# §3.2　向量与向量组的线性组合

数学系　梁卓滨

2017 - 2018 学年 I

# 向量

- $n$ 维行向量

$$\alpha = (a_1, a_2, \ldots, a_n)$$

# 向量

- $n$ 维行向量

$$\alpha = (a_1, a_2, \ldots, a_n)$$

即为：$1 \times n$ 的矩阵。

# 向量

- $n$ 维行向量

$$\alpha = (a_1, a_2, \ldots, a_n)$$

即为：$1 \times n$ 的矩阵。$a_i$ 称为 $\alpha$ 的第 $i$ 个分量。

# 向量

- $n$ 维行向量

$$\alpha = (a_1, a_2, \ldots, a_n)$$

即为：$1 \times n$ 的矩阵。$a_i$ 称为 $\alpha$ 的第 $i$ 个分量。

- $n$ 维列向量

$$\beta = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = (b_1, b_2, \ldots, b_n)^T$$

# 向量

- $n$ 维行向量

$$\alpha = (a_1, a_2, \ldots, a_n)$$

即为：$1 \times n$ 的矩阵。$a_i$ 称为 $\alpha$ 的第 $i$ 个分量。

- $n$ 维列向量

$$\beta = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = (b_1, b_2, \ldots, b_n)^T$$

即为：$n \times 1$ 的矩阵。

# 向量

- $n$ 维行向量

$$\alpha = (a_1, a_2, \ldots, a_n)$$

即为：$1 \times n$ 的矩阵。$a_i$ 称为 $\alpha$ 的第 $i$ 个分量。

- $n$ 维列向量

$$\beta = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = (b_1, b_2, \ldots, b_n)^T$$

即为：$n \times 1$ 的矩阵。$b_i$ 称为 $\beta$ 的第 $i$ 个分量。

# 向量

- $n$ 维行向量

$$\alpha = (a_1, a_2, \ldots, a_n)$$

即为：$1 \times n$ 的矩阵。$a_i$ 称为 $\alpha$ 的第 $i$ 个分量。

- $n$ 维列向量

$$\beta = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = (b_1, b_2, \ldots, b_n)^T$$

即为：$n \times 1$ 的矩阵。$b_i$ 称为 $\beta$ 的第 $i$ 个分量。

- 行向量、列向量统称向量。

# 向量

- $n$ 维行向量

$$\alpha = (a_1, a_2, \ldots, a_n)$$

即为：$1 \times n$ 的矩阵。$a_i$ 称为 $\alpha$ 的第 $i$ 个分量。

- $n$ 维列向量

$$\beta = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = (b_1, b_2, \ldots, b_n)^T$$

即为：$n \times 1$ 的矩阵。$b_i$ 称为 $\beta$ 的第 $i$ 个分量。

- 行向量、列向量统称向量。

根据上下文判断“向量”是行向量，或列向量

# 向量

- $n$ 维行向量

$$\alpha = (a_1, a_2, \ldots, a_n)$$

即为：$1 \times n$ 的矩阵。$a_i$ 称为 $\alpha$ 的第 $i$ 个分量。

- $n$ 维列向量

$$\beta = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = (b_1, b_2, \ldots, b_n)^T$$

即为：$n \times 1$ 的矩阵。$b_i$ 称为 $\beta$ 的第 $i$ 个分量。

- 行向量、列向量统称向量。

  根据上下文判断“向量”是行向量，或列向量

- 零向量 $O = (0, 0, \cdots, 0)$

# 向量的线性运算

- 设 $\alpha = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$，$\beta = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$，$k \in \mathbb{R}$，则

$\alpha + \beta =$ ， $\alpha - \beta =$ ， $k\alpha =$

# 向量的线性运算

- 设 $\alpha = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$，$\beta = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$，$k \in \mathbb{R}$，则

$$\alpha + \beta = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{pmatrix}, \quad \alpha - \beta = \qquad\qquad\qquad, \quad k\alpha =$$

# 向量的线性运算

- 设 $\alpha = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$，$\beta = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$，$k \in \mathbb{R}$，则

$$\alpha + \beta = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{pmatrix}, \quad \alpha - \beta = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \\ \vdots \\ a_n - b_n \end{pmatrix}, \quad k\alpha =$$

# 向量的线性运算

- 设 $\alpha = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$，$\beta = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$，$k \in \mathbb{R}$，则

$$\alpha + \beta = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{pmatrix}, \quad \alpha - \beta = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \\ \vdots \\ a_n - b_n \end{pmatrix}, \quad k\alpha = \begin{pmatrix} ka_1 \\ ka_2 \\ \vdots \\ ka_n \end{pmatrix}$$

# 向量的线性运算

- 设 $\alpha = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$，$\beta = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$，$k \in \mathbb{R}$，则

$$\alpha+\beta = \begin{pmatrix} a_1+b_1 \\ a_2+b_2 \\ \vdots \\ a_n+b_n \end{pmatrix}, \quad \alpha-\beta = \begin{pmatrix} a_1-b_1 \\ a_2-b_2 \\ \vdots \\ a_n-b_n \end{pmatrix}, \quad k\alpha = \begin{pmatrix} ka_1 \\ ka_2 \\ \vdots \\ ka_n \end{pmatrix}$$

- 行向量类似

# 线性组合问题

给定向量组

$$\alpha_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \alpha_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \ldots, \alpha_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

及一向量 $\beta = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$

# 线性组合问题

给定向量组

$$\alpha_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \alpha_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \ldots, \alpha_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

及一向量 $\beta = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$

问 是否存在数 $k_1, k_2, \ldots, k_n$ 使得：

$$\beta = k_1\alpha_1 + k_2\alpha_2 + \cdots + k_n\alpha_n\,?$$

# 线性组合问题

给定向量组

$$\alpha_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \alpha_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \ldots, \alpha_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

及一向量 $\beta = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$

问 是否存在数 $k_1, k_2, \ldots, k_n$ 使得：

$$\beta = k_1\alpha_1 + k_2\alpha_2 + \cdots + k_n\alpha_n\,?$$

如果能，则称 $\beta$ 是向量组 $\alpha_1, \alpha_2, \ldots, \alpha_n$ 的线性组合。

# 线性组合问题

例 判断 $\beta$ 能否由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式 $\beta = k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3$。

- (1) 问

$$\begin{array}{cccc} \beta & \alpha_1 & \alpha_2 & \alpha_3 \\ \begin{pmatrix} 2 \\ -7 \\ 5 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \end{array}$$

# 线性组合问题

例 判断 $\beta$ 能否由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式 $\beta = k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3$。

- (1) 问

$$\underset{\beta}{\begin{pmatrix} 2 \\ -7 \\ 5 \end{pmatrix}} = \underline{\quad}\underset{\alpha_1}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}} + \underline{\qquad}\underset{\alpha_2}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}} + \underline{\quad}\underset{\alpha_3}{\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}}$$

# 线性组合问题

例 判断 $\beta$ 能否由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式 $\beta = k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3$。

- (1) 问

$$\overset{\beta}{\begin{pmatrix} 2 \\ -7 \\ 5 \end{pmatrix}} = \underline{2} \overset{\alpha_1}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}} + \underline{\qquad} \overset{\alpha_2}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}} + \underline{\quad} \overset{\alpha_3}{\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}}$$

# 线性组合问题

例 判断 $\beta$ 能否由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式 $\beta = k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3$。

- (1) 问

$$\begin{array}{c} \beta \\ \begin{pmatrix} 2 \\ -7 \\ 5 \end{pmatrix} \end{array} = \underline{2} \begin{array}{c} \alpha_1 \\ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \end{array} + \underline{-7} \begin{array}{c} \alpha_2 \\ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \end{array} + \underline{\quad} \begin{array}{c} \alpha_3 \\ \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \end{array}$$

# 线性组合问题

例 判断 $\beta$ 能否由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式 $\beta = k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3$。

- (1) 问

$$\begin{array}{c} \beta \\ \begin{pmatrix} 2 \\ -7 \\ 5 \end{pmatrix} \end{array} = \underline{2} \begin{array}{c} \alpha_1 \\ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \end{array} + \underline{-7} \begin{array}{c} \alpha_2 \\ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \end{array} + \underline{\frac{5}{2}} \begin{array}{c} \alpha_3 \\ \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \end{array}$$

# 线性组合问题

例 判断 $\beta$ 能否由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式 $\beta = k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3$。

- (1) 问

$$\overset{\beta}{\begin{pmatrix} 2 \\ -7 \\ 5 \end{pmatrix}} = \underline{2}\overset{\alpha_1}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}} + \underline{-7}\overset{\alpha_2}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}} + \underline{\frac{5}{2}}\overset{\alpha_3}{\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}}$$

所以 $\beta = 2\alpha_1 - 7\alpha_2 + \frac{5}{2}\alpha_3$；

# 线性组合问题

例 判断 $\beta$ 能否由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式 $\beta = k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3$。

- (1) 问

$$\overset{\beta}{\begin{pmatrix} 2 \\ -7 \\ 5 \end{pmatrix}} = \underline{2}\overset{\alpha_1}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}} + \underline{-7}\overset{\alpha_2}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}} + \underline{\frac{5}{2}}\overset{\alpha_3}{\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}}$$

所以 $\beta = 2\alpha_1 - 7\alpha_2 + \frac{5}{2}\alpha_3$；$\beta$ 能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表出

# 线性组合问题

例 判断 $\beta$ 能否由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式 $\beta = k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3$。

- (1) 问

$$\begin{array}{c} \beta \\ \begin{pmatrix} 2 \\ -7 \\ 5 \end{pmatrix} \end{array} = \underline{2} \begin{array}{c} \alpha_1 \\ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \end{array} + \underline{-7} \begin{array}{c} \alpha_2 \\ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \end{array} + \underline{\frac{5}{2}} \begin{array}{c} \alpha_3 \\ \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \end{array}$$

所以 $\beta = 2\alpha_1 - 7\alpha_2 + \frac{5}{2}\alpha_3$；$\beta$ 能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表出

---

- (2) 问

$$\begin{array}{c} \beta \\ \begin{pmatrix} 2 \\ -7 \\ 5 \end{pmatrix} \end{array} \quad \begin{array}{c} \alpha_1 \\ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \end{array} \quad \begin{array}{c} \alpha_2 \\ \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \end{array} \quad \begin{array}{c} \alpha_3 \\ \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \end{array}$$

# 线性组合问题

例 判断 $\beta$ 能否由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式 $\beta = k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3$。

- (1) 问

$$\begin{array}{c}\beta\\ \begin{pmatrix}2\\-7\\5\end{pmatrix}\end{array} = \underline{2}\begin{array}{c}\alpha_1\\ \begin{pmatrix}1\\0\\0\end{pmatrix}\end{array} + \underline{-7}\begin{array}{c}\alpha_2\\ \begin{pmatrix}0\\1\\0\end{pmatrix}\end{array} + \underline{\frac{5}{2}}\begin{array}{c}\alpha_3\\ \begin{pmatrix}0\\0\\2\end{pmatrix}\end{array}$$

所以 $\beta = 2\alpha_1 - 7\alpha_2 + \frac{5}{2}\alpha_3$；$\beta$ 能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表出

---

- (2) 问

$$\begin{array}{c}\beta\\ \begin{pmatrix}2\\-7\\5\end{pmatrix}\end{array} = \underline{\quad}\begin{array}{c}\alpha_1\\ \begin{pmatrix}1\\0\\0\end{pmatrix}\end{array} + \underline{\quad}\begin{array}{c}\alpha_2\\ \begin{pmatrix}2\\3\\0\end{pmatrix}\end{array} + \underline{\quad}\begin{array}{c}\alpha_3\\ \begin{pmatrix}0\\2\\0\end{pmatrix}\end{array}$$

# 线性组合问题

例 判断 $\beta$ 能否由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式 $\beta = k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3$。

- (1) 问

$$\begin{array}{c}\beta\\ \begin{pmatrix}2\\-7\\5\end{pmatrix}\end{array} = \underline{2}\begin{array}{c}\alpha_1\\ \begin{pmatrix}1\\0\\0\end{pmatrix}\end{array} + \underline{-7}\begin{array}{c}\alpha_2\\ \begin{pmatrix}0\\1\\0\end{pmatrix}\end{array} + \underline{\frac{5}{2}}\begin{array}{c}\alpha_3\\ \begin{pmatrix}0\\0\\2\end{pmatrix}\end{array}$$

所以 $\beta = 2\alpha_1 - 7\alpha_2 + \frac{5}{2}\alpha_3$；$\beta$ 能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表出

---

- (2) 问

$$\begin{array}{c}\beta\\ \begin{pmatrix}2\\-7\\5\end{pmatrix}\end{array} \cancel{=} \underline{\quad}\begin{array}{c}\alpha_1\\ \begin{pmatrix}1\\0\\0\end{pmatrix}\end{array} + \underline{\quad}\begin{array}{c}\alpha_2\\ \begin{pmatrix}2\\3\\0\end{pmatrix}\end{array} + \underline{\quad}\begin{array}{c}\alpha_3\\ \begin{pmatrix}0\\2\\0\end{pmatrix}\end{array}$$

# 线性组合问题

例 判断 $\beta$ 能否由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式 $\beta = k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3$。

- (1) 问

$$\begin{array}{c} \beta \\ \begin{pmatrix} 2 \\ -7 \\ 5 \end{pmatrix} \end{array} = \underline{2} \begin{array}{c} \alpha_1 \\ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \end{array} + \underline{-7} \begin{array}{c} \alpha_2 \\ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \end{array} + \underline{\frac{5}{2}} \begin{array}{c} \alpha_3 \\ \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \end{array}$$

所以 $\beta = 2\alpha_1 - 7\alpha_2 + \frac{5}{2}\alpha_3$；$\beta$ 能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表出

---

- (2) 问

$$\begin{array}{c} \beta \\ \begin{pmatrix} 2 \\ -7 \\ 5 \end{pmatrix} \end{array} \cancel{=} \underline{\quad} \begin{array}{c} \alpha_1 \\ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \end{array} + \underline{\quad} \begin{array}{c} \alpha_2 \\ \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \end{array} + \underline{\quad} \begin{array}{c} \alpha_3 \\ \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \end{array}$$

所以 $\beta$ 不能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表出！

# 线性组合问题

**例** 设 $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$；及 $\beta = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix}$，问

$$\begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix} = \underline{\quad}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix} + \underline{\quad}\begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix} + \underline{\quad}\begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

即：$\beta$ 能否由 $\alpha_1, \alpha_2, \alpha_3$ 线性表出？如果能，线性表达式是什么？

# 线性组合问题

**例** 设 $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$；及 $\beta = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix}$，问

$$\begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix} = \underline{\quad} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix} + \underline{\quad} \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix} + \underline{\quad} \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

即：$\beta$ 能否由 $\alpha_1, \alpha_2, \alpha_3$ 线性表出？如果能，线性表达式是什么？

## 问题

- 一般地，如何判断 $\beta$ 能否由 $\alpha_1, \alpha_2, \ldots, \alpha_n$ 线性表出？

# 线性组合问题

**例** 设 $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$；及 $\beta = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix}$，问

$$\begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix} = \_\_ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix} + \_\_ \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix} + \_\_ \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

即：$\beta$ 能否由 $\alpha_1, \alpha_2, \alpha_3$ 线性表出？如果能，线性表达式是什么？

## 问题

- 一般地，如何判断 $\beta$ 能否由 $\alpha_1, \alpha_2, \ldots, \alpha_n$ 线性表出？
- 如果能线性表出，如何求出 $k_1, k_2, \ldots, k_n$ 使

$$k_1\alpha_1 + k_2\alpha_2 + \cdots + k_n\alpha_n = \beta\ ?$$

# 线性表示问题与线性方程组求解问题的联系

问题是否存在数 $k_1, k_2, \dots, k_n$ 使

$$\underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}_{\beta} = k_1 \underbrace{\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}}_{\alpha_1} + k_2 \underbrace{\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}}_{\alpha_2} + \cdots + k_n \underbrace{\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}}_{\alpha_n}$$

# 线性表示问题与线性方程组求解问题的联系

问题是否存在数 $k_1, k_2, \dots, k_n$ 使

$$\overset{\beta}{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}} = k_1 \overset{\alpha_1}{\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}} + k_2 \overset{\alpha_2}{\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}} + \cdots + k_n \overset{\alpha_n}{\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}}$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix}$$

# 线性表示问题与线性方程组求解问题的联系

问题是否存在数 $k_1, k_2, \dots, k_n$ 使

$$\overset{\beta}{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}} = k_1 \overset{\alpha_1}{\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}} + k_2 \overset{\alpha_2}{\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}} + \cdots + k_n \overset{\alpha_n}{\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}}$$

$$\begin{array}{c} \begin{array}{cccc} \alpha_1 & & & \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix}$$

# 线性表示问题与线性方程组求解问题的联系

问题是否存在数 $k_1, k_2, \ldots, k_n$ 使

$$\overset{\beta}{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}} = k_1 \overset{\alpha_1}{\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}} + k_2 \overset{\alpha_2}{\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}} + \cdots + k_n \overset{\alpha_n}{\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}}$$

$$\begin{array}{cccc} \alpha_1 & \alpha_2 & & \\ \end{array}$$
$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix}$$

# 线性表示问题与线性方程组求解问题的联系

问题是否存在数 $k_1, k_2, \ldots, k_n$ 使

$$\overset{\beta}{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}} = k_1 \overset{\alpha_1}{\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}} + k_2 \overset{\alpha_2}{\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}} + \cdots + k_n \overset{\alpha_n}{\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}}$$

$$\begin{array}{cccc} \alpha_1 & \alpha_2 & & \alpha_n \end{array}$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix}$$

# 线性表示问题与线性方程组求解问题的联系

问题是否存在数 $k_1, k_2, \ldots, k_n$ 使

$$\overset{\beta}{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}} = k_1 \overset{\alpha_1}{\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}} + k_2 \overset{\alpha_2}{\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}} + \cdots + k_n \overset{\alpha_n}{\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}}$$

等价于

$$\begin{array}{c} \begin{matrix} \alpha_1 & \alpha_2 & & \alpha_n \end{matrix} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} = \overset{\beta}{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}$$

# 线性表示问题与线性方程组求解问题的联系

问题是否存在数 $k_1, k_2, \ldots, k_n$ 使

$$\overset{\beta}{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}} = k_1 \overset{\alpha_1}{\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}} + k_2 \overset{\alpha_2}{\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}} + \cdots + k_n \overset{\alpha_n}{\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}}$$

等价于

$$\underbrace{\begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & & \alpha_n \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array}}_{A} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} = \overset{\beta}{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}$$

# 线性表示问题与线性方程组求解问题的联系

问题是否存在数 $k_1, k_2, \ldots, k_n$ 使

$$\overset{\beta}{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}} = k_1 \overset{\alpha_1}{\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}} + k_2 \overset{\alpha_2}{\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}} + \cdots + k_n \overset{\alpha_n}{\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}}$$

等价于

$$\underbrace{\begin{array}{cccc} \alpha_1 & \alpha_2 & & \alpha_n \\ \end{array}}_{}$$

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} = \overset{\beta}{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}} \quad \Longleftrightarrow \quad Ax = \beta$$

# 线性表示问题与线性方程组求解问题的联系

问题是否存在数 $k_1, k_2, \ldots, k_n$ 使

$$\overset{\beta}{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}} = k_1 \overset{\alpha_1}{\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}} + k_2 \overset{\alpha_2}{\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}} + \cdots + k_n \overset{\alpha_n}{\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}}$$

等价于

$$\underbrace{\begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & & \alpha_n \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array}}_{A} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} = \overset{\beta}{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}} \quad \Leftrightarrow \quad Ax = \beta$$

方程有解等价于

# 线性表示问题与线性方程组求解问题的联系

问题是否存在数 $k_1, k_2, \ldots, k_n$ 使

$$\overset{\beta}{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}} = k_1 \overset{\alpha_1}{\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}} + k_2 \overset{\alpha_2}{\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}} + \cdots + k_n \overset{\alpha_n}{\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}}$$

等价于

$$\underbrace{\begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & & \alpha_n \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array}}_{A} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} = \overset{\beta}{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}} \quad \Longleftrightarrow \quad Ax = \beta$$

方程有解等价于

$$r(A) = r(A \vdots \beta)$$

# 线性表示问题与线性方程组求解问题的联系

问题是否存在数 $k_1, k_2, \ldots, k_n$ 使

$$\overset{\beta}{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}} = k_1 \overset{\alpha_1}{\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}} + k_2 \overset{\alpha_2}{\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}} + \cdots + k_n \overset{\alpha_n}{\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}}$$

等价于

$$\underbrace{\begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & & \alpha_n \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array}}_{A} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} = \overset{\beta}{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}} \quad \Leftrightarrow \quad Ax = \beta$$

方程有解等价于

$$r(A) = r(A \vdots \beta) \quad \Leftrightarrow \quad r(\alpha_1 \alpha_2 \cdots \alpha_n) = r(\alpha_1 \alpha_2 \cdots \alpha_n \beta)$$

# 初等行变换求线性表示问题——例 1

例 判断 $\beta$ 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

(1)

$$(\alpha_1\ \ \alpha_2\ \ \alpha_3 \mid \beta) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{array}\right)$$

# 初等行变换求线性表示问题——例 1

例 判断 $\beta$ 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

(1)

$$
\left(\alpha_1\ \ \alpha_2\ \ \alpha_3 \,\middle|\, \beta\right) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{array}\right) \xrightarrow{\text{初等行变换}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)
$$

# 初等行变换求线性表示问题——例 1

例 判断 $\beta$ 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

(1)

$$(\,\alpha_1\ \ \alpha_2\ \ \alpha_3 \,|\, \beta\,) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{array}\right) \xrightarrow{\text{初等行变换}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

- 所以 $r(\alpha_1\alpha_2\alpha_3) =$ ， $r(\alpha_1\alpha_2\alpha_3\beta) =$ ，

# 初等行变换求线性表示问题——例 1

例 判断 $\beta$ 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

(1)

$$(\alpha_1 \ \ \alpha_2 \ \ \alpha_3 \mid \beta) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{array}\right) \xrightarrow{\text{初等行变换}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

- 所以 $r(\alpha_1\alpha_2\alpha_3) =$ ， $r(\alpha_1\alpha_2\alpha_3\beta) =$ ，

# 初等行变换求线性表示问题——例 1

例 判断 $\beta$ 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

(1)

$$(\alpha_1 \ \ \alpha_2 \ \ \alpha_3 \,|\, \beta) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{array}\right) \xrightarrow{\text{初等行变换}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

- 所以 $r(\alpha_1\alpha_2\alpha_3) = 3$，$r(\alpha_1\alpha_2\alpha_3\beta) =$ ，

# 初等行变换求线性表示问题——例 1

例 判断 $\beta$ 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

(1)

$$(\alpha_1 \ \ \alpha_2 \ \ \alpha_3 \mid \beta) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{array}\right) \xrightarrow{\text{初等行变换}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

- 所以 $r(\alpha_1\alpha_2\alpha_3) = 3$，$r(\alpha_1\alpha_2\alpha_3\beta) = 3$，

# 初等行变换求线性表示问题——例 1

例 判断 $\beta$ 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

(1)

$$(\alpha_1\ \ \alpha_2\ \ \alpha_3 \mid \beta) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{array}\right) \xrightarrow{\text{初等行变换}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

- 所以 $r(\alpha_1\alpha_2\alpha_3) = 3$，$r(\alpha_1\alpha_2\alpha_3\beta) = 3$，成立

$$r(\alpha_1\alpha_2\alpha_3) = r(\alpha_1\alpha_2\alpha_3\beta)$$

# 初等行变换求线性表示问题——例 1

例 判断 $\beta$ 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

(1)

$$(\,\alpha_1\ \ \alpha_2\ \ \alpha_3\,|\,\beta\,) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{array}\right) \xrightarrow{\text{初等行变换}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

- 所以 $r(\alpha_1\alpha_2\alpha_3) = 3$，$r(\alpha_1\alpha_2\alpha_3\beta) = 3$，成立

$$r(\alpha_1\alpha_2\alpha_3) = r(\alpha_1\alpha_2\alpha_3\beta)$$

$\beta$ 可由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示。

# 初等行变换求线性表示问题——例 1

例 判断 $\beta$ 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

(1)

$$(\alpha_1 \;\; \alpha_2 \;\; \alpha_3 \,|\, \beta) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{array}\right) \xrightarrow{\text{初等行变换}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

- 所以 $r(\alpha_1\alpha_2\alpha_3) = 3$，$r(\alpha_1\alpha_2\alpha_3\beta) = 3$，成立

$$r(\alpha_1\alpha_2\alpha_3) = r(\alpha_1\alpha_2\alpha_3\beta)$$

$\beta$ 可由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示。

# 初等行变换求线性表示问题——例 1

例 判断 $\beta$ 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

(1)

$$(\alpha_1\ \ \alpha_2\ \ \alpha_3 \mid \beta) = \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{pmatrix} \xrightarrow{\text{初等行变换}} \begin{array}{c} \begin{array}{cccc} \alpha_1' & & & \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

- 所以 $r(\alpha_1\alpha_2\alpha_3) = 3$，$r(\alpha_1\alpha_2\alpha_3\beta) = 3$，成立

$$r(\alpha_1\alpha_2\alpha_3) = r(\alpha_1\alpha_2\alpha_3\beta)$$

$\beta$ 可由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示。

# 初等行变换求线性表示问题——例 1

例 判断 $\beta$ 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

(1)

$$
(\,\alpha_1\ \ \alpha_2\ \ \alpha_3 \,|\, \beta\,) = \left(\begin{array}{rrr|r} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{array}\right) \xrightarrow{\text{初等行变换}} \begin{array}{c} \begin{array}{cccc} \alpha_1' & \alpha_2' & \phantom{0} & \phantom{-1} \end{array} \\ \left(\begin{array}{ccc|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{array}
$$

- 所以 $r(\alpha_1\alpha_2\alpha_3) = 3$，$r(\alpha_1\alpha_2\alpha_3\beta) = 3$，成立

$$
r(\alpha_1\alpha_2\alpha_3) = r(\alpha_1\alpha_2\alpha_3\beta)
$$

$\beta$ 可由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示。

# 初等行变换求线性表示问题——例 1

例 判断 $\beta$ 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

(1)

$$(\alpha_1 \ \ \alpha_2 \ \ \alpha_3 \,|\, \beta) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{array}\right) \xrightarrow{\text{初等行变换}} \begin{array}{cccc} \alpha_1' & \alpha_2' & \alpha_3' & \\ \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

- 所以 $r(\alpha_1\alpha_2\alpha_3) = 3$，$r(\alpha_1\alpha_2\alpha_3\beta) = 3$，成立

$$r(\alpha_1\alpha_2\alpha_3) = r(\alpha_1\alpha_2\alpha_3\beta)$$

$\beta$ 可由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示。

# 初等行变换求线性表示问题——例 1

例 判断 $\beta$ 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

(1)

$$(\,\alpha_1\ \ \alpha_2\ \ \alpha_3 \mid \beta\,) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{array}\right) \xrightarrow{\text{初等行变换}} \begin{array}{c} \begin{array}{cccc} \alpha_1' & \alpha_2' & \alpha_3' & \beta' \end{array} \\ \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

- 所以 $r(\alpha_1\alpha_2\alpha_3) = 3$，$r(\alpha_1\alpha_2\alpha_3\beta) = 3$，成立

$$r(\alpha_1\alpha_2\alpha_3) = r(\alpha_1\alpha_2\alpha_3\beta)$$

$\beta$ 可由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示。

# 初等行变换求线性表示问题——例 1

例 判断 $\beta$ 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

(1)

$$(\,\alpha_1\ \ \alpha_2\ \ \alpha_3\,|\,\beta\,) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{array}\right) \xrightarrow{\text{初等行变换}} \begin{array}{cccc} \alpha_1' & \alpha_2' & \alpha_3' & \beta' \\ \multicolumn{4}{c}{\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)} \end{array}$$

- 所以 $r(\alpha_1\alpha_2\alpha_3) = 3$，$r(\alpha_1\alpha_2\alpha_3\beta) = 3$，成立

$$r(\alpha_1\alpha_2\alpha_3) = r(\alpha_1\alpha_2\alpha_3\beta)$$

  $\beta$ 可由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示。

- 显然 $\beta' = \alpha_1' - \alpha_2' + \alpha_3'$，

# 初等行变换求线性表示问题——例 1

例 判断 $\beta$ 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

(1)

$$(\alpha_1 \ \ \alpha_2 \ \ \alpha_3 \,|\, \beta) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{array}\right) \xrightarrow{\text{初等行变换}} \begin{array}{c} \begin{array}{cccc} \alpha_1' & \alpha_2' & \alpha_3' & \beta' \end{array} \\ \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

- 所以 $r(\alpha_1\alpha_2\alpha_3) = 3$，$r(\alpha_1\alpha_2\alpha_3\beta) = 3$，成立

$$r(\alpha_1\alpha_2\alpha_3) = r(\alpha_1\alpha_2\alpha_3\beta)$$

  $\beta$ 可由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示。

- 显然 $\beta' = \alpha_1' - \alpha_2' + \alpha_3'$，是否也有 $\beta = \alpha_1 - \alpha_2 + \alpha_3$？

# 初等行变换求线性表示问题——例 1

例 判断 $\beta$ 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

(1)

$$(\alpha_1\ \ \alpha_2\ \ \alpha_3 \mid \beta) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{array}\right) \xrightarrow{\text{初等行变换}} \begin{array}{cccc} \alpha_1' & \alpha_2' & \alpha_3' & \beta' \\ \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

- 所以 $r(\alpha_1\alpha_2\alpha_3) = 3$，$r(\alpha_1\alpha_2\alpha_3\beta) = 3$，成立

$$r(\alpha_1\alpha_2\alpha_3) = r(\alpha_1\alpha_2\alpha_3\beta)$$

$\beta$ 可由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示。

- 显然 $\beta' = \alpha_1' - \alpha_2' + \alpha_3'$，是否也有 $\beta = \alpha_1 - \alpha_2 + \alpha_3$？ 是的

# 初等行变换求线性表示问题——例 1

例 判断 $\beta$ 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

(1)

$$(\alpha_1\ \ \alpha_2\ \ \alpha_3\,|\,\beta) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{array}\right) \xrightarrow{\text{初等行变换}} \begin{array}{cccc} \alpha_1' & \alpha_2' & \alpha_3' & \beta' \\ \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

- 所以 $r(\alpha_1\alpha_2\alpha_3) = 3$，$r(\alpha_1\alpha_2\alpha_3\beta) = 3$，成立

$$r(\alpha_1\alpha_2\alpha_3) = r(\alpha_1\alpha_2\alpha_3\beta)$$

$\beta$ 可由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示。

- 显然 $\beta' = \alpha_1' - \alpha_2' + \alpha_3'$，是否也有 $\beta = \alpha_1 - \alpha_2 + \alpha_3$？ 是的

注 可证明：作初等行变换不改变列与列之间的“线性关系”。

# 初等行变换求线性表示问题——例 2

例 判断 $\beta$ 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

(2)

$$(\,\alpha_1\ \ \alpha_2\ \ \alpha_3 \,|\, \beta\,) = \left(\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 2 & -1 & 3 & 3 \\ -1 & 1 & -2 & 0 \\ 5 & 1 & 4 & 11 \end{array}\right)$$

# 初等行变换求线性表示问题——例 2

例 判断 $\beta$ 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

(2)

$$(\alpha_1\ \ \alpha_2\ \ \alpha_3 \mid \beta) = \left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 2 & -1 & 3 & 3 \\ -1 & 1 & -2 & 0 \\ 5 & 1 & 4 & 11 \end{array}\right) \xrightarrow{\text{初等行变换}} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

# 初等行变换求线性表示问题——例 2

例 判断 $\beta$ 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

(2)

$$(\ \alpha_1 \ \ \alpha_2 \ \ \alpha_3 \,|\, \beta\ ) = \left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 2 & -1 & 3 & 3 \\ -1 & 1 & -2 & 0 \\ 5 & 1 & 4 & 11 \end{array}\right) \xrightarrow{\text{初等行变换}} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

- 所以 $r(\alpha_1\alpha_2\alpha_3) =$ ， $r(\alpha_1\alpha_2\alpha_3\beta) =$ ，

# 初等行变换求线性表示问题——例 2

例 判断 $\beta$ 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

(2)

$$(\alpha_1 \ \ \alpha_2 \ \ \alpha_3 \,|\, \beta) = \left(\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 2 & -1 & 3 & 3 \\ -1 & 1 & -2 & 0 \\ 5 & 1 & 4 & 11 \end{array}\right) \xrightarrow{\text{初等行变换}} \left(\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

- 所以 $r(\alpha_1\alpha_2\alpha_3) =$ ， $r(\alpha_1\alpha_2\alpha_3\beta) =$ ，

# 初等行变换求线性表示问题——例 2

例 判断 $\beta$ 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

(2)

$$(\alpha_1\ \ \alpha_2\ \ \alpha_3\,|\,\beta) = \left(\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 2 & -1 & 3 & 3 \\ -1 & 1 & -2 & 0 \\ 5 & 1 & 4 & 11 \end{array}\right) \xrightarrow{\text{初等行变换}} \left(\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

- 所以 $r(\alpha_1\alpha_2\alpha_3) = 2$，$r(\alpha_1\alpha_2\alpha_3\beta) =$ ，

# 初等行变换求线性表示问题——例 2

例 判断 $\beta$ 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

(2)

$$(\alpha_1\ \ \alpha_2\ \ \alpha_3 \mid \beta) = \left(\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 2 & -1 & 3 & 3 \\ -1 & 1 & -2 & 0 \\ 5 & 1 & 4 & 11 \end{array}\right) \xrightarrow{\text{初等行变换}} \left(\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

- 所以 $r(\alpha_1\alpha_2\alpha_3) = 2$，$r(\alpha_1\alpha_2\alpha_3\beta) = 3$，

# 初等行变换求线性表示问题——例 2

例 判断 $\beta$ 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

(2)

$$(\alpha_1\ \ \alpha_2\ \ \alpha_3 \mid \beta) = \left(\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 2 & -1 & 3 & 3 \\ -1 & 1 & -2 & 0 \\ 5 & 1 & 4 & 11 \end{array}\right) \xrightarrow{\text{初等行变换}} \left(\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

- 所以 $r(\alpha_1\alpha_2\alpha_3) = 2$，$r(\alpha_1\alpha_2\alpha_3\beta) = 3$，成立

$$r(\alpha_1\alpha_2\alpha_3) \neq r(\alpha_1\alpha_2\alpha_3\beta)$$

# 初等行变换求线性表示问题——例 2

例 判断 $\beta$ 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

(2)

$$(\,\alpha_1\ \ \alpha_2\ \ \alpha_3\,|\,\beta\,) = \left(\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 2 & -1 & 3 & 3 \\ -1 & 1 & -2 & 0 \\ 5 & 1 & 4 & 11 \end{array}\right) \xrightarrow{\text{初等行变换}} \left(\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

- 所以 $r(\alpha_1\alpha_2\alpha_3) = 2$，$r(\alpha_1\alpha_2\alpha_3\beta) = 3$，成立

$$r(\alpha_1\alpha_2\alpha_3) \neq r(\alpha_1\alpha_2\alpha_3\beta)$$

$\beta$ 不能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示。

# 初等行变换求线性表示问题——总结

问题 $\beta$ 能否由 $\alpha_1, \alpha_2, \ldots, \alpha_n$ 线性表示？若能，写出线性表示等式。

步骤

# 初等行变换求线性表示问题——总结

问题 $\beta$ 能否由 $\alpha_1, \alpha_2, \ldots, \alpha_n$ 线性表示？若能，写出线性表示等式。

步骤 作初等 行 变换：

$$\left( \alpha_1 \ \ \alpha_2 \ \ \cdots \ \ \alpha_n \,\middle|\, \beta \right)$$

# 初等行变换求线性表示问题——总结

问题 $\beta$ 能否由 $\alpha_1, \alpha_2, \ldots, \alpha_n$ 线性表示？若能，写出线性表示等式。

步骤 作初等行变换：

$$\left( \alpha_1 \;\; \alpha_2 \;\; \cdots \;\; \alpha_n \,\middle|\, \beta \right) \xrightarrow{\text{初等行变换}}$$

# 初等行变换求线性表示问题——总结

**问题** $\beta$ 能否由 $\alpha_1, \alpha_2, ..., \alpha_n$ 线性表示？若能，写出线性表示等式。

**步骤** 作初等行变换：

$$\left(\alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_n \,\middle|\, \beta\right) \xrightarrow{\text{初等行变换}} \underbrace{\left(\alpha_1' \quad \alpha_2' \quad \cdots \quad \alpha_n' \,\middle|\, \beta'\right)}_{\text{（简化）阶梯型矩阵}}$$

# 初等行变换求线性表示问题——总结

问题 $\beta$ 能否由 $\alpha_1, \alpha_2, \ldots, \alpha_n$ 线性表示？若能，写出线性表示等式。

步骤 作初等 行 变换：

$$\left(\alpha_1 \ \ \alpha_2 \ \ \cdots \ \ \alpha_n \,\middle|\, \beta\right) \xrightarrow{\text{初等行变换}} \underbrace{\left(\alpha_1' \ \ \alpha_2' \ \ \cdots \ \ \alpha_n' \,\middle|\, \beta'\right)}_{}\text{（简化）阶梯型矩阵}$$

1.

$$\beta\text{由}\alpha_1, \alpha_2, \ldots, \alpha_n\text{线性表示} \iff r(\alpha_1 \cdots \alpha_n) = r(\alpha_1 \cdots \alpha_n\, \beta)$$

# 初等行变换求线性表示问题——总结

问题 $\beta$ 能否由 $\alpha_1, \alpha_2, \ldots, \alpha_n$ 线性表示？若能，写出线性表示等式。

步骤 作初等 行 变换：

$$(\alpha_1 \ \ \alpha_2 \ \ \cdots \ \ \alpha_n \,|\, \beta) \xrightarrow{\text{初等行变换}} (\alpha_1' \ \ \alpha_2' \ \ \cdots \ \ \alpha_n' \,|\, \beta')_{\substack{\text{（简化）} \\ \text{阶梯型矩阵}}}$$

1.

$$\beta\text{由}\alpha_1, \alpha_2, \ldots, \alpha_n\text{线性表示} \iff r(\alpha_1 \cdots \alpha_n) = r(\alpha_1 \cdots \alpha_n\, \beta)$$

$$\Updownarrow$$

$$r(\alpha_1' \cdots \alpha_n') = r(\alpha_1' \cdots \alpha_n'\, \beta')$$

# 初等行变换求线性表示问题——总结

问题 $\beta$ 能否由 $\alpha_1, \alpha_2, \ldots, \alpha_n$ 线性表示？若能，写出线性表示等式。

步骤 作初等行变换：

$$(\alpha_1 \ \alpha_2 \ \cdots \ \alpha_n \mid \beta) \xrightarrow{\text{初等行变换}} (\alpha_1' \ \alpha_2' \ \cdots \ \alpha_n' \mid \beta') \ \text{（简化）阶梯型矩阵}$$

1.

$$\beta\text{由}\alpha_1, \alpha_2, \ldots, \alpha_n\text{线性表示} \iff r(\alpha_1 \cdots \alpha_n) = r(\alpha_1 \cdots \alpha_n \, \beta)$$

$$\Updownarrow$$

$$r(\alpha_1' \cdots \alpha_n') = r(\alpha_1' \cdots \alpha_n' \, \beta')$$

2. 行变换前后列与列的线性关系不变，即：

# 初等行变换求线性表示问题——总结

问题 $\beta$ 能否由 $\alpha_1, \alpha_2, \ldots, \alpha_n$ 线性表示？若能，写出线性表示等式。

步骤 作初等行变换：

$$(\alpha_1 \ \ \alpha_2 \ \ \cdots \ \ \alpha_n \mid \beta) \xrightarrow{\text{初等行变换}} (\alpha_1' \ \ \alpha_2' \ \ \cdots \ \ \alpha_n' \mid \beta')_{\substack{\text{（简化）} \\ \text{阶梯型矩阵}}}$$

1.

$$\beta\text{由}\alpha_1, \alpha_2, \ldots, \alpha_n\text{线性表示} \iff r(\alpha_1 \cdots \alpha_n) = r(\alpha_1 \cdots \alpha_n \, \beta)$$

$$\Updownarrow$$

$$r(\alpha_1' \cdots \alpha_n') = r(\alpha_1' \cdots \alpha_n' \, \beta')$$

2. 行变换前后列与列的线性关系不变，即：

$$\beta' = k_1\alpha_1' + \cdots + k_n\alpha_n' \quad \Rightarrow$$

# 初等行变换求线性表示问题——总结

问题 $\beta$ 能否由 $\alpha_1, \alpha_2, \ldots, \alpha_n$ 线性表示？若能，写出线性表示等式。

步骤 作初等行变换：

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \mid \beta \end{pmatrix} \xrightarrow{\text{初等行变换}} \begin{pmatrix} \alpha_1' & \alpha_2' & \cdots & \alpha_n' \mid \beta' \end{pmatrix}_{\text{阶梯型矩阵}}^{\text{（简化）}}$$

1.

$$\beta\text{由}\alpha_1, \alpha_2, \ldots, \alpha_n\text{线性表示} \quad \Leftrightarrow \quad r(\alpha_1 \cdots \alpha_n) = r(\alpha_1 \cdots \alpha_n\, \beta)$$

$$\Updownarrow$$

$$r(\alpha_1' \cdots \alpha_n') = r(\alpha_1' \cdots \alpha_n'\, \beta')$$

2. 行变换前后列与列的线性关系不变，即：

$$\beta' = k_1\alpha_1' + \cdots + k_n\alpha_n' \quad \Rightarrow \quad \beta = k_1\alpha_1 + \cdots + k_n\alpha_n$$

例 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示？

解

例 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示？

解

$$\begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \beta \end{array} \\ \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{array}\right) \end{array}$$

例 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示？

解

$$\begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \beta \end{array} \\ \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{array}\right) \end{array} \xrightarrow[r_4-2r_1]{r_3-r_1}$$

例 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$，$\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$，$\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示？

解

$$\begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \beta \end{array} \\ \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{array}\right) \end{array} \xrightarrow[r_4-2r_1]{r_3-r_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ & & & \\ & & & \end{array}\right)$$

**例** $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示？

**解**

$$\begin{array}{c} \quad\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \beta \\ \left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\1&1&0&0\\2&-2&1&5\end{array}\right)\end{array} \xrightarrow[r_4-2r_1]{r_3-r_1} \left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\0&-1&-3&-2\\ & & & \end{array}\right)$$

**例** $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$ 能否由 $\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$ 线性表示？

**解**

$$\begin{array}{c} \quad\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \beta \\ \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2\\ 0 & -1 & 2 & 3\\ 1 & 1 & 0 & 0\\ 2 & -2 & 1 & 5 \end{array}\right) \end{array} \xrightarrow[r_4-2r_1]{r_3-r_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2\\ 0 & -1 & 2 & 3\\ 0 & -1 & -3 & -2\\ 0 & -6 & -5 & 1 \end{array}\right)$$

**例** $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$，$\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$，$\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示？

**解**

$$\begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \beta \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{array}\right) \xrightarrow[r_4-2r_1]{r_3-r_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 0 & -1 & -3 & -2 \\ 0 & -6 & -5 & 1 \end{array}\right) \xrightarrow{(-1)\times r_2}$$

例 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示？

解

$$\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array}$$

$$\left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\1&1&0&0\\2&-2&1&5\end{array}\right)\xrightarrow[r_4-2r_1]{r_3-r_1}\left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)\xrightarrow{(-1)\times r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)$$

**例** $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$，$\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$，$\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示？

**解**

$$\begin{array}{c} \begin{matrix}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{matrix} \\ \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{array}\right) \end{array} \xrightarrow[r_4-2r_1]{r_3-r_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 0 & -1 & -3 & -2 \\ 0 & -6 & -5 & 1 \end{array}\right) \xrightarrow{(-1)\times r_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & -1 & -3 & -2 \\ 0 & -6 & -5 & 1 \end{array}\right)$$

$$\xrightarrow[r_4+6r_2]{r_3+r_2}$$

例 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示？

解

$$\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array}$$

$$\left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\1&1&0&0\\2&-2&1&5\end{array}\right)\xrightarrow[r_4-2r_1]{r_3-r_1}\left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)\xrightarrow{(-1)\times r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)$$

$$\xrightarrow[r_4+6r_2]{r_3+r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\&&&\\&&&\end{array}\right)$$

**例** $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示？

**解**

$$\begin{array}{c}\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array}\\ \left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\1&1&0&0\\2&-2&1&5\end{array}\right)\end{array}\xrightarrow[r_4-2r_1]{r_3-r_1}\left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)\xrightarrow{(-1)\times r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)$$

$$\xrightarrow[r_4+6r_2]{r_3+r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&-5&-5\\ & & & \end{array}\right)$$

例 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示？

解

$$\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array}$$

$$\left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\1&1&0&0\\2&-2&1&5\end{array}\right)\xrightarrow[r_4-2r_1]{r_3-r_1}\left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)\xrightarrow{(-1)\times r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)$$

$$\xrightarrow[r_4+6r_2]{r_3+r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&-5&-5\\0&0&-17&-17\end{array}\right)$$

例 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$，$\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$，$\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示？

解

$$\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array}$$

$$\left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\0 & -1 & 2 & 3\\1 & 1 & 0 & 0\\2 & -2 & 1 & 5\end{array}\right)\xrightarrow[r_4-2r_1]{r_3-r_1}\left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\0 & -1 & 2 & 3\\0 & -1 & -3 & -2\\0 & -6 & -5 & 1\end{array}\right)\xrightarrow{(-1)\times r_2}\left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\0 & 1 & -2 & -3\\0 & -1 & -3 & -2\\0 & -6 & -5 & 1\end{array}\right)$$

$$\xrightarrow[r_4+6r_2]{r_3+r_2}\left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\0 & 1 & -2 & -3\\0 & 0 & -5 & -5\\0 & 0 & -17 & -17\end{array}\right)\longrightarrow\left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\0 & 1 & -2 & -3\\0 & 0 & 1 & 1\\0 & 0 & 1 & 1\end{array}\right)$$

**例** $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示？

**解**

$$\begin{array}{c} \begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array} \\ \left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\1&1&0&0\\2&-2&1&5\end{array}\right)\end{array} \xrightarrow[r_4-2r_1]{r_3-r_1} \left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right) \xrightarrow{(-1)\times r_2} \left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)$$

$$\xrightarrow[r_4+6r_2]{r_3+r_2} \left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&-5&-5\\0&0&-17&-17\end{array}\right) \longrightarrow \left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&1&1\\0&0&1&1\end{array}\right) \xrightarrow{r_4-r_3}$$

**例** $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示？

**解** $\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \beta$

$$\left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\1&1&0&0\\2&-2&1&5\end{array}\right)\xrightarrow[r_4-2r_1]{r_3-r_1}\left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)\xrightarrow{(-1)\times r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)$$

$$\xrightarrow[r_4+6r_2]{r_3+r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&-5&-5\\0&0&-17&-17\end{array}\right)\longrightarrow\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&1&1\\0&0&1&1\end{array}\right)\xrightarrow{r_4-r_3}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&1&1\\0&0&0&0\end{array}\right)$$

**例** $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示？

**解**

$$\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array}$$

$$\left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\ 0 & -1 & 2 & 3\\ 1 & 1 & 0 & 0\\ 2 & -2 & 1 & 5\end{array}\right)\xrightarrow[r_4-2r_1]{r_3-r_1}\left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\ 0 & -1 & 2 & 3\\ 0 & -1 & -3 & -2\\ 0 & -6 & -5 & 1\end{array}\right)\xrightarrow{(-1)\times r_2}\left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\ 0 & 1 & -2 & -3\\ 0 & -1 & -3 & -2\\ 0 & -6 & -5 & 1\end{array}\right)$$

$$\xrightarrow[r_4+6r_2]{r_3+r_2}\left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\ 0 & 1 & -2 & -3\\ 0 & 0 & -5 & -5\\ 0 & 0 & -17 & -17\end{array}\right)\longrightarrow\left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\ 0 & 1 & -2 & -3\\ 0 & 0 & 1 & 1\\ 0 & 0 & 1 & 1\end{array}\right)\xrightarrow{r_4-r_3}\left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\ 0 & 1 & -2 & -3\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0\end{array}\right)$$

$$\xrightarrow[r_1-3r_3]{r_2-2r_3}$$

**例** $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$，$\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$，$\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示？

**解**

$$\begin{array}{c}\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array}\\ \left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\1&1&0&0\\2&-2&1&5\end{array}\right)\end{array}\xrightarrow[r_4-2r_1]{r_3-r_1}\left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)\xrightarrow{(-1)\times r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)$$

$$\xrightarrow[r_4+6r_2]{r_3+r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&-5&-5\\0&0&-17&-17\end{array}\right)\longrightarrow\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&1&1\\0&0&1&1\end{array}\right)\xrightarrow{r_4-r_3}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&1&1\\0&0&0&0\end{array}\right)$$

$$\xrightarrow[r_1-3r_3]{r_2-2r_3}\left(\begin{array}{ccc|c}&&&\\&&&\\0&0&1&1\\0&0&0&0\end{array}\right)$$

例 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示？

解

$$\begin{array}{c} \begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array} \\ \left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\0 & -1 & 2 & 3\\1 & 1 & 0 & 0\\2 & -2 & 1 & 5\end{array}\right)\end{array} \xrightarrow[r_4-2r_1]{r_3-r_1} \left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\0 & -1 & 2 & 3\\0 & -1 & -3 & -2\\0 & -6 & -5 & 1\end{array}\right) \xrightarrow{(-1)\times r_2} \left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\0 & 1 & -2 & -3\\0 & -1 & -3 & -2\\0 & -6 & -5 & 1\end{array}\right)$$

$$\xrightarrow[r_4+6r_2]{r_3+r_2} \left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\0 & 1 & -2 & -3\\0 & 0 & -5 & -5\\0 & 0 & -17 & -17\end{array}\right) \longrightarrow \left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\0 & 1 & -2 & -3\\0 & 0 & 1 & 1\\0 & 0 & 1 & 1\end{array}\right) \xrightarrow{r_4-r_3} \left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\0 & 1 & -2 & -3\\0 & 0 & 1 & 1\\0 & 0 & 0 & 0\end{array}\right)$$

$$\xrightarrow[r_1-3r_3]{r_2-2r_3} \left(\begin{array}{ccc|c}0 & 1 & 0 & -1\\0 & 0 & 1 & 1\\0 & 0 & 0 & 0\end{array}\right)$$

**例** $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示？

**解**

$$\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array}$$

$$\left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\1&1&0&0\\2&-2&1&5\end{array}\right)\xrightarrow[r_4-2r_1]{r_3-r_1}\left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)\xrightarrow{(-1)\times r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)$$

$$\xrightarrow[r_4+6r_2]{r_3+r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&-5&-5\\0&0&-17&-17\end{array}\right)\longrightarrow\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&1&1\\0&0&1&1\end{array}\right)\xrightarrow{r_4-r_3}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&1&1\\0&0&0&0\end{array}\right)$$

$$\xrightarrow[r_1-3r_3]{r_2-2r_3}\left(\begin{array}{ccc|c}1&2&0&-1\\0&1&0&-1\\0&0&1&1\\0&0&0&0\end{array}\right)$$

例 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$，$\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$，$\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示？

解 $\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \beta$

$$\left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\1&1&0&0\\2&-2&1&5\end{array}\right)\xrightarrow[r_4-2r_1]{r_3-r_1}\left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)\xrightarrow{(-1)\times r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)$$

$$\xrightarrow[r_4+6r_2]{r_3+r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&-5&-5\\0&0&-17&-17\end{array}\right)\longrightarrow\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&1&1\\0&0&1&1\end{array}\right)\xrightarrow{r_4-r_3}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&1&1\\0&0&0&0\end{array}\right)$$

$$\xrightarrow[r_1-3r_3]{r_2-2r_3}\left(\begin{array}{ccc|c}1&2&0&-1\\0&1&0&-1\\0&0&1&1\\0&0&0&0\end{array}\right)\xrightarrow{r_1-2r_2}$$

**例** $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示？

**解**

$$\begin{array}{c}\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array}\\ \left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\1&1&0&0\\2&-2&1&5\end{array}\right)\end{array}\xrightarrow[r_4-2r_1]{r_3-r_1}\left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)\xrightarrow{(-1)\times r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)$$

$$\xrightarrow[r_4+6r_2]{r_3+r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&-5&-5\\0&0&-17&-17\end{array}\right)\longrightarrow\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&1&1\\0&0&1&1\end{array}\right)\xrightarrow{r_4-r_3}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&1&1\\0&0&0&0\end{array}\right)$$

$$\xrightarrow[r_1-3r_3]{r_2-2r_3}\left(\begin{array}{ccc|c}1&2&0&-1\\0&1&0&-1\\0&0&1&1\\0&0&0&0\end{array}\right)\xrightarrow{r_1-2r_2}\left(\begin{array}{ccc|c}1&0&0&1\\0&1&0&-1\\0&0&1&1\\0&0&0&0\end{array}\right)$$

**例** $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示？

**解**

$$\begin{array}{c} \begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array} \\ \left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\1&1&0&0\\2&-2&1&5\end{array}\right)\end{array} \xrightarrow[r_4-2r_1]{r_3-r_1} \left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right) \xrightarrow{(-1)\times r_2} \left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)$$

$$\xrightarrow[r_4+6r_2]{r_3+r_2} \left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&-5&-5\\0&0&-17&-17\end{array}\right) \longrightarrow \left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&1&1\\0&0&1&1\end{array}\right) \xrightarrow{r_4-r_3} \left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&1&1\\0&0&0&0\end{array}\right)$$

$$\xrightarrow[r_1-3r_3]{r_2-2r_3} \left(\begin{array}{ccc|c}1&2&0&-1\\0&1&0&-1\\0&0&1&1\\0&0&0&0\end{array}\right) \xrightarrow{r_1-2r_2} \left(\begin{array}{ccc|c}1&0&0&1\\0&1&0&-1\\0&0&1&1\\0&0&0&0\end{array}\right)$$

例 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$，$\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$，$\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示？

解

$$\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array}$$

$$\left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\1&1&0&0\\2&-2&1&5\end{array}\right)\xrightarrow[r_4-2r_1]{r_3-r_1}\left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)\xrightarrow{(-1)\times r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)$$

$$\xrightarrow[r_4+6r_2]{r_3+r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&-5&-5\\0&0&-17&-17\end{array}\right)\longrightarrow\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&1&1\\0&0&1&1\end{array}\right)\xrightarrow{r_4-r_3}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&1&1\\0&0&0&0\end{array}\right)$$

$$\xrightarrow[r_1-3r_3]{r_2-2r_3}\left(\begin{array}{ccc|c}1&2&0&-1\\0&1&0&-1\\0&0&1&1\\0&0&0&0\end{array}\right)\xrightarrow{r_1-2r_2}\left(\begin{array}{ccc|c}1&0&0&1\\0&1&0&-1\\0&0&1&1\\0&0&0&0\end{array}\right)$$

所以 $r(\alpha_1\alpha_2\alpha_3)=r(\alpha_1\alpha_2\alpha_3\beta)$,

例 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示？

解

$$\begin{array}{c}\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array}\\ \left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\1&1&0&0\\2&-2&1&5\end{array}\right)\end{array}\xrightarrow[r_4-2r_1]{r_3-r_1}\left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)\xrightarrow{(-1)\times r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)$$

$$\xrightarrow[r_4+6r_2]{r_3+r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&-5&-5\\0&0&-17&-17\end{array}\right)\longrightarrow\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&1&1\\0&0&1&1\end{array}\right)\xrightarrow{r_4-r_3}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&1&1\\0&0&0&0\end{array}\right)$$

$$\xrightarrow[r_1-3r_3]{r_2-2r_3}\left(\begin{array}{ccc|c}1&2&0&-1\\0&1&0&-1\\0&0&1&1\\0&0&0&0\end{array}\right)\xrightarrow{r_1-2r_2}\left(\begin{array}{ccc|c}1&0&0&1\\0&1&0&-1\\0&0&1&1\\0&0&0&0\end{array}\right)$$

所以 $r(\alpha_1\alpha_2\alpha_3)=r(\alpha_1\alpha_2\alpha_3\beta)$，能线性表示

**例** $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$线性表示？

**解**

$$\begin{array}{c} \begin{matrix}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{matrix} \\ \left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\1&1&0&0\\2&-2&1&5\end{array}\right)\end{array} \xrightarrow[r_4-2r_1]{r_3-r_1} \left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right) \xrightarrow{(-1)\times r_2} \left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)$$

$$\xrightarrow[r_4+6r_2]{r_3+r_2} \left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&-5&-5\\0&0&-17&-17\end{array}\right) \longrightarrow \left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&1&1\\0&0&1&1\end{array}\right) \xrightarrow{r_4-r_3} \left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&0&1&1\\0&0&0&0\end{array}\right)$$

$$\xrightarrow[r_1-3r_3]{r_2-2r_3} \left(\begin{array}{ccc|c}1&2&0&-1\\0&1&0&-1\\0&0&1&1\\0&0&0&0\end{array}\right) \xrightarrow{r_1-2r_2} \left(\begin{array}{ccc|c}1&0&0&1\\0&1&0&-1\\0&0&1&1\\0&0&0&0\end{array}\right)$$

所以 $r(\alpha_1\alpha_2\alpha_3)=r(\alpha_1\alpha_2\alpha_3\beta)$，能线性表示，且 $\beta=\alpha_1-\alpha_2+\alpha_3$

例 $\beta=\begin{pmatrix}1\\2\\-1\\6\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}2\\-1\\1\\1\end{pmatrix}$, $\alpha_2=\begin{pmatrix}-1\\3\\-2\\4\end{pmatrix}$, $\alpha_3=\begin{pmatrix}4\\3\\0\\11\end{pmatrix}$线性表示？

解

**例** $\beta=\begin{pmatrix}1\\2\\-1\\6\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}2\\-1\\1\\1\end{pmatrix}$, $\alpha_2=\begin{pmatrix}-1\\3\\-2\\4\end{pmatrix}$, $\alpha_3=\begin{pmatrix}4\\3\\0\\11\end{pmatrix}$线性表示？

**解**

$$\begin{array}{c}
\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array}\\
\left(\begin{array}{ccc|c}
2 & -1 & 4 & 1\\
-1 & 3 & 3 & 2\\
1 & -2 & 0 & -1\\
1 & 4 & 11 & 6
\end{array}\right)
\end{array}$$

**例** $\beta=\begin{pmatrix}1\\2\\-1\\6\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}2\\-1\\1\\1\end{pmatrix}$, $\alpha_2=\begin{pmatrix}-1\\3\\-2\\4\end{pmatrix}$, $\alpha_3=\begin{pmatrix}4\\3\\0\\11\end{pmatrix}$线性表示？

**解**

$$\begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \beta \end{array} \\ \left(\begin{array}{ccc|c} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{array}\right) \end{array} \xrightarrow{r_1 \leftrightarrow r_3}$$

**例** $\beta=\begin{pmatrix}1\\2\\-1\\6\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}2\\-1\\1\\1\end{pmatrix}$，$\alpha_2=\begin{pmatrix}-1\\3\\-2\\4\end{pmatrix}$，$\alpha_3=\begin{pmatrix}4\\3\\0\\11\end{pmatrix}$线性表示？

**解**

$$\begin{array}{c}\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array}\\ \left(\begin{array}{ccc|c}2 & -1 & 4 & 1\\ -1 & 3 & 3 & 2\\ 1 & -2 & 0 & -1\\ 1 & 4 & 11 & 6\end{array}\right)\end{array} \xrightarrow{r_1\leftrightarrow r_3} \left(\begin{array}{ccc|c}1 & -2 & 0 & -1\\ -1 & 3 & 3 & 2\\ 2 & -1 & 4 & 1\\ 1 & 4 & 11 & 6\end{array}\right)$$

**例** $\beta=\begin{pmatrix}1\\2\\-1\\6\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}2\\-1\\1\\1\end{pmatrix}$, $\alpha_2=\begin{pmatrix}-1\\3\\-2\\4\end{pmatrix}$, $\alpha_3=\begin{pmatrix}4\\3\\0\\11\end{pmatrix}$线性表示？

**解**

$$\begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \beta \end{array} \\ \left(\begin{array}{ccc|c} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{array}\right) \end{array} \xrightarrow{r_1 \leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ -1 & 3 & 3 & 2 \\ 2 & -1 & 4 & 1 \\ 1 & 4 & 11 & 6 \end{array}\right)$$

$$\xrightarrow[\substack{r_3-2r_1 \\ r_4-r_1}]{r_2+r_1}$$

**例** $\beta=\begin{pmatrix}1\\2\\-1\\6\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}2\\-1\\1\\1\end{pmatrix}$, $\alpha_2=\begin{pmatrix}-1\\3\\-2\\4\end{pmatrix}$, $\alpha_3=\begin{pmatrix}4\\3\\0\\11\end{pmatrix}$线性表示？

**解**

$$\begin{array}{c} \begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array} \\ \left(\begin{array}{ccc|c} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{array}\right) \end{array} \xrightarrow{r_1\leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ -1 & 3 & 3 & 2 \\ 2 & -1 & 4 & 1 \\ 1 & 4 & 11 & 6 \end{array}\right)$$

$$\xrightarrow[\substack{r_3-2r_1\\ r_4-r_1}]{r_2+r_1} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ & & & \\ & & & \\ & & & \end{array}\right)$$

例 $\beta=\begin{pmatrix}1\\2\\-1\\6\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}2\\-1\\1\\1\end{pmatrix}$，$\alpha_2=\begin{pmatrix}-1\\3\\-2\\4\end{pmatrix}$，$\alpha_3=\begin{pmatrix}4\\3\\0\\11\end{pmatrix}$线性表示？

解

$$\begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \beta \end{array}$$

$$\left(\begin{array}{ccc|c} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{array}\right) \xrightarrow{r_1 \leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ -1 & 3 & 3 & 2 \\ 2 & -1 & 4 & 1 \\ 1 & 4 & 11 & 6 \end{array}\right)$$

$$\xrightarrow[\substack{r_3-2r_1\\ r_4-r_1}]{r_2+r_1} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ & & & \\ & & & \end{array}\right)$$

**例** $\beta=\begin{pmatrix}1\\2\\-1\\6\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}2\\-1\\1\\1\end{pmatrix}$, $\alpha_2=\begin{pmatrix}-1\\3\\-2\\4\end{pmatrix}$, $\alpha_3=\begin{pmatrix}4\\3\\0\\11\end{pmatrix}$线性表示？

**解**

$$\begin{array}{c}\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array}\\ \left(\begin{array}{ccc|c}2 & -1 & 4 & 1\\ -1 & 3 & 3 & 2\\ 1 & -2 & 0 & -1\\ 1 & 4 & 11 & 6\end{array}\right)\end{array} \xrightarrow{r_1\leftrightarrow r_3} \left(\begin{array}{ccc|c}1 & -2 & 0 & -1\\ -1 & 3 & 3 & 2\\ 2 & -1 & 4 & 1\\ 1 & 4 & 11 & 6\end{array}\right)$$

$$\xrightarrow[\substack{r_3-2r_1\\ r_4-r_1}]{r_2+r_1} \left(\begin{array}{ccc|c}1 & -2 & 0 & -1\\ 0 & 1 & 3 & 1\\ 0 & 3 & 4 & 3\\ & & & \end{array}\right)$$

**例** $\beta=\begin{pmatrix}1\\2\\-1\\6\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}2\\-1\\1\\1\end{pmatrix}$, $\alpha_2=\begin{pmatrix}-1\\3\\-2\\4\end{pmatrix}$, $\alpha_3=\begin{pmatrix}4\\3\\0\\11\end{pmatrix}$线性表示？

**解**

$$\begin{array}{c} \quad\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \beta \\ \left(\begin{array}{ccc|c} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{array}\right) \end{array} \xrightarrow{r_1\leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ -1 & 3 & 3 & 2 \\ 2 & -1 & 4 & 1 \\ 1 & 4 & 11 & 6 \end{array}\right)$$

$$\xrightarrow[\substack{r_3-2r_1\\ r_4-r_1}]{r_2+r_1} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 3 & 4 & 3 \\ 0 & 6 & 11 & 7 \end{array}\right)$$

**例** $\beta=\begin{pmatrix}1\\2\\-1\\6\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}2\\-1\\1\\1\end{pmatrix}$, $\alpha_2=\begin{pmatrix}-1\\3\\-2\\4\end{pmatrix}$, $\alpha_3=\begin{pmatrix}4\\3\\0\\11\end{pmatrix}$线性表示？

**解**

$$\begin{array}{c} \begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array} \\ \left(\begin{array}{ccc|c} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{array}\right) \end{array} \xrightarrow{r_1\leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ -1 & 3 & 3 & 2 \\ 2 & -1 & 4 & 1 \\ 1 & 4 & 11 & 6 \end{array}\right)$$

$$\xrightarrow[\substack{r_3-2r_1\\ r_4-r_1}]{r_2+r_1} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 3 & 4 & 3 \\ 0 & 6 & 11 & 7 \end{array}\right) \xrightarrow[r_4-6r_2]{r_3-3r_2}$$

**例** $\beta=\begin{pmatrix}1\\2\\-1\\6\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}2\\-1\\1\\1\end{pmatrix}$, $\alpha_2=\begin{pmatrix}-1\\3\\-2\\4\end{pmatrix}$, $\alpha_3=\begin{pmatrix}4\\3\\0\\11\end{pmatrix}$线性表示？

**解**

$$\begin{array}{c}\quad\alpha_1\quad\alpha_2\quad\alpha_3\quad\beta\\ \left(\begin{array}{ccc|c}2&-1&4&1\\-1&3&3&2\\1&-2&0&-1\\1&4&11&6\end{array}\right)\end{array}\xrightarrow{r_1\leftrightarrow r_3}\left(\begin{array}{ccc|c}1&-2&0&-1\\-1&3&3&2\\2&-1&4&1\\1&4&11&6\end{array}\right)$$

$$\xrightarrow[\substack{r_3-2r_1\\ r_4-r_1}]{r_2+r_1}\left(\begin{array}{ccc|c}1&-2&0&-1\\0&1&3&1\\0&3&4&3\\0&6&11&7\end{array}\right)\xrightarrow[r_4-6r_2]{r_3-3r_2}\left(\begin{array}{ccc|c}1&-2&0&-1\\0&1&3&1\\&&&\\&&&\end{array}\right)$$

**例** $\beta=\begin{pmatrix}1\\2\\-1\\6\end{pmatrix}$ 能否由 $\alpha_1=\begin{pmatrix}2\\-1\\1\\1\end{pmatrix}$, $\alpha_2=\begin{pmatrix}-1\\3\\-2\\4\end{pmatrix}$, $\alpha_3=\begin{pmatrix}4\\3\\0\\11\end{pmatrix}$ 线性表示？

**解**

$$\begin{array}{c}\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array}\\ \left(\begin{array}{ccc|c}2 & -1 & 4 & 1\\ -1 & 3 & 3 & 2\\ 1 & -2 & 0 & -1\\ 1 & 4 & 11 & 6\end{array}\right)\end{array} \xrightarrow{r_1\leftrightarrow r_3} \left(\begin{array}{ccc|c}1 & -2 & 0 & -1\\ -1 & 3 & 3 & 2\\ 2 & -1 & 4 & 1\\ 1 & 4 & 11 & 6\end{array}\right)$$

$$\xrightarrow[\substack{r_3-2r_1\\ r_4-r_1}]{r_2+r_1} \left(\begin{array}{ccc|c}1 & -2 & 0 & -1\\ 0 & 1 & 3 & 1\\ 0 & 3 & 4 & 3\\ 0 & 6 & 11 & 7\end{array}\right) \xrightarrow[r_4-6r_2]{r_3-3r_2} \left(\begin{array}{ccc|c}1 & -2 & 0 & -1\\ 0 & 1 & 3 & 1\\ 0 & 0 & -5 & 0\\ & & & \end{array}\right)$$

**例** $\beta=\begin{pmatrix}1\\2\\-1\\6\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}2\\-1\\1\\1\end{pmatrix}$, $\alpha_2=\begin{pmatrix}-1\\3\\-2\\4\end{pmatrix}$, $\alpha_3=\begin{pmatrix}4\\3\\0\\11\end{pmatrix}$线性表示？

**解**

$$\begin{array}{c}\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array}\\ \left(\begin{array}{ccc|c}2 & -1 & 4 & 1\\ -1 & 3 & 3 & 2\\ 1 & -2 & 0 & -1\\ 1 & 4 & 11 & 6\end{array}\right)\end{array} \xrightarrow{r_1\leftrightarrow r_3} \left(\begin{array}{ccc|c}1 & -2 & 0 & -1\\ -1 & 3 & 3 & 2\\ 2 & -1 & 4 & 1\\ 1 & 4 & 11 & 6\end{array}\right)$$

$$\xrightarrow[\substack{r_3-2r_1\\ r_4-r_1}]{r_2+r_1} \left(\begin{array}{ccc|c}1 & -2 & 0 & -1\\ 0 & 1 & 3 & 1\\ 0 & 3 & 4 & 3\\ 0 & 6 & 11 & 7\end{array}\right) \xrightarrow[r_4-6r_2]{r_3-3r_2} \left(\begin{array}{ccc|c}1 & -2 & 0 & -1\\ 0 & 1 & 3 & 1\\ 0 & 0 & -5 & 0\\ 0 & 0 & -7 & 1\end{array}\right)$$

**例** $\beta=\begin{pmatrix}1\\2\\-1\\6\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}2\\-1\\1\\1\end{pmatrix}$, $\alpha_2=\begin{pmatrix}-1\\3\\-2\\4\end{pmatrix}$, $\alpha_3=\begin{pmatrix}4\\3\\0\\11\end{pmatrix}$线性表示？

**解**

$$\begin{array}{c} \begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array} \\ \left(\begin{array}{ccc|c} 2 & -1 & 4 & 1\\ -1 & 3 & 3 & 2\\ 1 & -2 & 0 & -1\\ 1 & 4 & 11 & 6\end{array}\right)\end{array} \xrightarrow{r_1\leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1\\ -1 & 3 & 3 & 2\\ 2 & -1 & 4 & 1\\ 1 & 4 & 11 & 6\end{array}\right)$$

$$\xrightarrow[\substack{r_3-2r_1\\ r_4-r_1}]{r_2+r_1} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1\\ 0 & 1 & 3 & 1\\ 0 & 3 & 4 & 3\\ 0 & 6 & 11 & 7\end{array}\right) \xrightarrow[r_4-6r_2]{r_3-3r_2} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1\\ 0 & 1 & 3 & 1\\ 0 & 0 & -5 & 0\\ 0 & 0 & -7 & 1\end{array}\right)$$

$$\xrightarrow{-\frac{1}{5}\times r_3}$$

例 $\beta=\begin{pmatrix}1\\2\\-1\\6\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}2\\-1\\1\\1\end{pmatrix}$, $\alpha_2=\begin{pmatrix}-1\\3\\-2\\4\end{pmatrix}$, $\alpha_3=\begin{pmatrix}4\\3\\0\\11\end{pmatrix}$线性表示？

解

$$\begin{array}{c} \quad\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \beta \\ \left(\begin{array}{ccc|c} 2 & -1 & 4 & 1\\ -1 & 3 & 3 & 2\\ 1 & -2 & 0 & -1\\ 1 & 4 & 11 & 6 \end{array}\right) \end{array} \xrightarrow{r_1\leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1\\ -1 & 3 & 3 & 2\\ 2 & -1 & 4 & 1\\ 1 & 4 & 11 & 6 \end{array}\right)$$

$$\xrightarrow[\substack{r_3-2r_1\\ r_4-r_1}]{r_2+r_1} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1\\ 0 & 1 & 3 & 1\\ 0 & 3 & 4 & 3\\ 0 & 6 & 11 & 7 \end{array}\right) \xrightarrow[r_4-6r_2]{r_3-3r_2} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1\\ 0 & 1 & 3 & 1\\ 0 & 0 & -5 & 0\\ 0 & 0 & -7 & 1 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{5}\times r_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1\\ 0 & 1 & 3 & 1\\ 0 & 0 & 1 & 0\\ 0 & 0 & -7 & 1 \end{array}\right)$$

例 $\beta=\begin{pmatrix}1\\2\\-1\\6\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}2\\-1\\1\\1\end{pmatrix}$, $\alpha_2=\begin{pmatrix}-1\\3\\-2\\4\end{pmatrix}$, $\alpha_3=\begin{pmatrix}4\\3\\0\\11\end{pmatrix}$线性表示？

解

$$\begin{array}{c} \quad\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \beta \\ \left(\begin{array}{ccc|c} 2 & -1 & 4 & 1\\ -1 & 3 & 3 & 2\\ 1 & -2 & 0 & -1\\ 1 & 4 & 11 & 6 \end{array}\right) \end{array} \xrightarrow{r_1\leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1\\ -1 & 3 & 3 & 2\\ 2 & -1 & 4 & 1\\ 1 & 4 & 11 & 6 \end{array}\right)$$

$$\xrightarrow[\substack{r_3-2r_1\\ r_4-r_1}]{r_2+r_1} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1\\ 0 & 1 & 3 & 1\\ 0 & 3 & 4 & 3\\ 0 & 6 & 11 & 7 \end{array}\right) \xrightarrow[r_4-6r_2]{r_3-3r_2} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1\\ 0 & 1 & 3 & 1\\ 0 & 0 & -5 & 0\\ 0 & 0 & -7 & 1 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{5}\times r_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1\\ 0 & 1 & 3 & 1\\ 0 & 0 & 1 & 0\\ 0 & 0 & -7 & 1 \end{array}\right) \xrightarrow{r_4+7r_3}$$

例 $\beta=\begin{pmatrix}1\\2\\-1\\6\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}2\\-1\\1\\1\end{pmatrix}$, $\alpha_2=\begin{pmatrix}-1\\3\\-2\\4\end{pmatrix}$, $\alpha_3=\begin{pmatrix}4\\3\\0\\11\end{pmatrix}$线性表示？

解

$$\begin{array}{c}\quad\alpha_1\quad\alpha_2\quad\alpha_3\quad\beta\\ \left(\begin{array}{ccc|c}2&-1&4&1\\-1&3&3&2\\1&-2&0&-1\\1&4&11&6\end{array}\right)\end{array}\xrightarrow{r_1\leftrightarrow r_3}\left(\begin{array}{ccc|c}1&-2&0&-1\\-1&3&3&2\\2&-1&4&1\\1&4&11&6\end{array}\right)$$

$$\xrightarrow[\substack{r_3-2r_1\\r_4-r_1}]{r_2+r_1}\left(\begin{array}{ccc|c}1&-2&0&-1\\0&1&3&1\\0&3&4&3\\0&6&11&7\end{array}\right)\xrightarrow[r_4-6r_2]{r_3-3r_2}\left(\begin{array}{ccc|c}1&-2&0&-1\\0&1&3&1\\0&0&-5&0\\0&0&-7&1\end{array}\right)$$

$$\xrightarrow{-\frac{1}{5}\times r_3}\left(\begin{array}{ccc|c}1&-2&0&-1\\0&1&3&1\\0&0&1&0\\0&0&-7&1\end{array}\right)\xrightarrow{r_4+7r_3}\left(\begin{array}{ccc|c}1&-2&0&-1\\0&1&3&1\\0&0&1&0\\0&0&0&1\end{array}\right)$$

**例** $\beta=\begin{pmatrix}1\\2\\-1\\6\end{pmatrix}$ 能否由 $\alpha_1=\begin{pmatrix}2\\-1\\1\\1\end{pmatrix}$, $\alpha_2=\begin{pmatrix}-1\\3\\-2\\4\end{pmatrix}$, $\alpha_3=\begin{pmatrix}4\\3\\0\\11\end{pmatrix}$ 线性表示？

**解**

$$\begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \beta \end{array}$$

$$\left(\begin{array}{ccc|c} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{array}\right) \xrightarrow{r_1 \leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ -1 & 3 & 3 & 2 \\ 2 & -1 & 4 & 1 \\ 1 & 4 & 11 & 6 \end{array}\right)$$

$$\xrightarrow[\substack{r_3-2r_1 \\ r_4-r_1}]{r_2+r_1} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 3 & 4 & 3 \\ 0 & 6 & 11 & 7 \end{array}\right) \xrightarrow[r_4-6r_2]{r_3-3r_2} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & -7 & 1 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{5}\times r_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -7 & 1 \end{array}\right) \xrightarrow{r_4+7r_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

**例** $\beta=\begin{pmatrix}1\\2\\-1\\6\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}2\\-1\\1\\1\end{pmatrix}$, $\alpha_2=\begin{pmatrix}-1\\3\\-2\\4\end{pmatrix}$, $\alpha_3=\begin{pmatrix}4\\3\\0\\11\end{pmatrix}$线性表示？

**解**

$$\begin{array}{c}\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array}\\ \left(\begin{array}{ccc|c}2&-1&4&1\\-1&3&3&2\\1&-2&0&-1\\1&4&11&6\end{array}\right)\end{array} \xrightarrow{r_1\leftrightarrow r_3} \left(\begin{array}{ccc|c}1&-2&0&-1\\-1&3&3&2\\2&-1&4&1\\1&4&11&6\end{array}\right)$$

$$\xrightarrow[\substack{r_3-2r_1\\ r_4-r_1}]{r_2+r_1} \left(\begin{array}{ccc|c}1&-2&0&-1\\0&1&3&1\\0&3&4&3\\0&6&11&7\end{array}\right) \xrightarrow[r_4-6r_2]{r_3-3r_2} \left(\begin{array}{ccc|c}1&-2&0&-1\\0&1&3&1\\0&0&-5&0\\0&0&-7&1\end{array}\right)$$

$$\xrightarrow{-\frac{1}{5}\times r_3} \left(\begin{array}{ccc|c}1&-2&0&-1\\0&1&3&1\\0&0&1&0\\0&0&-7&1\end{array}\right) \xrightarrow{r_4+7r_3} \left(\begin{array}{ccc|c}1&-2&0&-1\\0&1&3&1\\0&0&1&0\\0&0&0&1\end{array}\right)$$

可见 $r(\alpha_1\alpha_2\alpha_3\beta)=4>3=r(\alpha_1\alpha_2\alpha_3)$,

例 $\beta=\begin{pmatrix}1\\2\\-1\\6\end{pmatrix}$能否由$\alpha_1=\begin{pmatrix}2\\-1\\1\\1\end{pmatrix}$, $\alpha_2=\begin{pmatrix}-1\\3\\-2\\4\end{pmatrix}$, $\alpha_3=\begin{pmatrix}4\\3\\0\\11\end{pmatrix}$线性表示？

解

$$\begin{array}{c}\quad\alpha_1\quad\alpha_2\quad\alpha_3\quad\beta\\ \left(\begin{array}{ccc|c}2&-1&4&1\\-1&3&3&2\\1&-2&0&-1\\1&4&11&6\end{array}\right)\end{array}\xrightarrow{r_1\leftrightarrow r_3}\left(\begin{array}{ccc|c}1&-2&0&-1\\-1&3&3&2\\2&-1&4&1\\1&4&11&6\end{array}\right)$$

$$\xrightarrow[\substack{r_3-2r_1\\r_4-r_1}]{r_2+r_1}\left(\begin{array}{ccc|c}1&-2&0&-1\\0&1&3&1\\0&3&4&3\\0&6&11&7\end{array}\right)\xrightarrow[r_4-6r_2]{r_3-3r_2}\left(\begin{array}{ccc|c}1&-2&0&-1\\0&1&3&1\\0&0&-5&0\\0&0&-7&1\end{array}\right)$$

$$\xrightarrow{-\frac{1}{5}\times r_3}\left(\begin{array}{ccc|c}1&-2&0&-1\\0&1&3&1\\0&0&1&0\\0&0&-7&1\end{array}\right)\xrightarrow{r_4+7r_3}\left(\begin{array}{ccc|c}1&-2&0&-1\\0&1&3&1\\0&0&1&0\\0&0&0&1\end{array}\right)$$

可见 $r(\alpha_1\alpha_2\alpha_3\beta)=4>3=r(\alpha_1\alpha_2\alpha_3)$，所以不能线性表示。

# 向量组的线性组合

**定义** 设有两个向量组

$$(A): \quad \alpha_1, \alpha_2, \ldots, \alpha_s$$

$$(B): \quad \beta_1, \beta_2, \ldots, \beta_t$$

# 向量组的线性组合

**定义** 设有两个向量组

$$(A):\quad \alpha_1, \alpha_2, \ldots, \alpha_s$$

$$(B):\quad \beta_1, \beta_2, \ldots, \beta_t$$

如果中 $(A)$ 中每一向量均可由 $(B)$ 线性表示，则称向量组 $(A)$ 可由向量组 $(B)$线性表示。

# 向量组的线性组合

**定义** 设有两个向量组

$$(A):\quad \alpha_1, \alpha_2, \ldots, \alpha_s$$

$$(B):\quad \beta_1, \beta_2, \ldots, \beta_t$$

如果中 $(A)$ 中每一向量均可由 $(B)$ 线性表示，则称向量组 $(A)$ 可由向量组 $(B)$线性表示。

---

**例** 向量组 $\alpha_1, \alpha_2$ 可由向量组 $\beta_1, \beta_2, \beta_3$ 线性表示

# 向量组的线性组合

**定义** 设有两个向量组

$$(A): \quad \alpha_1, \alpha_2, \ldots, \alpha_s$$

$$(B): \quad \beta_1, \beta_2, \ldots, \beta_t$$

如果中 $(A)$ 中每一向量均可由 $(B)$ 线性表示，则称向量组 $(A)$ 可由向量组 $(B)$线性表示。

---

**例** 向量组 $\alpha_1, \alpha_2$ 可由向量组 $\beta_1, \beta_2, \beta_3$ 线性表示

$$\xRightarrow{\text{不妨设}} \begin{cases} \alpha_1 = \quad \beta_1 + \quad \beta_2 + \quad \beta_3 \\ \alpha_2 = \quad \beta_1 + \quad \beta_2 + \quad \beta_3 \end{cases}$$

# 向量组的线性组合

定义 设有两个向量组

$$(A): \quad \alpha_1, \alpha_2, \ldots, \alpha_s$$

$$(B): \quad \beta_1, \beta_2, \ldots, \beta_t$$

如果中 $(A)$ 中每一向量均可由 $(B)$ 线性表示，则称向量组 $(A)$ 可由向量组 $(B)$线性表示。

---

例 向量组 $\alpha_1, \alpha_2$ 可由向量组 $\beta_1, \beta_2, \beta_3$ 线性表示

$$\xRightarrow{\text{不妨设}} \begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 + a_{31}\beta_3 \\ \alpha_2 = \qquad \beta_1 + \qquad \beta_2 + \qquad \beta_3 \end{cases}$$

# 向量组的线性组合

**定义** 设有两个向量组

$$(A):\quad \alpha_1, \alpha_2, \ldots, \alpha_s$$

$$(B):\quad \beta_1, \beta_2, \ldots, \beta_t$$

如果中 $(A)$ 中每一向量均可由 $(B)$ 线性表示，则称向量组 $(A)$ 可由向量组 $(B)$ 线性表示。

---

**例** 向量组 $\alpha_1, \alpha_2$ 可由向量组 $\beta_1, \beta_2, \beta_3$ 线性表示

$$\xRightarrow{\text{不妨设}} \begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 + a_{31}\beta_3 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 + a_{32}\beta_3 \end{cases}$$

# 向量组的线性组合

**定义** 设有两个向量组

$$(A):\quad \alpha_1, \alpha_2, \ldots, \alpha_s$$

$$(B):\quad \beta_1, \beta_2, \ldots, \beta_t$$

如果中 $(A)$ 中每一向量均可由 $(B)$ 线性表示，则称向量组 $(A)$ 可由向量组 $(B)$线性表示。

---

**例** 向量组 $\alpha_1, \alpha_2$ 可由向量组 $\beta_1, \beta_2, \beta_3$ 线性表示

$$\xRightarrow{\text{不妨设}} \begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 + a_{31}\beta_3 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 + a_{32}\beta_3 \end{cases}$$

$$\xRightarrow{\text{改写为}} (\alpha_1, \alpha_2) = (\beta_1, \beta_2, \beta_3)\left( \qquad \right)$$

# 向量组的线性组合

**定义** 设有两个向量组

$$(A): \quad \alpha_1, \alpha_2, \ldots, \alpha_s$$

$$(B): \quad \beta_1, \beta_2, \ldots, \beta_t$$

如果中 $(A)$ 中每一向量均可由 $(B)$ 线性表示，则称向量组 $(A)$ 可由向量组 $(B)$线性表示。

---

**例** 向量组 $\alpha_1, \alpha_2$ 可由向量组 $\beta_1, \beta_2, \beta_3$ 线性表示

$$\xrightarrow{\text{不妨设}} \begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 + a_{31}\beta_3 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 + a_{32}\beta_3 \end{cases}$$

$$\xrightarrow{\text{改写为}} (\alpha_1, \alpha_2) = (\beta_1, \beta_2, \beta_3) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

# 向量组的线性组合

**定义** 设有两个向量组

$$(A):\quad \alpha_1, \alpha_2, \ldots, \alpha_s$$

$$(B):\quad \beta_1, \beta_2, \ldots, \beta_t$$

如果中 $(A)$ 中每一向量均可由 $(B)$ 线性表示，则称向量组 $(A)$ 可由向量组 $(B)$ 线性表示。

---

**例** 向量组 $\alpha_1, \alpha_2$ 可由向量组 $\beta_1, \beta_2, \beta_3$ 线性表示

$$\xRightarrow{\text{不妨设}} \begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 + a_{31}\beta_3 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 + a_{32}\beta_3 \end{cases}$$

$$\xRightarrow{\text{改写为}} (\alpha_1, \alpha_2) = (\beta_1, \beta_2, \beta_3)\underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}}_{A}$$

# 向量组的线性组合

**定义** 设有两个向量组

$$(A):\quad \alpha_1, \alpha_2, \ldots, \alpha_s$$

$$(B):\quad \beta_1, \beta_2, \ldots, \beta_t$$

如果中 $(A)$ 中每一向量均可由 $(B)$ 线性表示，则称向量组 $(A)$ 可由向量组 $(B)$ 线性表示。

---

**例** 向量组 $\alpha_1, \alpha_2$ 可由向量组 $\beta_1, \beta_2, \beta_3$ 线性表示

$$\xRightarrow{\text{不妨设}} \begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 + a_{31}\beta_3 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 + a_{32}\beta_3 \end{cases}$$

$$\xRightarrow{\text{改写为}} (\alpha_1, \alpha_2) = (\beta_1, \beta_2, \beta_3)\underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}}_{A} = (\beta_1, \beta_2, \beta_3)A$$

注 1 一般地，向量组 $\alpha_1, \alpha_2, ..., \alpha_s$ 可由向量组 $\beta_1, \beta_2, ..., \beta_t$ 线性表示，当且仅当存在矩阵 $A_{t\times s}$ 满足：

注 1 一般地，向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 可由向量组 $\beta_1, \beta_2, \ldots, \beta_t$ 线性表示，当且仅当存在矩阵 $A_{t\times s}$ 满足：

$$(\alpha_1, \alpha_2, \ldots, \alpha_s) = (\beta_1, \beta_2, \ldots, \beta_t)A$$

注 1 一般地，向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 可由向量组 $\beta_1, \beta_2, \ldots, \beta_t$ 线性表示，当且仅当存在矩阵 $A_{t\times s}$ 满足：

$$
\begin{aligned}
(\alpha_1, \alpha_2, \ldots, \alpha_s) &= (\beta_1, \beta_2, \ldots, \beta_t)A \\
&= (\beta_1, \beta_2, \ldots, \beta_t)\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ a_{t1} & a_{t2} & \cdots & a_{ts} \end{pmatrix}
\end{aligned}
$$

注 1 一般地，向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 可由向量组 $\beta_1, \beta_2, \ldots, \beta_t$ 线性表示，当且仅当存在矩阵 $A_{t\times s}$ 满足：

$$
\begin{aligned}
(\alpha_1, \alpha_2, \ldots, \alpha_s) &= (\beta_1, \beta_2, \ldots, \beta_t)A \\
&= (\beta_1, \beta_2, \ldots, \beta_t)\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ a_{t1} & a_{t2} & \cdots & a_{ts} \end{pmatrix}
\end{aligned}
$$

这时 $A$ 的每一列表示线性组合的系数。

注 1 一般地，向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 可由向量组 $\beta_1, \beta_2, \ldots, \beta_t$ 线性表示，当且仅当存在矩阵 $A_{t\times s}$ 满足：

$$
\begin{aligned}
(\alpha_1, \alpha_2, \ldots, \alpha_s) &= (\beta_1, \beta_2, \ldots, \beta_t)A \\
&= (\beta_1, \beta_2, \ldots, \beta_t)\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ a_{t1} & a_{t2} & \cdots & a_{ts} \end{pmatrix}
\end{aligned}
$$

这时 $A$ 的每一列表示线性组合的系数。例如，

$$\alpha_j = \quad \beta_1 + \quad \beta_2 + \cdots + \quad \beta_t$$

注 1 一般地，向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 可由向量组 $\beta_1, \beta_2, \ldots, \beta_t$ 线性表示，当且仅当存在矩阵 $A_{t\times s}$ 满足：

$$\begin{aligned}(\alpha_1, \alpha_2, \ldots, \alpha_s) &= (\beta_1, \beta_2, \ldots, \beta_t)A \\ &= (\beta_1, \beta_2, \ldots, \beta_t)\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ a_{t1} & a_{t2} & \cdots & a_{ts} \end{pmatrix}\end{aligned}$$

这时 $A$ 的每一列表示线性组合的系数。例如，

$$\alpha_j = a_{1j}\beta_1 + a_{2j}\beta_2 + \cdots + a_{tj}\beta_t$$

其中的系数就是 $A$ 的第 $j$ 列 $\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{sj} \end{pmatrix}$。

注 1 一般地，向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 可由向量组 $\beta_1, \beta_2, \ldots, \beta_t$ 线性表示，当且仅当存在矩阵 $A_{t\times s}$ 满足：

$$\begin{aligned}(\alpha_1, \alpha_2, \ldots, \alpha_s) &= (\beta_1, \beta_2, \ldots, \beta_t)A \\ &= (\beta_1, \beta_2, \ldots, \beta_t)\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ a_{t1} & a_{t2} & \cdots & a_{ts} \end{pmatrix}\end{aligned}$$

这时 $A$ 的每一列表示线性组合的系数。例如，

$$\alpha_j = a_{1j}\beta_1 + a_{2j}\beta_2 + \cdots + a_{tj}\beta_t$$

其中的系数就是 $A$ 的第 $j$ 列 $\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{sj} \end{pmatrix}$。

注 2 若上述向量均为列向量，

注 1 一般地，向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 可由向量组 $\beta_1, \beta_2, \ldots, \beta_t$ 线性表示，当且仅当存在矩阵 $A_{t\times s}$ 满足：

$$\begin{aligned}\underbrace{(\alpha_1, \alpha_2, \ldots, \alpha_s)}_{P} &= (\beta_1, \beta_2, \ldots, \beta_t)A \\ &= (\beta_1, \beta_2, \ldots, \beta_t)\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ a_{t1} & a_{t2} & \cdots & a_{ts} \end{pmatrix}\end{aligned}$$

这时 $A$ 的每一列表示线性组合的系数。例如，

$$\alpha_j = a_{1j}\beta_1 + a_{2j}\beta_2 + \cdots + a_{tj}\beta_t$$

其中的系数就是 $A$ 的第 $j$ 列 $\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{sj} \end{pmatrix}$。

注 2 若上述向量均为列向量，

注 1 一般地，向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 可由向量组 $\beta_1, \beta_2, \ldots, \beta_t$ 线性表示，当且仅当存在矩阵 $A_{t\times s}$ 满足：

$$\underbrace{(\alpha_1, \alpha_2, \ldots, \alpha_s)}_{P} = \underbrace{(\beta_1, \beta_2, \ldots, \beta_t)}_{Q} A$$

$$= (\beta_1, \beta_2, \ldots, \beta_t) \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ a_{t1} & a_{t2} & \cdots & a_{ts} \end{pmatrix}$$

这时 $A$ 的每一列表示线性组合的系数。例如，

$$\alpha_j = a_{1j}\beta_1 + a_{2j}\beta_2 + \cdots + a_{tj}\beta_t$$

其中的系数就是 $A$ 的第 $j$ 列 $\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{sj} \end{pmatrix}$。

注 2 若上述向量均为列向量，

注 1 一般地，向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 可由向量组 $\beta_1, \beta_2, \ldots, \beta_t$ 线性表示，当且仅当存在矩阵 $A_{t\times s}$ 满足：

$$\underbrace{(\alpha_1, \alpha_2, \ldots, \alpha_s)}_{P} = \underbrace{(\beta_1, \beta_2, \ldots, \beta_t)}_{Q} A$$

$$= (\beta_1, \beta_2, \ldots, \beta_t) \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ a_{t1} & a_{t2} & \cdots & a_{ts} \end{pmatrix}$$

这时 $A$ 的每一列表示线性组合的系数。例如，

$$\alpha_j = a_{1j}\beta_1 + a_{2j}\beta_2 + \cdots + a_{tj}\beta_t$$

其中的系数就是 $A$ 的第 $j$ 列 $\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{sj} \end{pmatrix}$。

注 2 若上述向量均为列向量，则上式正好表示矩阵乘积：$P = QA$

**定理（向量组线性表示的传递性）** 假设向量组 $(A), (B), (C)$ 满足：$(A)$ 可由 $(B)$ 线性表示，$(B)$ 可由 $(C)$ 线性表示，则 $(A)$ 可由 $(C)$ 线性表示。

**定理（向量组线性表示的传递性）** 假设向量组 $(A)$, $(B)$, $(C)$ 满足：$(A)$ 可由 $(B)$ 线性表示，$(B)$ 可由 $(C)$ 线性表示，则 $(A)$ 可由 $(C)$ 线性表示。

**证明** 设向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 可由向量组 $\beta_1, \beta_2, \ldots, \beta_t$ 线性表示：

向量组 $\beta_1, \beta_2, \ldots, \beta_t$ 可由向量组 $\gamma_1, \gamma_2, \ldots, \gamma_k$ 线性表示：

**定理（向量组线性表示的传递性）** 假设向量组 $(A), (B), (C)$ 满足：$(A)$ 可由 $(B)$ 线性表示，$(B)$ 可由 $(C)$ 线性表示，则 $(A)$ 可由 $(C)$ 线性表示。

**证明** 设向量组 $\alpha_1, \alpha_2, \dots, \alpha_s$ 可由向量组 $\beta_1, \beta_2, \dots, \beta_t$ 线性表示：

$$(\alpha_1, \alpha_2, \dots, \alpha_s) = (\beta_1, \beta_2, \dots, \beta_t)A_{t\times s}.$$

向量组 $\beta_1, \beta_2, \dots, \beta_t$ 可由向量组 $\gamma_1, \gamma_2, \dots, \gamma_k$ 线性表示：

**定理（向量组线性表示的传递性）** 假设向量组 $(A), (B), (C)$ 满足：$(A)$ 可由 $(B)$ 线性表示，$(B)$ 可由 $(C)$ 线性表示，则 $(A)$ 可由 $(C)$ 线性表示。

**证明** 设向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 可由向量组 $\beta_1, \beta_2, \ldots, \beta_t$ 线性表示：

$$(\alpha_1, \alpha_2, \ldots, \alpha_s) = (\beta_1, \beta_2, \ldots, \beta_t)A_{t\times s}.$$

向量组 $\beta_1, \beta_2, \ldots, \beta_t$ 可由向量组 $\gamma_1, \gamma_2, \ldots, \gamma_k$ 线性表示：

$$(\beta_1, \beta_2, \ldots, \beta_t) = (\gamma_1, \gamma_2, \ldots, \gamma_k)B_{k\times t}.$$

**定理（向量组线性表示的传递性）** 假设向量组 $(A), (B), (C)$ 满足：$(A)$ 可由 $(B)$ 线性表示，$(B)$ 可由 $(C)$ 线性表示，则 $(A)$ 可由 $(C)$ 线性表示。

**证明** 设向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 可由向量组 $\beta_1, \beta_2, \ldots, \beta_t$ 线性表示：

$$(\alpha_1, \alpha_2, \ldots, \alpha_s) = (\beta_1, \beta_2, \ldots, \beta_t)A_{t\times s}.$$

向量组 $\beta_1, \beta_2, \ldots, \beta_t$ 可由向量组 $\gamma_1, \gamma_2, \ldots, \gamma_k$ 线性表示：

$$(\beta_1, \beta_2, \ldots, \beta_t) = (\gamma_1, \gamma_2, \ldots, \gamma_k)B_{k\times t}.$$

将第 2 式代入第 1 式，可得

**定理（向量组线性表示的传递性）** 假设向量组 (A), (B), (C) 满足：(A) 可由 (B) 线性表示，(B) 可由 (C) 线性表示，则 (A) 可由 (C) 线性表示。

**证明** 设向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 可由向量组 $\beta_1, \beta_2, \ldots, \beta_t$ 线性表示：

$$(\alpha_1, \alpha_2, \ldots, \alpha_s) = (\beta_1, \beta_2, \ldots, \beta_t)A_{t\times s}.$$

向量组 $\beta_1, \beta_2, \ldots, \beta_t$ 可由向量组 $\gamma_1, \gamma_2, \ldots, \gamma_k$ 线性表示：

$$(\beta_1, \beta_2, \ldots, \beta_t) = (\gamma_1, \gamma_2, \ldots, \gamma_k)B_{k\times t}.$$

将第 2 式代入第 1 式，可得

$(\alpha_1, \alpha_2, \ldots, \alpha_s) = (\gamma_1, \gamma_2, \ldots, \gamma_k)B_{k\times t}A_{t\times s}$

定理（向量组线性表示的传递性） 假设向量组 $(A), (B), (C)$ 满足：$(A)$ 可由 $(B)$ 线性表示，$(B)$ 可由 $(C)$ 线性表示，则 $(A)$ 可由 $(C)$ 线性表示。

证明 设向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 可由向量组 $\beta_1, \beta_2, \ldots, \beta_t$ 线性表示：

$$(\alpha_1, \alpha_2, \ldots, \alpha_s) = (\beta_1, \beta_2, \ldots, \beta_t)A_{t\times s}.$$

向量组 $\beta_1, \beta_2, \ldots, \beta_t$ 可由向量组 $\gamma_1, \gamma_2, \ldots, \gamma_k$ 线性表示：

$$(\beta_1, \beta_2, \ldots, \beta_t) = (\gamma_1, \gamma_2, \ldots, \gamma_k)B_{k\times t}.$$

将第 2 式代入第 1 式，可得

$$(\alpha_1, \alpha_2, \ldots, \alpha_s) = (\gamma_1, \gamma_2, \ldots, \gamma_k)\underbrace{B_{k\times t}A_{t\times s}}_{C_{k\times s}}$$

**定理（向量组线性表示的传递性）** 假设向量组 $(A), (B), (C)$ 满足：$(A)$ 可由 $(B)$ 线性表示，$(B)$ 可由 $(C)$ 线性表示，则 $(A)$ 可由 $(C)$ 线性表示。

**证明** 设向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 可由向量组 $\beta_1, \beta_2, \ldots, \beta_t$ 线性表示：

$$(\alpha_1, \alpha_2, \ldots, \alpha_s) = (\beta_1, \beta_2, \ldots, \beta_t)A_{t\times s}.$$

向量组 $\beta_1, \beta_2, \ldots, \beta_t$ 可由向量组 $\gamma_1, \gamma_2, \ldots, \gamma_k$ 线性表示：

$$(\beta_1, \beta_2, \ldots, \beta_t) = (\gamma_1, \gamma_2, \ldots, \gamma_k)B_{k\times t}.$$

将第 2 式代入第 1 式，可得

$$(\alpha_1, \alpha_2, \ldots, \alpha_s) = (\gamma_1, \gamma_2, \ldots, \gamma_k)\underbrace{B_{k\times t}A_{t\times s}}_{C_{k\times s}} = (\gamma_1, \gamma_2, \ldots, \gamma_k)C.$$

**定理（向量组线性表示的传递性）** 假设向量组 $(A), (B), (C)$ 满足：$(A)$ 可由 $(B)$ 线性表示，$(B)$ 可由 $(C)$ 线性表示，则 $(A)$ 可由 $(C)$ 线性表示。

**证明** 设向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 可由向量组 $\beta_1, \beta_2, \ldots, \beta_t$ 线性表示：

$$(\alpha_1, \alpha_2, \ldots, \alpha_s) = (\beta_1, \beta_2, \ldots, \beta_t)A_{t\times s}.$$

向量组 $\beta_1, \beta_2, \ldots, \beta_t$ 可由向量组 $\gamma_1, \gamma_2, \ldots, \gamma_k$ 线性表示：

$$(\beta_1, \beta_2, \ldots, \beta_t) = (\gamma_1, \gamma_2, \ldots, \gamma_k)B_{k\times t}.$$

将第 2 式代入第 1 式，可得

$$(\alpha_1, \alpha_2, \ldots, \alpha_s) = (\gamma_1, \gamma_2, \ldots, \gamma_k)\underbrace{B_{k\times t}A_{t\times s}}_{C_{k\times s}} = (\gamma_1, \gamma_2, \ldots, \gamma_k)C.$$

所以向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 可由向量组 $\gamma_1, \gamma_2, \ldots, \gamma_k$ 线性表示。

**定理（向量组线性表示的传递性）** 假设向量组 $(A), (B), (C)$ 满足：$(A)$ 可由 $(B)$ 线性表示，$(B)$ 可由 $(C)$ 线性表示，则 $(A)$ 可由 $(C)$ 线性表示。

**证明** 设向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 可由向量组 $\beta_1, \beta_2, \ldots, \beta_t$ 线性表示：

$$(\alpha_1, \alpha_2, \ldots, \alpha_s) = (\beta_1, \beta_2, \ldots, \beta_t)A_{t\times s}.$$

向量组 $\beta_1, \beta_2, \ldots, \beta_t$ 可由向量组 $\gamma_1, \gamma_2, \ldots, \gamma_k$ 线性表示：

$$(\beta_1, \beta_2, \ldots, \beta_t) = (\gamma_1, \gamma_2, \ldots, \gamma_k)B_{k\times t}.$$

将第 2 式代入第 1 式，可得

$$(\alpha_1, \alpha_2, \ldots, \alpha_s) = (\gamma_1, \gamma_2, \ldots, \gamma_k)\underbrace{B_{k\times t}A_{t\times s}}_{C_{k\times s}} = (\gamma_1, \gamma_2, \ldots, \gamma_k)C.$$

所以向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 可由向量组 $\gamma_1, \gamma_2, \ldots, \gamma_k$ 线性表示。

（并且，线性组合的系数就是矩阵 $C$ 的列。）

例 $\left.\begin{array}{r} \alpha_1, \alpha_2\text{由}\beta_1, \beta_2\text{线性表示} \\ \beta_1, \beta_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示} \end{array}\right\} \Rightarrow \alpha_1, \alpha_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示}$

例 $\left.\begin{array}{r}\alpha_1, \alpha_2\text{由}\beta_1, \beta_2\text{线性表示} \\ \beta_1, \beta_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示}\end{array}\right\} \Rightarrow \alpha_1, \alpha_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示}$

具体地，设

$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases}$$

例 $\left.\begin{array}{r}\alpha_1, \alpha_2\text{由}\beta_1, \beta_2\text{线性表示}\\ \beta_1, \beta_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示}\end{array}\right\} \Rightarrow \alpha_1, \alpha_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示}$

具体地，设

$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases}$$

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases}$$

例 $\left.\begin{array}{r} \alpha_1, \alpha_2\text{由}\beta_1, \beta_2\text{线性表示} \\ \beta_1, \beta_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示} \end{array}\right\} \Rightarrow \alpha_1, \alpha_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示}$

具体地，设

$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases}$$

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases}$$

则

$\alpha_1 =$

$\alpha_2 =$

例 $\left.\begin{array}{r}\alpha_1, \alpha_2\text{由}\beta_1, \beta_2\text{线性表示}\\ \beta_1, \beta_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示}\end{array}\right\} \Rightarrow \alpha_1, \alpha_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示}$

具体地，设

$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases}$$

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases}$$

则

$$\alpha_1 = a_{11}(\qquad\qquad\qquad\qquad) + a_{21}(\qquad\qquad\qquad\qquad)$$

$$\alpha_2 =$$

**例** $\left.\begin{array}{r}\alpha_1, \alpha_2\text{由}\beta_1, \beta_2\text{线性表示}\\ \beta_1, \beta_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示}\end{array}\right\} \Rightarrow \alpha_1, \alpha_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示}$

具体地，设

$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases}$$

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases}$$

则

$$\alpha_1 = a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(\qquad\qquad\qquad\qquad)$$

$$\alpha_2 =$$

**例** $\left.\begin{array}{r}\alpha_1, \alpha_2\text{由}\beta_1, \beta_2\text{线性表示}\\ \beta_1, \beta_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示}\end{array}\right\} \Rightarrow \alpha_1, \alpha_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示}$

具体地，设

$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases}$$

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases}$$

则

$$\alpha_1 = a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)$$

$$\alpha_2 =$$

例 $\left.\begin{array}{r}\alpha_1, \alpha_2\text{由}\beta_1, \beta_2\text{线性表示}\\ \beta_1, \beta_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示}\end{array}\right\} \Rightarrow \alpha_1, \alpha_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示}$

具体地，设

$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases}$$

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases}$$

则

$$\alpha_1 = a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)$$

$$= (\qquad\qquad\qquad)\gamma_1 + (\qquad\qquad\qquad)\gamma_2 + (\qquad\qquad\qquad)\gamma_3$$

$$\alpha_2 =$$

**例** $\left.\begin{array}{r}\alpha_1,\ \alpha_2\text{由}\beta_1,\ \beta_2\text{线性表示}\\ \beta_1,\ \beta_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示}\end{array}\right\} \Rightarrow \alpha_1,\ \alpha_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示}$

具体地，设

$$\begin{cases}\alpha_1 = a_{11}\beta_1 + a_{21}\beta_2\\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2\end{cases}$$

$$\begin{cases}\beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3\\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3\end{cases}$$

则

$$\begin{aligned}\alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)\\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (\qquad\qquad\qquad)\gamma_2 + (\qquad\qquad\qquad)\gamma_3\end{aligned}$$

$$\alpha_2 =$$

**例** $\left.\begin{array}{r}\alpha_1, \alpha_2\text{由}\beta_1, \beta_2\text{线性表示}\\ \beta_1, \beta_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示}\end{array}\right\} \Rightarrow \alpha_1, \alpha_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示}$

具体地，设

$$\begin{cases}\alpha_1 = a_{11}\beta_1 + a_{21}\beta_2\\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2\end{cases}$$

$$\begin{cases}\beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3\\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3\end{cases}$$

则

$$\begin{aligned}\alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)\\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (\qquad\qquad\qquad)\gamma_3\end{aligned}$$

$$\alpha_2 =$$

例 $\left.\begin{array}{r}\alpha_1,\ \alpha_2\text{由}\beta_1,\ \beta_2\text{线性表示}\\ \beta_1,\ \beta_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示}\end{array}\right\}\Rightarrow \alpha_1,\ \alpha_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示}$

具体地，设

$$\begin{cases}\alpha_1 = a_{11}\beta_1 + a_{21}\beta_2\\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2\end{cases}$$

$$\begin{cases}\beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3\\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3\end{cases}$$

则

$$\begin{aligned}\alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)\\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3\end{aligned}$$

$$\alpha_2 =$$

例 $\left.\begin{array}{r}\alpha_1,\alpha_2\text{由}\beta_1,\beta_2\text{线性表示}\\ \beta_1,\beta_2\text{由}\gamma_1,\gamma_2,\gamma_3\text{线性表示}\end{array}\right\}\Rightarrow \alpha_1,\alpha_2\text{由}\gamma_1,\gamma_2,\gamma_3\text{线性表示}$

具体地，设

$$\begin{cases}\alpha_1=a_{11}\beta_1+a_{21}\beta_2\\ \alpha_2=a_{12}\beta_1+a_{22}\beta_2\end{cases}$$

$$\begin{cases}\beta_1=b_{11}\gamma_1+b_{21}\gamma_2+b_{31}\gamma_3\\ \beta_2=b_{12}\gamma_1+b_{22}\gamma_2+b_{32}\gamma_3\end{cases}$$

则

$$\begin{aligned}\alpha_1&=a_{11}(b_{11}\gamma_1+b_{21}\gamma_2+b_{31}\gamma_3)+a_{21}(b_{12}\gamma_1+b_{22}\gamma_2+b_{32}\gamma_3)\\ &=(a_{11}b_{11}+a_{21}b_{12})\gamma_1+(a_{11}b_{21}+a_{21}b_{22})\gamma_2+(a_{11}b_{31}+a_{21}b_{32})\gamma_3\end{aligned}$$

$$\alpha_2=a_{12}(\qquad\qquad\qquad\qquad)+a_{22}(\qquad\qquad\qquad\qquad)$$

例 $\left.\begin{array}{r}\alpha_1, \alpha_2\text{由}\beta_1, \beta_2\text{线性表示} \\ \beta_1, \beta_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示}\end{array}\right\} \Rightarrow \alpha_1, \alpha_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示}$

具体地，设

$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases}$$

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases}$$

则

$$\alpha_1 = a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)$$

$$= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3$$

$$\alpha_2 = a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(\qquad\qquad\qquad\qquad)$$

例 $\left.\begin{array}{r}\alpha_1, \alpha_2\text{由}\beta_1, \beta_2\text{线性表示}\\ \beta_1, \beta_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示}\end{array}\right\} \Rightarrow \alpha_1, \alpha_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示}$

具体地，设

$$\begin{cases}\alpha_1 = a_{11}\beta_1 + a_{21}\beta_2\\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2\end{cases}$$

$$\begin{cases}\beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3\\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3\end{cases}$$

则

$$\begin{aligned}\alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)\\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3\end{aligned}$$

$$\alpha_2 = a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)$$

例 $\left.\begin{array}{r}\alpha_1, \alpha_2\text{由}\beta_1, \beta_2\text{线性表示} \\ \beta_1, \beta_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示}\end{array}\right\} \Rightarrow \alpha_1, \alpha_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示}$

具体地，设

$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases}$$

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases}$$

则

$$\begin{aligned} \alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3 \end{aligned}$$

$$\begin{aligned} \alpha_2 &= a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (\qquad\qquad\qquad)\gamma_1 + (\qquad\qquad\qquad)\gamma_2 + (\qquad\qquad\qquad)\gamma_3 \end{aligned}$$

**例** $\left.\begin{array}{r}\alpha_1, \alpha_2\text{由}\beta_1, \beta_2\text{线性表示} \\ \beta_1, \beta_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示}\end{array}\right\} \Rightarrow \alpha_1, \alpha_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示}$

具体地，设

$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases}$$

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases}$$

则

$$\begin{aligned} \alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3 \end{aligned}$$

$$\begin{aligned} \alpha_2 &= a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (\qquad\qquad\qquad)\gamma_2 + (\qquad\qquad\qquad)\gamma_3 \end{aligned}$$

例 $\left.\begin{array}{r}\alpha_1,\ \alpha_2\text{由}\beta_1,\ \beta_2\text{线性表示}\\ \beta_1,\ \beta_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示}\end{array}\right\}\Rightarrow \alpha_1,\ \alpha_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示}$

具体地，设

$$\begin{cases}\alpha_1 = a_{11}\beta_1 + a_{21}\beta_2\\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2\end{cases}$$

$$\begin{cases}\beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3\\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3\end{cases}$$

则

$$\begin{aligned}\alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)\\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3\end{aligned}$$

$$\begin{aligned}\alpha_2 &= a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)\\ &= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (\qquad\qquad\qquad)\gamma_3\end{aligned}$$

例 $\left.\begin{array}{r}\alpha_1, \alpha_2\text{由}\beta_1, \beta_2\text{线性表示}\\ \beta_1, \beta_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示}\end{array}\right\} \Rightarrow \alpha_1, \alpha_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示}$

具体地，设

$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases}$$

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases}$$

则

$$\begin{aligned} \alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3 \end{aligned}$$

$$\begin{aligned} \alpha_2 &= a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (a_{12}b_{31} + a_{22}b_{32})\gamma_3 \end{aligned}$$

例 $\left.\begin{array}{r}\alpha_1,\ \alpha_2\text{由}\beta_1,\ \beta_2\text{线性表示}\\ \beta_1,\ \beta_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示}\end{array}\right\}\Rightarrow \alpha_1,\ \alpha_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示}$

具体地，设

$$\begin{cases}\alpha_1 = a_{11}\beta_1 + a_{21}\beta_2\\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2\end{cases}$$

$$\begin{cases}\beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3\\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3\end{cases}$$

则

$$\begin{aligned}\alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)\\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3\\ &= c_{11}\gamma_1 +\end{aligned}$$

$$\begin{aligned}\alpha_2 &= a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)\\ &= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (a_{12}b_{31} + a_{22}b_{32})\gamma_3\end{aligned}$$

例 $\left.\begin{array}{r}\alpha_1, \alpha_2\text{由}\beta_1, \beta_2\text{线性表示}\\ \beta_1, \beta_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示}\end{array}\right\} \Rightarrow \alpha_1, \alpha_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示}$

具体地，设

$$\begin{cases}\alpha_1 = a_{11}\beta_1 + a_{21}\beta_2\\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2\end{cases}$$

$$\begin{cases}\beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3\\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3\end{cases}$$

则

$$\begin{aligned}\alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)\\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3\\ &= c_{11}\gamma_1 + c_{21}\gamma_2 +\\ \alpha_2 &= a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)\\ &= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (a_{12}b_{31} + a_{22}b_{32})\gamma_3\end{aligned}$$

例 $\left.\begin{array}{r}\alpha_1, \alpha_2\text{由}\beta_1, \beta_2\text{线性表示} \\ \beta_1, \beta_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示}\end{array}\right\} \Rightarrow \alpha_1, \alpha_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示}$

具体地，设

$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases}$$

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases}$$

则

$$\begin{aligned} \alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3 \\ &= c_{11}\gamma_1 + c_{21}\gamma_2 + c_{31}\gamma_3 \\ \alpha_2 &= a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (a_{12}b_{31} + a_{22}b_{32})\gamma_3 \end{aligned}$$

例 $\left.\begin{array}{r}\alpha_1, \alpha_2\text{由}\beta_1, \beta_2\text{线性表示}\\ \beta_1, \beta_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示}\end{array}\right\} \Rightarrow \alpha_1, \alpha_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示}$

具体地，设

$$\begin{cases}\alpha_1 = a_{11}\beta_1 + a_{21}\beta_2\\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2\end{cases}$$

$$\begin{cases}\beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3\\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3\end{cases}$$

则

$$\begin{aligned}\alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)\\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3\\ &= c_{11}\gamma_1 + c_{21}\gamma_2 + c_{31}\gamma_3\\ \alpha_2 &= a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)\\ &= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (a_{12}b_{31} + a_{22}b_{32})\gamma_3\\ &= c_{12}\gamma_1 +\end{aligned}$$

例 $\left.\begin{array}{r}\alpha_1, \alpha_2\text{由}\beta_1, \beta_2\text{线性表示}\\ \beta_1, \beta_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示}\end{array}\right\} \Rightarrow \alpha_1, \alpha_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示}$

具体地，设

$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases}$$

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases}$$

则

$$\begin{aligned} \alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3 \\ &= c_{11}\gamma_1 + c_{21}\gamma_2 + c_{31}\gamma_3 \\ \alpha_2 &= a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (a_{12}b_{31} + a_{22}b_{32})\gamma_3 \\ &= c_{12}\gamma_1 + c_{22}\gamma_2 + \end{aligned}$$

例 $\left.\begin{array}{r}\alpha_1, \alpha_2\text{由}\beta_1, \beta_2\text{线性表示} \\ \beta_1, \beta_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示}\end{array}\right\} \Rightarrow \alpha_1, \alpha_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示}$

具体地，设

$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases}$$

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases}$$

则

$$\begin{aligned} \alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3 \\ &= c_{11}\gamma_1 + c_{21}\gamma_2 + c_{31}\gamma_3 \\ \alpha_2 &= a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (a_{12}b_{31} + a_{22}b_{32})\gamma_3 \\ &= c_{12}\gamma_1 + c_{22}\gamma_2 + c_{32}\gamma_3 \end{aligned}$$

例 $\left.\begin{array}{r}\alpha_1, \alpha_2\text{由}\beta_1, \beta_2\text{线性表示} \\ \beta_1, \beta_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示}\end{array}\right\} \Rightarrow \alpha_1, \alpha_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示}$

具体地，设

$$\begin{cases}\alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2\end{cases}$$

$$\begin{cases}\beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3\end{cases}$$

则

$$\begin{aligned}\alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3 \\ &= c_{11}\gamma_1 + c_{21}\gamma_2 + c_{31}\gamma_3 \\ \alpha_2 &= a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (a_{12}b_{31} + a_{22}b_{32})\gamma_3 \\ &= c_{12}\gamma_1 + c_{22}\gamma_2 + c_{32}\gamma_3\end{aligned}$$

其中

$$(c_{ij}) = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{12} & a_{12}b_{11} + a_{22}b_{12} \\ a_{11}b_{21} + a_{21}b_{22} & a_{12}b_{21} + a_{22}b_{22} \\ a_{11}b_{31} + a_{21}b_{32} & a_{12}b_{31} + a_{22}b_{32} \end{pmatrix}$$

**例** $\left.\begin{array}{r}\alpha_1,\ \alpha_2\text{由}\beta_1,\ \beta_2\text{线性表示}\\ \beta_1,\ \beta_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示}\end{array}\right\}\Rightarrow \alpha_1,\ \alpha_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示}$

具体地，设

$$\begin{cases}\alpha_1 = a_{11}\beta_1 + a_{21}\beta_2\\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2\end{cases}$$

$$\begin{cases}\beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3\\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3\end{cases}$$

则

$$\begin{aligned}\alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)\\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3\\ &= c_{11}\gamma_1 + c_{21}\gamma_2 + c_{31}\gamma_3\\ \alpha_2 &= a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)\\ &= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (a_{12}b_{31} + a_{22}b_{32})\gamma_3\\ &= c_{12}\gamma_1 + c_{22}\gamma_2 + c_{32}\gamma_3\end{aligned}$$

其中

$$(c_{ij}) = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{12} & a_{12}b_{11} + a_{22}b_{12}\\ a_{11}b_{21} + a_{21}b_{22} & a_{12}b_{21} + a_{22}b_{22}\\ a_{11}b_{31} + a_{21}b_{32} & a_{12}b_{31} + a_{22}b_{32}\end{pmatrix} = \begin{pmatrix} b_{11} & b_{12}\\ b_{21} & b_{22}\\ b_{31} & b_{32}\end{pmatrix}\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$$

例 $\left.\begin{array}{r}\alpha_1,\ \alpha_2\text{由}\beta_1,\ \beta_2\text{线性表示}\\ \beta_1,\ \beta_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示}\end{array}\right\}\Rightarrow \alpha_1,\ \alpha_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示}$

具体地，设

$$\begin{cases}\alpha_1 = a_{11}\beta_1 + a_{21}\beta_2\\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2\end{cases} \Rightarrow (\alpha_1,\ \alpha_2) = (\beta_1,\ \beta_2)\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$$

$$\begin{cases}\beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3\\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3\end{cases}$$

则

$$\begin{aligned}\alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)\\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3\\ &= c_{11}\gamma_1 + c_{21}\gamma_2 + c_{31}\gamma_3\\ \alpha_2 &= a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)\\ &= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (a_{12}b_{31} + a_{22}b_{32})\gamma_3\\ &= c_{12}\gamma_1 + c_{22}\gamma_2 + c_{32}\gamma_3\end{aligned}$$

其中

$$(c_{ij}) = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{12} & a_{12}b_{11} + a_{22}b_{12}\\ a_{11}b_{21} + a_{21}b_{22} & a_{12}b_{21} + a_{22}b_{22}\\ a_{11}b_{31} + a_{21}b_{32} & a_{12}b_{31} + a_{22}b_{32}\end{pmatrix} = \begin{pmatrix} b_{11} & b_{12}\\ b_{21} & b_{22}\\ b_{31} & b_{32}\end{pmatrix}\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$$

例 $\left.\begin{array}{r}\alpha_1, \alpha_2\text{由}\beta_1, \beta_2\text{线性表示}\\ \beta_1, \beta_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示}\end{array}\right\} \Rightarrow \alpha_1, \alpha_2\text{由}\gamma_1, \gamma_2, \gamma_3\text{线性表示}$

具体地，设

$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases} \Rightarrow (\alpha_1, \alpha_2) = (\beta_1, \beta_2)\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases} \Rightarrow (\beta_1, \beta_2) = (\gamma_1, \gamma_2, \gamma_3)\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}$$

则

$$\begin{aligned} \alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3 \\ &= c_{11}\gamma_1 + c_{21}\gamma_2 + c_{31}\gamma_3 \\ \alpha_2 &= a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (a_{12}b_{31} + a_{22}b_{32})\gamma_3 \\ &= c_{12}\gamma_1 + c_{22}\gamma_2 + c_{32}\gamma_3 \end{aligned}$$

其中

$$(c_{ij}) = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{12} & a_{12}b_{11} + a_{22}b_{12} \\ a_{11}b_{21} + a_{21}b_{22} & a_{12}b_{21} + a_{22}b_{22} \\ a_{11}b_{31} + a_{21}b_{32} & a_{12}b_{31} + a_{22}b_{32} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

**例** $\left.\begin{array}{r}\alpha_1,\ \alpha_2\text{由}\beta_1,\ \beta_2\text{线性表示}\\ \beta_1,\ \beta_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示}\end{array}\right\}\Rightarrow \alpha_1,\ \alpha_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示}$

具体地，设

$$\begin{cases}\alpha_1=a_{11}\beta_1+a_{21}\beta_2\\ \alpha_2=a_{12}\beta_1+a_{22}\beta_2\end{cases}\Rightarrow(\alpha_1,\ \alpha_2)=(\beta_1,\ \beta_2)\underbrace{\begin{pmatrix}a_{11}&a_{12}\\ a_{21}&a_{22}\end{pmatrix}}_{A}$$

$$\begin{cases}\beta_1=b_{11}\gamma_1+b_{21}\gamma_2+b_{31}\gamma_3\\ \beta_2=b_{12}\gamma_1+b_{22}\gamma_2+b_{32}\gamma_3\end{cases}\Rightarrow(\beta_1,\ \beta_2)=(\gamma_1,\ \gamma_2,\ \gamma_3)\begin{pmatrix}b_{11}&b_{12}\\ b_{21}&b_{22}\\ b_{31}&b_{32}\end{pmatrix}$$

则

$$\begin{aligned}\alpha_1&=a_{11}(b_{11}\gamma_1+b_{21}\gamma_2+b_{31}\gamma_3)+a_{21}(b_{12}\gamma_1+b_{22}\gamma_2+b_{32}\gamma_3)\\ &=(a_{11}b_{11}+a_{21}b_{12})\gamma_1+(a_{11}b_{21}+a_{21}b_{22})\gamma_2+(a_{11}b_{31}+a_{21}b_{32})\gamma_3\\ &=c_{11}\gamma_1+c_{21}\gamma_2+c_{31}\gamma_3\\ \alpha_2&=a_{12}(b_{11}\gamma_1+b_{21}\gamma_2+b_{31}\gamma_3)+a_{22}(b_{12}\gamma_1+b_{22}\gamma_2+b_{32}\gamma_3)\\ &=(a_{12}b_{11}+a_{22}b_{12})\gamma_1+(a_{12}b_{21}+a_{22}b_{22})\gamma_2+(a_{12}b_{31}+a_{22}b_{32})\gamma_3\\ &=c_{12}\gamma_1+c_{22}\gamma_2+c_{32}\gamma_3\end{aligned}$$

其中

$$(c_{ij})=\begin{pmatrix}a_{11}b_{11}+a_{12}b_{12}&a_{12}b_{11}+a_{22}b_{12}\\ a_{11}b_{21}+a_{21}b_{22}&a_{12}b_{21}+a_{22}b_{22}\\ a_{11}b_{31}+a_{21}b_{32}&a_{12}b_{31}+a_{22}b_{32}\end{pmatrix}=\begin{pmatrix}b_{11}&b_{12}\\ b_{21}&b_{22}\\ b_{31}&b_{32}\end{pmatrix}\begin{pmatrix}a_{11}&a_{12}\\ a_{21}&a_{22}\end{pmatrix}$$

**例** $\left.\begin{array}{r}\alpha_1,\ \alpha_2\text{由}\beta_1,\ \beta_2\text{线性表示}\\ \beta_1,\ \beta_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示}\end{array}\right\}\Rightarrow \alpha_1,\ \alpha_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示}$

具体地，设

$$\begin{cases}\alpha_1 = a_{11}\beta_1 + a_{21}\beta_2\\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2\end{cases} \Rightarrow (\alpha_1,\ \alpha_2) = (\beta_1,\ \beta_2)\underbrace{\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}}_{A}$$

$$\begin{cases}\beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3\\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3\end{cases} \Rightarrow (\beta_1,\ \beta_2) = (\gamma_1,\ \gamma_2,\ \gamma_3)\underbrace{\begin{pmatrix} b_{11} & b_{12}\\ b_{21} & b_{22}\\ b_{31} & b_{32}\end{pmatrix}}_{B}$$

则

$$\begin{aligned}\alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)\\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3\\ &= c_{11}\gamma_1 + c_{21}\gamma_2 + c_{31}\gamma_3\end{aligned}$$

$$\begin{aligned}\alpha_2 &= a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)\\ &= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (a_{12}b_{31} + a_{22}b_{32})\gamma_3\\ &= c_{12}\gamma_1 + c_{22}\gamma_2 + c_{32}\gamma_3\end{aligned}$$

其中

$$(c_{ij}) = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{12} & a_{12}b_{11} + a_{22}b_{12}\\ a_{11}b_{21} + a_{21}b_{22} & a_{12}b_{21} + a_{22}b_{22}\\ a_{11}b_{31} + a_{21}b_{32} & a_{12}b_{31} + a_{22}b_{32}\end{pmatrix} = \begin{pmatrix} b_{11} & b_{12}\\ b_{21} & b_{22}\\ b_{31} & b_{32}\end{pmatrix}\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$$

例 $\left.\begin{array}{r}\alpha_1,\ \alpha_2\text{由}\beta_1,\ \beta_2\text{线性表示}\\ \beta_1,\ \beta_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示}\end{array}\right\}\Rightarrow \alpha_1,\ \alpha_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示}$

具体地，设

$$\begin{cases}\alpha_1 = a_{11}\beta_1 + a_{21}\beta_2\\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2\end{cases} \Rightarrow (\alpha_1,\ \alpha_2) = (\beta_1,\ \beta_2)\underbrace{\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}}_{A}$$

$$\begin{cases}\beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3\\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3\end{cases} \Rightarrow (\beta_1,\ \beta_2) = (\gamma_1,\ \gamma_2,\ \gamma_3)\underbrace{\begin{pmatrix} b_{11} & b_{12}\\ b_{21} & b_{22}\\ b_{31} & b_{32}\end{pmatrix}}_{B}$$

则

$$\begin{aligned}\alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)\\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3\\ &= c_{11}\gamma_1 + c_{21}\gamma_2 + c_{31}\gamma_3\\ \alpha_2 &= a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)\\ &= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (a_{12}b_{31} + a_{22}b_{32})\gamma_3\\ &= c_{12}\gamma_1 + c_{22}\gamma_2 + c_{32}\gamma_3\end{aligned}$$

其中

$$(c_{ij}) = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{12} & a_{12}b_{11} + a_{22}b_{12}\\ a_{11}b_{21} + a_{21}b_{22} & a_{12}b_{21} + a_{22}b_{22}\\ a_{11}b_{31} + a_{21}b_{32} & a_{12}b_{31} + a_{22}b_{32}\end{pmatrix} = \begin{pmatrix} b_{11} & b_{12}\\ b_{21} & b_{22}\\ b_{31} & b_{32}\end{pmatrix}\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix} = BA$$

定义 设有两个向量组

$$(A):\quad \alpha_1, \alpha_2, \ldots, \alpha_s$$

$$(B):\quad \beta_1, \beta_2, \ldots, \beta_t$$

如果 $(A)$ 与 $(B)$ 可相互线性表示，则称向量组 $(A)$ 与 $(B)$ 等价。