

向量组的极大无关组

$$\alpha_1, \alpha_2, \dots, \alpha_s$$

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$\alpha_1, \alpha_2, \dots, \alpha_s$ $\xrightarrow[\text{能被其余向量线性表示的向量}]{\text{逐个剔除}}$

向量组的极大无关组

$\alpha_1, \alpha_2, \dots, \alpha_s$ $\xrightarrow[\text{直到不能再剔除为止}]{\begin{array}{c} \text{逐个剔除} \\ \text{能被其余向量线性表示的向量} \end{array}}$

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例 求 $\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ 的一个极大无关组。

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解

$\alpha_1, \alpha_2, \alpha_3, \alpha_4$

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$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\text{剔除 } \alpha_4}$

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$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow[\text{剔除}\alpha_4]{\alpha_4 = 2\alpha_1 + 0\alpha_2 + 0\alpha_3}$$

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$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\text{剔除}\alpha_1}$

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$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\hspace{2cm}} \hspace{2cm}$$

剔除 α_2

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例 求 $\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ 的一个极大无关组。

解

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow[\text{剔除}\alpha_4]{\alpha_4 = 2\alpha_1 + 0\alpha_2 + 0\alpha_3} \alpha_1, \alpha_2, \alpha_3 \xrightarrow[\text{剔除}\alpha_3]{\alpha_3 = \alpha_1 + \alpha_2} \alpha_1, \alpha_2 \quad \begin{matrix} \text{极大} \\ \text{无关组} \end{matrix}$$

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow[\text{剔除}\alpha_1]{\alpha_1 = -\alpha_2 + \alpha_3 + 0\alpha_4} \alpha_2, \alpha_3, \alpha_4 \xrightarrow[\text{剔除}\alpha_2]{\alpha_2 = \alpha_3 - \frac{1}{2}\alpha_4} \alpha_3, \alpha_4 \quad \begin{matrix} \text{极大} \\ \text{无关组} \end{matrix}$$

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow[\text{剔除}\alpha_2]{\alpha_2 = 0\alpha_1 + \alpha_3 - \frac{1}{2}\alpha_4} \alpha_1, \alpha_3, \alpha_4 \xrightarrow{\hspace{1cm}} \hspace{1cm} \text{剔除}\alpha_4$$

向量组的极大无关组

$\alpha_1, \alpha_2, \dots, \alpha_s$ $\xrightarrow[\text{直到不能再剔除为止}]{\substack{\text{逐个剔除} \\ \text{能被其余向量线性表示的向量}}} \alpha_{j_1}, \alpha_{j_2}, \dots, \alpha_{j_r}$ (极大无关组)

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向量组的极大无关组

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还有其他极大无关组吗？

向量组的极大无关组

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还有其他极大无关组吗？

注 极大无关组不唯一！ 但极大无关所含向量个数唯一！

极大无关组的性质

定理 $\alpha_{j_1}, \alpha_{j_2}, \dots, \alpha_{j_r}$ 是 $\alpha_1, \alpha_2, \dots, \alpha_s$ 的极大无关组，当且仅当

- $\alpha_1, \alpha_2, \dots, \alpha_s$ 中每个向量都可由 $\alpha_{j_1}, \alpha_{j_2}, \dots, \alpha_{j_r}$ 线性表示
- $\alpha_{j_1}, \alpha_{j_2}, \dots, \alpha_{j_r}$ 线性无关

极大无关组的性质

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定理 极大无关组所包含向量的个数是唯一确定的。

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定理 极大无关组所包含向量的个数是唯一确定的。即：若

$$\alpha_{j_1}, \alpha_{j_2}, \dots, \alpha_{j_r}; \quad \beta_{k_1}, \beta_{k_2}, \dots, \beta_{k_t}$$

都是 $\alpha_1, \alpha_2, \dots, \alpha_s$ 的极大无关组, 则

极大无关组的性质

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例 设 $\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, 则极大无关组是:
 $\alpha_1, \alpha_2; \quad \alpha_1, \alpha_3; \quad \alpha_2, \alpha_3; \quad \alpha_2, \alpha_4; \quad \alpha_3, \alpha_4$

极大无关组的性质

定理 $\alpha_{j_1}, \alpha_{j_2}, \dots, \alpha_{j_r}$ 是 $\alpha_1, \alpha_2, \dots, \alpha_s$ 的极大无关组, 当且仅当

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$$\alpha_1, \alpha_2; \quad \alpha_1, \alpha_3; \quad \alpha_2, \alpha_3; \quad \alpha_2, \alpha_4; \quad \alpha_3, \alpha_4$$

可见, 每个极大无关组都由 2 个向量构成。

向量组的秩

定义 向量组 $\alpha_1, \alpha_2, \dots, \alpha_s$ 的极大无关组所包含向量的个数, 称向量组的**秩**, 记为:

$$r(\alpha_1, \alpha_2, \dots, \alpha_s)$$

向量组的秩

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 $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \underline{\hspace{2cm}}$

向量组的秩

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 $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \underline{2}$

向量组的秩

定义 向量组 $\alpha_1, \alpha_2, \dots, \alpha_s$ 的极大无关组所包含向量的个数, 称向量组的**秩**, 记为:

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 $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \underline{\quad 2 \quad}$

注 $r(\alpha_1, \alpha_2, \dots, \alpha_s) \leq s$ 且 $\leq m$ (维数)。

秩

设

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (\alpha_1 \alpha_2 \cdots \alpha_n)$$

$$r(\alpha_1, \alpha_2, \dots, \alpha_n)$$

秩

设

$$A_{m \times n} = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (\alpha_1 \alpha_2 \cdots \alpha_n)$$

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定理 $r(A) = r(\alpha_1, \alpha_2, \dots, \alpha_n)$

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定义

- $r(\alpha_1, \alpha_2, \dots, \alpha_n)$ 称为 A 的列秩;

定理 $r(A) = r(\alpha_1, \alpha_2, \dots, \alpha_n)$

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定义

- $r(\alpha_1, \alpha_2, \dots, \alpha_n)$ 称为 A 的列秩;
- $r(\beta_1, \beta_2, \dots, \beta_m)$ 称为 A 的行秩;

定理 $r(A) = r(\alpha_1, \alpha_2, \dots, \alpha_n)$

秩

设

$$A_{m \times n} = \begin{matrix} & \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \begin{matrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{matrix} & \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{matrix} = (\alpha_1 \ \alpha_2 \ \cdots \ \alpha_n) = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{pmatrix}$$

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- $r(\beta_1, \beta_2, \dots, \beta_m)$ 称为 A 的行秩;

定理 $r(A) = r(\alpha_1, \alpha_2, \dots, \alpha_n) = r(\beta_1, \beta_2, \dots, \beta_m)$

初等变换求极大无关组

问题 给出 m 维的向量组 $\alpha_1, \alpha_2, \dots, \alpha_n$, 如何求出其一组极大无关组?

步骤

初等变换求极大无关组

问题 给出 m 维的向量组 $\alpha_1, \alpha_2, \dots, \alpha_n$, 如何求出其一组极大无关组?

步骤

$$1. A_{m \times n} = \begin{matrix} & \alpha_1 & \alpha_2 & & \alpha_n \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{matrix}$$

初等变换求极大无关组

问题 给出 m 维的向量组 $\alpha_1, \alpha_2, \dots, \alpha_n$, 如何求出其一组极大无关组?

步骤

1.
$$A_{m \times n} = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \xrightarrow{\text{初等行变换}} \text{简化的阶梯型矩阵}$$

初等变换求极大无关组

问题 给出 m 维的向量组 $\alpha_1, \alpha_2, \dots, \alpha_n$, 如何求出其一组极大无关组?

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2. 通过简化的阶梯型矩阵, 求出 $r(A)$ 。

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利用 $r(\alpha_1, \alpha_2, \dots, \alpha_n) = r(A)$, 得出极大无关组所包含向量的个数

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4. 通过简化的阶梯型矩阵, 容易看出其余列如何用该选定极大无关组线性表示

例 1 求向量组 $\alpha_1 = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$ 的一个极大无关组; 并把其余向量用该极大无关组线性表示。

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解

	α_1	α_2	α_3	α_4
$\begin{pmatrix} 2 & 1 & 2 & 3 \\ 4 & 1 & 3 & 5 \\ 2 & 0 & 1 & 2 \end{pmatrix}$				

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解

$$\begin{array}{cccc} & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \left(\begin{array}{cccc} 2 & 1 & 2 & 3 \\ 4 & 1 & 3 & 5 \\ 2 & 0 & 1 & 2 \end{array} \right) & \xrightarrow[r_3-r_1]{r_2-2r_1} & & & \end{array}$$

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$$\xrightarrow[r_3-r_1]{r_2-2r_1} \begin{pmatrix} 2 & 1 & 2 & 3 \\ 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 \end{pmatrix} \rightarrow$$

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所以

- $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2;$

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- $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2$;
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所以

- $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2$;
- α_1, α_2 是极大无关组;
- $\alpha_3 = \frac{1}{2}\alpha_1 + \alpha_2, \quad \alpha_4 = \alpha_1 + \alpha_2$

例 2 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一

个极大无关组；并把其余向量用该极大无关组线性表示。

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极大无关组；并把其余向量用该极大无关组线性表示。

解

	α_1	α_2	α_3	α_4
$\begin{pmatrix}$	1	0	1	2
$\begin{pmatrix}$	2	1	1	4
$\begin{pmatrix}$	1	1	0	3
$\begin{pmatrix}$	0	2	-2	3
$\end{pmatrix}$				

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$$\begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \left(\begin{array}{cccc} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 0 & 2 & -2 & 3 \end{array} \right) & \xrightarrow[r_3-r_1]{r_2-2r_1} & \left(\begin{array}{cccc} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 3 \end{array} \right) & \xrightarrow[r_4-2r_2]{r_3-r_2} \end{array}$$

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解 $\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4$

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow[r_3-r_1]{r_2-2r_1} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow[r_4-2r_2]{r_3-r_2} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$
$$\xrightarrow[r_1-2r_3]{r_4-3r_3} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

例2 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解 $\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4$

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow[r_3-r_1]{r_2-2r_1} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow[r_4-2r_2]{r_3-r_2} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$\xrightarrow[r_1-2r_3]{r_4-3r_3} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

例 2 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{cccc}
 \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\
 \left(\begin{array}{cccc} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 0 & 2 & -2 & 3 \end{array} \right) & \xrightarrow[r_3-r_1]{r_2-2r_1} & \left(\begin{array}{cccc} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 3 \end{array} \right) & \xrightarrow[r_4-2r_2]{r_3-r_2} \left(\begin{array}{cccc} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{array} \right) \\
 & & \xrightarrow[r_1-2r_3]{r_4-3r_3} & \left(\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right)
 \end{array}$$

所以

例2 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一个极大无关组; 并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \left(\begin{array}{cccc} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 0 & 2 & -2 & 3 \end{array} \right) & \xrightarrow[r_3-r_1]{r_2-2r_1} & \left(\begin{array}{cccc} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 3 \end{array} \right) & \xrightarrow[r_4-2r_2]{r_3-r_2} \left(\begin{array}{cccc} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{array} \right) \\ & & \xrightarrow[r_1-2r_3]{r_4-3r_3} & \left(\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right) \end{array}$$

所以

• $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 3;$

例2 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一个

极大无关组；并把其余向量用该极大无关组线性表示。

解 $\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4$

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow[r_3-r_1]{r_2-2r_1} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow[r_4-2r_2]{r_3-r_2} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$
$$\xrightarrow[r_1-2r_3]{r_4-3r_3} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

所以

• $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 3;$

例2 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一个

极大无关组；并把其余向量用该极大无关组线性表示。

解 $\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4$

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow[r_3-r_1]{r_2-2r_1} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow[r_4-2r_2]{r_3-r_2} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$
$$\xrightarrow[r_1-2r_3]{r_4-3r_3} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

所以

- $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 3;$

- $\alpha_1, \alpha_2, \alpha_4$ 是极大无关组;

例2 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一个

极大无关组；并把其余向量用该极大无关组线性表示。

解 $\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4$

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow[r_3-r_1]{r_2-2r_1} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow[r_4-2r_2]{r_3-r_2} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$
$$\xrightarrow[r_1-2r_3]{r_4-3r_3} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

所以

- $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 3$;
- $\alpha_1, \alpha_2, \alpha_4$ 是极大无关组;
- $\alpha_3 = \alpha_1 - \alpha_2$

例 3 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

例 3 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个

极大无关组；并把其余向量用该极大无关组线性表示。

解 $\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4$
 $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix}$

例 3 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个

极大无关组；并把其余向量用该极大无关组线性表示。

解 $\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \xrightarrow[r_4-4r_1]{\begin{matrix} r_2-2r_1 \\ r_3-3r_1 \end{matrix}}$$

例3 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个

极大无关组；并把其余向量用该极大无关组线性表示。

解 $\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \xrightarrow[r_4-4r_1]{\begin{matrix} r_2-2r_1 \\ r_3-3r_1 \end{matrix}} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{pmatrix}$$

例3 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个

极大无关组；并把其余向量用该极大无关组线性表示。

解 $\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \xrightarrow[r_4-4r_1]{\begin{matrix} r_2-2r_1 \\ r_3-3r_1 \end{matrix}} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{pmatrix}$$

$$\xrightarrow[r_4-3r_2]{r_3-2r_2}$$

例3 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个

极大无关组；并把其余向量用该极大无关组线性表示。

解 $\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \xrightarrow[r_4-4r_1]{\begin{matrix} r_2-2r_1 \\ r_3-3r_1 \end{matrix}} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{pmatrix}$$
$$\xrightarrow[r_4-3r_2]{r_3-2r_2} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

例3 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个

极大无关组；并把其余向量用该极大无关组线性表示。

解 $\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \xrightarrow[r_4-4r_1]{\begin{matrix} r_2-2r_1 \\ r_3-3r_1 \end{matrix}} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{pmatrix}$$
$$\xrightarrow[r_4-3r_2]{r_3-2r_2} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

例3 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个

极大无关组；并把其余向量用该极大无关组线性表示。

解 $\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \xrightarrow[r_4-4r_1]{\begin{matrix} r_2-2r_1 \\ r_3-3r_1 \end{matrix}} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{pmatrix}$$
$$\xrightarrow[r_4-3r_2]{r_3-2r_2} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

例 3 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个

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解 $\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \xrightarrow[r_4-4r_1]{\begin{matrix} r_2-2r_1 \\ r_3-3r_1 \end{matrix}} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{pmatrix}$$

$$\xrightarrow[r_4-3r_2]{r_3-2r_2} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

所以

- $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2;$

例3 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个

极大无关组；并把其余向量用该极大无关组线性表示。

解 $\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \xrightarrow[r_4-4r_1]{\begin{matrix} r_2-2r_1 \\ r_3-3r_1 \end{matrix}} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{pmatrix}$$
$$\xrightarrow[r_4-3r_2]{r_3-2r_2} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

所以

• $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2;$

例3 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个

极大无关组；并把其余向量用该极大无关组线性表示。

解 $\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \xrightarrow[r_4-4r_1]{\begin{matrix} r_2-2r_1 \\ r_3-3r_1 \end{matrix}} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{pmatrix}$$
$$\xrightarrow[r_4-3r_2]{r_3-2r_2} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

所以

- $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2$;
- α_1, α_2 是极大无关组;

例3 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个

极大无关组；并把其余向量用该极大无关组线性表示。

解 $\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \xrightarrow[r_4-4r_1]{\begin{matrix} r_2-2r_1 \\ r_3-3r_1 \end{matrix}} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{pmatrix}$$
$$\xrightarrow[r_4-3r_2]{r_3-2r_2} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

所以

- $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2$;
- α_1, α_2 是极大无关组;
- $\alpha_3 = -\alpha_1 + 2\alpha_2$

例3 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个

极大无关组；并把其余向量用该极大无关组线性表示。

解 $\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \xrightarrow[r_4-4r_1]{\begin{matrix} r_2-2r_1 \\ r_3-3r_1 \end{matrix}} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{pmatrix}$$
$$\xrightarrow[r_4-3r_2]{r_3-2r_2} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

所以

- $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2$;
- α_1, α_2 是极大无关组;
- $\alpha_3 = -\alpha_1 + 2\alpha_2$, $\alpha_4 = -2\alpha_1 + 3\alpha_2$

例 假设向量组 $\alpha_1, \alpha_2, \dots, \alpha_s$ 可由 $\beta_1, \beta_2, \dots, \beta_t$ 线性表示, 则

$$r(\alpha_1, \alpha_2, \dots, \alpha_s) \leq r(\beta_1, \beta_2, \dots, \beta_t).$$

例 假设向量组 $\alpha_1, \alpha_2, \dots, \alpha_s$ 可由 $\beta_1, \beta_2, \dots, \beta_t$ 线性表示, 则

$$r(\alpha_1, \alpha_2, \dots, \alpha_s) \leq r(\beta_1, \beta_2, \dots, \beta_t).$$

证明 设

$$r_1 = r(\alpha_1, \alpha_2, \dots, \alpha_s),$$

$$r_2 = r(\beta_1, \beta_2, \dots, \beta_t),$$

例 假设向量组 $\alpha_1, \alpha_2, \dots, \alpha_s$ 可由 $\beta_1, \beta_2, \dots, \beta_t$ 线性表示, 则

$$r(\alpha_1, \alpha_2, \dots, \alpha_s) \leq r(\beta_1, \beta_2, \dots, \beta_t).$$

证明 设

$$r_1 = r(\alpha_1, \alpha_2, \dots, \alpha_s), \quad \alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_{r_1}} \text{ 是极大无关组}$$

$$r_2 = r(\beta_1, \beta_2, \dots, \beta_t),$$

例 假设向量组 $\alpha_1, \alpha_2, \dots, \alpha_s$ 可由 $\beta_1, \beta_2, \dots, \beta_t$ 线性表示, 则

$$r(\alpha_1, \alpha_2, \dots, \alpha_s) \leq r(\beta_1, \beta_2, \dots, \beta_t).$$

证明 设

$r_1 = r(\alpha_1, \alpha_2, \dots, \alpha_s), \quad \alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_{r_1}}$ 是极大无关组

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$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_C = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_A \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_B$$

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$$\delta_1 \underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_C = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_A \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_B \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$$

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$$\begin{array}{l} \delta_1 \\ \delta_2 \end{array} \underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_C = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_A \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_B \begin{array}{l} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{array}$$

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$$\begin{array}{c} \delta_1 \\ \delta_2 \\ \vdots \end{array} \underbrace{\begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1s} \\ C_{21} & C_{22} & \cdots & C_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ C_{m1} & C_{m2} & \cdots & C_{ms} \end{pmatrix}}_C = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_A \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_B \begin{array}{c} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{array}$$

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即

$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$$

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$$\Rightarrow \delta_1 = a_{11}\beta_1 + a_{12}\beta_2 + \cdots + a_{1n}\beta_n$$

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即

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即

$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$$

$$\Rightarrow \delta_1 = a_{11}\beta_1 + a_{12}\beta_2 + \cdots + a_{1n}\beta_n \quad \text{等等}$$

可见 $\delta_1, \dots, \delta_m$ 由 β_1, \dots, β_n 线性表示,

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即

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证明 设 $AB = C_{m \times s}$

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即

$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$$

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$$r(\delta_1, \dots, \delta_m) \leq r(\beta_1, \dots, \beta_n) = r(B)$$

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证明 设 $AB = C_{m \times s}$

$$\begin{matrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{matrix} \underbrace{\begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1s} \\ C_{21} & C_{22} & \cdots & C_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ C_{m1} & C_{m2} & \cdots & C_{ms} \end{pmatrix}}_C = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_A \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_B \begin{matrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{matrix}$$

即

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