第 11 周作业解答

练习 1. 求下列函数的偏导数：
(1) $z = \sin(xy) + \cos^2(xy)$; (2) $z = x^y \cdot \ln y$; (3) $u = \ln(xy + z)$; (4) $z = \tan\frac{x}{y}$

提示：可能要利用公式 $a^b = e^{\ln a^b} = e^{b\ln a}$

解 (1)

$$
\begin{aligned}
z_x &= \left(\sin(xy) + \cos^2(xy)\right)'_x = y\cos(xy) - 2y\cos(xy)\sin(xy)\\
z_y &= \left(\sin(xy) + \cos^2(xy)\right)'_y = x\cos(xy) - 2x\cos(xy)\sin(xy)
\end{aligned}
$$

(2)

$$
\begin{aligned}
z_x &= (x^y \cdot \ln y)'_x = \ln y \cdot (x^y)'_x = y\ln y \cdot x^{y-1}\\
z_y &= (x^y \cdot \ln y)'_y = (x^y)'_y \cdot \ln y + x^y \cdot (\ln y)'_y = x^y \ln x \cdot \ln y + x^y \cdot \frac{1}{y} = x^y[\ln x \cdot \ln y + \frac{1}{y}]
\end{aligned}
$$

(3)

$$
\begin{aligned}
u_x &= (\ln(xy+z))'_x = \frac{y}{xy+z}\\
u_y &= (\ln(xy+z))'_y = \frac{x}{xy+z}\\
u_z &= (\ln(xy+z))'_z = \frac{1}{xy+z}
\end{aligned}
$$

(4)

$$
\begin{aligned}
z_x &= \left(\tan\frac{x}{y}\right)'_x = \frac{1}{\cos^2(\frac{x}{y})} \cdot \left(\frac{x}{y}\right)'_x = \frac{1}{y\cos^2(\frac{x}{y})}\\
z_y &= \left(\tan\frac{x}{y}\right)'_y = \frac{1}{\cos^2(\frac{x}{y})} \cdot \left(\frac{x}{y}\right)'_y = -\frac{x}{y^2\cos^2(\frac{x}{y})}
\end{aligned}
$$

练习 2. 设某产品的生产函数为

$$
Q = 36KL - 2K^2 - 3L^2
$$

其中 Q 为产量，K，L 分别表示所需的资本和劳动力，求边际产量 $\frac{\partial Q}{\partial K}$ 和 $\frac{\partial Q}{\partial L}$。

解

$$
\begin{aligned}
\frac{\partial Q}{\partial K} &= \left(36KL - 2K^2 - 3L^2\right)'_K = 36L - 4K\\
\frac{\partial Q}{\partial L} &= \left(36KL - 2K^2 - 3L^2\right)'_L = 36K - 6L
\end{aligned}
$$

练习 3. 求 $z = x^3 + x^4y - y^3x$ 的全部二阶偏导数。

解

$$\frac{\partial z}{\partial x}=\left(x^3+x^4y-y^3x\right)'_x=\left(x^3\right)'_x+\left(x^4y\right)'_x-\left(y^3x\right)'_x=3x^2+4x^3y-y^3$$
$$\frac{\partial z}{\partial y}=\left(x^3+x^4y-y^3x\right)'_y=\left(x^3\right)'_y+\left(x^4y\right)'_y-\left(y^3x\right)'_y=x^4-3y^2x$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\left(3x^2+4x^3y-y^3\right)'_x=\left(3x^2\right)'_x+\left(4x^3y\right)'_x-\left(y^3\right)'_x=6x+12x^2y$$
$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\left(3x^2+4x^3y-y^3\right)'_y=\left(3x^2\right)'_y+\left(4x^3y\right)'_y-\left(y^3\right)'_y=4x^3-3y^2$$
$$\frac{\partial^2 z}{\partial y\partial x}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\left(x^4-3y^2x\right)'_x=\left(x^4\right)'_x-\left(3y^2x\right)'_x=4x^3-3y^2$$
$$\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\left(x^4-3y^2x\right)'_y=\left(x^4\right)'_y-\left(3y^2x\right)'_y=-6xy$$

练习 4. 设 $z=x\ln(x+y^2)$，求 dz。

解

$$z_x=\left(x\ln(x+y^2)\right)'_x=(x)'_x\cdot\ln(x+y^2)+x\cdot\left(\ln(x+y^2)\right)'_x=\ln(x+y^2)+\frac{x}{x+y^2}$$
$$z_y=\left(x\ln(x+y^2)\right)'_y=x\cdot\left(\ln(x+y^2)\right)'_y=x\cdot\frac{1}{x+y^2}\cdot(y^2)'_y=\frac{2xy}{x+y^2}$$

$$dz=z_xdx+z_ydy=\left(\ln(x+y^2)+\frac{x}{x+y^2}\right)dx+\frac{2xy}{x+y^2}dy$$

练习 5. 求函数 $z=\frac{y}{x}$ 当 $x=2$，$y=1$，$\Delta x=0.1$，$\Delta y=-0.2$ 时的全增量 Δz 和全微分 dz。

解

$$dz=z_xdx+z_ydy=-\frac{y}{x^2}dx+\frac{1}{x}dy.$$

将 $x=2$，$y=1$，$\Delta x=0.1$，$\Delta y=-0.2$ 代入，得到全微分

$$dz=-\frac{1}{4}\cdot 0.1+\frac{1}{2}\cdot(-0.2)=-0.125=-\frac{1}{8}.$$

而全增量 Δz 为

$$\Delta z=f(x+\Delta x,\,y+\Delta y)-f(x,\,y)=f(2+0.1,\,1-0.2)-f(2,\,1)=\frac{0.8}{2.1}-\frac{1}{2}=\frac{16-21}{42}=-\frac{5}{42}\approx-0.119047619$$

在此例中 Δz 与 dz 在精确到小数点后 1 位时是相等。

练习 6. 设函数 $z=e^{xy}$，而 $x=\sin t$，$y=\cos t$，求 $\frac{dz}{dt}$。

解法一：

$$\begin{aligned}\frac{dz}{dt}&=\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}\\&=\left(e^{xy}\right)'_x\cdot(\sin t)'_t+\left(e^{xy}\right)'_y\cdot(\cos t)'_t\\&=ye^{xy}\cos t-xe^{xy}\sin t\\&=\cos te^{\sin t\cos t}\cos t-\sin te^{\sin t\cos t}\sin t\\&=e^{\sin t\cos t}(\cos^2 t-\sin^2 t)\end{aligned}$$

解法二：因为

$$z = e^{xy} = e^{\sin t \cos t}$$

所以

$$\frac{dz}{dt} = \left(e^{\sin t \cos t}\right)_t' = e^{\sin t \cos t} \cdot (\sin t \cos t)_t' = e^{\sin t \cos t}(\cos^2 t - \sin^2 t)$$