第 9 章 c：多元复合函数的求导法则

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二元复合函数求导

设有二元函数 $z = f(u, v)$

- 设 $u = \varphi(t)$，$v = \psi(t)$，则 $z = f(\varphi(t), \psi(t))$

$$\begin{tikzcd}[row sep=small] & u \arrow[dr] & \\ z \arrow[ur] \arrow[dr] & & t \\ & v \arrow[ur] & \end{tikzcd}$$

  问 $\frac{dz}{dt} = ?$

- 设 $u = \varphi(x, y)$，$v = \psi(x, y)$，则 $z = f(\varphi(x, y), \psi(x, y))$

$$\begin{tikzcd}[row sep=small] & u \arrow[r] \arrow[dr] & x \\ z \arrow[ur] \arrow[dr] & & \\ & v \arrow[r] \arrow[ur] & y \end{tikzcd}$$

  问 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} = ?$

二元复合函数求导公式——中间变量是一元函数

公式 设 $z=f(u,v)$，$u=\varphi(t)$，$v=\psi(t)$，则 $z=f(\varphi(t),\psi(t))$ 的**全导数**

$$\frac{dz}{dt}=\frac{\partial z}{\partial u}\cdot\frac{du}{dt}+\frac{\partial z}{\partial v}\cdot\frac{dv}{dt}$$

$$\begin{tikzcd}
 & u \arrow[r, "\frac{du}{dt}"] & t \\
z \arrow[ur, "\frac{\partial z}{\partial u}"] \arrow[dr, "\frac{\partial z}{\partial v}"'] & & \\
 & v \arrow[r, "\frac{dv}{dt}"'] & t
\end{tikzcd}$$

例 设 $z=uv$，而 $u=e^{-t}$，$v=\sin t$，求全导数 $\frac{dz}{dt}$

解法一

$$\begin{aligned}\frac{dz}{dt}&=\frac{\partial z}{\partial u}\cdot\frac{du}{dt}+\frac{\partial z}{\partial v}\cdot\frac{dv}{dt}\\&=(uv)'_u\cdot(e^{-t})'_t+(uv)'_v\cdot(\sin t)'_t\\&=v\cdot(-e^{-t})+u\cdot\cos t\\&=\sin t\cdot(-e^{-t})+e^{-t}\cdot\cos t\\&=e^{-t}(\cos t-\sin t)\end{aligned}$$

解法二

$$\because\quad z=uv=e^{-t}\cdot\sin t$$

$$\begin{aligned}\therefore\quad\frac{dz}{dt}&=\frac{d}{dt}(e^{-t}\sin t)=(e^{-t})'_t\cdot\sin t+e^{-t}\cdot(\sin t)'_t\\&=(-e^{-t})\cdot\sin t+e^{-t}\cdot\cos t=e^{-t}(\cos t-\sin t)\end{aligned}$$

例 设 $z=\frac{y}{x}$，而 $x=e^t$，$y=1-e^{2t}$，求全导数 $\frac{dz}{dt}$

解

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}=(\frac{y}{x})'_x\cdot(e^t)'_t+(\frac{y}{x})'_y\cdot(1-e^{2t})'_t$$

$$=-\frac{y}{x^2}\cdot e^t+\frac{1}{x}\cdot(-2e^{2t})=-\frac{1-e^{2t}}{e^{2t}}\cdot e^t+\frac{1}{e^t}\cdot(-2e^{2t})$$

$$=-e^{-t}-e^t$$

$$\begin{tikzcd}
 & x \arrow[r, "\frac{dx}{dt}"] & t \\
z \arrow[ur, "\frac{\partial z}{\partial x}"] \arrow[dr, "\frac{\partial z}{\partial y}"'] & & \\
 & y \arrow[r, "\frac{dy}{dt}"'] & t
\end{tikzcd}$$

三元复合函数求导公式——中间变量是一元函数

公式 设 $z=f(u,v,w)$，$u=\varphi(t)$，$v=\psi(t)$，$w=\omega(t)$，则 $z=f(\varphi(t),\psi(t),\omega(t))$ 的全导数

$$\frac{dz}{dt}=\frac{\partial z}{\partial u}\cdot\frac{du}{dt}+\frac{\partial z}{\partial v}\cdot\frac{dv}{dt}+\frac{\partial z}{\partial w}\cdot\frac{dw}{dt}$$

$$\begin{tikzcd}
 & u \arrow[r, "\frac{du}{dt}"] & t \\
z \arrow[ur, "\frac{\partial z}{\partial u}"] \arrow[r, "\frac{\partial z}{\partial v}"] \arrow[dr, "\frac{\partial z}{\partial w}"] & v \arrow[r, "\frac{dv}{dt}"] & t \\
 & w \arrow[r, "\frac{dw}{dt}"] & t
\end{tikzcd}$$

二元复合函数求导公式——中间变量是多元函数

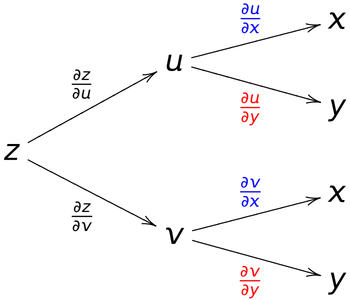
公式 设 $z=f(u,v)$，$u=\varphi(x,y)$，$v=\psi(x,y)$，则复合函数

$$z=f(\varphi(x,y),\psi(x,y))$$

的偏导数是：

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x},\qquad \frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

图示



例 设 $z=e^{2u}\sin v$，$u=x^3y$，$v=x^2+y^2$，求 $\frac{\partial z}{\partial x}$ 和 $\frac{\partial z}{\partial y}$

解
$$\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} \\
&= (e^{2u}\sin v)'_u\cdot(x^3y)'_x+(e^{2u}\sin v)'_v\cdot(x^2+y^2)'_x \\
&= 2e^{2u}\sin v\cdot 3x^2y+e^{2u}\cos v\cdot 2x \\
&= 2e^{2x^3y}\sin(x^2+y^2)\cdot 3x^2y+e^{2x^3y}\cos(x^2+y^2)\cdot 2x
\end{aligned}$$

$$\begin{aligned}
\frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} \\
&= (e^{2u}\sin v)'_u\cdot(x^3y)'_y+(e^{2u}\sin v)'_v\cdot(x^2+y^2)'_y \\
&= 2e^{2u}\sin v\cdot x^3+e^{2u}\cos v\cdot 2y \\
&= 2e^{2x^3y}\sin(x^2+y^2)\cdot x^3+e^{2x^3y}\cos(x^2+y^2)\cdot 2y
\end{aligned}$$

三元复合函数求导公式：举例

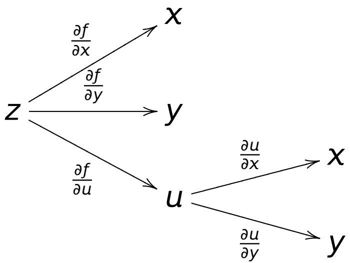
公式 设 $z=f(x,y,u)$，$u=u(x,y)$，则复合函数

$$z=f(x,y,u(x,y))$$

的偏导数是：

$$\frac{\partial z}{\partial x}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x},\qquad \frac{\partial z}{\partial y}=\frac{\partial f}{\partial y}+\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}$$

图示



例 设 $z=f(x^2-y^2, e^{xy})$，求 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

解 设 $z=f(u, v)$，$u=x^2-y^2$，$v=e^{xy}$，则

$$\frac{\partial z}{\partial x}=f_u\cdot u_x+f_v\cdot v_x=f_u\cdot(x^2-y^2)_x+f_v\cdot(e^{xy})_x=2xf_u+ye^{xy}f_v$$

$$\frac{\partial z}{\partial y}=f_u\cdot u_x+f_v\cdot v_x=f_u\cdot(x^2-y^2)_y+f_v\cdot(e^{xy})_y=-2yf_u+xe^{xy}f_v$$

例 设 $g=f(\frac{x}{y}, \frac{y}{z})$，求 $\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}$

解 设 $g=f(u, v)$，$u=\frac{x}{y}$，$v=\frac{y}{z}$，则

$$\frac{\partial g}{\partial x}=f_u\cdot u_x+f_v\cdot v_x=f_u\cdot(\frac{x}{y})_x+f_v\cdot(\frac{y}{z})_x=\frac{1}{y}f_u$$

$$\frac{\partial g}{\partial y}=f_u\cdot u_y+f_v\cdot v_y=f_u\cdot(\frac{x}{y})_y+f_v\cdot(\frac{y}{z})_y=-\frac{x}{y^2}f_u+\frac{1}{z}f_v$$

$$\frac{\partial g}{\partial z}=f_u\cdot u_z+f_v\cdot v_z=f_u\cdot(\frac{x}{y})_z+f_v\cdot(\frac{y}{z})_z=-\frac{y}{z^2}f_v$$

例 设 $g=f(x, xy, xyz)$，求 $\frac{\partial g}{\partial x}$, $\frac{\partial g}{\partial y}$, $\frac{\partial g}{\partial z}$

解 设 $g=f(u, v, w)$，$u=x$，$v=xy$，$w=xyz$，则

$$\frac{\partial g}{\partial x}=f_u\cdot u_x+f_v\cdot v_x+f_w\cdot w_x=f_u+yf_v+yzf_w$$

$$\frac{\partial g}{\partial y}=f_u\cdot u_y+f_v\cdot v_y+f_w\cdot w_y=xf_v+xzf_w$$

$$\frac{\partial g}{\partial z}=f_u\cdot u_z+f_v\cdot v_z+f_w\cdot w_z=xyf_w$$

复合函数的高阶导数

公式 设 $z=f(u,v)$，$u=u(x,y)$，$v=v(x,y)$，则复合函数

$$z=f(u(x,y),v(x,y))$$

的偏导数是：

$$z_x = z_u \cdot u_x + z_v \cdot v_x,$$

$$z_y = z_u \cdot u_y + z_v \cdot v_y,$$

$$\begin{aligned} z_{xx} &= (z_x)'_x = (z_u \cdot u_x + z_v \cdot v_x)'_x \\ &= (z_u)'_x \cdot u_x + z_u \cdot u_{xx} + (z_v)'_x \cdot v_x + z_v \cdot v_{xx} \\ &= (z_{uu} \cdot u_x + z_{uv} \cdot v_x) \cdot u_x + z_u \cdot u_{xx} + (z_{vu} \cdot u_x + z_{vv} \cdot v_x) \cdot v_x + z_v \cdot v_{xx} \\ &= z_{uu} u_x^2 + 2 z_{uv} u_x v_x + z_{vv} v_x^2 + z_u u_{xx} + z_v v_{xx} \end{aligned}$$

$$z_{xy} = ?$$

$$z_{yy} = ?$$

复合函数的高阶导数

公式 设 $z=f(u,v)$，$u=u(x,y)$，$v=v(x,y)$，则复合函数

$$z=f(u(x,y),v(x,y))$$

的偏导数是：

$$z_x = z_u \cdot u_x + z_v \cdot v_x,$$

$$z_y = z_u \cdot u_y + z_v \cdot v_y,$$

$$z_{xx} = z_{uu}u_x^2 + 2z_{uv}u_x v_x + z_{vv}v_x^2 + z_u u_{xx} + z_v v_{xx}$$

$$\begin{aligned} z_{xy} &= (z_x)'_y = (z_u \cdot u_x + z_v \cdot v_x)'_y \\ &= (z_u)'_y \cdot u_x + z_u \cdot u_{xy} + (z_v)'_y \cdot v_x + z_v \cdot v_{xy} \\ &= (z_{uu}\cdot u_y + z_{uv}\cdot v_y)\cdot u_x + z_u\cdot u_{xy} + (z_{vu}\cdot u_y + z_{vv}\cdot v_y)\cdot v_x + z_v \cdot v_{xy} \\ &= z_{uu}u_x u_y + z_{uv}(u_x v_y + u_y v_x) + z_{vv}v_x v_y + z_u u_{xy} + z_v v_{xy} \end{aligned}$$

$$z_{yy} = ?$$

复合函数的高阶导数

公式 设 $z=f(u,v)$，$u=u(x,y)$，$v=v(x,y)$，则复合函数

$$z=f(u(x,y),v(x,y))$$

的偏导数是：

$$z_x = z_u \cdot u_x + z_v \cdot v_x,$$

$$z_y = z_u \cdot u_y + z_v \cdot v_y,$$

$$z_{xx} = z_{uu}u_x^2 + 2z_{uv}u_x v_x + z_{vv}v_x^2 + z_u u_{xx} + z_v v_{xx}$$

$$z_{xy} = z_{uu}u_x u_y + z_{uv}(u_x v_y + u_y v_x) + z_{vv}v_x v_y + z_u u_{xy} + z_v v_{xy}$$

$$\begin{aligned} z_{yy} &= (z_y)'_y = (z_u \cdot u_y + z_v \cdot v_y)'_y \\ &= (z_u)'_y \cdot u_y + z_u \cdot u_{yy} + (z_v)'_y \cdot v_y + z_v \cdot v_{yy} \\ &= (z_{uu}\cdot u_y + z_{uv}\cdot v_y)\cdot u_y + z_u \cdot u_{yy} + (z_{vu}\cdot u_y + z_{vv}\cdot v_y)\cdot v_y + z_v \cdot v_{yy} \\ &= z_{uu}u_y^2 + 2z_{uv}u_y v_y + z_{vv}v_y^2 + z_u u_{yy} + z_v v_{yy} \end{aligned}$$

例 设 $z=f(xy^2, x^2y)$，求 $\frac{\partial^2 z}{\partial x \partial y}$

解 设 $z=f(u, v)$，$u=xy^2$，$v=x^2y$，则

$$\frac{\partial z}{\partial x}=f_u \cdot u_x + f_v \cdot v_x = f_u \cdot (xy^2)'_x + f_v \cdot (x^2y)'_x = y^2 f_u + 2xyf_v$$

$$\frac{\partial^2 z}{\partial x \partial y}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial y}\left(y^2 f_u + 2xyf_v\right)$$

$$= (y^2)'_y \cdot f_u + y^2 \cdot (f_u)'_y + (2xy)'_y \cdot f_v + 2xy \cdot (f_v)'_y$$

$$= 2yf_u + y^2 \cdot (f_{uu} \cdot u_y + f_{uv} \cdot v_y) + 2xf_v + 2xy \cdot (f_{vu} \cdot u_y + f_{vv} \cdot v_y)$$

$$= 2yf_u + y^2 \cdot (2xyf_{uu} + x^2 f_{uv}) + 2xf_v + 2xy \cdot (2xyf_{vu} + x^2 f_{vv})$$

$$= 2yf_u + 2xf_v + 2xy^3 f_{uu} + x^2y^2 f_{uv} + 4x^2y^2 f_{vu} + 2x^3y f_{vv}$$

$$= 2yf_u + 2xf_v + 2xy^3 f_{uu} + 5x^2y^2 f_{uv} + 2x^3y f_{vv}$$

例 设 $z=f(\sin x, \cos y, e^{x+y})$，求 $\frac{\partial^2 z}{\partial x \partial y}$

例 设 $z=f(\sin x, \cos y, e^{x+y})$，求 $\frac{\partial^2 z}{\partial x\partial y}$

解 设 $z=f(u, v, w)$，$u=\sin x$，$v=\cos y$，$w=e^{x+y}$，则

$$\frac{\partial z}{\partial x}=f_u\cdot u_x+f_v\cdot v_x+f_w\cdot w_x=f_u\cdot(\sin x)'_x+f_v\cdot 0+f_w\cdot(e^{x+y})'_x$$

$$=\cos x\cdot f_u+e^{x+y}f_w$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial y}\left(\cos x\cdot f_u+e^{x+y}f_w\right)$$

$$=\cos x\cdot(f_u)'_y+(e^{x+y})'_y\cdot f_w+e^{x+y}\cdot(f_w)'_y$$

$$=\cos x\cdot(f_{uu}\cdot u_y+f_{uv}\cdot v_y+f_{uw}\cdot w_y)$$

$$+e^{x+y}f_w+e^{x+y}\cdot(f_{wu}\cdot u_y+f_{wv}\cdot v_y+f_{ww}\cdot w_y)$$

$$=\cos x\cdot(-\sin y\cdot f_{uv}+e^{x+y}f_{uw})$$

$$+e^{x+y}f_w+e^{x+y}\cdot(-\sin y\cdot f_{wv}+e^{x+y}f_{ww})$$

$$=e^{x+y}f_w-\cos x\sin y\cdot f_{uv}+\cos x e^{x+y}f_{uw}-\sin y e^{x+y}f_{wv}+e^{2x+2y}f_{ww}$$