第 11 章 f: 高斯公式、斯托克斯公式

数学系 梁卓滨

2016-2017 **学年** II



Outline

1. 高斯公式

2. 斯托克斯公式

We are here now...

1. 高斯公式

2. 斯托克斯公式

定义 设
$$F = (P, Q, R)$$
 是空间中向量场,定义

$$\mathrm{div} F := \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

称为向量场 F 的散度。

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$$\operatorname{div} F = \frac{\partial}{\partial x}(x^2 + yz) + \frac{\partial}{\partial y}(y^2 + xz) + \frac{\partial}{\partial z}(z^2 + xy) = 2x + 2y + 2z.$$



$$\nabla \frac{1}{r}$$

$$\operatorname{div} \nabla \frac{1}{r}$$

$$\nabla \frac{1}{r} = (\frac{\partial}{\partial x} r^{-1}, \frac{\partial}{\partial y} r^{-1}, \frac{\partial}{\partial y} r^{-1})$$

$$\operatorname{div}\nabla\frac{1}{r}$$

$$\nabla \frac{1}{r} = \left(\frac{\partial}{\partial x}r^{-1}, \frac{\partial}{\partial y}r^{-1}, \frac{\partial}{\partial y}r^{-1}\right)$$
$$-r^{-2} \cdot r_{x}$$
$$\operatorname{div} \nabla \frac{1}{r}$$

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$$\operatorname{div} \nabla \frac{1}{r}$$

$$r_{x} = \frac{x}{r},$$

$$\nabla \frac{1}{r} = (\frac{\partial}{\partial x} r^{-1}, \frac{\partial}{\partial y} r^{-1}, \frac{\partial}{\partial y} r^{-1})$$

$$= (-r^{-2} \cdot r_{x}, -r^{-2} \cdot r_{y}, -r^{-2} \cdot r_{y})$$

$$\operatorname{div} \nabla \frac{1}{r}$$

$$r_{x} = \frac{x}{r}, \qquad r_{y} = \frac{y}{r}, \qquad r_{z} = \frac{z}{r},$$

$$\nabla \frac{1}{r} = (\frac{\partial}{\partial x}r^{-1}, \frac{\partial}{\partial y}r^{-1}, \frac{\partial}{\partial y}r^{-1})$$

$$= (-r^{-2} \cdot r_{x}, -r^{-2} \cdot r_{y}, -r^{-2} \cdot r_{y})$$

$$\operatorname{div} \nabla \frac{1}{r}$$

$$r_{X} = \frac{x}{r}, \qquad r_{y} = \frac{y}{r}, \qquad r_{z} = \frac{z}{r},$$

$$\nabla \frac{1}{r} = \left(\frac{\partial}{\partial x}r^{-1}, \frac{\partial}{\partial y}r^{-1}, \frac{\partial}{\partial y}r^{-1}\right)$$

$$= \left(-r^{-2} \cdot r_{x}, -r^{-2} \cdot r_{y}, -r^{-2} \cdot r_{y}\right) = \left(-\frac{x}{r^{3}}, -\frac{y}{r^{3}}, -\frac{z}{r^{3}}\right),$$

$$r \nabla \frac{1}{r^{3}}$$

$$\operatorname{div}\nabla\frac{1}{r}$$

$$r_{x} = \frac{x}{r}, \qquad r_{y} = \frac{y}{r}, \qquad r_{z} = \frac{z}{r},$$

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$$\operatorname{div} \nabla \frac{1}{r} = \frac{\partial}{\partial x} (-\frac{x}{r^{3}}) + \frac{\partial}{\partial y} (-\frac{y}{r^{3}}) + \frac{\partial}{\partial z} (-\frac{z}{r^{3}})$$

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$$(-\frac{1}{r^{3}} + \frac{3x^{2}}{r^{5}})$$

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$$= (-\frac{1}{r^{3}} + \frac{3x^{2}}{r^{5}}) + (-\frac{1}{r^{3}} + \frac{3x^{2}}{r^{5}}) + (-\frac{1}{r^{3}} + \frac{3x^{2}}{r^{5}})$$



$$r_{x} = \frac{x}{r}, \qquad r_{y} = \frac{y}{r}, \qquad r_{z} = \frac{z}{r},$$

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$$= (-r^{-2} \cdot r_{x}, -r^{-2} \cdot r_{y}, -r^{-2} \cdot r_{y}) = (-\frac{x}{r^{3}}, -\frac{y}{r^{3}}, -\frac{z}{r^{3}}),$$

$$\operatorname{div} \nabla \frac{1}{r} = \frac{\partial}{\partial x}(-\frac{x}{r^{3}}) + \frac{\partial}{\partial y}(-\frac{y}{r^{3}}) + \frac{\partial}{\partial z}(-\frac{z}{r^{3}})$$

$$= (-\frac{1}{r^{3}} + \frac{3x^{2}}{r^{5}}) + (-\frac{1}{r^{3}} + \frac{3x^{2}}{r^{5}}) + (-\frac{1}{r^{3}} + \frac{3x^{2}}{r^{5}})$$

$$= -\frac{3}{r^{3}} + \frac{3(x^{2} + y^{2} + z^{2})}{r^{5}}$$



解

$$r_{x} = \frac{x}{r}, \qquad r_{y} = \frac{y}{r}, \qquad r_{z} = \frac{z}{r},$$

$$\nabla \frac{1}{r} = (\frac{\partial}{\partial x} r^{-1}, \frac{\partial}{\partial y} r^{-1}, \frac{\partial}{\partial y} r^{-1})$$

$$= (-r^{-2} \cdot r_{x}, -r^{-2} \cdot r_{y}, -r^{-2} \cdot r_{y}) = (-\frac{x}{r^{3}}, -\frac{y}{r^{3}}, -\frac{z}{r^{3}}),$$

$$\operatorname{div} \nabla \frac{1}{r} = \frac{\partial}{\partial x} (-\frac{x}{r^{3}}) + \frac{\partial}{\partial y} (-\frac{y}{r^{3}}) + \frac{\partial}{\partial z} (-\frac{z}{r^{3}})$$

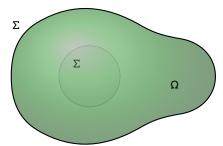
$$= (-\frac{1}{r^{3}} + \frac{3x^{2}}{r^{5}}) + (-\frac{1}{r^{3}} + \frac{3x^{2}}{r^{5}}) + (-\frac{1}{r^{3}} + \frac{3x^{2}}{r^{5}})$$

 $=-\frac{3}{x^3}+\frac{3(x^2+y^2+z^2)}{x^5}=0.$

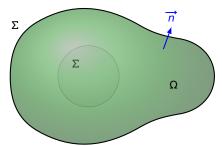




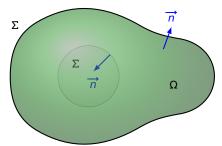
- 空间闭区域 Ω 的边界是分片光滑的闭曲面 Σ ,
- \overrightarrow{n} 是 Σ 的单位外法向量,



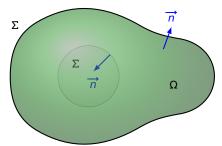
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- F = (P, Q, R) 是 Ω 中向量场,且 P, Q, R 具有一阶连续的偏导数,



定理(高斯公式) 假设

- 空间闭区域 Ω 的边界是分片光滑的闭曲面 Σ ,
- \overrightarrow{n} 是 Σ 的单位外法向量,
- F = (P, Q, R) 是 Ω 中向量场,且 P, Q, R 具有一阶连续的偏导数,

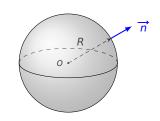
则

$$\iiint_{\Omega} \operatorname{div} F dv = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$

$$\sum_{\overrightarrow{n}} \bigcap_{\Omega} \operatorname{div} F dv = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$

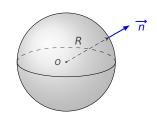
$$I = \iint_{\Sigma} 2x \, dy \, dz + y^2 \, dz \, dx + z^2 \, dx \, dy$$

其中定向曲面 Σ 是球面 $x^2 + y^2 + z^2 = R^2$, 定向取外侧



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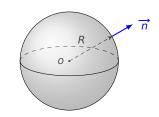


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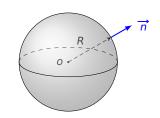


$$I = \iint_{\Sigma} F \cdot \overrightarrow{n} dS = \frac{\overline{\operatorname{sh}} \cdot \overline{\operatorname{sh}}}{\int \iint_{\Omega} \operatorname{div} F dv}$$



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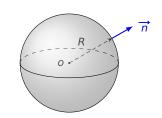


$$I = \underbrace{F = (2x, y^2, z^2)}_{\Gamma} \iint_{\Sigma} F \cdot \overrightarrow{n} dS = \underbrace{\overline{\text{s斯公式}}}_{\Omega} \iint_{\Omega} \operatorname{div} F dv$$



$$I = \iint_{\Sigma} 2x dy dz + y^2 dz dx + z^2 dx dy$$

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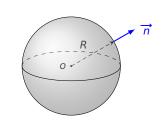


$$I \xrightarrow{F=(2x,y^2,z^2)} \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\overline{\text{synch}}} \iiint_{\Omega} \operatorname{div} F dv$$
$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} (2x) + \frac{\partial}{\partial y} (y^2) + \frac{\partial}{\partial z} (z^2) \right] dv$$



$$I = \iint_{\Sigma} 2x \, dy \, dz + y^2 \, dz \, dx + z^2 \, dx \, dy$$

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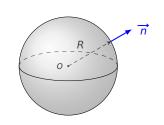


$$I = \frac{F = (2x, y^2, z^2)}{\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS} = \frac{\overrightarrow{\text{sin} \triangle x}}{\iint_{\Omega} \text{div} F \, dv}$$
$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} (2x) + \frac{\partial}{\partial y} (y^2) + \frac{\partial}{\partial z} (z^2) \right] dv = \iiint_{\Omega} (2 + y + z) \, dx \, dy \, dz$$



$$I = \iint_{\Sigma} 2x dy dz + y^2 dz dx + z^2 dx dy$$

其中定向曲面 Σ 是球面 $x^2 + y^2 + z^2 = R^2$, 定向取外侧



$$I = \frac{F = (2x, y^{2}, z^{2})}{\int \int_{\Sigma} F \cdot \overrightarrow{n} dS} = \frac{\overline{\text{sh} \triangle x}}{\int \int_{\Omega} \text{div} F dv}$$

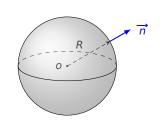
$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} (2x) + \frac{\partial}{\partial y} (y^{2}) + \frac{\partial}{\partial z} (z^{2}) \right] dv = \iiint_{\Omega} (2 + y + z) dx dy dz$$

$$= \frac{\overline{\text{sh} \triangle x}}{\int \int_{\Omega} 2 dy dz}$$



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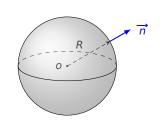
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$$= \frac{\overline{\text{sh} \triangle x}}{\int \int_{\Omega} 2 \, dy \, dz} = 2 \, \text{Vol}(\Omega)$$



$$I = \iint_{\Sigma} 2x dy dz + y^2 dz dx + z^2 dx dy$$

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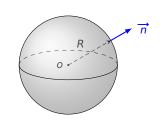
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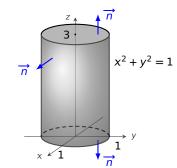
$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} (2x) + \frac{\partial}{\partial y} (y^{2}) + \frac{\partial}{\partial z} (z^{2}) \right] dv = \iiint_{\Omega} (2 + y + z) dx dy dz$$

$$= \frac{\overline{\text{sh} \triangle x}}{\int \int_{\Omega} 2 dy dz} = 2 \text{Vol}(\Omega) = \frac{8}{3} \pi R^{3}$$



$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$

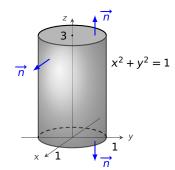
其中定向曲面 Σ 是右图柱体的边界曲面



$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$

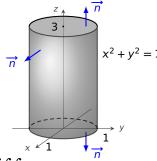
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$$I = \int \int_{\Sigma} F \cdot \overrightarrow{n} \, dS$$



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其中定向曲面 Σ 是右图柱体的边界曲面

$$x^{2} + y^{2} =$$

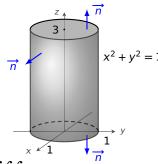
$$I = F = ((y-z)x, 0, x-y)$$

$$\iint_{\Sigma} F \cdot \overrightarrow{n} dS = \overline{\text{S斯公式}} \iiint_{\Omega} \text{div} F dv$$



$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$

其中定向曲面 Σ 是右图柱体的边界曲面



$$I = \frac{F = ((y-z)x, 0, x-y)}{\int \int_{\Sigma} F \cdot \overrightarrow{n} \, dS} = \frac{\overline{\text{synch}}}{\int \int_{\Omega} \text{div} F \, dv}$$
$$= \left[\int \int_{\Omega} \left[\frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv$$

$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$

其中定向曲面 Σ 是右图柱体的边界曲面

$$x^{2} + y^{2} = 1$$

解
$$I = \frac{F = ((y-z)x, 0, x-y)}{\int \int_{\Sigma} F \cdot \overrightarrow{n} dS} = \frac{\text{高斯公式}}{\int \int_{\Omega} \text{div} F dv}$$

$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dz$$



$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$

其中定向曲面 Σ 是右图柱体的边界曲面

$$x^{2} + y^{2} = 1$$

$$I = \frac{F = ((y-z)x, 0, x-y)}{\int \int_{\Sigma} F \cdot \overrightarrow{n} dS} = \frac{\overline{\sin} \Delta \overrightarrow{x}}{\int \int_{\Omega} \operatorname{div} F dv}$$

$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dz$$

$$= \int \left[\iint (y-z) dx y \right] dz$$



$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$

其中定向曲面 Σ 是右图柱体的边界曲面

$$x^{2} + y^{2} = 1$$

$$I = \frac{F = ((y-z)x, 0, x-y)}{\int \int_{\Sigma} F \cdot \overrightarrow{n} dS} = \frac{\overline{\text{sh}公式}}{\int \int_{\Omega} \text{div} F dv}$$

$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dz$$

$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial z} \right] dz$$
$$= \left[\iint_{\Omega} (y-z) dxy \right] dz$$



$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$

其中定向曲面 Σ 是右图柱体的边界曲面

$$\overrightarrow{D}$$

$$x^2 + y^2 =$$

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$$I \xrightarrow{F = ((y-z)x, 0, x-y)} \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\overline{\text{sh}} \triangle \underline{x}} \iiint_{\Omega} \operatorname{div} F dv$$
$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dz$$

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$$I \xrightarrow{F = ((y-z)x, 0, x-y)} \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\underline{a}\underline{y}} \iiint_{\Omega} \operatorname{div} F \, dv$$
$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z \, dx \, dy \, dz$$

$$= \int \left[\int \int (y-z)dxy \right] dz$$



$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$

其中定向曲面 Σ 是右图柱体的边界曲面

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$$I \xrightarrow{F = ((y-z)x, 0, x-y)} \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\underline{\text{sin}} \Delta \underline{x}} \iiint_{\Omega} \operatorname{div} F \, dv$$
$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z \, dx \, dy \, dz$$

$$= \int_{0}^{3} \left[\int \int (y-z)dxy \right] dz$$



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$$I = \frac{F = ((y - z)x, 0, x - y)}{\int \int_{\Sigma} F \cdot \overrightarrow{n} dS} = \frac{\overline{\operatorname{sh} \triangle x}}{\int \int_{\Omega} \operatorname{div} F dv}$$

$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} ((y - z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x - y) \right] dv = \iiint_{\Omega} y - z dx dy dz$$

$$= \int_{\Omega} \left[\iint_{\Omega} (y - z) dx y \right] dz$$

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$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$

其中定向曲面 Σ 是右图柱体的边界曲面

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$$= \int_{0}^{3} \left[\iint_{\Omega} (y-z) \, dx \, y \right] dz = \int_{0}^{3} \left[\iint_{\Omega} -z \, dx \, y \right] dz$$



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其中定向曲面 Σ 是右图柱体的边界曲面

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$$I \xrightarrow{F = ((y-z)x, 0, x-y)} \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{a}\underline{n} \underline{n}} \iiint_{\Omega} \operatorname{div} F dv$$
$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dz$$

 $= \int_0^3 \left[\iint_{\Omega} (y-z) dxy \right] dz = \int_0^3 \left[\iint_{\Omega} -z dxy \right] dz$

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$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$

其中定向曲面 Σ 是右图柱体的边界曲面

$$x^{2} + y^{2} = 1$$

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$$I = F = ((y-z)x, 0, x-y) = \iint_{\Sigma} F \cdot \overrightarrow{n} dS = \overline{\text{sin} \Delta t} = \iiint_{\Omega} \text{div} F dv$$

$$=\iiint_{\Omega} \left[\frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dz$$

$$= \int_{0}^{3} \left[\int_{0}^{3} ((y-z)x) dx \right] dz = \int_{0}^{3} \left[\int_{0}^{3} ((y-z)x) dx \right] dz = \int_{0}^{3} \left[\int_{0}^{3} ((y-z)x) dx \right] dz$$

$$= \int_{0}^{3} \left[\iint_{\Omega} (y-z) dxy \right] dz = \int_{0}^{3} \left[\iint_{\Omega} -z dxy \right] dz = \int_{0}^{3} \left[-z |D_{z}| \right] dz$$



 $I = \iint_{\mathbb{R}^{n}} (x - y) dx dy + (y - z) x dy dz$

$$x^{2} + y^{2} = 1$$

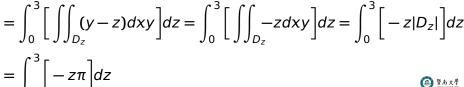
$$y$$

$$x = 1$$

$$y$$

其中定向曲面 Σ 是右图柱体的边界曲面 I = F = ((y-z)x, 0, x-y) $\iint_{\mathbb{R}} F \cdot \overrightarrow{n} \, dS = \overline{\text{sh} \Delta t} \iiint_{\mathbb{R}} \text{div} F dv$ $= \iiint_{\Omega} \left[\frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dz$

 $=\int_{-\infty}^{\infty} \left[-z\pi\right] dz$



例 计算 $I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$

$$I = \iint_{\Sigma} (x-y) dx dy + (y-z) x dy dz$$

其中定向曲面 Σ 是右图柱体的边界曲面
 $\underbrace{F = ((y-z)x, 0, x-y)}_{\times}$ $\int \int_{\Gamma} F \cdot \overrightarrow{n} dS = \underbrace{\frac{\overline{s} \times \Delta x}{\overline{s}}}_{\times}$ $\int \int \int_{\Gamma} div F dv$

解
$$\frac{F = ((y-z)x, 0, x-y)}{\int_{\Sigma} F \cdot \overrightarrow{n} dS} = \frac{\overline{\text{sin} \triangle x}}{\int_{\Omega} \text{div} F dv}$$

$$= \iiint_{\Omega} \frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) dv = \iiint_{\Omega} y - z dx dy dv$$

I = F = ((y-z)x, 0, x-y) $\iint_{\mathbb{R}} F \cdot \overrightarrow{n} \, dS = \overline{\text{sh} \Delta t} \iiint_{\mathbb{R}} \text{div} F dv$ $= \iiint_{\Omega} \left[\frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dz$

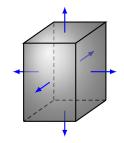
$$\iint_{\Omega} \operatorname{div} F dV$$

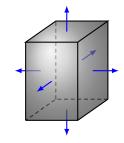
$$-y) dv = \iiint_{\Omega} y - z dx dy$$

$$-z dx y dz = \int_{\Omega} |-z| D_z|$$

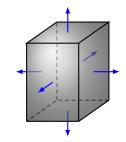
 $= \int_0^3 \left[\iint_{D_z} (y-z) dxy \right] dz = \int_0^3 \left[\iint_{D_z} -z dxy \right] dz = \int_0^3 \left[-z |D_z| \right] dz$

 $= \int_{0}^{3} \left[-z\pi \right] dz = -\frac{9}{2}\pi$



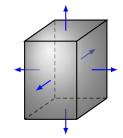


$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS$$



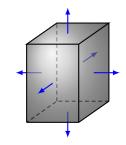
$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\underline{a} \underline{m} \triangle \underline{\exists}} \iiint_{\Omega} \mathrm{div} F dv$$





$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\underline{=}\underline{\text{MSL}}} \iiint_{\Omega} \operatorname{div} F \, dv$$
$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} (x - y^2) + \frac{\partial}{\partial y} (y) + \frac{\partial}{\partial z} (z^3) \right] \, dv$$

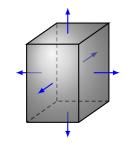




$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\underline{a}\underline{m}\underline{\omega}\underline{\omega}} \iiint_{\Omega} \operatorname{div} F \, dv$$

$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} (x - y^2) + \frac{\partial}{\partial y} (y) + \frac{\partial}{\partial z} (z^3) \right] dv$$

$$= \iiint_{\Omega} \left[(2 + 3z^2) \, dx \, dy \, dz \right]$$

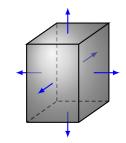


$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\underline{\underline{a}}\underline{\underline{m}}\underline{\underline{M}}\underline{\underline{M}}} \iiint_{\Omega} \operatorname{div} F \, dv$$

$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} (x - y^2) + \frac{\partial}{\partial y} (y) + \frac{\partial}{\partial z} (z^3) \right] \, dv$$

$$= \iiint_{\Omega} (2 + 3z^2) \, dx \, dy \, dz = \int_{\Omega} \left[\int_{\Omega} \left[\int_{\Omega} (2 + 3z^2) \, dz \right] \, dx \right] \, dy$$



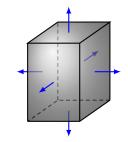


$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\underline{\underline{a}}\underline{\underline{m}}\underline{\underline{M}}\underline{\underline{M}}} \iiint_{\Omega} \operatorname{div} F \, dv$$

$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} (x - y^2) + \frac{\partial}{\partial y} (y) + \frac{\partial}{\partial z} (z^3) \right] \, dv$$

$$= \iiint_{\Omega} (2 + 3z^2) \, dx \, dy \, dz = \int_{0}^{1} \left[\int_{0}^{1} \left[\int_{0}^{1} (2 + 3z^2) \, dz \right] \, dx \right] \, dy$$



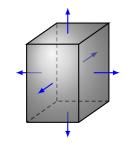


$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\underline{=}\underline{\text{M}} \times \underline{\text{M}}} \iiint_{\Omega} \operatorname{div} F \, dv$$

$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} (x - y^2) + \frac{\partial}{\partial y} (y) + \frac{\partial}{\partial z} (z^3) \right] \, dv$$

$$= \iiint_{\Omega} (2 + 3z^2) \, dx \, dy \, dz = \int_{0}^{1} \left[\int_{1}^{2} \left[\int_{1}^{2} (2 + 3z^2) \, dz \right] \, dx \right] \, dy$$



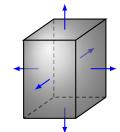


$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\underline{\underline{a}}\underline{\underline{m}}\underline{\underline{M}}\underline{\underline{M}}} \iiint_{\Omega} \operatorname{div} F \, dv$$

$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} (x - y^2) + \frac{\partial}{\partial y} (y) + \frac{\partial}{\partial z} (z^3) \right] \, dv$$

$$= \iiint_{\Omega} (2 + 3z^2) \, dx \, dy \, dz = \int_{0}^{1} \left[\int_{1}^{2} \left[\int_{1}^{4} (2 + 3z^2) \, dz \right] \, dx \right] \, dy$$





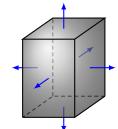
$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\underline{a}\underline{m}\underline{\omega}\underline{\omega}} \iiint_{\Omega} \operatorname{div} F \, dV$$

$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} (x - y^2) + \frac{\partial}{\partial y} (y) + \frac{\partial}{\partial z} (z^3) \right] dV$$

$$= \iiint_{\Omega} (2 + 3z^2) \, dx \, dy \, dz = \int_{0}^{1} \left[\int_{1}^{2} \left[\int_{1}^{4} (2 + 3z^2) \, dz \right] dx \right] dy$$

$$= \int_{0}^{1} 1 \, dy \cdot \int_{1}^{2} 1 \, dx \cdot \int_{1}^{4} (2 + 3z^2) \, dz$$





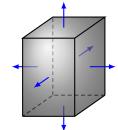
$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{a}\underline{m}\underline{\omega}\underline{\omega}} \iiint_{\Omega} \operatorname{div} F dV$$

$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} (x - y^2) + \frac{\partial}{\partial y} (y) + \frac{\partial}{\partial z} (z^3) \right] dV$$

$$= \iiint_{\Omega} (2 + 3z^2) dx dy dz = \int_{0}^{1} \left[\int_{1}^{2} \left[\int_{1}^{4} (2 + 3z^2) dz \right] dx \right] dy$$

$$= \int_{0}^{1} 1 dy \cdot \int_{1}^{2} 1 dx \cdot \int_{1}^{4} (2 + 3z^2) dz = 1 \cdot 1 \cdot (2z + z^3) \Big|_{1}^{4}$$





$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\underline{a}\underline{y} \underline{y}} \iiint_{\Omega} \operatorname{div} F \, dv$$

$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} (x - y^2) + \frac{\partial}{\partial y} (y) + \frac{\partial}{\partial z} (z^3) \right] \, dv$$

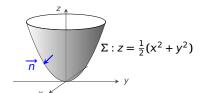
$$= \iiint_{\Omega} (2 + 3z^2) \, dx \, dy \, dz = \int_{0}^{1} \left[\int_{1}^{2} \left[\int_{1}^{4} (2 + 3z^2) \, dz \right] \, dx \right] \, dy$$

$$= \int_{0}^{1} 1 \, dy \cdot \int_{1}^{2} 1 \, dx \cdot \int_{1}^{4} (2 + 3z^2) \, dz = 1 \cdot 1 \cdot (2z + z^3) \Big|_{1}^{4} = 69$$

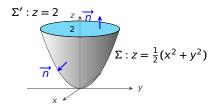


$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

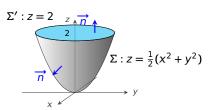
其中定向曲面 Σ 是抛物面的一部分, 取单位外法向量,如图:



$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$



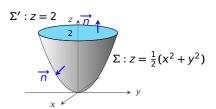
$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$



$$\iint_{\Sigma} (z^{2} + x) dy dz - z dx dy = \iint_{\Sigma} F \cdot \overrightarrow{n} dS,$$

$$\iint_{\Sigma'} (z^{2} + x) dy dz - z dx dy = \iint_{\Sigma'} F \cdot \overrightarrow{n} dS$$

$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

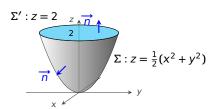


$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy = \iint_{\Sigma} F \cdot \overrightarrow{n} dS,$$

$$\iint_{\Sigma'} (z^2 + x) dy dz - z dx dy = \iint_{\Sigma'} F \cdot \overrightarrow{n} dS$$

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS + \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS = \iint_{\Sigma \cup \Sigma'} F \cdot \overrightarrow{n} \, dS$$

$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$



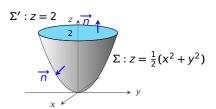
$$\iint_{\Sigma} (z^{2} + x) dy dz - z dx dy = \iint_{\Sigma} F \cdot \overrightarrow{n} dS,$$

$$\iint_{\Sigma'} (z^{2} + x) dy dz - z dx dy = \iint_{\Sigma'} F \cdot \overrightarrow{n} dS$$

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS + \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS = \iint_{\Sigma \cup \Sigma'} \operatorname{div} F \, dv$$



$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

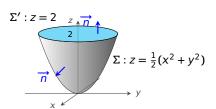


$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS + \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS = \iint_{\Sigma \cup \Sigma'} \operatorname{div} F \, dv \xrightarrow{\operatorname{div} F = 0}$$



$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

其中定向曲面 Σ 是抛物面的一部分, 取单位外法向量,如图:



解 知園村で七十回
$$Z$$
 、 別 Z O Z 相似 S 報区域 Ω D $介$, 应 布 同利 G X $\int_{\Sigma} (z^2 + x) dy dz - z dx dy = \int_{\Sigma} F \cdot \overrightarrow{n} dS$
$$\int_{\Sigma'} (z^2 + x) dy dz - z dx dy = \int_{\Sigma'} F \cdot \overrightarrow{n} dS$$

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS + \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS = \iint_{\Sigma \cup \Sigma'} \operatorname{div} F \, dv \xrightarrow{\operatorname{div} F = 0} 0.$$



$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

其中定向曲面 Σ 是抛物面的一部分, 取单位外法向量,如图:

$$\Sigma': z = 2$$

$$Z : z = \frac{1}{2}(x^2 + y^2)$$

$$Z : z = \frac{1}{2}(x^2 + y^2)$$

$$\iint_{\Sigma} (z^{2} + x) dy dz - z dx dy = \iint_{\Sigma} F \cdot \overrightarrow{n} dS,$$

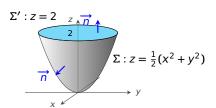
$$\iint_{\Sigma'} (z^{2} + x) dy dz - z dx dy = \iint_{\Sigma'} F \cdot \overrightarrow{n} dS \xrightarrow{F = (z^{2} + x, 0, -z)}$$

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS + \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS = \iint_{\Sigma \cup \Sigma'} \operatorname{div} F \, dv \xrightarrow{\operatorname{div} F = 0} 0.$$



$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

其中定向曲面 Σ 是抛物面的一部分, 取单位外法向量,如图:



 \mathbf{m} 如图补充平面 Σ' ,则 $\Sigma \cup \Sigma'$ 构成 3 维区域 Ω 边界,应用高斯公式:

$$\iint_{\Sigma} (z^{2} + x) dy dz - z dx dy = \iint_{\Sigma} F \cdot \overrightarrow{n} dS,$$

$$\iint_{\Sigma'} (z^{2} + x) dy dz - z dx dy = \iint_{\Sigma'} F \cdot \overrightarrow{n} dS \xrightarrow{F = (z^{2} + x, 0, -z)}{\overrightarrow{n} = (0, 0, 1)}$$

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS + \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS = \iint_{\Sigma \cup \Sigma'} \operatorname{div} F \, dv \xrightarrow{\operatorname{div} F = 0} 0.$$



$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

其中定向曲面 Σ 是抛物面的一部分, 取单位外法向量,如图:

$$\Sigma': z = 2$$

$$\Sigma: z = \frac{1}{2}(x^2 + y^2)$$

$$Y$$

 \mathbf{m} 如图补充平面 Σ' ,则 $\Sigma \cup \Sigma'$ 构成 3 维区域 Ω 边界,应用高斯公式:

$$\iint_{\Sigma} (z^{2} + x) dy dz - z dx dy = \iint_{\Sigma} F \cdot \overrightarrow{n} dS,$$

$$\iint_{\Sigma'} (z^{2} + x) dy dz - z dx dy = \iint_{\Sigma'} F \cdot \overrightarrow{n} dS \xrightarrow{F = (z^{2} + x, 0, -z)}{\overrightarrow{n} = (0, 0, 1)} \iint_{\Sigma'} -z dS$$

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS + \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS = \iint_{\Sigma \cup \Sigma'} \operatorname{div} F \, dv \xrightarrow{\operatorname{div} F = 0} 0.$$



$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

其中定向曲面 Σ 是抛物面的一部分, 取单位外法向量,如图:

$$\Sigma': z = 2$$

$$z \to \overline{D}$$

$$\Sigma: z = \frac{1}{2}(x^2 + y^2)$$

解 如图补充平面 Σ' ,则 $\Sigma \cup \Sigma'$ 构成 3 维区域 Ω 边界,应用高斯公式:

解 知图和记书面 Z ,则 Z O Z 和成 S 建区域
$$\Omega$$
 边外,应用高别公式:
$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy = \iint_{\Sigma} F \cdot \overrightarrow{n} dS,$$
$$\iint_{\Sigma'} (z^2 + x) dy dz - z dx dy = \iint_{\Sigma'} F \cdot \overrightarrow{n} dS \frac{F = (z^2 + x, 0, -z)}{\overrightarrow{n} = (0, 0, 1)} \iint_{\Sigma'} -z dS$$
$$= \iint_{\Sigma'} -2 dS$$

$$= \int \int_{\Sigma'} -2dS$$

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS + \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS = \iint_{\Sigma \cup \Sigma'} \operatorname{div} F \, dv \xrightarrow{\operatorname{div} F = 0} 0.$$



$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

其中定向曲面 Σ 是抛物面的一部分, 取单位外法向量,如图:

$$\Sigma': z = 2$$

$$\Sigma: z = \frac{1}{2}(x^2 + y^2)$$

 \mathbf{m} 如图补充平面 Σ' ,则 $\Sigma \cup \Sigma'$ 构成 3 维区域 Ω 边界,应用高斯公式:

解 如图称允十個
$$Z$$
 ,则 Z O Z 构成 S 维区域 Ω 边外,应用高利公式:
$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy = \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{F = (z^2 + x, 0, -z)} \iint_{\Sigma'} -z dS$$
$$= \iint_{\Sigma'} -2 dS = -2 \operatorname{Area}(\Sigma')$$

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS + \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS = \iint_{\Sigma \cup \Sigma'} \operatorname{div} F \, dv \xrightarrow{\operatorname{div} F = 0} 0.$$



$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

其中定向曲面 Σ 是抛物面的一部分, 取单位外法向量,如图:

$$\Sigma': z = 2$$

$$\Sigma: z = \frac{1}{2}(x^2 + y^2)$$

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解 如图和允许图
$$Z$$
 ,则 Z O Z 和成 S 维区域 Ω 边外,应用高利公式:
$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$
$$\iint_{\Sigma'} (z^2 + x) dy dz - z dx dy = \iint_{\Sigma'} F \cdot \overrightarrow{n} dS \frac{F = (z^2 + x, 0, -z)}{\overrightarrow{n} = (0, 0, 1)} \iint_{\Sigma'} -z dS$$
$$= \iint_{\Sigma'} -2 dS = -2 \operatorname{Area}(\Sigma') = -8\pi,$$

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS + \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS = \iint_{\Sigma \cup \Sigma'} \operatorname{div} F \, dv \xrightarrow{\operatorname{div} F = 0} 0.$$



其中定向曲面 Σ 是抛物面的一部分, 取单位外法向量,如图:

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 \mathbf{m} 如图补充平面 Σ' ,则 $\Sigma \cup \Sigma'$ 构成 3 维区域 Ω 边界,应用高斯公式:

$$\iint_{\Sigma} (z^{2} + x) dy dz - z dx dy = \iint_{\Sigma} F \cdot \overrightarrow{n} dS,$$

$$\iint_{\Sigma} (z^{2} + x) dy dz - z dx dy = \iint_{\Sigma'} F \cdot \overrightarrow{n} dS \frac{F = (z^{2} + x, 0, -z)}{\overrightarrow{n} = (0, 0, 1)} \iint_{\Sigma'} -z dS$$

$$= \iint_{\Sigma'} -2 dS = -2 \operatorname{Area}(\Sigma') = -8\pi,$$

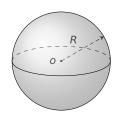
$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS + \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS = \iint_{\Sigma \cup \Sigma'} \operatorname{div} F \, dv \xrightarrow{\operatorname{div} F = 0} 0.$$

所以原积分等于 8π。



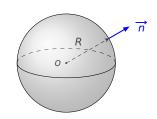
$$I = \iint_{\Sigma} (x^2 + y + z) dS$$

其中曲面 Σ 是球面 $x^2 + y^2 + z^2 = R^2$



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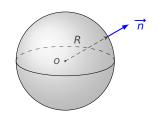
解

$$\iint_{\Sigma} (x^2 + y + z) dS$$

$$= \iint_{\Sigma} F \cdot \overrightarrow{n} dS \stackrel{\overline{a} \underline{m} \underline{\wedge} \underline{\wedge}}{\underline{\wedge}} \iiint_{\Omega} \operatorname{div} F dv$$

$$I = \iiint_{\Sigma} (x^2 + y + z) dS$$

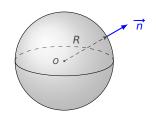
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解 球面单位外法向量
$$\overrightarrow{n} = \frac{1}{R}(x, y, z)$$
,所以
$$\iint_{\Sigma} (x^2 + y + z) dS$$

$$= \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\overline{\mathrm{sh} \otimes \mathbb{H}}} \iiint_{\Omega} \mathrm{div} F d\nu$$

$$I = \iint_{\Sigma} (x^2 + y + z) dS$$

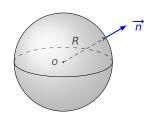


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解 球面单位外法向量
$$\overrightarrow{n} = \frac{1}{R}(x, y, z)$$
,所以
$$\iint_{\Sigma} (x^2 + y + z) dS \qquad (, ,) \cdot \frac{1}{R}(x, y, z)$$

$$= \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\overline{\text{nh} \Delta x}} \iiint_{\Omega} \text{div} F dv$$

$$I = \iint_{\Sigma} (x^2 + y + z) dS$$

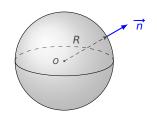


其中曲面 Σ 是球面
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解 球面单位外法向量
$$\overrightarrow{n} = \frac{1}{R}(x, y, z)$$
,所以
$$\iint_{\Sigma} (x^2 + y + z) dS \qquad R(x, 1, 1) \cdot \frac{1}{R}(x, y, z)$$

$$= \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{\text{S斯公式}}} \iiint_{\Omega} \text{div} F dv$$

$$I = \iint_{\Sigma} (x^2 + y + z) dS$$

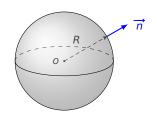


其中曲面 Σ 是球面 $x^2 + y^2 + z^2 = R^2$

解 球面单位外法向量
$$\overrightarrow{n} = \frac{1}{R}(x, y, z)$$
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$$\iint_{\Sigma} (x^2 + y + z) dS = \iint_{\Sigma} R(x, 1, 1) \cdot \frac{1}{R}(x, y, z) dS$$

$$= \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\overline{\text{sh}\Delta t}} \iiint_{\Omega} \text{div} F dV$$

$$I = \iint_{\Sigma} (x^2 + y + z) dS$$



其中曲面 Σ 是球面
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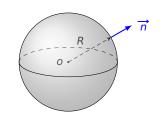
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$$= \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{a} \underline{n} \underline{n} \underline{n}} \iiint_{\Omega} \operatorname{div} F dv$$

$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} (Rx) + \frac{\partial}{\partial y} (R) + \frac{\partial}{\partial z} (R) \right] dv$$

$$I = \iint_{\Sigma} (x^2 + y + z) dS$$

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解 球面单位外法向量
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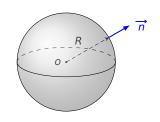
$$= \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{n} \underline{n} \underline{n} \underline{n}} \iiint_{\Omega} \operatorname{div} F dv$$

$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} (Rx) + \frac{\partial}{\partial y} (R) + \frac{\partial}{\partial z} (R) \right] dv$$

$$= \iiint_{\Omega} R dx dy dz$$

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$$\overrightarrow{n} = \frac{1}{R}(x, y, z)$$
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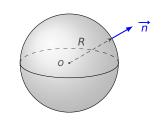
$$= \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{a} \underline{m} \triangle \underline{d}} \iiint_{\Omega} \underline{\mathrm{div}} F dv$$

$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} (Rx) + \frac{\partial}{\partial y} (R) + \frac{\partial}{\partial z} (R) \right] dv$$

$$= \iiint_{\Omega} R dx dy dz = R \mathrm{Vol}(\Omega)$$

$$I = \iint_{\Sigma} (x^2 + y + z) dS$$

其中曲面 Σ 是球面 $x^2 + y^2 + z^2 = R^2$

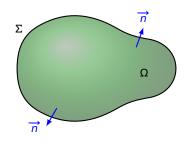


解 球面单位外法向量 $\overrightarrow{n} = \frac{1}{R}(x, y, z)$,所以 $\iint_{\mathbb{R}} (x^2 + y + z) dS = \iint_{\mathbb{R}} R(x, 1, 1) \cdot \frac{1}{R} (x, y, z) dS$ $=\iint_{-} F \cdot \overrightarrow{n} dS \xrightarrow{\stackrel{\text{Sh}}{=}} \iiint_{-} \operatorname{div} F dv$ $= \iiint_{\Omega} \left[\frac{\partial}{\partial x} (Rx) + \frac{\partial}{\partial y} (R) + \frac{\partial}{\partial z} (R) \right] dv$ $= \iiint_{\mathbb{R}} R dx dy dz = R Vol(\Omega) = \frac{8}{3} \pi R^4$

高斯公式
$$\iiint_{\Omega} \operatorname{div} F dv = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$



• 假设 F = (P, Q, R) 是流体的速度向量场,



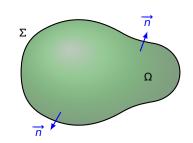
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 假设 F = (P, Q, R) 是流体的速度向 量场,则

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS$$

表示单位时间流向 Σ 外侧的通量。



高斯公式
$$\iiint_{\Omega} \operatorname{div} F dv = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$



假设 F = (P, Q, R) 是流体的速度向量场,则

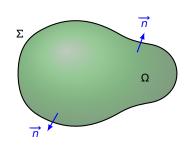
$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS$$

表示单位时间流向 Σ 外侧的通量。

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS > 0$$

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS < 0$$

高斯公式
$$\iiint_{\Omega} \operatorname{div} F dv = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$



 假设 F = (P, Q, R) 是流体的速度向 量场,则

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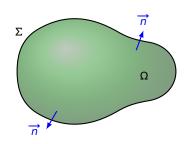
表示单位时间流向 Σ 外侧的通量。

• 进一步假设流体是不可压,则

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS > 0$$

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高斯公式
$$\iiint_{\Omega} \operatorname{div} F dv = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$



 假设 F = (P, Q, R) 是流体的速度向 量场,则

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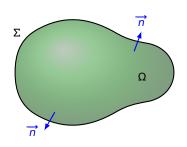
表示单位时间流向 Σ 外侧的通量。

• 进一步假设流体是不可压,则

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS > 0 \Rightarrow \Omega \text{ 内有 "source"}$$

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$$\iiint_{\Omega} \operatorname{div} F dv = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$



 假设 F = (P, Q, R) 是流体的速度向 量场,则

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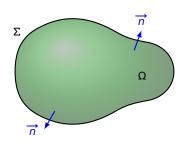
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高斯公式
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假设 F = (P, Q, R) 是流体的速度向量场,则

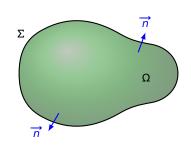
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注 高斯公式 $\iiint_{\Omega} \operatorname{div} F dv = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$ 表明: $\operatorname{div} F$ 反映这种

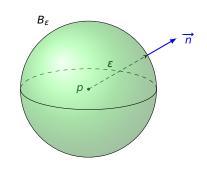
"source"和 "sink"的强度。



р.



散度 $\operatorname{div} \mathbf{F}$ 的物理解释 (2)

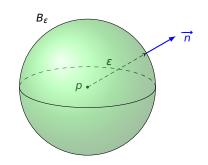




散度 $\operatorname{div} \mathbf{F}$ 的物理解释 (2)

$$\iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \iiint_{B_{\varepsilon}} \operatorname{div} F \, dv$$





散度 $\operatorname{div} \mathbf{F}$ 的物理解释 (2)

$$\iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$\iiint_{B_{\varepsilon}} \operatorname{div} F dV$$

$$= \operatorname{Vol}(B_{\varepsilon}) \operatorname{div} F(p_{\varepsilon})$$

$$\frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \iiint_{B_{\varepsilon}} \operatorname{div} F \, dv$$

$$= \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \operatorname{Vol}(B_{\varepsilon}) \operatorname{div} F(p_{\varepsilon})$$



$$\frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \iiint_{B_{\varepsilon}} \operatorname{div} F \, dv$$

$$= \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \operatorname{Vol}(B_{\varepsilon}) \operatorname{div} F(p_{\varepsilon})$$

$$= \operatorname{div} F(p_{\varepsilon})$$



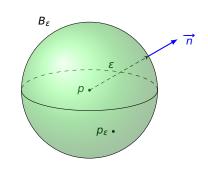
$$\lim_{\varepsilon \to 0^{+}} \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \iiint_{B_{\varepsilon}} \operatorname{div} F \, dv$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \operatorname{Vol}(B_{\varepsilon}) \operatorname{div} F(p_{\varepsilon})$$

$$= \lim_{\varepsilon \to 0^{+}} \operatorname{div} F(p_{\varepsilon})$$

$$\operatorname{div} F(p)$$



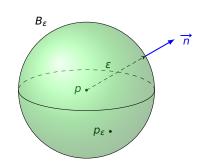
$$\lim_{\varepsilon \to 0^{+}} \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \iiint_{B_{\varepsilon}} \operatorname{div} F \, dv$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \operatorname{Vol}(B_{\varepsilon}) \operatorname{div} F(p_{\varepsilon})$$

$$= \lim_{\varepsilon \to 0^{+}} \operatorname{div} F(p_{\varepsilon})$$

$$= \operatorname{div} F(p)$$



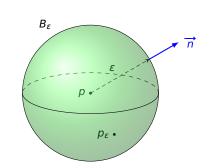
$$\lim_{\varepsilon \to 0^{+}} \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \iiint_{B_{\varepsilon}} \operatorname{div} F \, dv$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \operatorname{Vol}(B_{\varepsilon}) \operatorname{div} F(p_{\varepsilon})$$

$$= \lim_{\varepsilon \to 0^{+}} \operatorname{div} F(p_{\varepsilon})$$

$$= \operatorname{div} F(p)$$



- div F(p)>0 时,
- div*F*(*p*)<0 时,

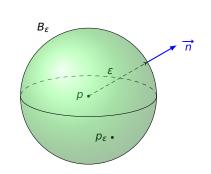
$$\lim_{\varepsilon \to 0^{+}} \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

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$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \operatorname{Vol}(B_{\varepsilon}) \operatorname{div} F(p_{\varepsilon})$$

$$= \lim_{\varepsilon \to 0^{+}} \operatorname{div} F(p_{\varepsilon})$$

$$= \operatorname{div} F(p)$$



- div*F*(*p*)>0 时, ∫∫_{∂B}, *F* · \overrightarrow{n} dS >0 (ε 充分小),
- div*F*(*p*)<0 时,



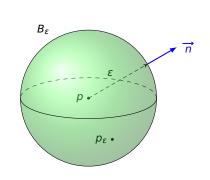
$$\lim_{\varepsilon \to 0^{+}} \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \iiint_{B_{\varepsilon}} \operatorname{div} F \, dv$$

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$$= \lim_{\varepsilon \to 0^{+}} \operatorname{div} F(p_{\varepsilon})$$

$$= \operatorname{div} F(p)$$



- $\operatorname{div} F(p) > 0$ 时, $\iint_{\partial B_{\epsilon}} F \cdot \overrightarrow{n} dS > 0$ (ϵ 充分小),说明 p 点是 source
- div*F(p)*<0 时,



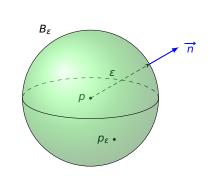
$$\lim_{\varepsilon \to 0^{+}} \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \iiint_{B_{\varepsilon}} \operatorname{div} F \, dv$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \operatorname{Vol}(B_{\varepsilon}) \operatorname{div} F(p_{\varepsilon})$$

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$$= \operatorname{div} F(p)$$



- $\operatorname{div} F(p) > 0$ 时, $\iint_{\partial B_{\epsilon}} F \cdot \overrightarrow{n} dS > 0$ (ϵ 充分小),说明 p 点是 source
- div*F*(*p*)<0 时,∫∫∂Bε *F* · *n* dS <0(ε 充分小),



散度 div**F 的物理解释** (2)

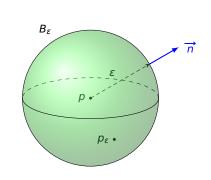
$$\lim_{\varepsilon \to 0^{+}} \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \iiint_{B_{\varepsilon}} \operatorname{div} F \, dv$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \operatorname{Vol}(B_{\varepsilon}) \operatorname{div} F(p_{\varepsilon})$$

$$= \lim_{\varepsilon \to 0^{+}} \operatorname{div} F(p_{\varepsilon})$$

$$= \operatorname{div} F(p)$$



- $\operatorname{div} F(p) > 0$ 时, $\iint_{\partial B_{\epsilon}} F \cdot \overrightarrow{n} \, dS > 0$ (ϵ 充分小),说明 p 点是 source
- $\operatorname{div} F(p) < 0$ 时, $\iint_{\partial B_{\epsilon}} F \cdot \overrightarrow{n} \, dS < 0$ (ϵ 充分小),说明 p 点是 sink



We are here now...

1. 高斯公式

2. 斯托克斯公式

定义 设
$$F = (P, Q, R)$$
 是空间中向量场,定义

$$\cot F := \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

定义 设
$$F = (P, Q, R)$$
 是空间中向量场, 定义

$$\cot F := \left| \begin{array}{ccc} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{array} \right| = \left(\left| \begin{array}{ccc} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{array} \right|, \qquad ,$$

定义 设 F = (P, Q, R) 是空间中向量场, 定义

$$\cot F := \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & O & R \end{vmatrix} = \left(\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{vmatrix}, \right)$$



定义 设 F = (P, Q, R) 是空间中向量场, 定义

$$\cot F := \left| \begin{array}{ccc} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{array} \right| = \left(\left| \begin{array}{ccc} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{array} \right|, - \left| \begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{array} \right|, \left| \begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{array} \right| \right)$$



定义 设 F = (P, Q, R) 是空间中向量场, 定义

$$\cot F := \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} \right)$$
$$= (R_y - Q_z, \qquad , \qquad)$$



定义 设
$$F = (P, Q, R)$$
 是空间中向量场, 定义

$$\cot F := \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} \right)$$
$$= (R_y - Q_z, P_z - R_x,)$$



定义 设 F = (P, Q, R) 是空间中向量场, 定义

$$\cot F := \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} \right)$$
$$= (R_y - Q_z, P_z - R_x, Q_x - P_y)$$



定义 设 F = (P, Q, R) 是空间中向量场, 定义

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$$= (R_y - Q_z, P_z - R_x, Q_x - P_y)$$

称为向量场 F 的旋度。

例 计算向量场 $F = (y, -x, e^{xz})$ 的旋度。



定义 设 F = (P, Q, R) 是空间中向量场, 定义

$$\cot F := \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} \right)$$
$$= (R_V - Q_Z, P_Z - R_X, Q_X - P_Y)$$

称为向量场 F 的旋度。

例 计算向量场 $F = (y, -x, e^{xz})$ 的旋度。

$$\cot F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & e^{xz} \end{vmatrix}$$

定义 设 F = (P, Q, R) 是空间中向量场, 定义

$$\cot F := \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} \right)$$
$$= (R_V - Q_Z, P_Z - R_X, Q_X - P_V)$$

称为向量场 F 的旋度。

例 计算向量场 $F = (v_1 - x_2)$ 的旋度。

$$\cot F = \begin{vmatrix}
\overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
y & -x & e^{xz}
\end{vmatrix} = \left(\begin{vmatrix}
\frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
-x & e^{xz}
\end{vmatrix}, -\begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\
y & e^{xz}
\end{vmatrix}, \begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\
y & -x
\end{vmatrix}\right)$$





定义 设
$$F = (P, Q, R)$$
 是空间中向量场, 定义

$$\cot F := \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} \right)$$
$$= (R_V - Q_Z, P_Z - R_X, Q_X - P_Y)$$

称为向量场 F 的旋度。

例 计算向量场 $F = (v_1 - x_2)$ 的旋度。

$$\begin{aligned}
\widehat{\mathbf{P}} & \underset{\text{cot}}{\mathbf{F}} = \begin{vmatrix}
\overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
y & -x & e^{XZ}
\end{vmatrix} = \left(\begin{vmatrix}
\frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
-x & e^{XZ}
\end{vmatrix}, -\begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\
y & e^{XZ}
\end{vmatrix}, \begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\
y & -x
\end{vmatrix}\right)$$

$$|y -x e^{\lambda z}|$$

定义 设
$$F = (P, Q, R)$$
 是空间中向量场, 定义

$$\cot F := \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \begin{pmatrix} \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} \end{pmatrix}$$
$$= (R_V - Q_Z, P_Z - R_X, Q_X - P_Y)$$

称为向量场 F 的旋度。

例 计算向量场 $F = (v_1 - x_2)$ 的旋度。

 $= (0, -ze^{xz},)$





定义 设
$$F = (P, Q, R)$$
 是空间中向量场, 定义

$$\cot F := \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} \right)$$
$$= (R_V - Q_Z, P_Z - R_X, Q_X - P_Y)$$

称为向量场 F 的旋度。

例 计算向量场 $F = (v_1 - x_2)$ 的旋度。

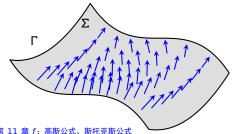
$$\frac{\mathbf{f}}{\cot F} = \begin{vmatrix}
\overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
y & -x & e^{xz}
\end{vmatrix} = \left(\begin{vmatrix}
\frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
-x & e^{xz}
\end{vmatrix}, -\begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\
y & e^{xz}
\end{vmatrix}, \begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\
y & -x
\end{vmatrix}\right)$$

$$=(0, -ze^{xz}, -2)$$

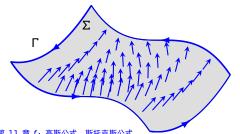


第 11 章 f: 高斯公式、斯托克斯公式

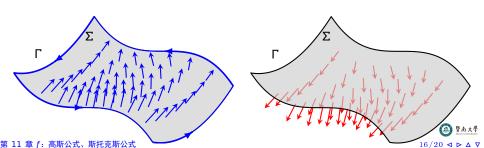
- Σ 是空间中分片光滑的定向曲面,选定单位外法向量 \overrightarrow{n} ,
- Γ 是 Σ 的边界, 且赋予 "边界定向",



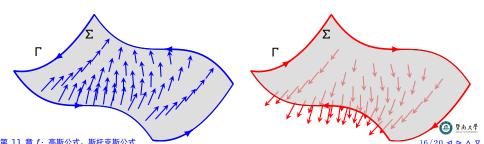
- Σ 是空间中分片光滑的定向曲面,选定单位外法向量 \overrightarrow{n} ,
- Γ 是 Σ 的边界, 且赋予 "边界定向",



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- F = (P, Q, R) 是空间向量场, 且 P, Q, R 具有一阶连续偏导数,

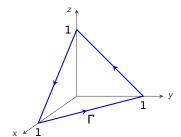


定理(斯托克斯公式) 假设

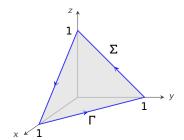
- Σ 是空间中分片光滑的定向曲面,选定单位外法向量 \overrightarrow{n} ,
- Γ是 Σ的边界, 且赋予"边界定向",
- F = (P, Q, R) 是空间向量场,且 P, Q, R 具有一阶连续偏导数,

则成立: $\iint_{\Sigma} \cot F \cdot \overrightarrow{n} \, dS = \int_{\Gamma} P dx + Q dy + R dz.$

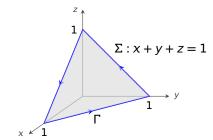
$$I = \int_{\Gamma} z dx + x dy + y dz$$



$$I = \int_{\Gamma} z dx + x dy + y dz$$

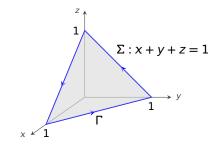


$$I = \int_{\Gamma} z dx + x dy + y dz$$



$$I = \int_{\Gamma} z dx + x dy + y dz$$

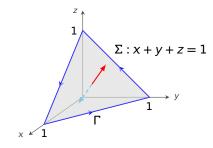
解设
$$F=(z,x,y)$$
,则



所以
$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \cot F \cdot \overrightarrow{n} dS$$

$$I = \int_{\Gamma} z dx + x dy + y dz$$

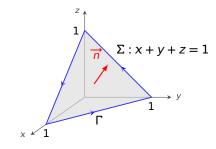
解设
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所以
$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \cot F \cdot \overrightarrow{n} dS$$

$$I = \int_{\Gamma} z dx + x dy + y dz$$

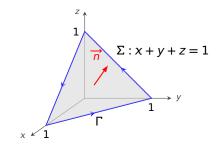
解设
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所以
$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \cot F \cdot \overrightarrow{n} dS$$

$$I = \int_{\Gamma} z dx + x dy + y dz$$

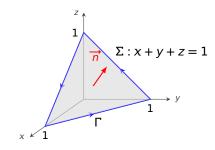
解设
$$F=(z,x,y)$$
,则



所以
$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \cot F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)}$$

$$I = \int_{\Gamma} z dx + x dy + y dz$$

$$\mathbf{H}$$
 设 $F = (z, x, y)$, 则
$$\cot F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix}$$

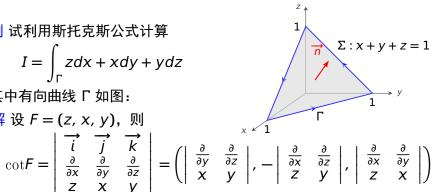


所以
$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \cot F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)}$$

$$I = \int_{\Gamma} z dx + x dy + y dz$$

解设
$$F = (z, x, y)$$
,则

$$\cot F = \begin{vmatrix} l & J & K \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & X & y \end{vmatrix} =$$



所以
$$\int zdx + xdy + ydz = \int \int \cot F \cdot \overrightarrow{n} dS = \frac{1}{\sqrt{3}}(1,1,1)$$

$$I = \int_{\Gamma} z dx + x dy + y dz$$

解设
$$F = (z, x, y)$$
,则

$$\cot F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \left(\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ z & y \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ z & x \end{vmatrix} \right)$$

所以
$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Gamma} \cot F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)}$$

$$\Sigma: x + y + z = 1$$

$$\sum_{x = 1}^{z} \sum_{y = 1}^{y} \sum_{z = 1}$$

$$I = \int_{\Gamma} z dx + x dy + y dz$$

解设
$$F = (z, x, y)$$
,则

$$\cot F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \left(\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ z & y \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ z & x \end{vmatrix} \right)$$

$$=(1, 1,)$$

所以
$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \cot F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)}$$

$$\Sigma: x + y + z = 1$$

$$X = 1$$

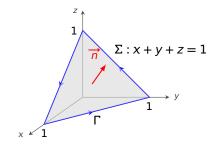
$$X$$

$$I = \int_{\Gamma} z dx + x dy + y dz$$

解设
$$F = (z, x, y)$$
,则

$$\cot F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \left(\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ z & y \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ z & x \end{vmatrix} \right)$$

$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \cot F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)}$$



$$I = \int_{\Gamma} z dx + x dy + y dz$$

解设
$$F = (z, x, y)$$
,则

$$\cot F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \left(\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ z & y \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ z & x \end{vmatrix} \right)$$

$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \cot F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)} \iint_{\Sigma} \sqrt{3} dS$$

$$\Sigma: x + y + z = 1$$

$$\frac{\partial}{\partial z} \left| - \frac{\partial}{\partial x} \frac{\partial}{\partial z} \right| \frac{\partial}{\partial x} \frac{\partial}{\partial y} \right|$$

$$I = \int_{\Gamma} z dx + x dy + y dz$$

解设
$$F = (z, x, y)$$
,则

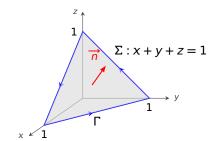
$$\cot F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \left(\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ z & y \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ z & x \end{vmatrix} \right)$$

$$=\left(\left|\begin{array}{c}\partial y\\X\end{array}\right|\right)$$

$$=(1, 1, 1)$$

所以
$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \cot F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)} \iint_{\Sigma} \sqrt{3} dS$$

$$=\sqrt{3}\mathrm{Area}(\Sigma)$$



$$I = \int_{\Gamma} z dx + x dy + y dz$$

解设
$$F = (z, x, y)$$
,则

$$\cot F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \left(\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ z & y \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ z & x \end{vmatrix} \right)$$

$$\Sigma: x + y + z = 1$$

$$\begin{bmatrix} \lambda & \lambda & \lambda \\ \lambda & \lambda \end{bmatrix} \quad \begin{bmatrix} \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda \end{bmatrix}$$

$$=(1, 1, 1)$$

所以
$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \cot F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)} \iint_{\Sigma} \sqrt{3} dS$$

$$= \sqrt{3} \operatorname{Area}(\Sigma) = \sqrt{3} \cdot \frac{1}{2} \cdot \sqrt{2} \cdot \sqrt{2} \cdot \sin \frac{\pi}{3}$$



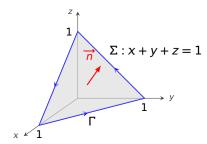
$$I = \int_{\Gamma} z dx + x dy + y dz$$

解设
$$F = (z, x, y)$$
,则

$$\cot F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \left(\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ z & y \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ z & x \end{vmatrix} \right)$$

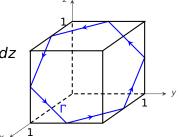
$$= \iint_{\Sigma} \cot F \cdot n' dS =$$

$$= \sqrt{3} \operatorname{Area}(\Sigma) = \sqrt{3} \cdot \frac{1}{2} \cdot \sqrt{2} \cdot \sqrt{2} \cdot \sin \frac{\pi}{3} = \frac{3}{2}$$

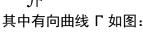


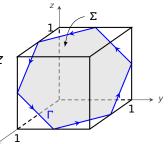
$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$

其中有向曲线 Γ 如图:



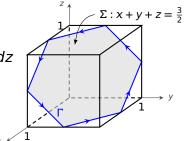
$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$





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其中有向曲线 Γ 如图:



$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$

其中有向曲线
$$\Gamma$$
 如图:
解设 $F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$, 则

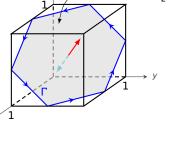
$$I = \iint_{-\infty} \cot F \cdot \overrightarrow{n} \, dS$$



$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$

解设
$$F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$
,则

所以
$$I = \iint_{-\infty} \cot F \cdot \overrightarrow{n} \, dS$$



$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$

解设
$$F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$
,则

所以
$$I = \iint_{-\infty} \cot F \cdot \overrightarrow{n} \, dS$$



 $\Sigma: x + y + z = \frac{3}{2}$

$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$

解设
$$F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$
, 则

所以
$$I = \iint_{-\infty} \cot F \cdot \overrightarrow{n} \, dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)}$$



$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$

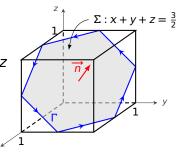
 \mathbf{H} 设 $F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$,则

其中有向曲线 Γ如图:

$$\cot F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix}$$

所以

$$I = \iint_{-\infty} \cot F \cdot \overrightarrow{n} \, dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)}$$



$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$

其中有向曲线 Γ如图:

解设
$$F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$
, 则
$$\cot F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix} = (-2y - 2z, \qquad , \qquad)$$

所以

$$I = \iint_{-\infty} \cot F \cdot \overrightarrow{n} \, dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)}$$



$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$

解设
$$F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$
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$$\cot F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix} = (-2y - 2z, -2z - 2x,$$

所以

$$I = \iint_{-\infty} \cot F \cdot \overrightarrow{n} \, dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1, 1, 1)}$$



$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$

解设
$$F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$
, 则

$$\cot F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix} = (-2y - 2z, -2z - 2x, -2x - 2y)$$

所以

$$I = \iint_{-\infty} \cot F \cdot \overrightarrow{n} \, dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1, 1, 1)}$$



$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$

其中有向曲线 Γ如图:

解设
$$F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$
, 则
$$\cot F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix} = (-2y - 2z, -2z - 2x, -2x - 2y)$$

所以

$$I = \iint_{\Gamma} \cot F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)} \frac{-4}{\sqrt{3}} \iint_{\Gamma} (x+y+z) dS$$



例 试利用斯托克斯公式计算
$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$

其中有向曲线 Γ 如图:

解设
$$F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$
, 则

$$\cot F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix} = (-2y - 2z, -2z - 2x, -2x - 2y)$$

$$I = \iint_{\Sigma} \cot F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)} \frac{-4}{\sqrt{3}} \iint_{\Sigma} (x+y+z) dS = \frac{-4}{\sqrt{3}} \iint_{\Sigma} \frac{3}{2} dS$$



解设 $F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$,则

例 试利用斯托克斯公式计算
$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$

 $\cot F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - v^2 \end{vmatrix} = (-2y - 2z, -2z - 2x, -2x - 2y)$

 $I = \iiint_{\Sigma} \cot F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)} \frac{-4}{\sqrt{3}} \iiint_{\Sigma} (x+y+z) dS = \frac{-4}{\sqrt{3}} \iiint_{\Sigma} \frac{3}{2} dS$





第 11 章 f:高斯公式、斯托克斯公式

 $=-2\sqrt{3}\mathrm{Area}(\Sigma)$

其中有向曲线 [如图:

 $I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$

解设
$$F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$
,则

解设
$$F = (y^2 - z^2, z^2)$$

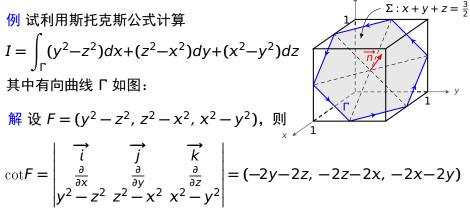
$$\cot F = \begin{bmatrix} l \\ \frac{\partial}{\partial x} \end{bmatrix}$$

$$F = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ y^2 - z^2 & z^2 - x \end{vmatrix}$$

$$y^2 \stackrel{\partial x}{-} z^2 z^2$$
f以

$$I = \iint_{\Sigma} \cot F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)} \frac{-4}{\sqrt{3}} \iint_{\Sigma} (x+y+z) dS = \frac{-4}{\sqrt{3}} \iint_{\Sigma} \frac{3}{2} dS$$

$$=-2\sqrt{3}\operatorname{Area}(\Sigma)$$





 $I = \int_{\mathbb{R}} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$

$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2)$$

其中有向曲线 [如图:

解设 $F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$,则

 $= -2\sqrt{3}\operatorname{Area}(\Sigma) = -2\sqrt{3}\cdot 6\cdot \frac{1}{2}\cdot \sqrt{\frac{1}{2}\cdot \sqrt{\frac{1}{2}\cdot \sin\frac{\pi}{3}}}$

 $\Sigma: x + y + z = \frac{3}{2}$

$$\cot F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix} = (-2y - 2z, -2z - 2x, -2x - 2y)$$
所以
$$I = \iint_{\Sigma} \cot F \cdot \overrightarrow{n} \, dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1, 1, 1)} \frac{-4}{\sqrt{3}} \iint_{\Sigma} (x + y + z) dS = \frac{-4}{\sqrt{3}} \iint_{\Sigma} \frac{3}{2} dS$$



 $I = \int_{\mathbb{R}} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$

解设 $F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$,则

 $=-2\sqrt{3}\operatorname{Area}(\Sigma)=-2\sqrt{3}\cdot 6\cdot \frac{1}{2}\cdot \sqrt{\frac{1}{2}\cdot \sqrt{\frac{1}{2}\cdot \sin\frac{\pi}{3}}}=-\frac{9}{2}$

 $I = \iint_{\Gamma} \cot F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)} \frac{-4}{\sqrt{3}} \iint_{\Gamma} (x+y+z) dS = \frac{-4}{\sqrt{3}} \iint_{\Gamma} \frac{3}{2} dS$

 $\cot F = \begin{vmatrix} \vec{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - v^2 \end{vmatrix} = (-2y - 2z, -2z - 2x, -2x - 2y)$

 $\Sigma : x + y + z = \frac{3}{2}$

其中有向曲线 [如图: