第 12 章 d: 函数展开成幂级数

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2016-2017 **学年** II



Outline

$$f(x) \stackrel{?}{=} a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

$$f(x) \neq a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots + a_n(x-x_0)^n + \cdots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

$$f(x) \stackrel{?}{=} a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

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$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

$$f^{(k)}(x) = \left[\sum_{n=0}^{\infty} a_n (x - x_0)^n\right]^{(k)}$$

$$f(x) \neq a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

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$$f^{(k)}(x) = \left[\sum_{n=0}^{\infty} a_n (x - x_0)^n\right]^{(k)} = \sum_{n=0}^{\infty} \left[a_n (x - x_0)^n\right]^{(k)}$$

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$$n \cdot (n-1) \cdots (n-k+1) \cdot (x-x_0)^{n-k}$$

$$f(x) \neq a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

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$$= \sum_{n=0}^{\infty} a_n \cdot n \cdot (n-1) \cdots (n-k+1) \cdot (x - x_0)^{n-k}$$

$$f(x) \neq a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

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$$= \sum_{n=k}^{\infty} a_n \cdot n \cdot (n-1) \cdots (n-k+1) \cdot (x - x_0)^{n-k}$$
$$= a_k \cdot k!$$

$$f(x) \stackrel{?}{=} a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

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$$= \sum_{n=k}^{\infty} a_n \cdot n \cdot (n-1) \cdots (n-k+1) \cdot (x - x_0)^{n-k}$$
$$= a_k \cdot k! + a_{k+1} \cdot (k+1) \cdots 2 \cdot (x - x_0)$$

$$f(x) \stackrel{?}{=} a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

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$$= \sum_{n=k}^{\infty} a_n \cdot n \cdot (n-1) \cdots (n-k+1) \cdot (x - x_0)^{n-k}$$

$$= a_k \cdot k! + a_{k+1} \cdot (k+1) \cdots 2 \cdot (x - x_0)$$

$$+ a_{k+2} \cdot (k+2) \cdots 3 \cdot (x - x_0)^2 + \cdots$$

$$f(x) \neq a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

证明 逐项求 k 次导得:

$$f^{(k)}(x) = \left[\sum_{n=0}^{\infty} a_n (x - x_0)^n\right]^{(k)} = \sum_{n=0}^{\infty} \left[a_n (x - x_0)^n\right]^{(k)}$$
$$= \sum_{n=k}^{\infty} a_n \cdot n \cdot (n-1) \cdots (n-k+1) \cdot (x - x_0)^{n-k}$$
$$= a_k \cdot k! + a_{k+1} \cdot (k+1) \cdots 2 \cdot (x - x_0)$$

 $+ a_{k+2} \cdot (k+2) \cdots 3 \cdot (x-x_0)^2 + \cdots$

取 $x = x_0$ 得 $a_k = \frac{1}{k!} f^{(k)}(x_0)$



$$f(x) \stackrel{?}{=} a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$



$$f(x) \neq a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots + a_n(x-x_0)^n + \dots$$

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注 1

 $f(x_0)$

$$f(x) \neq a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

$$f(x_0)$$
 $f'(x_0)$

$$f(x) \neq a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

$$f(x_0)$$
 $f'(x_0)$ $\frac{1}{2!}f''(x_0)$

$$f(x) \neq a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

$$f(x_0)$$
 $f'(x_0)$ $\frac{1}{2!}f''(x_0)$ \cdots $\frac{1}{n!}f^{(n)}(x_0)$

$$f(x) \neq a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

注 1 也就是, f(x) 至多能展成如下形式的幂级数:

$$f(x_0)+f'(x_0)(x-x_0)+\frac{1}{2!}f''(x_0)(x-x_0)^2+\cdots+\frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n+\cdots$$

$$f(x) \neq a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

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- 此级数称为 f(x) 在 x_0 处的 泰勒级数。

$$f(x) \stackrel{?}{=} a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots + a_n(x-x_0)^n + \cdots$$

性质 若 f(x) 能展成上述幂级数,则

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- 此级数称为 *f*(x) 在 x₀ 处的 泰勒级数。
 - 此级数称为 J(X) 在 X_0 处的 泰剌级数。
 - 级数前 n+1 项的部分和记为 p_n ,称为 n 次泰勒多项式

$$f(x) \stackrel{?}{=} a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

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- 此级数称为
$$f(x)$$
 在 x_0 处的 泰勒级数。

- 级数前 n+1 项的部分和记为 p_n ,称为 n 次泰勒多项式

注 2
$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0) (x - x_0)^n$$



$$f(x) \stackrel{?}{=} a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

 $a_n = \frac{1}{n!} f^{(n)}(x_0).$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = -\frac{1}{n!}f^{(n)}(x)$$

注 1 也就是,f(x) 至多能展成如下形式的幂级数:

- 此级数称为
$$f(x)$$
 在 x_0 处的 泰勒级数。

 $f(x_0)+f'(x_0)(x-x_0)+\frac{1}{2!}f''(x_0)(x-x_0)^2+\cdots+\frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n+\cdots$

- 级数前 n+1 项的部分和记为 p_n ,称为 n 次泰勒多项式

注 2
$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0) (x - x_0)^n \quad \Leftrightarrow \quad f(x) = \lim_{n \to \infty} p_n(x)$$

解 取
$$x_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

解 取
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当
$$f(x) = e^x$$
 时,

$$f(x) = f'(x) = f''(x) = f'''(x) = \dots = f^{(n)}(x) = e^x$$

解 取
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$$\Rightarrow f(0) = f'(0) = f''(0) = f'''(0) = \dots = f^{(n)}(0) = 1$$

解 取
$$x_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = e^x$$
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$$\Rightarrow f(0) = f'(0) = f''(0) = f'''(0) = \dots = f^{(n)}(0) = 1$$

⇒
$$\bar{x}$$
 \$\text{\$\pi\$}\$ \$\pi\$\$ \$\pi\$ \$\p

解 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = e^x$ 时,

$$f(x) = f'(x) = f''(x) = f'''(x) = \dots = f^{(n)}(x) = e^x$$

$$\Rightarrow f(0) = f'(0) = f''(0) = f'''(0) = \dots = f^{(n)}(0) = 1$$

$$\Rightarrow f(0) = f(0) = f(0) = f(0) = \cdots = f(0) = 1$$

$$= \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{3}$$

⇒
$$\bar{x}$$
 \$\text{\$\pi\$}\$ \$\tag{\pi}\$ \$\ta

注 n 次泰勒多项式是:

$$p_n(x) =$$

解 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = e^x$$
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⇒
$$\bar{x}$$
 \$\text{\$\pi\$}\$ \$\pi\$\$ \$\pi\$ \$\pi\$\$ \$\pi\$ \$\

注 n 次泰勒多项式是:

$$p_n(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots + \frac{1}{n!}x^n$$



解 取
$$x_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = \sin x$$
 时,

解 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = \sin x$ 时,

	$f^{(n)}(x)$	$f^{(n)}(0)$
n = 0, 4, 8	sin x	0
<i>n</i> = 1, 5, 9	cosx	1
n = 2, 6, 10	— sin <i>x</i>	0
n = 3, 7, 11	— cos x	-1

解 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = \sin x$ 时,

	$f^{(n)}(x) = \sin(x + \frac{n}{2}\pi)$	$f^{(n)}(0)$
n = 0, 4, 8	sin x	0
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	$f^{(n)}(x) = \sin(x + \frac{n}{2}\pi)$	$f^{(n)}(0) = \sin(\frac{n}{2}\pi)$
n = 0, 4, 8	sin x	0
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<i>n</i> = 3, 7, 11	— cos x	-1

所以泰勒级数是

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \cdots$$

 $\mathbf{m} \mathbf{n} \mathbf{x}_0 = \mathbf{0} \mathbf{n}$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = \sin x$ 时,

	$f^{(n)}(x) = \sin(x + \frac{n}{2}\pi)$	$f^{(n)}(0) = \sin(\frac{n}{2}\pi)$
n = 0, 4, 8	sin x	0
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$$x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\frac{1}{7!}x^7+\frac{1}{9!}x^9-\frac{1}{11!}x^{11}+\cdots+(-1)^m\frac{1}{(2m+1)!}x^{2m+1}+\cdots$$

展开成幂级数

$$x - \frac{1}{3!}x^{3} + \frac{1}{5!}x^{5} - \frac{1}{7!}x^{7} + \frac{1}{9!}x^{9} - \frac{1}{11!}x^{11} + \dots + (-1)^{m} \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

$$x - \frac{1}{3!}x^{3} + \frac{1}{5!}x^{5} - \frac{1}{7!}x^{7} + \frac{1}{9!}x^{9} - \frac{1}{11!}x^{11} + \dots + (-1)^{m} \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

$$p_1 = x$$
;

$$x - \frac{1}{3!}x^{3} + \frac{1}{5!}x^{5} - \frac{1}{7!}x^{7} + \frac{1}{9!}x^{9} - \frac{1}{11!}x^{11} + \dots + (-1)^{m} \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

$$p_1 = p_2 = x$$
;

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

$$p_1 = p_2 = x;$$

 $p_3 = x - \frac{1}{3!}x^3;$

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 $p_1 = p_2 = x$;

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

$$p_3 = p_4 = x - \frac{1}{3!}x^3;$$

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

sinx的n次泰勒多项式是:

$$p_1 = p_2 = x;$$

$$p_3 = p_4 = x - \frac{1}{3!}x^3;$$

$$p_5 = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5;$$

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

sinx的n次泰勒多项式是:

$$p_1 = p_2 = x;$$

$$p_3 = p_4 = x - \frac{1}{3!}x^3;$$

$$p_5 = p_6 = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5;$$

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

$$n_1 = n_2 = Y$$
:

 $p_1 = p_2 = x$;

$$p_3 = p_4 = x - \frac{1}{3!}x^3;$$

$$p_5 = p_6 = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5;$$

 p_{2m+1}



$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

 $p_1 = p_2 = x$;

 p_{2m+1}

sinx的n次泰勒多项式是:

 $p_3 = p_4 = x - \frac{1}{3!}x^3;$

 $p_5 = p_6 = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5;$

 $= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1}$

- sinx 的泰勒级数是:

$$\frac{1}{2}$$
 $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$

 $p_1 = p_2 = x$;

 $x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$

$$\frac{1}{3}$$
 $\frac{1}{5}$ $\frac{1}{5}$

sinx的n次泰勒多项式是:

 $p_3 = p_4 = x - \frac{1}{3!}x^3;$



 $p_5 = p_6 = x - \frac{1}{31}x^3 + \frac{1}{51}x^5;$

 $p_{2m+1} = p_{2m+2} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1}$

- sinx 的泰勒级数是:

解 取
$$x_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = \cos x$$
 时,

解 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = \cos x$ 时,

	$f^{(n)}(x)$	$f^{(n)}(0)$
n = 0, 4, 8	cosx	1
n = 1, 5, 9	— sin <i>x</i>	0
n = 2, 6, 10	— cos x	-1
n = 3, 7, 11	sin x	0

解 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = \cos x$ 时,

	$f^{(n)}(x) = \cos(x + \frac{n}{2}\pi)$	$f^{(n)}(0)$
n = 0, 4, 8	cosx	1
n = 1, 5, 9	— sin <i>x</i>	0
n = 2, 6, 10	— cos x	-1
n = 3, 7, 11	sin x	0

解 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = \cos x$ 时,

	$f^{(n)}(x) = \cos(x + \frac{n}{2}\pi)$	$f^{(n)}(0) = \cos(\frac{n}{2}\pi)$
n = 0, 4, 8	cosx	1
$n = 1, 5, 9 \dots$	— sin <i>x</i>	0
n = 2, 6, 10	— cos x	-1
<i>n</i> = 3, 7, 11	sin x	0

解 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = \cos x$ 时,

	$f^{(n)}(x) = \cos(x + \frac{n}{2}\pi)$	$f^{(n)}(0) = \cos(\frac{n}{2}\pi)$
n = 0, 4, 8	cosx	1
<i>n</i> = 1, 5, 9	— sin <i>x</i>	0
n = 2, 6, 10	— cos x	-1
n = 3, 7, 11	sin x	0

所以泰勒级数是

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \cdots$$



解 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = \cos x$ 时,

	$f^{(n)}(x) = \cos(x + \frac{n}{2}\pi)$	$f^{(n)}(0) = \cos(\frac{n}{2}\pi)$
n = 0, 4, 8	cosx	1
n = 1, 5, 9	— sin <i>x</i>	0
<i>n</i> = 2, 6, 10	— cos x	-1
<i>n</i> = 3, 7, 11	sin x	0

所以泰勒级数是

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$



$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

$$p_0 = 1;$$

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

$$p_0 = p_1 = 1$$
;

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

$$p_0 = p_1 = 1;$$

 $p_2 = 1 - \frac{1}{2!}x^2;$

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

$$p_0 = p_1 = 1;$$

 $p_2 = p_3 = 1 - \frac{1}{2!}x^2;$

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

$$p_0 = p_1 = 1;$$

$$p_2 = p_3 = 1 - \frac{1}{2!}x^2;$$

$$p_4 = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4;$$



$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

$$p_0 = p_1 = 1;$$

$$p_2 = p_3 = 1 - \frac{1}{2!}x^2;$$

$$p_4 = p_5 = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4;$$

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

• cos x 的 n 次泰勒多项式是:

$$p_0 = p_1 = 1;$$

$$p_2 = p_3 = 1 - \frac{1}{2!}x^2;$$

$$p_4 = p_5 = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4;$$
:

 $p_{2m}(x)$



$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

•
$$\cos x$$
 的 n 次泰勒多项式是:
$$p_0 = p_1 = 1;$$

 $p_2 = p_3 = 1 - \frac{1}{2!}x^2;$ $p_4 = p_5 = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4;$

$$p_4 = p_5 = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4;$$

 $=1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\cdots+(-1)^m\frac{1}{(2m)!}x^{2m}$

 $p_{2m}(x)$

COS X 的泰勒级数是:

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

cos x 的 n 次泰勒多项式是:

 $p_0 = p_1 = 1$:

$$p_2 = p_3 = 1 - \frac{1}{2!}x^2;$$

$$p_4 = p_5 = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4;$$

 $p_{2m}(x) = p_{2m+1}(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots + (-1)^m \frac{1}{(2m)!}x^{2m}$

解 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = \ln(1+x)$ 时,

解 取
$$x_0 = 0$$
 时,泰勒级数是 $f''(0)$ $f'''(0)$

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = \ln(1+x)$$
 时,
 $f = \ln(1+x)$, $f' = \frac{1}{1+x}$,

$$f = \ln(1+x), \quad f' = \frac{1}{1+x}$$

$$\mathbf{m} \mathbf{n} \mathbf{x}_0 = \mathbf{0} \mathbf{n}$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = \ln(1+x)$$
 时,
 $f = \ln(1+x)$, $f' = \frac{1}{1+x}$, $f'' = \frac{(-1)}{(1+x)^2}$,



解 取
$$x_0 = 0$$
 时,泰勒级数是
$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = \ln(1+x)$$
 时,
 $f = \ln(1+x)$, $f' = \frac{1}{1+x}$, $f'' = \frac{(-1)}{(1+x)^2}$, $f''' = \frac{2}{(1+x)^3}$,



解 取
$$x_0 = 0$$
 时,泰勒级数是
$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

$$f(0) + f'(0)x + \frac{f'(0)}{2!}x^2 + \frac{f'(0)}{3!}x^3 + \dots + \frac{f'(0)}{n!}x^n + \dots$$

$$\stackrel{\text{def}}{=} f(x) = \ln(1+x) \text{ pt},$$

$$f = \ln(1+x), \quad f' = \frac{1}{1+x}, \quad f'' = \frac{(-1)}{(1+x)^2}, \quad f''' = \frac{2}{(1+x)^3},$$

$$f^{(4)} = \frac{-2 \cdot 3}{(1+x)^4},$$

解 取
$$x_0 = 0$$
 时,泰勒级数是
$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

$$f(0) + f'(0)x + \frac{f'(0)}{2!}x^2 + \frac{f'(0)}{3!}x^3 + \dots + \frac{f'(0)}{n!}x^n + \dots$$

$$\stackrel{\text{def}}{=} f(x) = \ln(1+x) \text{ pt},$$

$$f = \ln(1+x), \quad f' = \frac{1}{1+x}, \quad f'' = \frac{(-1)}{(1+x)^2}, \quad f''' = \frac{2}{(1+x)^3},$$

$$f^{(4)} = \frac{-2 \cdot 3}{(1+x)^4}, \quad f^{(5)} = \frac{2 \cdot 3 \cdot 4}{(1+x)^5}, \dots,$$

$$\mathbf{H}$$
 取 $\mathbf{X}_0 = \mathbf{0}$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = \ln(1+x)$$
 时,

$$f = \ln(1+x), \quad f' = \frac{1}{1+x}, \quad f'' = \frac{(-1)}{(1+x)^2}, \quad f''' = \frac{2}{(1+x)^3},$$

$$f = \ln(1+x), \quad f' = \frac{1}{1+x}, \quad f'' = \frac{1}{(1+x)^2}, \quad f''' = \frac{1}{(1+x)^3},$$

$$f^{(4)} = \frac{-2 \cdot 3}{(1+x)^4}, \quad f^{(5)} = \frac{2 \cdot 3 \cdot 4}{(1+x)^5}, \dots, f^{(n)} = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}, \dots$$

解 取
$$x_0 = 0$$
 时,泰勒级数是
$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

$$f(0) + f'(0)x + \frac{f'(0)}{2!}x^2 + \frac{f'(0)}{3!}x^3 + \dots + \frac{f''(0)}{n!}x^n + \dots$$

当
$$f(x) = \ln(1+x)$$
 时,
 $f = \ln(1+x)$, $f' = \frac{1}{1+x}$, $f'' = \frac{(-1)}{(1+x)^2}$, $f''' = \frac{2}{(1+x)^3}$,

$$f^{(4)} = \frac{-2 \cdot 3}{(1+x)^4}, \quad f^{(5)} = \frac{2 \cdot 3 \cdot 4}{(1+x)^5}, \dots, f^{(n)} = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}, \dots$$

所以
$$\frac{1}{n!}f^{(n)}(0) = \frac{(-1)^{n-1}}{n}$$
,

例 求 $f(x) = \ln(1+x)$ 在 x = 0 处泰勒级数。

解 取
$$x_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = \ln(1+x)$$
 时,

$$f = \ln(1+x), \quad f' = \frac{1}{1+x}, \quad f'' = \frac{(-1)}{(1+x)^2}, \quad f''' = \frac{2}{(1+x)^3},$$

$$f^{(4)} = \frac{-2 \cdot 3}{(1+x)^4}, \quad f^{(5)} = \frac{2 \cdot 3 \cdot 4}{(1+x)^5}, \dots, f^{(n)} = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}, \dots$$

所以
$$\frac{1}{n!}f^{(n)}(0) = \frac{(-1)^{n-1}}{n}$$
,泰勒级数是
$$x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{1}{4}x^4 + \cdots$$



例 求 $f(x) = \ln(1+x)$ 在 x = 0 处泰勒级数。

解 取
$$x_0 = 0$$
 时,泰勒级数是
$$f''(0) = f'''(0) = f^{(n)}(0)$$

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = \ln(1+x)$$
 时,
 $f = \ln(1+x)$, $f' = \frac{1}{1+x}$, $f'' = \frac{(-1)}{(1+x)^2}$, $f''' = \frac{2}{(1+x)^3}$,

$$f^{(4)} = \frac{-2 \cdot 3}{(1+x)^4}, \quad f^{(5)} = \frac{2 \cdot 3 \cdot 4}{(1+x)^5}, \dots, f^{(n)} = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}, \dots$$
所以 $\frac{1}{n!} f^{(n)}(0) = \frac{(-1)^{n-1}}{n}, \,\,$ 泰勒级数是

$$x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots$$



解 取 $x_0 = 0$ 时,泰勒级数是

$$\mathbf{H}$$
 以 $X_0 = \mathbf{U}$ 的, 泰剌级数 $f''(\mathbf{0})$

$$f(0) + f'(0)x + \frac{f'(0)}{2!}$$

$$f(0) + f'(0)x + \frac{f'(0)}{2!}x$$

 $f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{2!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$

所以 $\frac{1}{n!}f^{(n)}(0) = \frac{(-1)^{n-1}}{n}$,泰勒级数是

$$f(0) + f'(0)x + \frac{f'(0)}{2!}x$$

当 $f(x) = \ln(1+x)$ 时,

例 求 $f(x) = \ln(1+x)$ 在 x = 0 处泰勒级数。

 $f = \ln(1+x), \quad f' = \frac{1}{1+x}, \quad f'' = \frac{(-1)}{(1+x)^2}, \quad f''' = \frac{2}{(1+x)^3},$

 $f^{(4)} = \frac{-2 \cdot 3}{(1+x)^4}, \quad f^{(5)} = \frac{2 \cdot 3 \cdot 4}{(1+x)^5}, \dots, f^{(n)} = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}, \dots$

 $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots$

解 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = (1+x)^{\alpha}$$
 时,

 \mathbf{H} 取 $\mathbf{x}_0 = \mathbf{0}$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = (1+x)^{\alpha}$$
 时,
 $f = (1+x)^{\alpha}$, $f' = \alpha(1+x)^{\alpha-1}$,

解 取
$$x_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = (1+x)^{\alpha}$$
 时,

$$f = (1+x)^{\alpha}, \quad f' = \alpha(1+x)^{\alpha-1}, \quad f'' = \alpha(\alpha-1)(1+x)^{\alpha-2},$$

 \mathbf{H} 取 $\mathbf{x}_0 = \mathbf{0}$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = (1+x)^{\alpha}$$
 时,

$$f = (1+x)^{\alpha}, \quad f' = \alpha(1+x)^{\alpha-1}, \quad f'' = \alpha(\alpha-1)(1+x)^{\alpha-2},$$

$$\ldots, f^{(n)} = \alpha(\alpha - 1) \cdots (\alpha - n + 1)(1 + x)^{\alpha - n}, \cdots$$

解 取
$$x_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = (1+x)^{\alpha}$$
 时,

$$f = (1+x)^{\alpha}, \quad f' = \alpha(1+x)^{\alpha-1}, \quad f'' = \alpha(\alpha-1)(1+x)^{\alpha-2},$$

$$\ldots, f^{(n)} = \alpha(\alpha - 1) \cdots (\alpha - n + 1)(1 + x)^{\alpha - n}, \cdots$$

所以
$$\frac{1}{n!}f^{(n)}(0) = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$$
,

解 取
$$x_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = (1+x)^{\alpha}$$
 时,

$$f = (1+x)^{\alpha}$$
, $f' = \alpha(1+x)^{\alpha-1}$, $f'' = \alpha(\alpha-1)(1+x)^{\alpha-2}$,

$$\ldots, f^{(n)} = \alpha(\alpha - 1) \cdots (\alpha - n + 1)(1 + x)^{\alpha - n}, \cdots$$

所以
$$\frac{1}{n!}f^{(n)}(0) = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$$
,泰勒级数是
$$1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots$$

解 取
$$x_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = (1+x)^{\alpha}$$
 时,

$$f = (1+x)^{\alpha}, \quad f' = \alpha(1+x)^{\alpha-1}, \quad f'' = \alpha(\alpha-1)(1+x)^{\alpha-2},$$

..., $f^{(n)} = \alpha(\alpha-1)\cdots(\alpha-n+1)(1+x)^{\alpha-n}, \cdots$

所以
$$\frac{1}{n!}f^{(n)}(0) = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$$
,泰勒级数是

$$1 + \alpha x + \frac{\alpha(\alpha - 1)}{2!} x^2 + \dots + \frac{\alpha(\alpha - 1) \cdots (\alpha - n + 1)}{n!} x^n + \dots$$



 \mathbf{m} 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{2!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = (1+x)^{\alpha}$ 时,

$$f = (1+x)^{\alpha}, \quad f' = \alpha(1+x)^{\alpha-1}, \quad f'' = \alpha(\alpha-1)(1+x)^{\alpha-2},$$

$$\ldots, f^{(n)} = \alpha(\alpha - 1) \cdots (\alpha - n + 1)(1 + x)^{\alpha - n}, \cdots$$

所以
$$\frac{1}{n!}f^{(n)}(0) = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$$
,泰勒级数是
$$1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n + \cdots$$

注 n 次泰勒多项式是:

$$p_n(x) =$$



 $f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{2!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$

 \mathbf{H} 取 $x_0 = 0$ 时,泰勒级数是

例 求 $f(x) = (1+x)^{\alpha}$ 在 x = 0 处的 n 次泰勒多项式 $p_n(x)$

 $1 + \alpha x + \frac{\alpha(\alpha - 1)}{2!} x^2 + \dots + \frac{\alpha(\alpha - 1) \cdots (\alpha - n + 1)}{2!} x^n + \dots$

所以 $\frac{1}{n!}f^{(n)}(0) = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$, 泰勒级数是

 $p_n(x) = 1 + \alpha x + \frac{\alpha(\alpha - 1)}{2!}x^2 + \dots + \frac{\alpha(\alpha - 1)\cdots(\alpha - n + 1)}{n!}x^n$

 $\dots, f^{(n)} = \alpha(\alpha - 1) \cdots (\alpha - n + 1)(1 + x)^{\alpha - n}, \dots$

注 *n* 次泰勒多项式是:

当 $f(x) = (1+x)^{\alpha}$ 时, $f = (1+x)^{\alpha}$, $f' = \alpha(1+x)^{\alpha-1}$, $f'' = \alpha(\alpha-1)(1+x)^{\alpha-2}$

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0) (x - x_0)^n$$

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0) (x - x_0)^n \iff f(x) = \lim_{n \to \infty} p_n(x)$$

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$$(\Leftrightarrow R_n(x) = f(x) - p_n(x))$$

$$\Leftrightarrow \lim_{n \to \infty} R_n(x) = 0$$

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$$(R_n(x) = f(x) - p_n(x))$$

$$\Leftrightarrow \lim_{n \to \infty} R_n(x) = 0$$

注 $R_n(x) = f(x) - p_n(x)$,或者 $f(x) = p_n(x) + R_n(x)$,刻画了原函数 f(x) 与其泰勒多项式 $p_n(x)$ 的差异。



回忆 泰勒中值定理 1 若 f 具有 n 阶导数,则

$$R_n(x) = o((x-x_0)^n).$$

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回忆 泰勒中值定理 1 若 f 具有 n 阶导数,则

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特别地,

$$f(x) = p_n(x) + R_n(x)$$

$$= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + o((x - x_0)^n)$$

 $e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + o(x^{n})$

$$(1+x)^{\alpha} = 1 + \alpha x$$

 $(1+x)^{\alpha} = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \dots + \frac{\alpha(\alpha-1) \cdots (\alpha-n+1)}{n!} x^n + o(x^n)$

 $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots + (-1)^{n-1}\frac{1}{n}x^n + o(x^n)$

 $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + o(x^{2m+2})$

 $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + o(x^{2m+1})$

例求 $\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$, $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1-x)]}$

例求
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
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例求
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$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$$

例求
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$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$$

$$= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3}$$

$$= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3}$$



例求
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
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$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$$

$$= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3} = \frac{1}{3}$$

例 求
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
, $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1-x)]}$

$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$$

$$\frac{1}{2}x^3 + o(x^4) = 1$$

$$\sin^{3} x \qquad x \to 0
= \lim_{x \to 0} \frac{\frac{1}{3}x^{3} + o(x^{4})}{x^{3}} = \frac{1}{3}$$

$$= \lim_{X \to 0} \frac{\frac{1}{3}X^3 + o(X^4)}{X^3} =$$

$$\cos X - e^{-\frac{X^2}{2}}$$

$$\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2 [x + \ln(1 - x)]}$$

$$\lim_{x\to 0} \frac{1}{x^2 [x + \ln(1-x)]}$$

$$= \lim_{x \to 0} \frac{\left[\frac{1}{x + \ln(1 - x)} \right]}{x^2 \left[x + \left(\frac{1}{x + \ln(1 - x)} \right) \right]}$$



例 求
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
, $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1-x)]}$

例 求
$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
, $\lim_{x \to 0} \frac{\cos x - e^{-2}}{x^2 [x + \ln(1 - x)]}$

$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$$

$$= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3} = \frac{1}{3}$$

$$\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2 [x + \ln(1 - x)]}$$

$$x \to 0 \quad x^{2} \left[x + \ln(1 - x) \right]$$

$$= \lim_{x \to 0} \frac{\left[1 - \frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[\frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[\frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[\frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[\frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[\frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[\frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[\frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[\frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[\frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[\frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[\frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[\frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[\frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[\frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[\frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[\frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[\frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[\frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[\frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[\frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[\frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[\frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[\frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[\frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[\frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[\frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[\frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[\frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[\frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[\frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[\frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[\frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[\frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[\frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[\frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[\frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[\frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[\frac{1}{2!} x^{2} + \frac{1}{4!} x^{4} + o(x^{5}) \right] - \left[$$

例 求
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
, $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1-x)]}$

$$\lim_{x \to 0} \frac{\sin^3 x}{\sin^3 x} + \lim_{x \to 0} \frac{x^2 [x + \ln(1 - x)]}{x^2 [x + \ln(1 - x)]}$$

$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$$

$$\begin{array}{ll}
\Rightarrow 0 & \sin^3 x & x \to 0 \\
& = \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3} = \frac{1}{3}
\end{array}$$

$$\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2 [x + \ln(1 - x)]}$$

$$\frac{1}{x^2} \frac{1}{x^2} \frac{1}$$

$$\frac{1}{2} x^2 + \frac{1}{2} x^4$$

 $= \lim_{x \to 0} \frac{\left[1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + o(x^5)\right] - \left[1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + o(x^4)\right]}{x^2 \left[x + \left(\frac{1}{2}x^2 + \frac{1}{8}x^4 + o(x^4)\right]\right]}$

例 求
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
, $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1-x)]}$

$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$$
$$= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3} = \frac{1}{3}$$

$$= \lim_{x \to 0} \frac{3}{x^3}$$

$$\cos x - e^{-\frac{x^2}{2}}$$

$$\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2 \left[x + \ln(1 - x) \right]}$$

$$= \lim_{x \to 0} \frac{\left[1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 + o(x^5) \right] - \left[1 - \frac{1}{2} x^2 + \frac{1}{8} x^4 + o(x^4) \right]}{x^2 \left[x + \left(-x - \frac{1}{2} x^2 + o(x^2) \right) \right]}$$



$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} =$$

$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$$
$$= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3} = \frac{1}{3}$$

$$\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2 [x + \ln(1 - x)]}$$

$$[x + ln($$

 $= \lim_{x \to 0} \frac{-\frac{1}{12}x^4 + o(x^4)}{-\frac{1}{2}x^4 + o(x^4)}$

 $= \lim_{x \to 0} \frac{\left[1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + o(x^5)\right] - \left[1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + o(x^4)\right]}{x^2 \left[x + \left(-x - \frac{1}{2}x^2 + o(x^2)\right)\right]}$

$$x^2[x]$$

例 求 $\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$, $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1-x)]}$ 解

 $\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$

 $\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2 [x + \ln(1 - x)]}$

 $= \lim_{x \to 0} \frac{\left[1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + o(x^5)\right] - \left[1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + o(x^4)\right]}{x^2 \left[x + \left(-x - \frac{1}{2}x^2 + o(x^2)\right)\right]}$

函数展开成幂级数

 $= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3} = \frac{1}{2}$

 $= \lim_{x \to 0} \frac{-\frac{1}{12}x^4 + o(x^4)}{-\frac{1}{2}x^4 + o(x^4)} = \lim_{x \to 0} \frac{-\frac{1}{12} + o(x^4)/x^4}{-\frac{1}{2} + o(x^4)/x^4}$

例 求
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
, $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x}{2}}}{x^2[x + \ln(1 - x)]}$

$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x\to 0} \frac{\left[x - \frac{1}{3!}x^3 + c^3\right]}{x^2[x + \ln(1 - x)]}$$

$$\lim_{x \to \infty} \frac{\sin x - x \cos x}{\sin x - x \cos x}$$

 $\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$ $= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3} = \frac{1}{2}$

$$\lim_{n \to \infty} \frac{\cos x - e^{n}}{n}$$

$$\lim_{x \to 0} \frac{\cos x - \cos x}{x^2 [x + \ln(x)]}$$

$$x \to 0 \quad X^{2} \left[X + \ln(1 - \frac{1}{2!} X^{2} + \frac{1}{2!} X^{2} + \frac{1}{2!} X^{2} + \frac{1}{2!} X^{2} \right]$$

$$= \lim_{x \to 0} \frac{\left[1 - \frac{1}{2!} X^{2} + \frac{1}{2!} X^{2} + \frac{1}{2!} X^{2} + \frac{1}{2!} X^{2} + \frac{1}{2!} X^{2} \right]}{x^{2} \left[1 - \frac{1}{2!} X^{2} + \frac{1}{2$$

函数展开成幂级数

 $\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2 [x + \ln(1 - x)]}$

 $= \lim_{x \to 0} \frac{\left[1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + o(x^5)\right] - \left[1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + o(x^4)\right]}{x^2 \left[x + \left(-x - \frac{1}{2}x^2 + o(x^2)\right)\right]}$

 $= \lim_{x \to 0} \frac{-\frac{1}{12}x^4 + o(x^4)}{-\frac{1}{2}x^4 + o(x^4)} = \lim_{x \to 0} \frac{-\frac{1}{12} + o(x^4)/x^4}{-\frac{1}{2} + o(x^4)/x^4} = \frac{1}{6}$

泰勒中值定理 2 若 f 具有 n+1 阶导数,则

$$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) (x - x_0)^{n+1}$$

其中 ξ 是 x_0 与 x 之间的某个值

$$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) (x - x_0)^{n+1}$$

$$(1-\theta)x_0+\theta x$$

其中 ξ 是 x_0 与 x 之间的某个值, $0 < \theta < 1$ 。

$$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) (x - x_0)^{n+1}$$

$$\stackrel{or}{=} \frac{1}{(n+1)!} f^{(n+1)}((1-\theta)x_0 + \theta x) (x - x_0)^{n+1},$$

其中 ξ 是 x_0 与 x 之间的某个值, $0 < \theta < 1$ 。

$$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) (x - x_0)^{n+1}$$

$$\stackrel{or}{=} \frac{1}{(n+1)!} f^{(n+1)}((1-\theta)x_0 + \theta x) (x - x_0)^{n+1},$$

其中 ξ 是 x_0 与 x 之间的某个值, $0 < \theta < 1$ 。

注

1. ξ (以及 θ) 不是固定不变的,而是随 x 和 n 的改变而变化。

$$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) (x - x_0)^{n+1}$$

$$\stackrel{or}{=} \frac{1}{(n+1)!} f^{(n+1)} ((1-\theta)x_0 + \theta x) (x - x_0)^{n+1},$$

其中 ξ 是 x_0 与 x 之间的某个值, $0 < \theta < 1$ 。

注

- 1. ξ (以及 θ) 不是固定不变的,而是随 x 和 n 的改变而变化。
- 2. 当 $x_0 = 0$ 时,则余项可写成

$$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1}, \qquad (0 < \theta < 1)$$



$$e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty).$$

$$e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty).$$

证明

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- $n \to \infty$

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 $\leq \frac{|\mathsf{X}|^{n+1}}{(n+1)!} \to 0$ (已知级数 $\sum_{|\mathsf{X}|^{n+1}}^{|\mathsf{X}|^{n+1}}$ 收敛,所以一般项 $\frac{|\mathsf{X}|^{n+1}}{(n+1)!} \to 0$)

— (n+1): (n+1):



$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

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性质 对任意 $x \in (-\infty, \infty)$, $\sin x$ 等于其泰勒级数。即

1 1 1 1 1 1 1

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• 至此,我们知道 e^x , $\sin x$, $\cos x$ 以及 $\frac{1}{1+x}$ 是等于其泰勒级数,即

$$e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty)$$

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• 利用上述结果,及逐项积分公式,可进一步求出

• 利用工处结果,及逐项积分公式,可进一步来出 $\ln(1+x)$, $\arctan x$

的幂级数展开。



$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$$



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证明 1. 幂级数的收敛域是 (-1, 1], 故上式至多对 $x \in (-1, 1]$ 成立。

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$$\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots + (-1)^n \frac{1}{2n+1}x^{2n+1} + \dots, \quad x \in [-1, 1]$$



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当 x ∈ [-1, 1] 时,利用逐项积分可得

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注 取 x=1,则得到

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$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$$



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$$\frac{1}{1+x} = 1 - x + x^{2} - x^{3} + \dots + (-1)^{n} x^{n} + \dots, x \in (-1, 1)$$

$$\ln(1+x) = x - \frac{1}{2}x^{2} + \frac{1}{3}x^{3} + \dots + (-1)^{n-1} \frac{1}{n}x^{n} + \dots, x \in (-1, 1]$$

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• 至此, 得出如下常用函数的幂级数展开式:

$$-x + x^2 - x^3 + \cdots + (-1)^{-1}$$

 $e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty)$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots + (-1)^n \frac{1}{(2n)!}x^{2n} + \dots, \ x \in (-\infty, \infty)$$



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 $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots + (-1)^n x^n + \dots, x \in (-1, 1)$

 $\sin x = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 + \dots + (-1)^n \frac{1}{(2n+1)!} x^{2n+1} + \dots, \ x \in (-\infty, \infty)$

 $\cos x = 1 - \tfrac{1}{2!} x^2 + \tfrac{1}{4!} x^4 - \tfrac{1}{6!} x^6 + \dots + (-1)^n \tfrac{1}{(2n)!} x^{2n} + \dots \,, \; x \in (-\infty, \infty)$

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 $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots + (-1)^n \frac{1}{2n+1}x^{2n+1} + \dots, x \in [-1,1]$

• 用上述结果, 及逐项求导、积分公式, 可求更多函数的泰勒级数展开

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$$

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所以当
$$x \in (-1, 1]$$
 时,
$$(1-x)\ln(1+x) = (1-x)\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n$$

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所以当
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$$\sum_{n=1}^{\infty} (-1)^{n-2} \frac{1}{n-1} x^n$$

$$)^{n-2}\frac{1}{n-1}x^n$$

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$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n - \sum_{n=2}^{\infty} (-1)^{n-2} \frac{1}{n-1} x^n$$

解 利用

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$$

所以当 $x \in (-1, 1]$ 时, $(1-x)\ln(1+x) = (1-x)\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n - \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^{n+1}$$

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解利用

 $(1-x)\ln(1+x) = (1-x)\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n$

所以当 $x \in (-1, 1]$ 时,

例 把函数 $f(x) = (1-x) \ln(1+x)$ 展开成 x 的幂级数。

 $= x + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{1}{n} x^n - \sum_{n=2}^{\infty} (-1)^{n-2} \frac{1}{n-1} x^n$

 $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$

 $=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}x^n-\sum_{n=2}^{\infty}(-1)^{n-2}\frac{1}{n-1}x^n$

 $=\sum_{n=0}^{\infty}(-1)^{n-1}\frac{1}{n}x^{n}-\sum_{n=0}^{\infty}(-1)^{n-1}\frac{1}{n}x^{n+1}$

 $=x+\sum_{n=0}^{\infty}\left(\frac{(-1)^{n-1}}{n}-\frac{(-1)^n}{n-1}\right)x^n$

解 利用

$$\cos t = 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6 + \dots + (-1)^n \frac{1}{(2n)!}t^{2n} + \dots, \ t \in (-\infty, \infty)$$

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$$

解 利用

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解 利用

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 $= \frac{1}{2} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!} x^{2n}$

 $=1+\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n2^{2n}}{(2n)!}x^{2n}$

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解 1. 注意到
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将 $\frac{1}{\sqrt{11}}$, $\frac{1}{\sqrt{12}}$ 分别展开成 (x + 4) 的幂级数:

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2. 利用
$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \dots + t^n + \dots, t \in (-1, 1)$$

将
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其中
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,即 $-6 < x < -2$ 。

解 1. 注意到
$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$
.

2. 利用 $\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots + t^n + \cdots$, $t \in (-1, 1)$

将
$$\frac{1}{x+1}$$
, $\frac{1}{x+2}$ 分别展开成 $(x+4)$ 的幂级数: 令 $t=x+4$, 则

* $\frac{1}{x+1} = \frac{1}{t-3} = \frac{1}{-3} \cdot \frac{1}{1-\frac{t}{3}} = -\frac{1}{3} \sum_{n=0}^{\infty} \frac{t^n}{3^n} = -\sum_{n=0}^{\infty} \frac{(x+4)^n}{3^{n+1}}$

其中
$$\left|\frac{x+4}{3}\right| = \left|\frac{t}{3}\right| < 1$$
,即 $-7 < x < -1$ 。

*
$$\frac{1}{x+2} = \frac{1}{t-2} = \frac{1}{-2} \cdot \frac{1}{1-\frac{t}{2}} = -\frac{1}{2} \sum_{n=0}^{\infty} \frac{t^n}{2^n} = -\sum_{n=0}^{\infty} \frac{(x+4)^n}{2^{n+1}}$$

其中
$$\left|\frac{x+4}{2}\right| = \left|\frac{t}{2}\right| < 1$$
,即 $-6 < x < -2$ 。

3. 所以-6 < x < -2时

解 1. 注意到 $\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$.

例 把函数 $f(x) = \frac{1}{x^2 + 3x + 2}$ 展开成 (x + 4) 的幂级数。

2.
$$\forall \exists t \in \mathbb{Z}_{x^2+3x+2}$$
 $(x+1)(x+2)$ $(x+1)(x+2)$

将
$$\frac{1}{x+1}$$
, $\frac{1}{x+2}$ 分别展开成 $(x+4)$ 的幂级数: 令 $t=x+4$, 则
* $\frac{1}{x+1} = \frac{1}{t-3} = \frac{1}{-3} \cdot \frac{1}{1-\frac{t}{3}} = -\frac{1}{3} \sum_{n=0}^{\infty} \frac{t^n}{3^n} = -\sum_{n=0}^{\infty} \frac{(x+4)^n}{3^{n+1}}$

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$$\frac{1}{x+2} = \frac{1}{t-2} = \frac{1}{-2} \cdot \frac{1}{1-\frac{t}{2}} = \frac{1}{1-\frac{$$

$$\times^{1/1}$$
 $_{2}$ $_{1}$ $_{12}$ $_{1}$ $_{1}$ $_{1}$

其中
$$\left|\frac{x+4}{2}\right| = \left|\frac{t}{2}\right| < 1$$
,即 $-6 < x < -2$ 。

*
$$\frac{1}{x+2} = \frac{1}{t-2} = \frac{1}{-2} \cdot \frac{1}{1-\frac{t}{2}} = -\frac{1}{2} \sum_{n=0}^{\infty} \frac{t^n}{2^n} = -\sum_{n=0}^{\infty} \frac{(x+4)^n}{2^{n+1}}$$

$$=-\frac{1}{2}\sum_{n=0}^{\infty}\frac{c}{2}$$

$$\frac{1}{2}\sum_{n=0}^{\infty}\frac{t^n}{2^n}=-\sum_{n=0}^{\infty}\frac{(x^n)^n}{2^n}$$

$$\sum_{n=0}^{\infty} \frac{(x+4)^n}{2^{n+1}}$$

 $\frac{1}{x^2 + 3x + 2} = \frac{1}{x + 1} - \frac{1}{x + 2} = \sum_{n=0}^{\infty} \left(\frac{1}{2^{n+1}} - \frac{1}{3^{n+1}} \right) (x + 4)_{0}^{n}$