

Outline of §1.4

二元线性方程组

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$$

的解是

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$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$$

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$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \quad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

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注

• $D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ 称为系数行列式

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- $D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ 称为系数行列式，上述公式隐含要求 $D \neq 0$
- D_i : 将 D 的第 i 列换成常数项 $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

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系数行列式

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系数行列式

对 n 元线性
方程组

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

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$$\begin{vmatrix} a_{11} & \cdots & a_{1j-1} & a_{1,j} & a_{1,j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2,j} & a_{2,j+1} & \cdots & a_{2n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{n,j} & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}$$

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令 $D_j = \begin{vmatrix} a_{11} & \cdots & a_{1j-1} & b_1 & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & b_2 & a_{2j+1} & \cdots & a_{2n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj-1} & b_n & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}$

克莱姆法则

定理（克莱姆法则） 线性方程组

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当系数行列式 $D \neq 0$ 时，方程具有唯一解：

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注 两个前提：(1) 未知元个数 = 方程个数；(2) 系数行列式 $D \neq 0$

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注 两个前提：(1) 未知元个数 = 方程个数；(2) 系数行列式 $D \neq 0$

(3) 若 $D = 0$ ，则方程有无穷多解或无解（以后详说）

克莱姆法则证明 (仅验证 $x_j = \frac{D_j}{D}$ 是解, 唯一性的证明要用到矩阵知识, 略去。)

$$x_j = \frac{D_j}{D}$$

验证第 k 条方程成立 ($k = 1, 2, \dots, n$):

$$a_{k1}x_1 + \dots + a_{kn}x_n =$$

b_k

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$$x_j = \frac{D_j}{D} = \frac{\begin{vmatrix} a_{11} & \cdots & a_{1j-1} & \color{red}{b_1} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & \color{red}{b_2} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & & \vdots & \color{red}{\vdots} & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj-1} & \color{red}{b_n} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}}$$

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$$a_{k1}x_1 + \cdots + a_{kn}x_n =$$

克莱姆法则证明 (仅验证 $x_j = \frac{D_j}{D}$ 是解, 唯一性的证明要用到矩阵知识, 略去。)

$$x_j = \frac{D_j}{D} = \frac{\begin{vmatrix} a_{11} & \cdots & a_{1j-1} & \color{red}{b_1} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & \color{red}{b_2} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & & \vdots & \color{red}{\vdots} & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj-1} & \color{red}{b_n} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}} = \frac{b_1 A_{1j} + b_2 \quad + \cdots + b_n}{D}$$

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$$\sum_{i=1}^n \color{red}{b_i} A_{ij}$$

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$$a_{k1}x_1 + \cdots + a_{kn}x_n = \sum_{j=1}^n a_{kj}x_j = \sum_{j=1}^n a_{kj} \left(\frac{1}{D} \sum_{i=1}^n b_i A_{ij} \right)$$

b_k

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$$\begin{aligned} a_{k1}x_1 + \cdots + a_{kn}x_n &= \sum_{j=1}^n a_{kj}x_j = \sum_{j=1}^n a_{kj} \left(\frac{1}{D} \sum_{i=1}^n b_i A_{ij} \right) = \frac{1}{D} \sum_{j=1}^n \sum_{i=1}^n a_{kj} b_i A_{ij} \\ &= \frac{1}{D} \sum_{i=1}^n \sum_{j=1}^n a_{kj} b_i A_{ij} = \frac{1}{D} \sum_{i=1}^n b_i \sum_{j=1}^n a_{kj} A_{ij} \quad b_k \sum_{j=1}^n a_{kj} A_{kj} \end{aligned}$$

克莱姆法则证明 (仅验证 $x_j = \frac{D_j}{D}$ 是解, 唯一性的证明要用到矩阵知识, 略去。)

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- $\begin{cases} x + y = 1 \\ x + y = 0 \end{cases}$, $D = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$

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- $\begin{cases} x + y = 1 \\ x + y = 0 \end{cases}$, $D = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$, 方程组包含矛盾方程，显然无解。

例 解线性方程组

$$\begin{cases} 2x_1 + x_2 - x_3 = 1 \\ 3x_1 - x_2 - x_3 = -2 \\ -x_1 + 2x_2 + x_3 = 6 \end{cases}$$

练习 解线性方程组

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提示 $D = 2, D_1 = 84, D_2 = 96, D_3 = 76$

齐次线性方程组

定理 齐次线性方程组

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = 0 \end{cases}$$

当系数行列式 $D \neq 0$ 时, 仅有零解 ($x_1 = x_2 = \cdots = x_n = 0$)

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另一方面, 因为 $D \neq 0$, 所以方程组有唯一解 (克莱姆法则)

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注

- 其实是充分必要条件: 仅有零解的充分必要条件是 $D \neq 0$

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注

- 其实是充分必要条件: 仅有零解的充分必要条件是 $D \neq 0$
- 若 $D = 0$, 方程有无穷多的解

例子

例 齐次方程组 $\begin{cases} x_1 - 2x_2 = 0 \\ 2x_1 - 4x_2 = 0 \end{cases}$ 的系数矩阵 $D = \begin{vmatrix} 1 & -2 \\ 2 & -4 \end{vmatrix}$

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例 判断线性方程组 $\begin{cases} 2x_1 + 3x_2 + 4x_3 + 5x_4 = 0 \\ 3x_1 + 4x_2 + 5x_3 + 5x_4 = 0 \\ 4x_1 + 5x_2 + 6x_3 + 6x_4 = 0 \\ 5x_1 + 6x_2 + 8x_3 + 9x_4 = 0 \end{cases}$ 是否只有零解

解

$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 5 & 6 & 8 & 9 \end{vmatrix}$$

解

$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 5 & 6 & 8 & 9 \end{vmatrix} \underline{\underline{r_4 - r_3}}$$

解

$$\left| \begin{array}{cccc} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 5 & 6 & 8 & 9 \end{array} \right| \xrightarrow{\underline{\underline{r_4 - r_3}}} \left| \begin{array}{cccc} & & & \\ & & & \\ & & & \\ 1 & 1 & 2 & 3 \end{array} \right|$$

解

$$\left| \begin{array}{cccc} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 5 & 6 & 8 & 9 \end{array} \right| \xrightarrow{\underline{\underline{r_4 - r_3}}} \left| \begin{array}{cccc} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 1 & 1 & 2 & 3 \end{array} \right|$$

解

$$\left| \begin{array}{cccc} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 5 & 6 & 8 & 9 \end{array} \right| \xrightarrow{\underline{\underline{r_4 - r_3}}} \left| \begin{array}{cccc} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 1 & 1 & 2 & 3 \end{array} \right| \xrightarrow{\underline{\underline{\begin{array}{l} c_2 - c_1 \\ c_3 - 2c_1 \\ c_4 - 3c_1 \end{array}}}}$$

解

$$\begin{array}{c|cccc} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 5 & 6 & 8 & 9 \end{array} \xrightarrow{\underline{\underline{r_4 - r_3}}} \begin{array}{c|cccc} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 1 & 1 & 2 & 3 \end{array} \xrightarrow{\begin{array}{c} \underline{\underline{c_2 - c_1}} \\ c_3 - 2c_1 \\ c_4 - 3c_1 \end{array}} \begin{array}{c|cccc} 2 & & & \\ 3 & & & \\ 4 & & & \\ 1 & & & \end{array}$$

解

$$\begin{array}{c|cccc} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 5 & 6 & 8 & 9 \end{array} \xrightarrow{\underline{\underline{r_4 - r_3}}} \begin{array}{c|cccc} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 1 & 1 & 2 & 3 \end{array} \xrightarrow{\begin{array}{c} \underline{\underline{c_2 - c_1}} \\ c_3 - 2c_1 \\ c_4 - 3c_1 \end{array}} \begin{array}{c|cc} 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 1 & 0 \end{array} \quad |$$

解

$$\begin{array}{c|cccc} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 5 & 6 & 8 & 9 \end{array} \xrightarrow{\underline{\underline{r_4 - r_3}}} \begin{array}{c|cccc} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 1 & 1 & 2 & 3 \end{array} \xrightarrow{\begin{array}{c} \underline{\underline{c_2 - c_1}} \\ c_3 - 2c_1 \\ c_4 - 3c_1 \end{array}} \begin{array}{c|ccc} 2 & 1 & 0 & \\ 3 & 1 & -1 & \\ 4 & 1 & -2 & \\ 1 & 0 & 0 & \end{array} \Bigg|$$

解

$$\left| \begin{array}{cccc} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 5 & 6 & 8 & 9 \end{array} \right| \xrightarrow{r_4 - r_3} \left| \begin{array}{cccc} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 1 & 1 & 2 & 3 \end{array} \right| \xrightarrow{\substack{c_2 - c_1 \\ c_3 - 2c_1 \\ c_4 - 3c_1}} \left| \begin{array}{cccc} 2 & 1 & 0 & -1 \\ 3 & 1 & -1 & -4 \\ 4 & 1 & -2 & -6 \\ 1 & 0 & 0 & 0 \end{array} \right|$$

解

$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 5 & 6 & 8 & 9 \end{vmatrix} \xrightarrow{r_4 - r_3} \begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 1 & 1 & 2 & 3 \end{vmatrix} \xrightarrow{\substack{c_2 - c_1 \\ c_3 - 2c_1 \\ c_4 - 3c_1}} \begin{vmatrix} 2 & 1 & 0 & -1 \\ 3 & 1 & -1 & -4 \\ 4 & 1 & -2 & -6 \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

$$= 1 \times (-1)^{4+1} \times \begin{vmatrix} 1 & 0 & -1 \\ 1 & -1 & -4 \\ 1 & -2 & -6 \end{vmatrix}$$

解

$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 5 & 6 & 8 & 9 \end{vmatrix} \xrightarrow{\underline{\underline{r_4 - r_3}}} \begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 1 & 1 & 2 & 3 \end{vmatrix} \xrightarrow{\begin{matrix} \underline{\underline{c_2 - c_1}} \\ \underline{\underline{c_3 - 2c_1}} \\ \underline{\underline{c_4 - 3c_1}} \end{matrix}} \begin{vmatrix} 2 & 1 & 0 & -1 \\ 3 & 1 & -1 & -4 \\ 4 & 1 & -2 & -6 \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

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解

$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 5 & 6 & 8 & 9 \end{vmatrix} \xrightarrow{\underline{\underline{r_4 - r_3}}} \begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 1 & 1 & 2 & 3 \end{vmatrix} \xrightarrow{\begin{matrix} \underline{\underline{c_2 - c_1}} \\ \underline{\underline{c_3 - 2c_1}} \\ \underline{\underline{c_4 - 3c_1}} \end{matrix}} \begin{vmatrix} 2 & 1 & 0 & -1 \\ 3 & 1 & -1 & -4 \\ 4 & 1 & -2 & -6 \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

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解

$$\begin{aligned}
 & \begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 5 & 6 & 8 & 9 \end{vmatrix} \xrightarrow{r_4 - r_3} \begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 1 & 1 & 2 & 3 \end{vmatrix} \xrightarrow{\substack{c_2 - c_1 \\ c_3 - 2c_1 \\ c_4 - 3c_1}} \begin{vmatrix} 2 & 1 & 0 & -1 \\ 3 & 1 & -1 & -4 \\ 4 & 1 & -2 & -6 \\ 1 & 0 & 0 & 0 \end{vmatrix} \\
 &= 1 \times (-1)^{4+1} \times \begin{vmatrix} 1 & 0 & -1 \\ 1 & -1 & -4 \\ 1 & -2 & -6 \end{vmatrix} \xrightarrow{c_3 + c_1} - \begin{vmatrix} 1 & 0 & 0 \\ 1 & -1 & -3 \\ 1 & -2 & -5 \end{vmatrix} \\
 &= - \begin{vmatrix} -1 & -3 \\ -2 & -5 \end{vmatrix}
 \end{aligned}$$

解

$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 5 & 6 & 8 & 9 \end{vmatrix} \xrightarrow{r_4 - r_3} \begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 1 & 1 & 2 & 3 \end{vmatrix} \xrightarrow{\substack{c_2 - c_1 \\ c_3 - 2c_1 \\ c_4 - 3c_1}} \begin{vmatrix} 2 & 1 & 0 & -1 \\ 3 & 1 & -1 & -4 \\ 4 & 1 & -2 & -6 \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

$$= 1 \times (-1)^{4+1} \times \begin{vmatrix} 1 & 0 & -1 \\ 1 & -1 & -4 \\ 1 & -2 & -6 \end{vmatrix} \xrightarrow{c_3 + c_1} - \begin{vmatrix} 1 & 0 & 0 \\ 1 & -1 & -3 \\ 1 & -2 & -5 \end{vmatrix}$$

$$= - \begin{vmatrix} -1 & -3 \\ -2 & -5 \end{vmatrix} = 1 \neq 0$$

解

$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 5 & 6 & 8 & 9 \end{vmatrix} \xrightarrow{r_4 - r_3} \begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 1 & 1 & 2 & 3 \end{vmatrix} \xrightarrow{\substack{c_2 - c_1 \\ c_3 - 2c_1 \\ c_4 - 3c_1}} \begin{vmatrix} 2 & 1 & 0 & -1 \\ 3 & 1 & -1 & -4 \\ 4 & 1 & -2 & -6 \\ 1 & 0 & 0 & 0 \end{vmatrix}$$
$$= 1 \times (-1)^{4+1} \times \begin{vmatrix} 1 & 0 & -1 \\ 1 & -1 & -4 \\ 1 & -2 & -6 \end{vmatrix} \xrightarrow{c_3 + c_1} - \begin{vmatrix} 1 & 0 & 0 \\ 1 & -1 & -3 \\ 1 & -2 & -5 \end{vmatrix}$$
$$= - \begin{vmatrix} -1 & -3 \\ -2 & -5 \end{vmatrix} = 1 \neq 0$$

所以齐次线性方程组有唯一解

练习 齐次线性方程组 $\begin{cases} kx_1 & & + x_4 = 0 \\ x_1 + 2x_2 & & - x_4 = 0 \\ (k+2)x_1 - x_2 & & + 4x_4 = 0 \\ 2x_1 + x_2 + 3x_3 + kx_4 = 0 \end{cases}$ 有非零解

的充分必要条件是 k 满足 _____

练习 齐次线性方程组 $\begin{cases} kx_1 & + x_4 = 0 \\ x_1 + 2x_2 & - x_4 = 0 \\ (k+2)x_1 - x_2 & + 4x_4 = 0 \\ 2x_1 + x_2 + 3x_3 + kx_4 = 0 \end{cases}$ 有非零解

的充分必要条件是 k 满足 _____

解

$$D = \begin{vmatrix} k & 0 & 0 & 1 \\ 1 & 2 & 0 & -1 \\ k+2 & -1 & 0 & 4 \\ 2 & 1 & 3 & k \end{vmatrix}$$

练习 齐次线性方程组 $\begin{cases} kx_1 & + x_4 = 0 \\ x_1 + 2x_2 & - x_4 = 0 \\ (k+2)x_1 - x_2 & + 4x_4 = 0 \\ 2x_1 + x_2 + 3x_3 + kx_4 = 0 \end{cases}$ 有非零解

的充分必要条件是 k 满足 _____

解

$$D = \begin{vmatrix} k & 0 & 0 & 1 \\ 1 & 2 & 0 & -1 \\ k+2 & -1 & 0 & 4 \\ 2 & 1 & 3 & k \end{vmatrix} = 3.$$

练习 齐次线性方程组 $\begin{cases} kx_1 & + x_4 = 0 \\ x_1 + 2x_2 & - x_4 = 0 \\ (k+2)x_1 - x_2 & + 4x_4 = 0 \\ 2x_1 + x_2 + 3x_3 + kx_4 = 0 \end{cases}$ 有非零解

的充分必要条件是 k 满足 _____

解

$$D = \begin{vmatrix} k & 0 & 0 & 1 \\ 1 & 2 & 0 & -1 \\ k+2 & -1 & 0 & 4 \\ 2 & 1 & 3 & k \end{vmatrix} = 3 \cdot (-1)^{3+4} \begin{vmatrix} k & 0 & 1 \\ 1 & 2 & -1 \\ k+2 & -1 & 4 \end{vmatrix}$$

练习 齐次线性方程组 $\begin{cases} kx_1 & + x_4 = 0 \\ x_1 + 2x_2 & - x_4 = 0 \\ (k+2)x_1 - x_2 & + 4x_4 = 0 \\ 2x_1 + x_2 + 3x_3 + kx_4 = 0 \end{cases}$ 有非零解

的充分必要条件是 k 满足 _____

解

$$D = \begin{vmatrix} k & 0 & 0 & 1 \\ 1 & 2 & 0 & -1 \\ k+2 & -1 & 0 & 4 \\ 2 & 1 & 3 & k \end{vmatrix} = 3 \cdot (-1)^{3+4} \begin{vmatrix} k & 0 & 1 \\ 1 & 2 & -1 \\ k+2 & -1 & 4 \end{vmatrix}$$

$$\underline{\underline{r_2 + r_1}}$$

练习 齐次线性方程组 $\begin{cases} kx_1 & + x_4 = 0 \\ x_1 + 2x_2 & - x_4 = 0 \\ (k+2)x_1 - x_2 & + 4x_4 = 0 \\ 2x_1 + x_2 + 3x_3 + kx_4 = 0 \end{cases}$ 有非零解

的充分必要条件是 k 满足 _____

解

$$D = \begin{vmatrix} k & 0 & 0 & 1 \\ 1 & 2 & 0 & -1 \\ k+2 & -1 & 0 & 4 \\ 2 & 1 & 3 & k \end{vmatrix} = 3 \cdot (-1)^{3+4} \begin{vmatrix} k & 0 & 1 \\ 1 & 2 & -1 \\ k+2 & -1 & 4 \end{vmatrix}$$

$$\xrightarrow{r_2+r_1} (-3) \cdot \begin{vmatrix} k & 0 & 1 \\ & & \end{vmatrix}$$

练习 齐次线性方程组 $\begin{cases} kx_1 & + x_4 = 0 \\ x_1 + 2x_2 & - x_4 = 0 \\ (k+2)x_1 - x_2 & + 4x_4 = 0 \\ 2x_1 + x_2 + 3x_3 + kx_4 = 0 \end{cases}$ 有非零解

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解

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$$\xrightarrow{r_2+r_1} (-3) \cdot \begin{vmatrix} k & 0 & 1 \\ k+1 & 2 & 0 \end{vmatrix}$$

练习 齐次线性方程组 $\begin{cases} kx_1 & + x_4 = 0 \\ x_1 + 2x_2 & - x_4 = 0 \\ (k+2)x_1 - x_2 & + 4x_4 = 0 \\ 2x_1 + x_2 + 3x_3 + kx_4 = 0 \end{cases}$ 有非零解

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$$\xrightarrow[r_3 - 4r_1]{r_2 + r_1} (-3) \cdot \begin{vmatrix} k & 0 & 1 \\ k+1 & 2 & 0 \end{vmatrix}$$

练习 齐次线性方程组 $\begin{cases} kx_1 & + x_4 = 0 \\ x_1 + 2x_2 & - x_4 = 0 \\ (k+2)x_1 - x_2 & + 4x_4 = 0 \\ 2x_1 + x_2 + 3x_3 + kx_4 = 0 \end{cases}$ 有非零解

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$$\xrightarrow[r_3-4r_1]{r_2+r_1} (-3) \cdot \begin{vmatrix} k & 0 & 1 \\ k+1 & 2 & 0 \\ -3k+2 & -1 & 0 \end{vmatrix}$$

练习 齐次线性方程组 $\begin{cases} kx_1 & & + x_4 = 0 \\ x_1 + 2x_2 & & - x_4 = 0 \\ (k+2)x_1 - x_2 & & + 4x_4 = 0 \\ 2x_1 + x_2 + 3x_3 + kx_4 = 0 \end{cases}$ 有非零解

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练习 齐次线性方程组 $\begin{cases} kx_1 & + x_4 = 0 \\ x_1 + 2x_2 & - x_4 = 0 \\ (k+2)x_1 - x_2 & + 4x_4 = 0 \\ 2x_1 + x_2 + 3x_3 + kx_4 = 0 \end{cases}$ 有非零解

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$$\xrightarrow[r_3 - 4r_1]{r_2 + r_1} (-3) \cdot \begin{vmatrix} k & 0 & 1 \\ k+1 & 2 & 0 \\ -3k+2 & -1 & 0 \end{vmatrix} = (-3) \cdot (-1)^{1+3} \cdot \begin{vmatrix} k+1 & 2 \\ -3k+2 & -1 \end{vmatrix}$$

$$= -3(5k-5)$$

练习 齐次线性方程组 $\begin{cases} kx_1 & + x_4 = 0 \\ x_1 + 2x_2 & - x_4 = 0 \\ (k+2)x_1 - x_2 & + 4x_4 = 0 \\ 2x_1 + x_2 + 3x_3 + kx_4 = 0 \end{cases}$ 有非零解

的充分必要条件是 k 满足 _____

解

$$D = \begin{vmatrix} k & 0 & 0 & 1 \\ 1 & 2 & 0 & -1 \\ k+2 & -1 & 0 & 4 \\ 2 & 1 & 3 & k \end{vmatrix} = 3 \cdot (-1)^{3+4} \begin{vmatrix} k & 0 & 1 \\ 1 & 2 & -1 \\ k+2 & -1 & 4 \end{vmatrix}$$

$$\begin{aligned} & \xrightarrow[r_3-4r_1]{r_2+r_1} (-3) \cdot \begin{vmatrix} k & 0 & 1 \\ k+1 & 2 & 0 \\ -3k+2 & -1 & 0 \end{vmatrix} = (-3) \cdot (-1)^{1+3} \cdot \begin{vmatrix} k+1 & 2 \\ -3k+2 & -1 \end{vmatrix} \\ & = -3(5k-5) \end{aligned}$$

有非零解当且仅当 $D = 0$,

练习 齐次线性方程组 $\begin{cases} kx_1 & & + x_4 = 0 \\ x_1 + 2x_2 & & - x_4 = 0 \\ (k+2)x_1 - x_2 & & + 4x_4 = 0 \\ 2x_1 + x_2 + 3x_3 + kx_4 = 0 \end{cases}$ 有非零解

的充分必要条件是 k 满足 _____

解

$$D = \begin{vmatrix} k & 0 & 0 & 1 \\ 1 & 2 & 0 & -1 \\ k+2 & -1 & 0 & 4 \\ 2 & 1 & 3 & k \end{vmatrix} = 3 \cdot (-1)^{3+4} \begin{vmatrix} k & 0 & 1 \\ 1 & 2 & -1 \\ k+2 & -1 & 4 \end{vmatrix}$$

$$\begin{aligned} & \xrightarrow[r_3-4r_1]{r_2+r_1} (-3) \cdot \begin{vmatrix} k & 0 & 1 \\ k+1 & 2 & 0 \\ -3k+2 & -1 & 0 \end{vmatrix} = (-3) \cdot (-1)^{1+3} \cdot \begin{vmatrix} k+1 & 2 \\ -3k+2 & -1 \end{vmatrix} \\ & = -3(5k-5) \end{aligned}$$

有非零解当且仅当 $D = 0$, 当且仅当 $k = 1$