第 06 周作业解答

练习 1. 求下列函数的全微分

$$(1)\quad z = xy + \frac{x}{y}; \qquad (2)\quad u = x^{yz}.$$

解 (1)

$$dz = z_x dx + z_y dy = \left(y + \frac{1}{y}\right) dx + \left(x - \frac{x}{y^2}\right) dy.$$

(2)

$$du = u_x dx + u_y dy + u_z dz = yzx^{yz-1} dx + zx^{yz} \ln x dy + yx^{yz} \ln x dz.$$

练习 2. 求函数 $z = \frac{y}{x}$ 当 $x = 2$，$y = 1$，$\Delta x = 0.1$，$\Delta y = -0.2$ 时的全增量和全微分。

解

$$dz = z_x dx + z_y dy = -\frac{y}{x^2} dx + \frac{1}{x} dy.$$

将 $x = 2$，$y = 1$，$\Delta x = 0.1$，$\Delta y = -0.2$ 代入，得到全微分

$$dz = -\frac{1}{4} \cdot 0.1 + \frac{1}{2} \cdot (-0.2) = -0.125 = -\frac{1}{8}.$$

而全增量 Δz 为

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x, y) = f(2+0.1, 1-0.2) - f(2, 1) = \frac{0.8}{2.1} - \frac{1}{2} = \frac{16-21}{42} = -\frac{5}{42} \approx -0.119047619$$

在此例中 Δz 与 dz 在精确到小数点后 1 位时是相等。

练习 3. （选择题）设函数 $f(x, y)$ 在点 $P(x_0, y_0)$ 的两个偏导数 $f_x(x_0, y_0)$ 都存在，则（ C ）

A $f(x, y)$ 在点 P 处连续；

B $f(x, y)$ 在点 P 处可微；

C $\lim_{x \to x_0} f(x, y_0)$ 及 $\lim_{y \to y_0} f(x_0, y)$ 都存在；

D $\lim_{(x, y) \to (x_0, y_0)} f(x, y)$ 存在.

练习 4. （选择题）二元函数 $f(x, y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y = (0, 0) \end{cases}$ 在点 $(0, 0)$ 处（ C ）

A 连续，偏导数存在；

B 连续，偏导数不存在；

C 不连续，偏导数存在；

D 不连续，偏导数不存在.

练习 5. （选择题）“$f_x(x_0, y_0)$ 与 $f_y(x_0, y_0)$ 均存在”是函数 $f(x, y)$ 在点 $P(x_0, y_0)$ 处连续的（ D ）条件。

A 充分非必要；

B 必要非充分；

C 充分且必要；

D 非充分非必要.

练习 6. 设 $z = \arctan(xy)$，$y = e^x$，求 $\frac{dz}{dx}$。

解设 $z = f(x, y)$，$y = e^x$。

$$\frac{dz}{dx} = f_x + f_y \cdot \frac{dy}{dx} = \frac{y}{1+x^2y^2} + \frac{x}{1+x^2y^2} \cdot e^x = \frac{y+xe^x}{1+x^2y^2} = \frac{e^x(1+x)}{1+x^2e^{2x}}.$$

练习 7. 设 $z = xy + xF(u)$，$u = \frac{y}{x}$，$F(u)$ 为可导函数，证明

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z + xy.$$

解

$$\begin{aligned}
\frac{\partial z}{\partial x} &= y + F(u) + xF'(u) \cdot \left(\frac{y}{x}\right)_x = y + F - \frac{y}{x}F', \\
\frac{\partial z}{\partial y} &= x + xF'(u) \cdot \left(\frac{y}{x}\right)_y = x + F', \\
x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} &= x\left(y + F - \frac{y}{x}F'\right) + y\left(x + F'\right) = 2xy + xF = xy + z.
\end{aligned}$$

练习 8. 求下列复合函数的一阶偏导数（假设 f 具有一阶连续偏导）：

$$(1) \quad z = f(x^2 - y^2, e^{xy}); \qquad (2) \quad u = f(\frac{x}{y}, \frac{y}{z}); \qquad (3) \quad u = f(x, xy, xyz).$$

解 (1)

$$\begin{aligned}
\frac{\partial z}{\partial x} &= f_1' \cdot (x^2 - y^2)_x + f_2' \cdot (e^{xy})_x = 2xf_1' + ye^{xy}f_2', \\
\frac{\partial z}{\partial y} &= f_1' \cdot (x^2 - y^2)_y + f_2' \cdot (e^{xy})_y = -2yf_1' + xe^{xy}f_2'.
\end{aligned}$$

(2)

$$\begin{aligned}
\frac{\partial u}{\partial x} &= f_1' \cdot (\frac{x}{y})_x = \frac{1}{y}f_1', \\
\frac{\partial u}{\partial y} &= f_1' \cdot (\frac{x}{y})_y + f_2' \cdot (\frac{y}{z})_y = -\frac{x}{y^2}f_1' + \frac{1}{z}f_2', \\
\frac{\partial u}{\partial z} &= f_2' \cdot (\frac{y}{z})_z = -\frac{y}{z^2}f_2'.
\end{aligned}$$

(3)

$$\begin{aligned}
\frac{\partial u}{\partial x} &= f_1' \cdot (x)_x + f_2' \cdot (xy)_x + f_3' \cdot (xyz)_x = f_1' + yf_2' + yzf_3', \\
\frac{\partial u}{\partial y} &= f_2' \cdot (xy)_y + f_3' \cdot (xyz)_y = xf_2' + xzf_3', \\
\frac{\partial u}{\partial z} &= f_3' \cdot (xyz)_z = xyf_3'.
\end{aligned}$$

练习 9. 求复合函数 $z = f(xy^2, x^2y)$ 的所有二阶偏导数。这里假设 f 具有二阶连续偏导数。

解

$$\begin{aligned}
z_x &= f_1' \cdot (xy^2)_x + f_2' \cdot (x^2y)_x = y^2 f_1' + 2xyf_2', \\
z_y &= f_1' \cdot (xy^2)_y + f_2' \cdot (x^2y)_y = 2xyf_1' + x^2 f_2',
\end{aligned}$$

$$\begin{aligned}
z_{xx} &= \left(y^2 f_1' + 2xyf_2'\right)_x = y^2 (f_1')_x + 2yf_2' + 2xy(f_2')_x \\
&= y^2 \left[f_{11}'' \cdot (xy^2)_x + f_{12}'' \cdot (x^2y)_x\right] + 2yf_2' + 2xy\left[f_{21}'' \cdot (xy^2)_x + f_{22}'' \cdot (x^2y)_x\right] \\
&= y^2 \left[y^2 f_{11}'' + 2xyf_{12}''\right] + 2yf_2' + 2xy\left[y^2 f_{21}'' + 2xyf_{22}''\right] \\
&= 2yf_2' + y^4 f_{11}'' + 4xy^3 f_{12}'' + 4x^2y^2 f_{22}'', \\
z_{yx} = z_{xy} &= \left(y^2 f_1' + 2xyf_2'\right)_y = 2yf_1' + y^2 (f_1')_y + 2xf_2' + 2xy(f_2')_y \\
&= 2yf_1' + y^2 \left[f_{11}'' \cdot (xy^2)_y + f_{12}'' \cdot (x^2y)_y\right] + 2xf_2' + 2xy\left[f_{21}'' \cdot (xy^2)_y + f_{22}'' \cdot (x^2y)_y\right] \\
&= 2yf_1' + y^2 \left[2xyf_{11}'' + x^2 f_{12}''\right] + 2xf_2' + 2xy\left[2xyf_{21}'' + x^2 f_{22}''\right] \\
&= 2yf_1' + 2xf_2' + 2xy^3 f_{11}'' + 5x^2y^2 f_{12}'' + 2x^3yf_{22}'', \\
z_{yy} &= \left(2xyf_1' + x^2 f_2'\right)_y = 2xf_1' + 2xy(f_1')_y + x^2 (f_2')_y \\
&= 2xf_1' + 2xy\left[f_{11}'' \cdot (xy^2)_y + f_{12}'' \cdot (x^2y)_y\right] + x^2\left[f_{21}'' \cdot (xy^2)_y + f_{22}'' \cdot (x^2y)_y\right] \\
&= 2xf_1' + 2xy\left[2xyf_{11}'' + x^2 f_{12}''\right] + x^2\left[2xyf_{21}'' + x^2 f_{22}''\right] \\
&= 2xf_1' + 4x^2y^2 f_{11}'' + 4x^3yf_{12}'' + x^4 f_{22}''.
\end{aligned}$$

练习 10. 设 $\ln\sqrt{x^2+y^2} = \arctan\frac{y}{x}$，求 $\frac{dy}{dx}$。

解令 $F(x, y) = \ln\sqrt{x^2+y^2} - \arctan\frac{y}{x}$。则方程相当于 $F(x, y) = 0$。所以

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{\frac{x}{x^2+y^2} - \frac{-\frac{y}{x^2}}{1+\frac{y^2}{x^2}}}{\frac{y}{x^2+y^2} - \frac{\frac{1}{x}}{1+\frac{y^2}{x^2}}} = \frac{x+y}{y-x}.$$

练习 11. 设 $\frac{x}{z} = \ln\frac{z}{y}$，求 $\frac{\partial z}{\partial x}$ 及 $\frac{\partial z}{\partial y}$。

解令 $F(x, y, z) = \frac{x}{z} - \ln\frac{z}{y}$。则方程相当于 $F(x, y, z) = 0$。所以

$$\begin{aligned}
z_x &= -\frac{F_x}{F_z} = -\frac{\frac{1}{z}}{-\frac{x}{z^2} - \frac{1}{z}} = \frac{z}{x+z}, \\
z_y &= -\frac{F_y}{F_z} = -\frac{\frac{1}{y}}{-\frac{x}{z^2} - \frac{1}{z}} = \frac{z^2}{y(x+z)}.
\end{aligned}$$

练习 12. 设 $x = x(y, z)$，$y = y(x, z)$，$z = z(x, y)$ 都是由方程 $F(x, y, z) = 0$ 所确定的具有连续偏导数的函数，证明

$$\frac{\partial x}{\partial y} \cdot \frac{\partial y}{\partial z} \cdot \frac{\partial z}{\partial x} = -1.$$

证明

$$\frac{\partial x}{\partial y} = -\frac{F_y}{F_x}$$
$$\frac{\partial y}{\partial z} = -\frac{F_z}{F_y}$$
$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

所以

$$\frac{\partial x}{\partial y} \cdot \frac{\partial y}{\partial z} \cdot \frac{\partial z}{\partial x} = \left(-\frac{F_y}{F_x}\right) \cdot \left(-\frac{F_z}{F_y}\right) \cdot \left(-\frac{F_x}{F_z}\right) = -1.$$

练习 13. 设 $z^3 - 3xyz = a^3$，求 $\frac{\partial^2 z}{\partial x \partial y}$。

解令 $F(x, y, z) = z^3 - 3xyz$。则方程相当于 $F(x, y, z) = 0$。所以

$$\begin{aligned}
z_x &= -\frac{F_x}{F_z} = -\frac{-3yz}{3z^2 - 3xy} = \frac{yz}{z^2 - xy}, \\
z_y &= -\frac{F_y}{F_z} = -\frac{-3xz}{3z^2 - 3xy} = \frac{xz}{z^2 - xy}, \\
z_{xy} &= \left(\frac{yz}{z^2 - xy}\right)_y = \frac{(yz)_y(z^2 - xy) - yz(z^2 - xy)_y}{(z^2 - xy)^2} = \frac{(z + yz_y)(z^2 - xy) - yz(2zz_y - x)}{(z^2 - xy)^2} \\
&= \frac{(z + y\frac{xz}{z^2 - xy})(z^2 - xy) - yz(2z\frac{xz}{z^2 - xy} - x)}{(z^2 - xy)^2} \\
&= \frac{z^5 - 2xyz^3 - x^2y^2z}{(z^2 - xy)^3}.
\end{aligned}$$