第 4 章 d：有理函数的积分

数学系　　梁卓滨

2019-2020 学年 I

Outline

有理函数

定义 设

$$P(x) = p_0 x^n + p_1 x^{n-1} + \cdots + p_{n-1} x + p_n$$

$$Q(x) = q_0 x^m + q_1 x^{m-1} + \cdots + q_{m-1} x + q_m$$

为多项式，并且没有公共零点（即 $P(x)$，$Q(x)$ 没有公因式），则

$$R(x) := \frac{P(x)}{Q(x)}$$

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真分式 $$\frac{4x+1}{x^2-6x+9}$$

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问题 本节讨论如下形式的有函数的积分：

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- $\Delta > 0 \Rightarrow x^2+px+q = (x-a)(x-b)$,
- $\Delta = 0 \Rightarrow x^2+px+q = (x-a)^2$,
- $\Delta < 0 \Rightarrow x^2+px+q$ 不可约,

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$$\frac{4x+1}{x^2-6x+9} = \frac{x+1}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2} \Rightarrow A = 4,\ B = 13$$

$\Delta = 0$ 情形

例 3 求不定积分 $\int \frac{4x+1}{x^2-6x+9} dx.$

$$\frac{A(x-3)+B}{(x-3)^2} = \frac{Ax-3A+B}{(x-3)^2} \Rightarrow \begin{cases} A = 4 \\ -3A + B = 1 \end{cases}$$

解

$$\frac{4x+1}{x^2-6x+9} = \frac{x+1}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2} \Rightarrow A = 4,\ B = 13$$

所以

$$原式 = 4\int \frac{1}{x-3} dx + 13 \int \frac{1}{(x-3)^2} dx$$

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$$原式 = 4\int \frac{1}{x-3}dx + 13\int \frac{1}{(x-3)^2}dx = 4\ln|x-3| - \frac{13}{x-3} + C$$

$\Delta < 0$ 情形

例 4 求不定积分 $\int \frac{x-3}{x^2-x+1} dx.$

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例 4 求不定积分 $\int \frac{x-3}{x^2-x+1}dx.$

解

$$\frac{x-3}{x^2-x+1} = \frac{A(x^2-x+1)'}{x^2-x+1} + \frac{B}{x^2-x+1}$$

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例 4 求不定积分 $\int \frac{x-3}{x^2-x+1} dx.$

$\frac{2Ax-A+B}{x^2-x+1}$

解

$$\frac{x-3}{x^2-x+1} = \frac{A(x^2-x+1)'}{x^2-x+1} + \frac{B}{x^2-x+1}$$

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例 4 求不定积分 $\int \frac{x-3}{x^2-x+1}dx$.

$\frac{2Ax-A+B}{x^2-x+1} \Rightarrow \begin{cases} 2A = 1 \\ -A + B = -3 \end{cases}$

解

$$\frac{x-3}{x^2-x+1} = \frac{A(x^2-x+1)'}{x^2-x+1} + \frac{B}{x^2-x+1}$$

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$$\text{原式} = \frac{1}{2}\int \frac{(x^2-x+1)'}{x^2-x+1}dx - \frac{5}{2}\int \frac{1}{x^2-x+1}dx$$

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$$\int \frac{(x^2-x+1)'}{x^2-x+1}dx = \int \frac{d(x^2-x+1)}{x^2-x+1}$$

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其中

$$\int \frac{1}{x^2 - x + 1} dx = \int \frac{1}{(x - \frac{1}{2})^2 + \frac{3}{4}} dx$$

$$\int \frac{1}{t^2 + 1} dt$$

其中

$$\int \frac{1}{x^2-x+1}dx = \int \frac{1}{(x-\frac{1}{2})^2+\frac{3}{4}}dx = \frac{4}{3}\int \frac{1}{\left(\frac{2}{\sqrt{3}}x-\frac{1}{\sqrt{3}}\right)^2+1}dx$$

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所以

$$原式 = \frac{1}{2}\int \frac{(x^2-x+1)'}{x^2-x+1}dx - \frac{5}{2}\int \frac{1}{x^2-x+1}dx$$

$$= \frac{1}{2}\ln|x^2-x+1| - \frac{5}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}x-\frac{1}{\sqrt{3}}\right)+C$$

例 5 求不定积分 $\int \frac{5x+6}{x^2+x+1} dx.$

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$$\frac{5x+6}{x^2+x+1} = \frac{A(x^2+x+1)'}{x^2+x+1} + \frac{B}{x^2+x+1}$$

例 5 求不定积分 $\int \frac{5x+6}{x^2+x+1}dx.$

$\frac{2Ax+A+B}{x^2+x+1}$

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例 5 求不定积分 $\int \frac{5x+6}{x^2+x+1}dx$.

$$\frac{2Ax+A+B}{x^2+x+1} \Rightarrow \begin{cases} 2A = 5 \\ A+B = 6 \end{cases}$$

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