

# 第 4 章 c: 实对称矩阵的特征值和特征向量

数学系 梁卓滨

2019-2020 学年 I

# 本节内容

- ◇ 向量的内积
- ♣ 正交向量组，施密特正交化方法
- ♥ 正交矩阵
- ♠ 对称矩阵可对角化

# 向量内积

**定义**  $\mathbb{R}^n$  中两个向量  $\alpha = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$  和  $\beta = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$  的**内积** 定义为:

$$\alpha^T \beta =$$

# 向量内积

**定义**  $\mathbb{R}^n$  中两个向量  $\alpha = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$  和  $\beta = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$  的**内积** 定义为:

$$\alpha^T \beta = (a_1 \ a_2 \ \cdots \ a_n) \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} =$$

# 向量内积

**定义**  $\mathbb{R}^n$  中两个向量  $\alpha = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$  和  $\beta = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$  的**内积** 定义为:

$$\alpha^T \beta = (a_1 \ a_2 \ \cdots \ a_n) \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

# 向量内积

**定义**  $\mathbb{R}^n$  中两个向量  $\alpha = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$  和  $\beta = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$  的**内积** 定义为:

$$\alpha^T \beta = (a_1 \ a_2 \ \cdots \ a_n) \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

**例**  $\mathbb{R}^4$  中两个向量  $\alpha = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 2 \end{pmatrix}$  和  $\beta = \begin{pmatrix} 2 \\ 0 \\ -1 \\ 3 \end{pmatrix}$  的内积是

# 向量内积

**定义**  $\mathbb{R}^n$  中两个向量  $\alpha = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$  和  $\beta = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$  的**内积**定义为:

$$\alpha^T \beta = (a_1 \ a_2 \ \cdots \ a_n) \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

**例**  $\mathbb{R}^4$  中两个向量  $\alpha = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 2 \end{pmatrix}$  和  $\beta = \begin{pmatrix} 2 \\ 0 \\ -1 \\ 3 \end{pmatrix}$  的内积是

$$\alpha^T \beta$$

# 向量内积

**定义**  $\mathbb{R}^n$  中两个向量  $\alpha = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$  和  $\beta = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$  的**内积**定义为:

$$\alpha^T \beta = (a_1 \ a_2 \ \cdots \ a_n) \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

**例**  $\mathbb{R}^4$  中两个向量  $\alpha = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 2 \end{pmatrix}$  和  $\beta = \begin{pmatrix} 2 \\ 0 \\ -1 \\ 3 \end{pmatrix}$  的内积是

$$\alpha^T \beta = (-1 \ 1 \ 0 \ 2) \begin{pmatrix} 2 \\ 0 \\ -1 \\ 3 \end{pmatrix}$$



# 向量内积

**定义**  $\mathbb{R}^n$  中两个向量  $\alpha = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$  和  $\beta = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$  的**内积** 定义为:

$$\alpha^T \beta = (a_1 \ a_2 \ \cdots \ a_n) \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

**例**  $\mathbb{R}^4$  中两个向量  $\alpha = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 2 \end{pmatrix}$  和  $\beta = \begin{pmatrix} 2 \\ 0 \\ -1 \\ 3 \end{pmatrix}$  的内积是

$$\begin{aligned} \alpha^T \beta &= (-1 \ 1 \ 0 \ 2) \begin{pmatrix} 2 \\ 0 \\ -1 \\ 3 \end{pmatrix} \\ &= (-1) \times 2 + 1 \times 0 + 0 \times (-1) + 2 \times 3 \end{aligned}$$

# 向量内积

**定义**  $\mathbb{R}^n$  中两个向量  $\alpha = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$  和  $\beta = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$  的**内积** 定义为:

$$\alpha^T \beta = (a_1 \ a_2 \ \cdots \ a_n) \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

**例**  $\mathbb{R}^4$  中两个向量  $\alpha = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 2 \end{pmatrix}$  和  $\beta = \begin{pmatrix} 2 \\ 0 \\ -1 \\ 3 \end{pmatrix}$  的内积是

$$\begin{aligned} \alpha^T \beta &= (-1 \ 1 \ 0 \ 2) \begin{pmatrix} 2 \\ 0 \\ -1 \\ 3 \end{pmatrix} \\ &= (-1) \times 2 + 1 \times 0 + 0 \times (-1) + 2 \times 3 = 4 \end{aligned}$$

# 内积性质

1.  $\alpha^T \beta = \beta^T \alpha$
2.  $(k\alpha)^T \beta = k\alpha^T \beta$ , ( $k$  是实数)
3.  $(\alpha + \beta)^T \gamma = \alpha^T \gamma + \beta^T \gamma$
4.  $\alpha^T \alpha \geq 0$ , 并且仅当  $\alpha = 0$  时,  $\alpha^T \alpha = 0$

# 内积性质

1.  $\alpha^T \beta = \beta^T \alpha$

# 内积性质

1.  $\alpha^T \beta = \beta^T \alpha$

**证明** 设  $\alpha = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$  和  $\beta = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ , 则

$$\alpha^T \beta =$$

$$\beta^T \alpha =$$

# 内积性质

1.  $\alpha^T \beta = \beta^T \alpha$

**证明** 设  $\alpha = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$  和  $\beta = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ , 则

$$\alpha^T \beta = (a_1 \ a_2 \ \cdots \ a_n) \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n,$$

$$\beta^T \alpha =$$

# 内积性质

1.  $\alpha^T \beta = \beta^T \alpha$

**证明** 设  $\alpha = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$  和  $\beta = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ , 则

$$\alpha^T \beta = (a_1 \ a_2 \ \cdots \ a_n) \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n,$$

$$\beta^T \alpha = (b_1 \ b_2 \ \cdots \ b_n) \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

# 内积性质

1.  $\alpha^T \beta = \beta^T \alpha$

**证明** 设  $\alpha = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$  和  $\beta = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ , 则

$$\alpha^T \beta = (a_1 \ a_2 \ \cdots \ a_n) \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n,$$

$$\beta^T \alpha = (b_1 \ b_2 \ \cdots \ b_n) \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = b_1 a_1 + b_2 a_2 + \cdots + b_n a_n.$$



# 内积性质

1.  $\alpha^T \beta = \beta^T \alpha$

**证明** 设  $\alpha = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$  和  $\beta = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ , 则

$$\alpha^T \beta = (a_1 \ a_2 \ \cdots \ a_n) \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n,$$

$$\beta^T \alpha = (b_1 \ b_2 \ \cdots \ b_n) \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = b_1 a_1 + b_2 a_2 + \cdots + b_n a_n.$$

所以  $\alpha^T \beta = \beta^T \alpha$

# 内积性质

1.  $\alpha^T \beta = \beta^T \alpha$

**证明** 设  $\alpha = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$  和  $\beta = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ , 则

$$\alpha^T \beta = (a_1 \ a_2 \ \cdots \ a_n) \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n,$$

$$\beta^T \alpha = (b_1 \ b_2 \ \cdots \ b_n) \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = b_1 a_1 + b_2 a_2 + \cdots + b_n a_n.$$

所以  $\alpha^T \beta = \beta^T \alpha$

**另证**  $\alpha^T \beta = (\alpha^T \beta)^T =$

# 内积性质

1.  $\alpha^T \beta = \beta^T \alpha$

**证明** 设  $\alpha = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$  和  $\beta = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ , 则

$$\alpha^T \beta = (a_1 \ a_2 \ \cdots \ a_n) \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n,$$

$$\beta^T \alpha = (b_1 \ b_2 \ \cdots \ b_n) \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = b_1 a_1 + b_2 a_2 + \cdots + b_n a_n.$$

所以  $\alpha^T \beta = \beta^T \alpha$

**另证**  $\alpha^T \beta = (\alpha^T \beta)^T = \beta^T (\alpha^T)^T =$

# 内积性质

1.  $\alpha^T \beta = \beta^T \alpha$

**证明** 设  $\alpha = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$  和  $\beta = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ , 则

$$\alpha^T \beta = (a_1 \ a_2 \ \cdots \ a_n) \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n,$$

$$\beta^T \alpha = (b_1 \ b_2 \ \cdots \ b_n) \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = b_1 a_1 + b_2 a_2 + \cdots + b_n a_n.$$

所以  $\alpha^T \beta = \beta^T \alpha$

**另证**  $\alpha^T \beta = (\alpha^T \beta)^T = \beta^T (\alpha^T)^T = \beta^T \alpha$

# 内积性质

- 2.  $(k\alpha)^T\beta = k\alpha^T\beta$ , ( $k$  是实数)
- 3.  $(\alpha + \beta)^T\gamma = \alpha^T\gamma + \beta^T\gamma$
- 4.  $\alpha^T\alpha \geq 0$ , 并且仅当  $\alpha = 0$  时,  $\alpha^T\alpha = 0$

## 证明

# 内积性质

- 2.  $(k\alpha)^T\beta = k\alpha^T\beta$ , ( $k$  是实数)
- 3.  $(\alpha + \beta)^T\gamma = \alpha^T\gamma + \beta^T\gamma$
- 4.  $\alpha^T\alpha \geq 0$ , 并且仅当  $\alpha = 0$  时,  $\alpha^T\alpha = 0$

## 证明

- 2. 显然

# 内积性质

- 2.  $(k\alpha)^T\beta = k\alpha^T\beta$ , ( $k$  是实数)
- 3.  $(\alpha + \beta)^T\gamma = \alpha^T\gamma + \beta^T\gamma$
- 4.  $\alpha^T\alpha \geq 0$ , 并且仅当  $\alpha = 0$  时,  $\alpha^T\alpha = 0$

## 证明

- 2. 显然
- 3.  $(\alpha + \beta)^T\gamma =$

# 内积性质

- 2.  $(k\alpha)^T\beta = k\alpha^T\beta$ , ( $k$  是实数)
- 3.  $(\alpha + \beta)^T\gamma = \alpha^T\gamma + \beta^T\gamma$
- 4.  $\alpha^T\alpha \geq 0$ , 并且仅当  $\alpha = 0$  时,  $\alpha^T\alpha = 0$

## 证明

- 2. 显然
- 3.  $(\alpha + \beta)^T\gamma = (\alpha^T + \beta^T)\gamma =$



# 内积性质

- 2.  $(k\alpha)^T\beta = k\alpha^T\beta$ , ( $k$  是实数)
- 3.  $(\alpha + \beta)^T\gamma = \alpha^T\gamma + \beta^T\gamma$
- 4.  $\alpha^T\alpha \geq 0$ , 并且仅当  $\alpha = 0$  时,  $\alpha^T\alpha = 0$

## 证明

- 2. 显然
- 3.  $(\alpha + \beta)^T\gamma = (\alpha^T + \beta^T)\gamma = \alpha^T\gamma + \beta^T\gamma$

# 内积性质

- 2.  $(k\alpha)^T\beta = k\alpha^T\beta$ , ( $k$  是实数)
- 3.  $(\alpha + \beta)^T\gamma = \alpha^T\gamma + \beta^T\gamma$
- 4.  $\alpha^T\alpha \geq 0$ , 并且仅当  $\alpha = 0$  时,  $\alpha^T\alpha = 0$

## 证明

- 2. 显然
- 3.  $(\alpha + \beta)^T\gamma = (\alpha^T + \beta^T)\gamma = \alpha^T\gamma + \beta^T\gamma$
- 4.  $\alpha^T\alpha = (a_1 \ a_2 \ \cdots \ a_n) \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = a_1^2 + a_2^2 + \cdots + a_n^2$

# 内积性质

- 2.  $(k\alpha)^T\beta = k\alpha^T\beta$ , ( $k$  是实数)
- 3.  $(\alpha + \beta)^T\gamma = \alpha^T\gamma + \beta^T\gamma$
- 4.  $\alpha^T\alpha \geq 0$ , 并且仅当  $\alpha = 0$  时,  $\alpha^T\alpha = 0$

## 证明

- 2. 显然
- 3.  $(\alpha + \beta)^T\gamma = (\alpha^T + \beta^T)\gamma = \alpha^T\gamma + \beta^T\gamma$
- 4.  $\alpha^T\alpha = (a_1 \ a_2 \ \cdots \ a_n) \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = a_1^2 + a_2^2 + \cdots + a_n^2 \geq 0$

# 向量长度

## 定义

$$\|\alpha\| := \sqrt{\alpha^T \alpha} = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

称为向量的 **长度** 或 **范数**。

# 向量长度

## 定义

$$\|\alpha\| := \sqrt{\alpha^T \alpha} = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

称为向量的 **长度** 或 **范数**。

---

**例** 求向量  $\alpha = \begin{pmatrix} -4 \\ -5 \\ 6 \end{pmatrix}$ ,  $\beta = \begin{pmatrix} -1 \\ 3 \\ 1 \\ 5 \end{pmatrix}$  的长度。

# 向量长度

## 定义

$$\|\alpha\| := \sqrt{\alpha^T \alpha} = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

称为向量的 **长度** 或 **范数**。

---

**例** 求向量  $\alpha = \begin{pmatrix} -4 \\ -5 \\ 6 \end{pmatrix}$ ,  $\beta = \begin{pmatrix} -1 \\ 3 \\ 1 \\ 5 \end{pmatrix}$  的长度。

**解**

$$\|\alpha\| =$$

$$\|\beta\| =$$

# 向量长度

## 定义

$$\|\alpha\| := \sqrt{\alpha^T \alpha} = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

称为向量的 **长度** 或 **范数**。

---

**例** 求向量  $\alpha = \begin{pmatrix} -4 \\ -5 \\ 6 \end{pmatrix}$ ,  $\beta = \begin{pmatrix} -1 \\ 3 \\ 1 \\ 5 \end{pmatrix}$  的长度。

**解**

$$\|\alpha\| = \sqrt{(-4)^2 + (-5)^2 + 6^2} =$$

$$\|\beta\| =$$

# 向量长度

## 定义

$$\|\alpha\| := \sqrt{\alpha^T \alpha} = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

称为向量的 **长度** 或 **范数**。

---

**例** 求向量  $\alpha = \begin{pmatrix} -4 \\ -5 \\ 6 \end{pmatrix}$ ,  $\beta = \begin{pmatrix} -1 \\ 3 \\ 1 \\ 5 \end{pmatrix}$  的长度。

## 解

$$\|\alpha\| = \sqrt{(-4)^2 + (-5)^2 + 6^2} = \sqrt{16 + 25 + 36} =$$

$$\|\beta\| =$$



# 向量长度

## 定义

$$\|\alpha\| := \sqrt{\alpha^T \alpha} = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

称为向量的 **长度** 或 **范数**。

---

**例** 求向量  $\alpha = \begin{pmatrix} -4 \\ -5 \\ 6 \end{pmatrix}$ ,  $\beta = \begin{pmatrix} -1 \\ 3 \\ 1 \\ 5 \end{pmatrix}$  的长度。

## 解

$$\|\alpha\| = \sqrt{(-4)^2 + (-5)^2 + 6^2} = \sqrt{16 + 25 + 36} = \sqrt{77}$$

$$\|\beta\| =$$

# 向量长度

## 定义

$$\|\alpha\| := \sqrt{\alpha^T \alpha} = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

称为向量的 **长度** 或 **范数**。

---

**例** 求向量  $\alpha = \begin{pmatrix} -4 \\ -5 \\ 6 \end{pmatrix}$ ,  $\beta = \begin{pmatrix} -1 \\ 3 \\ 1 \\ 5 \end{pmatrix}$  的长度。

## 解

$$\|\alpha\| = \sqrt{(-4)^2 + (-5)^2 + 6^2} = \sqrt{16 + 25 + 36} = \sqrt{77}$$

$$\|\beta\| = \sqrt{(-1)^2 + 3^2 + 1^2 + 5^2} =$$

# 向量长度

## 定义

$$\|\alpha\| := \sqrt{\alpha^T \alpha} = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

称为向量的 **长度** 或 **范数**。

---

**例** 求向量  $\alpha = \begin{pmatrix} -4 \\ -5 \\ 6 \end{pmatrix}$ ,  $\beta = \begin{pmatrix} -1 \\ 3 \\ 1 \\ 5 \end{pmatrix}$  的长度。

## 解

$$\|\alpha\| = \sqrt{(-4)^2 + (-5)^2 + 6^2} = \sqrt{16 + 25 + 36} = \sqrt{77}$$

$$\|\beta\| = \sqrt{(-1)^2 + 3^2 + 1^2 + 5^2} = \sqrt{1 + 9 + 1 + 25} =$$

# 向量长度

## 定义

$$\|\alpha\| := \sqrt{\alpha^T \alpha} = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

称为向量的 **长度** 或 **范数**。

---

**例** 求向量  $\alpha = \begin{pmatrix} -4 \\ -5 \\ 6 \end{pmatrix}$ ,  $\beta = \begin{pmatrix} -1 \\ 3 \\ 1 \\ 5 \end{pmatrix}$  的长度。

## 解

$$\|\alpha\| = \sqrt{(-4)^2 + (-5)^2 + 6^2} = \sqrt{16 + 25 + 36} = \sqrt{77}$$

$$\|\beta\| = \sqrt{(-1)^2 + 3^2 + 1^2 + 5^2} = \sqrt{1 + 9 + 1 + 25} = 6$$

- 向量的 **长度** 或 **范数**

$$\|\alpha\| := \sqrt{\alpha^T \alpha} = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

- 向量的 **长度** 或 **范数**

$$\|\alpha\| := \sqrt{\alpha^T \alpha} = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

## 长度性质

1.  $\|\alpha\| \geq 0$ , 并且仅当  $\alpha = 0$  时,  $\|\alpha\| = 0$

- 向量的 **长度** 或 **范数**

$$\|\alpha\| := \sqrt{\alpha^T \alpha} = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

## 长度性质

1.  $\|\alpha\| \geq 0$ ，并且仅当  $\alpha = 0$  时， $\|\alpha\| = 0$
2.  $\|k\alpha\| = |k| \cdot \|\alpha\|$ ，( $k$  是实数)

- 向量的 **长度** 或 **范数**

$$\|\alpha\| := \sqrt{\alpha^T \alpha} = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

## 长度性质

1.  $\|\alpha\| \geq 0$ , 并且仅当  $\alpha = 0$  时,  $\|\alpha\| = 0$
2.  $\|k\alpha\| = |k| \cdot \|\alpha\|$ , ( $k$  是实数)
3. 对任意向量  $\alpha, \beta$ , 都成立

$$|\alpha^T \beta| \leq \|\alpha\| \cdot \|\beta\|$$



- 向量的 **长度** 或 **范数**

$$\|\alpha\| := \sqrt{\alpha^T \alpha} = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

## 长度性质

1.  $\|\alpha\| \geq 0$ , 并且仅当  $\alpha = 0$  时,  $\|\alpha\| = 0$
2.  $\|k\alpha\| = |k| \cdot \|\alpha\|$ , ( $k$  是实数)
3. 对任意向量  $\alpha, \beta$ , 都成立

$$|\alpha^T \beta| \leq \|\alpha\| \cdot \|\beta\|$$

即

$$|a_1 b_1 + \cdots + a_n b_n| \leq \sqrt{a_1^2 + \cdots + a_n^2} \cdot \sqrt{b_1^2 + \cdots + b_n^2}$$

# 向量单位化

- **定义** 长度为 1 的向量称为 **单位向量**。

# 向量单位化

- **定义** 长度为 1 的向量称为 **单位向量**。
- **例** 向量

$$\alpha = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \quad \beta = \begin{pmatrix} 1/3 \\ 2/3 \\ -2/3 \end{pmatrix}, \quad \varepsilon_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i\text{-th}$$

都是单位向量

# 向量单位化

- **定义** 长度为 1 的向量称为 **单位向量**。

- **例** 向量

$$\alpha = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \quad \beta = \begin{pmatrix} 1/3 \\ 2/3 \\ -2/3 \end{pmatrix}, \quad \varepsilon_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i\text{-th}$$

都是单位向量

- 设  $\alpha \neq 0$ , 则  $\|\alpha\| \neq 0$ ,

# 向量单位化

- **定义** 长度为 1 的向量称为 **单位向量**。

- **例** 向量

$$\alpha = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \quad \beta = \begin{pmatrix} 1/3 \\ 2/3 \\ -2/3 \end{pmatrix}, \quad \varepsilon_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i\text{-th}$$

都是单位向量

- 设  $\alpha \neq 0$ , 则  $\|\alpha\| \neq 0$ , 向量  $\frac{1}{\|\alpha\|} \alpha$  是单位向量:

# 向量单位化

- **定义** 长度为 1 的向量称为 **单位向量**。

- **例** 向量

$$\alpha = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \quad \beta = \begin{pmatrix} 1/3 \\ 2/3 \\ -2/3 \end{pmatrix}, \quad \varepsilon_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i\text{-th}$$

都是单位向量

- 设  $\alpha \neq 0$ , 则  $\|\alpha\| \neq 0$ , 向量  $\frac{1}{\|\alpha\|}\alpha$  是单位向量:

$$\left\| \frac{1}{\|\alpha\|} \alpha \right\| = \frac{1}{\|\alpha\|} \|\alpha\| = 1$$

# 向量单位化

- **定义** 长度为 1 的向量称为 **单位向量**。

- **例** 向量

$$\alpha = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \quad \beta = \begin{pmatrix} 1/3 \\ 2/3 \\ -2/3 \end{pmatrix}, \quad \varepsilon_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i\text{-th}$$

都是单位向量

- 设  $\alpha \neq 0$ , 则  $\|\alpha\| \neq 0$ , 向量  $\frac{1}{\|\alpha\|}\alpha$  是单位向量:

$$\left\| \frac{1}{\|\alpha\|}\alpha \right\| = \frac{1}{\|\alpha\|}\|\alpha\| = 1$$

称  $\frac{1}{\|\alpha\|}\alpha$  为  $\alpha$  的 **单位化**

**例** 将下列向量单位化

$$\alpha = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \beta = \begin{pmatrix} 2 \\ 2 \\ 4 \\ 5 \end{pmatrix}$$



**例** 将下列向量单位化

$$\alpha = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \beta = \begin{pmatrix} 2 \\ 2 \\ 4 \\ 5 \end{pmatrix}$$

**解**

1.  $\|\alpha\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14},$

**例** 将下列向量单位化  $\alpha = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \beta = \begin{pmatrix} 2 \\ 2 \\ 4 \\ 5 \end{pmatrix}$

**解**

1.  $\|\alpha\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$ , 所以的  $\alpha$  单位化为:

$$\frac{1}{\|\alpha\|} \alpha = \frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{pmatrix}$$

**例** 将下列向量单位化  $\alpha = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \beta = \begin{pmatrix} 2 \\ 2 \\ 4 \\ 5 \end{pmatrix}$

**解**

1.  $\|\alpha\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$ , 所以的  $\alpha$  单位化为:

$$\frac{1}{\|\alpha\|} \alpha = \frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{pmatrix}$$

2.  $\|\beta\| = \sqrt{2^2 + 2^2 + 4^2 + 5^2} = \sqrt{49} = 7$ ,

例 将下列向量单位化

$$\alpha = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \beta = \begin{pmatrix} 2 \\ 2 \\ 4 \\ 5 \end{pmatrix}$$

解

1.  $\|\alpha\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$ , 所以的  $\alpha$  单位化为:

$$\frac{1}{\|\alpha\|} \alpha = \frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{pmatrix}$$

2.  $\|\beta\| = \sqrt{2^2 + 2^2 + 4^2 + 5^2} = \sqrt{49} = 7$ , 所以的  $\beta$  单位化为:

$$\frac{1}{\|\beta\|} \beta = \frac{1}{7} \begin{pmatrix} 2 \\ 2 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 2/7 \\ 2/7 \\ 4/7 \\ 5/7 \end{pmatrix}$$

# 向量正交

**定义** 若  $\alpha^T \beta = 0$ ，则称  $\alpha, \beta$  **正交**（或**垂直**）

# 向量正交

**定义** 若  $\alpha^T \beta = 0$ ，则称  $\alpha, \beta$  **正交**（或**垂直**）

**例** 零向量与任意向量正交：

$$0^T \alpha$$

## 向量正交

**定义** 若  $\alpha^T \beta = 0$ , 则称  $\alpha, \beta$  **正交** (或 **垂直**)

**例** 零向量与任意向量正交:

$$0^T \alpha = 0 \cdot a_1 + 0 \cdot a_2 + \cdots + 0 \cdot a_n = 0$$

# 向量正交

**定义** 若  $\alpha^T \beta = 0$ , 则称  $\alpha, \beta$  **正交** (或 **垂直**)

**例** 零向量与任意向量正交:

$$0^T \alpha = 0 \cdot a_1 + 0 \cdot a_2 + \cdots + 0 \cdot a_n = 0$$

**例**  $\alpha = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$  与  $\beta = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$  正交:



# 向量正交

**定义** 若  $\alpha^T \beta = 0$ , 则称  $\alpha, \beta$  **正交** (或 **垂直**)

**例** 零向量与任意向量正交:

$$0^T \alpha = 0 \cdot a_1 + 0 \cdot a_2 + \cdots + 0 \cdot a_n = 0$$

**例**  $\alpha = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$  与  $\beta = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$  正交:

$$\alpha^T \beta = 2 \times 1 + 4 \times 2 + 5 \times (-2) = 0$$

## 向量正交

**定义** 若  $\alpha^T \beta = 0$ , 则称  $\alpha, \beta$  **正交** (或 **垂直**)

**例** 零向量与任意向量正交:

$$0^T \alpha = 0 \cdot a_1 + 0 \cdot a_2 + \cdots + 0 \cdot a_n = 0$$

**例**  $\alpha = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$  与  $\beta = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$  正交:

$$\alpha^T \beta = 2 \times 1 + 4 \times 2 + 5 \times (-2) = 0$$

**例** 向量组  $\varepsilon_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ ,  $\varepsilon_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ ,  $\varepsilon_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$  中的向量两两正交:

## 向量正交

**定义** 若  $\alpha^T \beta = 0$ , 则称  $\alpha, \beta$  **正交** (或**垂直**)

**例** 零向量与任意向量正交:

$$0^T \alpha = 0 \cdot a_1 + 0 \cdot a_2 + \cdots + 0 \cdot a_n = 0$$

**例**  $\alpha = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$  与  $\beta = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$  正交:

$$\alpha^T \beta = 2 \times 1 + 4 \times 2 + 5 \times (-2) = 0$$

**例** 向量组  $\varepsilon_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ ,  $\varepsilon_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ ,  $\varepsilon_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$  中的向量两两正交:

$$\varepsilon_1^T \varepsilon_2 = 0, \quad \varepsilon_1^T \varepsilon_3 = 0, \quad \varepsilon_2^T \varepsilon_3 = 0$$

# 正交向量组

**定义** 若  $\mathbb{R}^n$  中向量组  $\alpha_1, \alpha_2, \dots, \alpha_s$  满足

1. 每个向量非零:  $\alpha_i \neq 0, i = 1, 2, \dots, s$
2. 两两正交:  $\alpha_i^T \alpha_j = 0, i \neq j$

即则称该向量组为 **正交向量组**。

# 正交向量组

**定义** 若  $\mathbb{R}^n$  中向量组  $\alpha_1, \alpha_2, \dots, \alpha_s$  满足

1. 每个向量非零:  $\alpha_i \neq 0, i = 1, 2, \dots, s$
2. 两两正交:  $\alpha_i^T \alpha_j = 0, i \neq j$

即则称该向量组为 **正交向量组**。

---

**例** 向量组  $\varepsilon_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \varepsilon_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \varepsilon_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$  是正交向量组

# 正交向量组

**定义** 若  $\mathbb{R}^n$  中向量组  $\alpha_1, \alpha_2, \dots, \alpha_s$  满足

1. 每个向量非零:  $\alpha_i \neq 0, i = 1, 2, \dots, s$
2. 两两正交:  $\alpha_i^T \alpha_j = 0, i \neq j$

即则称该向量组为 **正交向量组**。

---

**例** 向量组  $\varepsilon_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \varepsilon_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \varepsilon_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$  是正交向量组

**例** 向量组  $\alpha = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \beta = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}, \gamma = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$

# 正交向量组

**定义** 若  $\mathbb{R}^n$  中向量组  $\alpha_1, \alpha_2, \dots, \alpha_s$  满足

1. 每个向量非零:  $\alpha_i \neq 0, i = 1, 2, \dots, s$
2. 两两正交:  $\alpha_i^T \alpha_j = 0, i \neq j$

即则称该向量组为 **正交向量组**。

---

**例** 向量组  $\varepsilon_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \varepsilon_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \varepsilon_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$  是正交向量组

**例** 向量组  $\alpha = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \beta = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}, \gamma = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$  是正交向量组

**定理**  $\mathbb{R}^n$  中正交向量组  $\alpha_1, \alpha_2, \dots, \alpha_s$  一定线性无关。



**定理**  $\mathbb{R}^n$  中正交向量组  $\alpha_1, \alpha_2, \dots, \alpha_s$  一定线性无关。

**证明** 设

$$k_1\alpha_1 + k_2\alpha_2 + \cdots + k_s\alpha_s = 0$$

$$k_1 = k_2 = \cdots = k_s = 0$$

**定理**  $\mathbb{R}^n$  中正交向量组  $\alpha_1, \alpha_2, \dots, \alpha_s$  一定线性无关。

**证明** 设

$$k_1\alpha_1 + k_2\alpha_2 + \cdots + k_s\alpha_s = 0$$

则

$$0 = \alpha_i^T (k_1\alpha_1 + k_2\alpha_2 + \cdots + k_i\alpha_i + \cdots + k_s\alpha_s)$$

$$k_1 = k_2 = \cdots = k_s = 0$$

**定理**  $\mathbb{R}^n$  中正交向量组  $\alpha_1, \alpha_2, \dots, \alpha_s$  一定线性无关。

**证明** 设

$$k_1\alpha_1 + k_2\alpha_2 + \cdots + k_s\alpha_s = 0$$

则

$$0 = \alpha_i^T (k_1\alpha_1 + k_2\alpha_2 + \cdots + k_i\alpha_i + \cdots + k_s\alpha_s) \underline{\underline{\alpha_i^T \alpha_j = 0 \text{ for } i \neq j}}$$

$$k_1 = k_2 = \cdots = k_s = 0$$

**定理**  $\mathbb{R}^n$  中正交向量组  $\alpha_1, \alpha_2, \dots, \alpha_s$  一定线性无关。

**证明** 设

$$k_1\alpha_1 + k_2\alpha_2 + \cdots + k_s\alpha_s = 0$$

则

$$0 = \alpha_i^T (k_1\alpha_1 + k_2\alpha_2 + \cdots + k_i\alpha_i + \cdots + k_s\alpha_s) \xrightarrow{\alpha_i^T \alpha_j = 0 \text{ for } i \neq j} k_i \alpha_i^T \alpha_i$$

$$k_1 = k_2 = \cdots = k_s = 0$$

**定理**  $\mathbb{R}^n$  中正交向量组  $\alpha_1, \alpha_2, \dots, \alpha_s$  一定线性无关。

**证明** 设

$$k_1\alpha_1 + k_2\alpha_2 + \cdots + k_s\alpha_s = 0$$

则

$$0 = \alpha_i^T (k_1\alpha_1 + k_2\alpha_2 + \cdots + k_i\alpha_i + \cdots + k_s\alpha_s) \xrightarrow{\alpha_i^T \alpha_j = 0 \text{ for } i \neq j} k_i \underbrace{\alpha_i^T \alpha_i}_{\neq 0}$$

$$k_1 = k_2 = \cdots = k_s = 0$$

**定理**  $\mathbb{R}^n$  中正交向量组  $\alpha_1, \alpha_2, \dots, \alpha_s$  一定线性无关。

**证明** 设

$$k_1\alpha_1 + k_2\alpha_2 + \cdots + k_s\alpha_s = 0$$

则

$$0 = \alpha_i^T (k_1\alpha_1 + k_2\alpha_2 + \cdots + k_i\alpha_i + \cdots + k_s\alpha_s) \xrightarrow{\alpha_i^T\alpha_j=0 \text{ for } i \neq j} k_i \underbrace{\alpha_i^T \alpha_i}_{\neq 0}$$

所以  $k_i = 0$ 。

$$k_1 = k_2 = \cdots = k_s = 0$$

**定理**  $\mathbb{R}^n$  中正交向量组  $\alpha_1, \alpha_2, \dots, \alpha_s$  一定线性无关。

**证明** 设

$$k_1\alpha_1 + k_2\alpha_2 + \cdots + k_s\alpha_s = 0$$

则

$$0 = \alpha_i^T (k_1\alpha_1 + k_2\alpha_2 + \cdots + k_i\alpha_i + \cdots + k_s\alpha_s) \xrightarrow{\alpha_i^T \alpha_j = 0 \text{ for } i \neq j} k_i \underbrace{\alpha_i^T \alpha_i}_{\neq 0}$$

所以  $k_i = 0$ 。由  $i$  的任意性

$$k_1 = k_2 = \cdots = k_s = 0$$

# 正交化

$\alpha_1, \alpha_2, \dots, \alpha_s$  (线性无关)  $\longrightarrow \beta_1, \beta_2, \dots, \beta_s$  (等价, 两两正交)



# 正交化

$\alpha_1, \alpha_2, \dots, \alpha_s$ (线性无关)  $\xrightarrow{\text{正交化}}$   $\beta_1, \beta_2, \dots, \beta_s$ (等价, 两两正交)

# 正交化

$\alpha_1, \alpha_2, \dots, \alpha_s$  (线性无关)  $\xrightarrow{\text{正交化}}$   $\beta_1, \beta_2, \dots, \beta_s$  (等价, 两两正交)

实现正交化步骤 (施密特正交化方法):

$$\beta_1 =$$

$$\beta_2 =$$

$$\beta_3 =$$

$$\vdots$$

$$\beta_s =$$

# 正交化

$\alpha_1, \alpha_2, \dots, \alpha_s$  (线性无关)  $\xrightarrow{\text{正交化}}$   $\beta_1, \beta_2, \dots, \beta_s$  (等价, 两两正交)

实现正交化步骤 (施密特正交化方法):

$$\beta_1 = \alpha_1$$

$$\beta_2 =$$

$$\beta_3 =$$

$$\vdots$$

$$\beta_s =$$

# 正交化

$\alpha_1, \alpha_2, \dots, \alpha_s$  (线性无关)  $\xrightarrow{\text{正交化}}$   $\beta_1, \beta_2, \dots, \beta_s$  (等价, 两两正交)

实现正交化步骤 (施密特正交化方法):

$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \text{——} \beta_1$$

$$\beta_3 =$$

$$\vdots$$

$$\beta_s =$$

# 正交化

$\alpha_1, \alpha_2, \dots, \alpha_s$  (线性无关)  $\xrightarrow{\text{正交化}}$   $\beta_1, \beta_2, \dots, \beta_s$  (等价, 两两正交)

实现正交化步骤 (施密特正交化方法):

$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \frac{\langle \alpha_2, \beta_1 \rangle}{\|\beta_1\|^2} \beta_1$$

$$\beta_3 =$$

$$\vdots$$

$$\beta_s =$$

# 正交化

$\alpha_1, \alpha_2, \dots, \alpha_s$  (线性无关)  $\xrightarrow{\text{正交化}}$   $\beta_1, \beta_2, \dots, \beta_s$  (等价, 两两正交)

实现正交化步骤 (施密特正交化方法):

$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2^T \beta_1}{\|\beta_1\|^2} \beta_1$$

$$\beta_3 =$$

$$\vdots$$

$$\beta_s =$$

# 正交化

$\alpha_1, \alpha_2, \dots, \alpha_s$  (线性无关)  $\xrightarrow{\text{正交化}}$   $\beta_1, \beta_2, \dots, \beta_s$  (等价, 两两正交)

实现正交化步骤 (施密特正交化方法):

$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2^T \beta_1}{\|\beta_1\|^2} \beta_1$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3^T \beta_1}{\|\beta_1\|^2} \beta_1 - \frac{\alpha_3^T \beta_2}{\|\beta_2\|^2} \beta_2$$

$$\vdots$$

$$\beta_s =$$

# 正交化

$\alpha_1, \alpha_2, \dots, \alpha_s$  (线性无关)  $\xrightarrow{\text{正交化}}$   $\beta_1, \beta_2, \dots, \beta_s$  (等价, 两两正交)

实现正交化步骤 (施密特正交化方法):

$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2^T \beta_1}{\|\beta_1\|^2} \beta_1$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3^T \beta_1}{\|\beta_1\|^2} \beta_1 - \frac{\alpha_3^T \beta_2}{\|\beta_2\|^2} \beta_2$$

$$\vdots$$

$$\beta_s =$$



# 正交化

$\alpha_1, \alpha_2, \dots, \alpha_s$  (线性无关)  $\xrightarrow{\text{正交化}}$   $\beta_1, \beta_2, \dots, \beta_s$  (等价, 两两正交)

实现正交化步骤 (施密特正交化方法):

$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2^T \beta_1}{\|\beta_1\|^2} \beta_1$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3^T \beta_1}{\|\beta_1\|^2} \beta_1 - \frac{\alpha_3^T \beta_2}{\|\beta_2\|^2} \beta_2$$

$$\vdots$$

$$\beta_s =$$

# 正交化

$\alpha_1, \alpha_2, \dots, \alpha_s$  (线性无关)  $\xrightarrow{\text{正交化}}$   $\beta_1, \beta_2, \dots, \beta_s$  (等价, 两两正交)

实现正交化步骤 (施密特正交化方法):

$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2^T \beta_1}{\|\beta_1\|^2} \beta_1$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3^T \beta_1}{\|\beta_1\|^2} \beta_1 - \frac{\alpha_3^T \beta_2}{\|\beta_2\|^2} \beta_2$$

$$\vdots$$

$$\beta_s =$$

# 正交化

$\alpha_1, \alpha_2, \dots, \alpha_s$  (线性无关)  $\xrightarrow{\text{正交化}}$   $\beta_1, \beta_2, \dots, \beta_s$  (等价, 两两正交)

实现正交化步骤 (施密特正交化方法):

$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2^T \beta_1}{\|\beta_1\|^2} \beta_1$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3^T \beta_1}{\|\beta_1\|^2} \beta_1 - \frac{\alpha_3^T \beta_2}{\|\beta_2\|^2} \beta_2$$

$\vdots$

$$\beta_s =$$

# 正交化

$\alpha_1, \alpha_2, \dots, \alpha_s$  (线性无关)  $\xrightarrow{\text{正交化}}$   $\beta_1, \beta_2, \dots, \beta_s$  (等价, 两两正交)

实现正交化步骤 (施密特正交化方法):

$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2^T \beta_1}{\|\beta_1\|^2} \beta_1$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3^T \beta_1}{\|\beta_1\|^2} \beta_1 - \frac{\alpha_3^T \beta_2}{\|\beta_2\|^2} \beta_2$$

$\vdots$

$$\beta_s = \alpha_s - \frac{\alpha_s^T \beta_1}{\|\beta_1\|^2} \beta_1 - \frac{\alpha_s^T \beta_2}{\|\beta_2\|^2} \beta_2 - \dots - \frac{\alpha_s^T \beta_{s-1}}{\|\beta_{s-1}\|^2} \beta_{s-1}$$

# 正交化

$\alpha_1, \alpha_2, \dots, \alpha_s$  (线性无关)  $\xrightarrow{\text{正交化}}$   $\beta_1, \beta_2, \dots, \beta_s$  (等价, 两两正交)

实现正交化步骤 (施密特正交化方法):

$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2^T \beta_1}{\|\beta_1\|^2} \beta_1$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3^T \beta_1}{\|\beta_1\|^2} \beta_1 - \frac{\alpha_3^T \beta_2}{\|\beta_2\|^2} \beta_2$$

$\vdots$

$$\beta_s = \alpha_s - \frac{\alpha_s^T \beta_1}{\|\beta_1\|^2} \beta_1 - \frac{\alpha_s^T \beta_2}{\|\beta_2\|^2} \beta_2 - \dots - \frac{\alpha_s^T \beta_{s-1}}{\|\beta_{s-1}\|^2} \beta_{s-1}$$

# 正交化

$\alpha_1, \alpha_2, \dots, \alpha_s$  (线性无关)  $\xrightarrow{\text{正交化}}$   $\beta_1, \beta_2, \dots, \beta_s$  (等价, 两两正交)

实现正交化步骤 (施密特正交化方法):

$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2^T \beta_1}{\|\beta_1\|^2} \beta_1$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3^T \beta_1}{\|\beta_1\|^2} \beta_1 - \frac{\alpha_3^T \beta_2}{\|\beta_2\|^2} \beta_2$$

$\vdots$

$$\beta_s = \alpha_s - \frac{\alpha_s^T \beta_1}{\|\beta_1\|^2} \beta_1 - \frac{\alpha_s^T \beta_2}{\|\beta_2\|^2} \beta_2 - \dots - \frac{\alpha_s^T \beta_{s-1}}{\|\beta_{s-1}\|^2} \beta_{s-1}$$

# 正交化

$\alpha_1, \alpha_2, \dots, \alpha_s$ (线性无关)  $\xrightarrow{\text{正交化}}$   $\beta_1, \beta_2, \dots, \beta_s$ (等价, 两两正交)

实现正交化步骤 (施密特正交化方法):

$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2^T \beta_1}{\|\beta_1\|^2} \beta_1$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3^T \beta_1}{\|\beta_1\|^2} \beta_1 - \frac{\alpha_3^T \beta_2}{\|\beta_2\|^2} \beta_2$$

$\vdots$

$$\beta_s = \alpha_s - \frac{\alpha_s^T \beta_1}{\|\beta_1\|^2} \beta_1 - \frac{\alpha_s^T \beta_2}{\|\beta_2\|^2} \beta_2 - \dots - \frac{\alpha_s^T \beta_{s-1}}{\|\beta_{s-1}\|^2} \beta_{s-1}$$

# 正交化

$\alpha_1, \alpha_2, \dots, \alpha_s$ (线性无关)  $\xrightarrow{\text{正交化}}$   $\beta_1, \beta_2, \dots, \beta_s$ (等价, 两两正交)

实现正交化步骤 (施密特正交化方法):

$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2^T \beta_1}{\|\beta_1\|^2} \beta_1$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3^T \beta_1}{\|\beta_1\|^2} \beta_1 - \frac{\alpha_3^T \beta_2}{\|\beta_2\|^2} \beta_2$$

$\vdots$

$$\beta_s = \alpha_s - \frac{\alpha_s^T \beta_1}{\|\beta_1\|^2} \beta_1 - \frac{\alpha_s^T \beta_2}{\|\beta_2\|^2} \beta_2 - \dots - \frac{\alpha_s^T \beta_{s-1}}{\|\beta_{s-1}\|^2} \beta_{s-1}$$



# 正交化

$\alpha_1, \alpha_2, \dots, \alpha_s$ (线性无关)  $\xrightarrow{\text{正交化}}$   $\beta_1, \beta_2, \dots, \beta_s$ (等价, 两两正交)

实现正交化步骤 (施密特正交化方法):

$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2^T \beta_1}{\|\beta_1\|^2} \beta_1$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3^T \beta_1}{\|\beta_1\|^2} \beta_1 - \frac{\alpha_3^T \beta_2}{\|\beta_2\|^2} \beta_2$$

$\vdots$

$$\beta_s = \alpha_s - \frac{\alpha_s^T \beta_1}{\|\beta_1\|^2} \beta_1 - \frac{\alpha_s^T \beta_2}{\|\beta_2\|^2} \beta_2 - \dots - \frac{\alpha_s^T \beta_{s-1}}{\|\beta_{s-1}\|^2} \beta_{s-1}$$

# 正交化

$\alpha_1, \alpha_2, \dots, \alpha_s$  (线性无关)  $\xrightarrow{\text{正交化}}$   $\beta_1, \beta_2, \dots, \beta_s$  (等价, 两两正交)

实现正交化步骤 (施密特正交化方法):

$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2^T \beta_1}{\|\beta_1\|^2} \beta_1$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3^T \beta_1}{\|\beta_1\|^2} \beta_1 - \frac{\alpha_3^T \beta_2}{\|\beta_2\|^2} \beta_2$$

$\vdots$

$$\beta_s = \alpha_s - \frac{\alpha_s^T \beta_1}{\|\beta_1\|^2} \beta_1 - \frac{\alpha_s^T \beta_2}{\|\beta_2\|^2} \beta_2 - \dots - \frac{\alpha_s^T \beta_{s-1}}{\|\beta_{s-1}\|^2} \beta_{s-1}$$

# 正交化

$\alpha_1, \alpha_2, \dots, \alpha_s$  (线性无关)  $\xrightarrow{\text{正交化}}$   $\beta_1, \beta_2, \dots, \beta_s$  (等价, 两两正交)

实现正交化步骤 (施密特正交化方法):

$$\beta_1 = \alpha_1$$

$$\beta_1^T \beta_2 = \beta_1^T \left( \alpha_2 - \frac{\alpha_2^T \beta_1}{\|\beta_1\|^2} \beta_1 \right)$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3^T \beta_1}{\|\beta_1\|^2} \beta_1 - \frac{\alpha_3^T \beta_2}{\|\beta_2\|^2} \beta_2$$

$\vdots$

$$\beta_s = \alpha_s - \frac{\alpha_s^T \beta_1}{\|\beta_1\|^2} \beta_1 - \frac{\alpha_s^T \beta_2}{\|\beta_2\|^2} \beta_2 - \dots - \frac{\alpha_s^T \beta_{s-1}}{\|\beta_{s-1}\|^2} \beta_{s-1}$$

# 正交化

$\alpha_1, \alpha_2, \dots, \alpha_s$  (线性无关)  $\xrightarrow{\text{正交化}}$   $\beta_1, \beta_2, \dots, \beta_s$  (等价, 两两正交)

实现正交化步骤 (施密特正交化方法):

$$\beta_1 = \alpha_1$$

$$\beta_1^T \beta_2 = \beta_1^T \left( \alpha_2 - \frac{\alpha_2^T \beta_1}{\|\beta_1\|^2} \beta_1 \right) = \beta_1^T \alpha_2 - \frac{\alpha_2^T \beta_1}{\|\beta_1\|^2} \beta_1^T \beta_1$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3^T \beta_1}{\|\beta_1\|^2} \beta_1 - \frac{\alpha_3^T \beta_2}{\|\beta_2\|^2} \beta_2$$

$\vdots$

$$\beta_s = \alpha_s - \frac{\alpha_s^T \beta_1}{\|\beta_1\|^2} \beta_1 - \frac{\alpha_s^T \beta_2}{\|\beta_2\|^2} \beta_2 - \dots - \frac{\alpha_s^T \beta_{s-1}}{\|\beta_{s-1}\|^2} \beta_{s-1}$$

# 正交化

$\alpha_1, \alpha_2, \dots, \alpha_s$  (线性无关)  $\xrightarrow{\text{正交化}}$   $\beta_1, \beta_2, \dots, \beta_s$  (等价, 两两正交)

实现正交化步骤 (施密特正交化方法):

$$\beta_1 = \alpha_1$$

$$\beta_1^T \beta_2 = \beta_1^T \left( \alpha_2 - \frac{\alpha_2^T \beta_1}{\|\beta_1\|^2} \beta_1 \right) = \beta_1^T \alpha_2 - \frac{\alpha_2^T \beta_1}{\|\beta_1\|^2} \beta_1^T \beta_1 = 0$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3^T \beta_1}{\|\beta_1\|^2} \beta_1 - \frac{\alpha_3^T \beta_2}{\|\beta_2\|^2} \beta_2$$

$\vdots$

$$\beta_s = \alpha_s - \frac{\alpha_s^T \beta_1}{\|\beta_1\|^2} \beta_1 - \frac{\alpha_s^T \beta_2}{\|\beta_2\|^2} \beta_2 - \dots - \frac{\alpha_s^T \beta_{s-1}}{\|\beta_{s-1}\|^2} \beta_{s-1}$$

# 正交化

$\alpha_1, \alpha_2, \dots, \alpha_s$  (线性无关)  $\xrightarrow{\text{正交化}}$   $\beta_1, \beta_2, \dots, \beta_s$  (等价, 两两正交)

实现正交化步骤 (施密特正交化方法):

$$\beta_1 = \alpha_1$$

$$\beta_1^T \beta_2 = \beta_1^T \left( \alpha_2 - \frac{\alpha_2^T \beta_1}{\|\beta_1\|^2} \beta_1 \right) = \beta_1^T \alpha_2 - \frac{\alpha_2^T \beta_1}{\|\beta_1\|^2} \beta_1^T \beta_1 = 0$$

$$\beta_1^T \beta_3 = \beta_1^T \left( \alpha_3 - \frac{\alpha_3^T \beta_1}{\|\beta_1\|^2} \beta_1 - \frac{\alpha_3^T \beta_2}{\|\beta_2\|^2} \beta_2 \right)$$

$\vdots$

$$\beta_s = \alpha_s - \frac{\alpha_s^T \beta_1}{\|\beta_1\|^2} \beta_1 - \frac{\alpha_s^T \beta_2}{\|\beta_2\|^2} \beta_2 - \dots - \frac{\alpha_s^T \beta_{s-1}}{\|\beta_{s-1}\|^2} \beta_{s-1}$$

# 正交化

$\alpha_1, \alpha_2, \dots, \alpha_s$  (线性无关)  $\xrightarrow{\text{正交化}}$   $\beta_1, \beta_2, \dots, \beta_s$  (等价, 两两正交)

实现正交化步骤 (施密特正交化方法):

$$\beta_1 = \alpha_1$$

$$\beta_1^T \beta_2 = \beta_1^T \left( \alpha_2 - \frac{\alpha_2^T \beta_1}{\|\beta_1\|^2} \beta_1 \right) = \beta_1^T \alpha_2 - \frac{\alpha_2^T \beta_1}{\|\beta_1\|^2} \beta_1^T \beta_1 = 0$$

$$\beta_1^T \beta_3 = \beta_1^T \left( \alpha_3 - \frac{\alpha_3^T \beta_1}{\|\beta_1\|^2} \beta_1 - \frac{\alpha_3^T \beta_2}{\|\beta_2\|^2} \beta_2 \right) = \beta_1^T \alpha_3 - \frac{\alpha_3^T \beta_1}{\|\beta_1\|^2} \beta_1^T \beta_1$$

$\vdots$

$$\beta_s = \alpha_s - \frac{\alpha_s^T \beta_1}{\|\beta_1\|^2} \beta_1 - \frac{\alpha_s^T \beta_2}{\|\beta_2\|^2} \beta_2 - \dots - \frac{\alpha_s^T \beta_{s-1}}{\|\beta_{s-1}\|^2} \beta_{s-1}$$

# 正交化

$\alpha_1, \alpha_2, \dots, \alpha_s$  (线性无关)  $\xrightarrow{\text{正交化}}$   $\beta_1, \beta_2, \dots, \beta_s$  (等价, 两两正交)

实现正交化步骤 (施密特正交化方法):

$$\beta_1 = \alpha_1$$

$$\beta_1^T \beta_2 = \beta_1^T \left( \alpha_2 - \frac{\alpha_2^T \beta_1}{\|\beta_1\|^2} \beta_1 \right) = \beta_1^T \alpha_2 - \frac{\alpha_2^T \beta_1}{\|\beta_1\|^2} \beta_1^T \beta_1 = 0$$

$$\beta_1^T \beta_3 = \beta_1^T \left( \alpha_3 - \frac{\alpha_3^T \beta_1}{\|\beta_1\|^2} \beta_1 - \frac{\alpha_3^T \beta_2}{\|\beta_2\|^2} \beta_2 \right) = \beta_1^T \alpha_3 - \frac{\alpha_3^T \beta_1}{\|\beta_1\|^2} \beta_1^T \beta_1 = 0$$

$\vdots$

$$\beta_s = \alpha_s - \frac{\alpha_s^T \beta_1}{\|\beta_1\|^2} \beta_1 - \frac{\alpha_s^T \beta_2}{\|\beta_2\|^2} \beta_2 - \dots - \frac{\alpha_s^T \beta_{s-1}}{\|\beta_{s-1}\|^2} \beta_{s-1}$$



# 正交化

$\alpha_1, \alpha_2, \dots, \alpha_s$  (线性无关)  $\xrightarrow{\text{正交化}}$   $\beta_1, \beta_2, \dots, \beta_s$  (等价, 两两正交)

实现正交化步骤 (施密特正交化方法):

$$\beta_1 = \alpha_1$$

$$\beta_1^T \beta_2 = \beta_1^T \left( \alpha_2 - \frac{\alpha_2^T \beta_1}{\|\beta_1\|^2} \beta_1 \right) = \beta_1^T \alpha_2 - \frac{\alpha_2^T \beta_1}{\|\beta_1\|^2} \beta_1^T \beta_1 = 0$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3^T \beta_1}{\|\beta_1\|^2} \beta_1 - \frac{\alpha_3^T \beta_2}{\|\beta_2\|^2} \beta_2$$

$\vdots$

$$\beta_s = \alpha_s - \frac{\alpha_s^T \beta_1}{\|\beta_1\|^2} \beta_1 - \frac{\alpha_s^T \beta_2}{\|\beta_2\|^2} \beta_2 - \dots - \frac{\alpha_s^T \beta_{s-1}}{\|\beta_{s-1}\|^2} \beta_{s-1}$$

# 正交化

$\alpha_1, \alpha_2, \dots, \alpha_s$  (线性无关)  $\xrightarrow{\text{正交化}}$   $\beta_1, \beta_2, \dots, \beta_s$  (等价, 两两正交)

实现正交化步骤 (施密特正交化方法):

$$\beta_1 = \alpha_1$$

$$\beta_1^T \beta_2 = \beta_1^T \left( \alpha_2 - \frac{\alpha_2^T \beta_1}{\|\beta_1\|^2} \beta_1 \right) = \beta_1^T \alpha_2 - \frac{\alpha_2^T \beta_1}{\|\beta_1\|^2} \beta_1^T \beta_1 = 0$$

$$\beta_2^T \beta_3 = \beta_2^T \left( \alpha_3 - \frac{\alpha_3^T \beta_1}{\|\beta_1\|^2} \beta_1 - \frac{\alpha_3^T \beta_2}{\|\beta_2\|^2} \beta_2 \right)$$

$\vdots$

$$\beta_s = \alpha_s - \frac{\alpha_s^T \beta_1}{\|\beta_1\|^2} \beta_1 - \frac{\alpha_s^T \beta_2}{\|\beta_2\|^2} \beta_2 - \dots - \frac{\alpha_s^T \beta_{s-1}}{\|\beta_{s-1}\|^2} \beta_{s-1}$$

# 正交化

$\alpha_1, \alpha_2, \dots, \alpha_s$  (线性无关)  $\xrightarrow{\text{正交化}}$   $\beta_1, \beta_2, \dots, \beta_s$  (等价, 两两正交)

实现正交化步骤 (施密特正交化方法):

$$\beta_1 = \alpha_1$$

$$\beta_1^T \beta_2 = \beta_1^T \left( \alpha_2 - \frac{\alpha_2^T \beta_1}{\|\beta_1\|^2} \beta_1 \right) = \beta_1^T \alpha_2 - \frac{\alpha_2^T \beta_1}{\|\beta_1\|^2} \beta_1^T \beta_1 = 0$$

$$\beta_2^T \beta_3 = \beta_2^T \left( \alpha_3 - \frac{\alpha_3^T \beta_1}{\|\beta_1\|^2} \beta_1 - \frac{\alpha_3^T \beta_2}{\|\beta_2\|^2} \beta_2 \right) = \beta_2^T \alpha_3 - \frac{\alpha_3^T \beta_2}{\|\beta_2\|^2} \beta_2^T \beta_2$$

$\vdots$

$$\beta_s = \alpha_s - \frac{\alpha_s^T \beta_1}{\|\beta_1\|^2} \beta_1 - \frac{\alpha_s^T \beta_2}{\|\beta_2\|^2} \beta_2 - \dots - \frac{\alpha_s^T \beta_{s-1}}{\|\beta_{s-1}\|^2} \beta_{s-1}$$

# 正交化

$\alpha_1, \alpha_2, \dots, \alpha_s$  (线性无关)  $\xrightarrow{\text{正交化}}$   $\beta_1, \beta_2, \dots, \beta_s$  (等价, 两两正交)

实现正交化步骤 (施密特正交化方法):

$$\beta_1 = \alpha_1$$

$$\beta_1^T \beta_2 = \beta_1^T \left( \alpha_2 - \frac{\alpha_2^T \beta_1}{\|\beta_1\|^2} \beta_1 \right) = \beta_1^T \alpha_2 - \frac{\alpha_2^T \beta_1}{\|\beta_1\|^2} \beta_1^T \beta_1 = 0$$

$$\beta_2^T \beta_3 = \beta_2^T \left( \alpha_3 - \frac{\alpha_3^T \beta_1}{\|\beta_1\|^2} \beta_1 - \frac{\alpha_3^T \beta_2}{\|\beta_2\|^2} \beta_2 \right) = \beta_2^T \alpha_3 - \frac{\alpha_3^T \beta_2}{\|\beta_2\|^2} \beta_2^T \beta_2 = 0$$

$\vdots$

$$\beta_s = \alpha_s - \frac{\alpha_s^T \beta_1}{\|\beta_1\|^2} \beta_1 - \frac{\alpha_s^T \beta_2}{\|\beta_2\|^2} \beta_2 - \dots - \frac{\alpha_s^T \beta_{s-1}}{\|\beta_{s-1}\|^2} \beta_{s-1}$$

# 正交化

$\alpha_1, \alpha_2, \dots, \alpha_s$  (线性无关)  $\xrightarrow{\text{正交化}}$   $\beta_1, \beta_2, \dots, \beta_s$  (等价, 两两正交)

实现正交化步骤 (施密特正交化方法):

$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2^T \beta_1}{\|\beta_1\|^2} \beta_1$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3^T \beta_1}{\|\beta_1\|^2} \beta_1 - \frac{\alpha_3^T \beta_2}{\|\beta_2\|^2} \beta_2$$

$\vdots$

$$\beta_s = \alpha_s - \frac{\alpha_s^T \beta_1}{\|\beta_1\|^2} \beta_1 - \frac{\alpha_s^T \beta_2}{\|\beta_2\|^2} \beta_2 - \dots - \frac{\alpha_s^T \beta_{s-1}}{\|\beta_{s-1}\|^2} \beta_{s-1}$$

**例 1** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 3 \\ 3 \\ -1 \\ -1 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix}$  正交化

**例 1** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 3 \\ 3 \\ -1 \\ -1 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix}$  正交化

**解**

$$\beta_1 =$$

$$\beta_2 =$$

$$\beta_3 =$$

**例 1** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 3 \\ 3 \\ -1 \\ -1 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix}$  正交化

**解**

$$\beta_1 = \alpha_1$$

$$\beta_2 =$$

$$\beta_3 =$$



**例 1** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 3 \\ 3 \\ -1 \\ -1 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix}$  正交化

**解**

$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \text{——} \beta_1$$

$$\beta_3 =$$

**例 1** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 3 \\ 3 \\ -1 \\ -1 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix}$  正交化

**解**

$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \text{——} \beta_1$$

$$\beta_3 = \alpha_3 - \text{——} \beta_1 - \text{——} \beta_2$$

**例 1** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 3 \\ 3 \\ -1 \\ -1 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix}$  正交化

**解**

$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2^T \beta_1}{\|\beta_1\|^2} \beta_1$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3^T \beta_1}{\|\beta_1\|^2} \beta_1 - \frac{\alpha_3^T \beta_2}{\|\beta_2\|^2} \beta_2$$

**例 1** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 3 \\ 3 \\ -1 \\ -1 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix}$  正交化

**解**

$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2^T \beta_1}{\|\beta_1\|^2} \beta_1$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3^T \beta_1}{\|\beta_1\|^2} \beta_1 - \frac{\alpha_3^T \beta_2}{\|\beta_2\|^2} \beta_2$$

**例 1** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 3 \\ 3 \\ -1 \\ -1 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix}$  正交化

**解**

$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2^T \beta_1}{\|\beta_1\|^2} \beta_1$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3^T \beta_1}{\|\beta_1\|^2} \beta_1 - \frac{\alpha_3^T \beta_2}{\|\beta_2\|^2} \beta_2$$

**例 1** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 3 \\ 3 \\ -1 \\ -1 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix}$  正交化

**解**

$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \text{——} \beta_1$$

$$\beta_3 = \alpha_3 - \text{——} \beta_1 - \text{——} \beta_2$$

**例 1** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix}$  正交化

**解**

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$

**例 1** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix}$  正交化

**解**

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$



**例 1** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix}$  正交化

**解**

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix} - \frac{1}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$

**例 1** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix}$  正交化

**解**

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix} - \frac{4}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$

**例 1** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix}$  正交化

**解**

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix} - \frac{4}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$

**例 1** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 3 \\ 3 \\ -1 \\ -1 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix}$  正交化

**解**

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} 3 \\ 3 \\ -1 \\ -1 \end{pmatrix} - \frac{4}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -2 \\ -2 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$

$$= \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix} - \frac{10}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{10}{8} \begin{pmatrix} 2 \\ 2 \\ -2 \\ -2 \end{pmatrix}$$

**例 1** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 3 \\ 3 \\ -1 \\ -1 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix}$  正交化

**解**

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} 3 \\ 3 \\ -1 \\ -1 \end{pmatrix} - \frac{4}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -2 \\ -2 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$

$$= \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix} - \frac{1}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{4} \begin{pmatrix} 2 \\ 2 \\ -2 \\ -2 \end{pmatrix}$$

**例 1** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 3 \\ 3 \\ -1 \\ -1 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix}$  正交化

**解**

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} 3 \\ 3 \\ -1 \\ -1 \end{pmatrix} - \frac{4}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -2 \\ -2 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$

$$= \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix} - \frac{12}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{12}{8} \begin{pmatrix} 2 \\ 2 \\ -2 \\ -2 \end{pmatrix}$$

**例 1** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 3 \\ 3 \\ -1 \\ -1 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix}$  正交化

**解**

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} 3 \\ 3 \\ -1 \\ -1 \end{pmatrix} - \frac{4}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -2 \\ -2 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$

$$= \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix} - \frac{12}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{16} \begin{pmatrix} 2 \\ 2 \\ -2 \\ -2 \end{pmatrix}$$

**例 1** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 3 \\ 3 \\ -1 \\ -1 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix}$  正交化

**解**

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} 3 \\ 3 \\ -1 \\ -1 \end{pmatrix} - \frac{4}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -2 \\ -2 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$

$$= \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix} - \frac{12}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{-32}{16} \begin{pmatrix} 2 \\ 2 \\ -2 \\ -2 \end{pmatrix}$$



**例 1** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix}$  正交化

**解**

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix} - \frac{4}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$

$$= \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix} - \frac{12}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{-32}{16} \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

**例 2** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix}$  正交化

**例 2** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix}$  正交化

**解**

$$\beta_1 =$$

$$\beta_2 =$$

$$\beta_3 =$$

**例 2** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix}$  正交化

**解**

$$\beta_1 = \alpha_1$$

$$\beta_2 =$$

$$\beta_3 =$$

**例 2** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix}$  正交化

**解**

$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \text{——} \beta_1$$

$$\beta_3 =$$

**例 2** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix}$  正交化

**解**

$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \text{——} \beta_1$$

$$\beta_3 = \alpha_3 - \text{——} \beta_1 - \text{——} \beta_2$$

**例 2** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix}$  正交化

**解**

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \text{——} \beta_1$$

$$\beta_3 = \alpha_3 - \text{——} \beta_1 - \text{——} \beta_2$$

**例 2** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix}$  正交化

**解**

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$



**例 2** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix}$  正交化

**解**

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$

**例 2** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix}$  正交化

**解**

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix} - \frac{6}{3} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$

**例 2** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix}$  正交化

**解**

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix} - \frac{6}{3} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$

**例 2** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix}$  正交化

**解**

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix} - \frac{6}{3} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\begin{aligned} \beta_3 &= \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2 \\ &= \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix} - \frac{6}{3} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \frac{0}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix} \end{aligned}$$

**例 2** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix}$  正交化

**解**

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix} - \frac{6}{3} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$

$$= \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

**例 2** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix}$  正交化

**解**

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix} - \frac{6}{3} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$

$$= \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix} - \frac{6}{3} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

**例 2** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix}$  正交化

**解**

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix} - \frac{6}{3} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$

$$= \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix} - \frac{6}{3} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

**例 2** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix}$  正交化

**解**

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{(\alpha_2, \beta_1)}{(\beta_1, \beta_1)} \beta_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix} - \frac{6}{3} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{(\alpha_3, \beta_1)}{(\beta_1, \beta_1)} \beta_1 - \frac{(\alpha_3, \beta_2)}{(\beta_2, \beta_2)} \beta_2$$

$$= \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix} - \frac{6}{3} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \frac{0}{3} \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$



**例 2** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix}$  正交化

**解**

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix} - \frac{6}{3} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$

$$= \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix} - \frac{6}{3} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \frac{0}{3} \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

**例 3** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$  正交化

**例 3** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$  正交化

**解**

$$\beta_1 =$$

$$\beta_2 =$$

$$\beta_3 =$$

**例 3** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$  正交化

**解**

$$\beta_1 = \alpha_1$$

$$\beta_2 =$$

$$\beta_3 =$$

**例 3** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$  正交化

**解**

$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \text{——} \beta_1$$

$$\beta_3 =$$

**例 3** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$  正交化

**解**

$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \text{——} \beta_1$$

$$\beta_3 = \alpha_3 - \text{——} \beta_1 - \text{——} \beta_2$$

**例 3** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$  正交化

**解**

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$

**例 3** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$  正交化

**解**

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$



**例 3** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$  正交化

**解**

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} - \frac{1}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$

**例 3** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$  正交化

**解**

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} - \frac{4}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$

**例 3** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$  正交化

**解**

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} - \frac{4}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$

**例 3** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$  正交化

**解**

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} - \frac{4}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$

$$= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{-1}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

**例 3** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$  正交化

**解**

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\langle \alpha_2, \beta_1 \rangle}{\langle \beta_1, \beta_1 \rangle} \beta_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} - \frac{4}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\langle \alpha_3, \beta_1 \rangle}{\langle \beta_1, \beta_1 \rangle} \beta_1 - \frac{\langle \alpha_3, \beta_2 \rangle}{\langle \beta_2, \beta_2 \rangle} \beta_2$$

$$= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{0}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

**例 3** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$  正交化

**解**

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} - \frac{4}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$

$$= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

**例 3** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$  正交化

**解**

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} - \frac{4}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$

$$= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

**例 3** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$  正交化

**解**

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} - \frac{4}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$

$$= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{2}{2} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$



**例 3** 将线性无关组  $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$  正交化

**解**

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} - \frac{4}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$

$$= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{2}{2} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/4 \\ -1/4 \\ 3/4 \end{pmatrix}$$

# 正交矩阵

**定义** 设  $n$  阶矩阵  $Q$  满足  $Q^T Q = I_n$ , 则称  $Q$  是 **正交矩阵**。

# 正交矩阵

**定义** 设  $n$  阶矩阵  $Q$  满足  $Q^T Q = I_n$ , 则称  $Q$  是 **正交矩阵**。

**例**  $Q = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$  是正交矩阵:

# 正交矩阵

**定义** 设  $n$  阶矩阵  $Q$  满足  $Q^T Q = I_n$ , 则称  $Q$  是 **正交矩阵**。

**例**  $Q = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$  是正交矩阵:

$$Q^T Q = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

# 正交矩阵

**定义** 设  $n$  阶矩阵  $Q$  满足  $Q^T Q = I_n$ , 则称  $Q$  是 **正交矩阵**。

**例**  $Q = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$  是正交矩阵:

$$Q^T Q = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

# 正交矩阵

**定义** 设  $n$  阶矩阵  $Q$  满足  $Q^T Q = I_n$ , 则称  $Q$  是 **正交矩阵**。

**例**  $Q = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$  是正交矩阵:

$$Q^T Q = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

## 性质

1. 若  $Q$  为正交矩阵, 则  $|Q| = 1$  或  $|Q| = -1$ ;

# 正交矩阵

**定义** 设  $n$  阶矩阵  $Q$  满足  $Q^T Q = I_n$ , 则称  $Q$  是 **正交矩阵**。

**例**  $Q = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$  是正交矩阵:

$$Q^T Q = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

## 性质

1. 若  $Q$  为正交矩阵, 则  $|Q| = 1$  或  $|Q| = -1$ ;

## 证明

1.  $Q^T Q = I_n \Rightarrow$

# 正交矩阵

**定义** 设  $n$  阶矩阵  $Q$  满足  $Q^T Q = I_n$ , 则称  $Q$  是 **正交矩阵**。

**例**  $Q = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$  是正交矩阵:

$$Q^T Q = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

## 性质

1. 若  $Q$  为正交矩阵, 则  $|Q| = 1$  或  $|Q| = -1$ ;

## 证明

1.  $Q^T Q = I_n \Rightarrow |I_n| = |Q^T Q|$



# 正交矩阵

**定义** 设  $n$  阶矩阵  $Q$  满足  $Q^T Q = I_n$ , 则称  $Q$  是 **正交矩阵**。

**例**  $Q = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$  是正交矩阵:

$$Q^T Q = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

## 性质

1. 若  $Q$  为正交矩阵, 则  $|Q| = 1$  或  $|Q| = -1$ ;

## 证明

1.  $Q^T Q = I_n \Rightarrow 1 = |I_n| = |Q^T Q|$

# 正交矩阵

**定义** 设  $n$  阶矩阵  $Q$  满足  $Q^T Q = I_n$ , 则称  $Q$  是 **正交矩阵**。

**例**  $Q = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$  是正交矩阵:

$$Q^T Q = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

## 性质

1. 若  $Q$  为正交矩阵, 则  $|Q| = 1$  或  $|Q| = -1$ ;

## 证明

1.  $Q^T Q = I_n \Rightarrow 1 = |I_n| = |Q^T Q| = |Q^T| \cdot |Q|$

# 正交矩阵

**定义** 设  $n$  阶矩阵  $Q$  满足  $Q^T Q = I_n$ , 则称  $Q$  是 **正交矩阵**。

**例**  $Q = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$  是正交矩阵:

$$Q^T Q = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

## 性质

1. 若  $Q$  为正交矩阵, 则  $|Q| = 1$  或  $|Q| = -1$ ;

## 证明

1.  $Q^T Q = I_n \Rightarrow 1 = |I_n| = |Q^T Q| = |Q^T| \cdot |Q| = |Q|^2$

# 正交矩阵

**定义** 设  $n$  阶矩阵  $Q$  满足  $Q^T Q = I_n$ , 则称  $Q$  是 **正交矩阵**。

**例**  $Q = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$  是正交矩阵:

$$Q^T Q = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

## 性质

1. 若  $Q$  为正交矩阵, 则  $|Q| = 1$  或  $|Q| = -1$ ;

## 证明

1.  $Q^T Q = I_n \Rightarrow 1 = |I_n| = |Q^T Q| = |Q^T| \cdot |Q| = |Q|^2 \Rightarrow |Q| = \pm 1$

# 正交矩阵

**定义** 设  $n$  阶矩阵  $Q$  满足  $Q^T Q = I_n$ , 则称  $Q$  是 **正交矩阵**。

**例**  $Q = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$  是正交矩阵:

$$Q^T Q = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

## 性质

1. 若  $Q$  为正交矩阵, 则  $|Q| = 1$  或  $|Q| = -1$ ;
2. 若  $Q$  为正交矩阵, 则  $Q$  可逆, 且  $Q^{-1} = Q^T$ ;

## 证明

1.  $Q^T Q = I_n \Rightarrow 1 = |I_n| = |Q^T Q| = |Q^T| \cdot |Q| = |Q|^2 \Rightarrow |Q| = \pm 1$

# 正交矩阵

**定义** 设  $n$  阶矩阵  $Q$  满足  $Q^T Q = I_n$ , 则称  $Q$  是 **正交矩阵**。

**例**  $Q = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$  是正交矩阵:

$$Q^T Q = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

## 性质

1. 若  $Q$  为正交矩阵, 则  $|Q| = 1$  或  $|Q| = -1$ ;
2. 若  $Q$  为正交矩阵, 则  $Q$  可逆, 且  $Q^{-1} = Q^T$ ;

## 证明

1.  $Q^T Q = I_n \Rightarrow 1 = |I_n| = |Q^T Q| = |Q^T| \cdot |Q| = |Q|^2 \Rightarrow |Q| = \pm 1$
2. 显然

# 正交矩阵

**定义** 设  $n$  阶矩阵  $Q$  满足  $Q^T Q = I_n$ , 则称  $Q$  是 **正交矩阵**。

**例**  $Q = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$  是正交矩阵:

$$Q^T Q = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

## 性质

1. 若  $Q$  为正交矩阵, 则  $|Q| = 1$  或  $|Q| = -1$ ;
2. 若  $Q$  为正交矩阵, 则  $Q$  可逆, 且  $Q^{-1} = Q^T$ ;
3. 若  $P, Q$  为正交矩阵, 则  $PQ$  也是正交矩阵。

## 证明

1.  $Q^T Q = I_n \Rightarrow 1 = |I_n| = |Q^T Q| = |Q^T| \cdot |Q| = |Q|^2 \Rightarrow |Q| = \pm 1$
2. 显然

# 正交矩阵

**定义** 设  $n$  阶矩阵  $Q$  满足  $Q^T Q = I_n$ , 则称  $Q$  是 **正交矩阵**。

**例**  $Q = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$  是正交矩阵:

$$Q^T Q = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

## 性质

1. 若  $Q$  为正交矩阵, 则  $|Q| = 1$  或  $|Q| = -1$ ;
2. 若  $Q$  为正交矩阵, 则  $Q$  可逆, 且  $Q^{-1} = Q^T$ ;
3. 若  $P, Q$  为正交矩阵, 则  $PQ$  也是正交矩阵。

## 证明

1.  $Q^T Q = I_n \Rightarrow 1 = |I_n| = |Q^T Q| = |Q^T| \cdot |Q| = |Q|^2 \Rightarrow |Q| = \pm 1$
2. 显然
3.  $(PQ)^T(PQ) =$



# 正交矩阵

**定义** 设  $n$  阶矩阵  $Q$  满足  $Q^T Q = I_n$ , 则称  $Q$  是 **正交矩阵**。

**例**  $Q = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$  是正交矩阵:

$$Q^T Q = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

## 性质

1. 若  $Q$  为正交矩阵, 则  $|Q| = 1$  或  $|Q| = -1$ ;
2. 若  $Q$  为正交矩阵, 则  $Q$  可逆, 且  $Q^{-1} = Q^T$ ;
3. 若  $P, Q$  为正交矩阵, 则  $PQ$  也是正交矩阵。

## 证明

1.  $Q^T Q = I_n \Rightarrow 1 = |I_n| = |Q^T Q| = |Q^T| \cdot |Q| = |Q|^2 \Rightarrow |Q| = \pm 1$
2. 显然
3.  $(PQ)^T (PQ) = Q^T P^T P Q =$

# 正交矩阵

**定义** 设  $n$  阶矩阵  $Q$  满足  $Q^T Q = I_n$ , 则称  $Q$  是 **正交矩阵**。

**例**  $Q = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$  是正交矩阵:

$$Q^T Q = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

## 性质

1. 若  $Q$  为正交矩阵, 则  $|Q| = 1$  或  $|Q| = -1$ ;
2. 若  $Q$  为正交矩阵, 则  $Q$  可逆, 且  $Q^{-1} = Q^T$ ;
3. 若  $P, Q$  为正交矩阵, 则  $PQ$  也是正交矩阵。

## 证明

1.  $Q^T Q = I_n \Rightarrow 1 = |I_n| = |Q^T Q| = |Q^T| \cdot |Q| = |Q|^2 \Rightarrow |Q| = \pm 1$
2. 显然
3.  $(PQ)^T (PQ) = Q^T P^T P Q = Q^T I_n Q =$

# 正交矩阵

**定义** 设  $n$  阶矩阵  $Q$  满足  $Q^T Q = I_n$ , 则称  $Q$  是 **正交矩阵**。

**例**  $Q = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$  是正交矩阵:

$$Q^T Q = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

## 性质

1. 若  $Q$  为正交矩阵, 则  $|Q| = 1$  或  $|Q| = -1$ ;
2. 若  $Q$  为正交矩阵, 则  $Q$  可逆, 且  $Q^{-1} = Q^T$ ;
3. 若  $P, Q$  为正交矩阵, 则  $PQ$  也是正交矩阵。

## 证明

1.  $Q^T Q = I_n \Rightarrow 1 = |I_n| = |Q^T Q| = |Q^T| \cdot |Q| = |Q|^2 \Rightarrow |Q| = \pm 1$
2. 显然
3.  $(PQ)^T (PQ) = Q^T P^T P Q = Q^T I_n Q = Q^T Q =$

# 正交矩阵

**定义** 设  $n$  阶矩阵  $Q$  满足  $Q^T Q = I_n$ , 则称  $Q$  是 **正交矩阵**。

**例**  $Q = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$  是正交矩阵:

$$Q^T Q = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

## 性质

1. 若  $Q$  为正交矩阵, 则  $|Q| = 1$  或  $|Q| = -1$ ;
2. 若  $Q$  为正交矩阵, 则  $Q$  可逆, 且  $Q^{-1} = Q^T$ ;
3. 若  $P, Q$  为正交矩阵, 则  $PQ$  也是正交矩阵。

## 证明

1.  $Q^T Q = I_n \Rightarrow 1 = |I_n| = |Q^T Q| = |Q^T| \cdot |Q| = |Q|^2 \Rightarrow |Q| = \pm 1$
2. 显然
3.  $(PQ)^T (PQ) = Q^T P^T P Q = Q^T I_n Q = Q^T Q = I_n$

# 正交矩阵

**定理**  $n$  阶矩阵  $Q$  是正交矩阵的充分必要条件是： $Q$  的列（行）向量组是单位正交向量组。

# 正交矩阵

**定理**  $n$  阶矩阵  $Q$  是正交矩阵的充分必要条件是： $Q$  的列（行）向量组是单位正交向量组。

**证明** 设  $Q = (\alpha_1 \alpha_2 \dots \alpha_n)$ ，则

$$Q^T Q = \begin{pmatrix} \alpha_1^T \\ \alpha_2^T \\ \vdots \\ \alpha_n^T \end{pmatrix} (\alpha_1 \alpha_2 \dots \alpha_n)$$

# 正交矩阵

**定理**  $n$  阶矩阵  $Q$  是正交矩阵的充分必要条件是： $Q$  的列（行）向量组是单位正交向量组。

**证明** 设  $Q = (\alpha_1 \alpha_2 \dots \alpha_n)$ ，则

$$Q^T Q = \begin{pmatrix} \alpha_1^T \\ \alpha_2^T \\ \vdots \\ \alpha_n^T \end{pmatrix} (\alpha_1 \alpha_2 \dots \alpha_n) = \begin{pmatrix} & & & \\ & & & \\ & & & \\ & & & \end{pmatrix}$$

# 正交矩阵

**定理**  $n$  阶矩阵  $Q$  是正交矩阵的充分必要条件是： $Q$  的列（行）向量组是单位正交向量组。

**证明** 设  $Q = (\alpha_1 \alpha_2 \dots \alpha_n)$ ，则

$$Q^T Q = \begin{pmatrix} \alpha_1^T \\ \alpha_2^T \\ \vdots \\ \alpha_n^T \end{pmatrix} (\alpha_1 \alpha_2 \dots \alpha_n) = \begin{pmatrix} \alpha_1^T \alpha_1 & & \\ & \ddots & \\ & & \alpha_n^T \alpha_n \end{pmatrix}$$



# 正交矩阵

**定理**  $n$  阶矩阵  $Q$  是正交矩阵的充分必要条件是： $Q$  的列（行）向量组是单位正交向量组。

**证明** 设  $Q = (\alpha_1 \alpha_2 \dots \alpha_n)$ ，则

$$Q^T Q = \begin{pmatrix} \alpha_1^T \\ \alpha_2^T \\ \vdots \\ \alpha_n^T \end{pmatrix} (\alpha_1 \alpha_2 \dots \alpha_n) = \begin{pmatrix} \alpha_1^T \alpha_1 & \alpha_1^T \alpha_2 & & \\ & \alpha_2^T \alpha_1 & \alpha_2^T \alpha_2 & \\ & & & \ddots \\ & & & & \alpha_n^T \alpha_n \end{pmatrix}$$

# 正交矩阵

**定理**  $n$  阶矩阵  $Q$  是正交矩阵的充分必要条件是： $Q$  的列（行）向量组是单位正交向量组。

**证明** 设  $Q = (\alpha_1 \alpha_2 \dots \alpha_n)$ ，则

$$Q^T Q = \begin{pmatrix} \alpha_1^T \\ \alpha_2^T \\ \vdots \\ \alpha_n^T \end{pmatrix} (\alpha_1 \alpha_2 \dots \alpha_n) = \begin{pmatrix} \alpha_1^T \alpha_1 & \alpha_1^T \alpha_2 & \cdots & \alpha_1^T \alpha_n \\ \alpha_2^T \alpha_1 & \alpha_2^T \alpha_2 & \cdots & \alpha_2^T \alpha_n \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_n^T \alpha_1 & \alpha_n^T \alpha_2 & \cdots & \alpha_n^T \alpha_n \end{pmatrix}$$

# 正交矩阵

**定理**  $n$  阶矩阵  $Q$  是正交矩阵的充分必要条件是： $Q$  的列（行）向量组是单位正交向量组。

**证明** 设  $Q = (\alpha_1 \alpha_2 \dots \alpha_n)$ ，则

$$Q^T Q = \begin{pmatrix} \alpha_1^T \\ \alpha_2^T \\ \vdots \\ \alpha_n^T \end{pmatrix} (\alpha_1 \alpha_2 \dots \alpha_n) = \begin{pmatrix} \alpha_1^T \alpha_1 & \alpha_1^T \alpha_2 & \cdots & \alpha_1^T \alpha_n \\ \alpha_2^T \alpha_1 & & & \\ & & & \\ & & & \end{pmatrix}$$

# 正交矩阵

**定理**  $n$  阶矩阵  $Q$  是正交矩阵的充分必要条件是： $Q$  的列（行）向量组是单位正交向量组。

**证明** 设  $Q = (\alpha_1 \alpha_2 \dots \alpha_n)$ ，则

$$Q^T Q = \begin{pmatrix} \alpha_1^T \\ \alpha_2^T \\ \vdots \\ \alpha_n^T \end{pmatrix} (\alpha_1 \alpha_2 \dots \alpha_n) = \begin{pmatrix} \alpha_1^T \alpha_1 & \alpha_1^T \alpha_2 & \cdots & \alpha_1^T \alpha_n \\ \alpha_2^T \alpha_1 & \alpha_2^T \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_n^T \alpha_n \end{pmatrix}$$

# 正交矩阵

**定理**  $n$  阶矩阵  $Q$  是正交矩阵的充分必要条件是： $Q$  的列（行）向量组是单位正交向量组。

**证明** 设  $Q = (\alpha_1 \alpha_2 \dots \alpha_n)$ ，则

$$Q^T Q = \begin{pmatrix} \alpha_1^T \\ \alpha_2^T \\ \vdots \\ \alpha_n^T \end{pmatrix} (\alpha_1 \alpha_2 \dots \alpha_n) = \begin{pmatrix} \alpha_1^T \alpha_1 & \alpha_1^T \alpha_2 & \cdots & \alpha_1^T \alpha_n \\ \alpha_2^T \alpha_1 & \alpha_2^T \alpha_2 & \cdots & \alpha_2^T \alpha_n \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_n^T \alpha_1 & \alpha_n^T \alpha_2 & \cdots & \alpha_n^T \alpha_n \end{pmatrix}$$

# 正交矩阵

**定理**  $n$  阶矩阵  $Q$  是正交矩阵的充分必要条件是： $Q$  的列（行）向量组是单位正交向量组。

**证明** 设  $Q = (\alpha_1 \alpha_2 \dots \alpha_n)$ ，则

$$Q^T Q = \begin{pmatrix} \alpha_1^T \\ \alpha_2^T \\ \vdots \\ \alpha_n^T \end{pmatrix} (\alpha_1 \alpha_2 \dots \alpha_n) = \begin{pmatrix} \alpha_1^T \alpha_1 & \alpha_1^T \alpha_2 & \cdots & \alpha_1^T \alpha_n \\ \alpha_2^T \alpha_1 & \alpha_2^T \alpha_2 & \cdots & \alpha_2^T \alpha_n \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_n^T \alpha_1 & \alpha_n^T \alpha_2 & \cdots & \alpha_n^T \alpha_n \end{pmatrix}$$

# 正交矩阵

**定理**  $n$  阶矩阵  $Q$  是正交矩阵的充分必要条件是： $Q$  的列（行）向量组是单位正交向量组。

**证明** 设  $Q = (\alpha_1 \alpha_2 \dots \alpha_n)$ ，则

$$Q^T Q = \begin{pmatrix} \alpha_1^T \\ \alpha_2^T \\ \vdots \\ \alpha_n^T \end{pmatrix} (\alpha_1 \alpha_2 \dots \alpha_n) = \begin{pmatrix} \alpha_1^T \alpha_1 & \alpha_1^T \alpha_2 & \cdots & \alpha_1^T \alpha_n \\ \alpha_2^T \alpha_1 & \alpha_2^T \alpha_2 & \cdots & \alpha_2^T \alpha_n \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_n^T \alpha_1 & \alpha_n^T \alpha_2 & \cdots & \alpha_n^T \alpha_n \end{pmatrix}$$

# 正交矩阵

**定理**  $n$  阶矩阵  $Q$  是正交矩阵的充分必要条件是： $Q$  的列（行）向量组是单位正交向量组。

**证明** 设  $Q = (\alpha_1 \alpha_2 \dots \alpha_n)$ ，则

$$Q^T Q = \begin{pmatrix} \alpha_1^T \\ \alpha_2^T \\ \vdots \\ \alpha_n^T \end{pmatrix} (\alpha_1 \alpha_2 \dots \alpha_n) = \begin{pmatrix} \alpha_1^T \alpha_1 & \alpha_1^T \alpha_2 & \cdots & \alpha_1^T \alpha_n \\ \alpha_2^T \alpha_1 & \alpha_2^T \alpha_2 & \cdots & \alpha_2^T \alpha_n \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_n^T \alpha_1 & \alpha_n^T \alpha_2 & \cdots & \alpha_n^T \alpha_n \end{pmatrix}$$



# 正交矩阵

**定理**  $n$  阶矩阵  $Q$  是正交矩阵的充分必要条件是： $Q$  的列（行）向量组是单位正交向量组。

**证明** 设  $Q = (\alpha_1 \alpha_2 \dots \alpha_n)$ ，则

$$Q^T Q = \begin{pmatrix} \alpha_1^T \\ \alpha_2^T \\ \vdots \\ \alpha_n^T \end{pmatrix} (\alpha_1 \alpha_2 \dots \alpha_n) = \begin{pmatrix} \alpha_1^T \alpha_1 & \alpha_1^T \alpha_2 & \cdots & \alpha_1^T \alpha_n \\ \alpha_2^T \alpha_1 & \alpha_2^T \alpha_2 & \cdots & \alpha_2^T \alpha_n \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_n^T \alpha_1 & \alpha_n^T \alpha_2 & \cdots & \alpha_n^T \alpha_n \end{pmatrix}$$

所以

$$Q^T Q = I$$

# 正交矩阵

**定理**  $n$  阶矩阵  $Q$  是正交矩阵的充分必要条件是： $Q$  的列（行）向量组是单位正交向量组。

**证明** 设  $Q = (\alpha_1 \alpha_2 \dots \alpha_n)$ ，则

$$Q^T Q = \begin{pmatrix} \alpha_1^T \\ \alpha_2^T \\ \vdots \\ \alpha_n^T \end{pmatrix} (\alpha_1 \alpha_2 \dots \alpha_n) = \begin{pmatrix} \alpha_1^T \alpha_1 & \alpha_1^T \alpha_2 & \cdots & \alpha_1^T \alpha_n \\ \alpha_2^T \alpha_1 & \alpha_2^T \alpha_2 & \cdots & \alpha_2^T \alpha_n \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_n^T \alpha_1 & \alpha_n^T \alpha_2 & \cdots & \alpha_n^T \alpha_n \end{pmatrix}$$

所以

$$Q^T Q = I \iff \begin{cases} \alpha_i^T \alpha_i = 1, \\ \alpha_i^T \alpha_j = 0, \end{cases}$$

# 正交矩阵

**定理**  $n$  阶矩阵  $Q$  是正交矩阵的充分必要条件是： $Q$  的列（行）向量组是单位正交向量组。

**证明** 设  $Q = (\alpha_1 \alpha_2 \dots \alpha_n)$ ，则

$$Q^T Q = \begin{pmatrix} \alpha_1^T \\ \alpha_2^T \\ \vdots \\ \alpha_n^T \end{pmatrix} (\alpha_1 \alpha_2 \dots \alpha_n) = \begin{pmatrix} \alpha_1^T \alpha_1 & \alpha_1^T \alpha_2 & \cdots & \alpha_1^T \alpha_n \\ \alpha_2^T \alpha_1 & \alpha_2^T \alpha_2 & \cdots & \alpha_2^T \alpha_n \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_n^T \alpha_1 & \alpha_n^T \alpha_2 & \cdots & \alpha_n^T \alpha_n \end{pmatrix}$$

所以

$$Q^T Q = I \iff \begin{cases} \alpha_i^T \alpha_i = 1, & (i = 1, 2, \dots, n) \\ \alpha_i^T \alpha_j = 0, & (i \neq j; i, j = 1, 2, \dots, n) \end{cases}$$

**例** 验证下列矩阵是否正交矩阵：

$$A_1 = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{pmatrix}, \quad A_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{pmatrix},$$

**例** 验证下列矩阵是否正交矩阵：

$$A_1 = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{pmatrix}, \quad A_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{pmatrix},$$

**提示** 验证：列向量组是单位正交向量组

**例** 验证下列矩阵是否正交矩阵：

$$A_1 = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{pmatrix}, \quad A_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{pmatrix},$$

**提示** 验证：列向量组是单位正交向量组

**答案**  $A_1$  是正交矩阵

**例** 验证下列矩阵是否正交矩阵：

$$A_1 = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{pmatrix}, \quad A_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{pmatrix},$$

**提示** 验证：列向量组是单位正交向量组

**答案**  $A_1$  是正交矩阵， $A_2$  不是正交矩阵

# 实对称矩阵的特征值和特征向量

- 对任意  $n$  阶方阵:

1. 一定有  $n$  个特征值 (计算重数, 复数域内), 可能有非实数特征值
2. 不一定能对角化



# 实对称矩阵的特征值和特征向量

● 对任意  $n$  阶方阵：

1. 一定有  $n$  个特征值（计算重数，复数域内），可能有非实数特征值
2. 不一定能对角化

例  $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  的特征值方程是

$$0 = |\lambda I - A| =$$

# 实对称矩阵的特征值和特征向量

● 对任意  $n$  阶方阵:

1. 一定有  $n$  个特征值 (计算重数, 复数域内), 可能有非实数特征值
2. 不一定能对角化

例  $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  的特征值方程是

$$0 = |\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ 1 & \lambda \end{vmatrix} =$$

# 实对称矩阵的特征值和特征向量

● 对任意  $n$  阶方阵:

1. 一定有  $n$  个特征值 (计算重数, 复数域内), 可能有非实数特征值
2. 不一定能对角化

例  $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  的特征值方程是

$$0 = |\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ 1 & \lambda \end{vmatrix} = \lambda^2 + 1$$

# 实对称矩阵的特征值和特征向量

● 对任意  $n$  阶方阵：

1. 一定有  $n$  个特征值（计算重数，复数域内），可能有非实数特征值
2. 不一定能对角化

**例**  $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  的特征值方程是

$$0 = |\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ 1 & \lambda \end{vmatrix} = \lambda^2 + 1$$

所以特征值是  $\lambda_1 = -\sqrt{-1}$ ,  $\lambda_2 = \sqrt{-1}$ 。

# 实对称矩阵的特征值和特征向量

- 对任意  $n$  阶方阵:

1. 一定有  $n$  个特征值 (计算重数, 复数域内), 可能有非实数特征值
2. 不一定能对角化

**例**  $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  的特征值方程是

$$0 = |\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ 1 & \lambda \end{vmatrix} = \lambda^2 + 1$$

所以特征值是  $\lambda_1 = -\sqrt{-1}$ ,  $\lambda_2 = \sqrt{-1}$ 。

- 对实对称矩阵, 总成立:

1. **定理** 实对称矩阵的特征值都是实数。
2. **定理** 实对称矩阵一定可以对角化。

**定理** 实对称矩阵一定可以对角化。

**定理** 实对称矩阵一定可以对角化。

也就是：设  $A$  为实对称矩阵，则一定存在可逆矩阵  $P$ ，使得

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{pmatrix}$$

**定理** 实对称矩阵一定可以对角化。

也就是：设  $A$  为实对称矩阵，则一定存在可逆矩阵  $P$ ，使得

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

事实上，还可以进一步要求  $P$  是正交矩阵：



**定理** 实对称矩阵一定可以对角化。

也就是：设  $A$  为实对称矩阵，则一定存在可逆矩阵  $P$ ，使得

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{pmatrix}$$

事实上，还可以进一步要求  $P$  是正交矩阵：

**定理** 设  $A$  为实对称矩阵，则一定存在正交矩阵  $Q$ ，使得

$$Q^{-1}AQ = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{pmatrix}$$

**定理** 实对称矩阵一定可以对角化。

也就是：设  $A$  为实对称矩阵，则一定存在可逆矩阵  $P$ ，使得

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{pmatrix}$$

事实上，还可以进一步要求  $P$  是正交矩阵：

**定理** 设  $A$  为实对称矩阵，则一定存在正交矩阵  $Q$ ，使得

$$Q^{-1}AQ = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{pmatrix}$$

**注** 由于正交矩阵满足  $Q^{-1} = Q^T$ ，

**定理** 实对称矩阵一定可以对角化。

也就是：设  $A$  为实对称矩阵，则一定存在可逆矩阵  $P$ ，使得

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{pmatrix}$$

事实上，还可以进一步要求  $P$  是正交矩阵：

**定理** 设  $A$  为实对称矩阵，则一定存在正交矩阵  $Q$ ，使得

$$Q^{-1}AQ = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{pmatrix}$$

**注** 由于正交矩阵满足  $Q^{-1} = Q^T$ ，上述等价于  $Q^T A Q = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{pmatrix}$

**定理** 实对称矩阵的对应于不同特征值的特征向量正交。

**定理** 实对称矩阵的对应于不同特征值的特征向量正交。

**证明** 设  $A$  为实对称矩阵,  $\lambda_1 \neq \lambda_2$  为两特征值,  $\alpha_1, \alpha_2$  为相应特征向量,

$$\alpha_2^T \alpha_1 = 0$$

**定理** 实对称矩阵的对应于不同特征值的特征向量正交。

**证明** 设  $A$  为实对称矩阵,  $\lambda_1 \neq \lambda_2$  为两特征值,  $\alpha_1, \alpha_2$  为相应特征向量, 则

$$A\alpha_1 = \lambda_1\alpha_1$$

$$A\alpha_2 = \lambda_2\alpha_2$$

$$\alpha_2^T \alpha_1 = 0$$

**定理** 实对称矩阵的对应于不同特征值的特征向量正交。

**证明** 设  $A$  为实对称矩阵,  $\lambda_1 \neq \lambda_2$  为两特征值,  $\alpha_1, \alpha_2$  为相应特征向量, 则

$$A\alpha_1 = \lambda_1\alpha_1 \quad \Rightarrow \quad \alpha_2^T A\alpha_1 = \lambda_1 \alpha_2^T \alpha_1$$

$$A\alpha_2 = \lambda_2\alpha_2$$

$$\alpha_2^T \alpha_1 = 0$$

**定理** 实对称矩阵的对应于不同特征值的特征向量正交。

**证明** 设  $A$  为实对称矩阵,  $\lambda_1 \neq \lambda_2$  为两特征值,  $\alpha_1, \alpha_2$  为相应特征向量, 则

$$A\alpha_1 = \lambda_1\alpha_1 \quad \Rightarrow \quad \alpha_2^T A\alpha_1 = \lambda_1 \alpha_2^T \alpha_1$$

$$A\alpha_2 = \lambda_2\alpha_2 \quad \Rightarrow \quad \alpha_1^T A\alpha_2 = \lambda_2 \alpha_1^T \alpha_2$$

$$\alpha_2^T \alpha_1 = 0$$



**定理** 实对称矩阵的对应于不同特征值的特征向量正交。

**证明** 设  $A$  为实对称矩阵,  $\lambda_1 \neq \lambda_2$  为两特征值,  $\alpha_1, \alpha_2$  为相应特征向量, 则

$$\begin{aligned} A\alpha_1 &= \lambda_1\alpha_1 \quad \Rightarrow \quad \alpha_2^T A\alpha_1 = \lambda_1 \alpha_2^T \alpha_1 \\ A\alpha_2 &= \lambda_2\alpha_2 \quad \Rightarrow \quad \alpha_1^T A\alpha_2 = \lambda_2 \alpha_1^T \alpha_2 \end{aligned}$$

$$\alpha_2^T \alpha_1 = 0$$

**定理** 实对称矩阵的对应于不同特征值的特征向量正交。

**证明** 设  $A$  为实对称矩阵,  $\lambda_1 \neq \lambda_2$  为两特征值,  $\alpha_1, \alpha_2$  为相应特征向量, 则

$$\begin{aligned} A\alpha_1 &= \lambda_1\alpha_1 \quad \Rightarrow \quad \alpha_2^T A\alpha_1 = \lambda_1 \alpha_2^T \alpha_1 \\ A\alpha_2 &= \lambda_2\alpha_2 \quad \Rightarrow \quad \alpha_1^T A\alpha_2 = \lambda_2 \alpha_1^T \alpha_2 \end{aligned}$$

$$\alpha_2^T \alpha_1 = 0$$

**定理** 实对称矩阵的对应于不同特征值的特征向量正交。

**证明** 设  $A$  为实对称矩阵,  $\lambda_1 \neq \lambda_2$  为两特征值,  $\alpha_1, \alpha_2$  为相应特征向量, 则

$$\begin{aligned} A\alpha_1 &= \lambda_1\alpha_1 \quad \Rightarrow \quad \alpha_2^T A\alpha_1 = \lambda_1 \alpha_2^T \alpha_1 \\ A\alpha_2 &= \lambda_2\alpha_2 \quad \Rightarrow \quad \alpha_1^T A\alpha_2 = \lambda_2 \alpha_1^T \alpha_2 \end{aligned}$$

注意  $\alpha_2^T A\alpha_1 = (\alpha_2^T A\alpha_1)^T =$

$$\alpha_1^T A\alpha_2 = 0$$

**定理** 实对称矩阵的对应于不同特征值的特征向量正交。

**证明** 设  $A$  为实对称矩阵,  $\lambda_1 \neq \lambda_2$  为两特征值,  $\alpha_1, \alpha_2$  为相应特征向量, 则

$$\begin{aligned} A\alpha_1 &= \lambda_1\alpha_1 \quad \Rightarrow \quad \alpha_2^T A\alpha_1 = \lambda_1 \alpha_2^T \alpha_1 \\ A\alpha_2 &= \lambda_2\alpha_2 \quad \Rightarrow \quad \alpha_1^T A\alpha_2 = \lambda_2 \alpha_1^T \alpha_2 \end{aligned}$$

注意  $\alpha_2^T A\alpha_1 = (\alpha_2^T A\alpha_1)^T = \alpha_1^T A^T (\alpha_2^T)^T =$

$$\alpha_2^T \alpha_1 = 0$$

**定理** 实对称矩阵的对应于不同特征值的特征向量正交。

**证明** 设  $A$  为实对称矩阵,  $\lambda_1 \neq \lambda_2$  为两特征值,  $\alpha_1, \alpha_2$  为相应特征向量, 则

$$\begin{aligned} A\alpha_1 &= \lambda_1\alpha_1 \quad \Rightarrow \quad \alpha_2^T A\alpha_1 = \lambda_1 \alpha_2^T \alpha_1 \\ A\alpha_2 &= \lambda_2\alpha_2 \quad \Rightarrow \quad \alpha_1^T A\alpha_2 = \lambda_2 \alpha_1^T \alpha_2 \end{aligned}$$

注意  $\alpha_2^T A\alpha_1 = (\alpha_2^T A\alpha_1)^T = \alpha_1^T A^T (\alpha_2^T)^T = \alpha_1^T A\alpha_2$

$$\alpha_2^T \alpha_1 = 0$$

**定理** 实对称矩阵的对应于不同特征值的特征向量正交。

**证明** 设  $A$  为实对称矩阵,  $\lambda_1 \neq \lambda_2$  为两特征值,  $\alpha_1, \alpha_2$  为相应特征向量, 则

$$\begin{aligned} A\alpha_1 &= \lambda_1\alpha_1 \quad \Rightarrow \quad \alpha_2^T A\alpha_1 = \lambda_1 \alpha_2^T \alpha_1 \\ A\alpha_2 &= \lambda_2\alpha_2 \quad \Rightarrow \quad \alpha_1^T A\alpha_2 = \lambda_2 \alpha_1^T \alpha_2 \end{aligned}$$

注意  $\alpha_2^T A\alpha_1 = (\alpha_2^T A\alpha_1)^T = \alpha_1^T A^T (\alpha_2^T)^T = \alpha_1^T A\alpha_2$ , 两式相减得

$$0 = (\lambda_1 - \lambda_2)\alpha_2^T \alpha_1$$

$$\alpha_2^T \alpha_1 = 0$$

**定理** 实对称矩阵的对应于不同特征值的特征向量正交。

**证明** 设  $A$  为实对称矩阵,  $\lambda_1 \neq \lambda_2$  为两特征值,  $\alpha_1, \alpha_2$  为相应特征向量, 则

$$\begin{aligned} A\alpha_1 &= \lambda_1\alpha_1 \quad \Rightarrow \quad \alpha_2^T A\alpha_1 = \lambda_1 \alpha_2^T \alpha_1 \\ A\alpha_2 &= \lambda_2\alpha_2 \quad \Rightarrow \quad \alpha_1^T A\alpha_2 = \lambda_2 \alpha_1^T \alpha_2 \end{aligned}$$

注意  $\alpha_2^T A\alpha_1 = (\alpha_2^T A\alpha_1)^T = \alpha_1^T A^T (\alpha_2^T)^T = \alpha_1^T A\alpha_2$ , 两式相减得

$$0 = (\lambda_1 - \lambda_2)\alpha_2^T \alpha_1$$

由于  $\lambda_1 \neq \lambda_2$ , 所以

$$\alpha_2^T \alpha_1 = 0$$

**定理** 设  $A$  为实对称矩阵，则存在正交矩阵  $Q$ ，使得  $Q^{-1}AQ$  为对角矩阵。



**定理** 设  $A$  为实对称矩阵，则存在正交矩阵  $Q$ ，使得  $Q^{-1}AQ$  为对角矩阵。

## 解释示意图

不同 特征值	重 数	正交化	单位化
$\lambda_1$	$n_1$		
$\lambda_2$	$n_2$		
$\vdots$	$\vdots$		
$\lambda_s$	$n_s$		
共 $n$			
$ \lambda I - A  = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_s)^{n_s}$			

**定理** 设  $A$  为实对称矩阵, 则存在正交矩阵  $Q$ , 使得  $Q^{-1}AQ$  为对角矩阵。

## 解释示意图

不同 特征值	重 数	$(\lambda_i I - A)x = 0$ 基础解系	正交化	单位化
$\lambda_1$	$n_1$			
$\lambda_2$	$n_2$			
$\vdots$	$\vdots$			
$\lambda_s$	$n_s$			
共 $n$				
$ \lambda I - A  = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_s)^{n_s}$				

**定理** 设  $A$  为实对称矩阵, 则存在正交矩阵  $Q$ , 使得  $Q^{-1}AQ$  为对角矩阵。

## 解释示意图

不同 特征值	重 数	$(\lambda_i I - A)x = 0$ 基础解系	正交化	单位化
$\lambda_1$	$n_1$	$\alpha_1^{(1)}, \dots, \alpha_{n_1}^{(1)}$		
$\lambda_2$	$n_2$			
$\vdots$	$\vdots$			
$\lambda_s$	$n_s$			
共 $n$				
$ \lambda I - A  = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \dots (\lambda - \lambda_s)^{n_s}$				

**定理** 设  $A$  为实对称矩阵, 则存在正交矩阵  $Q$ , 使得  $Q^{-1}AQ$  为对角矩阵。

## 解释示意图

不同 特征值	重 数	$(\lambda_i I - A)x = 0$ 基础解系	正交化	单位化
$\lambda_1$	$n_1$	$\alpha_1^{(1)}, \dots, \alpha_{n_1}^{(1)}$		
$\lambda_2$	$n_2$	$\alpha_1^{(2)}, \dots, \alpha_{n_2}^{(2)}$		
$\vdots$	$\vdots$			
$\lambda_s$	$n_s$			
共 $n$				
$ \lambda I - A  = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \dots (\lambda - \lambda_s)^{n_s}$				

**定理** 设  $A$  为实对称矩阵, 则存在正交矩阵  $Q$ , 使得  $Q^{-1}AQ$  为对角矩阵。

## 解释示意图

不同 特征值	重 数	$(\lambda_i I - A)x = 0$ 基础解系	正交化	单位化
$\lambda_1$	$n_1$	$\alpha_1^{(1)}, \dots, \alpha_{n_1}^{(1)}$		
$\lambda_2$	$n_2$	$\alpha_1^{(2)}, \dots, \alpha_{n_2}^{(2)}$		
$\vdots$	$\vdots$	$\vdots$		
$\lambda_s$	$n_s$	$\alpha_1^{(s)}, \dots, \alpha_{n_s}^{(s)}$		
共 $n$				
$ \lambda I - A  = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \dots (\lambda - \lambda_s)^{n_s}$				

**定理** 设  $A$  为实对称矩阵，则存在正交矩阵  $Q$ ，使得  $Q^{-1}AQ$  为对角矩阵。

## 解释示意图

不同 特征值	重 数	$(\lambda_i I - A)x = 0$ 基础解系	正交化	单位化
$\lambda_1$	$n_1$	$\alpha_1^{(1)}, \dots, \alpha_{n_1}^{(1)}$		
$\lambda_2$	$n_2$	$\alpha_1^{(2)}, \dots, \alpha_{n_2}^{(2)}$		
$\vdots$	$\vdots$	$\vdots$		
$\lambda_s$	$n_s$	$\alpha_1^{(s)}, \dots, \alpha_{n_s}^{(s)}$		
共 $n$		共 $n$ 个无关特征向量		
$ \lambda I - A  = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \dots (\lambda - \lambda_s)^{n_s}$				

**定理** 设  $A$  为实对称矩阵, 则存在正交矩阵  $Q$ , 使得  $Q^{-1}AQ$  为对角矩阵。

## 解释示意图

不同 特征值	重 数	$(\lambda_i I - A)x = 0$ 基础解系	正交化	单位化
$\lambda_1$	$n_1$	$\alpha_1^{(1)}, \dots, \alpha_{n_1}^{(1)}$		
$\lambda_2$	$n_2$	$\alpha_1^{(2)}, \dots, \alpha_{n_2}^{(2)}$		
$\vdots$	$\vdots$	$\vdots$		
$\lambda_s$	$n_s$	$\alpha_1^{(s)}, \dots, \alpha_{n_s}^{(s)}$		
共 $n$		共 $n$ 个无关特征向量		
$ \lambda I - A  = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \dots (\lambda - \lambda_s)^{n_s}$				

● 令  $P = (\alpha_1^{(1)}, \dots, \alpha_{n_s}^{(n_s)})$ , 则  $P^{-1}AP = \Lambda$ 。

**定理** 设  $A$  为实对称矩阵, 则存在正交矩阵  $Q$ , 使得  $Q^{-1}AQ$  为对角矩阵。

## 解释示意图

不同 特征值	重 数	$(\lambda_i I - A)x = 0$ 基础解系	正交化	单位化
$\lambda_1$	$n_1$	$\alpha_1^{(1)}, \dots, \alpha_{n_1}^{(1)}$		
$\lambda_2$	$n_2$	$\alpha_1^{(2)}, \dots, \alpha_{n_2}^{(2)}$		
$\vdots$	$\vdots$	$\vdots$		
$\lambda_s$	$n_s$	$\alpha_1^{(s)}, \dots, \alpha_{n_s}^{(s)}$		
共 $n$		共 $n$ 个无关特征向量		
$ \lambda I - A  = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \dots (\lambda - \lambda_s)^{n_s}$				

- 令  $P = (\alpha_1^{(1)}, \dots, \alpha_{n_s}^{(n_s)})$ , 则  $P^{-1}AP = \Lambda$ 。但一般地,  $P$  不是正交矩阵。



**定理** 设  $A$  为实对称矩阵, 则存在正交矩阵  $Q$ , 使得  $Q^{-1}AQ$  为对角矩阵。

## 解释示意图

不同 特征值	重 数	$(\lambda_i I - A)x = 0$ 基础解系	正交化	单位化
$\lambda_1$	$n_1$	$\alpha_1^{(1)}, \dots, \alpha_{n_1}^{(1)}$	$\Rightarrow \beta_1^{(1)}, \dots, \beta_{n_1}^{(1)}$	
$\lambda_2$	$n_2$	$\alpha_1^{(2)}, \dots, \alpha_{n_2}^{(2)}$		
$\vdots$	$\vdots$	$\vdots$		
$\lambda_s$	$n_s$	$\alpha_1^{(s)}, \dots, \alpha_{n_s}^{(s)}$		
共 $n$		共 $n$ 个无关特征向量		
$ \lambda I - A  = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \dots (\lambda - \lambda_s)^{n_s}$				

- 令  $P = (\alpha_1^{(1)}, \dots, \alpha_{n_s}^{(n_s)})$ , 则  $P^{-1}AP = \Lambda$ 。但一般地,  $P$  不是正交矩阵。

**定理** 设  $A$  为实对称矩阵, 则存在正交矩阵  $Q$ , 使得  $Q^{-1}AQ$  为对角矩阵。

## 解释示意图

不同 特征值	重 数	$(\lambda_i I - A)x = 0$ 基础解系	正交化	单位化
$\lambda_1$	$n_1$	$\alpha_1^{(1)}, \dots, \alpha_{n_1}^{(1)}$	$\Rightarrow \beta_1^{(1)}, \dots, \beta_{n_1}^{(1)}$	$\Rightarrow \gamma_1^{(1)}, \dots, \gamma_{n_1}^{(1)}$
$\lambda_2$	$n_2$	$\alpha_1^{(2)}, \dots, \alpha_{n_2}^{(2)}$		
$\vdots$	$\vdots$	$\vdots$		
$\lambda_s$	$n_s$	$\alpha_1^{(s)}, \dots, \alpha_{n_s}^{(s)}$		
共 $n$		共 $n$ 个无关特征向量		

$$|\lambda I - A| = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \dots (\lambda - \lambda_s)^{n_s}$$

- 令  $P = (\alpha_1^{(1)}, \dots, \alpha_{n_s}^{(n_s)})$ , 则  $P^{-1}AP = \Lambda$ 。但一般地,  $P$  不是正交矩阵。

**定理** 设  $A$  为实对称矩阵, 则存在正交矩阵  $Q$ , 使得  $Q^{-1}AQ$  为对角矩阵。

## 解释示意图

不同 特征值	重 数	$(\lambda_i I - A)x = 0$ 基础解系	正交化	单位化
$\lambda_1$	$n_1$	$\alpha_1^{(1)}, \dots, \alpha_{n_1}^{(1)}$	$\Rightarrow \beta_1^{(1)}, \dots, \beta_{n_1}^{(1)}$	$\Rightarrow \gamma_1^{(1)}, \dots, \gamma_{n_1}^{(1)}$
$\lambda_2$	$n_2$	$\alpha_1^{(2)}, \dots, \alpha_{n_2}^{(2)}$	$\Rightarrow \beta_1^{(2)}, \dots, \beta_{n_2}^{(2)}$	
$\vdots$	$\vdots$	$\vdots$		
$\lambda_s$	$n_s$	$\alpha_1^{(s)}, \dots, \alpha_{n_s}^{(s)}$		
共 $n$		共 $n$ 个无关特征向量		

$$|\lambda I - A| = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \dots (\lambda - \lambda_s)^{n_s}$$

- 令  $P = (\alpha_1^{(1)}, \dots, \alpha_{n_s}^{(n_s)})$ , 则  $P^{-1}AP = \Lambda$ 。但一般地,  $P$  不是正交矩阵。

**定理** 设  $A$  为实对称矩阵, 则存在正交矩阵  $Q$ , 使得  $Q^{-1}AQ$  为对角矩阵。

## 解释示意图

不同特征值	重数	$(\lambda_i I - A)x = 0$ 基础解系	正交化	单位化
$\lambda_1$	$n_1$	$\alpha_1^{(1)}, \dots, \alpha_{n_1}^{(1)}$	$\Rightarrow \beta_1^{(1)}, \dots, \beta_{n_1}^{(1)}$	$\Rightarrow \gamma_1^{(1)}, \dots, \gamma_{n_1}^{(1)}$
$\lambda_2$	$n_2$	$\alpha_1^{(2)}, \dots, \alpha_{n_2}^{(2)}$	$\Rightarrow \beta_1^{(2)}, \dots, \beta_{n_2}^{(2)}$	$\Rightarrow \gamma_1^{(2)}, \dots, \gamma_{n_2}^{(2)}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
$\lambda_s$	$n_s$	$\alpha_1^{(s)}, \dots, \alpha_{n_s}^{(s)}$		
共 $n$		共 $n$ 个无关特征向量		

$$|\lambda I - A| = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \dots (\lambda - \lambda_s)^{n_s}$$

- 令  $P = (\alpha_1^{(1)}, \dots, \alpha_{n_s}^{(n_s)})$ , 则  $P^{-1}AP = \Lambda$ 。但一般地,  $P$  不是正交矩阵。

**定理** 设  $A$  为实对称矩阵, 则存在正交矩阵  $Q$ , 使得  $Q^{-1}AQ$  为对角矩阵。

### 解释示意图

不同 特征值	重 数	$(\lambda_i I - A)x = 0$ 基础解系	正交化	单位化
$\lambda_1$	$n_1$	$\alpha_1^{(1)}, \dots, \alpha_{n_1}^{(1)}$	$\Rightarrow \beta_1^{(1)}, \dots, \beta_{n_1}^{(1)}$	$\Rightarrow \gamma_1^{(1)}, \dots, \gamma_{n_1}^{(1)}$
$\lambda_2$	$n_2$	$\alpha_1^{(2)}, \dots, \alpha_{n_2}^{(2)}$	$\Rightarrow \beta_1^{(2)}, \dots, \beta_{n_2}^{(2)}$	$\Rightarrow \gamma_1^{(2)}, \dots, \gamma_{n_2}^{(2)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\lambda_s$	$n_s$	$\alpha_1^{(s)}, \dots, \alpha_{n_s}^{(s)}$	$\Rightarrow \beta_1^{(s)}, \dots, \beta_{n_s}^{(s)}$	
共 $n$		共 $n$ 个无关特征向量		

$$|\lambda I - A| = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \dots (\lambda - \lambda_s)^{n_s}$$

- 令  $P = (\alpha_1^{(1)}, \dots, \alpha_{n_s}^{(n_s)})$ , 则  $P^{-1}AP = \Lambda$ 。但一般地,  $P$  不是正交矩阵。

**定理** 设  $A$  为实对称矩阵, 则存在正交矩阵  $Q$ , 使得  $Q^{-1}AQ$  为对角矩阵。

### 解释示意图

不同 特征值	重 数	$(\lambda_i I - A)x = 0$ 基础解系	正交化	单位化
$\lambda_1$	$n_1$	$\alpha_1^{(1)}, \dots, \alpha_{n_1}^{(1)}$	$\Rightarrow \beta_1^{(1)}, \dots, \beta_{n_1}^{(1)}$	$\Rightarrow \gamma_1^{(1)}, \dots, \gamma_{n_1}^{(1)}$
$\lambda_2$	$n_2$	$\alpha_1^{(2)}, \dots, \alpha_{n_2}^{(2)}$	$\Rightarrow \beta_1^{(2)}, \dots, \beta_{n_2}^{(2)}$	$\Rightarrow \gamma_1^{(2)}, \dots, \gamma_{n_2}^{(2)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\lambda_s$	$n_s$	$\alpha_1^{(s)}, \dots, \alpha_{n_s}^{(s)}$	$\Rightarrow \beta_1^{(s)}, \dots, \beta_{n_s}^{(s)}$	$\Rightarrow \gamma_1^{(s)}, \dots, \gamma_{n_s}^{(s)}$
共 $n$		共 $n$ 个无关特征向量		

$$|\lambda I - A| = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \dots (\lambda - \lambda_s)^{n_s}$$

- 令  $P = (\alpha_1^{(1)}, \dots, \alpha_{n_s}^{(n_s)})$ , 则  $P^{-1}AP = \Lambda$ 。但一般地,  $P$  不是正交矩阵。

**定理** 设  $A$  为实对称矩阵, 则存在正交矩阵  $Q$ , 使得  $Q^{-1}AQ$  为对角矩阵。

### 解释示意图

不同特征值	重数	$(\lambda_i I - A)x = 0$ 基础解系	正交化	单位化
$\lambda_1$	$n_1$	$\alpha_1^{(1)}, \dots, \alpha_{n_1}^{(1)}$	$\Rightarrow \beta_1^{(1)}, \dots, \beta_{n_1}^{(1)}$	$\Rightarrow \gamma_1^{(1)}, \dots, \gamma_{n_1}^{(1)}$
$\lambda_2$	$n_2$	$\alpha_1^{(2)}, \dots, \alpha_{n_2}^{(2)}$	$\Rightarrow \beta_1^{(2)}, \dots, \beta_{n_2}^{(2)}$	$\Rightarrow \gamma_1^{(2)}, \dots, \gamma_{n_2}^{(2)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\lambda_s$	$n_s$	$\alpha_1^{(s)}, \dots, \alpha_{n_s}^{(s)}$	$\Rightarrow \beta_1^{(s)}, \dots, \beta_{n_s}^{(s)}$	$\Rightarrow \gamma_1^{(s)}, \dots, \gamma_{n_s}^{(s)}$
共 $n$		共 $n$ 个无关特征向量		构成单位正交特征向量

$$|\lambda I - A| = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \dots (\lambda - \lambda_s)^{n_s}$$

- 令  $P = (\alpha_1^{(1)}, \dots, \alpha_{n_s}^{(n_s)})$ , 则  $P^{-1}AP = \Lambda$ 。但一般地,  $P$  不是正交矩阵。

**定理** 设  $A$  为实对称矩阵, 则存在正交矩阵  $Q$ , 使得  $Q^{-1}AQ$  为对角矩阵。

## 解释示意图

不同特征值	重数	$(\lambda_i I - A)x = 0$ 基础解系	正交化	单位化
$\lambda_1$	$n_1$	$\alpha_1^{(1)}, \dots, \alpha_{n_1}^{(1)}$	$\Rightarrow \beta_1^{(1)}, \dots, \beta_{n_1}^{(1)}$	$\Rightarrow \gamma_1^{(1)}, \dots, \gamma_{n_1}^{(1)}$
$\lambda_2$	$n_2$	$\alpha_1^{(2)}, \dots, \alpha_{n_2}^{(2)}$	$\Rightarrow \beta_1^{(2)}, \dots, \beta_{n_2}^{(2)}$	$\Rightarrow \gamma_1^{(2)}, \dots, \gamma_{n_2}^{(2)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\lambda_s$	$n_s$	$\alpha_1^{(s)}, \dots, \alpha_{n_s}^{(s)}$	$\Rightarrow \beta_1^{(s)}, \dots, \beta_{n_s}^{(s)}$	$\Rightarrow \gamma_1^{(s)}, \dots, \gamma_{n_s}^{(s)}$
共 $n$		共 $n$ 个无关特征向量		构成单位正交特征向量

$$|\lambda I - A| = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \dots (\lambda - \lambda_s)^{n_s}$$

- 令  $P = (\alpha_1^{(1)}, \dots, \alpha_{n_s}^{(n_s)})$ , 则  $P^{-1}AP = \Lambda$ 。但一般地,  $P$  不是正交矩阵。
- 令  $Q = (\gamma_1^{(1)}, \dots, \gamma_{n_s}^{(n_s)})$ ,



**定理** 设  $A$  为实对称矩阵, 则存在正交矩阵  $Q$ , 使得  $Q^{-1}AQ$  为对角矩阵。

## 解释示意图

不同特征值	重数	$(\lambda_i I - A)x = 0$ 基础解系	正交化	单位化
$\lambda_1$	$n_1$	$\alpha_1^{(1)}, \dots, \alpha_{n_1}^{(1)}$	$\Rightarrow \beta_1^{(1)}, \dots, \beta_{n_1}^{(1)}$	$\Rightarrow \gamma_1^{(1)}, \dots, \gamma_{n_1}^{(1)}$
$\lambda_2$	$n_2$	$\alpha_1^{(2)}, \dots, \alpha_{n_2}^{(2)}$	$\Rightarrow \beta_1^{(2)}, \dots, \beta_{n_2}^{(2)}$	$\Rightarrow \gamma_1^{(2)}, \dots, \gamma_{n_2}^{(2)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\lambda_s$	$n_s$	$\alpha_1^{(s)}, \dots, \alpha_{n_s}^{(s)}$	$\Rightarrow \beta_1^{(s)}, \dots, \beta_{n_s}^{(s)}$	$\Rightarrow \gamma_1^{(s)}, \dots, \gamma_{n_s}^{(s)}$
共 $n$		共 $n$ 个无关特征向量		构成单位正交特征向量

$$|\lambda I - A| = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \dots (\lambda - \lambda_s)^{n_s}$$

- 令  $P = (\alpha_1^{(1)}, \dots, \alpha_{n_s}^{(n_s)})$ , 则  $P^{-1}AP = \Lambda$ 。但一般地,  $P$  不是正交矩阵。
- 令  $Q = (\gamma_1^{(1)}, \dots, \gamma_{n_s}^{(n_s)})$ , 则  $Q^{-1}AQ = \Lambda$ ,

**定理** 设  $A$  为实对称矩阵, 则存在正交矩阵  $Q$ , 使得  $Q^{-1}AQ$  为对角矩阵。

## 解释示意图

不同特征值	重数	$(\lambda_i I - A)x = 0$ 基础解系	正交化	单位化
$\lambda_1$	$n_1$	$\alpha_1^{(1)}, \dots, \alpha_{n_1}^{(1)}$	$\Rightarrow \beta_1^{(1)}, \dots, \beta_{n_1}^{(1)}$	$\Rightarrow \gamma_1^{(1)}, \dots, \gamma_{n_1}^{(1)}$
$\lambda_2$	$n_2$	$\alpha_1^{(2)}, \dots, \alpha_{n_2}^{(2)}$	$\Rightarrow \beta_1^{(2)}, \dots, \beta_{n_2}^{(2)}$	$\Rightarrow \gamma_1^{(2)}, \dots, \gamma_{n_2}^{(2)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\lambda_s$	$n_s$	$\alpha_1^{(s)}, \dots, \alpha_{n_s}^{(s)}$	$\Rightarrow \beta_1^{(s)}, \dots, \beta_{n_s}^{(s)}$	$\Rightarrow \gamma_1^{(s)}, \dots, \gamma_{n_s}^{(s)}$
共 $n$		共 $n$ 个无关特征向量		构成单位正交特征向量

$$|\lambda I - A| = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \dots (\lambda - \lambda_s)^{n_s}$$

- 令  $P = (\alpha_1^{(1)}, \dots, \alpha_{n_s}^{(n_s)})$ , 则  $P^{-1}AP = \Lambda$ 。但一般地,  $P$  不是正交矩阵。
- 令  $Q = (\gamma_1^{(1)}, \dots, \gamma_{n_s}^{(n_s)})$ , 则  $Q^{-1}AQ = \Lambda$ , 并且  $Q$  是正交矩阵。

**例 1** 求矩阵  $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$  的特征值与特征向量。

$$Q^{-1}AQ = \begin{pmatrix} * & \\ & * \end{pmatrix}$$

**例 1** 求矩阵  $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$  的特征值与特征向量。

**解**

- 特征方程:  $0 = |\lambda I - A|$

$$Q^{-1}AQ = \begin{pmatrix} * & \\ & * \end{pmatrix}$$

**例 1** 求矩阵  $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$  的特征值与特征向量。

**解**

● 特征方程:  $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} =$

$$Q^{-1}AQ = \begin{pmatrix} * & \\ & * \end{pmatrix}$$

**例 1** 求矩阵  $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$  的特征值与特征向量。

**解**

- 特征方程:  $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^2 - 1$

$$Q^{-1}AQ = \begin{pmatrix} * & \\ & * \end{pmatrix}$$

**例 1** 求矩阵  $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$  的特征值与特征向量。

**解**

- 特征方程:  $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$

$$Q^{-1}AQ = \begin{pmatrix} * & \\ & * \end{pmatrix}$$

**例 1** 求矩阵  $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$  的特征值与特征向量。

**解**

- 特征方程:  $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$

- $\lambda_1 = 1$

- $\lambda_2 = 3$

$$Q^{-1}AQ = \begin{pmatrix} * & \\ & * \end{pmatrix}$$



**例 1** 求矩阵  $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$  的特征值与特征向量。

**解**

- 特征方程:  $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$

- $\lambda_1 = 1$

- $\lambda_2 = 3$

$$Q^{-1}AQ = \begin{pmatrix} 1 & \\ & 3 \end{pmatrix}$$

**例 1** 求矩阵  $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$  的特征值与特征向量。

**解**

- 特征方程:  $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$

- $\lambda_1 = 1$ , 求解  $(\lambda_1 I - A)x = 0$ :

- $\lambda_2 = 3$ , 求解  $(\lambda_2 I - A)x = 0$ :

$$Q^{-1}AQ = \begin{pmatrix} 1 & \\ & 3 \end{pmatrix}$$

**例 1** 求矩阵  $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$  的特征值与特征向量。

**解**

- 特征方程:  $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$
- $\lambda_1 = 1$ , 求解  $(\lambda_1 I - A)x = 0$ :

$$(1I - A : 0) = \left( \begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array} \right)$$

- $\lambda_2 = 3$ , 求解  $(\lambda_2 I - A)x = 0$ :

$$Q^{-1}AQ = \begin{pmatrix} 1 & \\ & 3 \end{pmatrix}$$

**例 1** 求矩阵  $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$  的特征值与特征向量。

**解**

- 特征方程:  $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$
- $\lambda_1 = 1$ , 求解  $(\lambda_1 I - A)x = 0$ :

$$(1I - A : 0) = \left( \begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array} \right) \rightarrow \left( \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right)$$

- $\lambda_2 = 3$ , 求解  $(\lambda_2 I - A)x = 0$ :

$$Q^{-1}AQ = \begin{pmatrix} 1 & \\ & 3 \end{pmatrix}$$

**例 1** 求矩阵  $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$  的特征值与特征向量。

**解**

- 特征方程:  $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$

- $\lambda_1 = 1$ , 求解  $(\lambda_1 I - A)x = 0$ :

$$(1I - A : 0) = \left( \begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array} \right) \rightarrow \left( \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad x_1 + x_2 = 0$$

- $\lambda_2 = 3$ , 求解  $(\lambda_2 I - A)x = 0$ :

$$Q^{-1}AQ = \begin{pmatrix} 1 & \\ & 3 \end{pmatrix}$$

**例 1** 求矩阵  $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$  的特征值与特征向量。

**解**

- 特征方程:  $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$

- $\lambda_1 = 1$ , 求解  $(\lambda_1 I - A)x = 0$ :

$$(1I - A : 0) = \left( \begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array} \right) \rightarrow \left( \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad \begin{array}{l} x_1 + x_2 = 0 \\ \downarrow \\ x_1 = -x_2 \end{array}$$

- $\lambda_2 = 3$ , 求解  $(\lambda_2 I - A)x = 0$ :

$$Q^{-1}AQ = \begin{pmatrix} 1 & \\ & 3 \end{pmatrix}$$

**例 1** 求矩阵  $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$  的特征值与特征向量。

**解**

- 特征方程:  $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$

- $\lambda_1 = 1$ , 求解  $(\lambda_1 I - A)x = 0$ :

$$(1I - A : 0) = \left( \begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array} \right) \rightarrow \left( \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad \begin{array}{l} x_1 + x_2 = 0 \\ \downarrow \\ x_1 = -x_2 \end{array}$$

基础解系:  $\alpha_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

- $\lambda_2 = 3$ , 求解  $(\lambda_2 I - A)x = 0$ :

$$Q^{-1}AQ = \begin{pmatrix} 1 & \\ & 3 \end{pmatrix}$$

**例 1** 求矩阵  $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$  的特征值与特征向量。

**解**

- 特征方程:  $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$

- $\lambda_1 = 1$ , 求解  $(\lambda_1 I - A)x = 0$ :

$$(1I - A : 0) = \left( \begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array} \right) \rightarrow \left( \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad \begin{array}{l} x_1 + x_2 = 0 \\ \downarrow \\ x_1 = -x_2 \end{array}$$

基础解系:  $\alpha_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

- $\lambda_2 = 3$ , 求解  $(\lambda_2 I - A)x = 0$ :

$$Q^{-1}AQ = \begin{pmatrix} 1 & \\ & 3 \end{pmatrix}$$



**例 1** 求矩阵  $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$  的特征值与特征向量。

**解**

- 特征方程:  $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$

- $\lambda_1 = 1$ , 求解  $(\lambda_1 I - A)x = 0$ :

$$(1I - A : 0) = \left( \begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array} \right) \rightarrow \left( \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad \begin{array}{l} x_1 + x_2 = 0 \\ \downarrow \\ x_1 = -x_2 \end{array}$$

基础解系:  $\alpha_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

- $\lambda_2 = 3$ , 求解  $(\lambda_2 I - A)x = 0$ :

$$(3I - A : 0) = \left( \begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array} \right)$$

$$Q^{-1}AQ = \begin{pmatrix} 1 & \\ & 3 \end{pmatrix}$$

**例 1** 求矩阵  $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$  的特征值与特征向量。

**解**

- 特征方程:  $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$

- $\lambda_1 = 1$ , 求解  $(\lambda_1 I - A)x = 0$ :

$$(1I - A : 0) = \left( \begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array} \right) \rightarrow \left( \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad \begin{array}{l} x_1 + x_2 = 0 \\ \downarrow \\ x_1 = -x_2 \end{array}$$

基础解系:  $\alpha_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

- $\lambda_2 = 3$ , 求解  $(\lambda_2 I - A)x = 0$ :

$$(3I - A : 0) = \left( \begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array} \right) \rightarrow \left( \begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right)$$

$$Q^{-1}AQ = \begin{pmatrix} 1 & \\ & 3 \end{pmatrix}$$

**例 1** 求矩阵  $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$  的特征值与特征向量。

**解**

● 特征方程:  $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$

●  $\lambda_1 = 1$ , 求解  $(\lambda_1 I - A)x = 0$ :

$$(1I - A : 0) = \left( \begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array} \right) \rightarrow \left( \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad \begin{array}{l} x_1 + x_2 = 0 \\ \downarrow \\ x_1 = -x_2 \end{array}$$

基础解系:  $\alpha_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

●  $\lambda_2 = 3$ , 求解  $(\lambda_2 I - A)x = 0$ :

$$(3I - A : 0) = \left( \begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array} \right) \rightarrow \left( \begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad x_1 - x_2 = 0$$

$$Q^{-1}AQ = \begin{pmatrix} 1 & \\ & 3 \end{pmatrix}$$

**例 1** 求矩阵  $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$  的特征值与特征向量。

**解**

● 特征方程:  $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$

●  $\lambda_1 = 1$ , 求解  $(\lambda_1 I - A)x = 0$ :

$$(1I - A : 0) = \left( \begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array} \right) \rightarrow \left( \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad \begin{array}{l} x_1 + x_2 = 0 \\ \downarrow \\ x_1 = -x_2 \end{array}$$

基础解系:  $\alpha_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

●  $\lambda_2 = 3$ , 求解  $(\lambda_2 I - A)x = 0$ :

$$(3I - A : 0) = \left( \begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array} \right) \rightarrow \left( \begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad \begin{array}{l} x_1 - x_2 = 0 \\ \downarrow \\ x_1 = x_2 \end{array}$$

$$Q^{-1}AQ = \begin{pmatrix} 1 & \\ & 3 \end{pmatrix}$$

**例 1** 求矩阵  $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$  的特征值与特征向量。

**解**

● 特征方程:  $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$

●  $\lambda_1 = 1$ , 求解  $(\lambda_1 I - A)x = 0$ :

$$(1I - A : 0) = \left( \begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array} \right) \rightarrow \left( \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad \begin{array}{l} x_1 + x_2 = 0 \\ \downarrow \\ x_1 = -x_2 \end{array}$$

基础解系:  $\alpha_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

●  $\lambda_2 = 3$ , 求解  $(\lambda_2 I - A)x = 0$ :

$$(3I - A : 0) = \left( \begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array} \right) \rightarrow \left( \begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad \begin{array}{l} x_1 - x_2 = 0 \\ \downarrow \\ x_1 = x_2 \end{array}$$

基础解系:  $\alpha_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$Q^{-1}AQ = \begin{pmatrix} 1 & \\ & 3 \end{pmatrix}$$

**例 1** 求矩阵  $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$  的特征值与特征向量。

**解**

- 特征方程:  $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$

- $\lambda_1 = 1$ , 求解  $(\lambda_1 I - A)x = 0$ :

$$(1I - A : 0) = \left( \begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array} \right) \rightarrow \left( \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad \begin{array}{l} x_1 + x_2 = 0 \\ \downarrow \\ x_1 = -x_2 \end{array}$$

基础解系:  $\alpha_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

- $\lambda_2 = 3$ , 求解  $(\lambda_2 I - A)x = 0$ :

$$(3I - A : 0) = \left( \begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array} \right) \rightarrow \left( \begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad \begin{array}{l} x_1 - x_2 = 0 \\ \downarrow \\ x_1 = x_2 \end{array}$$

基础解系:  $\alpha_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$Q^{-1}AQ = \begin{pmatrix} 1 & \\ & 3 \end{pmatrix}$$

**例 1** 求矩阵  $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$  的特征值与特征向量。

**解**

● 特征方程:  $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$

●  $\lambda_1 = 1$ , 求解  $(\lambda_1 I - A)x = 0$ :

$$(1I - A : 0) = \left( \begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array} \right) \rightarrow \left( \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad \begin{array}{l} x_1 + x_2 = 0 \\ \downarrow \\ x_1 = -x_2 \end{array}$$

基础解系:  $\alpha_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

●  $\lambda_2 = 3$ , 求解  $(\lambda_2 I - A)x = 0$ :

$$(3I - A : 0) = \left( \begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array} \right) \rightarrow \left( \begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad \begin{array}{l} x_1 - x_2 = 0 \\ \downarrow \\ x_1 = x_2 \end{array}$$

基础解系:  $\alpha_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

所以取  $Q = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$ , 则  $Q^{-1}AQ = \begin{pmatrix} 1 & \\ & 3 \end{pmatrix}$

**例 2** 求矩阵  $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$  的特征值与特征向量。

$$Q^{-1}AQ = \begin{pmatrix} * & \\ & * \end{pmatrix}$$



**例 2** 求矩阵  $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$  的特征值与特征向量。

**解**

- 特征方程:  $0 = |\lambda I - A|$

$$Q^{-1}AQ = \begin{pmatrix} * & \\ & * \end{pmatrix}$$

**例 2** 求矩阵  $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$  的特征值与特征向量。

**解**

● 特征方程:  $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} =$

$$Q^{-1}AQ = \begin{pmatrix} * & \\ & * \end{pmatrix}$$

**例 2** 求矩阵  $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$  的特征值与特征向量。

**解**

- 特征方程:  $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2 - (-2)^2$

$$Q^{-1}AQ = \begin{pmatrix} * & \\ & * \end{pmatrix}$$

**例 2** 求矩阵  $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$  的特征值与特征向量。

**解**

- 特征方程:  $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = (\lambda + 1)(\lambda - 3)$

$$Q^{-1}AQ = \begin{pmatrix} * & \\ & * \end{pmatrix}$$

**例 2** 求矩阵  $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$  的特征值与特征向量。

**解**

- 特征方程:  $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = (\lambda + 1)(\lambda - 3)$

- $\lambda_1 = -1$

- $\lambda_2 = 3$

$$Q^{-1}AQ = \begin{pmatrix} * & \\ & * \end{pmatrix}$$

**例 2** 求矩阵  $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$  的特征值与特征向量。

**解**

- 特征方程:  $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = (\lambda + 1)(\lambda - 3)$

- $\lambda_1 = -1$

- $\lambda_2 = 3$

$$Q^{-1}AQ = \begin{pmatrix} -1 & \\ & 3 \end{pmatrix}$$

**例 2** 求矩阵  $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$  的特征值与特征向量。

**解**

- 特征方程:  $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = (\lambda + 1)(\lambda - 3)$
- $\lambda_1 = -1$ , 求解  $(\lambda_1 I - A)x = 0$ :

- $\lambda_2 = 3$ , 求解  $(\lambda_2 I - A)x = 0$ :

$$Q^{-1}AQ = \begin{pmatrix} -1 & \\ & 3 \end{pmatrix}$$

**例 2** 求矩阵  $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$  的特征值与特征向量。

**解**

- 特征方程:  $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = (\lambda + 1)(\lambda - 3)$
- $\lambda_1 = -1$ , 求解  $(\lambda_1 I - A)x = 0$ :

$$(-I - A : 0) = \left( \begin{array}{cc|c} -2 & -2 & 0 \\ -2 & -2 & 0 \end{array} \right)$$

- $\lambda_2 = 3$ , 求解  $(\lambda_2 I - A)x = 0$ :

$$Q^{-1}AQ = \begin{pmatrix} -1 & \\ & 3 \end{pmatrix}$$



**例 2** 求矩阵  $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$  的特征值与特征向量。

**解**

- 特征方程:  $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = (\lambda + 1)(\lambda - 3)$
- $\lambda_1 = -1$ , 求解  $(\lambda_1 I - A)x = 0$ :

$$(-I - A : 0) = \left( \begin{array}{cc|c} -2 & -2 & 0 \\ -2 & -2 & 0 \end{array} \right) \rightarrow \left( \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right)$$

- $\lambda_2 = 3$ , 求解  $(\lambda_2 I - A)x = 0$ :

$$Q^{-1}AQ = \begin{pmatrix} -1 & \\ & 3 \end{pmatrix}$$

**例 2** 求矩阵  $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$  的特征值与特征向量。

**解**

- 特征方程:  $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = (\lambda + 1)(\lambda - 3)$
- $\lambda_1 = -1$ , 求解  $(\lambda_1 I - A)x = 0$ :

$$(-I - A : 0) = \left( \begin{array}{cc|c} -2 & -2 & 0 \\ -2 & -2 & 0 \end{array} \right) \rightarrow \left( \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad x_1 + x_2 = 0$$

- $\lambda_2 = 3$ , 求解  $(\lambda_2 I - A)x = 0$ :

$$Q^{-1}AQ = \begin{pmatrix} -1 & \\ & 3 \end{pmatrix}$$

**例 2** 求矩阵  $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$  的特征值与特征向量。

**解**

- 特征方程:  $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = (\lambda + 1)(\lambda - 3)$
- $\lambda_1 = -1$ , 求解  $(\lambda_1 I - A)x = 0$ :

$$(-I - A : 0) = \left( \begin{array}{cc|c} -2 & -2 & 0 \\ -2 & -2 & 0 \end{array} \right) \rightarrow \left( \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad \begin{array}{l} x_1 + x_2 = 0 \\ \downarrow \\ x_1 = -x_2 \end{array}$$

- $\lambda_2 = 3$ , 求解  $(\lambda_2 I - A)x = 0$ :

$$Q^{-1}AQ = \begin{pmatrix} -1 & \\ & 3 \end{pmatrix}$$

**例 2** 求矩阵  $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$  的特征值与特征向量。

**解**

- 特征方程:  $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = (\lambda + 1)(\lambda - 3)$

- $\lambda_1 = -1$ , 求解  $(\lambda_1 I - A)x = 0$ :

$$(-I - A : 0) = \left( \begin{array}{cc|c} -2 & -2 & 0 \\ -2 & -2 & 0 \end{array} \right) \rightarrow \left( \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad \begin{array}{l} x_1 + x_2 = 0 \\ \downarrow \\ x_1 = -x_2 \end{array}$$

基础解系:  $\alpha_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

- $\lambda_2 = 3$ , 求解  $(\lambda_2 I - A)x = 0$ :

$$Q^{-1}AQ = \begin{pmatrix} -1 & \\ & 3 \end{pmatrix}$$

**例 2** 求矩阵  $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$  的特征值与特征向量。

**解**

- 特征方程:  $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = (\lambda + 1)(\lambda - 3)$

- $\lambda_1 = -1$ , 求解  $(\lambda_1 I - A)x = 0$ :

$$(-I - A : 0) = \left( \begin{array}{cc|c} -2 & -2 & 0 \\ -2 & -2 & 0 \end{array} \right) \rightarrow \left( \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad \begin{array}{l} x_1 + x_2 = 0 \\ \downarrow \\ x_1 = -x_2 \end{array}$$

基础解系:  $\alpha_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

- $\lambda_2 = 3$ , 求解  $(\lambda_2 I - A)x = 0$ :

$$Q^{-1}AQ = \begin{pmatrix} -1 & \\ & 3 \end{pmatrix}$$

**例 2** 求矩阵  $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$  的特征值与特征向量。

**解**

- 特征方程:  $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = (\lambda + 1)(\lambda - 3)$

- $\lambda_1 = -1$ , 求解  $(\lambda_1 I - A)x = 0$ :

$$(-I - A : 0) = \left( \begin{array}{cc|c} -2 & -2 & 0 \\ -2 & -2 & 0 \end{array} \right) \rightarrow \left( \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad \begin{array}{l} x_1 + x_2 = 0 \\ \downarrow \\ x_1 = -x_2 \end{array}$$

基础解系:  $\alpha_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

- $\lambda_2 = 3$ , 求解  $(\lambda_2 I - A)x = 0$ :

$$(3I - A : 0) = \left( \begin{array}{cc|c} 2 & -2 & 0 \\ -2 & 2 & 0 \end{array} \right)$$

$$Q^{-1}AQ = \begin{pmatrix} -1 & \\ & 3 \end{pmatrix}$$

**例 2** 求矩阵  $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$  的特征值与特征向量。

**解**

- 特征方程:  $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = (\lambda + 1)(\lambda - 3)$

- $\lambda_1 = -1$ , 求解  $(\lambda_1 I - A)x = 0$ :

$$(-I - A : 0) = \left( \begin{array}{cc|c} -2 & -2 & 0 \\ -2 & -2 & 0 \end{array} \right) \rightarrow \left( \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad \begin{array}{l} x_1 + x_2 = 0 \\ \downarrow \\ x_1 = -x_2 \end{array}$$

基础解系:  $\alpha_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

- $\lambda_2 = 3$ , 求解  $(\lambda_2 I - A)x = 0$ :

$$(3I - A : 0) = \left( \begin{array}{cc|c} 2 & -2 & 0 \\ -2 & 2 & 0 \end{array} \right) \rightarrow \left( \begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right)$$

$$Q^{-1}AQ = \begin{pmatrix} -1 & \\ & 3 \end{pmatrix}$$

**例 2** 求矩阵  $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$  的特征值与特征向量。

**解**

- 特征方程:  $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = (\lambda + 1)(\lambda - 3)$

- $\lambda_1 = -1$ , 求解  $(\lambda_1 I - A)x = 0$ :

$$(-I - A : 0) = \left( \begin{array}{cc|c} -2 & -2 & 0 \\ -2 & -2 & 0 \end{array} \right) \rightarrow \left( \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad \begin{array}{l} x_1 + x_2 = 0 \\ \downarrow \\ x_1 = -x_2 \end{array}$$

基础解系:  $\alpha_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

- $\lambda_2 = 3$ , 求解  $(\lambda_2 I - A)x = 0$ :

$$(3I - A : 0) = \left( \begin{array}{cc|c} 2 & -2 & 0 \\ -2 & 2 & 0 \end{array} \right) \rightarrow \left( \begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad x_1 - x_2 = 0$$

$$Q^{-1}AQ = \begin{pmatrix} -1 & \\ & 3 \end{pmatrix}$$



**例 2** 求矩阵  $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$  的特征值与特征向量。

**解**

● 特征方程:  $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = (\lambda + 1)(\lambda - 3)$

●  $\lambda_1 = -1$ , 求解  $(\lambda_1 I - A)x = 0$ :

$$(-I - A : 0) = \left( \begin{array}{cc|c} -2 & -2 & 0 \\ -2 & -2 & 0 \end{array} \right) \rightarrow \left( \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad \begin{array}{l} x_1 + x_2 = 0 \\ \downarrow \\ x_1 = -x_2 \end{array}$$

基础解系:  $\alpha_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

●  $\lambda_2 = 3$ , 求解  $(\lambda_2 I - A)x = 0$ :

$$(3I - A : 0) = \left( \begin{array}{cc|c} 2 & -2 & 0 \\ -2 & 2 & 0 \end{array} \right) \rightarrow \left( \begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad \begin{array}{l} x_1 - x_2 = 0 \\ \downarrow \\ x_1 = x_2 \end{array}$$

$$Q^{-1}AQ = \begin{pmatrix} -1 & \\ & 3 \end{pmatrix}$$

**例 2** 求矩阵  $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$  的特征值与特征向量。

**解**

● 特征方程:  $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = (\lambda + 1)(\lambda - 3)$

●  $\lambda_1 = -1$ , 求解  $(\lambda_1 I - A)x = 0$ :

$$(-I - A : 0) = \left( \begin{array}{cc|c} -2 & -2 & 0 \\ -2 & -2 & 0 \end{array} \right) \rightarrow \left( \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad \begin{array}{l} x_1 + x_2 = 0 \\ \downarrow \\ x_1 = -x_2 \end{array}$$

基础解系:  $\alpha_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

●  $\lambda_2 = 3$ , 求解  $(\lambda_2 I - A)x = 0$ :

$$(3I - A : 0) = \left( \begin{array}{cc|c} 2 & -2 & 0 \\ -2 & 2 & 0 \end{array} \right) \rightarrow \left( \begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad \begin{array}{l} x_1 - x_2 = 0 \\ \downarrow \\ x_1 = x_2 \end{array}$$

基础解系:  $\alpha_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$Q^{-1}AQ = \begin{pmatrix} -1 & \\ & 3 \end{pmatrix}$$

**例 2** 求矩阵  $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$  的特征值与特征向量。

**解**

- 特征方程:  $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = (\lambda + 1)(\lambda - 3)$

- $\lambda_1 = -1$ , 求解  $(\lambda_1 I - A)x = 0$ :

$$(-I - A : 0) = \left( \begin{array}{cc|c} -2 & -2 & 0 \\ -2 & -2 & 0 \end{array} \right) \rightarrow \left( \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad \begin{array}{l} x_1 + x_2 = 0 \\ \downarrow \\ x_1 = -x_2 \end{array}$$

基础解系:  $\alpha_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

- $\lambda_2 = 3$ , 求解  $(\lambda_2 I - A)x = 0$ :

$$(3I - A : 0) = \left( \begin{array}{cc|c} 2 & -2 & 0 \\ -2 & 2 & 0 \end{array} \right) \rightarrow \left( \begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad \begin{array}{l} x_1 - x_2 = 0 \\ \downarrow \\ x_1 = x_2 \end{array}$$

基础解系:  $\alpha_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$Q^{-1}AQ = \begin{pmatrix} -1 & \\ & 3 \end{pmatrix}$$

**例 2** 求矩阵  $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$  的特征值与特征向量。

**解**

● 特征方程:  $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = (\lambda + 1)(\lambda - 3)$

●  $\lambda_1 = -1$ , 求解  $(\lambda_1 I - A)x = 0$ :

$$(-I - A : 0) = \left( \begin{array}{cc|c} -2 & -2 & 0 \\ -2 & -2 & 0 \end{array} \right) \rightarrow \left( \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad \begin{array}{l} x_1 + x_2 = 0 \\ \downarrow \\ x_1 = -x_2 \end{array}$$

基础解系:  $\alpha_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

●  $\lambda_2 = 3$ , 求解  $(\lambda_2 I - A)x = 0$ :

$$(3I - A : 0) = \left( \begin{array}{cc|c} 2 & -2 & 0 \\ -2 & 2 & 0 \end{array} \right) \rightarrow \left( \begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad \begin{array}{l} x_1 - x_2 = 0 \\ \downarrow \\ x_1 = x_2 \end{array}$$

基础解系:  $\alpha_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

所以取  $Q = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$ , 则  $Q^{-1}AQ = \begin{pmatrix} -1 & \\ & 3 \end{pmatrix}$

**例 1**  $A = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 3 \end{pmatrix}$

$$Q^{-1}AQ = \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix}$$

**例 1**  $A = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 3 \end{pmatrix}$ , 特征方程:  $0 = |\lambda I - A| = (\lambda + 1)(\lambda - 2)(\lambda - 5)$

$$Q^{-1}AQ = \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix}$$

**例 1**  $A = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 3 \end{pmatrix}$ , 特征方程:  $0 = |\lambda I - A| = (\lambda + 1)(\lambda - 2)(\lambda - 5)$

- $\lambda_1 = -1,$

- $\lambda_2 = 2,$

- $\lambda_3 = 5,$

$$Q^{-1}AQ = \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix}$$

**例 1**  $A = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 3 \end{pmatrix}$ , 特征方程:  $0 = |\lambda I - A| = (\lambda + 1)(\lambda - 2)(\lambda - 5)$

- $\lambda_1 = -1,$

- $\lambda_2 = 2,$

- $\lambda_3 = 5,$

$$Q^{-1}AQ = \begin{pmatrix} -1 & & \\ & 2 & \\ & & 5 \end{pmatrix}$$



**例 1**  $A = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 3 \end{pmatrix}$ , 特征方程:  $0 = |\lambda I - A| = (\lambda + 1)(\lambda - 2)(\lambda - 5)$

- $\lambda_1 = -1$ , 特征向量  $\alpha_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$

- $\lambda_2 = 2$ ,

- $\lambda_3 = 5$ ,

$$Q^{-1}AQ = \begin{pmatrix} -1 & & \\ & 2 & \\ & & 5 \end{pmatrix}$$

**例 1**  $A = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 3 \end{pmatrix}$ , 特征方程:  $0 = |\lambda I - A| = (\lambda + 1)(\lambda - 2)(\lambda - 5)$

- $\lambda_1 = -1$ , 特征向量  $\alpha_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$
- $\lambda_2 = 2$ , 特征向量  $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$
- $\lambda_3 = 5$ ,

$$Q^{-1}AQ = \begin{pmatrix} -1 & & \\ & 2 & \\ & & 5 \end{pmatrix}$$

**例 1**  $A = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 3 \end{pmatrix}$ , 特征方程:  $0 = |\lambda I - A| = (\lambda + 1)(\lambda - 2)(\lambda - 5)$

- $\lambda_1 = -1$ , 特征向量  $\alpha_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$

- $\lambda_2 = 2$ , 特征向量  $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$

- $\lambda_3 = 5$ , 特征向量  $\alpha_3 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

$$Q^{-1}AQ = \begin{pmatrix} -1 & & \\ & 2 & \\ & & 5 \end{pmatrix}$$

**例 1**  $A = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 3 \end{pmatrix}$ , 特征方程:  $0 = |\lambda I - A| = (\lambda + 1)(\lambda - 2)(\lambda - 5)$

•  $\lambda_1 = -1$ , 特征向量  $\alpha_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_1 = \begin{pmatrix} 2/3 \\ 2/3 \\ 1/3 \end{pmatrix}$

•  $\lambda_2 = 2$ , 特征向量  $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$

•  $\lambda_3 = 5$ , 特征向量  $\alpha_3 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

$$Q^{-1}AQ = \begin{pmatrix} -1 & & \\ & 2 & \\ & & 5 \end{pmatrix}$$

**例 1**  $A = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 3 \end{pmatrix}$ , 特征方程:  $0 = |\lambda I - A| = (\lambda + 1)(\lambda - 2)(\lambda - 5)$

•  $\lambda_1 = -1$ , 特征向量  $\alpha_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_1 = \begin{pmatrix} 2/3 \\ 2/3 \\ 1/3 \end{pmatrix}$

•  $\lambda_2 = 2$ , 特征向量  $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_2 = \begin{pmatrix} 2/3 \\ -1/3 \\ -2/3 \end{pmatrix}$

•  $\lambda_3 = 5$ , 特征向量  $\alpha_3 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

$$Q^{-1}AQ = \begin{pmatrix} -1 & & \\ & 2 & \\ & & 5 \end{pmatrix}$$

**例 1**  $A = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 3 \end{pmatrix}$ , 特征方程:  $0 = |\lambda I - A| = (\lambda + 1)(\lambda - 2)(\lambda - 5)$

•  $\lambda_1 = -1$ , 特征向量  $\alpha_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_1 = \begin{pmatrix} 2/3 \\ 2/3 \\ 1/3 \end{pmatrix}$

•  $\lambda_2 = 2$ , 特征向量  $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_2 = \begin{pmatrix} 2/3 \\ -1/3 \\ -2/3 \end{pmatrix}$

•  $\lambda_3 = 5$ , 特征向量  $\alpha_3 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_3 = \begin{pmatrix} 1/3 \\ -2/3 \\ 2/3 \end{pmatrix}$

$$Q^{-1}AQ = \begin{pmatrix} -1 & & \\ & 2 & \\ & & 5 \end{pmatrix}$$

**例 1**  $A = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 3 \end{pmatrix}$ , 特征方程:  $0 = |\lambda I - A| = (\lambda + 1)(\lambda - 2)(\lambda - 5)$

•  $\lambda_1 = -1$ , 特征向量  $\alpha_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_1 = \begin{pmatrix} 2/3 \\ 2/3 \\ 1/3 \end{pmatrix}$

•  $\lambda_2 = 2$ , 特征向量  $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_2 = \begin{pmatrix} 2/3 \\ -1/3 \\ -2/3 \end{pmatrix}$

•  $\lambda_3 = 5$ , 特征向量  $\alpha_3 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_3 = \begin{pmatrix} 1/3 \\ -2/3 \\ 2/3 \end{pmatrix}$

所以取  $Q = \underbrace{\begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ 2/3 & 2/3 & 1/3 \\ 2/3 & -1/3 & -2/3 \\ 1/3 & -2/3 & 2/3 \end{pmatrix}}_{Q: \text{正交阵}}$ , 则  $Q^{-1}AQ = \begin{pmatrix} -1 & & \\ & 2 & \\ & & 5 \end{pmatrix}$

**例 2**  $A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{pmatrix}$

$$Q^{-1}AQ = \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix}$$



**例 2**  $A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{pmatrix}$ , 特征方程:  $0 = |\lambda I - A| = (\lambda - 1)^2(\lambda - 10)$

$$Q^{-1}AQ = \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix}$$

**例 2**  $A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{pmatrix}$ , 特征方程:  $0 = |\lambda I - A| = (\lambda - 1)^2(\lambda - 10)$

- $\lambda_1 = 1$  (二重)

- $\lambda_3 = 10$

$$Q^{-1}AQ = \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix}$$

**例 2**  $A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{pmatrix}$ , 特征方程:  $0 = |\lambda I - A| = (\lambda - 1)^2(\lambda - 10)$

- $\lambda_1 = 1$  (二重)

- $\lambda_3 = 10$

$$Q^{-1}AQ = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 10 \end{pmatrix}$$

**例 2**  $A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{pmatrix}$ , 特征方程:  $0 = |\lambda I - A| = (\lambda - 1)^2(\lambda - 10)$

•  $\lambda_1 = 1$  (二重), 特征向量

$$\begin{cases} \alpha_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \\ \alpha_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \end{cases}$$

•  $\lambda_3 = 10$

$$Q^{-1}AQ = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 10 \end{pmatrix}$$

**例 2**  $A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{pmatrix}$ , 特征方程:  $0 = |\lambda I - A| = (\lambda - 1)^2(\lambda - 10)$

•  $\lambda_1 = 1$  (二重), 特征向量

$$\begin{cases} \alpha_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \\ \alpha_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \end{cases}$$

•  $\lambda_3 = 10$ , 特征向量  $\alpha_3 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$

$$Q^{-1}AQ = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 10 \end{pmatrix}$$

**例 2**  $A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{pmatrix}$ , 特征方程:  $0 = |\lambda I - A| = (\lambda - 1)^2(\lambda - 10)$

•  $\lambda_1 = 1$  (二重), 特征向量

$$\begin{cases} \alpha_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \\ \alpha_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \end{cases} \xrightarrow{\text{正交化}} \begin{cases} \beta_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \\ \beta_2 = \begin{pmatrix} 2/5 \\ 4/5 \\ 1 \end{pmatrix} \end{cases}$$

•  $\lambda_3 = 10$ , 特征向量  $\alpha_3 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$

$$Q^{-1}AQ = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 10 \end{pmatrix}$$

**例 2**  $A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{pmatrix}$ , 特征方程:  $0 = |\lambda I - A| = (\lambda - 1)^2(\lambda - 10)$

•  $\lambda_1 = 1$  (二重), 特征向量

$$\left\{ \begin{array}{l} \alpha_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \\ \alpha_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \end{array} \right. \xrightarrow{\text{正交化}} \left\{ \begin{array}{l} \beta_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \\ \beta_2 = \begin{pmatrix} 2/5 \\ 4/5 \\ 1 \end{pmatrix} \end{array} \right. \xrightarrow{\text{单位化}} \left\{ \begin{array}{l} \gamma_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \\ \gamma_2 = \frac{5}{3\sqrt{5}} \begin{pmatrix} 2/5 \\ 4/5 \\ 1 \end{pmatrix} \end{array} \right.$$

•  $\lambda_3 = 10$ , 特征向量  $\alpha_3 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$

$$Q^{-1}AQ = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 10 \end{pmatrix}$$

**例 2**  $A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{pmatrix}$ , 特征方程:  $0 = |\lambda I - A| = (\lambda - 1)^2(\lambda - 10)$

•  $\lambda_1 = 1$  (二重), 特征向量

$$\begin{cases} \alpha_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \\ \alpha_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \end{cases} \xrightarrow{\text{正交化}} \begin{cases} \beta_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \\ \beta_2 = \begin{pmatrix} 2/5 \\ 4/5 \\ 1 \end{pmatrix} \end{cases} \xrightarrow{\text{单位化}} \begin{cases} \gamma_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \\ \gamma_2 = \frac{5}{3\sqrt{5}} \begin{pmatrix} 2/5 \\ 4/5 \\ 1 \end{pmatrix} \end{cases}$$

•  $\lambda_3 = 10$ , 特征向量  $\alpha_3 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_3 = \begin{pmatrix} 1/3 \\ 2/3 \\ -2/3 \end{pmatrix}$

$$Q^{-1}AQ = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 10 \end{pmatrix}$$



**例 2**  $A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{pmatrix}$ , 特征方程:  $0 = |\lambda I - A| = (\lambda - 1)^2(\lambda - 10)$

•  $\lambda_1 = 1$  (二重), 特征向量

$$\begin{cases} \alpha_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \\ \alpha_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \end{cases} \xrightarrow{\text{正交化}} \begin{cases} \beta_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \\ \beta_2 = \begin{pmatrix} 2/5 \\ 4/5 \\ 1 \end{pmatrix} \end{cases} \xrightarrow{\text{单位化}} \begin{cases} \gamma_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \\ \gamma_2 = \frac{5}{3\sqrt{5}} \begin{pmatrix} 2/5 \\ 4/5 \\ 1 \end{pmatrix} \end{cases}$$

•  $\lambda_3 = 10$ , 特征向量  $\alpha_3 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_3 = \begin{pmatrix} 1/3 \\ 2/3 \\ -2/3 \end{pmatrix}$

$\gamma_1 \qquad \gamma_2 \qquad \gamma_3$

所以取  $Q = \underbrace{\begin{pmatrix} -2/\sqrt{5} & 2/3\sqrt{5} & 1/3 \\ 1/\sqrt{5} & 4/3\sqrt{5} & 2/3 \\ 0 & \sqrt{5}/3 & -2/3 \end{pmatrix}}_{Q: \text{正交阵}}$ , 则  $Q^{-1}AQ = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 10 \end{pmatrix}$

**例 3**  $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix},$

$$Q^{-1}AQ = \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix}$$

**例 3**  $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$ , 特征方程:  $0 = |\lambda I - A| =$

$$Q^{-1}AQ = \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix}$$

**例 3**  $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$ , 特征方程:  $0 = |\lambda I - A| = (\lambda + 1)^2(\lambda - 5)$  ► Det

$$Q^{-1}AQ = \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix}$$

**例 3**  $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$ , 特征方程:  $0 = |\lambda I - A| = (\lambda + 1)^2(\lambda - 5)$  ► Det

- $\lambda_1 = -1$  (二重)

- $\lambda_2 = 5$

$$Q^{-1}AQ = \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix}$$

**例 3**  $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$ , 特征方程:  $0 = |\lambda I - A| = (\lambda + 1)^2(\lambda - 5)$  ► Det

- $\lambda_1 = -1$  (二重)

- $\lambda_2 = 5$

$$Q^{-1}AQ = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 5 \end{pmatrix}$$

**例 3**  $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$ , 特征方程:  $0 = |\lambda I - A| = (\lambda + 1)^2(\lambda - 5)$  [▶ Det](#)

•  $\lambda_1 = -1$  (二重), 特征向量: [▶ Detail](#)

$$\left\{ \alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right.$$

$$\left. \alpha_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

•  $\lambda_2 = 5$

$$Q^{-1}AQ = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 5 \end{pmatrix}$$

**例 3**  $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$ , 特征方程:  $0 = |\lambda I - A| = (\lambda + 1)^2(\lambda - 5)$  [▶ Det](#)

•  $\lambda_1 = -1$  (二重), 特征向量: [▶ Detail](#)

$$\left\{ \alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right.$$

$$\left. \alpha_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

•  $\lambda_2 = 5$ , 特征向量: [▶ Det](#)  $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$Q^{-1}AQ = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 5 \end{pmatrix}$$



**例 3**  $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$ , 特征方程:  $0 = |\lambda I - A| = (\lambda + 1)^2(\lambda - 5)$  [▶ Det](#)

- $\lambda_1 = -1$  (二重), 特征向量: [▶ Detail](#)

$$\begin{cases} \alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \\ \alpha_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \end{cases} \xrightarrow[\text{▶ Det}]{\text{正交化}}$$

- $\lambda_2 = 5$ , 特征向量: [▶ Det](#)  $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$Q^{-1}AQ = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 5 \end{pmatrix}$$

**例 3**  $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$ , 特征方程:  $0 = |\lambda I - A| = (\lambda + 1)^2(\lambda - 5)$  [▶ Det](#)

•  $\lambda_1 = -1$  (二重), 特征向量: [▶ Detail](#)

$$\begin{cases} \alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \\ \alpha_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \end{cases} \xrightarrow[\text{▶ Det}]{\text{正交化}} \begin{cases} \beta_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \\ \beta_2 = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix} \end{cases}$$

•  $\lambda_2 = 5$ , 特征向量: [▶ Det](#)  $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$Q^{-1}AQ = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 5 \end{pmatrix}$$

**例 3**  $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$ , 特征方程:  $0 = |\lambda I - A| = (\lambda + 1)^2(\lambda - 5)$  [▶ Det](#)

•  $\lambda_1 = -1$  (二重), 特征向量: [▶ Detail](#)

$$\begin{cases} \alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \\ \alpha_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \end{cases} \xrightarrow[\text{▶ Det}]{\text{正交化}} \begin{cases} \beta_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \\ \beta_2 = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix} \end{cases} \xrightarrow{\text{单位化}} \begin{cases} \gamma_1 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \\ \gamma_2 = \begin{pmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix} \end{cases}$$

•  $\lambda_2 = 5$ , 特征向量: [▶ Det](#)  $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$Q^{-1}AQ = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 5 \end{pmatrix}$$

**例 3**  $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$ , 特征方程:  $0 = |\lambda I - A| = (\lambda + 1)^2(\lambda - 5)$  [▶ Det](#)

•  $\lambda_1 = -1$  (二重), 特征向量: [▶ Detail](#)

$$\begin{cases} \alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \\ \alpha_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \end{cases} \xrightarrow[\text{▶ Det}]{\text{正交化}} \begin{cases} \beta_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \\ \beta_2 = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix} \end{cases} \xrightarrow{\text{单位化}} \begin{cases} \gamma_1 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \\ \gamma_2 = \begin{pmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix} \end{cases}$$

•  $\lambda_2 = 5$ , 特征向量: [▶ Det](#)  $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_3 = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$

$$Q^{-1}AQ = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 5 \end{pmatrix}$$

**例 3**  $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$ , 特征方程:  $0 = |\lambda I - A| = (\lambda + 1)^2(\lambda - 5)$  ▶ Det

•  $\lambda_1 = -1$  (二重), 特征向量: ▶ Detail

$$\begin{cases} \alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \\ \alpha_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \end{cases} \xrightarrow[\text{▶ Det}]{\text{正交化}} \begin{cases} \beta_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \\ \beta_2 = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix} \end{cases} \xrightarrow{\text{单位化}} \begin{cases} \gamma_1 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \\ \gamma_2 = \begin{pmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix} \end{cases}$$

•  $\lambda_2 = 5$ , 特征向量: ▶ Det  $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_3 = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$

$\gamma_1 \qquad \gamma_2 \qquad \gamma_3$

取  $Q = \underbrace{\begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}}_{Q: \text{正交阵}}, \text{ 则 } Q^{-1}AQ = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 5 \end{pmatrix}$

—The End—

- 求解特征方程

$$0 = |\lambda I - A| =$$

- 求解特征方程

$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ -2 & -2 & \lambda - 1 \end{vmatrix}$$



- 求解特征方程

$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ -2 & -2 & \lambda - 1 \end{vmatrix}$$

$$\underline{\underline{r_3 - r_2}}$$

● 求解特征方程

$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ -2 & -2 & \lambda - 1 \end{vmatrix}$$

$$\underline{\underline{r_3 - r_2}} \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -\lambda - 1 & \lambda + 1 \end{vmatrix}$$

● 求解特征方程

$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ -2 & -2 & \lambda - 1 \end{vmatrix}$$

$$\xrightarrow{r_3 - r_2} \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -\lambda - 1 & \lambda + 1 \end{vmatrix}$$

$$= (\lambda + 1) \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -1 & 1 \end{vmatrix}$$

● 求解特征方程

$$\begin{aligned} 0 = |\lambda I - A| &= \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ -2 & -2 & \lambda - 1 \end{vmatrix} \\ &\xrightarrow{r_3 - r_2} \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -\lambda - 1 & \lambda + 1 \end{vmatrix} \\ &= (\lambda + 1) \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -1 & 1 \end{vmatrix} \xrightarrow{c_2 + c_3} \end{aligned}$$

● 求解特征方程

$$\begin{aligned} 0 = |\lambda I - A| &= \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ -2 & -2 & \lambda - 1 \end{vmatrix} \\ &\xrightarrow{r_3 - r_2} \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -\lambda - 1 & \lambda + 1 \end{vmatrix} \\ &= (\lambda + 1) \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -1 & 1 \end{vmatrix} \xrightarrow{c_2 + c_3} (\lambda + 1) \begin{vmatrix} \lambda - 1 & -4 & -2 \\ -2 & \lambda - 3 & -2 \\ 0 & 0 & 1 \end{vmatrix} \end{aligned}$$

● 求解特征方程

$$\begin{aligned} 0 &= |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ -2 & -2 & \lambda - 1 \end{vmatrix} \\ &\xrightarrow{r_3 - r_2} \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -\lambda - 1 & \lambda + 1 \end{vmatrix} \\ &= (\lambda + 1) \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -1 & 1 \end{vmatrix} \xrightarrow{c_2 + c_3} (\lambda + 1) \begin{vmatrix} \lambda - 1 & -4 & -2 \\ -2 & \lambda - 3 & -2 \\ 0 & 0 & 1 \end{vmatrix} \\ &= (\lambda + 1) \begin{vmatrix} \lambda - 1 & -4 \\ -2 & \lambda - 3 \end{vmatrix} \end{aligned}$$

● 求解特征方程

$$\begin{aligned}0 &= |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ -2 & -2 & \lambda - 1 \end{vmatrix} \\&\xrightarrow{r_3 - r_2} \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -\lambda - 1 & \lambda + 1 \end{vmatrix} \\&= (\lambda + 1) \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -1 & 1 \end{vmatrix} \xrightarrow{c_2 + c_3} (\lambda + 1) \begin{vmatrix} \lambda - 1 & -4 & -2 \\ -2 & \lambda - 3 & -2 \\ 0 & 0 & 1 \end{vmatrix} \\&= (\lambda + 1) \begin{vmatrix} \lambda - 1 & -4 \\ -2 & \lambda - 3 \end{vmatrix} \\&= (\lambda + 1)(\lambda^2 - 4\lambda - 5)\end{aligned}$$

● 求解特征方程

$$\begin{aligned}0 &= |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ -2 & -2 & \lambda - 1 \end{vmatrix} \\&\xrightarrow{r_3 - r_2} \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -\lambda - 1 & \lambda + 1 \end{vmatrix} \\&= (\lambda + 1) \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -1 & 1 \end{vmatrix} \xrightarrow{c_2 + c_3} (\lambda + 1) \begin{vmatrix} \lambda - 1 & -4 & -2 \\ -2 & \lambda - 3 & -2 \\ 0 & 0 & 1 \end{vmatrix} \\&= (\lambda + 1) \begin{vmatrix} \lambda - 1 & -4 \\ -2 & \lambda - 3 \end{vmatrix} \\&= (\lambda + 1)(\lambda^2 - 4\lambda - 5) \\&= (\lambda + 1)^2(\lambda - 5)\end{aligned}$$



- 当  $\lambda_1 = -1$ , 求解  $(\lambda_1 I - A)x = 0$ :

$$(-I - A : 0) =$$

- 当  $\lambda_1 = -1$ , 求解  $(\lambda_1 I - A)x = 0$ :

$$(-I - A : 0) = \left( \begin{array}{ccc|c} -2 & -2 & -2 & 0 \\ -2 & -2 & -2 & 0 \\ -2 & -2 & -2 & 0 \end{array} \right) \rightarrow$$

- 当  $\lambda_1 = -1$ , 求解  $(\lambda_1 I - A)x = 0$ :

$$(-I - A : 0) = \left( \begin{array}{ccc|c} -2 & -2 & -2 & 0 \\ -2 & -2 & -2 & 0 \\ -2 & -2 & -2 & 0 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

- 当  $\lambda_1 = -1$ , 求解  $(\lambda_1 I - A)x = 0$ :

$$(-I - A : 0) = \left( \begin{array}{ccc|c} -2 & -2 & -2 & 0 \\ -2 & -2 & -2 & 0 \\ -2 & -2 & -2 & 0 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

所以

$$x_1 + x_2 + x_3 = 0$$

- 当  $\lambda_1 = -1$ , 求解  $(\lambda_1 I - A)x = 0$ :

$$(-I - A : 0) = \left( \begin{array}{ccc|c} -2 & -2 & -2 & 0 \\ -2 & -2 & -2 & 0 \\ -2 & -2 & -2 & 0 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

所以

$$x_1 + x_2 + x_3 = 0 \quad \Rightarrow \quad x_1 = -x_2 - x_3$$

- 当  $\lambda_1 = -1$ , 求解  $(\lambda_1 I - A)x = 0$ :

$$(-I - A : 0) = \left( \begin{array}{ccc|c} -2 & -2 & -2 & 0 \\ -2 & -2 & -2 & 0 \\ -2 & -2 & -2 & 0 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

所以

$$x_1 + x_2 + x_3 = 0 \quad \Rightarrow \quad x_1 = -x_2 - x_3$$

$$\text{基础解系: } \alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

► Back

- 当  $\lambda_1 = -1$ , 求解  $(\lambda_1 I - A)x = 0$ :

$$(-I - A : 0) = \left( \begin{array}{ccc|c} -2 & -2 & -2 & 0 \\ -2 & -2 & -2 & 0 \\ -2 & -2 & -2 & 0 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

所以

$$x_1 + x_2 + x_3 = 0 \quad \Rightarrow \quad x_1 = -x_2 - x_3$$

基础解系:  $\alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

► Back

- 当  $\lambda_1 = -1$ , 求解  $(\lambda_1 I - A)x = 0$ :

$$(-I - A : 0) = \left( \begin{array}{ccc|c} -2 & -2 & -2 & 0 \\ -2 & -2 & -2 & 0 \\ -2 & -2 & -2 & 0 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

所以

$$x_1 + x_2 + x_3 = 0 \quad \Rightarrow \quad x_1 = -x_2 - x_3$$

基础解系:  $\alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$



- 当  $\lambda_2 = 5$ , 求解  $(\lambda_2 I - A)x = 0$ :

$$(5I - A : 0) =$$

- 当  $\lambda_2 = 5$ , 求解  $(\lambda_2 I - A)x = 0$ :

$$(5I - A : 0) = \left( \begin{array}{ccc|c} 4 & -2 & -2 & 0 \\ -2 & 4 & -2 & 0 \\ -2 & -2 & 4 & 0 \end{array} \right)$$

- 当  $\lambda_2 = 5$ , 求解  $(\lambda_2 I - A)x = 0$ :

$$(5I - A : 0) = \left( \begin{array}{ccc|c} 4 & -2 & -2 & 0 \\ -2 & 4 & -2 & 0 \\ -2 & -2 & 4 & 0 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array} \right)$$

- 当  $\lambda_2 = 5$ , 求解  $(\lambda_2 I - A)x = 0$ :

$$(5I - A : 0) = \left( \begin{array}{ccc|c} 4 & -2 & -2 & 0 \\ -2 & 4 & -2 & 0 \\ -2 & -2 & 4 & 0 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array} \right)$$

$$\xrightarrow{r_1 \leftrightarrow r_3}$$

- 当  $\lambda_2 = 5$ , 求解  $(\lambda_2 I - A)x = 0$ :

$$(5I - A : 0) = \left( \begin{array}{ccc|c} 4 & -2 & -2 & 0 \\ -2 & 4 & -2 & 0 \\ -2 & -2 & 4 & 0 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array} \right)$$
$$\xrightarrow{r_1 \leftrightarrow r_3} \left( \begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 1 & -2 & 1 & 0 \\ 2 & -1 & -1 & 0 \end{array} \right)$$

- 当  $\lambda_2 = 5$ , 求解  $(\lambda_2 I - A)x = 0$ :

$$\begin{aligned}(5I - A : 0) &= \left( \begin{array}{ccc|c} 4 & -2 & -2 & 0 \\ -2 & 4 & -2 & 0 \\ -2 & -2 & 4 & 0 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array} \right) \\ &\xrightarrow{r_1 \leftrightarrow r_3} \left( \begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 1 & -2 & 1 & 0 \\ 2 & -1 & -1 & 0 \end{array} \right) \xrightarrow[r_3 - 2r_1]{r_2 - r_1} \left( \begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & -3 & 3 & 0 \end{array} \right)\end{aligned}$$

- 当  $\lambda_2 = 5$ , 求解  $(\lambda_2 I - A)x = 0$ :

$$\begin{aligned}(5I - A : 0) &= \left( \begin{array}{ccc|c} 4 & -2 & -2 & 0 \\ -2 & 4 & -2 & 0 \\ -2 & -2 & 4 & 0 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array} \right) \\ &\xrightarrow{r_1 \leftrightarrow r_3} \left( \begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 1 & -2 & 1 & 0 \\ 2 & -1 & -1 & 0 \end{array} \right) \xrightarrow[r_3 - 2r_1]{r_2 - r_1} \left( \begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & -3 & 3 & 0 \end{array} \right) \\ &\rightarrow \left( \begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)\end{aligned}$$

- 当  $\lambda_2 = 5$ , 求解  $(\lambda_2 I - A)x = 0$ :

$$(5I - A : 0) = \left( \begin{array}{ccc|c} 4 & -2 & -2 & 0 \\ -2 & 4 & -2 & 0 \\ -2 & -2 & 4 & 0 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array} \right)$$

$$\xrightarrow{r_1 \leftrightarrow r_3} \left( \begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 1 & -2 & 1 & 0 \\ 2 & -1 & -1 & 0 \end{array} \right) \xrightarrow[r_3 - 2r_1]{r_2 - r_1} \left( \begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & -3 & 3 & 0 \end{array} \right)$$

$$\rightarrow \left( \begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \xrightarrow{r_1 - r_2} \left( \begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$



- 当  $\lambda_2 = 5$ , 求解  $(\lambda_2 I - A)x = 0$ :

$$(5I - A : 0) = \left( \begin{array}{ccc|c} 4 & -2 & -2 & 0 \\ -2 & 4 & -2 & 0 \\ -2 & -2 & 4 & 0 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array} \right)$$

$$\xrightarrow{r_1 \leftrightarrow r_3} \left( \begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 1 & -2 & 1 & 0 \\ 2 & -1 & -1 & 0 \end{array} \right) \xrightarrow[r_3 - 2r_1]{r_2 - r_1} \left( \begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & -3 & 3 & 0 \end{array} \right)$$

$$\rightarrow \left( \begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \xrightarrow{r_1 - r_2} \left( \begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

所以 
$$\begin{cases} x_1 & -x_3 = 0 \end{cases}$$

- 当  $\lambda_2 = 5$ , 求解  $(\lambda_2 I - A)x = 0$ :

$$(5I - A : 0) = \left( \begin{array}{ccc|c} 4 & -2 & -2 & 0 \\ -2 & 4 & -2 & 0 \\ -2 & -2 & 4 & 0 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array} \right)$$

$$\xrightarrow{r_1 \leftrightarrow r_3} \left( \begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 1 & -2 & 1 & 0 \\ 2 & -1 & -1 & 0 \end{array} \right) \xrightarrow[r_3 - 2r_1]{r_2 - r_1} \left( \begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & -3 & 3 & 0 \end{array} \right)$$

$$\rightarrow \left( \begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \xrightarrow{r_1 - r_2} \left( \begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

所以 
$$\begin{cases} x_1 - x_3 = 0 \\ x_2 - x_3 = 0 \end{cases}$$

- 当  $\lambda_2 = 5$ , 求解  $(\lambda_2 I - A)x = 0$ :

$$(5I - A : 0) = \left( \begin{array}{ccc|c} 4 & -2 & -2 & 0 \\ -2 & 4 & -2 & 0 \\ -2 & -2 & 4 & 0 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array} \right)$$

$$\xrightarrow{r_1 \leftrightarrow r_3} \left( \begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 1 & -2 & 1 & 0 \\ 2 & -1 & -1 & 0 \end{array} \right) \xrightarrow[r_3 - 2r_1]{r_2 - r_1} \left( \begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & -3 & 3 & 0 \end{array} \right)$$

$$\rightarrow \left( \begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \xrightarrow{r_1 - r_2} \left( \begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

所以

$$\begin{cases} x_1 - x_3 = 0 \\ x_2 - x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = x_3 \\ x_2 = x_3 \end{cases}$$

- 当  $\lambda_2 = 5$ , 求解  $(\lambda_2 I - A)x = 0$ :

$$(5I - A : 0) = \left( \begin{array}{ccc|c} 4 & -2 & -2 & 0 \\ -2 & 4 & -2 & 0 \\ -2 & -2 & 4 & 0 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array} \right)$$

$$\xrightarrow{r_1 \leftrightarrow r_3} \left( \begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 1 & -2 & 1 & 0 \\ 2 & -1 & -1 & 0 \end{array} \right) \xrightarrow[r_3 - 2r_1]{r_2 - r_1} \left( \begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & -3 & 3 & 0 \end{array} \right)$$

$$\rightarrow \left( \begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \xrightarrow{r_1 - r_2} \left( \begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

所以 
$$\begin{cases} x_1 - x_3 = 0 \\ x_2 - x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = x_3 \\ x_2 = x_3 \end{cases}$$

基础解系:  $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

- 当  $\lambda_2 = 5$ , 求解  $(\lambda_2 I - A)x = 0$ :

$$(5I - A : 0) = \left( \begin{array}{ccc|c} 4 & -2 & -2 & 0 \\ -2 & 4 & -2 & 0 \\ -2 & -2 & 4 & 0 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array} \right)$$

$$\xrightarrow{r_1 \leftrightarrow r_3} \left( \begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 1 & -2 & 1 & 0 \\ 2 & -1 & -1 & 0 \end{array} \right) \xrightarrow[r_3 - 2r_1]{r_2 - r_1} \left( \begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & -3 & 3 & 0 \end{array} \right)$$

$$\rightarrow \left( \begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \xrightarrow{r_1 - r_2} \left( \begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

所以 
$$\begin{cases} x_1 - x_3 = 0 \\ x_2 - x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = x_3 \\ x_2 = x_3 \end{cases}$$

基础解系:  $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

将线性无关组  $\alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \alpha_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$  正交化:

► Back

将线性无关组  $\alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$  正交化:

$$\beta_1 =$$

$$\beta_2 =$$

► Back

将线性无关组  $\alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$  正交化:

$$\beta_1 = \alpha_1$$

$$\beta_2 =$$

► Back



将线性无关组  $\alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$  正交化:

$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \text{———} \beta_1$$

► Back

将线性无关组  $\alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$  正交化:

$$\beta_1 = \alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \text{———} \beta_1$$

► Back

将线性无关组  $\alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$  正交化:

$$\beta_1 = \alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}$$

► Back

将线性无关组  $\alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$  正交化:

$$\beta_1 = \alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

► Back

将线性无关组  $\alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$  正交化:

$$\beta_1 = \alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

► Back

将线性无关组  $\alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$  正交化:

$$\beta_1 = \alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}$$

► Back