第 10 章 b：二重积分的计算

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如何计算二重积分：

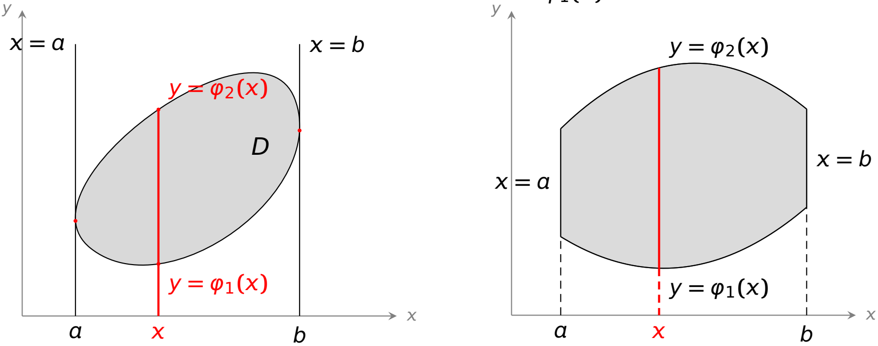
- 一般方法 化二重积分为“累次积分”：

$$\iint_D f(x,y)d\sigma = \iint_D f(x,y)dxdy = \int_*^* \left[\int_*^* f(x,y)dx\right]dy$$
$$= \int_*^* \left[\int_*^* f(x,y)dy\right]dx$$

- 问题：如何确定积分上下限？

固定 x，先对 y 积分

$$\iint_D f(x,y)dxdy = \int_a^b \left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy\right]dx$$



注 上述区域 D 可以表示成

$$D = \{(x,y) | \varphi_1(x) \le y \le \varphi_2(x), a \le x \le b\}$$

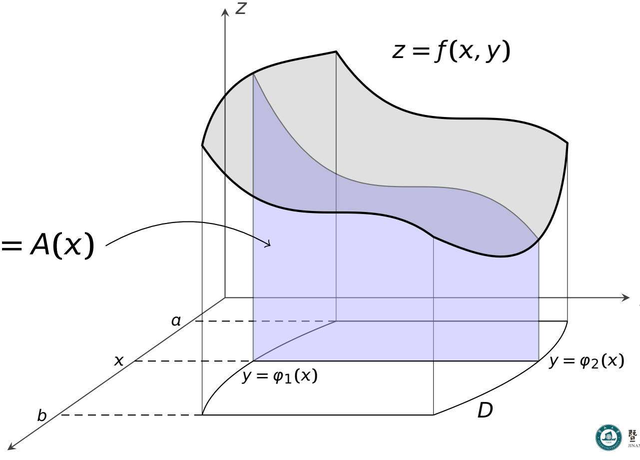
称为 X-型区域。

二次积分化为累次积分：几何解释

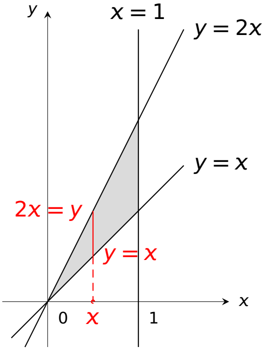
- 设 $D = \{(x, y) | \varphi_1(x) \le y \le \varphi_2(x), a \le x \le b\}$，则

$$\iint_D f(x, y)d\sigma = V = \int_a^b A(x)dx = \int_a^b \left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)dy \right] dx$$

$\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)dy = A(x)$



例 计算 $\iint_D xydxdy$，其中 D 是由直线 $y=2x$，$y=x$ 和 $x=1$ 所围成区域。



解

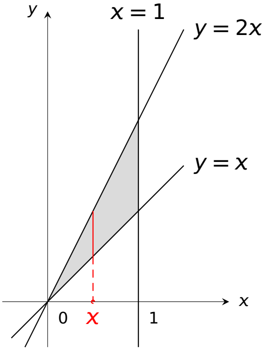
$$\iint_D xydxdy = \int_0^1 \left[\int_x^{2x} xydy\right]dx$$

$$= \int_0^1 \left[\frac{1}{2}xy^2\Big|_x^{2x}\right]dx = \int_0^1 \frac{3}{2}x^3dx = \frac{3}{8}x^4\Big|_0^1 = \frac{3}{8}$$

注 D 是 X-型区域，可以表示为

$$D = \{(x, y) | x \le y \le 2x,\ 0 \le x \le 1\}$$

例 计算 $\iint_D e^{x+y}dxdy$，其中 D 是由直线 $y=2x$，$y=x$ 和 $x=1$ 所围成区域。

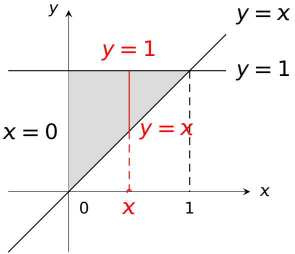


解

$$\iint_D e^{x+y}dxdy = \int_0^1 \left[\int_x^{2x} e^{x+y}dy\right]dx = \int_0^1 \left[e^{x+y}\Big|_x^{2x}\right]dx$$

$$= \int_0^1 e^{3x} - e^{2x}dx = \frac{1}{3}e^{3x} - \frac{1}{2}e^{2x}\Big|_0^1 = \frac{1}{3}e^3 - \frac{1}{2}e^2 + \frac{1}{6}$$

例 计算 $\iint_D (2x+6y)dxdy$，其中 D 是由直线 $x=0$，$y=1$ 和 $y=x$ 所围成区域。



解

$$\iint_D (2x+6y)dxdy = \int_0^1 \left[\int_x^1 (2x+6y)dy\right]dx$$

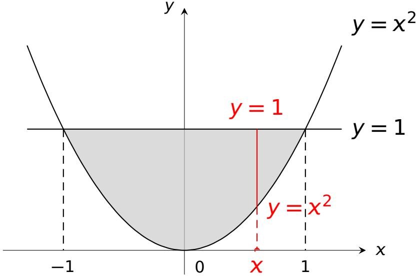
$$= \int_0^1 \left[2xy + 3y^2\Big|_x^1\right]dx = \int_0^1 -5x^2 + 2x + 3dx$$

$$= -\frac{5}{3}x^3 + x^2 + 3x\Big|_0^1 = \frac{7}{3}$$

注 D 是 X-型区域，可以表示为

$$D = \{(x,y) | x \le y \le 1, 0 \le x \le 1\}$$

例 计算 $\iint_D x^2 y dxdy$，其中 D 是由曲线 $y=x^2$ 和直线 $y=1$ 所围成区域。



解

$$\iint_D x^2 y dxdy = \int_{-1}^{1}\Big[\int_{x^2}^{1} x^2 y dy\Big]dx = \int_{-1}^{1}\Big[\frac{1}{2}x^2y^2\Big|_{x^2}^{1}\Big]dx$$

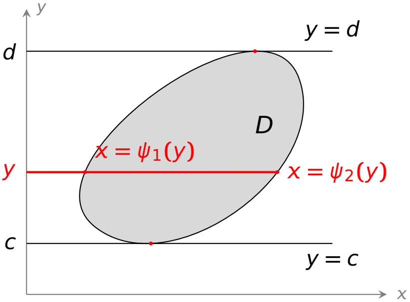
$$= \int_{-1}^{1}\frac{1}{2}x^2(1-x^4)dx = \frac{1}{2}\Big(\frac{1}{3}x^3-\frac{1}{7}x^7\Big)\Big|_{-1}^{1} = \frac{4}{21}$$

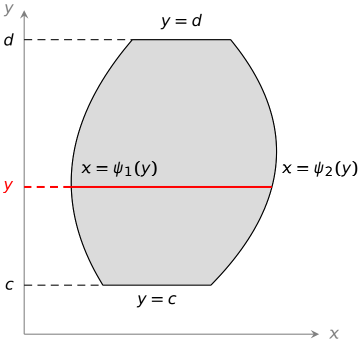
注 D 是 X-型区域，可以表示为

$$D=\{(x,y)|x^2\le y\le 1, -1\le x\le 1\}$$

固定 y，先对 x 积分

$$\iint_D f(x,y)dxdy = \int_c^d \left[\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx\right]dy$$



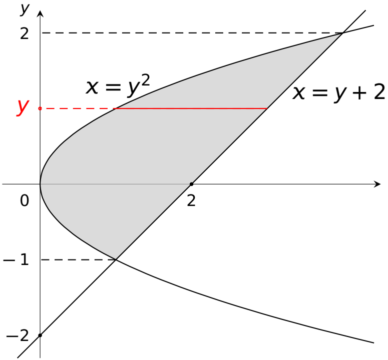


注 上述区域 D 可以表示成

$$D = \{(x,y) | \psi_1(y) \le x \le \psi_2(y), c \le y \le d\}$$

称为 Y-型区域。

例 计算 $\iint_D xydxdy$，其中是由抛物线 $x=y^2$ 和直线 $y=x-2$ 所围成区域。



解

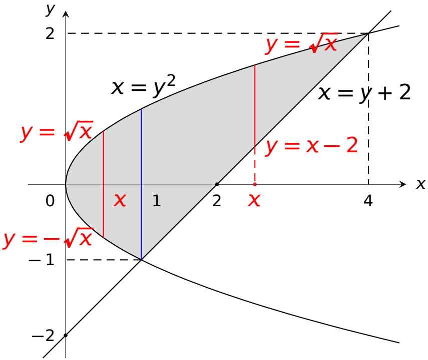
$$\text{原式} = \int_{-1}^{2}\left[\int_{y^2}^{y+2} xydx\right]dy = \int_{-1}^{2}\left[\frac{1}{2}x^2y\Big|_{y^2}^{y+2}\right]dy$$

$$= \int_{-1}^{2}\frac{1}{2}y\left[(y+2)^2-y^4\right]dy = \frac{1}{2}\int_{-1}^{2}-y^5+y^3+4y^2+4ydy = \frac{45}{8}$$

注 D 是 X-型区域，可以表示为

$$D=\{(x,y)|\,x^2\le y\le 1, -1\le x\le 1\}$$

例 计算 $\iint_D xydxdy$，其中是由抛物线 $x=y^2$ 和直线 $y=x-2$ 所围成区域。

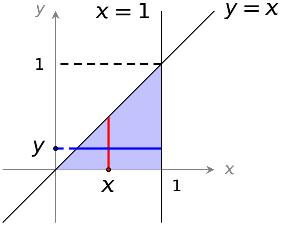


解

$$\text{原式} = \int \left[\int xydy \right] dx$$

$$= \int_0^1 \left[\int_{-\sqrt{x}}^{\sqrt{x}} xydy \right] dx + \int_1^4 \left[\int_{x-2}^{\sqrt{x}} xydy \right] dx = \cdots$$

例 计算 $\iint_D e^{x^2}dxdy$，其中 D 是由 $y=x$，$x=1$，x 轴所围成的区域



解法一 固定 x，先对 y 积分：

$$\iint_D e^{x^2}dxdy = \int_0^1 \left[\int_0^x e^{x^2}dy\right]dx = \int_0^1 \left[e^{x^2}y\Big|_0^x\right]dx$$

$$= \int_0^1 xe^{x^2}dx = \frac{1}{2}e^{x^2}\Big|_0^1 = \frac{1}{2}e - \frac{1}{2}$$

解法二 固定 y，先对 x 积分：

$$\iint_D e^{x^2}dxdy = \int_0^1 \left[\int_y^1 e^{x^2}dx\right]dy = \cdots\cdots \text{积不出}$$

注 选择恰当的积分次序，才能算出二重积分！

交换积分次序

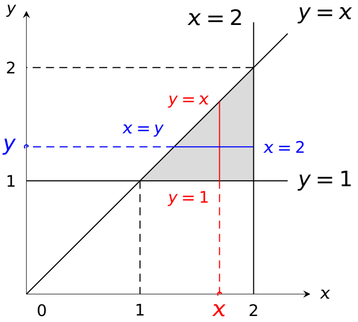
区域 D 同时是

- X-型区域:

  $D = \{(x, y) | 1 \le y \le x, 1 \le x \le 2\}$

- Y-型区域:

  $D = \{(x, y) | y \le x \le 2, 1 \le y \le 2\}$



$$\iint_D f(x,y)dx = \int_1^2 \left[\int_1^x f(x,y)dy\right]dx = \int_0^1 \left[\int_0^y f(x,y)dx\right]dy$$

问题 1. $\int_0^1 \left[\int_0^y f(x,y)dx\right]dy = \int_*^* \left[\int_*^* f(x,y)dy\right]dx,$

2. $\int_1^2 \left[\int_1^x f(x,y)dy\right]dx = \int_*^* \left[\int_*^* f(x,y)dx\right]dy.$

例 补充积分限 1. $\int_1^e\left[\int_0^{\ln x} f(x,y)dy\right]dx = \int_*^*\left[\int_*^* f(x,y)dx\right]dy,$

2. $\int_{-1}^1\left[\int_0^{\sqrt{1-x^2}} f(x,y)dy\right]dx = \int_*^*\left[\int_*^* f(x,y)dx\right]dy.$

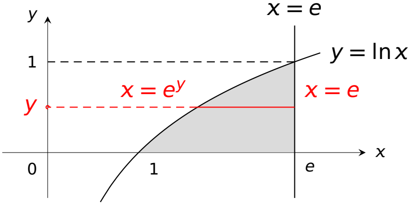
解 1. 因为

$$D=\{(x,y)\,|\,0\le y\le \ln x,\ 1\le x\le e\}$$

所以

$$\int_1^e\left[\int_0^{\ln x} f(x,y)dy\right]dx$$

$$=\int_0^1\left[\int_{e^y}^{e} f(x,y)dx\right]dy$$



例 补充积分限 1. $\int_1^e\left[\int_0^{\ln x} f(x,y)dy\right]dx = \int_*^*\left[\int_*^* f(x,y)dx\right]dy,$

2. $\int_{-1}^1\left[\int_0^{\sqrt{1-x^2}} f(x,y)dy\right]dx = \int_*^*\left[\int_*^* f(x,y)dx\right]dy.$

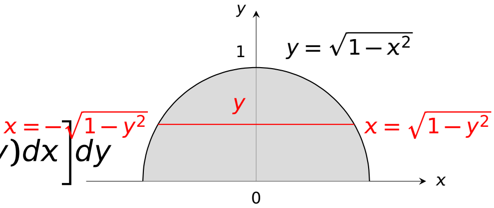
解 2. 因为

$$D = \{(x,y) | 0 \le y \le \sqrt{1-x^2}, -1 \le x \le 1\}$$

所以

$$\int_{-1}^1\left[\int_0^{\sqrt{1-x^2}} f(x,y)dy\right]dx$$

$$= \int_0^1\left[\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)dx\right]dy$$



例 补充积分限 $\int_0^2\left[\int_{y^2}^{2y} f(x,y)dx\right]dy = \int_*^*\left[\int_*^* f(x,y)dy\right]dx.$

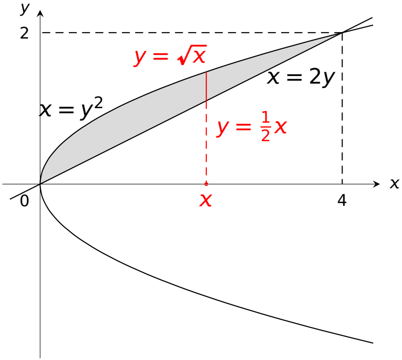
解 因为

$D = \{(x,y) | y^2 \le x \le 2y,\ 0 \le y \le 2\}$

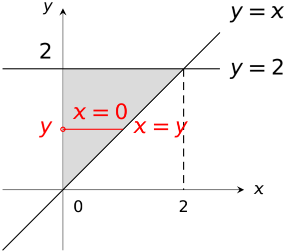
所以

$$\int_0^2\left[\int_{y^2}^{2y} f(x,y)dx\right]dy$$

$$= \int_0^4\left[\int_{\frac{1}{2}x}^{\sqrt{x}} f(x,y)dy\right]dx$$



例 计算 $\int_0^2 \left[\int_x^2 e^{-y^2} dy \right] dx$



解 1. $D = \{(x, y) | x \le y \le 2, 0 \le x \le 2\}$.

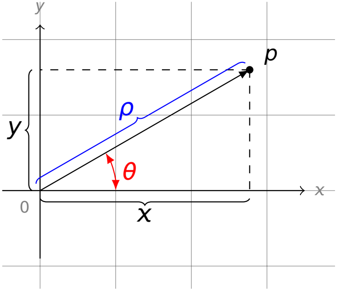
2. 交换积分次序：

$$\int_0^2 \left[\int_x^2 e^{-y^2} dy \right] dx = \iint_D e^{-y^2} dx dy = \int_0^2 \left[\int_0^y e^{-y^2} dx \right] dy$$

$$= \int_0^2 \left[e^{-y^2} x \Big|_0^y \right] dy = \int_0^2 e^{-y^2} y dy = -\frac{1}{2} e^{-y^2} \Big|_0^2$$

$$= \frac{1}{2}(1 - e^{-4})$$

回顾极坐标



- 直角坐标 (x, y)，极坐标 (ρ, θ) 的转换：

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$

- 注
  - 圆周的方程是 $\rho = \rho_0$
  - 射线的方程是 $\theta = \theta_0$

如下情形，不妨引入极坐标：

- 函数 $f(x, y)$ 在极坐标下，能够简化，如

$$f_1(x,y) = e^{-x^2-y^2} = e^{-\rho^2}; \quad f_2(x,y) = \ln(1+x^2+y^2) = \ln(1+\rho^2)$$

$$f_3(x,y) = \sqrt{4a^2 - x^2 - y^2} = \sqrt{4a^2 - \rho^2}$$

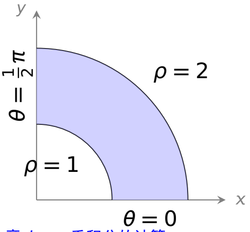
- 点集 D 在极坐标下的表示，显得简单

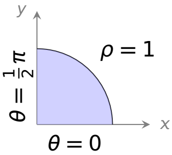
例 用极坐标表示以下的闭区域：

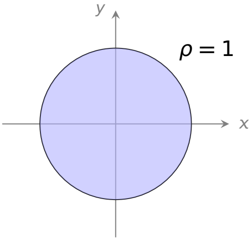
1. D_1 是由圆周 $x^2+y^2=1$ 和 $x^2+y^2=4$ 在第一象限围成的区域
2. D_2 是由圆周 $x^2+y^2=1$ 在第一象限所围成的闭区域
3. D_3 是由圆周 $x^2+y^2=1$ 所围成的闭区域

解

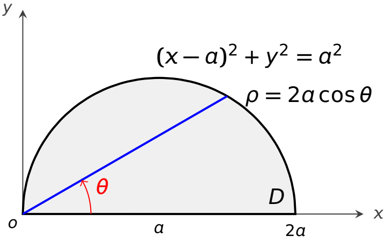
1. $D_1=\{(\rho,\theta)|\,1\le\rho\le 2,\ 0\le\theta\le\frac{\pi}{2}\}$.
2. $D_2=\{(\rho,\theta)|\,0\le\rho\le 1,\ 0\le\theta\le\frac{\pi}{2}\}$.
3. $D_3=\{(\rho,\theta)|\,0\le\rho\le 1,\ 0\le\theta\le 2\pi\}$.







例 用极坐标表示右图区域 D



解 1. 先把圆弧的方程用极坐标改写：

$$(x-a)^2+y^2=a^2 \quad \Rightarrow \quad x^2-2ax+y^2=0$$

$$\xRightarrow[y=\rho\sin\theta]{x=\rho\cos\theta} \quad \rho^2-2a\rho\cos\theta=0$$

$$\Rightarrow \quad \rho=2a\cos\theta$$

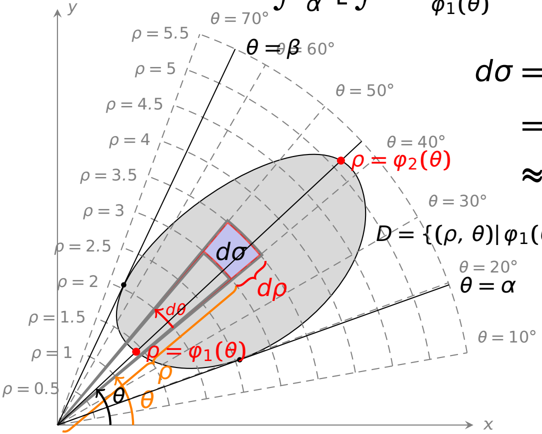
2. 所以

$$D=\{(\rho,\ \theta)\,|\,0\le\rho\le 2a\cos\theta,\ 0\le\theta\le\frac{\pi}{2}\}.$$

极坐标下计算二重积分

$$\iint_D f(x,y)d\sigma \xlongequal[y=\rho\sin\theta]{x=\rho\cos\theta} \iint_D f(\rho\cos\theta,\rho\sin\theta)\rho d\rho d\theta$$

$$= \int_\alpha^\beta \left[\int_{\varphi_1(\theta)}^{\varphi_2(\theta)} f(\rho\cos\theta,\rho\sin\theta)\rho d\rho \right] d\theta$$



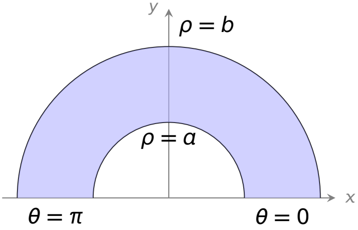
$$d\sigma = \pi(\rho+d\rho)^2\cdot\frac{d\theta}{2\pi} - \pi\rho^2\cdot\frac{d\theta}{2\pi}$$

$$= \rho d\rho d\theta + \frac{1}{2}d\rho^2 d\theta$$

$$\approx \rho d\rho d\theta$$

$$D = \{(\rho,\theta) | \varphi_1(\theta) \le \rho \le \varphi_2(\theta), \alpha \le \theta \le \beta\}$$

例 计算 $\iint_D \sqrt{x^2+y^2}dxdy$，其中区域 D 如右图所示



解 区域 D 用极坐标表示是：

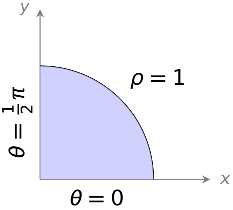
$$D = \{(\rho, \theta) | a \le \rho \le b, 0 \le \theta \le \pi\}$$

所以

$$原式 \overset{x=\rho\cos\theta}{\underset{y=\rho\sin\theta}{=\!=\!=\!=}} \iint_D \rho\cdot\rho d\rho d\theta = \int_0^{\pi}\Big[\int_a^b \rho^2 d\rho\Big]d\theta$$

$$= \pi\left(\frac{1}{3}\rho^3\Big|_a^b\right) = \frac{\pi}{3}(b^3 - a^3)$$

例 计算 $\iint_D \ln(1+x^2+y^2)dxdy$，其中区域 D 如右图所示



解 区域 D 用极坐标表示是:

$$D = \{(\rho,\ \theta) | 0 \le \rho \le 1,\ 0 \le \theta \le \frac{1}{2}\pi\}$$

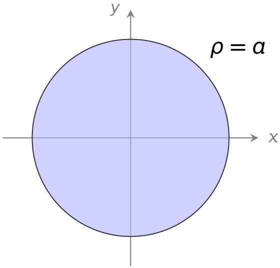
所以

$$原式 \xlongequal[y=\rho\sin\theta]{x=\rho\cos\theta} \iint_D \ln(1+\rho^2)\cdot\rho d\rho d\theta$$

$$= \int_0^{\frac{1}{2}\pi}\Big[\int_0^1 \ln(1+\rho^2)\cdot\rho d\rho\Big]d\theta \xlongequal{u=1+\rho^2} \int_0^{\frac{1}{2}\pi}\Big[\int_1^2 \ln u\cdot\frac{1}{2}du\Big]$$

$$= \frac{\pi}{2}\cdot\frac{1}{2}\Big[u\ln u\Big|_1^2 - \int_1^2 u d\ln u\Big] = \frac{\pi}{2}\cdot\frac{1}{2}\Big[2\ln 2 - 1\Big] = \frac{\pi}{4}(2\ln 2 - 1)$$

例 计算 $\iint_D e^{-x^2-y^2}dxdy$，其中区域 D 如右图所示



解 区域 D 用极坐标表示是：

$$D = \{(\rho, \theta) | \, 0 \le \rho \le a, \, 0 \le \theta \le 2\pi\}$$

所以

$$原式 \xlongequal[y=\rho\sin\theta]{x=\rho\cos\theta} \iint_D e^{-\rho^2} \cdot \rho d\rho d\theta = \int_0^{2\pi} \left[\int_0^a e^{-\rho^2} \cdot \rho d\rho \right] d\theta$$

$$\xlongequal{u=\rho^2} 2\pi \left[\int_0^{a^2} e^{-u} \cdot \frac{1}{2} du \right] = 2\pi \cdot \frac{1}{2} \left[-e^{-u} \Big|_0^{a^2} \right] = (1 - e^{-a^2})\pi$$

例 计算 $\iint_D x^2 dxdy$，其中区域 D 为圆域 $x^2+y^2\le 1$

解法一

原式 $\xlongequal[y=\rho\sin\theta]{x=\rho\cos\theta} \iint_D \rho^2\cos^2\theta\cdot\rho d\rho d\theta = \int_0^{2\pi}\left[\int_0^1 \rho^3\cos^2\theta d\rho\right]d\theta$

$$= \int_0^{2\pi}\cos^2\theta\left[\int_0^1 \rho^3 d\rho\right]d\theta = \left[\int_0^1 \rho^3 d\rho\right]\cdot\left[\int_0^{2\pi}\cos^2\theta d\theta\right]$$

$$= \frac{1}{4}\cdot\left[\int_0^{2\pi}\frac{1}{2}(\cos 2\theta+1)d\theta\right] = \frac{1}{4}\pi$$

解法二 由对称性，$\iint_D x^2 dxdy = \iint_D y^2 dxdy$，所以

$$\iint_D x^2 dxdy = \frac{1}{2}\iint_D (x^2+y^2)dxdy \xlongequal[y=\rho\sin\theta]{x=\rho\cos\theta} \frac{1}{2}\iint_D \rho^2\cdot\rho d\rho d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left[\int_0^1 \rho^3 d\rho\right]d\theta = \pi\cdot\int_0^1 \rho^3 d\rho = \frac{\pi}{4}$$

注 如何根据对称性说明 $\iint_D x^2 dxdy = \iint_D y^2 dxdy$?

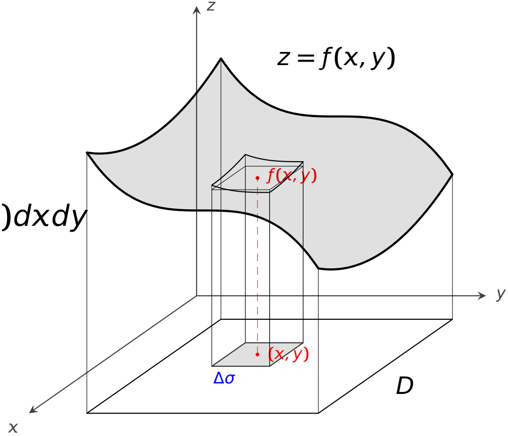
这是:

$$\begin{aligned}\iint_D x^2 dxdy &= \iint_{\{x^2+y^2\le 1\}} x^2 dxdy \\ &= \iint_{\{y^2+x^2\le 1\}} y^2 dydx = \iint_D y^2 dxdy\end{aligned}$$

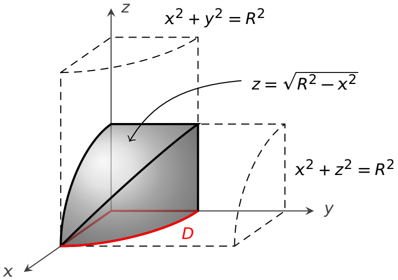
曲顶柱体体积

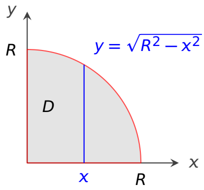
曲顶柱体的体积：

$$V = \iint_D f(x, y)d\sigma = \iint_D f(x, y)dxdy$$



例 求两个底圆半径均为 R 的直交圆柱面所围成的立体体积。



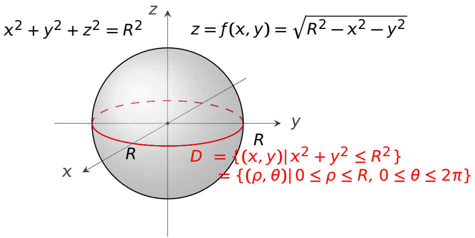


解

$$V=8\iint_D \sqrt{R^2-x^2}\,dxdy=8\int_0^R\left[\int_0^{\sqrt{R^2-x^2}}\sqrt{R^2-x^2}\,dy\right]d$$

$$=8\int_0^R\left[R^2-x^2\right]dx=8(R^2x-\frac{1}{3}x^3)\Big|_0^R=\frac{16}{3}R^3$$

例 求半径为 R 的球的体积。



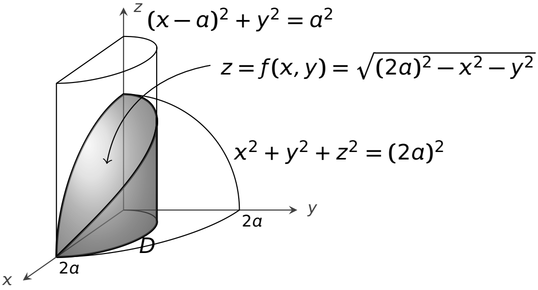
解

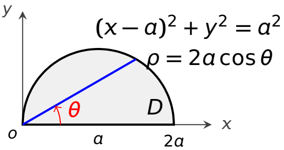
$$V=2\iint_D\sqrt{R^2-x^2-y^2}dxdy \xlongequal[y=\rho\sin\theta]{x=\rho\cos\theta} 2\iint_D\sqrt{R^2-\rho^2}\cdot\rho d\rho d\theta$$

$$=2\int_0^{2\pi}\Big[\int_0^R\sqrt{R^2-\rho^2}\cdot\rho d\rho\Big]d\theta=4\pi\int_0^R\sqrt{R^2-\rho^2}\cdot\rho d\rho$$

$$\xlongequal{u=R^2-\rho^2} 4\pi\int_{R^2}^{0}u^{\frac{1}{2}}\cdot(-\frac{1}{2})du=2\pi\int_0^{R^2}u^{\frac{1}{2}}du=2\pi\cdot\frac{2}{3}u^{\frac{3}{2}}\Big|_0^{R^2}=\frac{4}{3}\pi R$$

例 求球体 $x^2+y^2+z^2\le(2a)^2$ 被圆柱 $(x-a)^2+y^2=a^2$（$a>0$）所截得的立体的体积。





解

$$V=4\iint_D\sqrt{4a^2-x^2-y^2}\,dxdy \xlongequal[y=\rho\sin\theta]{x=\rho\cos\theta} 4\iint_D\sqrt{4a^2-\rho^2}\cdot\rho d\rho d\theta$$

$$=4\int_0^{\frac{\pi}{2}}\left[\int_0^{2a\cos\theta}\sqrt{4a^2-\rho^2}\cdot\rho d\rho\right]d\theta$$

$$V=4\int_0^{\frac{\pi}{2}}\Big[\int_0^{2a\cos\theta}\sqrt{4a^2-\rho^2}\cdot\rho d\rho\Big]d\theta$$

$$\xlongequal{u=4a^2-\rho^2}4\int_0^{\frac{\pi}{2}}\Big[\int_{4a^2}^{4a^2\sin^2\theta}u^{\frac{1}{2}}\cdot(-\frac{1}{2})du\Big]d\theta$$

$$=\frac{4}{3}\int_0^{\frac{\pi}{2}}\Big[u^{\frac{3}{2}}\Big|_{4a^2\sin^2\theta}^{4a^2}\Big]d\theta=\frac{4}{3}\cdot 8a^3\int_0^{\frac{\pi}{2}}(1-\sin^3\theta)d\theta$$

其中

$$\int_0^{\frac{\pi}{2}}\sin^3\theta d\theta=\int_0^{\frac{\pi}{2}}\sin^2\theta\cdot\sin\theta d\theta=-\int_0^{\frac{\pi}{2}}(1-\cos^2\theta)d\cos\theta$$

$$\xlongequal{u=\cos\theta}-\int_1^0(1-u^2)du=-(u-\frac{1}{3}u^3)\Big|_1^0=\frac{2}{3}$$

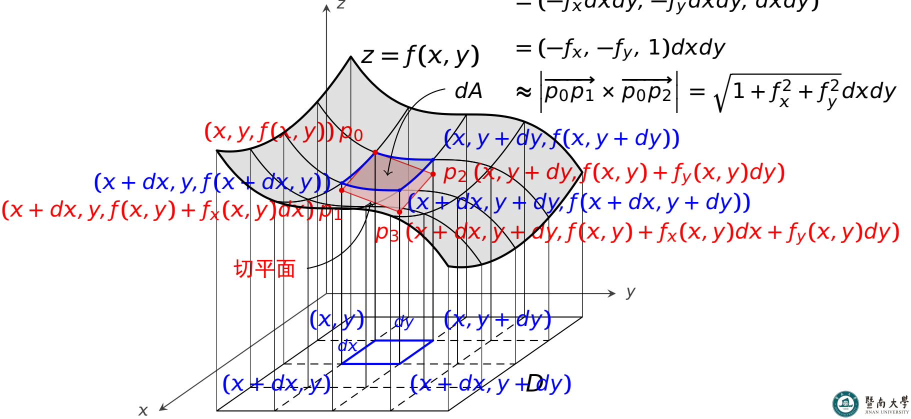
所以 $V=\frac{32}{3}a^3\Big[\frac{\pi}{2}-\frac{2}{3}\Big]$

曲面的面积

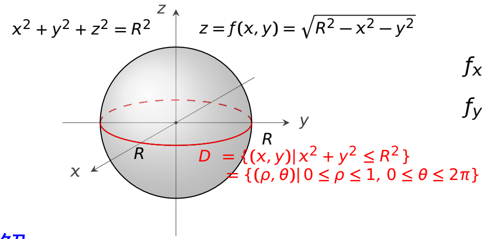
$$A=\iint_D \sqrt{1+f_x(x,y)^2+f_y(x,y)^2}dxdy$$

$$\begin{aligned}
\overrightarrow{p_0p_1}\times\overrightarrow{p_0p_2} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ dx & 0 & f_xdx \\ 0 & dy & f_ydy \end{vmatrix} \\
&= (-f_xdxdy, -f_ydxdy, dxdy) \\
&= (-f_x, -f_y, 1)dxdy
\end{aligned}$$

$$dA \approx \left|\overrightarrow{p_0p_1}\times\overrightarrow{p_0p_2}\right| = \sqrt{1+f_x^2+f_y^2}dxdy$$



例 求半径为 R 的球面的表面积。



$$f_x=\frac{-x}{\sqrt{R^2-x^2-y^2}},\quad f_y=\frac{-y}{\sqrt{R^2-x^2-y^2}}\Rightarrow 1+f_x^2+f_y^2=\frac{R^2}{R^2-x^2-y^2}$$

解

$$A=2\iint_D\sqrt{1+f_x^2+f_y^2}dxdy=2\iint_D\frac{R}{\sqrt{R^2-x^2-y^2}}dxdy$$

$$\xlongequal[y=\rho\sin\theta]{x=\rho\cos\theta}2\iint_D\frac{R}{\sqrt{R^2-\rho^2}}\cdot\rho d\rho d\theta=2\int_0^{2\pi}\Big[\int_0^R\frac{R}{\sqrt{R^2-\rho^2}}\cdot\rho d\rho\Big]$$

$$=4\pi R\int_0^R\frac{\rho}{\sqrt{R^2-\rho^2}}d\rho\xlongequal{u=R^2-\rho^2}4\pi R\int_{R^2}^0 u^{-\frac{1}{2}}\cdot(-\frac{1}{2})du=4\pi R^2$$