approx dz$，

所以

$$f(x+\Delta x, y+\Delta y)\approx f(x,y)+dz$$

例 计算 $(1.04)^{2.02}$ 的近似值。

解 设 $z=f(x,y)=x^y$，则

$(1.04)^{2.02}=\quad (1+0.04)^{2+0.02}\quad \approx 1^2+dz\,(1$

而

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=yx^{y-1}dx+x^y\ln x\,dy$$

将 $(x,y)=(1,2)$ 及 $dx=\Delta x=0.04$、$dy=\Delta y=0.02$ 代入得：

$$dz=2\cdot 1^1\cdot 0.04+1^2\cdot\ln 1\cdot 0.02=0.08$$

所以 $(1.04)^{2.02}\approx dz+1=0.08+1=1.08$

可微、偏导数存在、连续的区别与联系

设有二元函数 $z = f(x, y)$

- 在点 (x_0, y_0) 处存在偏导数 $\frac{\partial z}{\partial x}(x_0, y_0)$, $\frac{\partial z}{\partial y}(x_0, y_0)$ $\not\Rightarrow$ f 在点 (x_0, y_0) 处连续

- 在点 (x_0, y_0) 处存在可微 $\Rightarrow$ f 在点 (x_0, y_0) 处连续，且存在偏导数 $\frac{\partial z}{\partial x}(x_0, y_0)$, $\frac{\partial z}{\partial y}(x_0, y_0)$

- 在点 (x_0, y_0) 附近存在偏导数 $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$，且偏导数 $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ 在点 (x_0, y_0) 处连续 $\Rightarrow$ 在点 (x_0, y_0) 处可微