

第 10 周作业解答

练习 1. 设函数 $z = u + v$, 而 $u = x + y$, $v = xy$, 求 $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ 。

解法一:

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} \\ &= (u + v)'_u \cdot (x + y)'_x + (u + v)'_v \cdot (xy)'_x \\ &= 1 + y\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} \\ &= (u + v)'_u \cdot (x + y)'_y + (u + v)'_v \cdot (xy)'_y \\ &= 1 + x\end{aligned}$$

解法二: 因为

$$z = u + v = x + y + xy$$

所以

$$\frac{\partial z}{\partial x} = (x + y + xy)'_x = 1 + y$$

$$\frac{\partial z}{\partial y} = (x + y + xy)'_y = 1 + x$$

练习 2. 设函数 $z = u^2 \ln v$, 而 $u = \frac{x}{y}$, $v = 2x - 3y$, 求 $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ 。

解法一：

$$\begin{aligned}
 \frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} \\
 &= (u^2 \ln v)'_u \cdot \left(\frac{x}{y}\right)'_x + (u^2 \ln v)'_v \cdot (2x - 3y)'_x \\
 &= 2u \ln v \cdot \frac{1}{y} + u^2 \cdot \frac{1}{v} \cdot 2 \\
 &= 2 \cdot \frac{x}{y} \cdot \ln(2x - 3y) \cdot \frac{1}{y} + \frac{x^2}{y^2} \cdot \frac{1}{2x - 3y} \cdot 2 \\
 &= \frac{2x}{y^2} \ln(2x - 3y) + \frac{2x^2}{y^2(2x - 3y)}
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} \\
 &= (u^2 \ln v)'_u \cdot \left(\frac{x}{y}\right)'_y + (u^2 \ln v)'_v \cdot (2x - 3y)'_y \\
 &= 2u \ln v \cdot \left(-\frac{x}{y^2}\right) + u^2 \cdot \frac{1}{v} \cdot (-3) \\
 &= 2 \cdot \frac{x}{y} \cdot \ln(2x - 3y) \cdot \left(-\frac{x}{y^2}\right) + \frac{x^2}{y^2} \cdot \frac{1}{2x - 3y} \cdot (-3) \\
 &= -\frac{2x^2}{y^3} \ln(2x - 3y) - \frac{3x^2}{y^2(2x - 3y)}
 \end{aligned}$$

解法二：因为

$$z = u^2 \ln v = \frac{x^2 \ln(2x - 3y)}{y^2}$$

所以

$$\begin{aligned}
 \frac{\partial z}{\partial x} &= \left(\frac{x^2 \ln(2x - 3y)}{y^2}\right)'_x = \frac{2x}{y^2} \ln(2x - 3y) + \frac{2x^2}{y^2(2x - 3y)} \\
 \frac{\partial z}{\partial y} &= \left(\frac{x^2 \ln(2x - 3y)}{y^2}\right)'_y = -\frac{2x^2}{y^3} \ln(2x - 3y) - \frac{3x^2}{y^2(2x - 3y)}
 \end{aligned}$$

练习 3. 设 $y = y(x)$ 满足 $x \sin y + xy + 2 = 0$, 求 $\frac{dy}{dx}$ 。

解法一：令 $F(x, y) = x \sin y + xy + 2$, 则 $y = y(x)$ 满足 $F(x, y) = 0$, 所以

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{(x \sin y + xy + 2)'_x}{(x \sin y + xy + 2)'_y} = -\frac{\sin y + y}{x \cos y + x}$$

解法二：对

$$x \sin y(x) + xy(x) + 2 = 0$$

两边求导得

$$\begin{aligned}
 0 &= \frac{d}{dx} (x \sin y(x) + xy(x) + 2) \\
 &= \sin y(x) + x \cdot (\sin y(x))' + y(x) + xy'(x) \\
 &= (\sin y + y) + (x \cos y + x)y'
 \end{aligned}$$

所以

$$y' = -\frac{\sin y + y}{x \cos y + x}.$$

练习 4. 设 $z = z(x, y)$ 由方程 $e^{-xy} - 2z + e^{-z} = 0$ 确定, 求 $\frac{\partial z}{\partial x}$ 和 $\frac{\partial z}{\partial y}$ 。

解令 $F(x, y, z) = e^{-xy} - 2z + e^{-z}$, 则

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{-ye^{-xy}}{-2 - e^{-z}} = -\frac{ye^{-xy}}{2 + e^{-z}}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{-xe^{-xy}}{-2 - e^{-z}} = -\frac{xe^{-xy}}{2 + e^{-z}}.$$

练习 5. 设 $z = z(x, y)$ 由方程 $x^2 + y^2 + z^2 - 3xyz = 0$ 确定, 求 dz 。

解令 $F(x, y, z) = x^2 + y^2 + z^2 - 3xyz$, 则

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2x - 3yz}{2z - 3xy}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{2y - 3xz}{2z - 3xy}.$$

所以

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = -\frac{2x - 3yz}{2z - 3xy} dx - \frac{2y - 3xz}{2z - 3xy} dy.$$

练习 6. 求下列各函数的极值

1. $f(x, y) = 4(x - y) - x^2 - y^2$

2. $f(x, y) = x^3 + 3xy^2 - 15x - 12y$

解 1. (I) 先求驻点。求解方程组

$$\begin{cases} f_x = 4 - 2x = 0 \\ f_y = -4 - 2y = 0 \end{cases}$$

得 $(x, y) = (2, -2)$ 。

(II) 判断驻点是否极值点。计算二阶偏导数

$$f_{xx} = -2, \quad f_{xy} = 0, \quad f_{yy} = -2$$

可求出判别式 $P(x, y) = f_{xx}f_{yy} - f_{xy}^2 = 4$ 。

	$(2, -2)$
$P(x, y)$	$4 > 0$
f_{xx}	$-2 < 0$
是否极值点	极大值点
极值 $f(x, y)$	8

2. (I) 先求驻点。求解方程组

$$\begin{cases} f_x = 3x^2 + 3y^2 - 15 = 0 \\ f_y = 6xy - 12 = 0 \end{cases}$$

得 $(x, y) = (-2, -1), (-1, -2), (1, 2), (2, 1)$ 。

(II) 判断驻点是否极值点。计算二阶偏导数

$$f_{xx} = 6x, \quad f_{xy} = 6y, \quad f_{yy} = 6x$$

可求出判别式 $P(x, y) = f_{xx}f_{yy} - f_{xy}^2 = 36x^2 - 36y^2$ 。

	$(-2, -1)$	$(-1, -2)$	$(1, 2)$	$(2, 1)$
$P(x, y)$	$108 > 0$	$-108 < 0$	$-108 < 0$	$108 > 0$
f_{xx}	$-12 < 0$			$12 > 0$
是否极值点	极大值点	\ominus	\ominus	极小值点
极值 $f(x, y)$	28			-28

注： $(-2, -1)$ 不是最大值点。显然 $f(10, 0), f(100, 0), f(1000, 0) \dots$ 的值都比 $f(-2, -1)$ 大。想一想，这当中是否矛盾？同样， $(2, 1)$ 不是最小值点。