第 12 周作业解答

练习 1. 设函数 $z = u + v$，而 $u = x + y$，$v = xy$，求 $\frac{\partial z}{\partial x}$，$\frac{\partial z}{\partial y}$。

解法一：

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} \\ &= (u+v)'_u\cdot(x+y)'_x + (u+v)'_v\cdot(xy)'_x \\ &= 1 + y\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} \\ &= (u+v)'_u\cdot(x+y)'_y + (u+v)'_v\cdot(xy)'_y \\ &= 1 + x\end{aligned}$$

解法二：因为

$$z = u + v = x + y + xy$$

所以

$$\frac{\partial z}{\partial x} = (x + y + xy)'_x = 1 + y$$

$$\frac{\partial z}{\partial y} = (x + y + xy)'_y = 1 + x$$

练习 2. 设函数 $z = u^2\ln v$，而 $u = \frac{x}{y}$，$v = 2x - 3y$，求 $\frac{\partial z}{\partial x}$，$\frac{\partial z}{\partial y}$。

解法一：

$$
\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}\\
&= \left(u^2\ln v\right)_u'\cdot\left(\frac{x}{y}\right)_x'+\left(u^2\ln v\right)_v'\cdot(2x-3y)_x'\\
&= 2u\ln v\cdot\frac{1}{y}+u^2\cdot\frac{1}{v}\cdot 2\\
&= 2\cdot\frac{x}{y}\cdot\ln(2x-3y)\cdot\frac{1}{y}+\frac{x^2}{y^2}\cdot\frac{1}{2x-3y}\cdot 2\\
&= \frac{2x}{y^2}\ln(2x-3y)+\frac{2x^2}{y^2(2x-3y)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}\\
&= \left(u^2\ln v\right)_u'\cdot\left(\frac{x}{y}\right)_y'+\left(u^2\ln v\right)_v'\cdot(2x-3y)_y'\\
&= 2u\ln v\cdot(-\frac{x}{y^2})+u^2\cdot\frac{1}{v}\cdot(-3)\\
&= 2\cdot\frac{x}{y}\cdot\ln(2x-3y)\cdot(-\frac{x}{y^2})+\frac{x^2}{y^2}\cdot\frac{1}{2x-3y}\cdot(-3)\\
&= -\frac{2x^2}{y^3}\ln(2x-3y)-\frac{3x^2}{y^2(2x-3y)}
\end{aligned}
$$

解法二：因为

$$z=u^2\ln v=\frac{x^2\ln(2x-3y)}{y^2}$$

所以

$$\frac{\partial z}{\partial x}=\left(\frac{x^2\ln(2x-3y)}{y^2}\right)_x'=\frac{2x}{y^2}\ln(2x-3y)+\frac{2x^2}{y^2(2x-3y)}$$

$$\frac{\partial z}{\partial y}=\left(\frac{x^2\ln(2x-3y)}{y^2}\right)_y'=-\frac{2x^2}{y^3}\ln(2x-3y)-\frac{3x^2}{y^2(2x-3y)}$$

练习 3. 设 $y=y(x)$ 满足 $x\sin y+xy+2=0$，求 $\frac{dy}{dx}$。

解法一：令 $F(x,y)=x\sin y+xy+2$，则 $y=y(x)$ 满足 $F(x,y)=0$，所以

$$\frac{dy}{dx}=-\frac{F_x}{F_y}=-\frac{(x\sin y+xy+2)_x'}{(x\sin y+xy+2)_y'}=-\frac{\sin y+y}{x\cos y+x}$$

解法二：对

$$x\sin y(x)+xy(x)+2=0$$

两边求导得

$$
\begin{aligned}
0 &= \frac{d}{dx}\left(x\sin y(x)+xy(x)+2\right)\\
&= \sin y(x)+x\cdot(\sin y(x))'+y(x)+xy'(x)\\
&= (\sin y+y)+(x\cos y+x)y'
\end{aligned}
$$

所以

$$y' = -\frac{\sin y + y}{x\cos y + x}.$$

练习 4. 设 $z = z(x,\, y)$ 由方程 $e^{-xy} - 2z + e^{-z} = 0$ 确定，求 $\frac{\partial z}{\partial x}$ 和 $\frac{\partial z}{\partial y}$。

解令 $F(x,\, y,\, z) = e^{-xy} - 2z + e^{-z}$，则

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{-ye^{-xy}}{-2-e^{-z}} = -\frac{ye^{-xy}}{2+e^{-z}}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{-xe^{-xy}}{-2-e^{-z}} = -\frac{xe^{-xy}}{2+e^{-z}}.$$

练习 5. 设 $z = z(x,\, y)$ 由方程 $x^2 + y^2 + z^2 - 3xyz = 0$ 确定，求 dz。

解令 $F(x,\, y,\, z) = x^2 + y^2 + z^2 - 3xyz$，则

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2x-3yz}{2z-3xy}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{2y-3xz}{2z-3xy}.$$

所以

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = -\frac{2x-3yz}{2z-3xy}dx - \frac{2y-3xz}{2z-3xy}dy.$$

练习 6. 求下列各函数的极值

1. $f(x,\, y) = 4(x-y) - x^2 - y^2$
2. $f(x,\, y) = x^3 + 3xy^2 - 15x - 12y$

解 1. (I) 先求驻点。求解方程组

$$\begin{cases} f_x = 4 - 2x = 0 \\ f_y = -4 - 2y = 0 \end{cases}$$

得 $(x,\, y) = (2,\, -2)$。

(II) 判断驻点是否极值点。计算二阶偏导数

$$f_{xx} = -2, \quad f_{xy} = 0, \quad f_{yy} = -2$$

可求出判别式 $P(x,\, y) = f_{xx}f_{yy} - f_{xy}^2 = 4$。

	$(2,\, -2)$
$P(x,\, y)$	$4 > 0$
f_{xx}	$-2 < 0$
是否极值点	极大值点
极值 $f(x,\, y)$	8

2. (I) 先求驻点。求解方程组

$$\begin{cases} f_x = 3x^2 + 3y^2 - 15 = 0 \\ f_y = 6xy - 12 = 0 \end{cases}$$

得 $(x,\, y) = (-2,\, -1),\, (-1,\, -2),\, (1,\, 2),\, (2,\, 1)$。

(II) 判断驻点是否极值点。计算二阶偏导数

$$f_{xx} = 6x, \quad f_{xy} = 6y, \quad f_{yy} = 6x$$

可求出判别式 $P(x,\, y) = f_{xx}f_{yy} - f_{xy}^2 = 36x^2 - 36y^2$。

	$(-2,\,-1)$	$(-1,\,-2)$	$(1,\,2)$	$(2,\,1)$
$P(x,\,y)$	$108>0$	$-108<0$	$-108<0$	$108>0$
f_{xx}	$-12<0$			$12>0$
是否极值点	极大值点	☹	☹	极小值点
极值 $f(x,\,y)$	28			-28

注：$(-2,\,-1)$ 不是最大值点。显然 $f(10,\,0)$, $f(100,\,0)$, $f(1000,\,0)\ldots$ 的值都比 $f(-2,\,-1)$ 大。想一想，这当中是否矛盾？同样，$(2,\,1)$ 不是最小值点。