第 11 章 d:对面积的曲面积分

数学系 梁卓滨

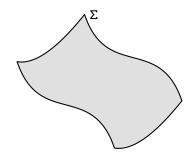
2018-2019 学年 II





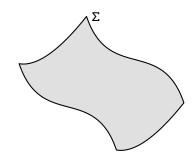
假设

- Σ 为空间中曲面
- 密度为 μ
- 质量为 m



假设

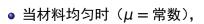
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- 当材料均匀时(μ=常数),

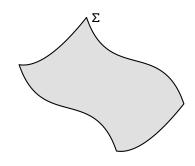


• 当材料非均匀时($\mu = \mu(x, y, z)$ 为 Σ 上函数),

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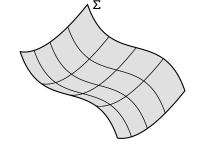


$$m = \mu \cdot Area(\Sigma)$$

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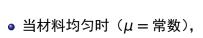


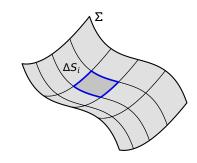
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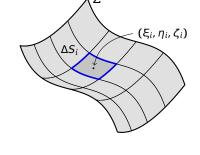




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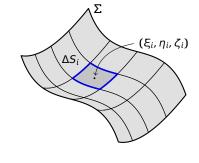


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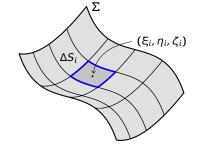
● 当材料均匀时(µ=常数),

$$m = \mu \cdot Area(\Sigma)$$

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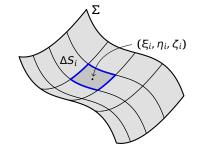
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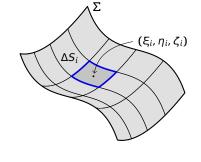
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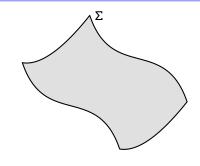
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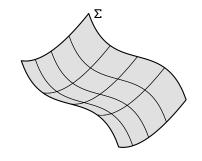
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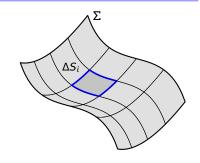
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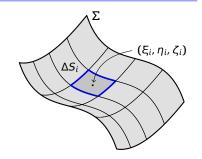
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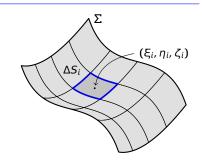
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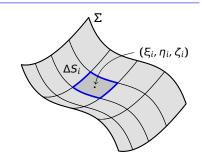
$$\sum_{i=1}^n f(\xi_i, \, \eta_i, \, \zeta_i) \Delta S_i$$



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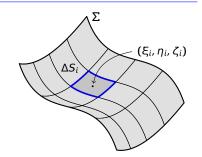


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若

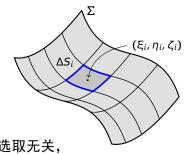
• 极限 $\lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_i, \eta_i, \zeta_i) \Delta S_i$ 存在,



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- 且该极限与 Σ 的划分、(ξ_i , η_i , ζ_i) 的选取无关,

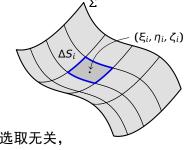


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则定义

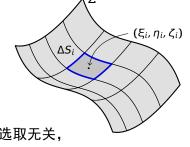
$$\iint_{\Sigma} f(x, y, z) dS = \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_i, \eta_i, \zeta_i) \Delta S_i$$

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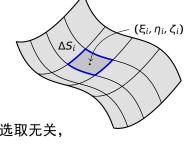
称为 f(x, y, z) 在 Σ 上对面积的曲面积分。

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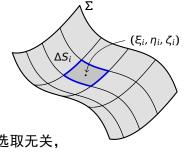
称为 f(x, y, z) 在 Σ 上对面积的曲面积分。 dS 称为面积元素。

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注 对面积曲面积分的定义式与二重积分的类似,故性质也类似

• 存在性 若 f(x, y, z) 在有界曲面 Σ 上连续,则

$$\iint_{\Sigma} f(x, y, z) dS$$

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• 线性性 $\iint_{\Sigma} (\alpha f + \beta g) dS = \alpha \iint_{\Sigma} f dS + \beta \iint_{\Sigma} g dS$

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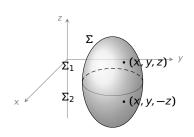
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- $\iint_{\Sigma} 1dS = \text{Area}(\Sigma)$
- 若 $f(x, y, z) \leq g(x, y, z)$,则

$$\iint_{\Sigma} f(x, y, z) dS \le \iint_{\Sigma} g(x, y, z) dS$$

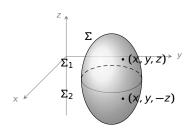


性质 设曲面 Σ 关于 xoy 坐标面对称,



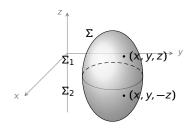
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• 若f(x, y, z) 关于z是奇函数(即: f(x, y, -z) = -f(x, y, z)),则



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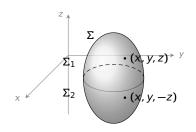


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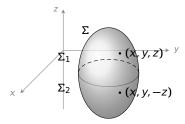
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$$\iint_{\Sigma} f(x, y, z) dS = 2 \iint_{\Sigma_1} f(x, y, z) dS = 2 \iint_{\Sigma_2} f(x, y, z) dS$$





例 设曲面 Σ 为上半球面 $x^2 + y^2 + z^2 = \alpha^2$ ($z \ge 0$); Σ_1 为 Σ 在第一卦

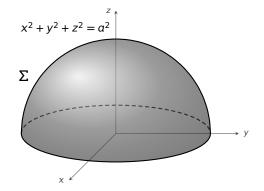
限的部分。则有(

(A)
$$\iint_{\Sigma} x dS = 4 \iint_{\Sigma_1} x dS$$

(B)
$$\iint_{\Sigma} y dS = 4 \iint_{\Sigma_1} y dS$$

(C)
$$\iint_{\Sigma} z dS = 4 \iint_{\Sigma_1} z dS$$

(D)
$$\iint_{\Sigma} xyzdS = 4 \iint_{\Sigma_1} xyzdS$$



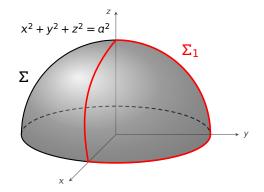
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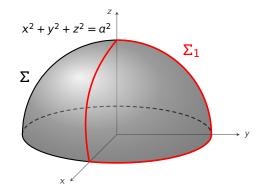
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$$\iint_{\Sigma} x^2 dS = \iint_{\Sigma} y^2 dS = \iint_{\Sigma} z^2 dS$$

解由对称性:

所以

$$\iint_{\Sigma} x^2 dS = \iint_{\Sigma} y^2 dS = \iint_{\Sigma} z^2 dS$$

$$\iint_{\Sigma} (x^2 + y^2) dS = 2 \iint_{\Sigma} x^2 dS$$

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所以
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$$= \frac{2}{3} \left[\iint_{\Sigma} x^2 dS + \iint_{\Sigma} y^2 dS + \iint_{\Sigma} z^2 dS \right]$$

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$$= \frac{2}{3} \iint_{\Sigma} x^2 + y^2 + z^2 dS$$

$$= \frac{2}{3} \iint_{\Sigma} R^2 dS$$

$$\iint_{\Sigma} x^2 dS = \iint_{\Sigma} y^2 dS = \iint_{\Sigma} z^2 dS$$

所以
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$$= \frac{2}{3} \left[\iint_{\Sigma} x^2 dS + \iint_{\Sigma} y^2 dS + \iint_{\Sigma} z^2 dS \right]$$

$$= \frac{2}{3} \iint_{\Sigma} x^2 + y^2 + z^2 dS$$

$$= \frac{2}{3} \iint_{\Sigma} R^2 dS = \frac{2}{3} R^2 \text{Area}(\Sigma)$$

$$\iint_{\Sigma} x^2 dS = \iint_{\Sigma} y^2 dS = \iint_{\Sigma} z^2 dS$$

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$$= \frac{2}{3} \iiint_{\pi} R^2 dS = \frac{2}{3} R^2 \text{Area}(\Sigma) = \frac{2}{3} R^2 \cdot 4\pi R^2$$



$$\iint_{\Sigma} x^2 dS = \iint_{\Sigma} y^2 dS = \iint_{\Sigma} z^2 dS$$

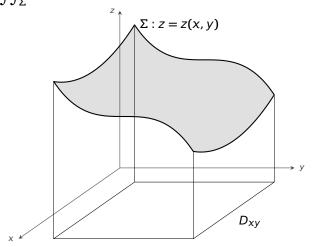
所以
$$\iint_{\mathbb{R}} (x^2 + y^2) dS = 2 \iint_{\mathbb{R}} x^2 dS$$

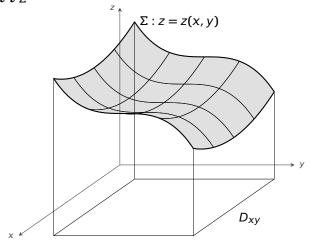
$$= \frac{2}{3} \left[\iint_{\Sigma} x^2 dS + \iint_{\Sigma} y^2 dS + \iint_{\Sigma} z^2 dS \right]$$

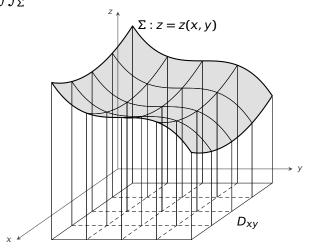
$$= \frac{2}{3} \iint_{\Sigma} x^2 + y^2 + z^2 dS$$

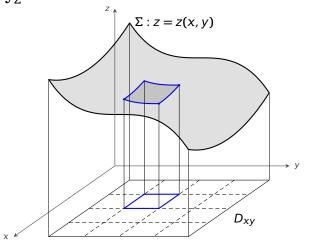
$$+z^2dS$$

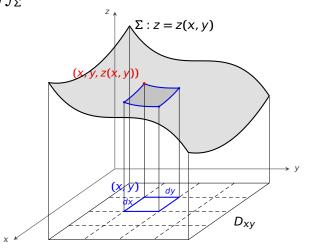
$$= \frac{2}{3} \iint_{\Sigma} R^2 dS = \frac{2}{3} R^2 \text{Area}(\Sigma) = \frac{2}{3} R^2 \cdot 4\pi R^2 = \frac{8}{3} \pi R^4$$

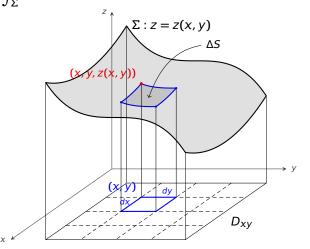


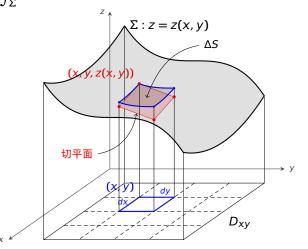


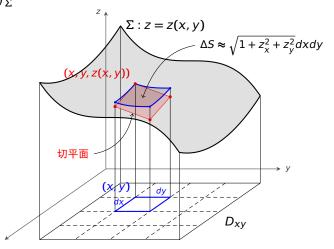


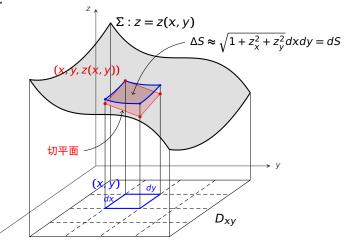




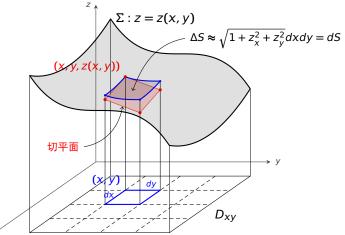




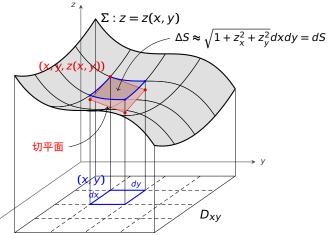




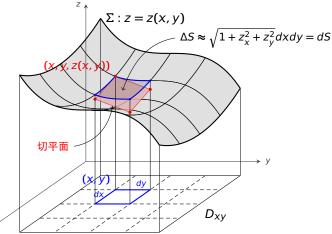
$$\iint_{\Sigma} f(x, y, z) dS = f(x, y, z(x, y)) \cdot \sqrt{1 + z_x^2 + z_y^2} dx dy$$



$$\iint_{\Sigma} f(x, y, z) dS = f(x, y, z(x, y)) \cdot \sqrt{1 + z_x^2 + z_y^2} dx dy$$

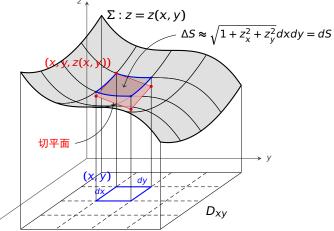


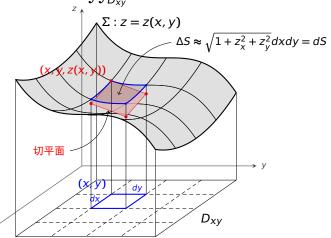
$$\iint_{\Sigma} f(x, y, z) dS = \sum f(x, y, z(x, y)) \cdot \sqrt{1 + z_{x}^{2} + z_{y}^{2}} dx dy$$





$$\iint_{\Sigma} f(x, y, z) dS = \lim_{z \to \infty} \sum_{x \to z} f(x, y, z(x, y)) \cdot \sqrt{1 + z_{x}^{2} + z_{y}^{2}} dx dy$$





$$\iint_{\Sigma} f(x, y, z) dS = \iint_{D_{xy}} f(x, y, z(x, y)) \cdot \sqrt{1 + z_x^2 + z_y^2} dx dy$$

- 假设 Σ 是二元函数 y = y(x, z), $(x, z) \in D_{xz}$ 的图形,则 $\iint_{\Sigma} f(x, y, z) dS =$
- 假设 Σ 是二元函数 x = x(y, z), $(y, z) \in D_{yz}$ 的图形,则 $\iint_{\Sigma} f(x, y, z) dS =$



- 假设 Σ 是二元函数 y = y(x, z), $(x, z) \in D_{xz}$ 的图形,则 $\iint_{\Sigma} f(x, y, z) dS = f(x, y(x, z), z)$
- 假设 Σ 是二元函数 x = x(y, z), $(y, z) \in D_{yz}$ 的图形,则 $\iint_{\Sigma} f(x, y, z) dS =$



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- 假设 Σ 是二元函数 z = z(x, y), $(x, y) \in D_{xy}$ 的图形,则 $\iint_{\Sigma} f(x, y, z) dS = \iint_{D_{xy}} f(x, y, z(x, y)) \cdot \sqrt{1 + z_x^2 + z_y^2} dx dy$
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- 假设 Σ 是二元函数 x = x(y, z), $(y, z) \in D_{yz}$ 的图形,则 $\iint_{\Sigma} f(x, y, z) dS = f(x(y, z), y, z)$



- 假设 Σ 是二元函数 z = z(x, y), $(x, y) \in D_{xy}$ 的图形,则 $\iint_{\Sigma} f(x, y, z) dS = \iint_{D_{xy}} f(x, y, z(x, y)) \cdot \sqrt{1 + z_x^2 + z_y^2} dx dy$
- 假设 Σ 是二元函数 y = y(x, z), $(x, z) \in D_{xz}$ 的图形,则 $\iint_{\Sigma} f(x, y, z) dS = \iint_{D_{xz}} f(x, y(x, z), z) \cdot \sqrt{1 + y_x^2 + y_z^2} dx dz$
- 假设 Σ 是二元函数 x = x(y, z), $(y, z) \in D_{yz}$ 的图形,则 $\iint_{\Sigma} f(x, y, z) dS = f(x(y, z), y, z) \cdot \sqrt{1 + x_y^2 + x_z^2} dy dz$



- 假设 Σ 是二元函数 z = z(x, y), $(x, y) \in D_{xy}$ 的图形,则 $\iint_{\Sigma} f(x, y, z) dS = \iint_{D_{xy}} f(x, y, z(x, y)) \cdot \sqrt{1 + z_x^2 + z_y^2} dx dy$
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• 假设 Σ 是二元函数 z = z(x, y), $(x, y) \in D_{xy}$ 的图形,则 $\iint_{\Sigma} f(x, y, z) dS = \iint_{D_{xy}} f(x, y, z(x, y)) \cdot \sqrt{1 + z_x^2 + z_y^2} dx dy$

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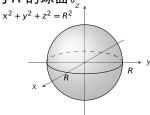
注 对于复杂的曲面 Σ ,尝试将其分解成若干部分 $\Sigma_1, \dots, \Sigma_n$,每一部分

 Σ_k 都分别是某个二元函数的图形



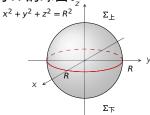
例 1 将对面积的曲面积分 $\iint_{\Sigma} f(x, y, z) dS$ 转换为重积分,其中 Σ 是球

心在原点,半径为 R 的球面。 $_{z}$



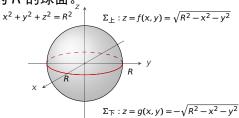
例 1 将对面积的曲面积分 $\iint_{\Sigma} f(x, y, z) dS$ 转换为重积分,其中 Σ 是球

心在原点,半径为R的球面。 $_{_{2}}$

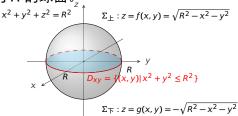


例 1 将对面积的曲面积分 $\iint_{\Sigma} f(x, y, z) dS$ 转换为重积分,其中 Σ 是球

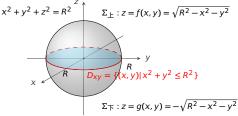
心在原点,半径为R的球面。



心在原点,半径为R的球面。



心在原点,半径为R的球面。 $_{z}$



$$\iint_{\Sigma} f(x, y, z) dS = \iint_{\Sigma_{+}} f(x, y, z) dS + \iint_{\Sigma_{\mp}} f(x, y, z) dS$$

心在原点,半径为R的球面。 $_{z}$

$$\Sigma_{\pm} : z = f(x, y) = \sqrt{R^2 - x^2 - y^2}$$

$$\sum_{\pm} : z = f(x, y) = \sqrt{R^2 - x^2 - y^2}$$

$$\sum_{\pm} : z = f(x, y) = \sqrt{R^2 - x^2 - y^2}$$

$$\sum_{\pm} : z = g(x, y) = -\sqrt{R^2 - x^2 - y^2}$$

$$\iint_{\Sigma} f(x, y, z) dS = \iint_{\Sigma_{\pm}} f(x, y, z) dS + \iint_{\Sigma_{\mp}} f(x, y, z) dS$$
$$= \iint_{D_{xy}} f(x, y, \sqrt{R^2 - x^2 - y^2}) \cdot \sqrt{1 + f_x^2 + f_y^2} dx dy$$

心在原点,半径为
$$R$$
 的球面。 $_{Z_{\pm}:z=f(x,y)=\sqrt{R^2-x^2-y^2}}$ $\Sigma_{\pm}:z=f(x,y)=\sqrt{R^2-x^2-y^2}$ $\Sigma_{\pm}:z=g(x,y)=\sqrt{R^2-x^2-y^2}$ $\Sigma_{\pm}:z=g(x,y)=\sqrt{R^2-x^2-y^2}$

$$\iint_{\Sigma} f(x, y, z) dS = \iint_{\Sigma_{\pm}} f(x, y, z) dS + \iint_{\Sigma_{\mp}} f(x, y, z) dS
= \iint_{D_{xy}} f(x, y, \sqrt{R^2 - x^2 - y^2}) \cdot \sqrt{1 + f_x^2 + f_y^2} dx dy
+ \iint_{D_{xy}} f(x, y, -\sqrt{R^2 - x^2 - y^2}) \cdot \sqrt{1 + g_x^2 + g_y^2} dx dy$$



心在原点,半径为
$$R$$
的球面。
$$z^2 + y^2 + z^2 = R^2$$

$$\sum_{\pm} : z = f(x, y) = \sqrt{R^2 - x^2 - y^2}$$

$$\sum_{\pm} : z = g(x, y) = \sqrt{R^2 - x^2 - y^2}$$

$$\sum_{\pm} : z = g(x, y) = -\sqrt{R^2 - x^2 - y^2}$$

$$\begin{split} \iint_{\Sigma} f(x, y, z) dS &= \iint_{\Sigma_{\pm}} f(x, y, z) dS + \iint_{\Sigma_{\mp}} f(x, y, z) dS \\ &= \iint_{D_{xy}} f(x, y, \sqrt{R^2 - x^2 - y^2}) \cdot \frac{R}{\sqrt{R^2 - x^2 - y^2}} dx dy \\ &+ \iint_{D} f(x, y, -\sqrt{R^2 - x^2 - y^2}) \cdot \sqrt{1 + g_x^2 + g_y^2} dx dy \end{split}$$

心在原点,半径为
$$R$$
 的球面。 $z^2 + y^2 + z^2 = R^2$ $\Sigma_{\perp} : z = f(x,y) = \sqrt{R^2 - x^2 - y^2}$ $\Sigma_{\perp} : z = g(x,y) = \sqrt{R^2 - x^2 - y^2}$

$$\iint_{\Sigma} f(x, y, z) dS = \iint_{\Sigma_{\pm}} f(x, y, z) dS + \iint_{\Sigma_{\mp}} f(x, y, z) dS
= \iint_{D_{xy}} f(x, y, \sqrt{R^2 - x^2 - y^2}) \cdot \frac{R}{\sqrt{R^2 - x^2 - y^2}} dx dy
+ \iint_{D_{xy}} f(x, y, -\sqrt{R^2 - x^2 - y^2}) \cdot \frac{R}{\sqrt{R^2 - x^2 - y^2}} dx dy$$



例 1 将对面积的曲面积分 $\iint_{\Sigma} f(x, y, z) dS$ 转换为重积分,其中 Σ 是球 心在原点,半径为R的球面。 $_{_{g}}$

$$\Sigma_{\pm} : z = f(x, y) = \sqrt{R^2 - x^2 - y^2}$$

$$\Sigma_{\pm} : z = f(x, y) = \sqrt{R^2 - x^2 - y^2}$$

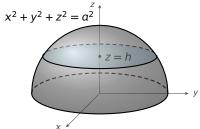
$$\Sigma_{\mp} : z = g(x, y) = -\sqrt{R^2 - x^2 - y^2}$$

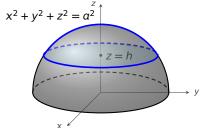
$$\iint_{\Sigma} f(x, y, z) dS = \iint_{\Sigma_{\pm}} f(x, y, z) dS + \iint_{\Sigma_{\mp}} f(x, y, z) dS$$

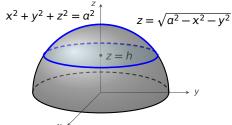
$$= \iint_{D_{xy}} f(x, y, \sqrt{R^2 - x^2 - y^2}) \cdot \frac{R}{\sqrt{R^2 - x^2 - y^2}} dx dy$$

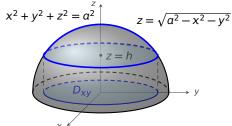
$$+ \iint_{D_{xy}} f(x, y, -\sqrt{R^2 - x^2 - y^2}) \cdot \frac{R}{\sqrt{R^2 - x^2 - y^2}} dx dy$$

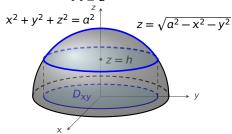
$$= \iint_{D_{xy}} f(x, y, \sqrt{R^2 - x^2 - y^2}) \cdot \frac{\pi}{\sqrt{R^2 - x^2 - y^2}} dxdy$$
$$+ \iint_{D_{xy}} f(x, y, -\sqrt{R^2 - x^2 - y^2}) \cdot \frac{R}{\sqrt{R^2 - x^2 - y^2}} dxdy$$

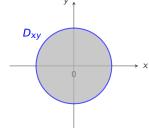


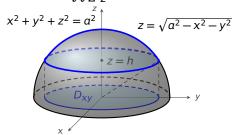


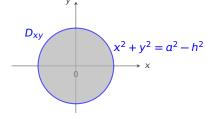


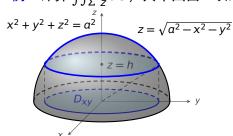


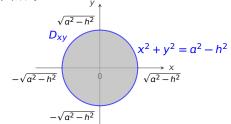


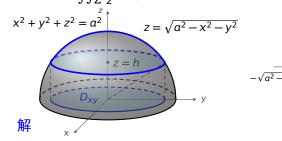


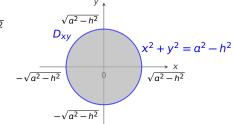






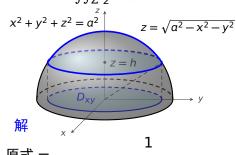




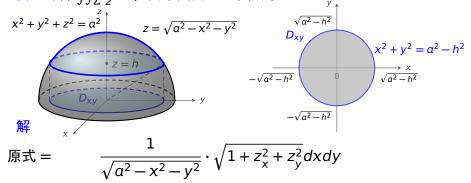


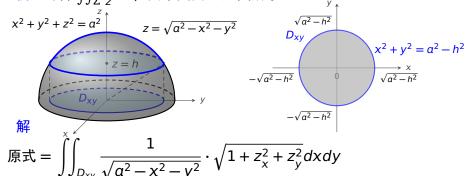
原式 =

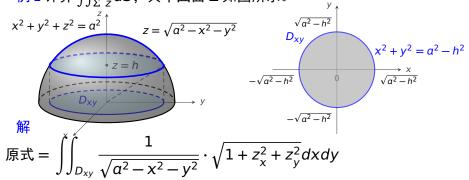




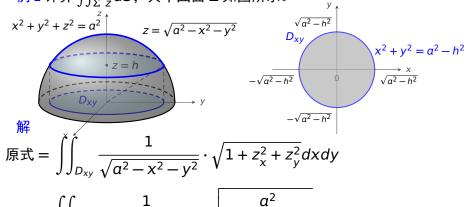
原式 = $\frac{1}{\sqrt{a^2 - x^2 - y^2}}$





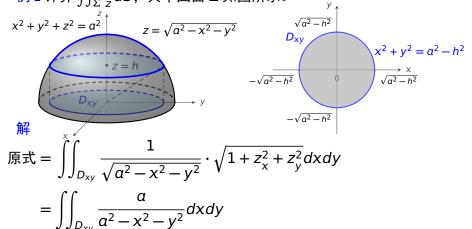


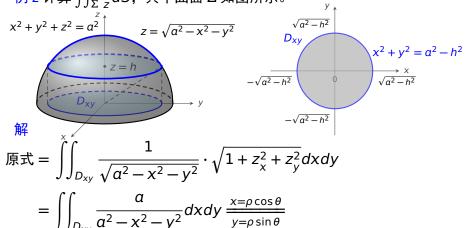
$$\cdot \sqrt{\frac{a^2}{a^2 - x^2 - y^2}}$$

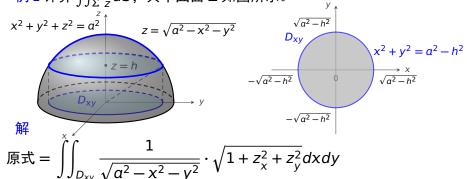


$$= \iint_{D_{xy}} \frac{1}{\sqrt{a^2 - x^2 - y^2}} \cdot \sqrt{\frac{a^2}{a^2 - x^2 - y^2}} dx dy$$



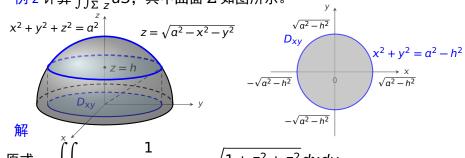






$$= \iint_{D_{xy}} \sqrt{a^2 - x^2 - y^2} dx dy \frac{x = \rho \cos \theta}{y = \rho \sin \theta} \iint_{D_{xy}} \frac{a}{a^2 - \rho^2} \cdot \rho d\rho d\theta$$

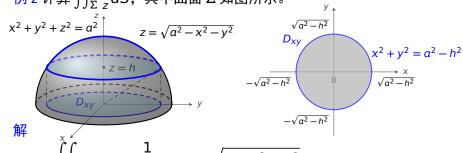
$$= \iint_{D_{xy}} \frac{d}{a^2 - x^2 - y^2} dx dy \frac{\frac{x = \rho \cos \theta}{y = \rho \sin \theta}}{\int_{D_{xy}} \frac{d}{a^2 - \rho^2} \cdot \rho d\rho d\theta}$$



原式 =
$$\iint_{D_{xy}} \frac{1}{\sqrt{a^2 - x^2 - y^2}} \cdot \sqrt{1 + z_x^2 + z_y^2} dx dy$$
$$= \iint_{D_{xy}} \frac{a}{a^2 - x^2 - y^2} dx dy \frac{x = \rho \cos \theta}{y = \rho \sin \theta} \iint_{D_{xy}} \frac{a}{a^2 - \rho^2} \cdot \rho d\rho d\theta$$

$$= \int \left[\int \frac{a}{a^2 - \rho^2} \cdot \rho d\rho \right] d\theta$$





原式 =
$$\iint_{D_{xy}} \frac{1}{\sqrt{a^2 - x^2 - y^2}} \cdot \sqrt{1 + z_x^2 + z_y^2} dx dy$$
$$= \iint_{D_{xy}} \frac{a}{a^2 - x^2 - y^2} dx dy \frac{x = \rho \cos \theta}{y = \rho \sin \theta} \iint_{D_{xy}} \frac{a}{a^2 - \rho^2} \cdot \rho d\rho d\theta$$

$$= \int_0^{2\pi} \left[\int \frac{\alpha}{\alpha^2 - \rho^2} \cdot \rho d\rho \right] d\theta$$



$$x^{2} + y^{2} + z^{2} = a^{2}$$

$$z = \sqrt{a^{2} - x^{2} - y^{2}}$$

$$pxy$$

$$z = h$$

$$-\sqrt{a^{2} - h^{2}}$$

$$Dxy$$

$$x^{2} + y^{2} = a^{2} - h^{2}$$

$$x^{2} + y^{2} = a^{2} - h^{2}$$

$$x^{2} + y^{2} = a^{2} - h^{2}$$

$$\sqrt{a^{2} - h^{2}}$$

$$\sqrt$$

$$= \int_0^{2\pi} \left[\int_0^{\sqrt{\alpha^2 - h^2}} \frac{\alpha}{\alpha^2 - \rho^2} \cdot \rho d\rho \right] d\theta$$



$$x^{2} + y^{2} + z^{2} = a^{2}$$

$$z = \sqrt{a^{2} - x^{2} - y^{2}}$$

$$z = \sqrt{a^{2} - x^{2} - y^{2}}$$

$$x^{2} + y^{2} + z^{2} = a^{2} - h^{2}$$

$$x^{2} + y^{2} = a^{2} - h^{2}$$

原式 =
$$\iint_{D_{xy}} \frac{1}{\sqrt{a^2 - x^2 - y^2}} \cdot \sqrt{1 + z_x^2 + z_y^2} dx dy$$

$$= \iint_{D_{xy}} \frac{a}{a^2 - x^2 - y^2} dx dy \frac{x = \rho \cos \theta}{y = \rho \sin \theta} \iint_{D_{xy}} \frac{a}{a^2 - \rho^2} \cdot \rho d\rho d\theta$$

$$= \int_0^{2\pi} \left[\int_0^{\sqrt{a^2 - h^2}} \frac{a}{a^2 - \rho^2} \cdot \rho d\rho \right] d\theta = 2\pi.$$

$$x^{2} + y^{2} + z^{2} = a^{2}$$

$$z = \sqrt{a^{2} - x^{2} - y^{2}}$$

$$z = \sqrt{a^{2} - x^{2} - y^{2}}$$

$$y$$

$$D_{xy}$$

$$x^{2} + y^{2} = a^{2} - h^{2}$$

$$\sqrt{a^{2} - h^{2}}$$

原式 =
$$\iint_{D_{xy}} \frac{1}{\sqrt{a^2 - x^2 - y^2}} \cdot \sqrt{1 + z_x^2 + z_y^2} dx dy$$
$$= \iint_{D_{xy}} \frac{a}{a^2 - x^2 - y^2} dx dy \frac{x = \rho \cos \theta}{y = \rho \sin \theta} \iint_{D_{xy}} \frac{a}{a^2 - \rho^2} \cdot \rho d\rho d\theta$$

$$= \int_0^{2\pi} \left[\int_0^{\sqrt{\alpha^2 - h^2}} \frac{a}{\alpha^2 - \rho^2} \cdot \rho d\rho \right] d\theta \xrightarrow{u = \alpha^2 - \rho^2} 2\pi \cdot$$



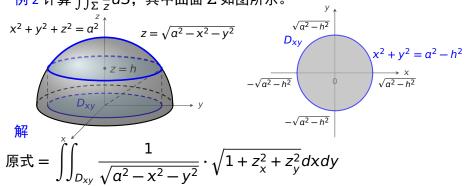
解
$$z = \sqrt{a^2 - x^2 - y^2}$$
 $z = \sqrt{a^2 - x^2 - y^2}$ $z = \sqrt{a^2 - h^2}$ $z = \sqrt{a^2 - h^2}$

原式 =
$$\iint_{D_{xy}} \frac{1}{\sqrt{a^2 - x^2 - y^2}} \cdot \sqrt{1 + z_x^2 + z_y^2} dx dy$$

$$= \iint_{D_{xy}} \frac{a}{a^2 - x^2 - y^2} dx dy \frac{x = \rho \cos \theta}{y = \rho \sin \theta} \iint_{D_{xy}} \frac{a}{a^2 - \rho^2} \cdot \rho d\rho d\theta$$

$$= \int_0^{2\pi} \left[\int_0^{\sqrt{a^2 - h^2}} \frac{a}{a^2 - \rho^2} \cdot \rho d\rho \right] d\theta \xrightarrow{u = a^2 - \rho^2} 2\pi \cdot \frac{a}{u}$$





$$\iint_{D_{xy}} \sqrt{a^2 - x^2 - y^2} \sqrt{x} \sqrt{x}$$

$$= \iint_{D_{xy}} \frac{a}{a^2 - x^2 - y^2} dx dy \frac{x = \rho \cos \theta}{y = \rho \sin \theta} \iint_{D_{xy}} \frac{a}{a^2 - \rho^2} \cdot \rho d\rho d\theta$$

$$= \int_{D_{xy}}^{2\pi} \frac{a^2 - x^2 - y^2}{a^2 - x^2 - y^2} \frac{dxdy}{y = \rho \sin \theta} \int_{D_{xy}}^{2\pi} \frac{a^2 - \rho^2}{a^2 - \rho^2} \frac{d\rho d\theta}{d\theta}$$
$$= \int_{0}^{2\pi} \left[\int_{0}^{\sqrt{a^2 - h^2}} \frac{a}{a^2 - \rho^2} \cdot \rho d\rho \right] d\theta \xrightarrow{u = a^2 - \rho^2} 2\pi \cdot \frac{a}{u} \cdot (-\frac{1}{2}) du$$



解
原式 =
$$\int_{D_{xy}}^{x^2 + y^2 + z^2} \int_{D_{xy}}^{z^2 - y^2} \int_{-\sqrt{a^2 - h^2}}^{\sqrt{a^2 - h^2}} \int_{D_{xy}}^{x^2 + y^2 = a^2 - h^2}$$

$$\iint_{D_{xy}} \sqrt{a^2 - x^2 - y^2} \sqrt{x} \sqrt{x}$$

$$= \iint_{D_{xy}} \frac{a}{a^2 - x^2 - y^2} dx dy \frac{x = \rho \cos \theta}{y = \rho \sin \theta} \iint_{D_{xy}} \frac{a}{a^2 - \rho^2} \cdot \rho d\rho d\theta$$

$$= \int_{D_{xy}}^{2\pi} \frac{a^2 - x^2 - y^2}{a^2 - x^2 - y^2} dx dy \frac{1}{y = \rho \sin \theta} \int_{D_{xy}}^{2\pi} \frac{1}{a^2 - \rho^2} d\rho d\theta$$

$$= \int_{0}^{2\pi} \left[\int_{0}^{\sqrt{a^2 - h^2}} \frac{a}{a^2 - \rho^2} \cdot \rho d\rho \right] d\theta \frac{u = a^2 - \rho^2}{a^2 - \rho^2} 2\pi \cdot \int_{a^2}^{h^2} \frac{a}{u} \cdot (-\frac{1}{2}) du$$



例 2 计算 $\iint_{\Sigma} \frac{1}{z} dS$, 其中曲面 Σ 如图所示。 $x^2 + y^2 + z^2 = a^2 \uparrow$ $\sqrt{a^2 - h^2}$ $z = \sqrt{a^2 - x^2 - y^2}$ $x^2 + v^2 = a^2 - h^2$

$$z = \sqrt{a^2 - x^2 - y^2}$$

$$z = \sqrt{a^2 - x^2 - y^2}$$

$$x^2 + y^2 = a^2 - h$$

$$x^2 + y^2 = a^2 - h$$

$$\sqrt{a^2 - h^2}$$

$$\sqrt{a^2 - h^2}$$

$$\sqrt{a^2 - h^2}$$

$$\sqrt{1 + z_x^2 + z_y^2} dx dy$$

$$= \iint_{D_{xy}} \frac{a}{a^2 - x^2 - y^2} dx dy \xrightarrow{\frac{x = \rho \cos \theta}{y = \rho \sin \theta}} \iint_{D_{xy}} \frac{a}{a^2 - \rho^2} \cdot \rho d\rho d\theta$$

$$= \int_0^{2\pi} \left[\int_0^{\sqrt{a^2 - h^2}} \frac{a}{a^2 - \rho^2} \cdot \rho d\rho \right] d\theta \xrightarrow{u = a^2 - \rho^2} 2\pi \cdot \int_{a^2}^{h^2} \frac{a}{u} \cdot (-\frac{1}{2}) du$$

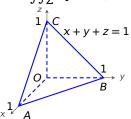
$$= -\pi \alpha \ln u \Big|_{a^2}^{h^2}$$

例 2 计算 $\iint_{\Sigma} \frac{1}{z} dS$, 其中曲面 Σ 如图所示。 $x^2 + y^2 + z^2 = a^2$ $z = \sqrt{a^2 - x^2 - y^2}$ D_{xy} $x^2 + y^2 = a^2 - h^2$ $-\sqrt{a^2 - h^2}$ $\sqrt{a^2 - h^2}$

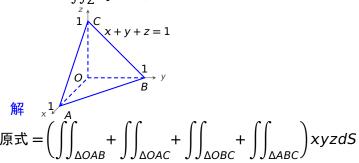
 $= \iint_{D_{xy}} \frac{a}{a^2 - x^2 - y^2} dx dy \xrightarrow{\frac{x = \rho \cos \theta}{y = \rho \sin \theta}} \iint_{D_{xy}} \frac{a}{a^2 - \rho^2} \cdot \rho d\rho d\theta$ $= \int_0^{2\pi} \left[\int_0^{\sqrt{a^2 - h^2}} \frac{a}{a^2 - \rho^2} \cdot \rho d\rho \right] d\theta \xrightarrow{u = a^2 - \rho^2} 2\pi \cdot \int_{a^2}^{h^2} \frac{a}{u} \cdot (-\frac{1}{2}) du$

 $= -\pi a \ln u \Big|_{a^2}^{h^2} = 2\pi a \ln \frac{a}{h}$

例 3 计算 $\iint_{\Sigma} xyzdS$, 其中曲面 Σ 如图所示。



例 3 计算 $\iint_{\Sigma} xyzdS$, 其中曲面 Σ 如图所示。

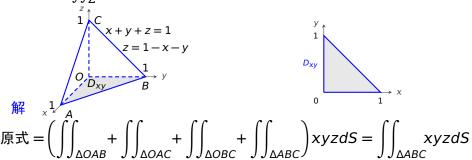


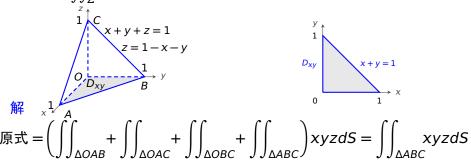
$$\mathbf{p}$$
 展式 = $\left(\iint_{\Delta OAB} + \iint_{\Delta OAC} + \iint_{\Delta OBC} + \iint_{\Delta ABC} xyzdS = \iint_{\Delta ABC} xyzdS \right)$

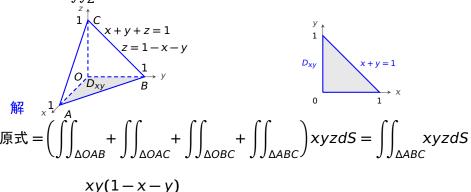
原式 =
$$\left(\iint_{\Delta OAB} + \iint_{\Delta OAC} + \iint_{\Delta OBC} + \iint_{\Delta ABC} xyzdS = \iint_{\Delta ABC} xyzdS$$

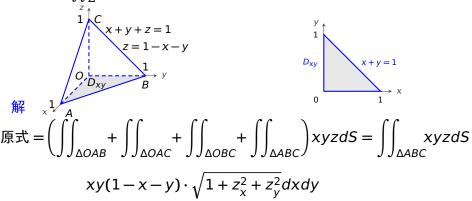
$$x + y + z = 1$$
 $z = 1 - x - y$

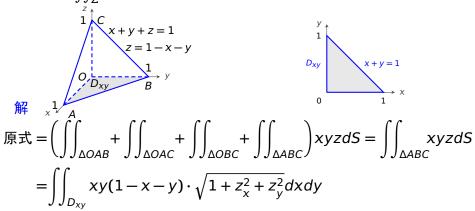
原式 = $\left(\iint_{\Delta OAB} + \iint_{\Delta OAC} + \iint_{\Delta OBC} + \iint_{\Delta ABC} \right) xyzdS = \iint_{\Delta ABC} xyzdS$

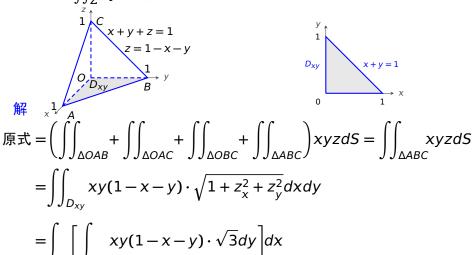




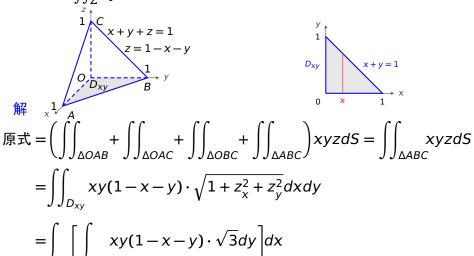




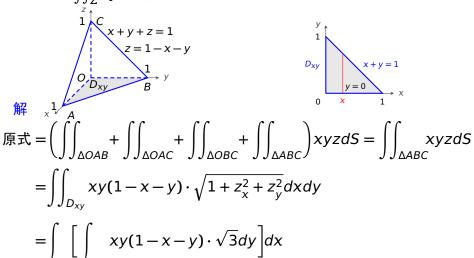




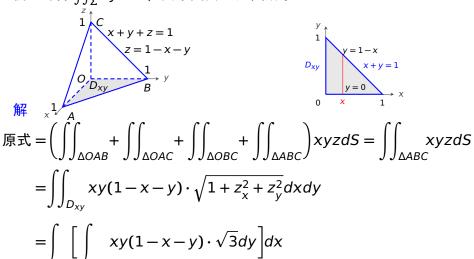




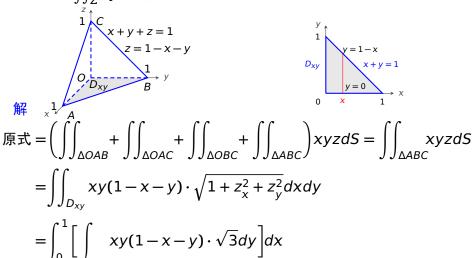




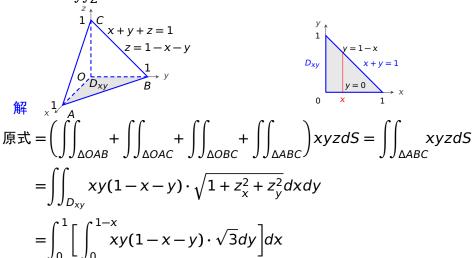












解
$$x^{1}A$$

原式 = $\left(\iint_{\Delta OAB} + \iint_{\Delta OAC} + \iint_{\Delta OBC} + \iint_{\Delta ABC}\right) xyzdS = \iint_{\Delta ABC} xyzdS$

$$= \iint_{\Delta OAB} \int_{\Delta OAC} \int_{\Delta OBC} \int_{\Delta OBC} \int_{\Delta ABC} \int_{\Delta ABC} \int_{\Delta ABC} \int_{\Delta OBC} \int_$$



解
$$x^{1}A$$

原式 = $\left(\iint_{\Delta OAB} + \iint_{\Delta OAC} + \iint_{\Delta OBC} + \iint_{\Delta ABC}\right) xyzdS = \iint_{\Delta ABC} xyzdS$

$$= \iint_{D_{xy}} xy(1-x-y) \cdot \sqrt{1+z_x^2+z_y^2} dx dy$$

$$= \int_0^1 \left[\int_0^{1-x} xy(1-x-y) \cdot \sqrt{3} dy \right] dx$$

$$= x \left[(1-x) \frac{y^2}{2} - \frac{1}{3} y^3 \right]_0^{1-x}$$



解
$$x^{1}$$
 A

原式 = $\left(\iint_{\Delta OAB} + \iint_{\Delta OAC} + \iint_{\Delta OBC} + \iint_{\Delta ABC}\right) xyzdS = \iint_{\Delta ABC} xyzdS$

$$= \iint_{D_{xy}} xy(1-x-y) \cdot \sqrt{1+z_{x}^{2}+z_{y}^{2}} dxdy$$

$$= \int_{0}^{1} \left[\int_{0}^{1-x} xy(1-x-y) \cdot \sqrt{3} dy\right] dx$$

$$= \sqrt{3} \int_{0}^{1} x \left[(1-x)\frac{y^{2}}{2} - \frac{1}{3}y^{3}\right]_{0}^{1-x} dx$$



$$= \iint_{D_{xy}} xy(1-x-y) \cdot \sqrt{1+z_x^2+z_y^2} dxdy$$
$$= \int_0^1 \left[\int_0^{1-x} xy(1-x-y) \cdot \sqrt{3} dy \right] dx$$

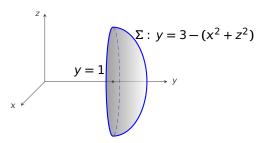
 $= \sqrt{3} \int_{0}^{1} x \left[(1-x) \frac{y^{2}}{2} - \frac{1}{3} y^{3} \right]_{0}^{1-x} dx = \sqrt{3} \int_{0}^{1} \frac{1}{6} x (1-x)^{3} dx$

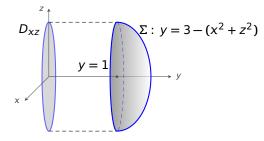
 $=\sqrt{3}\int_{0}^{1}x\left[(1-x)\frac{y^{2}}{2}-\frac{1}{3}y^{3}\right]_{0}^{1-x}dx=\sqrt{3}\int_{0}^{1}\frac{1}{6}x(1-x)^{3}dx=\frac{\sqrt{3}}{120}$

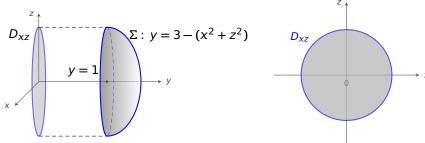
$$= \iint_{D_{xy}} xy(1-x-y) \cdot \sqrt{1+z_{x}^{2}+z_{y}^{2}} dx dy$$
$$= \int_{0}^{1} \left[\int_{0}^{1-x} xy(1-x-y) \cdot \sqrt{3} dy \right] dx$$

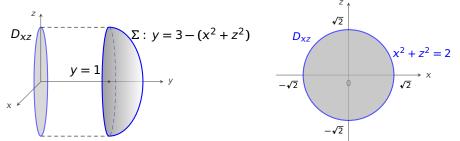
11章 d:对面积的曲面积分

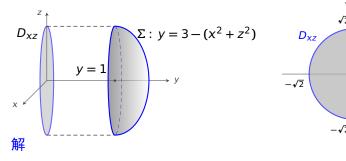
13 ⊲ ⊳ Δ

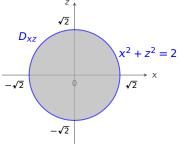




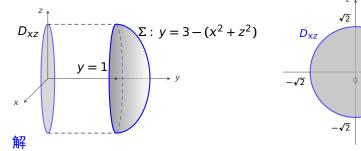


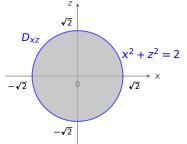


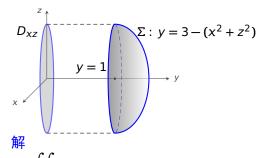


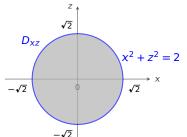


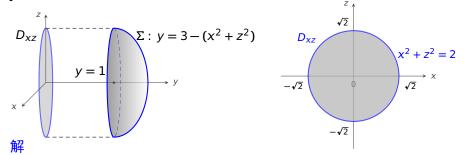
I =



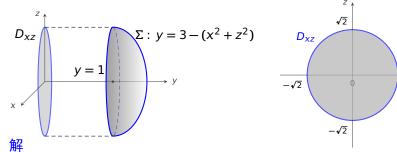








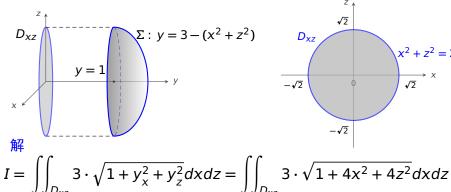
 $I = \iint_{D_{xz}} 3 \cdot \sqrt{1 + y_x^2 + y_z^2} dx dz = \iint_{D_{xz}} 3 \cdot \sqrt{1 + 4x^2 + 4z^2} dx dz$



 $I = \iint_{D_{xz}} 3 \cdot \sqrt{1 + y_x^2 + y_z^2} dx dz = \iint_{D_{xz}} 3 \cdot \sqrt{1 + 4x^2 + 4z^2} dx dz$ $\underbrace{x = \rho \cos \theta}$

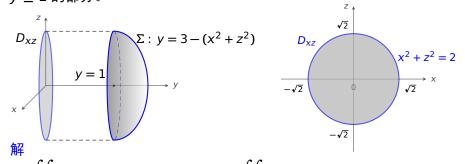
$$\frac{x=\rho\cos\theta}{z=\rho\sin\theta}$$





$$\frac{x = \rho \cos \theta}{z = \rho \sin \theta} \iint_{D_{\text{cur}}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho d\theta$$

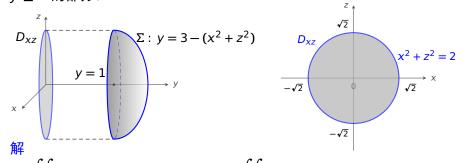




$$I = \iint_{D_{xz}} 3 \cdot \sqrt{1 + y_x^2 + y_z^2} dx dz = \iint_{D_{xz}} 3 \cdot \sqrt{1 + 4x^2 + 4z^2} dx dz$$

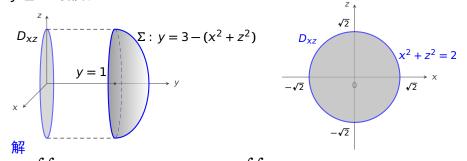
$$\frac{x = \rho \cos \theta}{z = \rho \sin \theta} \iint_{D_{xz}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho d\theta = \int_{D_{xz}} \left[\int_{D_{xz}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho \right] d\theta$$





$$I = \iint_{D_{XZ}} 3 \cdot \sqrt{1 + y_X^2 + y_Z^2} dx dz = \iint_{D_{XZ}} 3 \cdot \sqrt{1 + 4x^2 + 4z^2} dx dz$$

$$\frac{x = \rho \cos \theta}{z = \rho \sin \theta} \iint_{D_{XZ}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho d\theta = \int_{0}^{2\pi} \left[\int_{0}^{2\pi} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho \right] d\theta$$



$$I = \iint_{D_{xz}} 3 \cdot \sqrt{1 + y_x^2 + y_z^2} dx dz = \iint_{D_{xz}} 3 \cdot \sqrt{1 + 4x^2 + 4z^2} dx dz$$

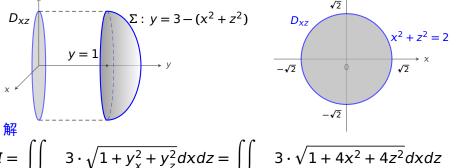
$$\frac{x = \rho \cos \theta}{z = \rho \sin \theta} \iint_{D_{xz}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho d\theta = \int_{0}^{2\pi} \left[\int_{0}^{\sqrt{2}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho \right] d\theta$$

$$D_{XZ}$$

$$y = 1$$

$$y =$$





$$\mathbf{H}$$

$$I = \iint_{D_{xz}} 3 \cdot \sqrt{1 + y_x^2 + y_z^2} dx dz = \iint_{D_{xz}} 3 \cdot \sqrt{1 + 4x^2 + 4z^2} dx dz$$

$$\underline{x = \rho \cos \theta} \left[\int_{0}^{\infty} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho d\theta = \int_{0}^{2\pi} \left[\int_{0}^{\sqrt{2}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho \right] d\theta$$

$$\mathbf{R}$$

$$I = \iint_{D_{xz}} 3 \cdot \sqrt{1 + y_x^2 + y_z^2} dx dz = \iint_{D_{xz}} 3 \cdot \sqrt{1 + 4x^2 + 4z^2} dx dz$$

$$\frac{x = \rho \cos \theta}{z = \rho \sin \theta} \iint_{D_{xz}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho d\theta = \int_{0}^{2\pi} \left[\int_{0}^{\sqrt{2}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho \right] d\theta$$

$$\frac{z=\rho\sin\theta}{I} \int J_{Dxz} \qquad J_0 \quad L J_0 \qquad J_0$$

$$\frac{u=1+4\rho^2}{I} = 2\pi \cdot I_0$$

例 4 计算 $I = \iint_{\Sigma} (x^2 + z^2 + y) dS$, 其中 Σ 是曲面 $y = 3 - (x^2 + z^2)$ 在

$$\sum : y = 3 - (x^{2} + z^{2})$$

$$y = 1$$

 $I = \iint_{D} 3 \cdot \sqrt{1 + y_x^2 + y_z^2} dx dz = \iint_{D} 3 \cdot \sqrt{1 + 4x^2 + 4z^2} dx dz$

 $\frac{x = \rho \cos \theta}{z = \rho \sin \theta} \iint_{D} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho d\theta = \int_{0}^{2\pi} \left[\int_{0}^{\sqrt{2}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho \right] d\theta$



$$y = 1$$

$$-\sqrt{2}$$

$$= \iint_{D_{XZ}} 3 \cdot \sqrt{1 + y_X^2 + y_Z^2} dx dz = \iint_{D_{XZ}} 3 \cdot \sqrt{1 + 4x^2 + 4z^2} dx dz$$

$$\mathbf{H}$$

$$I = \iint_{D_{xz}} 3 \cdot \sqrt{1 + y_x^2 + y_z^2} dx dz = \iint_{D_{xz}} 3 \cdot \sqrt{1 + 4x^2 + 4z^2} dx dz$$

$$\frac{x = \rho \cos \theta}{z = \rho \sin \theta} \iint_{D_{xz}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho d\theta = \int_0^{2\pi} \left[\int_0^{\sqrt{2}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho \right] d\theta$$

$$\frac{z=\rho\sin\theta}{J} \int_{D_{xz}} \int_{D_{xz$$

$$\frac{x = \rho \cos \theta}{z = \rho \sin \theta} \iint_{D_{xz}} 3\sqrt{1 + 4\rho^2 \cdot \rho d\rho} d\theta = \int_0^{\infty} \left[\int_0^{\infty} 3\sqrt{1 + 4\rho^2 \cdot \rho d\rho} \right] d\theta$$

$$\frac{u = 1 + 4\rho^2}{2\pi \cdot \rho^2} 2\pi \cdot \left[\int_0^{\infty} 3\sqrt{u} \cdot \frac{1}{8} du \right] d\theta$$

$$\int_{D_{XZ}} 3 \cdot \sqrt{1 + y_X^2 + y_Z^2} dx dz = \iint_{D_{XZ}} 3 \cdot \sqrt{1 + 4x^2 + 4z^2} dx dz$$

$$\mathbf{H}$$

$$I = \iint_{D_{xz}} 3 \cdot \sqrt{1 + y_x^2 + y_z^2} dx dz = \iint_{D_{xz}} 3 \cdot \sqrt{1 + 4x^2 + 4z^2} dx dz$$

$$\frac{x = \rho \cos \theta}{1 - 2\sin \theta} \iint_{D_{xz}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho d\theta = \int_{D_{xz}}^{2\pi} \left[\int_{0}^{\sqrt{2}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho \right] d\theta$$

$$\begin{aligned}
&= \iint_{D_{XZ}} 3 \cdot \sqrt{1 + y_X^2 + y_Z^2} dx dz = \iint_{D_{XZ}} 3 \cdot \sqrt{1 + 4x^2 + 4z^2} dx dz \\
&= \underbrace{\sum_{z=\rho \sin \theta}^{x=\rho \cos \theta}} \iint_{D_{XZ}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho d\theta = \int_{0}^{2\pi} \left[\int_{0}^{\sqrt{2}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho \right] d\theta
\end{aligned}$$

$$\frac{x = \rho \cos \theta}{z = \rho \sin \theta} \iint_{D_{xz}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho d\theta = \int_0^{2\pi} \left[\int_0^{\sqrt{2}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho \right] d\theta$$

$$\frac{u = 1 + 4\rho^2}{2\pi} \left[2\pi \cdot \int_0^9 3\sqrt{u} \cdot \frac{1}{8} du \right]_0^9 d\theta$$

$$\frac{u = 1 + 4\rho^2}{2\pi} \left[\frac{1}{2} \pi u^{\frac{3}{2}} \right]_0^9 d\theta$$

例 4 计算 $I = \int_{\Sigma} (x^2 + z^2 + y) dS$,其中 Σ 是曲面 $y = 3 - (x^2 + z^2)$ 在

 $\frac{x = \rho \cos \theta}{z = \rho \sin \theta} \iint_{D_{x,z}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho d\theta = \int_{0}^{2\pi} \left[\int_{0}^{\sqrt{2}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho \right] d\theta$

$$\frac{u=1+4\rho^2}{2\pi} 2\pi \cdot \int_1^9 3\sqrt{u} \cdot \frac{1}{8} du = \frac{1}{2}\pi u^{\frac{3}{2}} \Big|_1^9 = 13\pi$$