第 09 周作业解答

练习 1. 求下列函数的偏导数：

(1) $z = \sin(xy) + \cos^2(xy)$; (2) $z = x^y \cdot \ln y$; (3) $u = \ln(xy + z)$; (4) $z = \tan \frac{x}{y}$

提示：可能要利用公式 $a^b = e^{\ln a^b} = e^{b \ln a}$

解 (1)

$$z_x = \left(\sin(xy) + \cos^2(xy)\right)'_x = y\cos(xy) - 2y\cos(xy)\sin(xy)$$
$$z_y = \left(\sin(xy) + \cos^2(xy)\right)'_y = x\cos(xy) - 2x\cos(xy)\sin(xy)$$

(2)

$$z_x = (x^y \cdot \ln y)'_x = \ln y \cdot (x^y)'_x = y \ln y \cdot x^{y-1}$$
$$z_y = (x^y \cdot \ln y)'_y = (x^y)'_y \cdot \ln y + x^y \cdot (\ln y)'_y = x^y \ln x \cdot \ln y + x^y \cdot \frac{1}{y} = x^y[\ln x \cdot \ln y + \frac{1}{y}]$$

(3)

$$u_x = (\ln(xy + z))'_x = \frac{y}{xy + z}$$
$$u_y = (\ln(xy + z))'_y = \frac{x}{xy + z}$$
$$u_z = (\ln(xy + z))'_z = \frac{1}{xy + z}$$

(4)

$$z_x = \left(\tan \frac{x}{y}\right)'_x = \frac{1}{\cos^2(\frac{x}{y})} \cdot \left(\frac{x}{y}\right)'_x = \frac{1}{y\cos^2(\frac{x}{y})}$$
$$z_y = \left(\tan \frac{x}{y}\right)'_y = \frac{1}{\cos^2(\frac{x}{y})} \cdot \left(\frac{x}{y}\right)'_y = -\frac{x}{y^2\cos^2(\frac{x}{y})}$$

练习 2. 设某产品的生产函数为

$$Q = 36KL - 2K^2 - 3L^2$$

其中 Q 为产量，K，L 分别表示所需的资本和劳动力，求边际产量 $\frac{\partial Q}{\partial K}$ 和 $\frac{\partial Q}{\partial L}$。

解

$$\frac{\partial Q}{\partial K} = \left(36KL - 2K^2 - 3L^2\right)'_K = 36L - 4K$$
$$\frac{\partial Q}{\partial L} = \left(36KL - 2K^2 - 3L^2\right)'_L = 36K - 6L$$

练习 3. 求 $z = x^3 + x^4y - y^3x$ 的全部二阶偏导数。

解

$$
\begin{aligned}
\frac{\partial z}{\partial x} &= \left(x^3+x^4y-y^3x\right)_x' = \left(x^3\right)_x' + \left(x^4y\right)_x' - \left(y^3x\right)_x' = 3x^2+4x^3y-y^3 \\
\frac{\partial z}{\partial y} &= \left(x^3+x^4y-y^3x\right)_y' = \left(x^3\right)_y' + \left(x^4y\right)_y' - \left(y^3x\right)_y' = x^4-3y^2x
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(3x^2+4x^3y-y^3\right)_x' = \left(3x^2\right)_x' + \left(4x^3y\right)_x' - \left(y^3\right)_x' = 6x+12x^2y \\
\frac{\partial^2 z}{\partial x\partial y} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \left(3x^2+4x^3y-y^3\right)_y' = \left(3x^2\right)_y' + \left(4x^3y\right)_y' - \left(y^3\right)_y' = 4x^3-3y^2 \\
\frac{\partial^2 z}{\partial y\partial x} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \left(x^4-3y^2x\right)_x' = \left(x^4\right)_x' - \left(3y^2x\right)_x' = 4x^3-3y^2 \\
\frac{\partial^2 z}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(x^4-3y^2x\right)_y' = \left(x^4\right)_y' - \left(3y^2x\right)_y' = -6xy
\end{aligned}
$$

练习 4. 设 $z=x\ln(x+y^2)$，求 dz。

解

$$
\begin{aligned}
z_x &= \left(x\ln(x+y^2)\right)_x' = (x)_x'\cdot\ln(x+y^2) + x\cdot\left(\ln(x+y^2)\right)_x' = \ln(x+y^2)+\frac{x}{x+y^2} \\
z_y &= \left(x\ln(x+y^2)\right)_y' = x\cdot\left(\ln(x+y^2)\right)_y' = x\cdot\frac{1}{x+y^2}\cdot(y^2)_y' = \frac{2xy}{x+y^2}
\end{aligned}
$$

$$
dz = z_x dx + z_y dy = \left(\ln(x+y^2)+\frac{x}{x+y^2}\right)dx + \frac{2xy}{x+y^2}dy
$$

练习 5. 求函数 $z=\frac{y}{x}$ 当 $x=2$，$y=1$，$\Delta x=0.1$，$\Delta y=-0.2$ 时的全增量 Δz 和全微分 dz。

解

$$
dz = z_x dx + z_y dy = -\frac{y}{x^2}dx + \frac{1}{x}dy.
$$

将 $x=2$，$y=1$，$\Delta x=0.1$，$\Delta y=-0.2$ 代入，得到全微分

$$
dz = -\frac{1}{4}\cdot 0.1 + \frac{1}{2}\cdot(-0.2) = -0.125 = -\frac{1}{8}.
$$

而全增量 Δz 为

$$
\Delta z = f(x+\Delta x,\, y+\Delta y) - f(x,\, y) = f(2+0.1,\, 1-0.2) - f(2,\, 1) = \frac{0.8}{2.1} - \frac{1}{2} = \frac{16-21}{42} = -\frac{5}{42} \approx -0.119047619
$$

在此例中 Δz 与 dz 在精确到小数点后 1 位时是相等。

练习 6. 设函数 $z=e^{xy}$，而 $x=\sin t$，$y=\cos t$，求 $\frac{dz}{dt}$。

解法一：

$$
\begin{aligned}
\frac{dz}{dt} &= \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt} \\
&= \left(e^{xy}\right)_x'\cdot(\sin t)_t' + \left(e^{xy}\right)_y'\cdot(\cos t)_t' \\
&= ye^{xy}\cos t - xe^{xy}\sin t \\
&= \cos t e^{\sin t\cos t}\cos t - \sin t e^{\sin t\cos t}\sin t \\
&= e^{\sin t\cos t}(\cos^2 t - \sin^2 t)
\end{aligned}
$$

解法二：因为

$$z = e^{xy} = e^{\sin t \cos t}$$

所以

$$\frac{dz}{dt} = \left(e^{\sin t \cos t}\right)_t' = e^{\sin t \cos t} \cdot (\sin t \cos t)_t' = e^{\sin t \cos t}(\cos^2 t - \sin^2 t)$$