第 08 周作业解答

练习 1. 求解线性方程组 $\begin{cases} x_1+ & 2x_2+ & x_3+ & x_4+ & x_5 = & 1 \\ 2x_1+ & 4x_2+ & 3x_3+ & x_4+ & x_5 = & 3 \\ -x_1- & 2x_2+ & x_3+ & 3x_4- & 3x_5 = & 7 \\ & & 2x_3+ & 5x_4- & 2x_5 = & 9 \end{cases}$ 的通解。

解对增广矩阵作初等行变换：

$$
(A\vdots b)=\left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{array}\right) \xrightarrow[r_3+r_1]{r_2-2r_1} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{array}\right) \xrightarrow[r_4-2r_2]{r_3-2r_2} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 6 & 0 & 6 \\ 0 & 0 & 0 & 7 & 0 & 7 \end{array}\right)
$$

$$
\xrightarrow[\frac{1}{7}\times r_4]{\frac{1}{6}\times r_3} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right) \xrightarrow{r_4-r_3} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[r_1-r_3]{r_2+r_3} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)
$$

$$
\xrightarrow{r_1-r_2} \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)
$$

可见 $r(A)=r(A\vdots b)=3<5$，所以原方程组有无穷多的解，包含 $5-3=2$ 个自由变量。事实上，通过上述简化的阶梯型矩阵，可知原方程等价于

$$
\begin{cases} x_1+ \quad 2x_2 \quad\quad\quad + \ 2x_5 = -2 \\ \quad\quad\quad x_3 \quad - \ x_5 = 2 \\ \quad\quad\quad\quad x_4 \quad\quad = 1 \end{cases} \Rightarrow \begin{cases} x_1=-2-2x_2-2x_5 \\ x_3=2+x_5 \\ x_4=1 \end{cases}
$$

所以通解是

$$
\begin{cases} x_1=-2-2c_1-2c_2 \\ x_2=c_1 \\ x_3=2+c_2 \\ x_4=1 \\ x_5=c_2 \end{cases} \qquad (c_1,\ c_2\text{为任意常数})
$$

用向量形式表示则是

$$
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 2 \\ 1 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}
$$

练习 2. 《九章算术》卷八为“方程”，试解其中第八题：

今有賣牛二羊五以買一十三豕有餘錢一千賣牛三豕三以買九羊錢適足賣六羊八豕以買五牛錢不足六百問牛羊豕價各幾何

解设牛价 x，羊价 y，豕价 z，则

$$\begin{cases} 2x+5y=13z+1000 \\ 3x+3z=9y \\ 6y+8z+600=5x \end{cases}$$

求解方程如下：

$$(A\vdots b)=\left(\begin{array}{ccc|c} 2 & 5 & -13 & 1000 \\ 3 & -9 & 3 & 0 \\ -5 & 6 & 8 & -600 \end{array}\right)\rightarrow\left(\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 2 & 5 & -13 & 1000 \\ -5 & 6 & 8 & -600 \end{array}\right)\rightarrow\left(\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 11 & -15 & 1000 \\ 0 & -9 & 13 & -600 \end{array}\right)$$

$$\xrightarrow{r_2+r_3}\left(\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 2 & -2 & 400 \\ 0 & -9 & 13 & -600 \end{array}\right)\rightarrow\left(\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 1 & -1 & 200 \\ 0 & -9 & 13 & -600 \end{array}\right)\rightarrow\left(\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 1 & -1 & 200 \\ 0 & 0 & 4 & 1200 \end{array}\right)$$

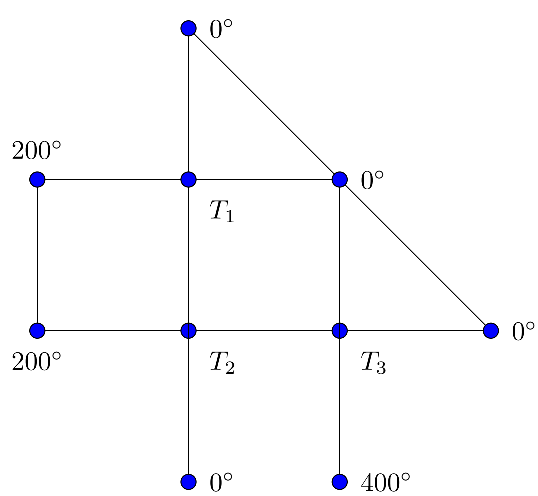
$$\rightarrow\left(\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 1 & -1 & 200 \\ 0 & 0 & 1 & 300 \end{array}\right)\rightarrow\left(\begin{array}{ccc|c} 1 & -3 & 0 & -300 \\ 0 & 1 & 0 & 500 \\ 0 & 0 & 1 & 300 \end{array}\right)\rightarrow\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1200 \\ 0 & 1 & 0 & 500 \\ 0 & 0 & 1 & 300 \end{array}\right)$$

所以 $x=1200, y=500, z=300$。

练习 3. In a grid of wires, the temperature at exterior mesh points is maintained at constant values (in $^\circ C$), as shown in the accompanying figure. When the grid is in thermal equilibrium, the temperature T at each interior mesh point is the average of the temperatures at the four adjacent points. For example,

$$T_2=\frac{T_3+T_1+200+0}{4}.$$

Find the temperatures T_1, T_2 and T_3 when the grid is in thermal equilibrium.



Solution.

$$\begin{cases} 4T_1=200+T_2 \\ 4T_2=200+T_1+T_3 \\ 4T_3=T_2+400 \end{cases}$$

Then

$$(A\vdots b)=\left(\begin{array}{ccc|c}4&-1&0&200\\-1&4&-1&200\\0&-1&4&400\end{array}\right)\to\left(\begin{array}{ccc|c}1&-4&1&-200\\4&-1&0&200\\0&-1&4&400\end{array}\right)\to\left(\begin{array}{ccc|c}1&-4&1&-200\\0&15&-4&1000\\0&1&-4&-400\end{array}\right)\to\left(\begin{array}{ccc|c}1&-4&1&-200\\0&1&-4&-400\\0&15&-4&1000\end{array}\right)$$

$$\to\left(\begin{array}{ccc|c}1&-4&1&-200\\0&1&-4&-400\\0&0&56&7000\end{array}\right)\to\left(\begin{array}{ccc|c}1&-4&1&-200\\0&1&-4&-400\\0&0&1&125\end{array}\right)\to\left(\begin{array}{ccc|c}1&-4&0&-325\\0&1&0&100\\0&0&1&125\end{array}\right)\to\left(\begin{array}{ccc|c}1&0&0&75\\0&1&0&100\\0&0&1&125\end{array}\right)$$

So $T_1=75°, T_2=100°$ and $T_3=125°$.

练习 4. 问 k 取何值时，方程组 $\begin{cases}x_1+\quad x_2+\quad kx_3= & 4\\-x_1+\quad kx_2+\quad x_3= & k^2\\x_1-\quad x_2+\quad 2x_3= & -4\end{cases}$ 有唯一解、无穷多解、无解。并且有解时，求出全部解。

解对增广矩阵作初等行变换：

$$(A\vdots b)=\left(\begin{array}{ccc|c}1&1&k&4\\-1&k&1&k^2\\1&-1&2&-4\end{array}\right)\xrightarrow[r_3-r_1]{r_2+r_1}\left(\begin{array}{ccc|c}1&1&k&4\\0&k+1&k+1&k^2+4\\0&-2&2-k&-8\end{array}\right)\xrightarrow{r_3\leftrightarrow r_2}\left(\begin{array}{ccc|c}1&1&k&4\\0&-2&2-k&-8\\0&k+1&k+1&k^2+4\end{array}\right)$$

$$\xrightarrow{-\frac{1}{2}\times r_2}\left(\begin{array}{ccc|c}1&1&k&4\\0&1&\frac{1}{2}k-1&4\\0&k+1&k+1&k^2+4\end{array}\right)\xrightarrow[r_1-r_2]{r_3-(k+1)\times r_2}\left(\begin{array}{ccc|c}1&0&\frac{1}{2}k+1&0\\0&1&\frac{1}{2}k-1&4\\0&0&\frac{1}{2}(k+1)(4-k)&k(k-4)\end{array}\right)$$

- 当 $k\neq-1$ 且 $k\neq4$ 时，$r(A)=r(A\vdots b)=3=$ 未知量个数，方程组有唯一解。此时

$$(A\vdots b)\longrightarrow\left(\begin{array}{ccc|c}1&0&\frac{1}{2}k+1&0\\0&1&\frac{1}{2}k-1&4\\0&0&\frac{1}{2}(k+1)(4-k)&k(k-4)\end{array}\right)\xrightarrow{\frac{2}{(k+1)(4-k)}\times r_3}\left(\begin{array}{ccc|c}1&0&\frac{1}{2}k+1&0\\0&1&\frac{1}{2}k-1&4\\0&0&1&-\frac{2k}{k+1}\end{array}\right)$$

$$\xrightarrow[r_2-(\frac{1}{2}k-1)\times r_3]{r_1-(\frac{1}{2}k+1)\times r_3}\left(\begin{array}{ccc|c}1&0&0&\frac{k(k+2)}{k+1}\\0&1&0&\frac{k^2+2k+4}{k+1}\\0&0&1&-\frac{2k}{k+1}\end{array}\right)$$

所以

$$\begin{cases}x_1=\frac{k^2+2k}{k+1}\\x_2=\frac{k^2+2k+4}{k+1}\\x_3=-\frac{2k}{k+1}\end{cases}$$

- 当 $k=-1$ 时

$$(A\vdots b)\longrightarrow\left(\begin{array}{ccc|c}1&0&\frac{1}{2}k+1&0\\0&1&\frac{1}{2}k-1&4\\0&0&\frac{1}{2}(k+1)(4-k)&k(k-4)\end{array}\right)\longrightarrow\left(\begin{array}{ccc|c}1&0&\frac{1}{2}&0\\0&1&-\frac{3}{2}&4\\0&0&0&5\end{array}\right)$$

可见 $r(A)=2<3=r(A\vdots b)$，此时方程无解。

- 当 $k=4$ 时

$$(A\vdots b)\longrightarrow\left(\begin{array}{ccc|c}1&0&\frac{1}{2}k+1&0\\0&1&\frac{1}{2}k-1&4\\0&0&\frac{1}{2}(k+1)(4-k)&k(k-4)\end{array}\right)\longrightarrow\left(\begin{array}{ccc|c}1&0&3&0\\0&1&1&4\\0&0&0&0\end{array}\right)$$

可见 $r(A)=r(A\vdots b)=2<$ 未知量个数3，方程组有无穷多的解，包含 $3-2=1$ 个自由变量。事实上，通过上述简化的阶梯型矩阵，可知原方程等价于

$$\begin{cases} x_1 \qquad + \ 3x_3 = \ 0 \\ \qquad x_2 + \quad x_3 = \ 4 \end{cases} \Rightarrow \begin{cases} x_1 = -3x_3 \\ x_3 = 4 - x_3 \end{cases}$$

所以通解是

$$\begin{cases} x_1 = -3c \\ x_2 = 4 - c \qquad (c\text{为任意常数}) \\ x_3 = c \end{cases}$$

用向量形式表示则是

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} + c\begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}$$

练习 5. 问 $\beta=\begin{pmatrix} 2 \\ 0 \\ 3 \\ -1 \\ 3 \end{pmatrix}$ 是否能由向量组 $\alpha_1=\begin{pmatrix} 0 \\ 2 \\ 1 \\ 5 \\ -1 \end{pmatrix}$，$\alpha_2=\begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}$，$\alpha_3=\begin{pmatrix} 1 \\ 1 \\ 0 \\ 4 \\ -1 \end{pmatrix}$ 线性表示？若能，写出其中一个线性组合的表达式。

解

$$\left(\begin{array}{ccc|c} \alpha_1 & \alpha_2 & \alpha_3 & \beta \end{array}\right) = \left(\begin{array}{ccc|c} 0 & 1 & 1 & 2 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 3 \\ 5 & 2 & 4 & -1 \\ -1 & 1 & -1 & 3 \end{array}\right) \xrightarrow{r_1 \leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 5 & 2 & 4 & -1 \\ -1 & 1 & -1 & 3 \end{array}\right) \xrightarrow[\substack{r_4-5r_1 \\ r_5+r_1}]{r_2-2r_1} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & -3 & 1 & -6 \\ 0 & 1 & 1 & 2 \\ 0 & -8 & 4 & -16 \\ 0 & 3 & -1 & 6 \end{array}\right)$$

$$\xrightarrow[\frac{1}{4}\times r_4]{r_2 \leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & -3 & 1 & -6 \\ 0 & -2 & 1 & -4 \\ 0 & 3 & -1 & 6 \end{array}\right) \xrightarrow[\substack{r_4+2r_2 \\ r_5-3r_2}]{r_3+3r_2} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right) \xrightarrow[\substack{\frac{1}{3}\times r_4 \\ -\frac{1}{4}\times r_5}]{\frac{1}{4}\times r_3} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[\substack{r_5-r_3 \\ r_2-r_3}]{r_4-r_3} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{r_1-2r_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

可见 $r(\alpha_1\alpha_2\alpha_3)=r(\alpha_1\alpha_2\alpha_3\beta)$，所以 β 能由 α_1, α_2, α_3。并且从最后简化的阶梯型矩阵容易看出：

$$\beta = -\alpha_1 + 2\alpha_2 + 0\alpha_3 = -\alpha_1 + 2\alpha_2.$$

练习 6. 问向量组 $\alpha_1=\begin{pmatrix} 3 \\ -1 \\ 2 \\ 3 \end{pmatrix}$，$\alpha_2=\begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix}$，$\alpha_3=\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ 是否线性相关？若线性相关，写出它们的一个相关表达式。

解

$$
\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 0 \\ -1 & 1 & 0 \\ 2 & 2 & 1 \\ 3 & 0 & 1 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{pmatrix} -1 & 1 & 0 \\ 3 & 1 & 0 \\ 2 & 2 & 1 \\ 3 & 0 & 1 \end{pmatrix} \xrightarrow[\substack{r_3+2r_1 \\ r_4+3r_1}]{r_2+3r_1} \begin{pmatrix} -1 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & 4 & 1 \\ 0 & 3 & 1 \end{pmatrix}
$$

$$
\xrightarrow{\frac{1}{4}\times r_2} \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 3 & 1 \end{pmatrix} \xrightarrow[r_4-3r_2]{r_3-4r_2} \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{r_4-r_3} \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}
$$

可见 $r(\alpha_1\alpha_2\alpha_3) = 3 =$ 向量个数，所以 $\alpha_1, \alpha_2, \alpha_3$ 线性无关。