第 06 周作业解答

练习 1. 求下列函数的所有二阶偏导数

$$(1)\quad z=\arctan\frac{y}{x};\qquad (2)\quad z=y^x.$$

解 (1)

$$\begin{aligned}
z_x &= \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(\frac{y}{x}\right)_x = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2},\\
z_y &= \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(\frac{y}{x}\right)_y = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x} = \frac{x}{x^2+y^2},
\end{aligned}$$

$$\begin{aligned}
z_{xx} &= \left(-\frac{y}{x^2+y^2}\right)_x = \frac{2xy}{(x^2+y^2)^2},\\
z_{xy} &= \left(-\frac{y}{x^2+y^2}\right)_y = -\frac{(x^2+y^2)-2y^2}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2},\\
z_{yx} &= \left(\frac{x}{x^2+y^2}\right)_x = \frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2},\\
z_{yy} &= \left(\frac{x}{x^2+y^2}\right)_y = -\frac{2xy}{(x^2+y^2)^2}.
\end{aligned}$$

(2)

$$\begin{aligned}
z_x &= (y^x)_x = y^x\ln y,\\
z_y &= (y^x)_y = xy^{x-1},\\
z_{xx} &= (y^x\ln y)_x = y^x(\ln y)^2,\\
z_{xy} &= (y^x\ln y)_y = xy^{x-1}\ln y + y^{x-1} = y^{x-1}(1+x\ln y),\\
z_{yx} &= (xy^{x-1})_x = y^{x-1} + xy^{x-1}\ln y = y^{x-1}(1+x\ln y),\\
z_{yy} &= (xy^{x-1})_y = x(x-1)y^{x-2}.
\end{aligned}$$

练习 2. 求下列函数的全微分

$$(1)\quad z=xy+\frac{x}{y};\qquad (2)\quad u=x^{yz}.$$

解 (1)

$$dz = z_x dx + z_y dy = \left(y+\frac{1}{y}\right)dx + \left(x-\frac{x}{y^2}\right)dy.$$

(2)

$$du = u_x dx + u_y dy + u_z dz = yzx^{yz-1}dx + zx^{yz}\ln x dy + yx^{yz}\ln x dz.$$

练习 3. 求函数 $z=\frac{y}{x}$ 当 $x=2$，$y=1$，$\Delta x=0.1$，$\Delta y=-0.2$ 时的全增量和全微分。

解

$$dz=z_x dx+z_y dy=-\frac{y}{x^2}dx+\frac{1}{x}dy.$$

将 $x=2$，$y=1$，$\Delta x=0.1$，$\Delta y=-0.2$ 代入，得到全微分

$$dz=-\frac{1}{4}\cdot 0.1+\frac{1}{2}\cdot(-0.2)=-0.125=-\frac{1}{8}.$$

而全增量 Δz 为

$$\Delta z=f(x+\Delta x,\,y+\Delta y)-f(x,\,y)=f(2+0.1,\,1-0.2)-f(2,\,1)=\frac{0.8}{2.1}-\frac{1}{2}=\frac{16-21}{42}=-\frac{5}{42}\approx-0.119047619$$

在此例中 Δz 与 dz 在精确到小数点后 1 位时是相等。

练习 4. （选择题）设函数 $f(x,\,y)$ 在点 $P(x_0,\,y_0)$ 的两个偏导数 $f_x(x_0,\,y_0)$ 都存在，则（ C ）

A $f(x,\,y)$ 在点 P 处连续；

B $f(x,\,y)$ 在点 P 处可微；

C $\lim_{x\to x_0} f(x,\,y_0)$ 及 $\lim_{y\to y_0} f(x_0,\,y)$ 都存在；

D $\lim_{(x,\,y)\to(x_0,\,y_0)} f(x,\,y)$ 存在.

练习 5. （选择题）二元函数 $f(x,\,y)=\begin{cases}\frac{xy}{x^2+y^2}, & (x,\,y)\neq(0,\,0)\\ 0, & (x,\,y=(0,\,0)\end{cases}$ 在点 $(0,\,0)$ 处（ C ）

A 连续，偏导数存在；

B 连续，偏导数不存在；

C 不连续，偏导数存在；

D 不连续，偏导数不存在.

练习 6. （选择题）"$f_x(x_0,\,y_0)$ 与 $f_y(x_0,\,y_0)$ 均存在" 是函数 $f(x,\,y)$ 在点 $P(x_0,\,y_0)$ 处连续的（ D ）条件。

A 充分非必要；

B 必要非充分；

C 充分且必要；

D 非充分非必要.

练习 7. 设 $z=\arctan(xy)$，$y=e^x$，求 $\frac{dz}{dx}$。

解设 $z = f(x, y)$，$y = e^x$。

$$\frac{dz}{dx} = f_x + f_y \cdot \frac{dy}{dx} = \frac{y}{1+x^2y^2} + \frac{x}{1+x^2y^2} \cdot e^x = \frac{y + xe^x}{1+x^2y^2} = \frac{e^x(1+x)}{1+x^2e^{2x}}.$$

练习 8. 设 $z = xy + xF(u)$，$u = \frac{y}{x}$，$F(u)$ 为可导函数，证明

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z + xy.$$

解

$$\begin{aligned}
\frac{\partial z}{\partial x} &= y + F(u) + xF'(u) \cdot \left(\frac{y}{x}\right)_x = y + F - \frac{y}{x}F', \\
\frac{\partial z}{\partial y} &= x + xF'(u) \cdot \left(\frac{y}{x}\right)_y = x + F', \\
x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} &= x\left(y + F - \frac{y}{x}F'\right) + y\left(x + F'\right) = 2xy + xF = xy + z.
\end{aligned}$$

练习 9. 求下列函数的一阶偏导数（其中 f 具有一阶连续偏导）：

$$(1) \quad z = f(x^2 - y^2, e^{xy}); \qquad (2) \quad u = f(\frac{x}{y}, \frac{y}{z}); \qquad (3) \quad u = f(x, xy, xyz).$$

解 (1)

$$\begin{aligned}
\frac{\partial z}{\partial x} &= f_1' \cdot (x^2 - y^2)_x + f_2' \cdot (e^{xy})_x = 2xf_1' + ye^{xy}f_2', \\
\frac{\partial z}{\partial y} &= f_1' \cdot (x^2 - y^2)_y + f_2' \cdot (e^{xy})_y = -2yf_1' + xe^{xy}f_2'.
\end{aligned}$$

(2)

$$\begin{aligned}
\frac{\partial u}{\partial x} &= f_1' \cdot (\frac{x}{y})_x = \frac{1}{y}f_1', \\
\frac{\partial u}{\partial y} &= f_1' \cdot (\frac{x}{y})_y + f_2' \cdot (\frac{y}{z})_y = -\frac{x}{y^2}f_1' + \frac{1}{z}f_2', \\
\frac{\partial u}{\partial z} &= f_2' \cdot (\frac{y}{z})_z = -\frac{y}{z^2}f_2'.
\end{aligned}$$

(3)

$$\begin{aligned}
\frac{\partial u}{\partial x} &= f_1' \cdot (x)_x + f_2' \cdot (xy)_x + f_3' \cdot (xyz)_x = f_1' + yf_2' + yzf_3', \\
\frac{\partial u}{\partial y} &= f_2' \cdot (xy)_y + f_3' \cdot (xyz)_y = xf_2' + xzf_3', \\
\frac{\partial u}{\partial z} &= f_3' \cdot (xyz)_z = xyf_3'.
\end{aligned}$$

练习 10. 求复合函数 $z = f(xy^2, x^2y)$ 的所有二阶偏导数。这里假设 f 具有二阶连续偏导数。

解

$$\begin{aligned}
z_x &= f_1' \cdot (xy^2)_x + f_2' \cdot (x^2y)_x = y^2f_1' + 2xyf_2', \\
z_y &= f_1' \cdot (xy^2)_y + f_2' \cdot (x^2y)_y = 2xyf_1' + x^2f_2',
\end{aligned}$$

$$
\begin{aligned}
z_{xx} &= \left(y^2 f_1' + 2xy f_2'\right)_x = y^2 (f_1')_x + 2y f_2' + 2xy (f_2')_x \\
&= y^2 \left[f_{11}'' \cdot (xy^2)_x + f_{12}'' \cdot (x^2 y)_x\right] + 2y f_2' + 2xy \left[f_{21}'' \cdot (xy^2)_x + f_{22}'' \cdot (x^2 y)_x\right] \\
&= y^2 \left[y^2 f_{11}'' + 2xy f_{12}''\right] + 2y f_2' + 2xy \left[y^2 f_{21}'' + 2xy f_{22}''\right] \\
&= 2y f_2' + y^4 f_{11}'' + 4xy^3 f_{12}'' + 4x^2 y^2 f_{22}'',
\end{aligned}
$$

$$
\begin{aligned}
z_{yx} = z_{xy} &= \left(y^2 f_1' + 2xy f_2'\right)_y = 2y f_1' + y^2 (f_1')_y + 2x f_2' + 2xy (f_2')_y \\
&= 2y f_1' + y^2 \left[f_{11}'' \cdot (xy^2)_y + f_{12}'' \cdot (x^2 y)_y\right] + 2x f_2' + 2xy \left[f_{21}'' \cdot (xy^2)_y + f_{22}'' \cdot (x^2 y)_y\right] \\
&= 2y f_1' + y^2 \left[2xy f_{11}'' + x^2 f_{12}''\right] + 2x f_2' + 2xy \left[2xy f_{21}'' + x^2 f_{22}''\right] \\
&= 2y f_1' + 2x f_2' + 2xy^3 f_{11}'' + 5x^2 y^2 f_{12}'' + 2x^3 y f_{22}'',
\end{aligned}
$$

$$
\begin{aligned}
z_{yy} &= \left(2xy f_1' + x^2 f_2'\right)_y = 2x f_1' + 2xy (f_1')_y + x^2 (f_2')_y \\
&= 2x f_1' + 2xy \left[f_{11}'' \cdot (xy^2)_y + f_{12}'' \cdot (x^2 y)_y\right] + x^2 \left[f_{21}'' \cdot (xy^2)_y + f_{22}'' \cdot (x^2 y)_y\right] \\
&= 2x f_1' + 2xy \left[2xy f_{11}'' + x^2 f_{12}''\right] + x^2 \left[2xy f_{21}'' + x^2 f_{22}''\right] \\
&= 2x f_1' + 4x^2 y^2 f_{11}'' + 4x^3 y f_{12}'' + x^4 f_{22}''.
\end{aligned}
$$

练习 11. 设 $\ln\sqrt{x^2+y^2} = \arctan\frac{y}{x}$，求 $\frac{dy}{dx}$。

解令 $F(x, y) = \ln\sqrt{x^2+y^2} - \arctan\frac{y}{x}$。则方程相当于 $F(x, y) = 0$。所以

$$
\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{\frac{x}{x^2+y^2} - \frac{-\frac{y}{x^2}}{1+\frac{y^2}{x^2}}}{\frac{y}{x^2+y^2} - \frac{\frac{1}{x}}{1+\frac{y^2}{x^2}}} = \frac{x+y}{y-x}.
$$

练习 12. 设 $\frac{x}{z} = \ln\frac{z}{y}$，求 $\frac{\partial z}{\partial x}$ 及 $\frac{\partial z}{\partial y}$。

解令 $F(x, y, z) = \frac{x}{z} - \ln\frac{z}{y}$。则方程相当于 $F(x, y, z) = 0$。所以

$$
\begin{aligned}
z_x &= -\frac{F_x}{F_z} = -\frac{\frac{1}{z}}{-\frac{x}{z^2} - \frac{1}{z}} = \frac{z}{x+z}, \\
z_y &= -\frac{F_y}{F_z} = -\frac{\frac{1}{y}}{-\frac{x}{z^2} - \frac{1}{z}} = \frac{z^2}{y(x+z)}.
\end{aligned}
$$

练习 13. 设 $x = x(y, z)$，$y = y(x, z)$，$z = z(x, y)$ 都是由方程 $F(x, y, z) = 0$ 所确定的具有连续偏导数的函数，证明

$$
\frac{\partial x}{\partial y} \cdot \frac{\partial y}{\partial z} \cdot \frac{\partial z}{\partial x} = -1.
$$

证明

$$
\begin{aligned}
\frac{\partial x}{\partial y} &= -\frac{F_y}{F_x} \\
\frac{\partial y}{\partial z} &= -\frac{F_z}{F_y} \\
\frac{\partial z}{\partial x} &= -\frac{F_x}{F_z}
\end{aligned}
$$

所以

$$
\frac{\partial x}{\partial y} \cdot \frac{\partial y}{\partial z} \cdot \frac{\partial z}{\partial x} = \left(-\frac{F_y}{F_x}\right) \cdot \left(-\frac{F_z}{F_y}\right) \cdot \left(-\frac{F_x}{F_z}\right) = -1.
$$

练习 14. 设 $z^3-3xyz=a^3$，求 $\frac{\partial^2 z}{\partial x\partial y}$。

解令 $F(x,y,z)=z^3-3xyz-a^3$。则方程相当于 $F(x,y,z)=0$。所以

$$
\begin{aligned}
z_x &= -\frac{F_x}{F_z} = -\frac{-3yz}{3z^2-3xy} = \frac{yz}{z^2-xy} \xlongequal{\text{或}} \frac{yz^2}{z^3-xyz} = \frac{yz^2}{2xyz+a^3},\\
z_{xy} &= \left(\frac{z}{2x}\right)_y = \frac{z_y}{2x} = \frac{1}{2x}\cdot\left(-\frac{F_y}{F_z}\right) = -\frac{1}{2x}\cdot\frac{-3xz}{3z^2-3xy} = \frac{1}{2}\cdot\frac{z}{z^2-xy}\\
&\xlongequal{\text{或}} \frac{1}{2}\cdot\frac{z^2}{z^3-xyz} = \frac{1}{2}\cdot\frac{z^2}{2xyz+a^3}.
\end{aligned}
$$