§5.1　二次型与对称矩阵

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二次型：引例

二元二次齐次多项式

$$f(x_1, x_2) = 6x_1^2 + 4x_1x_2 - 2x_2^2$$

二次型：引例

二元二次齐次多项式

$$f(x_1, x_2) = 6x_1^2 + 4x_1x_2 - 2x_2^2 = (x_1, x_2)\begin{pmatrix} 6 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

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$$= (6x_1 + 2x_2, 2x_1 - 2x_2)$$

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一般地，

$$f(x_1, x_2) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$$

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$$f(x_1,x_2)=6x_1^2+4x_1x_2-2x_2^2=(x_1,x_2)\begin{pmatrix}6&2\\2&-2\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$$

一般地，

$$\begin{aligned}f(x_1,x_2)&=a_{11}x_1^2+2a_{12}x_1x_2+a_{22}x_2^2\\&=(x_1,x_2)\begin{pmatrix}a_{11}&a_{12}\\a_{12}&a_{22}\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}\\&=(a_{11}x_1+a_{12}x_2,\ a_{12}x_1+a_{22}x_2)\begin{pmatrix}x_1\\x_2\end{pmatrix}\end{aligned}$$

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例

$$f(x_1, x_2) = -3x_1^2 + 2x_1x_2 + 5x_2^2 =$$

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例

$$f(x_1, x_2) = -3x_1^2 + 2x_1x_2 + 5x_2^2 = (x_1, x_2)\begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

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例

$$f(x_1,x_2)=-3x_1^2+2x_1x_2+5x_2^2=(x_1,x_2)\begin{pmatrix}-3&\\&\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$$

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例

$$f(x_1, x_2) = -3x_1^2 + 2x_1x_2 + 5x_2^2 = (x_1, x_2)\begin{pmatrix} -3 & 1 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

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$$f(x_1,x_2,x_3)=a_{11}x_1^2+a_{22}x_2^2+a_{33}x_3^2 \\ +2a_{12}x_1x_2+2a_{13}x_1x_3+2a_{23}x_2x_3$$

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\begin{aligned}
f(x_1, x_2, x_3) &= a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 \\
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$$f(x_1, x_2, x_3) = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3$$

$$= \underbrace{(x_1, x_2, x_3)}_{x^T} \underbrace{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{x} = x^T A x$$

例

$$f(x_1, x_2, x_3) = x_1x_2 + x_1x_3 + 2x_3^2 - 2x_2x_3$$

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f(x_1, x_2, x_3) = x_1x_2 + x_1x_3 + 2x_3^2 - 2x_2x_3
$$

$$
= (x_1, x_2, x_3) \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}
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$$f(x_1,x_2,x_3)=a_{11}x_1^2+a_{22}x_2^2+a_{33}x_3^2$$
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f(x_1,x_2,x_3) &= x_1x_2 + x_1x_3 + 2x_3^2 - 2x_2x_3 \\
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例

$$f(x_1,x_2,x_3)=x_1x_2+x_1x_3+2x_3^2-2x_2x_3$$
$$=(x_1,x_2,x_3)\begin{pmatrix} 0 & & \\ & 0 & \\ & & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

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$$f(x_1, x_2, x_3) = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3$$

$$= \underbrace{(x_1, x_2, x_3)}_{x^T} \underbrace{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{x} = x^T A x$$

例

$$f(x_1, x_2, x_3) = x_1x_2 + x_1x_3 + 2x_3^2 - 2x_2x_3$$

$$= (x_1, x_2, x_3) \begin{pmatrix} 0 & \frac{1}{2} & \\ & 0 & \\ & & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

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例

$$f(x_1,x_2,x_3)=x_1x_2+x_1x_3+2x_3^2-2x_2x_3$$

$$=(x_1,x_2,x_3)\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ & 0 & \\ & & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

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\begin{aligned}
f(x_1, x_2, x_3) &= a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 \\
&\quad + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3 \\
&= \underbrace{(x_1, x_2, x_3)}_{x^T} \underbrace{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{x} = x^T A x
\end{aligned}
$$

例

$$
\begin{aligned}
f(x_1, x_2, x_3) &= x_1x_2 + x_1x_3 + 2x_3^2 - 2x_2x_3 \\
&= (x_1, x_2, x_3) \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ & 0 & -1 \\ & & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}
\end{aligned}
$$

二次型：引例

三元二次齐次多项式

$$f(x_1,x_2,x_3)=a_{11}x_1^2+a_{22}x_2^2+a_{33}x_3^2$$
$$+2a_{12}x_1x_2+2a_{13}x_1x_3+2a_{23}x_2x_3$$
$$=\underbrace{(x_1,x_2,x_3)}_{x^T}\underbrace{\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{12}&a_{22}&a_{23}\\a_{13}&a_{23}&a_{33}\end{pmatrix}}_{A}\underbrace{\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}}_{x}=x^TAx$$

例

$$f(x_1,x_2,x_3)=x_1x_2+x_1x_3+2x_3^2-2x_2x_3$$
$$=(x_1,x_2,x_3)\begin{pmatrix}0&\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&0&-1\\\frac{1}{2}&-1&2\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$

二次型：引例

例 给定二次型，写出对称矩阵 A：

$$f(x_1, x_2, x_3) = x_1^2 + x_1x_2 + 3x_1x_3 + 2x_2^2 + 4x_2x_3 + x_3^2$$

二次型：引例

例 给定二次型，写出对称矩阵 A：

$$f(x_1, x_2, x_3) = x_1^2 + x_1x_2 + 3x_1x_3 + 2x_2^2 + 4x_2x_3 + x_3^2$$

例 给定对称矩阵 A，写出相应二次型：

$$f(x_1, x_2, x_3) = (x_1, x_2, x_3)\begin{pmatrix} -1 & 1 & \frac{1}{2} \\ 1 & 2 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$=$$

二次型：引例

例 给定二次型，写出对称矩阵 A：

$$f(x_1, x_2, x_3) = x_1^2 + x_1x_2 + 3x_1x_3 + 2x_2^2 + 4x_2x_3 + x_3^2$$

$$= (x_1, x_2, x_3)\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

例 给定对称矩阵 A，写出相应二次型：

$$f(x_1, x_2, x_3) = (x_1, x_2, x_3)\begin{pmatrix} -1 & 1 & \frac{1}{2} \\ 1 & 2 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$=$$

二次型：引例

例 给定二次型，写出对称矩阵 A：

$$f(x_1, x_2, x_3) = x_1^2 + x_1x_2 + 3x_1x_3 + 2x_2^2 + 4x_2x_3 + x_3^2$$

$$= (x_1, x_2, x_3)\begin{pmatrix} 1 & & \\ & & \\ & & \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

例 给定对称矩阵 A，写出相应二次型：

$$f(x_1, x_2, x_3) = (x_1, x_2, x_3)\begin{pmatrix} -1 & 1 & \frac{1}{2} \\ 1 & 2 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$=$$

二次型：引例

例 给定二次型，写出对称矩阵 A：

$$f(x_1, x_2, x_3) = x_1^2 + x_1x_2 + 3x_1x_3 + 2x_2^2 + 4x_2x_3 + x_3^2$$

$$= (x_1, x_2, x_3)\begin{pmatrix} 1 & & \\ & 2 & \\ & & \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

例 给定对称矩阵 A，写出相应二次型：

$$f(x_1, x_2, x_3) = (x_1, x_2, x_3)\begin{pmatrix} -1 & 1 & \frac{1}{2} \\ 1 & 2 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$=$$

二次型：引例

例 给定二次型，写出对称矩阵 A：

$$f(x_1,x_2,x_3)=x_1^2+x_1x_2+3x_1x_3+2x_2^2+4x_2x_3+x_3^2$$

$$=(x_1,x_2,x_3)\begin{pmatrix}1 & & \\ & 2 & \\ & & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$

例 给定对称矩阵 A，写出相应二次型：

$$f(x_1,x_2,x_3)=(x_1,x_2,x_3)\begin{pmatrix}-1 & 1 & \frac{1}{2}\\ 1 & 2 & 0\\ \frac{1}{2} & 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$

$$=$$

二次型：引例

例 给定二次型，写出对称矩阵 A：

$$f(x_1, x_2, x_3) = x_1^2 + x_1x_2 + 3x_1x_3 + 2x_2^2 + 4x_2x_3 + x_3^2$$

$$= (x_1, x_2, x_3)\begin{pmatrix} 1 & \frac{1}{2} & \\ & 2 & \\ & & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

例 给定对称矩阵 A，写出相应二次型：

$$f(x_1, x_2, x_3) = (x_1, x_2, x_3)\begin{pmatrix} -1 & 1 & \frac{1}{2} \\ 1 & 2 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$=$$

二次型：引例

例 给定二次型，写出对称矩阵 A：

$$f(x_1,x_2,x_3)=x_1^2+x_1x_2+3x_1x_3+2x_2^2+4x_2x_3+x_3^2$$

$$=(x_1,x_2,x_3)\begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ & 2 & \\ & & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

例 给定对称矩阵 A，写出相应二次型：

$$f(x_1,x_2,x_3)=(x_1,x_2,x_3)\begin{pmatrix} -1 & 1 & \frac{1}{2} \\ 1 & 2 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$=$$

二次型：引例

例 给定二次型，写出对称矩阵 A：

$$f(x_1, x_2, x_3) = x_1^2 + x_1x_2 + 3x_1x_3 + 2x_2^2 + 4x_2x_3 + x_3^2$$

$$= (x_1, x_2, x_3)\begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ & 2 & 2 \\ & & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

例 给定对称矩阵 A，写出相应二次型：

$$f(x_1, x_2, x_3) = (x_1, x_2, x_3)\begin{pmatrix} -1 & 1 & \frac{1}{2} \\ 1 & 2 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$=$$

二次型：引例

例 给定二次型，写出对称矩阵 A：

$$f(x_1,x_2,x_3)=x_1^2+x_1x_2+3x_1x_3+2x_2^2+4x_2x_3+x_3^2$$

$$=(x_1,x_2,x_3)\begin{pmatrix}1&\frac{1}{2}&\frac{3}{2}\\ \frac{1}{2}&2&2\\ \frac{3}{2}&2&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$

例 给定对称矩阵 A，写出相应二次型：

$$f(x_1,x_2,x_3)=(x_1,x_2,x_3)\begin{pmatrix}-1&1&\frac{1}{2}\\ 1&2&0\\ \frac{1}{2}&0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$

$$=$$

二次型：引例

例 给定二次型，写出对称矩阵 A：

$$f(x_1, x_2, x_3) = x_1^2 + x_1x_2 + 3x_1x_3 + 2x_2^2 + 4x_2x_3 + x_3^2$$

$$= (x_1, x_2, x_3)\begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & 2 & 2 \\ \frac{3}{2} & 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

例 给定对称矩阵 A，写出相应二次型：

$$f(x_1, x_2, x_3) = (x_1, x_2, x_3)\begin{pmatrix} -1 & 1 & \frac{1}{2} \\ 1 & 2 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= -x_1^2$$

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二次型：引例

例 给定二次型，写出对称矩阵 A：

$$f(x_1, x_2, x_3) = x_1^2 + x_1x_2 + 3x_1x_3 + 2x_2^2 + 4x_2x_3 + x_3^2$$

$$= (x_1, x_2, x_3)\begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & 2 & 2 \\ \frac{3}{2} & 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

例 给定对称矩阵 A，写出相应二次型：

$$f(x_1, x_2, x_3) = (x_1, x_2, x_3)\begin{pmatrix} -1 & 1 & \frac{1}{2} \\ 1 & 2 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= -x_1^2 + 2x_2^2$$

二次型：引例

例 给定二次型，写出对称矩阵 A：

$$f(x_1, x_2, x_3) = x_1^2 + x_1x_2 + 3x_1x_3 + 2x_2^2 + 4x_2x_3 + x_3^2$$

$$= (x_1, x_2, x_3)\begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & 2 & 2 \\ \frac{3}{2} & 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

例 给定对称矩阵 A，写出相应二次型：

$$f(x_1, x_2, x_3) = (x_1, x_2, x_3)\begin{pmatrix} -1 & 1 & \frac{1}{2} \\ 1 & 2 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= -x_1^2 + 2x_2^2 + 0x_3^2$$

二次型：引例

例 给定二次型，写出对称矩阵 A：

$$f(x_1,x_2,x_3)=x_1^2+x_1x_2+3x_1x_3+2x_2^2+4x_2x_3+x_3^2$$

$$=(x_1,x_2,x_3)\begin{pmatrix}1&\frac{1}{2}&\frac{3}{2}\\ \frac{1}{2}&2&2\\ \frac{3}{2}&2&1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}$$

例 给定对称矩阵 A，写出相应二次型：

$$f(x_1,x_2,x_3)=(x_1,x_2,x_3)\begin{pmatrix}-1&1&\frac{1}{2}\\ 1&2&0\\ \frac{1}{2}&0&0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}$$

$$=-x_1^2+2x_2^2+0x_3^2+2\cdot 1\cdot x_1x_2$$

二次型：引例

例 给定二次型，写出对称矩阵 A：

$$f(x_1, x_2, x_3) = x_1^2 + x_1x_2 + 3x_1x_3 + 2x_2^2 + 4x_2x_3 + x_3^2$$

$$= (x_1, x_2, x_3)\begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & 2 & 2 \\ \frac{3}{2} & 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

例 给定对称矩阵 A，写出相应二次型：

$$f(x_1, x_2, x_3) = (x_1, x_2, x_3)\begin{pmatrix} -1 & 1 & \frac{1}{2} \\ 1 & 2 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= -x_1^2 + 2x_2^2 + 0x_3^2 + 2\cdot 1\cdot x_1x_2 + 2\cdot\frac{1}{2}\cdot x_1x_3$$

二次型：引例

例 给定二次型，写出对称矩阵 A：

$$f(x_1, x_2, x_3) = x_1^2 + x_1x_2 + 3x_1x_3 + 2x_2^2 + 4x_2x_3 + x_3^2$$

$$= (x_1, x_2, x_3)\begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & 2 & 2 \\ \frac{3}{2} & 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

例 给定对称矩阵 A，写出相应二次型：

$$f(x_1, x_2, x_3) = (x_1, x_2, x_3)\begin{pmatrix} -1 & 1 & \frac{1}{2} \\ 1 & 2 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= -x_1^2 + 2x_2^2 + 0x_3^2 + 2\cdot 1\cdot x_1x_2 + 2\cdot\frac{1}{2}\cdot x_1x_3 + 2\cdot 0\cdot x_2x_3$$

二次型：引例

例 给定二次型，写出对称矩阵 A：

$$f(x_1, x_2, x_3) = x_1^2 + x_1x_2 + 3x_1x_3 + 2x_2^2 + 4x_2x_3 + x_3^2$$

$$= (x_1, x_2, x_3)\begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & 2 & 2 \\ \frac{3}{2} & 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

例 给定对称矩阵 A，写出相应二次型：

$$f(x_1, x_2, x_3) = (x_1, x_2, x_3)\begin{pmatrix} -1 & 1 & \frac{1}{2} \\ 1 & 2 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= -x_1^2 + 2x_2^2 + 0x_3^2 + 2\cdot 1\cdot x_1x_2 + 2\cdot\frac{1}{2}\cdot x_1x_3 + 2\cdot 0\cdot x_2x_3$$

$$= -x_1^2 + 2x_2^2 + 2x_1x_3 + x_1x_3$$

二次型

定义 n 元二次型

$$
\begin{aligned}
f(x_1, x_2, \ldots, x_n) = & a_{11}x_1^2 + 2a_{12}x_1x_2 + \cdots + 2a_{1n}x_1x_n \\
& + a_{22}x_2^2 \qquad + \cdots + 2a_{2n}x_2x_n \\
& + \cdots\cdots \\
& \qquad\qquad\qquad + a_{nn}x_n^2
\end{aligned}
$$

二次型

定义 n 元二次型

$$
\begin{aligned}
f(x_1, x_2, \ldots, x_n) = & a_{11}x_1^2 + 2a_{12}x_1x_2 + \cdots + 2a_{1n}x_1x_n \\
& + a_{22}x_2^2 \qquad + \cdots + 2a_{2n}x_2x_n \\
& + \cdots\cdots \\
& \qquad\qquad\qquad + a_{nn}x_n^2 \\
= & (x_1, x_2, \ldots, x_n) \begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \\ & & \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}
\end{aligned}
$$

二次型

定义 n 元二次型

$$
\begin{aligned}
f(x_1, x_2, \ldots, x_n) = & a_{11}x_1^2 + 2a_{12}x_1x_2 + \cdots + 2a_{1n}x_1x_n \\
& + a_{22}x_2^2 \qquad + \cdots + 2a_{2n}x_2x_n \\
& + \cdots\cdots \\
& \qquad\qquad + a_{nn}x_n^2 \\
= (x_1, x_2, \ldots, x_n) & \begin{pmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}
\end{aligned}
$$

二次型

定义 n 元二次型

$$
\begin{aligned}
f(x_1, x_2, \ldots, x_n) = & a_{11}x_1^2 + 2a_{12}x_1x_2 + \cdots + 2a_{1n}x_1x_n \\
& + a_{22}x_2^2 \qquad + \cdots + 2a_{2n}x_2x_n \\
& + \cdots\cdots \\
& \qquad\qquad + a_{nn}x_n^2 \\
= & (x_1, x_2, \ldots, x_n) \begin{pmatrix} a_{11} & a_{12} & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}
\end{aligned}
$$

二次型

定义 n 元二次型

$$
\begin{aligned}
f(x_1, x_2, \ldots, x_n) = & a_{11}x_1^2 + 2a_{12}x_1x_2 + \cdots + 2a_{1n}x_1x_n \\
& + a_{22}x_2^2 \qquad + \cdots + 2a_{2n}x_2x_n \\
& + \cdots\cdots \\
& \qquad\qquad + a_{nn}x_n^2 \\
= & (x_1, x_2, \ldots, x_n) \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}
\end{aligned}
$$

二次型

定义 n 元二次型

$$
\begin{aligned}
f(x_1, x_2, \ldots, x_n) = & a_{11}x_1^2 + 2a_{12}x_1x_2 + \cdots + 2a_{1n}x_1x_n \\
& + a_{22}x_2^2 \qquad + \cdots + 2a_{2n}x_2x_n \\
& + \cdots\cdots \\
& \qquad\qquad + a_{nn}x_n^2 \\
= & (x_1, x_2, \ldots, x_n) \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ & & & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}
\end{aligned}
$$

二次型

定义 n 元二次型

$$
\begin{aligned}
f(x_1, x_2, \ldots, x_n) = & a_{11}x_1^2 + 2a_{12}x_1x_2 + \cdots + 2a_{1n}x_1x_n \\
& + a_{22}x_2^2 \qquad + \cdots + 2a_{2n}x_2x_n \\
& + \cdots\cdots \\
& \qquad\qquad + a_{nn}x_n^2 \\
= & (x_1, x_2, \ldots, x_n) \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}
\end{aligned}
$$

二次型

定义 n 元二次型

$$\begin{aligned}
f(x_1, x_2, \ldots, x_n) = & a_{11}x_1^2 + 2a_{12}x_1x_2 + \cdots + 2a_{1n}x_1x_n \\
& + a_{22}x_2^2 \qquad + \cdots + 2a_{2n}x_2x_n \\
& + \cdots\cdots \\
& \qquad\qquad + a_{nn}x_n^2 \\
= & \underbrace{(x_1, x_2, \ldots, x_n)}_{x^T} \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{x}
\end{aligned}$$

二次型

定义 n 元二次型

$$
\begin{aligned}
f(x_1, x_2, \ldots, x_n) = & a_{11}x_1^2 + 2a_{12}x_1x_2 + \cdots + 2a_{1n}x_1x_n \\
& + a_{22}x_2^2 \qquad + \cdots + 2a_{2n}x_2x_n \\
& + \cdots\cdots \\
& \qquad\qquad\qquad + a_{nn}x_n^2 \\
= & \underbrace{(x_1, x_2, \ldots, x_n)}_{x^T} \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{x} \\
= & x^T A x
\end{aligned}
$$

二次型

定义 n 元二次型

$$
\begin{aligned}
f(x_1, x_2, \ldots, x_n) = & a_{11}x_1^2 + 2a_{12}x_1x_2 + \cdots + 2a_{1n}x_1x_n \\
& + a_{22}x_2^2 \qquad + \cdots + 2a_{2n}x_2x_n \\
& + \cdots\cdots \\
& \qquad\qquad\qquad + a_{nn}x_n^2 \\
= & \underbrace{(x_1, x_2, \ldots, x_n)}_{x^T} \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{x} \\
= & x^T A x
\end{aligned}
$$

注 n 元二次型与对称矩阵，是一一对应

给定二次型

$$
\begin{aligned}
f(x_1, x_2, \ldots, x_n) = & a_{11}x_1^2 + 2a_{12}x_1x_2 + \cdots + 2a_{1n}x_1x_n \\
& + a_{22}x_2^2 \qquad\qquad + \cdots + 2a_{2n}x_2x_n \\
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& \qquad\qquad\qquad\qquad\quad + a_{nn}x_n^2
\end{aligned}
$$

给定二次型

$$
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\end{aligned}
$$

作变量代换：

$$
\begin{cases}
x_1 = c_{11}y_1 + \cdots + c_{1n}y_n \\
\vdots \\
x_n = c_{n1}y_1 + \cdots + c_{nn}y_n
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$$\begin{cases} x_1 = c_{11}y_1 + \cdots + c_{1n}y_n \\ \vdots \\ x_n = c_{n1}y_1 + \cdots + c_{nn}y_n \end{cases}$$

代入二次型 $f(x_1, x_2, \ldots, x_n)$ 得

$$f = \text{关于} y_1, \cdots, y_n \text{ 的二次型}$$

给定二次型

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f(x_1, x_2, \dots, x_n) = & a_{11}x_1^2 + 2a_{12}x_1x_2 + \cdots + 2a_{1n}x_1x_n \\
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作变量代换：（要求 C 是可逆矩阵，所以可以反解出 y ）

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\Leftrightarrow
\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}
$$

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$$\Leftrightarrow x = Cy$$

代入二次型 $f(x_1, x_2, \ldots, x_n)$ 得

$$f = \text{关于} y_1, \cdots, y_n \text{ 的二次型}$$

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作变量代换：（要求 C 是可逆矩阵，所以可以反解出 $y = C^{-1}x$）

$$\begin{cases} x_1 = c_{11}y_1 + \cdots + c_{1n}y_n \\ \vdots \\ x_n = c_{n1}y_1 + \cdots + c_{nn}y_n \end{cases} \Leftrightarrow \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$\Leftrightarrow x = Cy$$

代入二次型 $f(x_1, x_2, \ldots, x_n)$ 得

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\Leftrightarrow \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}
$$

$$\Leftrightarrow x = Cy$$

代入二次型 $f(x_1, x_2, \ldots, x_n)$ 得

$$f = \text{关于} y_1, \cdots, y_n \text{ 的二次型}$$

问 ：在新变量 $y_1, y_2, \cdots, y_n$ 下，f 的表达式最多能化简到怎样的程度？

上述问题等价于问：给定对称矩阵 A，尝试找出可逆矩阵 C 使得

$$C^T A C$$

尽可能简单？

线性变换：引例

$f(x_1,x_2)=x_1^2+4x_1x_2-3x_2^2$

线性变换：引例

$$f(x_1,x_2)=x_1^2+4x_1x_2-3x_2^2$$

变量 $\Downarrow$ 代换

$$\begin{cases} x_1 = y_1 - 2y_2 \\ x_2 = \qquad\quad y_2 \end{cases}$$

$$\Downarrow$$

线性变换：引例

$$f(x_1, x_2) = x_1^2 + 4x_1x_2 - 3x_2^2$$

$$\text{变量} \Downarrow \text{代换}$$

$$\begin{cases} x_1 = y_1 - 2y_2 \\ x_2 = y_2 \end{cases}$$

$$\Downarrow$$

$$f =$$

线性变换：引例

$$f(x_1, x_2) = x_1^2 + 4x_1x_2 - 3x_2^2$$

$$\text{变量} \Downarrow \text{代换}$$

$$\begin{cases} x_1 = y_1 - 2y_2 \\ x_2 = \qquad\quad y_2 \end{cases}$$

$$\Downarrow$$

$$f = (y_1 - 2y_2)^2$$

线性变换：引例

$$f(x_1,x_2)=x_1^2+4x_1x_2-3x_2^2$$

$$\text{变量}\Downarrow\text{代换}$$

$$\begin{cases} x_1 = y_1 - 2y_2 \\ x_2 = y_2 \end{cases}$$

$$\Downarrow$$

$$f=(y_1-2y_2)^2$$

$$+4(y_1-2y_2)y_2$$

线性变换：引例

$$f(x_1,x_2)=x_1^2+4x_1x_2-3x_2^2$$

变量$\Downarrow$代换

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变量 $\Downarrow$ 代换

$$\begin{cases} x_1 = y_1 - 2y_2 \\ x_2 = \qquad\quad\; y_2 \end{cases}$$

$$\Downarrow$$

$$f = (y_1 - 2y_2)^2$$

$$+4(y_1 - 2y_2)y_2 - 3y_2^2$$

$$= y_1^2 - 7y_2^2$$

线性变换：引例

$$f(x_1,x_2)=x_1^2+4x_1x_2-3x_2^2 \Leftrightarrow f(x_1,x_2)=(x_1,x_2)\begin{pmatrix}1 & 2\\ 2 & -3\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

变量 $\Downarrow$ 代换

$$\begin{cases} x_1 = y_1 - 2y_2 \\ x_2 = \qquad\quad y_2 \end{cases}$$

$$\Downarrow$$

$$f=(y_1-2y_2)^2$$

$$+4(y_1-2y_2)y_2-3y_2^2$$

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变量 $\Downarrow$ 代换

$$\begin{cases}x_1=y_1-2y_2\\x_2=\qquad\quad y_2\end{cases}$$

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$$\begin{aligned}f&=(y_1-2y_2)^2\\&\quad+4(y_1-2y_2)y_2-3y_2^2\\&=y_1^2-7y_2^2\end{aligned}$$

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变量 $\Downarrow$ 代换　　　　变量 $\Downarrow$ 代换

$$\begin{cases}x_1=y_1-2y_2\\x_2=\quad\quad y_2\end{cases} \Leftrightarrow \begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}1&-2\\0&1\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix}$$

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$$\begin{aligned}f&=(y_1-2y_2)^2\\&\quad+4(y_1-2y_2)y_2-3y_2^2\\&=y_1^2-7y_2^2\end{aligned}$$

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变量 $\Downarrow$ 代换 　　　　变量 $\Downarrow$ 代换

$$\begin{cases}x_1=y_1-2y_2\\x_2=\quad\quad y_2\end{cases} \Leftrightarrow \begin{pmatrix}x_1\\x_2\end{pmatrix}=\underbrace{\begin{pmatrix}1&-2\\0&1\end{pmatrix}}_{C}\begin{pmatrix}y_1\\y_2\end{pmatrix}$$

$$\Downarrow$$

$$f=(y_1-2y_2)^2$$
$$+4(y_1-2y_2)y_2-3y_2^2$$
$$=y_1^2-7y_2^2$$

线性变换：引例

$$f(x_1,x_2)=x_1^2+4x_1x_2-3x_2^2 \Leftrightarrow f(x_1,x_2)=(x_1,x_2)\underbrace{\begin{pmatrix}1&2\\2&-3\end{pmatrix}}_{A}\begin{pmatrix}x_1\\x_2\end{pmatrix}=x^TAx$$

变量 $\Downarrow$ 代换 　　　　　　　　变量 $\Downarrow$ 代换

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$$\Downarrow$$

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变量 $\Downarrow$ 代换   变量 $\Downarrow$ 代换

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$x=Cy$

$x^T=y^TC^T$

$\Downarrow$

$$f=(y_1-2y_2)^2+4(y_1-2y_2)y_2-3y_2^2$$

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$\Downarrow$ $\Downarrow$

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$$f=(y_1,\ y_2)C^TAC\begin{pmatrix}y_1\\y_2\end{pmatrix}$$

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$$f(x_1,x_2)=x_1^2+4x_1x_2-3x_2^2 \Leftrightarrow f(x_1,x_2)=(x_1,x_2)\underbrace{\begin{pmatrix}1&2\\2&-3\end{pmatrix}}_{A}\begin{pmatrix}x_1\\x_2\end{pmatrix}=x^TAx$$

变量 $\Downarrow$ 代换　　　　变量 $\Downarrow$ 代换

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$x=Cy$
$x^T=y^TC^T$

$\Downarrow$　　　　$\Downarrow$

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$$f=(y_1,\ y_2)C^TAC\begin{pmatrix}y_1\\y_2\end{pmatrix}=(y_1,\ y_2)\begin{pmatrix}\quad&\quad\\\quad&\quad\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix}$$

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$$f(x_1,x_2)=x_1^2+4x_1x_2-3x_2^2 \Leftrightarrow f(x_1,x_2)=(x_1,x_2)\underbrace{\begin{pmatrix}1&2\\2&-3\end{pmatrix}}_{A}\begin{pmatrix}x_1\\x_2\end{pmatrix}=x^TAx$$

变量 $\Downarrow$ 代换　　　　变量 $\Downarrow$ 代换

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$x=Cy$

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$\Downarrow$　　　　$\Downarrow$

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$$f=(y_1,\ y_2)C^TAC\begin{pmatrix}y_1\\y_2\end{pmatrix}=(y_1,\ y_2)\begin{pmatrix}1&0\\0&-7\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix}$$

线性变换：引例

$$f(x_1,x_2)=x_1^2+4x_1x_2-3x_2^2 \Leftrightarrow f(x_1,x_2)=(x_1,x_2)\underbrace{\begin{pmatrix}1&2\\2&-3\end{pmatrix}}_{A}\begin{pmatrix}x_1\\x_2\end{pmatrix}=x^TAx$$

变量 $\Downarrow$ 代换　　　　变量 $\Downarrow$ 代换

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$x=Cy$

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$\Downarrow$　　　　$\Downarrow$

$$f=(y_1-2y_2)^2+4(y_1-2y_2)y_2-3y_2^2=y_1^2-7y_2^2$$

$$f=(y_1,\,y_2)C^TAC\begin{pmatrix}y_1\\y_2\end{pmatrix}=(y_1,\,y_2)\begin{pmatrix}1&0\\0&-7\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix}$$

注

1. $C^TAC=\begin{pmatrix}1&0\\-2&1\end{pmatrix}\begin{pmatrix}1&2\\2&-3\end{pmatrix}\begin{pmatrix}1&-2\\0&1\end{pmatrix}=\begin{pmatrix}1&0\\0&-7\end{pmatrix}$

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$$f(x_1,x_2)=x_1^2+4x_1x_2-3x_2^2 \Leftrightarrow f(x_1,x_2)=(x_1,x_2)\underbrace{\begin{pmatrix}1&2\\2&-3\end{pmatrix}}_{A}\begin{pmatrix}x_1\\x_2\end{pmatrix}=x^TAx$$

变量 $\Downarrow$ 代换 　　　　变量 $\Downarrow$ 代换

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$x=Cy$

$x^T=y^TC^T$

$\Downarrow$ 　　　　$\Downarrow$

$$f=(y_1-2y_2)^2+4(y_1-2y_2)y_2-3y_2^2=y_1^2-7y_2^2$$

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注

1. $C^TAC=\begin{pmatrix}1&0\\-2&1\end{pmatrix}\begin{pmatrix}1&2\\2&-3\end{pmatrix}\begin{pmatrix}1&-2\\0&1\end{pmatrix}=\begin{pmatrix}1&0\\0&-7\end{pmatrix}$

2. 变量代换可逆：

线性变换：引例

$$f(x_1,x_2)=x_1^2+4x_1x_2-3x_2^2 \Leftrightarrow f(x_1,x_2)=(x_1,x_2)\underbrace{\begin{pmatrix}1&2\\2&-3\end{pmatrix}}_{A}\begin{pmatrix}x_1\\x_2\end{pmatrix}=x^TAx$$

变量 $\Downarrow$ 代换 　　　　　　　　变量 $\Downarrow$ 代换

$$\begin{cases}x_1=y_1-2y_2\\x_2=\qquad\quad y_2\end{cases} \Leftrightarrow \begin{pmatrix}x_1\\x_2\end{pmatrix}=\underbrace{\begin{pmatrix}1&-2\\0&1\end{pmatrix}}_{C}\begin{pmatrix}y_1\\y_2\end{pmatrix}$$

$x=Cy$

$x^T=y^TC^T$

$\Downarrow$ 　　　　　　　　$\Downarrow$

$$\begin{aligned}f&=(y_1-2y_2)^2\\&\quad+4(y_1-2y_2)y_2-3y_2^2\\&=y_1^2-7y_2^2\end{aligned}$$

$$\begin{aligned}f&=(y_1,\ y_2)C^TAC\begin{pmatrix}y_1\\y_2\end{pmatrix}\\&=(y_1,\ y_2)\begin{pmatrix}1&0\\0&-7\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix}\end{aligned}$$

注

1. $C^TAC=\begin{pmatrix}1&0\\-2&1\end{pmatrix}\begin{pmatrix}1&2\\2&-3\end{pmatrix}\begin{pmatrix}1&-2\\0&1\end{pmatrix}=\begin{pmatrix}1&0\\0&-7\end{pmatrix}$
2. 变量代换可逆：$\begin{pmatrix}y_1\\y_2\end{pmatrix}=\begin{pmatrix}1&-2\\0&1\end{pmatrix}^{-1}\begin{pmatrix}x_1\\x_2\end{pmatrix}=$

线性变换：引例

$$f(x_1,x_2)=x_1^2+4x_1x_2-3x_2^2 \Leftrightarrow f(x_1,x_2)=(x_1,x_2)\underbrace{\begin{pmatrix}1&2\\2&-3\end{pmatrix}}_{A}\begin{pmatrix}x_1\\x_2\end{pmatrix}=x^TAx$$

变量 $\Downarrow$ 代换 　　　　变量 $\Downarrow$ 代换

$$\begin{cases}x_1=y_1-2y_2\\x_2=y_2\end{cases} \Leftrightarrow \begin{pmatrix}x_1\\x_2\end{pmatrix}=\underbrace{\begin{pmatrix}1&-2\\0&1\end{pmatrix}}_{C}\begin{pmatrix}y_1\\y_2\end{pmatrix}$$

$x=Cy$
$x^T=y^TC^T$

$\Downarrow$ 　　　　$\Downarrow$

$$f=(y_1-2y_2)^2+4(y_1-2y_2)y_2-3y_2^2=y_1^2-7y_2^2$$

$$f=(y_1,\ y_2)C^TAC\begin{pmatrix}y_1\\y_2\end{pmatrix}=(y_1,\ y_2)\begin{pmatrix}1&0\\0&-7\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix}$$

注

1. $C^TAC=\begin{pmatrix}1&0\\-2&1\end{pmatrix}\begin{pmatrix}1&2\\2&-3\end{pmatrix}\begin{pmatrix}1&-2\\0&1\end{pmatrix}=\begin{pmatrix}1&0\\0&-7\end{pmatrix}$
2. 变量代换可逆：$\begin{pmatrix}y_1\\y_2\end{pmatrix}=\begin{pmatrix}1&-2\\0&1\end{pmatrix}^{-1}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}1&2\\0&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$

定理　对任意n元二次型

$$f(x_1,\cdots,x_n)=a_{11}x_1^2+\cdots+2a_{1n}x_1x_n \\ +\cdots\cdots \\ +a_{nn}x_n^2$$

一定存在非退化线性变换

$$\begin{cases} x_1=c_{11}y_1+\cdots+c_{1n}y_n \\ \vdots \\ x_n=c_{n1}y_1+\cdots+c_{nn}y_n \end{cases}$$

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$$f = d_1y_1^2+\cdots+d_ry_r^2 \quad \text{标准形}$$

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$$x=Cy,\quad x^T=y^TC^T$$

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其中$d_1,\cdots,d_r\neq 0$

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$$f=y^TC^TACy$$

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$$f=y^T\boxed{C^TAC}y \qquad \boxed{B}$$

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注

- $r=$;

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$$x=Cy,\quad x^T=y^TC^T$$

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$$\begin{pmatrix}d_1&&&&\\&\ddots&&&\\&&d_r&&\\&&&0&\\&&&&\ddots\\&&&&&0\end{pmatrix}$$

其中$d_1,\cdots,d_r\neq 0$

注

- $r=r(B)$;

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注

- $r=r(B)=r(A)$；d_i 具体取值不唯一

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注

- $r=r(B)=r(A)$；d_i 具体取值不唯一
- 可以证明 $d_1,\ldots,d_r$ 中正、负数的个数唯一：

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- 可以证明 $d_1,\ldots,d_r$ 中正、负数的个数唯一：
  1. 正惯性指标：$d_1,\ldots,d_r$ 中正数的个数

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其中$d_1,\cdots,d_r\neq 0$

$$\begin{pmatrix} d_1 & & & & \\ & \ddots & & & \\ & & d_r & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix}$$

注

- $r=r(B)=r(A)$；d_i 具体取值不唯一
- 可以证明 $d_1,\ldots,d_r$ 中正、负数的个数唯一：
  1. 正惯性指标：$d_1,\ldots,d_r$ 中正数的个数
  2. 负惯性指标：$d_1,\ldots,d_r$ 中负数的个数

配方法化二次型为标准形

- 想法： $a^2 + 2ab =$

配方法化二次型为标准形

- 想法：$$a^2 + 2ab = a^2 + 2ab + b^2 - b^2 =$$

配方法化二次型为标准形

- 想法： $a^2+2ab=a^2+2ab+b^2-b^2=(a+b)^2-b^2$

配方法化二次型为标准形

- 想法： $a^2+2ab=a^2+2ab+b^2-b^2=(a+b)^2-b^2$

  $a^2+2ab+2ac=$

配方法化二次型为标准形

- 想法： $a^2+2ab=a^2+2ab+b^2-b^2=(a+b)^2-b^2$

$$a^2+2ab+2ac=a^2+2a(b+c)$$

$$=$$

配方法化二次型为标准形

- 想法：$$a^2+2ab=a^2+2ab+b^2-b^2=(a+b)^2-b^2$$

$$\begin{aligned} a^2+2ab+2ac &= a^2+2a(b+c) \\ &= a^2+2a(b+c)+(b+c)^2-(b+c)^2 \\ &= \end{aligned}$$

配方法化二次型为标准形

- 想法： $$a^2+2ab=a^2+2ab+b^2-b^2=(a+b)^2-b^2$$

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配方法化二次型为标准形

- 想法：$$a^2+2ab=a^2+2ab+b^2-b^2=(a+b)^2-b^2$$

$$\begin{aligned} a^2+2ab+2ac &= a^2+2a(b+c) \\ &= a^2+2a(b+c)+(b+c)^2-(b+c)^2 \\ &= (a+b+c)^2-(b+c)^2 \end{aligned}$$

例 配方法化二次型为标准形

$$f=x_1^2+2x_1x_2+2x_1x_3+2x_2^2+4x_2x_3+x_3^2$$

$$=$$

配方法化二次型为标准形

- 想法：$$a^2+2ab=a^2+2ab+b^2-b^2=(a+b)^2-b^2$$

$$\begin{aligned} a^2+2ab+2ac &= a^2+2a(b+c) \\ &= a^2+2a(b+c)+(b+c)^2-(b+c)^2 \\ &= (a+b+c)^2-(b+c)^2 \end{aligned}$$

例 配方法化二次型为标准形

$$\begin{aligned} f &= x_1^2+2x_1x_2+2x_1x_3+2x_2^2+4x_2x_3+x_3^2 \\ &= x_1^2+2x_1(x_2+x_3) \end{aligned}$$

配方法化二次型为标准形

- 想法：$$a^2+2ab=a^2+2ab+b^2-b^2=(a+b)^2-b^2$$

$$\begin{aligned}a^2+2ab+2ac&=a^2+2a(b+c)\\&=a^2+2a(b+c)+(b+c)^2-(b+c)^2\\&=(a+b+c)^2-(b+c)^2\end{aligned}$$

例 配方法化二次型为标准形

$$\begin{aligned}f&=x_1^2+2x_1x_2+2x_1x_3+2x_2^2+4x_2x_3+x_3^2\\&=x_1^2+2x_1(x_2+x_3)+(x_2+x_3)^2-(x_2+x_3)^2\end{aligned}$$

配方法化二次型为标准形

- 想法：$a^2+2ab=a^2+2ab+b^2-b^2=(a+b)^2-b^2$

$$
\begin{aligned}
a^2+2ab+2ac &= a^2+2a(b+c)\\
&= a^2+2a(b+c)+(b+c)^2-(b+c)^2\\
&= (a+b+c)^2-(b+c)^2
\end{aligned}
$$

例 配方法化二次型为标准形

$$
\begin{aligned}
f &= x_1^2+2x_1x_2+2x_1x_3+2x_2^2+4x_2x_3+x_3^2\\
&= x_1^2+2x_1(x_2+x_3)+(x_2+x_3)^2-(x_2+x_3)^2+2x_2^2+4x_2x_3+x_3^2\\
&=
\end{aligned}
$$

配方法化二次型为标准形

- 想法： $a^2+2ab=a^2+2ab+b^2-b^2=(a+b)^2-b^2$

$$
\begin{aligned}
a^2+2ab+2ac &= a^2+2a(b+c) \\
&= a^2+2a(b+c)+(b+c)^2-(b+c)^2 \\
&= (a+b+c)^2-(b+c)^2
\end{aligned}
$$

例 配方法化二次型为标准形

$$
\begin{aligned}
f &= x_1^2+2x_1x_2+2x_1x_3+2x_2^2+4x_2x_3+x_3^2 \\
&= x_1^2+2x_1(x_2+x_3)+(x_2+x_3)^2-(x_2+x_3)^2+2x_2^2+4x_2x_3+x_3^2 \\
&= (x_1+x_2+x_3)^2+
\end{aligned}
$$

配方法化二次型为标准形

- 想法：$$a^2+2ab=a^2+2ab+b^2-b^2=(a+b)^2-b^2$$

$$\begin{aligned} a^2+2ab+2ac &= a^2+2a(b+c) \\ &= a^2+2a(b+c)+(b+c)^2-(b+c)^2 \\ &= (a+b+c)^2-(b+c)^2 \end{aligned}$$

例 配方法化二次型为标准形

$$\begin{aligned} f &= x_1^2+2x_1x_2+2x_1x_3+2x_2^2+4x_2x_3+x_3^2 \\ &= x_1^2+2x_1(x_2+x_3)+(x_2+x_3)^2-(x_2+x_3)^2+2x_2^2+4x_2x_3+x_3^2 \\ &= (x_1+x_2+x_3)^2+x_2^2+2x_2x_3 \\ &= \end{aligned}$$

配方法化二次型为标准形

- 想法： $a^2+2ab=a^2+2ab+b^2-b^2=(a+b)^2-b^2$

$$
\begin{aligned}
a^2+2ab+2ac &= a^2+2a(b+c)\\
&= a^2+2a(b+c)+(b+c)^2-(b+c)^2\\
&= (a+b+c)^2-(b+c)^2
\end{aligned}
$$

例 配方法化二次型为标准形

$$
\begin{aligned}
f &= x_1^2+2x_1x_2+2x_1x_3+2x_2^2+4x_2x_3+x_3^2\\
&= x_1^2+2x_1(x_2+x_3)+(x_2+x_3)^2-(x_2+x_3)^2+2x_2^2+4x_2x_3+x_3^2\\
&= (x_1+x_2+x_3)^2+x_2^2+2x_2x_3\\
&= (x_1+x_2+x_3)^2+x_2^2+2x_2x_3+x_3^2-x_3^2\\
&=
\end{aligned}
$$

配方法化二次型为标准形

- 想法： $a^2+2ab=a^2+2ab+b^2-b^2=(a+b)^2-b^2$

$$
\begin{aligned}
a^2+2ab+2ac &= a^2+2a(b+c)\\
&= a^2+2a(b+c)+(b+c)^2-(b+c)^2\\
&= (a+b+c)^2-(b+c)^2
\end{aligned}
$$

例 配方法化二次型为标准形

$$
\begin{aligned}
f &= x_1^2+2x_1x_2+2x_1x_3+2x_2^2+4x_2x_3+x_3^2\\
&= x_1^2+2x_1(x_2+x_3)+(x_2+x_3)^2-(x_2+x_3)^2+2x_2^2+4x_2x_3+x_3^2\\
&= (x_1+x_2+x_3)^2+x_2^2+2x_2x_3\\
&= (x_1+x_2+x_3)^2+x_2^2+2x_2x_3+x_3^2-x_3^2\\
&= (x_1+x_2+x_3)^2+(x_2+x_3)^2-x_3^2
\end{aligned}
$$

例 配方法化二次型为标准形

$$\begin{aligned}f =& x_1^2 + 2x_1x_2 + 2x_1x_3 + 2x_2^2 + 4x_2x_3 + x_3^2 \\ =& x_1^2+2x_1(x_2+x_3)+(x_2+x_3)^2-(x_2+x_3)^2+2x_2^2+4x_2x_3+x_3^2 \\ =&(x_1 + x_2 + x_3)^2 + x_2^2 + 2x_2x_3 \\ =&(x_1 + x_2 + x_3)^2 + x_2^2 + 2x_2x_3 + x_3^2 - x_3^2 \\ =&(x_1 + x_2 + x_3)^2 + (x_2 + x_3)^2 - x_3^2\end{aligned}$$

例 配方法化二次型为标准形

$$
\begin{aligned}
f &= x_1^2 + 2x_1x_2 + 2x_1x_3 + 2x_2^2 + 4x_2x_3 + x_3^2 \\
&= x_1^2 + 2x_1(x_2+x_3) + (x_2+x_3)^2 - (x_2+x_3)^2 + 2x_2^2 + 4x_2x_3 + x_3^2 \\
&= (x_1 + x_2 + x_3)^2 + x_2^2 + 2x_2x_3 \\
&= (x_1 + x_2 + x_3)^2 + x_2^2 + 2x_2x_3 + x_3^2 - x_3^2 \\
&= (x_1 + x_2 + x_3)^2 + (x_2 + x_3)^2 - x_3^2
\end{aligned}
$$

作线性变量代换

$$
\begin{cases}
y_1 = x_1 + x_2 + x_3 \\
y_2 = \qquad x_2 + x_3 \\
y_3 = \qquad\qquad x_3
\end{cases}
$$

例 配方法化二次型为标准形

$$f = x_1^2 + 2x_1x_2 + 2x_1x_3 + 2x_2^2 + 4x_2x_3 + x_3^2$$

$$= x_1^2 + 2x_1(x_2+x_3) + (x_2+x_3)^2 - (x_2+x_3)^2 + 2x_2^2 + 4x_2x_3 + x_3^2$$

$$= (x_1 + x_2 + x_3)^2 + x_2^2 + 2x_2x_3$$

$$= (x_1 + x_2 + x_3)^2 + x_2^2 + 2x_2x_3 + x_3^2 - x_3^2$$

$$= (x_1 + x_2 + x_3)^2 + (x_2 + x_3)^2 - x_3^2$$

作线性变量代换

$$\begin{cases} y_1 = x_1 + x_2 + x_3 \\ y_2 = \phantom{x_1 +{}} x_2 + x_3 \\ y_3 = \phantom{x_1 + x_2 +{}} x_3 \end{cases}$$

则

$$f = y_1^2 + y_2^2 - y_3^2$$

例 配方法化二次型为标准形

$$
\begin{aligned}
f =& x_1^2 + 2x_1x_2 + 2x_1x_3 + 2x_2^2 + 4x_2x_3 + x_3^2 \\
=& x_1^2+2x_1(x_2+x_3)+(x_2+x_3)^2-(x_2+x_3)^2+2x_2^2+4x_2x_3+x_3^2 \\
=& (x_1 + x_2 + x_3)^2 + x_2^2 + 2x_2x_3 \\
=& (x_1 + x_2 + x_3)^2 + x_2^2 + 2x_2x_3 + x_3^2 - x_3^2 \\
=& (x_1 + x_2 + x_3)^2 + (x_2 + x_3)^2 - x_3^2
\end{aligned}
$$

作线性变量代换

$$
\begin{cases} y_1 = x_1 + x_2 + x_3 \\ y_2 = \qquad x_2 + x_3 \\ y_3 = \qquad\qquad x_3 \end{cases} \Rightarrow \begin{cases} x_1 = \\ x_2 = \\ x_3 = \end{cases}
$$

则

$$f = y_1^2 + y_2^2 - y_3^2$$

例 配方法化二次型为标准形

$$
\begin{aligned}
f =& x_1^2+2x_1x_2+2x_1x_3+2x_2^2+4x_2x_3+x_3^2\\
=& x_1^2+2x_1(x_2+x_3)+(x_2+x_3)^2-(x_2+x_3)^2+2x_2^2+4x_2x_3+x_3^2\\
=& (x_1+x_2+x_3)^2+x_2^2+2x_2x_3\\
=& (x_1+x_2+x_3)^2+x_2^2+2x_2x_3+x_3^2-x_3^2\\
=& (x_1+x_2+x_3)^2+(x_2+x_3)^2-x_3^2
\end{aligned}
$$

作线性变量代换

$$
\begin{cases}
y_1 = x_1+x_2+x_3\\
y_2 = \qquad x_2+x_3\\
y_3 = \qquad\qquad x_3
\end{cases}
\Rightarrow
\begin{cases}
x_1 = \\
x_2 = \\
x_3 = \qquad\qquad y_3
\end{cases}
$$

则

$$f = y_1^2+y_2^2-y_3^2$$

例 配方法化二次型为标准形

$$
\begin{aligned}
f = & x_1^2+2x_1x_2+2x_1x_3+2x_2^2+4x_2x_3+x_3^2 \\
= & x_1^2+2x_1(x_2+x_3)+(x_2+x_3)^2-(x_2+x_3)^2+2x_2^2+4x_2x_3+x_3^2 \\
= & (x_1+x_2+x_3)^2+x_2^2+2x_2x_3 \\
= & (x_1+x_2+x_3)^2+x_2^2+2x_2x_3+x_3^2-x_3^2 \\
= & (x_1+x_2+x_3)^2+(x_2+x_3)^2-x_3^2
\end{aligned}
$$

作线性变量代换

$$
\begin{cases} y_1 = x_1+x_2+x_3 \\ y_2 = \qquad x_2+x_3 \\ y_3 = \qquad\qquad x_3 \end{cases} \Rightarrow \begin{cases} x_1 = \\ x_2 = \qquad y_2-y_3 \\ x_3 = \qquad\qquad y_3 \end{cases}
$$

则

$$
f = y_1^2+y_2^2-y_3^2
$$

例 配方法化二次型为标准形

$$
\begin{aligned}
f =& x_1^2 + 2x_1x_2 + 2x_1x_3 + 2x_2^2 + 4x_2x_3 + x_3^2 \\
=& x_1^2+2x_1(x_2+x_3)+(x_2+x_3)^2-(x_2+x_3)^2+2x_2^2+4x_2x_3+x_3^2 \\
=& (x_1 + x_2 + x_3)^2 + x_2^2 + 2x_2x_3 \\
=& (x_1 + x_2 + x_3)^2 + x_2^2 + 2x_2x_3 + x_3^2 - x_3^2 \\
=& (x_1 + x_2 + x_3)^2 + (x_2 + x_3)^2 - x_3^2
\end{aligned}
$$

作线性变量代换

$$
\begin{cases} y_1 = x_1 + x_2 + x_3 \\ y_2 = x_2 + x_3 \\ y_3 = x_3 \end{cases} \Rightarrow \begin{cases} x_1 = y_1 - y_2 \\ x_2 = y_2 - y_3 \\ x_3 = y_3 \end{cases}
$$

则

$$
f = y_1^2 + y_2^2 - y_3^2
$$

例 配方法化二次型为标准形

$$
\begin{aligned}
f =& x_1^2 + 2x_1x_2 + 2x_1x_3 + 2x_2^2 + 4x_2x_3 + x_3^2 \\
=& x_1^2 + 2x_1(x_2+x_3) + (x_2+x_3)^2 - (x_2+x_3)^2 + 2x_2^2 + 4x_2x_3 + x_3^2 \\
=& (x_1 + x_2 + x_3)^2 + x_2^2 + 2x_2x_3 \\
=& (x_1 + x_2 + x_3)^2 + x_2^2 + 2x_2x_3 + x_3^2 - x_3^2 \\
=& (x_1 + x_2 + x_3)^2 + (x_2 + x_3)^2 - x_3^2
\end{aligned}
$$

作线性变量代换

$$
\begin{cases} y_1 = x_1 + x_2 + x_3 \\ y_2 = \quad\quad x_2 + x_3 \\ y_3 = \quad\quad\quad\quad x_3 \end{cases} \Rightarrow \begin{cases} x_1 = y_1 - y_2 \\ x_2 = \quad\quad y_2 - y_3 \\ x_3 = \quad\quad\quad\quad y_3 \end{cases} \Rightarrow x = \left(\quad\quad\quad\quad \right) y
$$

则

$$
f = y_1^2 + y_2^2 - y_3^2
$$

例 配方法化二次型为标准形

$$
\begin{aligned}
f =& x_1^2 + 2x_1x_2 + 2x_1x_3 + 2x_2^2 + 4x_2x_3 + x_3^2 \\
=& x_1^2 + 2x_1(x_2+x_3) + (x_2+x_3)^2 - (x_2+x_3)^2 + 2x_2^2 + 4x_2x_3 + x_3^2 \\
=& (x_1 + x_2 + x_3)^2 + x_2^2 + 2x_2x_3 \\
=& (x_1 + x_2 + x_3)^2 + x_2^2 + 2x_2x_3 + x_3^2 - x_3^2 \\
=& (x_1 + x_2 + x_3)^2 + (x_2 + x_3)^2 - x_3^2
\end{aligned}
$$

作线性变量代换

$$
\begin{cases} y_1 = x_1 + x_2 + x_3 \\ y_2 = \qquad x_2 + x_3 \\ y_3 = \qquad\qquad x_3 \end{cases} \Rightarrow \begin{cases} x_1 = y_1 - y_2 \\ x_2 = \qquad y_2 - y_3 \\ x_3 = \qquad\qquad y_3 \end{cases} \Rightarrow x = \begin{pmatrix} 1 & -1 & 0 \\ & & \\ & & \end{pmatrix} y
$$

则

$$
f = y_1^2 + y_2^2 - y_3^2
$$

例 配方法化二次型为标准形

$$
\begin{aligned}
f =& x_1^2 + 2x_1x_2 + 2x_1x_3 + 2x_2^2 + 4x_2x_3 + x_3^2 \\
=& x_1^2 + 2x_1(x_2+x_3) + (x_2+x_3)^2 - (x_2+x_3)^2 + 2x_2^2 + 4x_2x_3 + x_3^2 \\
=& (x_1 + x_2 + x_3)^2 + x_2^2 + 2x_2x_3 \\
=& (x_1 + x_2 + x_3)^2 + x_2^2 + 2x_2x_3 + x_3^2 - x_3^2 \\
=& (x_1 + x_2 + x_3)^2 + (x_2 + x_3)^2 - x_3^2
\end{aligned}
$$

作线性变量代换

$$
\begin{cases} y_1 = x_1 + x_2 + x_3 \\ y_2 = \qquad x_2 + x_3 \\ y_3 = \qquad\qquad x_3 \end{cases} \Rightarrow \begin{cases} x_1 = y_1 - y_2 \\ x_2 = \qquad y_2 - y_3 \\ x_3 = \qquad\qquad y_3 \end{cases} \Rightarrow x = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} y
$$

则

$$
f = y_1^2 + y_2^2 - y_3^2
$$

例 配方法化二次型为标准形

$$
\begin{aligned}
f =& x_1^2 + 2x_1x_2 + 2x_1x_3 + 2x_2^2 + 4x_2x_3 + x_3^2 \\
=& x_1^2+2x_1(x_2+x_3)+(x_2+x_3)^2-(x_2+x_3)^2+2x_2^2+4x_2x_3+x_3^2 \\
=& (x_1 + x_2 + x_3)^2 + x_2^2 + 2x_2x_3 \\
=& (x_1 + x_2 + x_3)^2 + x_2^2 + 2x_2x_3 + x_3^2 - x_3^2 \\
=& (x_1 + x_2 + x_3)^2 + (x_2 + x_3)^2 - x_3^2
\end{aligned}
$$

作线性变量代换

$$
\begin{cases} y_1 = x_1 + x_2 + x_3 \\ y_2 = \quad\ \ x_2 + x_3 \\ y_3 = \qquad\quad\ x_3 \end{cases} \Rightarrow \begin{cases} x_1 = y_1 - y_2 \\ x_2 = \quad\ \ y_2 - y_3 \\ x_3 = \qquad\quad\ y_3 \end{cases} \Rightarrow x = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} y
$$

则

$$
f = y_1^2 + y_2^2 - y_3^2
$$

例 配方法化二次型为标准形

$$
\begin{aligned}
f =& x_1^2 + 2x_1x_2 + 2x_1x_3 + 2x_2^2 + 4x_2x_3 + x_3^2 \\
=& x_1^2+2x_1(x_2+x_3)+(x_2+x_3)^2-(x_2+x_3)^2+2x_2^2+4x_2x_3+x_3^2 \\
=& (x_1 + x_2 + x_3)^2 + x_2^2 + 2x_2x_3 \\
=& (x_1 + x_2 + x_3)^2 + x_2^2 + 2x_2x_3 + x_3^2 - x_3^2 \\
=& (x_1 + x_2 + x_3)^2 + (x_2 + x_3)^2 - x_3^2
\end{aligned}
$$

作线性变量代换

$$
\begin{cases} y_1 = x_1 + x_2 + x_3 \\ y_2 = \quad\ \ x_2 + x_3 \\ y_3 = \qquad\quad\ \ x_3 \end{cases} \Rightarrow \begin{cases} x_1 = y_1 - y_2 \\ x_2 = \qquad y_2 - y_3 \\ x_3 = \qquad\qquad y_3 \end{cases} \Rightarrow x = \underbrace{\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}}_{C\text{：可逆}} y
$$

则

$$
f = y_1^2 + y_2^2 - y_3^2
$$

例 配方法化二次型为标准形

$$
\begin{aligned}
f &= x_1^2 + 2x_1x_2 + 2x_1x_3 + 2x_2^2 + 4x_2x_3 + x_3^2 \\
&= x_1^2 + 2x_1(x_2+x_3) + (x_2+x_3)^2 - (x_2+x_3)^2 + 2x_2^2 + 4x_2x_3 + x_3^2 \\
&= (x_1+x_2+x_3)^2 + x_2^2 + 2x_2x_3 \\
&= (x_1+x_2+x_3)^2 + x_2^2 + 2x_2x_3 + x_3^2 - x_3^2 \\
&= (x_1+x_2+x_3)^2 + (x_2+x_3)^2 - x_3^2
\end{aligned}
$$

作线性变量代换

$$
\begin{cases} y_1 = x_1 + x_2 + x_3 \\ y_2 = \quad\quad x_2 + x_3 \\ y_3 = \quad\quad\quad\quad x_3 \end{cases}
\Rightarrow
\begin{cases} x_1 = y_1 - y_2 \\ x_2 = \quad\quad y_2 - y_3 \\ x_3 = \quad\quad\quad\quad y_3 \end{cases}
\Rightarrow x = \underbrace{\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}}_{C\text{：可逆}} y
$$

则

$$
f = y_1^2 + y_2^2 - y_3^2
$$

注 正惯性指标 = ；负惯性指标 =

例 配方法化二次型为标准形

$$\begin{aligned}
f =&x_1^2 + 2x_1x_2 + 2x_1x_3 + 2x_2^2 + 4x_2x_3 + x_3^2 \\
=&x_1^2+2x_1(x_2+x_3)+(x_2+x_3)^2-(x_2+x_3)^2+2x_2^2+4x_2x_3+x_3^2 \\
=&(x_1+x_2+x_3)^2 + x_2^2 + 2x_2x_3 \\
=&(x_1+x_2+x_3)^2 + x_2^2 + 2x_2x_3 + x_3^2 - x_3^2 \\
=&(x_1+x_2+x_3)^2 + (x_2+x_3)^2 - x_3^2
\end{aligned}$$

作线性变量代换

$$\begin{cases} y_1 = x_1 + x_2 + x_3 \\ y_2 = \qquad\; x_2 + x_3 \\ y_3 = \qquad\qquad\;\; x_3 \end{cases} \Rightarrow \begin{cases} x_1 = y_1 - y_2 \\ x_2 = \qquad\; y_2 - y_3 \\ x_3 = \qquad\qquad\;\; y_3 \end{cases} \Rightarrow x = \underbrace{\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}}_{C\text{：可逆}} y$$

则

$$f = y_1^2 + y_2^2 - y_3^2$$

注 正惯性指标 $= 2$；负惯性指标 $=$

例 配方法化二次型为标准形

$$f = x_1^2 + 2x_1x_2 + 2x_1x_3 + 2x_2^2 + 4x_2x_3 + x_3^2$$
$$= x_1^2 + 2x_1(x_2+x_3) + (x_2+x_3)^2 - (x_2+x_3)^2 + 2x_2^2 + 4x_2x_3 + x_3^2$$
$$= (x_1 + x_2 + x_3)^2 + x_2^2 + 2x_2x_3$$
$$= (x_1 + x_2 + x_3)^2 + x_2^2 + 2x_2x_3 + x_3^2 - x_3^2$$
$$= (x_1 + x_2 + x_3)^2 + (x_2 + x_3)^2 - x_3^2$$

作线性变量代换

$$\begin{cases} y_1 = x_1 + x_2 + x_3 \\ y_2 = \quad\quad x_2 + x_3 \\ y_3 = \quad\quad\quad\quad x_3 \end{cases} \Rightarrow \begin{cases} x_1 = y_1 - y_2 \\ x_2 = \quad\quad y_2 - y_3 \\ x_3 = \quad\quad\quad\quad y_3 \end{cases} \Rightarrow x = \underbrace{\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}}_{C\text{：可逆}} y$$

则
$$f = y_1^2 + y_2^2 - y_3^2$$

注 正惯性指标 $= 2$；负惯性指标 $= 1$

小结

$$f = x_1^2 + 2x_1x_2 + 2x_1x_3 + 2x_2^2 + 4x_2x_3 + x_3^2$$

配方法

$$= (x_1 + x_2 + x_3)^2 + (x_2 + x_3)^2 - x_3^2$$

小结

$$f = x_1^2 + 2x_1x_2 + 2x_1x_3 + 2x_2^2 + 4x_2x_3 + x_3^2$$

配方法

$$= (x_1 + x_2 + x_3)^2 + (x_2 + x_3)^2 - x_3^2$$

$$= y_1^2 + y_2^2 - y_3^2$$

小结

$$f = x_1^2 + 2x_1x_2 + 2x_1x_3 + 2x_2^2 + 4x_2x_3 + x_3^2$$

配方法

$$= (x_1 + x_2 + x_3)^2 + (x_2 + x_3)^2 - x_3^2$$

变量代换 $y = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} x$

$$= y_1^2 + y_2^2 - y_3^2$$

小结

$$f = x_1^2 + 2x_1x_2 + 2x_1x_3 + 2x_2^2 + 4x_2x_3 + x_3^2$$

配方法

$$= (x_1 + x_2 + x_3)^2 + (x_2 + x_3)^2 - x_3^2$$

变量代换 $y = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} x \Rightarrow x = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} y$

$$= y_1^2 + y_2^2 - y_3^2$$

小结

$$f = x_1^2 + 2x_1x_2 + 2x_1x_3 + 2x_2^2 + 4x_2x_3 + x_3^2$$

配方法

$$= (x_1 + x_2 + x_3)^2 + (x_2 + x_3)^2 - x_3^2$$

变量代换 $y = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} x \Rightarrow x = \underbrace{\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}}_{C} y$

$$= y_1^2 + y_2^2 - y_3^2$$

小结

$$f = x_1^2 + 2x_1x_2 + 2x_1x_3 + 2x_2^2 + 4x_2x_3 + x_3^2$$

配方法

$$= (x_1 + x_2 + x_3)^2 + (x_2 + x_3)^2 - x_3^2$$

变量代换 $y = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} x \Rightarrow x = \underbrace{\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}}_{C} y$

$$= y_1^2 + y_2^2 - y_3^2$$

小结

$$f = x_1^2 + 2x_1x_2 + 2x_1x_3 + 2x_2^2 + 4x_2x_3 + x_3^2 \quad = x^TAx$$

配方法

$$= (x_1 + x_2 + x_3)^2 + (x_2 + x_3)^2 - x_3^2$$

变量代换 $y = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} x \Rightarrow x = \underbrace{\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}}_{C} y$

$$= y_1^2 + y_2^2 - y_3^2$$

小结

$$f = x_1^2 + 2x_1x_2 + 2x_1x_3 + 2x_2^2 + 4x_2x_3 + x_3^2 \qquad = x^TAx$$

配方法

$$= (x_1 + x_2 + x_3)^2 + (x_2 + x_3)^2 - x_3^2$$

变量代换 $y = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} x \Rightarrow x = \underbrace{\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}}_{C} y$

$$= y_1^2 + y_2^2 - y_3^2 \qquad = y^TC^TACy$$

小结

$$f = x_1^2 + 2x_1x_2 + 2x_1x_3 + 2x_2^2 + 4x_2x_3 + x_3^2 \qquad = x^T A x$$

配方法

$$= (x_1 + x_2 + x_3)^2 + (x_2 + x_3)^2 - x_3^2$$

变量代换 $y = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} x \Rightarrow x = \underbrace{\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}}_{C} y$

$$= y_1^2 + y_2^2 - y_3^2 \qquad = y^T \boxed{C^T A C} y$$

小结

$$f = x_1^2 + 2x_1x_2 + 2x_1x_3 + 2x_2^2 + 4x_2x_3 + x_3^2 \quad = x^TAx$$

配方法

$$= (x_1 + x_2 + x_3)^2 + (x_2 + x_3)^2 - x_3^2$$

变量代换 $y = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} x \Rightarrow x = \underbrace{\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}}_{C} y$

$$= y_1^2 + y_2^2 - y_3^2 \quad = y^T \boxed{C^TAC} y$$

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

小结

$$f = x_1^2 + 2x_1x_2 + 2x_1x_3 + 2x_2^2 + 4x_2x_3 + x_3^2 \quad = x^TAx$$

配方法

$$= (x_1 + x_2 + x_3)^2 + (x_2 + x_3)^2 - x_3^2$$

变量代换 $y = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} x \Rightarrow x = \underbrace{\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}}_{C} y$

$$= y_1^2 + y_2^2 - y_3^2 \quad = y^T \boxed{C^TAC} y$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}}_{C^T} \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}}_{C} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

小结

$$f = x_1^2 + 2x_1x_2 + 2x_1x_3 + 2x_2^2 + 4x_2x_3 + x_3^2 \quad = x^TAx$$

配方法

$$= (x_1 + x_2 + x_3)^2 + (x_2 + x_3)^2 - x_3^2$$

变量代换 $y = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} x \Rightarrow x = \underbrace{\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}}_{C} y$

$$= y_1^2 + y_2^2 - y_3^2 \quad = y^T \boxed{C^TAC} y$$

特别地，找到了可逆阵 C，使得

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}}_{C^T} \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}}_{C} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

例 配方法化二次型为标准形

$$f = x_1^2 + 4x_1x_2 + 4x_1x_3 + 2x_2^2 + 8x_2x_3 + 4x_3^2$$

$$=$$

例 配方法化二次型为标准形

$$f = x_1^2 + 4x_1x_2 + 4x_1x_3 + 2x_2^2 + 8x_2x_3 + 4x_3^2$$

$$= x_1^2 + 2x_1 \cdot (2x_2 + 2x_3)$$

例 配方法化二次型为标准形

$$\begin{aligned} f =& x_1^2 + 4x_1x_2 + 4x_1x_3 + 2x_2^2 + 8x_2x_3 + 4x_3^2 \\ =& x_1^2 + 2x_1 \cdot (2x_2 + 2x_3) + (2x_2 + 2x_3)^2 - (2x_2 + 2x_3)^2 \end{aligned}$$

例 配方法化二次型为标准形

$$f = x_1^2 + 4x_1x_2 + 4x_1x_3 + 2x_2^2 + 8x_2x_3 + 4x_3^2$$

$$= x_1^2 + 2x_1 \cdot (2x_2 + 2x_3) + (2x_2 + 2x_3)^2 - (2x_2 + 2x_3)^2$$

$$+ 2x_2^2 + 8x_2x_3 + 4x_3^2$$

例 配方法化二次型为标准形

$$
\begin{aligned}
f =& x_1^2 + 4x_1x_2 + 4x_1x_3 + 2x_2^2 + 8x_2x_3 + 4x_3^2 \\
=& x_1^2 + 2x_1 \cdot (2x_2 + 2x_3) + (2x_2 + 2x_3)^2 - (2x_2 + 2x_3)^2 \\
& + 2x_2^2 + 8x_2x_3 + 4x_3^2 \\
=& (x_1 + 2x_2 + 2x_3)^2
\end{aligned}
$$

例 配方法化二次型为标准形

$$\begin{aligned}
f =& x_1^2 + 4x_1x_2 + 4x_1x_3 + 2x_2^2 + 8x_2x_3 + 4x_3^2 \\
=& x_1^2 + 2x_1\cdot(2x_2+2x_3) + (2x_2+2x_3)^2 - (2x_2+2x_3)^2 \\
& + 2x_2^2 + 8x_2x_3 + 4x_3^2 \\
=& (x_1 + 2x_2 + 2x_3)^2 - 2x_2^2
\end{aligned}$$

例 配方法化二次型为标准形

$$
\begin{aligned}
f =& x_1^2 + 4x_1x_2 + 4x_1x_3 + 2x_2^2 + 8x_2x_3 + 4x_3^2 \\
=& x_1^2 + 2x_1 \cdot (2x_2 + 2x_3) + (2x_2 + 2x_3)^2 - (2x_2 + 2x_3)^2 \\
& + 2x_2^2 + 8x_2x_3 + 4x_3^2 \\
=& (x_1 + 2x_2 + 2x_3)^2 - 2x_2^2
\end{aligned}
$$

作线性变量代换

$$
\begin{cases}
y_1 = x_1 + 2x_2 + 2x_3 \\
y_2 = \qquad\quad x_2 \\
\end{cases}
$$

例 配方法化二次型为标准形

$$
\begin{aligned}
f &= x_1^2 + 4x_1x_2 + 4x_1x_3 + 2x_2^2 + 8x_2x_3 + 4x_3^2 \\
&= x_1^2 + 2x_1 \cdot (2x_2 + 2x_3) + (2x_2 + 2x_3)^2 - (2x_2 + 2x_3)^2 \\
&\quad + 2x_2^2 + 8x_2x_3 + 4x_3^2 \\
&= (x_1 + 2x_2 + 2x_3)^2 - 2x_2^2
\end{aligned}
$$

作线性变量代换

$$
\begin{cases}
y_1 = x_1 + 2x_2 + 2x_3 \\
y_2 = \qquad\quad x_2 \\
\end{cases}
$$

则

$$f = y_1^2 - 2y_2^2$$

例 配方法化二次型为标准形

$$f = x_1^2 + 4x_1x_2 + 4x_1x_3 + 2x_2^2 + 8x_2x_3 + 4x_3^2$$

$$= x_1^2 + 2x_1 \cdot (2x_2 + 2x_3) + (2x_2 + 2x_3)^2 - (2x_2 + 2x_3)^2$$

$$+ 2x_2^2 + 8x_2x_3 + 4x_3^2$$

$$= (x_1 + 2x_2 + 2x_3)^2 - 2x_2^2$$

作线性变量代换

$$\begin{cases} y_1 = x_1 + 2x_2 + 2x_3 \\ y_2 = x_2 \\ y_3 = x_3 \end{cases}$$

则

$$f = y_1^2 - 2y_2^2$$

例 配方法化二次型为标准形

$$f = x_1^2 + 4x_1x_2 + 4x_1x_3 + 2x_2^2 + 8x_2x_3 + 4x_3^2$$
$$= x_1^2 + 2x_1 \cdot (2x_2 + 2x_3) + (2x_2 + 2x_3)^2 - (2x_2 + 2x_3)^2$$
$$+ 2x_2^2 + 8x_2x_3 + 4x_3^2$$
$$= (x_1 + 2x_2 + 2x_3)^2 - 2x_2^2$$

作线性变量代换

$$\begin{cases} y_1 = x_1 + 2x_2 + 2x_3 \\ y_2 = \quad\quad\quad x_2 \\ y_3 = \quad\quad\quad\quad\quad x_3 \end{cases} \Rightarrow \begin{cases} \\ \\ \end{cases}$$

则

$$f = y_1^2 - 2y_2^2$$

例 配方法化二次型为标准形

$$f = x_1^2 + 4x_1x_2 + 4x_1x_3 + 2x_2^2 + 8x_2x_3 + 4x_3^2$$

$$= x_1^2 + 2x_1 \cdot (2x_2 + 2x_3) + (2x_2 + 2x_3)^2 - (2x_2 + 2x_3)^2$$

$$+ 2x_2^2 + 8x_2x_3 + 4x_3^2$$

$$= (x_1 + 2x_2 + 2x_3)^2 - 2x_2^2$$

作线性变量代换

$$\begin{cases} y_1 = x_1 + 2x_2 + 2x_3 \\ y_2 = \qquad\quad x_2 \\ y_3 = \qquad\qquad\quad x_3 \end{cases} \Rightarrow \begin{cases} \\ \\ x_3 = \qquad\qquad\qquad y_3 \end{cases}$$

则

$$f = y_1^2 - 2y_2^2$$

例 配方法化二次型为标准形

$$
\begin{aligned}
f =& x_1^2 + 4x_1x_2 + 4x_1x_3 + 2x_2^2 + 8x_2x_3 + 4x_3^2 \\
=& x_1^2 + 2x_1 \cdot (2x_2 + 2x_3) + (2x_2 + 2x_3)^2 - (2x_2 + 2x_3)^2 \\
& + 2x_2^2 + 8x_2x_3 + 4x_3^2 \\
=& (x_1 + 2x_2 + 2x_3)^2 - 2x_2^2
\end{aligned}
$$

作线性变量代换

$$
\begin{cases} y_1 = x_1 + 2x_2 + 2x_3 \\ y_2 = \quad x_2 \\ y_3 = \quad x_3 \end{cases} \Rightarrow \begin{cases} x_2 = \quad y_2 \\ x_3 = \quad y_3 \end{cases}
$$

则

$$
f = y_1^2 - 2y_2^2
$$

例 配方法化二次型为标准形

$$
\begin{aligned}
f =& x_1^2 + 4x_1x_2 + 4x_1x_3 + 2x_2^2 + 8x_2x_3 + 4x_3^2 \\
=& x_1^2 + 2x_1 \cdot (2x_2 + 2x_3) + (2x_2 + 2x_3)^2 - (2x_2 + 2x_3)^2 \\
& + 2x_2^2 + 8x_2x_3 + 4x_3^2 \\
=& (x_1 + 2x_2 + 2x_3)^2 - 2x_2^2
\end{aligned}
$$

作线性变量代换

$$
\begin{cases} y_1 = x_1 + 2x_2 + 2x_3 \\ y_2 = \quad x_2 \\ y_3 = \quad x_3 \end{cases} \Rightarrow \begin{cases} x_1 = y_1 - 2y_2 - 2y_3 \\ x_2 = \quad y_2 \\ x_3 = \quad y_3 \end{cases}
$$

则

$$
f = y_1^2 - 2y_2^2
$$

例 配方法化二次型为标准形

$$\begin{aligned} f &= x_1^2 + 4x_1x_2 + 4x_1x_3 + 2x_2^2 + 8x_2x_3 + 4x_3^2 \\ &= x_1^2 + 2x_1 \cdot (2x_2 + 2x_3) + (2x_2 + 2x_3)^2 - (2x_2 + 2x_3)^2 \\ &\quad + 2x_2^2 + 8x_2x_3 + 4x_3^2 \\ &= (x_1 + 2x_2 + 2x_3)^2 - 2x_2^2 \end{aligned}$$

作线性变量代换

$$\begin{cases} y_1 = x_1 + 2x_2 + 2x_3 \\ y_2 = x_2 \\ y_3 = x_3 \end{cases} \Rightarrow \begin{cases} x_1 = y_1 - 2y_2 - 2y_3 \\ x_2 = y_2 \\ x_3 = y_3 \end{cases} \Rightarrow x = \underbrace{\begin{pmatrix} 1 & -2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{C\text{：可逆}} y$$

则

$$f = y_1^2 - 2y_2^2$$

例 配方法化二次型为标准形

$$
\begin{aligned}
f =& x_1^2+4x_1x_2+4x_1x_3+2x_2^2+8x_2x_3+4x_3^2 \\
=& x_1^2+2x_1\cdot(2x_2+2x_3)+(2x_2+2x_3)^2-(2x_2+2x_3)^2 \\
& +2x_2^2+8x_2x_3+4x_3^2 \\
=& (x_1+2x_2+2x_3)^2-2x_2^2
\end{aligned}
$$

作线性变量代换

$$
\begin{cases} y_1 = x_1+2x_2+2x_3 \\ y_2 = \qquad\quad x_2 \\ y_3 = \qquad\qquad\quad x_3 \end{cases} \Rightarrow \begin{cases} x_1 = y_1-2y_2-2y_3 \\ x_2 = \qquad\quad y_2 \\ x_3 = \qquad\qquad\quad y_3 \end{cases} \Rightarrow x = \underbrace{\begin{pmatrix} 1 & -2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{C\text{：可逆}} y
$$

则

$$
f = y_1^2-2y_2^2
$$

注 正惯性指标 = ；负惯性指标 =

例 配方法化二次型为标准形

$$\begin{aligned} f =& x_1^2 + 4x_1x_2 + 4x_1x_3 + 2x_2^2 + 8x_2x_3 + 4x_3^2 \\ =& x_1^2+2x_1\cdot(2x_2+2x_3)+(2x_2+2x_3)^2-(2x_2+2x_3)^2 \\ & +2x_2^2+8x_2x_3+4x_3^2 \\ =& (x_1+2x_2+2x_3)^2-2x_2^2 \end{aligned}$$

作线性变量代换

$$\begin{cases} y_1 = x_1+2x_2+2x_3 \\ y_2 = \qquad\quad x_2 \\ y_3 = \qquad\qquad\quad x_3 \end{cases} \Rightarrow \begin{cases} x_1 = y_1-2y_2-2y_3 \\ x_2 = \qquad\quad y_2 \\ x_3 = \qquad\qquad\quad y_3 \end{cases} \Rightarrow x = \underbrace{\begin{pmatrix} 1 & -2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{C\text{：可逆}} y$$

则

$$f = y_1^2 - 2y_2^2$$

注 正惯性指标 $=1$；负惯性指标 $=$

例 配方法化二次型为标准形

$$
\begin{aligned}
f =& x_1^2 + 4x_1x_2 + 4x_1x_3 + 2x_2^2 + 8x_2x_3 + 4x_3^2 \\
=& x_1^2 + 2x_1\cdot(2x_2+2x_3) + (2x_2+2x_3)^2 - (2x_2+2x_3)^2 \\
& + 2x_2^2 + 8x_2x_3 + 4x_3^2 \\
=& (x_1 + 2x_2 + 2x_3)^2 - 2x_2^2
\end{aligned}
$$

作线性变量代换

$$
\begin{cases} y_1 = x_1 + 2x_2 + 2x_3 \\ y_2 = \quad\quad x_2 \\ y_3 = \quad\quad\quad\quad x_3 \end{cases} \Rightarrow \begin{cases} x_1 = y_1 - 2y_2 - 2y_3 \\ x_2 = \quad\quad y_2 \\ x_3 = \quad\quad\quad\quad y_3 \end{cases} \Rightarrow x = \underbrace{\begin{pmatrix} 1 & -2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{C\text{：可逆}} y
$$

则

$$
f = y_1^2 - 2y_2^2
$$

注 正惯性指标 = 1；负惯性指标 = 1

小结

$$f = x_1^2 + 4x_1x_2 + 4x_1x_3 + 2x_2^2 + 8x_2x_3 + 4x_3^2$$

配方法

$$= (x_1 + 2x_2 + 2x_3)^2 - 2x_2^2$$

小结

$$f = x_1^2 + 4x_1x_2 + 4x_1x_3 + 2x_2^2 + 8x_2x_3 + 4x_3^2$$

配方法

$$= (x_1 + 2x_2 + 2x_3)^2 - 2x_2^2$$

$$= y_1^2 - 2y_2^2$$

小结

$$f = x_1^2 + 4x_1x_2 + 4x_1x_3 + 2x_2^2 + 8x_2x_3 + 4x_3^2$$

配方法

$$= (x_1 + 2x_2 + 2x_3)^2 - 2x_2^2$$

变量代换 $y = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} x$

$$= y_1^2 - 2y_2^2$$

小结

$$f = x_1^2 + 4x_1x_2 + 4x_1x_3 + 2x_2^2 + 8x_2x_3 + 4x_3^2$$

配方法

$$= (x_1 + 2x_2 + 2x_3)^2 - 2x_2^2$$

变量代换 $y = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} x \Rightarrow x = \begin{pmatrix} 1 & -2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} y$

$$= y_1^2 - 2y_2^2$$

小结

$$f = x_1^2 + 4x_1x_2 + 4x_1x_3 + 2x_2^2 + 8x_2x_3 + 4x_3^2$$

配方法

$$= (x_1 + 2x_2 + 2x_3)^2 - 2x_2^2$$

变量代换 $y = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} x \Rightarrow x = \underbrace{\begin{pmatrix} 1 & -2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{C} y$

$$= y_1^2 - 2y_2^2$$

小结

$$f = x_1^2 + 4x_1x_2 + 4x_1x_3 + 2x_2^2 + 8x_2x_3 + 4x_3^2$$

配方法

$$= (x_1 + 2x_2 + 2x_3)^2 - 2x_2^2$$

变量代换 $y = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} x \Rightarrow x = \underbrace{\begin{pmatrix} 1 & -2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{C} y$

$$= y_1^2 - 2y_2^2$$

小结

$$f = x_1^2 + 4x_1x_2 + 4x_1x_3 + 2x_2^2 + 8x_2x_3 + 4x_3^2 = x^TAx$$

配方法

$$= (x_1 + 2x_2 + 2x_3)^2 - 2x_2^2$$

变量代换 $y = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} x \Rightarrow x = \underbrace{\begin{pmatrix} 1 & -2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{C} y$

$$= y_1^2 - 2y_2^2$$

小结

$$f = x_1^2 + 4x_1x_2 + 4x_1x_3 + 2x_2^2 + 8x_2x_3 + 4x_3^2 \quad = x^TAx$$

配方法

$$= (x_1 + 2x_2 + 2x_3)^2 - 2x_2^2$$

变量代换 $y = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} x \Rightarrow x = \underbrace{\begin{pmatrix} 1 & -2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{C} y$

$$= y_1^2 - 2y_2^2 \quad = y^TC^TACy$$

小结

$$f = x_1^2 + 4x_1x_2 + 4x_1x_3 + 2x_2^2 + 8x_2x_3 + 4x_3^2 \quad = x^TAx$$

配方法

$$= (x_1 + 2x_2 + 2x_3)^2 - 2x_2^2$$

变量代换 $y = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} x \Rightarrow x = \underbrace{\begin{pmatrix} 1 & -2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{C} y$

$$= y_1^2 - 2y_2^2 \quad = y^T\boxed{C^TAC}y$$

小结

$$f = x_1^2 + 4x_1x_2 + 4x_1x_3 + 2x_2^2 + 8x_2x_3 + 4x_3^2 \qquad = x^TAx$$

配方法

$$= (x_1 + 2x_2 + 2x_3)^2 - 2x_2^2$$

变量代换 $y = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} x \Rightarrow x = \underbrace{\begin{pmatrix} 1 & -2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{C} y$

$$= y_1^2 - 2y_2^2 \qquad = y^T\boxed{C^TAC}y$$

$$\begin{pmatrix} 1 & & \\ & -2 & \\ & & 0 \end{pmatrix}$$

小结

$$f = x_1^2 + 4x_1x_2 + 4x_1x_3 + 2x_2^2 + 8x_2x_3 + 4x_3^2 = x^T A x$$

配方法

$$= (x_1 + 2x_2 + 2x_3)^2 - 2x_2^2$$

变量代换 $y = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} x \Rightarrow x = \underbrace{\begin{pmatrix} 1 & -2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{C} y$

$$= y_1^2 - 2y_2^2 = y^T \boxed{C^T A C} y$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}}_{C^T} \underbrace{\begin{pmatrix} 1 & 2 & 2 \\ 2 & 2 & 4 \\ 2 & 4 & 4 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 1 & -2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{C} = \begin{pmatrix} 1 & & \\ & -2 & \\ & & 0 \end{pmatrix}$$

小结

$$f = x_1^2 + 4x_1x_2 + 4x_1x_3 + 2x_2^2 + 8x_2x_3 + 4x_3^2 \quad = x^TAx$$

配方法

$$= (x_1 + 2x_2 + 2x_3)^2 - 2x_2^2$$

变量代换 $y = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} x \Rightarrow x = \underbrace{\begin{pmatrix} 1 & -2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{C} y$

$$= y_1^2 - 2y_2^2 \quad = y^T\boxed{C^TAC}y$$

特别地，找到了可逆阵 C，使得

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}}_{C^T} \underbrace{\begin{pmatrix} 1 & 2 & 2 \\ 2 & 2 & 4 \\ 2 & 4 & 4 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 1 & -2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{C} = \begin{pmatrix} 1 & & \\ & -2 & \\ & & 0 \end{pmatrix}$$

例 配方法化二次型为标准形

$$f = x_1x_2 - x_1x_3 + 2x_2x_3 + x_3^2$$

$$=$$

例 配方法化二次型为标准形

$$f = x_1x_2 - x_1x_3 + 2x_2x_3 + x_3^2$$

$$= x_1x_2 \qquad + 2x_3(-\frac{1}{2}x_1 + x_2) + x_3^2$$

例 配方法化二次型为标准形

$$f = x_1x_2 - x_1x_3 + 2x_2x_3 + x_3^2$$

$$= x_1x_2 - (-\frac{1}{2}x_1 + x_2)^2 + (-\frac{1}{2}x_1 + x_2)^2 + 2x_3(-\frac{1}{2}x_1 + x_2) + x_3^2$$

例 配方法化二次型为标准形

$$f = x_1x_2 - x_1x_3 + 2x_2x_3 + x_3^2$$

$$= x_1x_2 - (-\frac{1}{2}x_1 + x_2)^2 + (-\frac{1}{2}x_1 + x_2)^2 + 2x_3(-\frac{1}{2}x_1 + x_2) + x_3^2$$

$$= \qquad\qquad + (-\frac{1}{2}x_1 + x_2 + x_3)^2$$

例 配方法化二次型为标准形

$$f = x_1x_2 - x_1x_3 + 2x_2x_3 + x_3^2$$

$$= x_1x_2 - (-\frac{1}{2}x_1 + x_2)^2 + (-\frac{1}{2}x_1 + x_2)^2 + 2x_3(-\frac{1}{2}x_1 + x_2) + x_3^2$$

$$= -\frac{1}{4}x_1^2 + 2x_1x_2 - x_2^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2$$

例 配方法化二次型为标准形

$$f = x_1x_2 - x_1x_3 + 2x_2x_3 + x_3^2$$

$$= x_1x_2 - (-\frac{1}{2}x_1 + x_2)^2 + (-\frac{1}{2}x_1 + x_2)^2 + 2x_3(-\frac{1}{2}x_1 + x_2) + x_3^2$$

$$= -\frac{1}{4}x_1^2 + 2x_1x_2 - x_2^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2$$

$$= -\frac{1}{4}x_1^2 + \qquad\qquad + 2x_1x_2 - x_2^2$$

例 配方法化二次型为标准形

$$f = x_1x_2 - x_1x_3 + 2x_2x_3 + x_3^2$$

$$= x_1x_2 - (-\frac{1}{2}x_1 + x_2)^2 + (-\frac{1}{2}x_1 + x_2)^2 + 2x_3(-\frac{1}{2}x_1 + x_2) + x_3^2$$

$$= -\frac{1}{4}x_1^2 + 2x_1x_2 - x_2^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2$$

$$= -\frac{1}{4}x_1^2 + x_1^2 - x_1^2 + 2x_1x_2 - x_2^2$$

例 配方法化二次型为标准形

$$
\begin{aligned}
f =& x_1x_2 - x_1x_3 + 2x_2x_3 + x_3^2 \\
=& x_1x_2 - (-\frac{1}{2}x_1 + x_2)^2 + (-\frac{1}{2}x_1 + x_2)^2 + 2x_3(-\frac{1}{2}x_1 + x_2) + x_3^2 \\
=& -\frac{1}{4}x_1^2 + 2x_1x_2 - x_2^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2 \\
=& -\frac{1}{4}x_1^2 + x_1^2 - x_1^2 + 2x_1x_2 - x_2^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2
\end{aligned}
$$

例 配方法化二次型为标准形

$$f = x_1x_2 - x_1x_3 + 2x_2x_3 + x_3^2$$

$$= x_1x_2 - (-\frac{1}{2}x_1 + x_2)^2 + (-\frac{1}{2}x_1 + x_2)^2 + 2x_3(-\frac{1}{2}x_1 + x_2) + x_3^2$$

$$= -\frac{1}{4}x_1^2 + 2x_1x_2 - x_2^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2$$

$$= -\frac{1}{4}x_1^2 + x_1^2 - x_1^2 + 2x_1x_2 - x_2^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2$$

$$= \quad -(x_1 - x_2)^2$$

例 配方法化二次型为标准形

$$f = x_1x_2 - x_1x_3 + 2x_2x_3 + x_3^2$$

$$= x_1x_2 - (-\frac{1}{2}x_1 + x_2)^2 + (-\frac{1}{2}x_1 + x_2)^2 + 2x_3(-\frac{1}{2}x_1 + x_2) + x_3^2$$

$$= -\frac{1}{4}x_1^2 + 2x_1x_2 - x_2^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2$$

$$= -\frac{1}{4}x_1^2 + x_1^2 - x_1^2 + 2x_1x_2 - x_2^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2$$

$$= \frac{3}{4}x_1^2 - (x_1 - x_2)^2$$

例 配方法化二次型为标准形

$$
\begin{aligned}
f =& x_1x_2 - x_1x_3 + 2x_2x_3 + x_3^2 \\
=& x_1x_2 - (-\frac{1}{2}x_1 + x_2)^2 + (-\frac{1}{2}x_1 + x_2)^2 + 2x_3(-\frac{1}{2}x_1 + x_2) + x_3^2 \\
=& -\frac{1}{4}x_1^2 + 2x_1x_2 - x_2^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2 \\
=& -\frac{1}{4}x_1^2 + x_1^2 - x_1^2 + 2x_1x_2 - x_2^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2 \\
=& \frac{3}{4}x_1^2 - (x_1 - x_2)^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2
\end{aligned}
$$

例 配方法化二次型为标准形

$$f = x_1x_2 - x_1x_3 + 2x_2x_3 + x_3^2$$

$$= x_1x_2 - (-\frac{1}{2}x_1 + x_2)^2 + (-\frac{1}{2}x_1 + x_2)^2 + 2x_3(-\frac{1}{2}x_1 + x_2) + x_3^2$$

$$= -\frac{1}{4}x_1^2 + 2x_1x_2 - x_2^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2$$

$$= -\frac{1}{4}x_1^2 + x_1^2 - x_1^2 + 2x_1x_2 - x_2^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2$$

$$= \frac{3}{4}x_1^2 - (x_1 - x_2)^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2 = \frac{3}{4}y_1^2 - y_2^2 + y_3^2$$

例 配方法化二次型为标准形

$$
\begin{aligned}
f &= x_1x_2 - x_1x_3 + 2x_2x_3 + x_3^2 \\
&= x_1x_2 - (-\frac{1}{2}x_1 + x_2)^2 + (-\frac{1}{2}x_1 + x_2)^2 + 2x_3(-\frac{1}{2}x_1 + x_2) + x_3^2 \\
&= -\frac{1}{4}x_1^2 + 2x_1x_2 - x_2^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2 \\
&= -\frac{1}{4}x_1^2 + x_1^2 - x_1^2 + 2x_1x_2 - x_2^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2 \\
&= \frac{3}{4}x_1^2 - (x_1 - x_2)^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2 = \frac{3}{4}y_1^2 - y_2^2 + y_3^2
\end{aligned}
$$

作线性变量代换

$$
\begin{cases}
y_1 = x_1 \\
y_2 = x_1 - x_2 \\
y_3 = -\frac{1}{2}x_1 + x_2 + x_3
\end{cases}
$$

例 配方法化二次型为标准形

$$f = x_1x_2 - x_1x_3 + 2x_2x_3 + x_3^2$$

$$= x_1x_2 - (-\frac{1}{2}x_1 + x_2)^2 + (-\frac{1}{2}x_1 + x_2)^2 + 2x_3(-\frac{1}{2}x_1 + x_2) + x_3^2$$

$$= -\frac{1}{4}x_1^2 + 2x_1x_2 - x_2^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2$$

$$= -\frac{1}{4}x_1^2 + x_1^2 - x_1^2 + 2x_1x_2 - x_2^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2$$

$$= \frac{3}{4}x_1^2 - (x_1 - x_2)^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2 = \frac{3}{4}y_1^2 - y_2^2 + y_3^2$$

作线性变量代换

$$\begin{cases} y_1 = x_1 \\ y_2 = x_1 - x_2 \\ y_3 = -\frac{1}{2}x_1 + x_2 + x_3 \end{cases} \Rightarrow \begin{cases} x_1 = y_1 \\ \\ \\ \end{cases}$$

例 配方法化二次型为标准形

$$f = x_1x_2 - x_1x_3 + 2x_2x_3 + x_3^2$$

$$= x_1x_2 - (-\frac{1}{2}x_1 + x_2)^2 + (-\frac{1}{2}x_1 + x_2)^2 + 2x_3(-\frac{1}{2}x_1 + x_2) + x_3^2$$

$$= -\frac{1}{4}x_1^2 + 2x_1x_2 - x_2^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2$$

$$= -\frac{1}{4}x_1^2 + x_1^2 - x_1^2 + 2x_1x_2 - x_2^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2$$

$$= \frac{3}{4}x_1^2 - (x_1 - x_2)^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2 = \frac{3}{4}y_1^2 - y_2^2 + y_3^2$$

作线性变量代换

$$\begin{cases} y_1 = x_1 \\ y_2 = x_1 - x_2 \\ y_3 = -\frac{1}{2}x_1 + x_2 + x_3 \end{cases} \Rightarrow \begin{cases} x_1 = y_1 \\ x_2 = y_1 - y_2 \end{cases}$$

例 配方法化二次型为标准形

$$f = x_1x_2 - x_1x_3 + 2x_2x_3 + x_3^2$$

$$= x_1x_2 - (-\frac{1}{2}x_1 + x_2)^2 + (-\frac{1}{2}x_1 + x_2)^2 + 2x_3(-\frac{1}{2}x_1 + x_2) + x_3^2$$

$$= -\frac{1}{4}x_1^2 + 2x_1x_2 - x_2^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2$$

$$= -\frac{1}{4}x_1^2 + x_1^2 - x_1^2 + 2x_1x_2 - x_2^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2$$

$$= \frac{3}{4}x_1^2 - (x_1 - x_2)^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2 = \frac{3}{4}y_1^2 - y_2^2 + y_3^2$$

作线性变量代换

$$\begin{cases} y_1 = x_1 \\ y_2 = x_1 - x_2 \\ y_3 = -\frac{1}{2}x_1 + x_2 + x_3 \end{cases} \Rightarrow \begin{cases} x_1 = y_1 \\ x_2 = y_1 - y_2 \\ x_3 = -\frac{1}{2}y_1 + y_2 + y_3 \end{cases}$$

例 配方法化二次型为标准形

$$f = x_1x_2 - x_1x_3 + 2x_2x_3 + x_3^2$$

$$= x_1x_2 - (-\frac{1}{2}x_1 + x_2)^2 + (-\frac{1}{2}x_1 + x_2)^2 + 2x_3(-\frac{1}{2}x_1 + x_2) + x_3^2$$

$$= -\frac{1}{4}x_1^2 + 2x_1x_2 - x_2^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2$$

$$= -\frac{1}{4}x_1^2 + x_1^2 - x_1^2 + 2x_1x_2 - x_2^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2$$

$$= \frac{3}{4}x_1^2 - (x_1 - x_2)^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2 = \frac{3}{4}y_1^2 - y_2^2 + y_3^2$$

作线性变量代换

$$\begin{cases} y_1 = x_1 \\ y_2 = x_1 - x_2 \\ y_3 = -\frac{1}{2}x_1 + x_2 + x_3 \end{cases} \Rightarrow \begin{cases} x_1 = y_1 \\ x_2 = y_1 - y_2 \\ x_3 = -\frac{1}{2}y_1 + y_2 + y_3 \end{cases} \Rightarrow x = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -\frac{1}{2} & 1 & 1 \end{pmatrix}}_{C\text{：可逆}} y$$

小结

$$f = x_1x_2 - x_1x_3 + 2x_2x_3 + x_3^2$$

配方法

$$= \frac{3}{4}x_1^2 - (x_1 - x_2)^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2$$

小结

$$f = x_1x_2 - x_1x_3 + 2x_2x_3 + x_3^2$$

配方法

$$= \frac{3}{4}x_1^2 - (x_1 - x_2)^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2$$

$$= \frac{3}{4}y_1^2 - y_2^2 + y_3^2$$

小结

$$f = x_1x_2 - x_1x_3 + 2x_2x_3 + x_3^2$$

配方法

$$= \frac{3}{4}x_1^2 - (x_1 - x_2)^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2$$

变量代换 $y = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -\frac{1}{2} & 1 & 1 \end{pmatrix} x$

$$= \frac{3}{4}y_1^2 - y_2^2 + y_3^2$$

小结

$$f = x_1x_2 - x_1x_3 + 2x_2x_3 + x_3^2$$

配方法

$$= \frac{3}{4}x_1^2 - (x_1 - x_2)^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2$$

变量代换 $y = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -\frac{1}{2} & 1 & 1 \end{pmatrix} x \Rightarrow x = \begin{pmatrix} 1 & -2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} y$

$$= \frac{3}{4}y_1^2 - y_2^2 + y_3^2$$

小结

$$f = x_1x_2 - x_1x_3 + 2x_2x_3 + x_3^2$$

配方法

$$= \frac{3}{4}x_1^2 - (x_1 - x_2)^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2$$

变量代换 $y = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -\frac{1}{2} & 1 & 1 \end{pmatrix} x \Rightarrow x = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -\frac{1}{2} & 1 & 1 \end{pmatrix}}_{C} y$

$$= \frac{3}{4}y_1^2 - y_2^2 + y_3^2$$

小结

$$f = x_1x_2 - x_1x_3 + 2x_2x_3 + x_3^2$$

配方法

$$= \frac{3}{4}x_1^2 - (x_1 - x_2)^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2$$

变量代换 $y = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -\frac{1}{2} & 1 & 1 \end{pmatrix} x \Rightarrow x = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -\frac{1}{2} & 1 & 1 \end{pmatrix}}_{C} y$

$$= \frac{3}{4}y_1^2 - y_2^2 + y_3^2$$

小结

$$f = x_1x_2 - x_1x_3 + 2x_2x_3 + x_3^2 = x^TAx$$

配方法

$$= \frac{3}{4}x_1^2 - (x_1 - x_2)^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2$$

变量代换 $y = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -\frac{1}{2} & 1 & 1 \end{pmatrix} x \Rightarrow x = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -\frac{1}{2} & 1 & 1 \end{pmatrix}}_{C} y$

$$= \frac{3}{4}y_1^2 - y_2^2 + y_3^2$$

小结

$$f = x_1x_2 - x_1x_3 + 2x_2x_3 + x_3^2 \qquad = x^TAx$$

配方法

$$= \frac{3}{4}x_1^2 - (x_1 - x_2)^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2$$

变量代换 $y = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -\frac{1}{2} & 1 & 1 \end{pmatrix} x \Rightarrow x = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -\frac{1}{2} & 1 & 1 \end{pmatrix}}_{C} y$

$$= \frac{3}{4}y_1^2 - y_2^2 + y_3^2 \qquad = y^TC^TACy$$

小结

$$f = x_1x_2 - x_1x_3 + 2x_2x_3 + x_3^2 \qquad = x^TAx$$

配方法

$$= \frac{3}{4}x_1^2 - (x_1 - x_2)^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2$$

变量代换 $y = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -\frac{1}{2} & 1 & 1 \end{pmatrix} x \Rightarrow x = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -\frac{1}{2} & 1 & 1 \end{pmatrix}}_{C} y$

$$= \frac{3}{4}y_1^2 - y_2^2 + y_3^2 \qquad = y^T\boxed{C^TAC}y$$

小结

$$f = x_1x_2 - x_1x_3 + 2x_2x_3 + x_3^2 \quad = x^TAx$$

配方法

$$= \frac{3}{4}x_1^2 - (x_1 - x_2)^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2$$

变量代换 $y = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -\frac{1}{2} & 1 & 1 \end{pmatrix} x \Rightarrow x = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -\frac{1}{2} & 1 & 1 \end{pmatrix}}_{C} y$

$$= \frac{3}{4}y_1^2 - y_2^2 + y_3^2 \quad = y^T\boxed{C^TAC}y$$

$$\begin{pmatrix} \frac{3}{4} & & \\ & -1 & \\ & & 1 \end{pmatrix}$$

小结

$$f = x_1x_2 - x_1x_3 + 2x_2x_3 + x_3^2 \qquad = x^TAx$$

配方法

$$= \frac{3}{4}x_1^2 - (x_1 - x_2)^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2$$

变量代换 $y = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -\frac{1}{2} & 1 & 1 \end{pmatrix} x \Rightarrow x = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -\frac{1}{2} & 1 & 1 \end{pmatrix}}_{C} y$

$$= \frac{3}{4}y_1^2 - y_2^2 + y_3^2 \qquad = y^T\boxed{C^TAC}y$$

$$\underbrace{\begin{pmatrix} 1 & 1 & -\frac{1}{2} \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}_{C^T} \underbrace{\begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 0 & 1 \\ -\frac{1}{2} & 1 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -\frac{1}{2} & 1 & 1 \end{pmatrix}}_{C} = \begin{pmatrix} \frac{3}{4} & & \\ & -1 & \\ & & 1 \end{pmatrix}$$

小结

$$f = x_1x_2 - x_1x_3 + 2x_2x_3 + x_3^2 \qquad = x^TAx$$

配方法

$$= \frac{3}{4}x_1^2 - (x_1 - x_2)^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2$$

变量代换 $y = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -\frac{1}{2} & 1 & 1 \end{pmatrix} x \Rightarrow x = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -\frac{1}{2} & 1 & 1 \end{pmatrix}}_{C} y$

$$= \frac{3}{4}y_1^2 - y_2^2 + y_3^2 \qquad = y^T\boxed{C^TAC}y$$

特别地，找到了可逆阵 C，使得

$$\underbrace{\begin{pmatrix} 1 & 1 & -\frac{1}{2} \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}_{C^T} \underbrace{\begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 0 & 1 \\ -\frac{1}{2} & 1 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -\frac{1}{2} & 1 & 1 \end{pmatrix}}_{C} = \begin{pmatrix} \frac{3}{4} & & \\ & -1 & \\ & & 1 \end{pmatrix}$$

二次型的规范形

$$f = x_1^2 + 4x_1x_2 + 4x_1x_3 + 2x_2^2 + 8x_2x_3 + 4x_3^2$$

配方法

$$= (x_1 + 2x_2 + 2x_3)^2 - 2x_2^2$$

二次型的规范形

$$f = x_1^2 + 4x_1x_2 + 4x_1x_3 + 2x_2^2 + 8x_2x_3 + 4x_3^2$$

配方法

$$= (x_1 + 2x_2 + 2x_3)^2 - 2x_2^2$$

$$(\sqrt{2}x_2)^2$$

二次型的规范形

$$f = x_1^2 + 4x_1x_2 + 4x_1x_3 + 2x_2^2 + 8x_2x_3 + 4x_3^2$$

配方法

$$= (x_1 + 2x_2 + 2x_3)^2 - 2x_2^2$$

$$= (x_1 + 2x_2 + 2x_3)^2 - (\sqrt{2}x_2)^2$$

二次型的规范形

$$f = x_1^2 + 4x_1x_2 + 4x_1x_3 + 2x_2^2 + 8x_2x_3 + 4x_3^2$$

配方法

$$= (x_1 + 2x_2 + 2x_3)^2 - 2x_2^2$$

$$= (x_1 + 2x_2 + 2x_3)^2 - (\sqrt{2}x_2)^2$$

$$= y_1^2 - y_2^2$$

二次型的规范形

$$f = x_1^2 + 4x_1x_2 + 4x_1x_3 + 2x_2^2 + 8x_2x_3 + 4x_3^2$$

配方法

$$= (x_1 + 2x_2 + 2x_3)^2 - 2x_2^2$$

$$= (x_1 + 2x_2 + 2x_3)^2 - (\sqrt{2}x_2)^2$$

变量代换 $y = \begin{pmatrix} 1 & 2 & 2 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} x$

$$= y_1^2 - y_2^2$$

二次型的规范形

$$f = x_1^2 + 4x_1x_2 + 4x_1x_3 + 2x_2^2 + 8x_2x_3 + 4x_3^2$$

配方法

$$= (x_1 + 2x_2 + 2x_3)^2 - 2x_2^2$$

$$= (x_1 + 2x_2 + 2x_3)^2 - (\sqrt{2}x_2)^2$$

变量代换 $y = \begin{pmatrix} 1 & 2 & 2 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} x \Rightarrow x = \begin{pmatrix} 1 & -\sqrt{2} & -2 \\ 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} y$

$$= y_1^2 - y_2^2$$

二次型的规范形

$$f = x_1^2 + 4x_1x_2 + 4x_1x_3 + 2x_2^2 + 8x_2x_3 + 4x_3^2$$

配方法

$$= (x_1 + 2x_2 + 2x_3)^2 - 2x_2^2$$

$$= (x_1 + 2x_2 + 2x_3)^2 - (\sqrt{2}x_2)^2$$

变量代换 $y = \begin{pmatrix} 1 & 2 & 2 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} x \Rightarrow x = \underbrace{\begin{pmatrix} 1 & -\sqrt{2} & -2 \\ 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{C} y$

$$= y_1^2 - y_2^2$$

二次型的规范形

$$f = x_1^2 + 4x_1x_2 + 4x_1x_3 + 2x_2^2 + 8x_2x_3 + 4x_3^2$$

配方法

$$= (x_1 + 2x_2 + 2x_3)^2 - 2x_2^2$$

$$= (x_1 + 2x_2 + 2x_3)^2 - (\sqrt{2}x_2)^2$$

变量代换 $y = \begin{pmatrix} 1 & 2 & 2 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} x \Rightarrow x = \underbrace{\begin{pmatrix} 1 & -\sqrt{2} & -2 \\ 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{C} y$

$$= y_1^2 - y_2^2$$

$$\begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$$

二次型的规范形

$$f = x_1^2 + 4x_1x_2 + 4x_1x_3 + 2x_2^2 + 8x_2x_3 + 4x_3^2$$

配方法

$$= (x_1 + 2x_2 + 2x_3)^2 - 2x_2^2$$

$$= (x_1 + 2x_2 + 2x_3)^2 - (\sqrt{2}x_2)^2$$

变量代换 $y = \begin{pmatrix} 1 & 2 & 2 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} x \Rightarrow x = \underbrace{\begin{pmatrix} 1 & -\sqrt{2} & -2 \\ 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{C} y$

$$= y_1^2 - y_2^2$$

特别地，找到了可逆阵 C，使得

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -\sqrt{2} & 1/\sqrt{2} & 0 \\ -2 & 0 & 1 \end{pmatrix}}_{C^T} \underbrace{\begin{pmatrix} 1 & 2 & 2 \\ 2 & 2 & 4 \\ 2 & 4 & 4 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 1 & -\sqrt{2} & -2 \\ 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{C} = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$$

二次型的规范形

$$f = x_1x_2 - x_1x_3 + 2x_2x_3 + x_3^2$$

配方法

$$= \frac{3}{4}x_1^2 - (x_1 - x_2)^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2$$

二次型的规范形

$$f = x_1x_2 - x_1x_3 + 2x_2x_3 + x_3^2$$

配方法

$$= \frac{3}{4}x_1^2 - (x_1 - x_2)^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2$$

$$= (\frac{\sqrt{3}}{2}x_1)^2$$

二次型的规范形

$$f = x_1x_2 - x_1x_3 + 2x_2x_3 + x_3^2$$

配方法

$$= \frac{3}{4}x_1^2 - (x_1 - x_2)^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2$$

$$= (\frac{\sqrt{3}}{2}x_1)^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2$$

二次型的规范形

$$f = x_1x_2 - x_1x_3 + 2x_2x_3 + x_3^2$$

配方法

$$= \frac{3}{4}x_1^2 - (x_1 - x_2)^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2$$

$$= (\frac{\sqrt{3}}{2}x_1)^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2 - (x_1 - x_2)^2$$

二次型的规范形

$$f = x_1x_2 - x_1x_3 + 2x_2x_3 + x_3^2$$

配方法

$$= \frac{3}{4}x_1^2 - (x_1 - x_2)^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2$$

$$= (\frac{\sqrt{3}}{2}x_1)^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2 - (x_1 - x_2)^2 = y_1^2 + y_2^2 - y_3^2$$

二次型的规范形

$$f = x_1x_2 - x_1x_3 + 2x_2x_3 + x_3^2$$

配方法

$$= \frac{3}{4}x_1^2 - (x_1 - x_2)^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2$$

$$= (\frac{\sqrt{3}}{2}x_1)^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2 - (x_1 - x_2)^2 = y_1^2 + y_2^2 - y_3^2$$

变量代换 $y = \begin{pmatrix} \sqrt{3}/2 & 0 & 0 \\ -1/2 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix} x$

二次型的规范形

$$f = x_1x_2 - x_1x_3 + 2x_2x_3 + x_3^2$$

配方法

$$= \frac{3}{4}x_1^2 - (x_1 - x_2)^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2$$

$$= (\frac{\sqrt{3}}{2}x_1)^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2 - (x_1 - x_2)^2 = y_1^2 + y_2^2 - y_3^2$$

变量代换 $y = \begin{pmatrix} \sqrt{3}/2 & 0 & 0 \\ -1/2 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix} x \Rightarrow x = \begin{pmatrix} 2/\sqrt{3} & 0 & 0 \\ 1/\sqrt{3} & -1 & 1 \\ 2/\sqrt{3} & -1 & 0 \end{pmatrix} y$

二次型的规范形

$$f = x_1x_2 - x_1x_3 + 2x_2x_3 + x_3^2$$

配方法

$$= \frac{3}{4}x_1^2 - (x_1 - x_2)^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2$$

$$= (\frac{\sqrt{3}}{2}x_1)^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2 - (x_1 - x_2)^2 = y_1^2 + y_2^2 - y_3^2$$

变量代换 $y = \begin{pmatrix} \sqrt{3}/2 & 0 & 0 \\ -1/2 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix} x \Rightarrow x = \underbrace{\begin{pmatrix} 2/\sqrt{3} & 0 & 0 \\ 1/\sqrt{3} & -1 & 1 \\ 2/\sqrt{3} & -1 & 0 \end{pmatrix}}_{C} y$

二次型的规范形

$$f = x_1x_2 - x_1x_3 + 2x_2x_3 + x_3^2$$

配方法

$$= \frac{3}{4}x_1^2 - (x_1 - x_2)^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2$$

$$= (\frac{\sqrt{3}}{2}x_1)^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2 - (x_1 - x_2)^2 = y_1^2 + y_2^2 - y_3^2$$

变量代换 $y = \begin{pmatrix} \sqrt{3}/2 & 0 & 0 \\ -1/2 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix} x \Rightarrow x = \underbrace{\begin{pmatrix} 2/\sqrt{3} & 0 & 0 \\ 1/\sqrt{3} & -1 & 1 \\ 2/\sqrt{3} & -1 & 0 \end{pmatrix}}_{C} y$

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

二次型的规范形

$$f = x_1x_2 - x_1x_3 + 2x_2x_3 + x_3^2$$

配方法

$$= \frac{3}{4}x_1^2 - (x_1 - x_2)^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2$$

$$= (\frac{\sqrt{3}}{2}x_1)^2 + (-\frac{1}{2}x_1 + x_2 + x_3)^2 - (x_1 - x_2)^2 = y_1^2 + y_2^2 - y_3^2$$

变量代换 $y = \begin{pmatrix} \sqrt{3}/2 & 0 & 0 \\ -1/2 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix} x \Rightarrow x = \underbrace{\begin{pmatrix} 2/\sqrt{3} & 0 & 0 \\ 1/\sqrt{3} & -1 & 1 \\ 2/\sqrt{3} & -1 & 0 \end{pmatrix}}_{C} y$

特别地，找到了可逆阵 C，使得

$$\underbrace{\begin{pmatrix} 2/\sqrt{3} & 1/\sqrt{3} & 2/\sqrt{3} \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix}}_{C^T} \underbrace{\begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 0 & 1 \\ -\frac{1}{2} & 1 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 2/\sqrt{3} & 0 & 0 \\ 1/\sqrt{3} & -1 & 1 \\ 2/\sqrt{3} & -1 & 0 \end{pmatrix}}_{C} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

二次型的规范形

定理 任意二次型 $f(x_1, \ldots, x_n)$ 都可以通过非退化线性变换

$$x = Cy$$

化为

$$f = y_1^2 + \cdots + y_p^2 - y_{p+1}^2 - \cdots - y_r^2$$

二次型的规范形

定理 任意二次型 $f(x_1, ..., x_n)$ 都可以通过非退化线性变换

$$x = Cy$$

化为

$$f = y_1^2 + \cdots + y_p^2 - y_{p+1}^2 - \cdots - y_r^2$$

$$\begin{pmatrix} I_p & & \\ & -I_{r-p} & \\ & & O \end{pmatrix}$$

二次型的规范形

定理 任意二次型 $f(x_1, \ldots, x_n)$ 都可以通过非退化线性变换

$$x = Cy$$

化为

$$f = y_1^2 + \cdots + y_p^2 - y_{p+1}^2 - \cdots - y_r^2$$

$$A \qquad \begin{pmatrix} I_p & & \\ & -I_{r-p} & \\ & & O \end{pmatrix}$$

二次型的规范形

定理 任意二次型 $f(x_1,...,x_n)$ 都可以通过非退化线性变换

$$x = Cy$$

化为

$$f = y_1^2 + \cdots + y_p^2 - y_{p+1}^2 - \cdots - y_r^2$$

$$C^TAC = \begin{pmatrix} I_p & & \\ & -I_{r-p} & \\ & & O \end{pmatrix}$$

二次型的规范形

定理 任意二次型 $f(x_1, \ldots, x_n)$ 都可以通过非退化线性变换

$$x = Cy$$

化为

$$f = y_1^2 + \cdots + y_p^2 - y_{p+1}^2 - \cdots - y_r^2$$

也就是，任意对称矩阵 A，都存在可逆矩阵 C，使得

$$C^T AC = \begin{pmatrix} I_p & & \\ & -I_{r-p} & \\ & & O \end{pmatrix}$$

二次型的规范形

定理 任意二次型 $f(x_1,\dots,x_n)$ 都可以通过非退化线性变换

$$x = Cy$$

化为

$$f = y_1^2 + \cdots + y_p^2 - y_{p+1}^2 - \cdots - y_r^2$$

也就是，任意对称矩阵 A，都存在可逆矩阵 C，使得

$$C^TAC = \begin{pmatrix} I_p & & \\ & -I_{r-p} & \\ & & O \end{pmatrix}$$

注

- $r = r(A)$，$p =$ 正惯性指标，$r - p =$ 负惯性指标

二次型的规范形

定理 任意二次型 $f(x_1, \dots, x_n)$ 都可以通过非退化线性变换

$$x = Cy$$

化为

$$f = y_1^2 + \cdots + y_p^2 - y_{p+1}^2 - \cdots - y_r^2$$

也就是，任意对称矩阵 A，都存在可逆矩阵 C，使得

$$C^T AC = \begin{pmatrix} I_p & & \\ & -I_{r-p} & \\ & & O \end{pmatrix}$$

注

- $r = r(A)$，$p =$ 正惯性指标，$r - p =$ 负惯性指标
- p 是由 A 唯一确定的

合同，合同的等价条件

定义 设 A, B 为两个 n 阶方阵，若存在可逆 n 阶方阵 C，使得

$$C^T A C = B$$

则称 A合同于B，记为 $A \simeq B$

合同，合同的等价条件

定义 设 A, B 为两个 n 阶方阵，若存在可逆 n 阶方阵 C，使得

$$C^T AC = B$$

则称 A合同于B，记为 $A \simeq B$

定理 任意对称矩阵 A，都成立

$$A \simeq \begin{pmatrix} I_p & & \\ & -I_{r-p} & \\ & & O \end{pmatrix}$$

合同，合同的等价条件

定义 设 A, B 为两个 n 阶方阵，若存在可逆 n 阶方阵 C，使得

$$C^TAC = B$$

则称 A 合同于 B，记为 $A \simeq B$

定理 任意对称矩阵 A，都成立

$$A \simeq \begin{pmatrix} I_p & & \\ & -I_{r-p} & \\ & & O \end{pmatrix}$$

定理 设 A, B 为对称矩阵，则 $A \simeq B$ 的充分必要条件是 A, B 具有相同的规范形（也就是，秩、正惯性指标都相等）