第12章 d: 函数展开成幂级数

数学系 梁卓滨

2018-2019 学年 II





$$f(x) \stackrel{?}{=} a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

$$f(x) \stackrel{?}{=} a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

$$f(x) \stackrel{?}{=} a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

$$f^{(k)}(x) = \left[\sum_{n=0}^{\infty} a_n (x - x_0)^n\right]^{(k)}$$

$$f(x) \stackrel{?}{=} a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

$$f^{(k)}(x) = \left[\sum_{n=0}^{\infty} a_n (x - x_0)^n\right]^{(k)} = \sum_{n=0}^{\infty} \left[a_n (x - x_0)^n\right]^{(k)}$$

$$f(x) \stackrel{?}{=} a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

$$f^{(k)}(x) = \left[\sum_{n=0}^{\infty} a_n (x - x_0)^n\right]^{(k)} = \sum_{n=0}^{\infty} \left[a_n (x - x_0)^n\right]^{(k)}$$
$$= \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} a_n (x - x_0)^n =$$

$$f(x) \stackrel{?}{=} a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

$$f^{(k)}(x) = \left[\sum_{n=0}^{\infty} a_n (x - x_0)^n\right]^{(k)} = \sum_{n=0}^{\infty} \left[a_n (x - x_0)^n\right]^{(k)}$$
$$= \sum_{n=0}^{\infty} a_n \cdot n \cdot (n-1) \cdots (n-k+1) \cdot (x - x_0)^{n-k}$$

$$f(x) \stackrel{?}{=} a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

$$f^{(k)}(x) = \left[\sum_{n=0}^{\infty} a_n (x - x_0)^n\right]^{(k)} = \sum_{n=0}^{\infty} \left[a_n (x - x_0)^n\right]^{(k)}$$
$$= \sum_{n=k}^{\infty} a_n \cdot n \cdot (n-1) \cdots (n-k+1) \cdot (x - x_0)^{n-k}$$
$$= a_k \cdot k!$$

$$f(x) \stackrel{?}{=} a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

$$f^{(k)}(x) = \left[\sum_{n=0}^{\infty} a_n (x - x_0)^n\right]^{(k)} = \sum_{n=0}^{\infty} \left[a_n (x - x_0)^n\right]^{(k)}$$
$$= \sum_{n=k}^{\infty} a_n \cdot n \cdot (n-1) \cdots (n-k+1) \cdot (x - x_0)^{n-k}$$
$$= a_k \cdot k! + a_{k+1} \cdot (k+1) \cdots 2 \cdot (x - x_0)$$

$$f(x) \stackrel{?}{=} a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

$$f^{(k)}(x) = \left[\sum_{n=0}^{\infty} a_n (x - x_0)^n\right]^{(k)} = \sum_{n=0}^{\infty} \left[a_n (x - x_0)^n\right]^{(k)}$$

$$= \sum_{n=k}^{\infty} a_n \cdot n \cdot (n-1) \cdots (n-k+1) \cdot (x - x_0)^{n-k}$$

$$= a_k \cdot k! + a_{k+1} \cdot (k+1) \cdots 2 \cdot (x - x_0)$$

$$+ a_{k+2} \cdot (k+2) \cdots 3 \cdot (x - x_0)^2 + \cdots$$

$$f(x) \stackrel{?}{=} a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots + a_n(x-x_0)^n + \dots$$

性质 若f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

证明 逐项求 k 次导得:

$$f^{(k)}(x) = \left[\sum_{n=0}^{\infty} a_n (x - x_0)^n\right]^{(k)} = \sum_{n=0}^{\infty} \left[a_n (x - x_0)^n\right]^{(k)}$$

$$= \sum_{n=k}^{\infty} a_n \cdot n \cdot (n-1) \cdots (n-k+1) \cdot (x - x_0)^{n-k}$$

$$= a_k \cdot k! + a_{k+1} \cdot (k+1) \cdots 2 \cdot (x - x_0)$$

$$+ a_{k+2} \cdot (k+2) \cdots 3 \cdot (x - x_0)^2 + \cdots$$

取 $x = x_0$ 得 $a_k = \frac{1}{k!} f^{(k)}(x_0)$



$$f(x) \stackrel{?}{=} a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

$$f(x) \stackrel{?}{=} a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

$$f(x_0)$$

$$f(x) \stackrel{?}{=} a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

$$f(x_0) f'(x_0)$$

$$f(x) \stackrel{?}{=} a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

$$f(x_0)$$
 $f'(x_0)$ $\frac{1}{2!}f''(x_0)$



$$f(x) \stackrel{?}{=} a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

$$f(x_0)$$
 $f'(x_0)$ $\frac{1}{2!}f''(x_0)$ \cdots $\frac{1}{n!}f^{(n)}(x_0)$

$$f(x) \stackrel{?}{=} a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

注 1 也就是,f(x) 至多能展成如下形式的幂级数:

$$f(x_0)+f'(x_0)(x-x_0)+\frac{1}{2!}f''(x_0)(x-x_0)^2+\cdots+\frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n+\cdots$$

$$f(x) \stackrel{?}{=} a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

注1也就是, f(x) 至多能展成如下形式的幂级数:

$$f(x_0)+f'(x_0)(x-x_0)+\frac{1}{2!}f''(x_0)(x-x_0)^2+\cdots+\frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n+\cdots$$

 (x_0) $(x_$

$$f(x) \stackrel{?}{=} a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

注1也就是, f(x) 至多能展成如下形式的幂级数:

$$f(x_0)+f'(x_0)(x-x_0)+\frac{1}{2!}f''(x_0)(x-x_0)^2+\cdots+\frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n+\cdots$$

- 此级数称为 f(x) 在 x_0 处的 泰勒级数。
- 记 p_n 为所有次数 ≤ n 的项之和(部分和),称为 n 次泰勒多项式

$$f(x) \stackrel{?}{=} a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

 $a_n = \frac{1}{n!} f^{(n)}(x_0).$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x)$$

注 1 也就是,f(x) 至多能展成如下形式的幂级数:

$$f(x_0)+f'(x_0)(x-x_0)+\frac{1}{2!}f''(x_0)(x-x_0)^2+\cdots+\frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n+\cdots$$

- 此级数称为 f(x) 在 x_0 处的 泰勒级数。
 - 记 p_n 为所有次数 ≤ n 的项之和(部分和),称为 n 次泰勒多项式

注 2
$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0) (x - x_0)^n$$

$$f(x) \neq a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

注 1 也就是,f(x) 至多能展成如下形式的幂级数:

$$f(x_0)+f'(x_0)(x-x_0)+\frac{1}{2!}f''(x_0)(x-x_0)^2+\cdots+\frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n+\cdots$$

- 此级数称为 $f(x)$ 在 x_0 处的 泰勒级数。

- 记 p_n 为所有次数 ≤ n 的项之和(部分和),称为 n 次泰勒多项式

注 2
$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0) (x - x_0)^n \quad \Leftrightarrow \quad f(x) = \lim_{n \to \infty} p_n(x)$$

解 取
$$x_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

解 取
$$x_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = e^x$$
时,

$$f(x) = f'(x) = f''(x) = f'''(x) = \dots = f^{(n)}(x) = e^x$$

解 取
$$x_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = e^x$$
时,

$$f(x) = f'(x) = f''(x) = f'''(x) = \dots = f^{(n)}(x) = e^x$$

$$\Rightarrow f(0) = f'(0) = f''(0) = f'''(0) = \dots = f^{(n)}(0) = 1$$

解 取
$$x_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = e^x$$
时,

$$f(x) = f'(x) = f''(x) = f'''(x) = \cdots = f^{(n)}(x) = e^x$$

$$\Rightarrow f(0) = f'(0) = f''(0) = f'''(0) = \dots = f^{(n)}(0) = 1$$

⇒
$$\bar{x}$$
 \$\text{\$\pi\$}\$ \$\pi\$\$ \$\pi\$ \$\p

解 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = e^x$$
时,

$$f(x) = f'(x) = f''(x) = f'''(x) = \dots = f^{(n)}(x) = e^x$$

$$\Rightarrow f(0) = f'(0) = f''(0) = f'''(0) = \dots = f^{(n)}(0) = 1$$

⇒
$$\$$$
 勒级数: $1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots + \frac{1}{n!}x^n + \dots$

注 n 次泰勒多项式是:

$$p_n(x) =$$



解取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = e^x$ 时,

$$f(x) = f'(x) = f''(x) = f'''(x) = \dots = f^{(n)}(x) = e^{x}$$

$$\Rightarrow f(0) = f'(0) = f''(0) = f'''(0) = \dots = f^{(n)}(0) = 1$$

⇒
$$\bar{x}$$
 \$\text{\$\shappa\$}\$ \$\tag{\text{\$\pi\$}}\$ \$\text{\$\pi\$}\$ \$\text{\$\pi\$}\$

 $\frac{3}{2!} + \frac{3!}{3!} + \cdots + \frac{n!}{n!} + \cdots$

注 n 次泰勒多项式是:

$$p_n(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots + \frac{1}{n!}x^n$$

例 2 求 $f(x) = \sin x$ 在 x = 0 处的泰勒级数。

$$\mathbf{H}$$
 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = \sin x$$
 时,

解 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = \sin x$ 时,

	$f^{(n)}(x)$	$f^{(n)}(0)$
n = 0, 4, 8	sin x	0
<i>n</i> = 1, 5, 9	cosx	1
n = 2, 6, 10	— sin <i>x</i>	0
n = 3, 7, 11	— cos x	-1

解 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = \sin x$ 时,

	$f^{(n)}(x) = \sin(x + \frac{n}{2}\pi)$	$f^{(n)}(0)$
n = 0, 4, 8	sin x	0
<i>n</i> = 1, 5, 9	cosx	1
n = 2, 6, 10	— sin <i>x</i>	0
n = 3, 7, 11	— cos x	-1

解 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = \sin x$ 时,

	$f^{(n)}(x) = \sin(x + \frac{n}{2}\pi)$	$f^{(n)}(0) = \sin(\frac{n}{2}\pi)$
n = 0, 4, 8	sin x	0
<i>n</i> = 1, 5, 9	cosx	1
n = 2, 6, 10	— sin <i>x</i>	0
n = 3, 7, 11	— cos x	-1

 \mathbf{H} 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = \sin x$ 时,

	$f^{(n)}(x) = \sin(x + \frac{n}{2}\pi)$	$f^{(n)}(0) = \sin(\frac{n}{2}\pi)$
n = 0, 4, 8	sin x	0
$n = 1, 5, 9 \dots$	cosx	1
n = 2, 6, 10	— sin <i>x</i>	0
n = 3, 7, 11	— cos x	-1

所以泰勒级数是

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \cdots$$

例 2 求 $f(x) = \sin x$ 在 x = 0 处的泰勒级数。

 \mathbf{H} 取 $\mathbf{x}_0 = \mathbf{0}$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = \sin x$ 时,

	$f^{(n)}(x) = \sin(x + \frac{n}{2}\pi)$	$f^{(n)}(0) = \sin(n\pi)$
	$\int \nabla f(x) = \sin(x + \frac{1}{2}\pi)$	$\int \sqrt{(0)} = \sin(\frac{\pi}{2}n)$
n = 0, 4, 8	sin x	0
$n = 1, 5, 9 \dots$	cosx	1
<i>n</i> = 2, 6, 10	— sin <i>x</i>	0
<i>n</i> = 3, 7, 11	- cos <i>x</i>	-1

所以泰勒级数是

所以泰朝级数是
$$x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\frac{1}{7!}x^7+\frac{1}{9!}x^9-\frac{1}{11!}x^{11}+\cdots+(-1)^m\frac{1}{(2m+1)!}x^{2m+1}+\cdots$$

sin x 的泰勒级数是:

$$x - \frac{1}{3!}x^{3} + \frac{1}{5!}x^{5} - \frac{1}{7!}x^{7} + \frac{1}{9!}x^{9} - \frac{1}{11!}x^{11} + \dots + (-1)^{m} \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

sin x 的 n 次泰勒多项式是:

$$x - \frac{1}{3!}x^{3} + \frac{1}{5!}x^{5} - \frac{1}{7!}x^{7} + \frac{1}{9!}x^{9} - \frac{1}{11!}x^{11} + \dots + (-1)^{m} \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

$$p_1 = x$$
;

$$x - \frac{1}{3!}x^{3} + \frac{1}{5!}x^{5} - \frac{1}{7!}x^{7} + \frac{1}{9!}x^{9} - \frac{1}{11!}x^{11} + \dots + (-1)^{m} \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

$$p_1 = p_2 = x$$
;

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

$$p_1 = p_2 = x;$$

 $p_3 = x - \frac{1}{3!}x^3;$

 $p_1 = p_2 = x$;

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

$$p_3 = p_4 = x - \frac{1}{3!}x^3;$$

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

$$p_1 = p_2 = x;$$

$$p_3 = p_4 = x - \frac{1}{3!}x^3;$$

$$p_5 = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5;$$

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

$$p_1 = p_2 = x;$$

$$p_3 = p_4 = x - \frac{1}{3!}x^3;$$

$$p_5 = p_6 = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5;$$

$$x - \frac{1}{3!}x^{3} + \frac{1}{5!}x^{5} - \frac{1}{7!}x^{7} + \frac{1}{9!}x^{9} - \frac{1}{11!}x^{11} + \dots + (-1)^{m} \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

sin x 的 n 次泰勒多项式是:

 $p_1 = p_2 = x$;

$$p_3 = p_4 = x - \frac{1}{3!}x^3;$$

$$p_5 = p_6 = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5;$$

$$\vdots$$

 p_{2m+1}



$$\frac{1}{2}$$
 $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{7}{2}$

 $p_1 = p_2 = x$;

 p_{2m+1}

 $x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$

$$1_{x_{3}}$$
 $1_{x_{5}}$ $1_{x_{7}}$

sin x 的 n 次泰勒多项式是:

 $p_3 = p_4 = x - \frac{1}{3!}x^3;$

 $p_5 = p_6 = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5;$

 $= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1}$

$$\frac{1}{2}$$
 $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$

 $p_1 = p_2 = x$;

 $x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$

sin x 的 n 次泰勒多项式是:

 $p_3 = p_4 = x - \frac{1}{3!}x^3;$

 $p_5 = p_6 = x - \frac{1}{2!}x^3 + \frac{1}{5!}x^5;$

 $p_{2m+1} = p_{2m+2} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1}$

- sin x 的泰勒级数是:

解 取
$$x_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = \cos x$$
时,

解 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = \cos x$ 时,

	$f^{(n)}(x)$	$f^{(n)}(0)$
n = 0, 4, 8	cosx	1
<i>n</i> = 1, 5, 9	— sin <i>x</i>	0
n = 2, 6, 10	— cos x	-1
n = 3, 7, 11	sin x	0

解 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = \cos x$ 时,

	$f^{(n)}(x) = \cos(x + \frac{n}{2}\pi)$	f ⁽ⁿ⁾ (0)
n = 0, 4, 8	cosx	1
n = 1, 5, 9	— sin <i>x</i>	0
n = 2, 6, 10	— cos <i>x</i>	-1
<i>n</i> = 3, 7, 11	sin x	0

解 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = \cos x$ 时,

	$f^{(n)}(x) = \cos(x + \frac{n}{2}\pi)$	$f^{(n)}(0) = \cos(\frac{n}{2}\pi)$
n = 0, 4, 8	cosx	1
$n = 1, 5, 9 \dots$	— sin <i>x</i>	0
n = 2, 6, 10	— cos <i>x</i>	-1
n = 3, 7, 11	sin x	0

$$\mathbf{H}$$
 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = \cos x$ 时,

	$f^{(n)}(x) = \cos(x + \frac{n}{2}\pi)$	$f^{(n)}(0) = \cos(\frac{n}{2}\pi)$
n = 0, 4, 8	cosx	1
n = 1, 5, 9	— sin <i>x</i>	0
n = 2, 6, 10	— cos x	-1
<i>n</i> = 3, 7, 11	sin x	0

所以泰勒级数是

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \cdots$$



解 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = \cos x$ 时,

	$f^{(n)}(x) = \cos(x + \frac{n}{2}\pi)$	$f^{(n)}(0) = \cos(\frac{n}{2}\pi)$
n = 0, 4, 8	cosx	1
n = 1, 5, 9	— sin <i>x</i>	0
<i>n</i> = 2, 6, 10	— cos x	-1
<i>n</i> = 3, 7, 11	sin x	0

所以泰勒级数是

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$



$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

$$p_0 = 1;$$

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

$$p_0 = p_1 = 1$$
;

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

$$p_0 = p_1 = 1;$$

 $p_2 = 1 - \frac{1}{2!}x^2;$

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

● cos x 的 n 次泰勒多项式是:

$$p_2 = p_3 = 1 - \frac{1}{2!}x^2;$$

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

• cos x 的 n 次泰勒多项式是:

$$p_2 = p_3 = 1 - \frac{1}{2!}x^2;$$

$$p_4 = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4;$$



$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

● cos x 的 n 次泰勒多项式是:

$$p_2 = p_3 = 1 - \frac{1}{2!}x^2;$$

 $p_4 = p_5 = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4;$

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

cos x 的 n 次泰勒多项式是:

 $p_0 = p_1 = 1$;

$$p_{2} = p_{3} = 1 - \frac{1}{2!}x^{2};$$

$$p_{4} = p_{5} = 1 - \frac{1}{2!}x^{2} + \frac{1}{4!}x^{4};$$
:

 $p_{2m}(x)$



$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

cos x 的 n 次泰勒多项式是:

 $p_0 = p_1 = 1$;

$$p_2 = p_3 = 1 - \frac{1}{2!}x^2;$$

$$p_4 = p_5 = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4;$$

 $=1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\cdots+(-1)^m\frac{1}{(2m)!}x^{2m}$







 $p_{2m}(x)$

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

 $p_{2m}(x) = p_{2m+1}(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots + (-1)^m \frac{1}{(2m)!}x^{2m}$

cos x 的 n 次泰勒多项式是:

$$p_2 = p_3 = 1 - \frac{1}{2!}x^2;$$

$$p_4 = p_5 = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4;$$





 $\mathbf{H} \mathbf{H} \mathbf{X}_0 = \mathbf{0} \mathbf{H}$,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = \ln(1+x)$$
时,

解 取
$$x_0 = 0$$
 时,泰勒级数是
$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = \ln(1+x)$$
时,
 $f = \ln(1+x)$, $f' = \frac{1}{1+x}$,

解 取
$$x_0 = 0$$
 时,泰勒级数是
$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = \ln(1+x)$$
时,
 $f = \ln(1+x)$, $f' = \frac{1}{1+x}$, $f'' = \frac{(-1)}{(1+x)^2}$,

解 取
$$x_0 = 0$$
 时,泰勒级数是
$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = \ln(1+x)$$
时,
 $f = \ln(1+x)$, $f' = \frac{1}{1+x}$, $f'' = \frac{(-1)}{(1+x)^2}$, $f''' = \frac{2}{(1+x)^3}$,

解 取
$$x_0 = 0$$
 时,泰勒级数是
$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

$$f^{(4)} = \frac{-2 \cdot 3}{(1+x)^4},$$

解 取
$$x_0 = 0$$
 时,泰勒级数是
$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

$$f^{(4)} = \frac{-2 \cdot 3}{(1+x)^4}, \quad f^{(5)} = \frac{2 \cdot 3 \cdot 4}{(1+x)^5}, \dots$$

 \mathbf{H} 取 $\mathbf{x}_0 = \mathbf{0}$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = \ln(1+x)$ 时, $f = \ln(1+x)$, $f' = \frac{1}{1+x}$, $f'' = \frac{(-1)}{(1+x)^2}$, $f''' = \frac{2}{(1+x)^3}$,

$$f = \ln(1+x), \quad f' = \frac{1}{1+x}, \quad f'' = \frac{(-1)^2}{(1+x)^2}, \quad f''' = \frac{1}{(1+x)^3},$$
$$f^{(4)} = \frac{-2 \cdot 3}{(1+x)^4}, \quad f^{(5)} = \frac{2 \cdot 3 \cdot 4}{(1+x)^5}, \dots, f^{(n)} = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}, \dots$$

解 取
$$x_0 = 0$$
 时,泰勒级数是
$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

$$f(0) + f'(0)x + \frac{f'(0)}{2!}x^2 + \frac{f'(0)}{3!}x^3 + \dots + \frac{f'(0)}{n!}x^n + \dots$$

当
$$f(x) = \ln(1+x)$$
 时,
 $f = \ln(1+x)$, $f' = \frac{1}{1+x}$, $f'' = \frac{(-1)}{(1+x)^2}$, $f''' = \frac{2}{(1+x)^3}$,
 $f^{(4)} = \frac{-2 \cdot 3}{(1+x)^4}$, $f^{(5)} = \frac{2 \cdot 3 \cdot 4}{(1+x)^5}$, ..., $f^{(n)} = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$, ...

所以
$$\frac{1}{n!}f^{(n)}(0) = \frac{(-1)^{n-1}}{n}$$
,

例 $4 \, \bar{x} f(x) = \ln(1+x) \, \bar{x} = 0 \,$ 处泰勒级数。

m 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = \ln(1+x)$$
时,

$$f = \ln(1+x), \quad f' = \frac{1}{1+x}, \quad f'' = \frac{(-1)}{(1+x)^2}, \quad f''' = \frac{2}{(1+x)^3},$$

$$f^{(4)} = \frac{-2 \cdot 3}{(1+x)^4}, \quad f^{(5)} = \frac{2 \cdot 3 \cdot 4}{(1+x)^5}, \dots, f^{(n)} = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}, \dots$$

$$\text{MU} \frac{1}{n!} f^{(n)}(0) = \frac{(-1)^{n-1}}{n}, \quad \text{\$$ $\overline{4}$ $\overline{$$

所以
$$\frac{1}{n!} f^{(n)}(0) = \frac{1}{n}$$
 , 泰朝级数是
$$x - \frac{1}{2} x^2 + \frac{1}{3} x^3 - \frac{1}{4} x^4 + \cdots$$

例 $4 \, \bar{x} f(x) = \ln(1+x) \, \bar{x} = 0 \,$ 处泰勒级数。

解 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = \ln(1+x)$ 时,

$$f = \ln(1+x), \quad f' = \frac{1}{1+x}, \quad f'' = \frac{(-1)}{(1+x)^2}, \quad f''' = \frac{2}{(1+x)^3},$$

$$f^{(4)} = \frac{-2 \cdot 3}{(1+x)^4}, \quad f^{(5)} = \frac{2 \cdot 3 \cdot 4}{(1+x)^5}, \dots, f^{(n)} = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}, \dots$$

$$x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots$$

 \mathbf{H} 取 $\mathbf{x}_0 = \mathbf{0}$ 时,泰勒级数是

解 取
$$X_0 = 0$$
 的, 泰剌级数 $f''(0)$

 $f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$

当 $f(x) = \ln(1+x)$ 时,

所以 $\frac{1}{n!}f^{(n)}(0) = \frac{(-1)^{n-1}}{n}$,泰勒级数是

例 4 求 $f(x) = \ln(1 + x)$ 在 x = 0 处泰勒级数。

 $f^{(4)} = \frac{-2 \cdot 3}{(1+x)^4}, \quad f^{(5)} = \frac{2 \cdot 3 \cdot 4}{(1+x)^5}, \dots, f^{(n)} = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}, \dots$

 $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots$

 $f = \ln(1+x), \quad f' = \frac{1}{1+x}, \quad f'' = \frac{(-1)}{(1+x)^2}, \quad f''' = \frac{2}{(1+x)^3},$

注 n 次泰勒多项式是: $p_n(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots + (-1)^{n-1}\frac{1}{n}x^n$

解 取
$$x_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = (1+x)^{\alpha}$$
时,

解 取
$$x_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = (1+x)^{\alpha}$$
 时,
 $f = (1+x)^{\alpha}$, $f' = \alpha(1+x)^{\alpha-1}$,

$$\mathbf{H}$$
 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = (1+x)^{\alpha}$$
时,

$$f = (1+x)^{\alpha}$$
, $f' = \alpha(1+x)^{\alpha-1}$, $f'' = \alpha(\alpha-1)(1+x)^{\alpha-2}$,

$$\mathbf{H}$$
 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = (1+x)^{\alpha}$$
 时,

$$f = (1+x)^{\alpha}, \quad f' = \alpha(1+x)^{\alpha-1}, \quad f'' = \alpha(\alpha-1)(1+x)^{\alpha-2},$$

$$\ldots, f^{(n)} = \alpha(\alpha - 1) \cdots (\alpha - n + 1)(1 + x)^{\alpha - n}, \cdots$$

$$\mathbf{H}$$
 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = (1+x)^{\alpha}$$
 时,

$$f = (1+x)^{\alpha}, \quad f' = \alpha(1+x)^{\alpha-1}, \quad f'' = \alpha(\alpha-1)(1+x)^{\alpha-2},$$

$$\ldots, f^{(n)} = \alpha(\alpha - 1) \cdots (\alpha - n + 1)(1 + x)^{\alpha - n}, \cdots$$

所以
$$\frac{1}{n!}f^{(n)}(0) = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$$
,

解
$$\mathbf{x}_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = (1+x)^{\alpha}$$
 时,

$$f = (1+x)^{\alpha}$$
, $f' = \alpha(1+x)^{\alpha-1}$, $f'' = \alpha(\alpha-1)(1+x)^{\alpha-2}$,

$$\ldots, f^{(n)} = \alpha(\alpha - 1) \cdots (\alpha - n + 1)(1 + x)^{\alpha - n}, \cdots$$

所以
$$\frac{1}{n!}f^{(n)}(0) = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$$
,泰勒级数是

$$1 + \alpha x + \frac{\alpha(\alpha - 1)}{2!}x^2 + \cdots$$

解 取
$$x_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = (1+x)^{\alpha}$$
 时,

$$f = (1+x)^{\alpha}, \quad f' = \alpha(1+x)^{\alpha-1}, \quad f'' = \alpha(\alpha-1)(1+x)^{\alpha-2},$$

$$\ldots, f^{(n)} = \alpha(\alpha - 1) \cdots (\alpha - n + 1)(1 + x)^{\alpha - n}, \cdots$$

所以
$$\frac{1}{n!}f^{(n)}(0) = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$$
, 泰勒级数是

$$1 + \alpha x + \frac{\alpha(\alpha - 1)}{2!}x^2 + \dots + \frac{\alpha(\alpha - 1)\cdots(\alpha - n + 1)}{n!}x^n + \dots$$

 $\mathbf{m} \mathbf{n} \mathbf{x}_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{2!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = (1+x)^{\alpha}$ 时,

$$f = (1+x)^{\alpha}, \quad f' = \alpha(1+x)^{\alpha-1}, \quad f'' = \alpha(\alpha-1)(1+x)^{\alpha-2},$$

$$\ldots, f^{(n)} = \alpha(\alpha - 1) \cdots (\alpha - n + 1)(1 + x)^{\alpha - n}, \cdots$$

所以
$$\frac{1}{n!}f^{(n)}(0) = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$$
,泰勒级数是
$$\alpha(\alpha-1) = \alpha(\alpha-1)\cdots(\alpha-n+1)$$

$$1 + \alpha x + \frac{\alpha(\alpha - 1)}{2!}x^2 + \dots + \frac{\alpha(\alpha - 1)\cdots(\alpha - n + 1)}{n!}x^n + \dots$$

注 n 次泰勒多项式是:

$$p_n(x) =$$



 $\mathbf{H} \mathbf{H} \mathbf{X}_0 = \mathbf{0} \mathbf{H}$,泰勒级数是

当 $f(x) = (1+x)^{\alpha}$ 时,

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

例 5 求 $f(x) = (1 + x)^{\alpha}$ 在 x = 0 处的 n 次泰勒多项式 $p_n(x)$

 $f = (1+x)^{\alpha}$, $f' = \alpha(1+x)^{\alpha-1}$, $f'' = \alpha(\alpha-1)(1+x)^{\alpha-2}$ $f^{(n)} = \alpha(\alpha - 1) \cdots (\alpha - n + 1)(1 + x)^{\alpha - n}, \cdots$

$$\dots, f^{(n)} = \alpha(\alpha - 1) \cdots (\alpha - n + 1)(1 + x)^{\alpha - n}, \dots$$
所以 $\frac{1}{n!} f^{(n)}(0) = \frac{\alpha(\alpha - 1) \cdots (\alpha - n + 1)}{n!}$,泰勒级数是

所以
$$\frac{1}{n!}f^{(n)}(0) = \frac{\alpha(\alpha-1)}{n!}$$
, 泰朝级数是
$$1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \dots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n + \dots$$

注 n 次泰勒多项式是: $p_n(x) = 1 + \alpha x + \frac{\alpha(\alpha - 1)}{2!}x^2 + \dots + \frac{\alpha(\alpha - 1)\cdots(\alpha - n + 1)}{n!}x^n$

回到问题:对哪些 x, f(x) 等于其泰勒级数 $\sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x_0)(x-x_0)^k$?

$$f(x) \stackrel{\overline{\text{ϕ h - diz} \#}}{====} p_n(x) + R_n(x)$$

回到问题: 对哪些 x, f(x) 等于其泰勒级数 $\sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x_0)(x-x_0)^k$?

想法

$$f(x) \stackrel{\overline{\text{δ}} \text{η-$diz}}{=\!=\!=\!=\!=} p_n(x) + R_n(x)$$

- 1. x 在 f 的定义域中; 并且
- 2. x 是泰勒级数 $\sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x_0)(x-x_0)^k$ 的收敛点;并且

回到问题:对哪些
$$x$$
, $f(x)$ 等于其泰勒级数 $\sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x_0)(x-x_0)^k$?

想法

$$f(x) \xrightarrow{\frac{\pi}{n} + \text{dic} 2} p_n(x) + R_n(x) \quad \Rightarrow \quad f(x) = \lim_{n \to \infty} p_n(x) + \lim_{n \to \infty} R_n(x)$$

- 1. x 在 f 的定义域中; 并且
- 2. x 是泰勒级数 $\sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x_0) (x x_0)^k$ 的收敛点;并且

回到问题:对哪些
$$x$$
, $f(x)$ 等于其泰勒级数 $\sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x_0)(x-x_0)^k$?

想法

$$f(x) \xrightarrow{\overline{\$} \text{ ψ-fice}} p_n(x) + R_n(x) \quad \Rightarrow \quad f(x) = \lim_{n \to \infty} p_n(x) + \lim_{n \to \infty} R_n(x)$$
如果

- 1. x 在 f 的定义域中; 并且
- 2. x 是泰勒级数 $\sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x_0) (x x_0)^k$ 的收敛点;并且
- $\lim_{n\to\infty}R_n(x)=0$

回到问题:对哪些
$$x$$
, $f(x)$ 等于其泰勒级数 $\sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x_0)(x-x_0)^k$?

想法

$$f(x) \xrightarrow{\frac{\pi}{n} + \text{dic}} p_n(x) + R_n(x) \Rightarrow f(x) = \lim_{n \to \infty} p_n(x) + \lim_{n \to \infty} R_n(x)$$
如果

- 1. x 在 f 的定义域中; 并且
- 2. x 是泰勒级数 $\sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x_0)(x-x_0)^k$ 的收敛点;并且
- $3. \lim_{n\to\infty} R_n(x) = 0$

则对此x成立

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x_0) (x - x_0)^k$$



回忆 泰勒中值定理 1 若 f 具有 n 阶导数,则

$$R_n(x) = o((x-x_0)^n).$$

回忆 泰勒中值定理 1 若 f 具有 n 阶导数,则

$$R_n(x) = o((x-x_0)^n).$$

特别地,

$$f(x) = p_n(x) + R_n(x)$$

回忆 泰勒中值定理 1 若 f 具有 n 阶导数,则

$$R_n(x) = o((x-x_0)^n).$$

特别地,

$$f(x) = p_n(x) + R_n(x)$$

$$= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + o((x - x_0)^n)$$

 $e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + o(x^{n})$

$$(1+x)^{\alpha}$$

$$(1+x)^{\alpha}=1$$

 $(1+x)^{\alpha} = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \dots + \frac{\alpha(\alpha-1) \cdots (\alpha-n+1)}{n!} x^n + o(x^n)$

$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots + (-1)^{n-1}\frac{1}{n}x^n + o(x^n)$

 $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + o(x^{2m+2})$

 $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + o(x^{2m+1})$





例求 $\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$, $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1-x)]}$

例求
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
, $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1-x)]}$

$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$

例求
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
, $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1-x)]}$

$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$$



例求
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
, $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1-x)]}$

$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$$

$$= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3}$$

$$= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3}$$

例求
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
, $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1-x)]}$

$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$$

$$\lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{1}$$

$$= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3} = \frac{1}{3}$$



例求
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
, $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1-x)]}$

例求
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
, $\lim_{x\to 0} \frac{\cos x - e^{-2}}{x^2 [x + \ln(1 - x)]}$
 $\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x\to 0} \frac{\left[x - \frac{1}{3!}x^3 + \frac{1}$

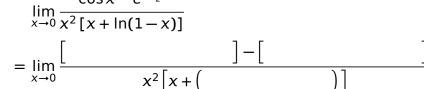
$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$$
$$= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3} = \frac{1}{3}$$

$$= \lim_{x \to 0} \frac{\overline{3}x^3 + b(x^4)}{x^3} = \frac{1}{3}$$

$$\cos x - e^{-\frac{x^2}{2}}$$

$$= \lim_{x \to 0} \frac{1}{x^3} = \frac{1}{x^3}$$

$$\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{\sin \frac{x^3}{2}} = \frac{1}{x^3}$$



例求
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
, $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1-x)]}$

$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$$

$$= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3} = \frac{1}{3}$$

$$\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1 - x)]}$$

$$\lim_{x \to 0} \frac{1}{x^2 [x + \ln(1 - x)]}$$

$$= \lim_{x \to 0} \left[- \lim_{x \to 0}$$

$$= \lim_{x \to 0} \frac{\left[\frac{1}{x^2 \left[x + \ln(1 - x) \right]} - \left[\frac{1}{x^2 \left[x + \left(-x - \frac{1}{2}x^2 + o(x^2) \right) \right]} \right]}$$

例求
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
, $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1-x)]}$

$$\lim_{x \to 0} \frac{\sin^3 x}{\sin^3 x}, \frac{\sin^3 x}{x^2 [x + \ln(1 - x)]}$$

$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$$

$$\frac{1}{2}x^3 + o(x^4) = 1$$

$$= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3} = \frac{1}{3}$$

$$\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2 [x + \ln(1 - x)]}$$

$$= \lim_{x \to 0} \frac{\left[1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + o(x^5)\right] - \left[\frac{1}{2!}x^2 + \frac{1}{4!}x^4 + o(x^5)\right]}{x^2 [x + \left(-x - \frac{1}{2}x^2 + o(x^2)\right)]}$$

例求
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
, $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1-x)]}$

例 求
$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
, $\lim_{x \to 0} \frac{\cos x - e^{-2}}{x^2 [x + \ln(1 - x)]}$

$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$$

$$= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3} = \frac{1}{3}$$

$$\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2 [x + \ln(1 - x)]}$$

$$-e^{-\frac{1}{2}}$$

$$1(1-x)$$

$$\overline{(-x)}$$

$$\frac{1}{4!}x^4 + o(x^5) - \left[1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{2}x^4 + o(x^5)\right]$$

$$= \lim_{x \to 0} \frac{\left[1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + o(x^5)\right] - \left[1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + o(x^4)\right]}{x^2 \left[x + \left(-x - \frac{1}{2}x^2 + o(x^2)\right)\right]}$$

例求
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
, $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1-x)]}$

$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$$
$$= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3} = \frac{1}{3}$$

 $\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2 [x + \ln(1 - x)]}$ $= \lim_{x \to 0} \frac{\left[1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + o(x^5)\right] - \left[1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + o(x^4)\right]}{x^2 \left[x + \left(-x - \frac{1}{2}x^2 + o(x^2)\right)\right]}$

$$= \lim_{x \to 0} \frac{x^2 \left[x \right]}{-\frac{1}{2}x^4 + o(x^4)}$$





例求
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
, $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1-x)]}$

 $\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$ $= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3} = \frac{1}{2}$

$$\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2 [x + \ln(1 - x)]}$$

 $= \lim_{x \to 0} \frac{\left[1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + o(x^5)\right] - \left[1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + o(x^4)\right]}{x^2 \left[x + \left(-x - \frac{1}{2}x^2 + o(x^2)\right)\right]}$

$$= \lim_{x \to 0} \frac{-\frac{1}{12}x^4 + o(x^4)}{-\frac{1}{2}x^4 + o(x^4)} = \lim_{x \to 0} \frac{-\frac{1}{12} + o(x^4)/x^4}{-\frac{1}{2} + o(x^4)/x^4}$$

$$\frac{\sin x - x \cos x}{\sin^3 x}$$

例求 $\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$, $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2 [x + \ln(1-x)]}$

 $\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$ $= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3} = \frac{1}{3}$

 $\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2 [x + \ln(1 - x)]}$

 $= \lim_{x \to 0} \frac{-\frac{1}{12}x^4 + o(x^4)}{-\frac{1}{2}x^4 + o(x^4)} = \lim_{x \to 0} \frac{-\frac{1}{12} + o(x^4)/x^4}{-\frac{1}{2} + o(x^4)/x^4} = \frac{1}{6}$

 $= \lim_{x \to 0} \frac{\left[1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + o(x^5)\right] - \left[1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + o(x^4)\right]}{x^2 \left[x + \left(-x - \frac{1}{2}x^2 + o(x^2)\right)\right]}$

d: 函数展开成幂级数

泰勒中值定理 2 若 f 具有 n+1 阶导数,则

$$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) (x - x_0)^{n+1}$$

其中 ξ 是 x_0 与 x 之间的某个值

泰勒中值定理 2 若 f 具有 n+1 阶导数,则

$$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) (x - x_0)^{n+1}$$

$$(1-\theta)x_0 + \theta x$$

其中 ξ 是 x_0 与 x 之间的某个值, $0 < \theta < 1$ 。

泰勒中值定理 2 若 f 具有 n+1 阶导数,则

$$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) (x - x_0)^{n+1}$$

$$\stackrel{or}{=} \frac{1}{(n+1)!} f^{(n+1)}((1-\theta)x_0 + \theta x) (x - x_0)^{n+1},$$

其中 ξ 是 x_0 与 x 之间的某个值, $0 < \theta < 1$ 。

泰勒中值定理 2 若 f 具有 n + 1 阶导数,则

$$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) (x - x_0)^{n+1}$$

$$\stackrel{or}{=} \frac{1}{(n+1)!} f^{(n+1)}((1-\theta)x_0 + \theta x) (x - x_0)^{n+1},$$

其中 ξ 是 x_0 与 x 之间的某个值, $0 < \theta < 1$ 。

注

1. ξ (以及 θ) 不是固定不变的,而是随 x 和 n 的改变而变化。

泰勒中值定理 2 若 f 具有 n+1 阶导数,则

$$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) (x - x_0)^{n+1}$$

$$\stackrel{or}{=} \frac{1}{(n+1)!} f^{(n+1)}((1-\theta)x_0 + \theta x) (x - x_0)^{n+1},$$

其中 ξ 是 x_0 与 x 之间的某个值, $0 < \theta < 1$ 。

注

- 1. ξ (以及 θ) 不是固定不变的,而是随 x 和 n 的改变而变化。
- 2. 当 $x_0 = 0$ 时,则余项可写成

$$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1}, \qquad (0 < \theta < 1)$$



$$e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty).$$

$$e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty).$$

证明

1. 只需证明对任意 x,成立 $\lim_{n\to\infty} R_n(x) = 0$ 。

$$e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty).$$

- 1. 只需证明对任意 x,成立 $\lim_{n\to\infty} R_n(x) = 0$ 。
- 2. 由泰勒中值定理 2,

$$|R_n(x)| = \left| \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1} \right|$$

$$e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty).$$

- 1. 只需证明对任意 x,成立 $\lim_{n\to\infty} R_n(x) = 0$ 。
- 2. 由泰勒中值定理 2,

$$|R_n(x)| = \left| \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1} \right| = \left| \frac{e^{\theta x} x^{n+1}}{(n+1)!} \right|$$

$$e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty).$$

- 1. 只需证明对任意 x,成立 $\lim_{n \to \infty} R_n(x) = 0$ 。
- 2. 由泰勒中值定理 2.

$$|R_n(x)| = \left| \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1} \right| = \left| \frac{e^{\theta x} x^{n+1}}{(n+1)!} \right| \le \frac{|x|^{n+1}}{(n+1)!} e^{|x|}$$

$$e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty).$$

- 1. 只需证明对任意 x,成立 $\lim_{n \to \infty} R_n(x) = 0$ 。
- 2. 由泰勒中值定理 2.

$$|R_n(x)| = \left| \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1} \right| = \left| \frac{e^{\theta x} x^{n+1}}{(n+1)!} \right| \le \frac{|x|^{n+1}}{(n+1)!} e^{|x|} \to 0$$



$$e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty).$$

证明

- 1. 只需证明对任意 x,成立 $\lim_{n \to \infty} R_n(x) = 0$ 。
- 2. 由泰勒中值定理 2.

$$|R_n(x)| = \left| \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1} \right| = \left| \frac{e^{\theta x} x^{n+1}}{(n+1)!} \right| \le \frac{|x|^{n+1}}{(n+1)!} e^{|x|} \to 0$$
(日知级数 $\sum \frac{|x|^{n+1}}{n} |x|^{n+1} = 0$)

(已知级数 $\sum_{(n+1)}^{|x|^{n+1}}$ 收敛,所以一般项 $\frac{|x|^{n+1}}{(n+1)!} \to 0$)

$$e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty).$$

证明

- 1. 只需证明对任意 x, 成立 $\lim_{n \to \infty} R_n(x) = 0$ 。
- 2. 由泰勒中值定理 2.

$$|R_n(x)| = \left| \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1} \right| = \left| \frac{e^{\theta x} x^{n+1}}{(n+1)!} \right| \le \frac{|x|^{n+1}}{(n+1)!} e^{|x|} \to 0$$

$$(已知级数 \sum_{n=1}^{|x|^{n+1}} |\psi_n|, \quad \text{MU} - \text{MV} \frac{|x|^{n+1}}{(n+1)!} \to 0)$$

注 $e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$



$$e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty).$$

证明

- 1. 只需证明对任意 x, 成立 $\lim_{n \to \infty} R_n(x) = 0$ 。
- 2. 由泰勒中值定理 2.

$$\begin{aligned} |R_n(x)| &= \left| \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1} \right| = \left| \frac{e^{\theta x} x^{n+1}}{(n+1)!} \right| \le \frac{|x|^{n+1}}{(n+1)!} e^{|x|} \to 0 \\ & (\exists \text{ Hand } \sum \frac{|x|^{n+1}}{(n+1)!} \text{ Wad, } \text{ Mind } \frac{|x|^{n+1}}{(n+1)!} \to 0) \end{aligned}$$

注 $e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots = \sum_{n=0}^{\infty} \frac{1}{n!}$



$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

其中
$$x$$
 ∈ $(-\infty, \infty)$ 。

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

其中x ∈ $(-\infty, \infty)$ 。

证明

1. 只需证明对任意 x,成立 $\lim_{n\to\infty} R_n(x) = 0$ 。

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

其中 $x \in (-\infty, \infty)$ 。

- 1. 只需证明对任意 x,成立 $\lim_{n\to\infty} R_n(x) = 0$ 。
- 2. 由泰勒中值定理 2,

$$|R_n(x)| = \left| \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1} \right|$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

其中 $x \in (-\infty, \infty)$ 。

- 1. 只需证明对任意 x,成立 $\lim_{n\to\infty} R_n(x) = 0$ 。
- 2. 由泰勒中值定理 2,

$$|R_n(x)| = \left| \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1} \right| = \left| \frac{\sin\left(\theta x + \frac{n+1}{2}\pi\right) x^{n+1}}{(n+1)!} \right|$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

其中x ∈ $(-\infty, \infty)$ 。

- 1. 只需证明对任意 x,成立 $\lim_{n\to\infty} R_n(x) = 0$ 。
- 2. 由泰勒中值定理 2.

$$|R_n(x)| = \left| \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1} \right| = \left| \frac{\sin\left(\theta x + \frac{n+1}{2}\pi\right) x^{n+1}}{(n+1)!} \right|$$

$$\leq \frac{|x|^{n+1}}{(n+1)!}$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

其中x ∈ $(-\infty, \infty)$ 。

- 1. 只需证明对任意 x,成立 $\lim_{n\to\infty} R_n(x) = 0$ 。
- 2. 由泰勒中值定理 2.

$$|R_n(x)| = \left| \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1} \right| = \left| \frac{\sin\left(\theta x + \frac{n+1}{2}\pi\right) x^{n+1}}{(n+1)!} \right|$$

$$\leq \frac{|x|^{n+1}}{(n+1)!} \to 0$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

其中 $x \in (-\infty, \infty)$ 。

证明

- 1. 只需证明对任意 x, 成立 $\lim_{n\to\infty} R_n(x) = 0$ 。
- 2. 由泰勒中值定理 2.

$$|R_n(x)| = \left| \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1} \right| = \left| \frac{\sin\left(\theta x + \frac{n+1}{2}\pi\right) x^{n+1}}{(n+1)!} \right|$$

 $\leq \frac{|X|^{n+1}}{(n+1)!} \to 0$

(已知级数 $\sum \frac{|x|^{n+1}}{(n+1)!}$ 收敛,所以一般项 $\frac{|x|^{n+1}}{(n+1)!} \to 0$)



$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

其中 $x \in (-\infty, \infty)$ 。

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

其中x ∈ $(-\infty, \infty)$ 。

证明

1. 只需证明对任意 x,成立 $\lim_{n\to\infty} R_n(x) = 0$ 。

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

其中 $x \in (-\infty, \infty)$ 。

- 1. 只需证明对任意 x,成立 $\lim_{n\to\infty} R_n(x) = 0$ 。
- 2. 由泰勒中值定理 2,

$$|R_n(x)| = \left| \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1} \right|$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

其中 $x \in (-\infty, \infty)$ 。

- 1. 只需证明对任意 x,成立 $\lim_{n\to\infty} R_n(x) = 0$ 。
- 2. 由泰勒中值定理 2,

$$|R_n(x)| = \left| \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1} \right| = \left| \frac{\cos\left(\theta x + \frac{n+1}{2}\pi\right) x^{n+1}}{(n+1)!} \right|$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

其中 $x \in (-\infty, \infty)$ 。

- 1. 只需证明对任意 x,成立 $\lim_{n\to\infty} R_n(x) = 0$ 。
- 2. 由泰勒中值定理 2,

$$|R_n(x)| = \left| \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1} \right| = \left| \frac{\cos\left(\theta x + \frac{n+1}{2}\pi\right) x^{n+1}}{(n+1)!} \right|$$

$$\leq \frac{|x|^{n+1}}{(n+1)!}$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

其中x ∈ $(-\infty, \infty)$ 。

- 1. 只需证明对任意 x,成立 $\lim_{n\to\infty} R_n(x) = 0$ 。
- 2. 由泰勒中值定理 2.

$$|R_n(x)| = \left| \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1} \right| = \left| \frac{\cos\left(\theta x + \frac{n+1}{2}\pi\right) x^{n+1}}{(n+1)!} \right|$$

$$\leq \frac{|x|^{n+1}}{(n+1)!} \to 0$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

其中x ∈ $(-\infty, \infty)$ 。

证明

- 1. 只需证明对任意 x,成立 $\lim_{n\to\infty} R_n(x) = 0$ 。
- 2. 由泰勒中值定理 2.

$$|R_n(x)| = \left| \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1} \right| = \left| \frac{\cos\left(\theta x + \frac{n+1}{2}\pi\right) x^{n+1}}{(n+1)!} \right|$$

$$\leq \frac{|x|^{n+1}}{(n+1)!} \to 0$$

(已知级数 $\sum \frac{|x|^{n+1}}{(n+1)!}$ 收敛,所以一般项 $\frac{|x|^{n+1}}{(n+1)!} \to 0$)



$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$$

证明

1. 级数收敛域 (-1, 1], 函数定义域 (-1, ∞), 故上式至多对 $x \in (-1, 1]$ 成立。

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$$

- 1. 级数收敛域 (-1, 1], 函数定义域 (-1, ∞), 故上式至多对 $x \in (-1, 1]$ 成立。
- 2. 再证明对任意 $x \in (-1, 1]$, 成立 $\lim_{n \to \infty} R_n(x) = 0$ 。

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$$

- 1. 级数收敛域 (-1, 1], 函数定义域 (-1, ∞), 故上式至多对 $x \in (-1, 1]$ 成立。
- 2. 再证明对任意 $x \in (-1, 1]$, 成立 $\lim_{n \to \infty} R_n(x) = 0$ 。
- 3. 由泰勒中值定理 2,

$$|R_n(x)| = \left| \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1} \right|$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$$

- 1. 级数收敛域 (-1, 1], 函数定义域 (-1, ∞), 故上式至多对 $x \in (-1, 1]$ 成立。
- 2. 再证明对任意 $x \in (-1, 1]$, 成立 $\lim_{n \to \infty} R_n(x) = 0$.
- 3. 由泰勒中值定理 2.

$$|R_n(x)| = \left| \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1} \right| = \left| \frac{1}{(n+1)!} \cdot \frac{(-1)^n n!}{(1+\theta x)^{n+1}} x^{n+1} \right|$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$$

- 1. 级数收敛域 (-1, 1], 函数定义域 (-1, ∞), 故上式至多对 $x \in (-1, 1]$ 成立。
- 2. 再证明对任意 $x \in (-1, 1]$, 成立 $\lim_{n \to \infty} R_n(x) = 0$.
- 3. 由泰勒中值定理 2.

$$|R_n(x)| = \left| \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1} \right| = \left| \frac{1}{(n+1)!} \cdot \frac{(-1)^n n!}{(1+\theta x)^{n+1}} x^{n+1} \right|$$

$$\leq \left| \frac{x^{n+1}}{(1+n)(1+\theta x)^{n+1}} \right|$$



$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$$

证明

- 1. 级数收敛域 (-1, 1], 函数定义域 (-1, ∞), 故上式至多对 $x \in (-1, 1]$ 成立。

 - 2. 再证明对任意 $x \in (-1, 1]$, 成立 $\lim_{n \to \infty} R_n(x) = 0$ 。

$$|R_n(x)| = \left| \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1} \right| = \left| \frac{1}{(n+1)!} \cdot \frac{(-1)^n n!}{(1+\theta x)^{n+1}} x^{n+1} \right|$$

3. 由泰勒中值定理 2.

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$$

证明

- 1. 级数收敛域 (-1, 1], 函数定义域 (-1, ∞), 故上式至多对 $x \in (-1, 1]$ 成立。
- 2. 再证明对任意 $x \in (-1, 1]$, 成立 $\lim_{n \to \infty} R_n(x) = 0$ 。

$$|R_n(x)| = \left| \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1} \right| = \left| \frac{1}{(n+1)!} \cdot \frac{(-1)^n n!}{(1+\theta x)^{n+1}} x^{n+1} \right|$$

(当 $x \ge 0$ 时, $\frac{1}{1+\theta x} \le 1$; 当 $x \in (-1,0)$ 时, $\frac{1}{1+\theta x} \le \frac{1}{1+x}$)

$$\leq \left|\frac{x^{n+1}}{(1+n)(1+\theta x)^{n+1}}\right| \leq \max\left(1, \frac{1}{1+x}\right) \frac{1}{n+1}$$

3. 由泰勒中值定理 2.

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$$

证明

- 1. 级数收敛域 (-1, 1], 函数定义域 (-1, ∞), 故上式至多对 $x \in (-1, 1]$ 成立。

 - 2. 再证明对任意 $x \in (-1, 1]$, 成立 $\lim_{n \to \infty} R_n(x) = 0$ 。
 - 3. 由泰勒中值定理 2.
- $|R_n(x)| = \left| \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1} \right| = \left| \frac{1}{(n+1)!} \cdot \frac{(-1)^n n!}{(1+\theta x)^{n+1}} x^{n+1} \right|$

 $\leq \left| \frac{x^{n+1}}{(1+n)(1+\theta x)^{n+1}} \right| \leq \max\left(1, \frac{1}{1+x}\right) \frac{1}{n+1} \to 0$

(当
$$x \ge 0$$
时, $\frac{1}{1+\theta x} \le 1$;当 $x \in (-1,0)$ 时, $\frac{1}{1+\theta x} \le \frac{1}{1+x}$) 資 學 点 大學 函数展开成幂级数

• 至此,我们知道 e^x , $\sin x$, $\cos x$ 以及 $\frac{1}{1+x}$ 是等于其泰勒级数,即

$$e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty)$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots + (-1)^n \frac{1}{(2n+1)!}x^{2n+1} + \dots, x \in (-\infty, \infty)$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots + (-1)^n \frac{1}{(2n)!}x^{2n} + \dots, \quad x \in (-\infty, \infty)$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1} \frac{1}{n}x^n + \dots, \quad x \in (-1, 1]$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots + (-1)^n x^n + \dots, x \in (-1, 1)$$

暨南大学

• 至此,我们知道 e^x , $\sin x$, $\cos x$ 以及 $\frac{1}{1+x}$ 是等于其泰勒级数,即

$$e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty)$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots + (-1)^n \frac{1}{(2n+1)!}x^{2n+1} + \dots, x \in (-\infty, \infty)$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots + (-1)^n \frac{1}{(2n)!}x^{2n} + \dots, x \in (-\infty, \infty)$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1,1]$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots + (-1)^n x^n + \dots, \quad x \in (-1,1]$$

● 利用最后一式,及逐项积分公式,可进一步求出 arctan x 的幂级数 展开。



性质 成立 $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots + (-1)^n \frac{1}{2n+1}x^{2n+1} + \dots, x \in [-1,1].$

性质成立 $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots + (-1)^n \frac{1}{2n+1}x^{2n+1} + \dots, x \in [-1,1].$

证明 1. 幂级数的收敛域是 [-1, 1],故上式至多对 $x \in [-1, 1]$ 成立。

性质 成立 $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots + (-1)^n \frac{1}{2n+1}x^{2n+1} + \dots, x \in [-1,1].$

证明 1. 幂级数的收敛域是 [-1, 1],故上式至多对 $x \in [-1, 1]$ 成立。

2.

 $\arctan x = \int_{0}^{\infty} \frac{1}{1+t^2} dt$



性质成立 $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots + (-1)^n \frac{1}{2n+1}x^{2n+1} + \dots, x \in [-1,1].$

证明 1. 幂级数的收敛域是 [-1, 1],故上式至多对 $x \in [-1, 1]$ 成立。

2.
$$\arctan x = \int_{0}^{x} \frac{1}{1+t^{2}} dt = \int_{0}^{x} \sum_{n=0}^{\infty} (-1)^{n} t^{2n} dt$$

性质 成立 $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots + (-1)^n \frac{1}{2n+1}x^{2n+1} + \dots, x \in [-1,1].$ 证明 1. 幂级数的收敛域是 [-1, 1],故上式至多对 $x \in [-1, 1]$ 成立。

2.
$$\arctan x = \int_0^x \frac{1}{1+t^2} dt = \int_0^x \sum_{n=0}^\infty (-1)^n t^{2n} dt = \sum_{n=0}^\infty (-1)^n \int_0^x t^{2n} dt$$



性质 成立 $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots + (-1)^n \frac{1}{2n+1}x^{2n+1} + \dots, x \in [-1,1].$

证明 1. 幂级数的收敛域是 [-1, 1], 故上式至多对 $x \in [-1, 1]$ 成立。

2.

 $\arctan x = \int_0^x \frac{1}{1+t^2} dt = \int_0^x \sum_{n=0}^\infty (-1)^n t^{2n} dt = \sum_{n=0}^\infty (-1)^n \int_0^x t^{2n} dt$ $=\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1}$



性质成立 $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots + (-1)^n \frac{1}{2n+1}x^{2n+1} + \dots, x \in [-1,1].$

证明 1. 幂级数的收敛域是 [-1, 1], 故上式至多对 $x \in [-1, 1]$ 成立。

2. 当
$$x \in (-1, 1)$$
 时,利用逐项积分可得
$$\arctan x = \int_0^x \frac{1}{1+t^2} dt = \int_0^x \sum_{n=0}^\infty (-1)^n t^{2n} dt = \sum_{n=0}^\infty (-1)^n \int_0^x t^{2n} dt$$

$$\operatorname{dictan} x = \int_{0}^{\infty} \frac{1}{1+t^{2}} dt = \int_{0}^{\infty} \sum_{n=0}^{\infty} (-1)^{n} t^{-n} dt = \sum_{n=0}^{\infty} (-1)^{n} \int_{0}^{\infty} t^{-n} dt$$
$$= \sum_{n=0}^{\infty} (-1)^{n} \frac{1}{2n+1} x^{2n+1}$$

 $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots + (-1)^n \frac{1}{2n+1}x^{2n+1} + \dots, x \in [-1,1].$

证明 1. 幂级数的收敛域是 [-1, 1],故上式至多对 $x \in [-1, 1]$ 成立。

2. 当
$$x \in (-1, 1)$$
 时,利用逐项积分可得
$$\arctan x = \int_0^x \frac{1}{1+t^2} dt = \int_0^x \sum_{n=0}^{\infty} (-1)^n t^{2n} dt = \sum_{n=0}^{\infty} (-1)^n \int_0^x t^{2n} dt$$

$$=\sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}x^{2n+1}$$
2 注音到 $\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n+1}}$ 收納域具[_1_1_1] 由连续性 当 $x=1$

3. 注意到 $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$ 收敛域是 [-1, 1], 由连续性, 当 $x = \pm 1$ 时也

有
$$\operatorname{arctan} x = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1}.$$



(如f(1) =

2. 当 $x \in (-1, 1)$ 时,利用逐项积分可得 $\arctan x = \int_0^x \frac{1}{1+t^2} dt = \int_0^x \sum_{r=0}^{\infty} (-1)^r t^{2r} dt = \sum_{r=0}^{\infty} (-1)^r \int_0^x t^{2r} dt$

性质成立 $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots + (-1)^n \frac{1}{2n+1}x^{2n+1} + \dots, x \in [-1,1].$

证明 1. 幂级数的收敛域是 [-1, 1], 故上式至多对 $x \in [-1, 1]$ 成立。

$$\arctan x = \int_0^\infty \frac{1}{1+t^2} dt = \int_0^\infty \sum_{n=0}^\infty (-1)^n t^{2n} dt = \sum_{n=0}^\infty (-1)^n \int_0^\infty t^{2n} dt$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1}$$

 $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1}.$

3. 注意到
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$
 收敛域是 [-1, 1], 由连续性, 当 $x = \pm 1$ 时也

= S(1)

21/26 < ▶ △ ▽

有

证明 1. 幂级数的收敛域是 [-1, 1],故上式至多对 $x \in [-1, 1]$ 成立。 2. 当 $x \in (-1, 1)$ 时,利用逐项积分可得

性质成立 $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots + (-1)^n \frac{1}{2n+1}x^{2n+1} + \dots, x \in [-1,1].$

 $\arctan x = \int_0^x \frac{1}{1+t^2} dt = \int_0^x \sum_{r=0}^{\infty} (-1)^r t^{2r} dt = \sum_{r=0}^{\infty} (-1)^r \int_0^x t^{2r} dt$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1}$$

$$\stackrel{\infty}{=} \sum_{n=0}^{\infty} (-1)^n x^{2n+1} || x || x^{2n+1} = 1$$

3. 注意到 $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$ 收敛域是 [-1, 1], 由连续性, 当 $x = \pm 1$ 时也

= S(1)

 $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1}.$

 $(\text{如}f(1) = \lim_{x \to 1^{-}} \operatorname{arctan} x$ 21/26 < ▷ △ ▽

有

 $(\text{如}f(1) = \lim_{x \to 1^{-}} \operatorname{arctan} x$

$$=\sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}x^{2n+1}$$

3. 注意到 $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$ 收敛域是 [-1, 1], 由连续性, 当 $x = \pm 1$ 时也

2. 当 $x \in (-1, 1)$ 时,利用逐项积分可得

 $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1}.$

 $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots + (-1)^n \frac{1}{2n+1}x^{2n+1} + \dots, x \in [-1,1].$

证明 1. 幂级数的收敛域是 [-1, 1],故上式至多对 $x \in [-1, 1]$ 成立。

 $\arctan x = \int_0^x \frac{1}{1+t^2} dt = \int_0^x \sum_{r=0}^\infty (-1)^r t^{2r} dt = \sum_{r=0}^\infty (-1)^r \int_0^x t^{2r} dt$

 $\lim_{x\to 1^-} S(x) = S(1)$

21/26 < ▷ △ ▽

 $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots + (-1)^n \frac{1}{2n+1}x^{2n+1} + \dots, x \in [-1,1].$

性质 成立

第 12 章 d:函数展开成幂级数

证明 1. 幂级数的收敛域是
$$[-1,1]$$
,故上式至多对 $x \in [-1,1]$ 成立。

2. 当
$$x \in (-1, 1)$$
 时,利用逐项积分可得
$$\arctan x = \int_0^x \frac{1}{1+t^2} dt = \int_0^x \sum_{n=0}^\infty (-1)^n t^{2n} dt = \sum_{n=0}^\infty (-1)^n \int_0^x t^{2n} dt$$

$$\int_{0}^{\infty} 1 + t^{2} \int_{0}^{\infty} \sum_{n=0}^{\infty} (-1)^{n} \frac{1}{2n+1} x^{2n+1}$$

3. 注意到 $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$ 收敛域是 [-1, 1], 由连续性, 当 $x = \pm 1$ 时也

3. 注意到
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$
 收敛域是 $[-1, 1]$, 由连续性, 当 $x = \pm 1$ 时也有
$$\operatorname{arctan} x = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1}.$$

 $(\inf(1) = \lim_{x \to 1^{-}} \arctan x \quad \lim_{x \to 1^{-}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = \lim_{x \to 1^{-}} S(x) = S(1))$

 $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots + (-1)^n \frac{1}{2n+1}x^{2n+1} + \dots, x \in [-1,1].$

性质 成立

证明 1. 幂级数的收敛域是 [-1, 1],故上式至多对 $x \in [-1, 1]$ 成立。

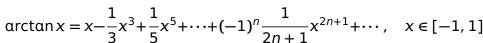
2. 当 $x \in (-1, 1)$ 时,利用逐项积分可得 $\arctan x = \int_0^x \frac{1}{1+t^2} dt = \int_0^x \sum_{r=0}^{\infty} (-1)^r t^{2r} dt = \sum_{r=0}^{\infty} (-1)^r \int_0^x t^{2r} dt$

 $=\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{2n+1}x^{2n+1}$

3. 注意到 $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$ 收敛域是 [-1, 1], 由连续性, 当 $x = \pm 1$ 时也

 $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1}.$ 有

 $(\inf(1) = \lim_{x \to 1^{-}} \arctan x = \lim_{x \to 1^{-}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = \lim_{x \to 1^{-}} S(x) = S(1))$





注 取
$$x = 1$$
,则得到

 $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots + (-1)^n \frac{1}{2n+1}x^{2n+1} + \dots, \quad x \in [-1, 1]$

$$\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots + (-1)^n \frac{1}{2n+1}x^{2n+1} + \dots, \quad x \in [-1, 1]$$

注 取
$$x = 1$$
,则得到

注 取
$$x = 1$$
,则得到

$$\pi$$
 ,则得到 π , π ,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$$

• 至此,我们知道 e^x , $\sin x$, $\cos x$ 以及 $\frac{1}{1+x}$ 是等于其泰勒级数,即

$$e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty)$$

$$e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty)$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots + (-1)^n \frac{1}{(2n+1)!}x^{2n+1} + \dots, x \in (-\infty, \infty)$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots + (-1)^n \frac{1}{(2n)!}x^{2n} + \dots, x \in (-\infty, \infty)$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1]$$

$$\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots + (-1)^n \frac{1}{2n+1}x^{2n+1} + \dots, \quad x \in [-1, 1]$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots + (-1)^n x^n + \dots, x \in (-1, 1)$$



• 至此,我们知道 e^x , $\sin x$, $\cos x$ 以及 $\frac{1}{1+x}$ 是等于其泰勒级数,即

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots + \frac{1}{n!}x^n + \dots, \quad x \in (-\infty, \infty)$$

 $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1,1]$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots + (-1)^n \frac{1}{(2n+1)!}x^{2n+1} + \dots, x \in (-\infty, \infty)$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots + (-1)^n \frac{1}{(2n)!}x^{2n} + \dots, x \in (-\infty, \infty)$$

$$\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots + (-1)^n \frac{1}{2n+1}x^{2n+1} + \dots, x \in [-1,1]$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots + (-1)^n x^n + \dots, x \in (-1,1)$$

• 用上述结果,及逐项求导、积分公式,可求更多函数的泰勒级数展开



$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$$

所以当
$$x \in (-1, 1]$$
时,
$$(1-x)\ln(1+x) = (1-x)\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n$$

解利用

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$$

所以当
$$x \in (-1, 1]$$
时,
$$(1-x)\ln(1+x) = (1-x)\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n$$
$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n - \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^{n+1}$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1} \frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$$

所以当
$$x \in (-1, 1]$$
时,
$$(1-x)\ln(1+x) = (1-x)\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n - \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^{n+1}$$
$$\sum_{n=2}^{\infty} (-1)^{n-2} \frac{1}{n-1} x^n$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$$

所以当
$$x \in (-1, 1]$$
时,
$$(1-x)\ln(1+x) = (1-x)\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n$$
$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n - \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^{n+1}$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n - \sum_{n=2}^{\infty} (-1)^{n-2} \frac{1}{n-1} x^n$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$$

所以当
$$x \in (-1, 1]$$
时,
$$(1-x)\ln(1+x) = (1-x)\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n - \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^{n+1}$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n - \sum_{n=2}^{\infty} (-1)^{n-2} \frac{1}{n-1} x^n$$

$$= x + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$$

所以当
$$x \in (-1, 1]$$
时,
$$(1-x)\ln(1+x) = (1-x)\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n - \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^{n+1}$$
$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n - \sum_{n=2}^{\infty} (-1)^{n-2} \frac{1}{n-1} x^n$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n - \sum_{n=2}^{\infty} (-1)^{n-2} \frac{1}{n-1} x^n$$

$$= x + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{1}{n} x^n - \sum_{n=2}^{\infty} (-1)^{n-2} \frac{1}{n-1} x^n$$

 $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1} \frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$

所以当
$$x \in (-1, 1]$$
 时, ∞ 1 ∞

 $(1-x)\ln(1+x) = (1-x)\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n$

例 1 把函数 $f(x) = (1-x)\ln(1+x)$ 展开成 x 的幂级数。

 $=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}x^n-\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}x^{n+1}$

 $=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}x^n-\sum_{n=2}^{\infty}(-1)^{n-2}\frac{1}{n-1}x^n$

$$= x + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{1}{n} x^n - \sum_{n=2}^{\infty} (-1)^{n-2} \frac{1}{n-1} x^n$$

$$= x + \sum_{n=2}^{\infty} \left(\frac{(-1)^{n-1}}{n} - \frac{(-1)^n}{n-1} \right) x^n$$



解利用

$$\cos t = 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6 + \dots + (-1)^n \frac{1}{(2n)!}t^{2n} + \dots, \ t \in (-\infty, \infty)$$

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$$

$$\cos t = 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6 + \dots + (-1)^n \frac{1}{(2n)!}t^{2n} + \dots, \ t \in (-\infty, \infty)$$

所以当
$$x \in (-\infty, \infty)$$
时,

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$$
$$= \frac{1}{2} + \frac{1}{2}\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} (2x)^{2n}$$

解利用

$$\cos t = 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6 + \dots + (-1)^n \frac{1}{(2n)!}t^{2n} + \dots, \ t \in (-\infty, \infty)$$

所以当
$$x \in (-\infty, \infty)$$
时,

$$\cos^{2} x = \frac{1}{2} + \frac{1}{2} \cos 2x$$

$$= \frac{1}{2} + \frac{1}{2} \sum_{n=0}^{\infty} (-1)^{n} \frac{1}{(2n)!} (2x)^{2n}$$

$$= \frac{1}{2} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^{n} 2^{2n}}{(2n)!} x^{2n}$$

解利用

$$\cos t = 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6 + \dots + (-1)^n \frac{1}{(2n)!}t^{2n} + \dots, \ t \in (-\infty, \infty)$$









所以当 $x \in (-\infty, \infty)$ 时,

 $= \frac{1}{2} + \frac{1}{2} \sum_{n=1}^{\infty} (-1)^n \frac{1}{(2n)!} (2x)^{2n}$

$$= \frac{1}{2} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!} x^{2n}$$
$$= 1 + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!} x^{2n}$$

 $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$

解 1. 注意到
$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$
.

解 1. 注意到
$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$
.

2. 利用 $\frac{1}{1-t} = 1 + t + t^2 + t^3 + \dots + t^n + \dots$, $t \in (-1, 1)$ 将 $\frac{1}{\sqrt{11}}$, $\frac{1}{\sqrt{12}}$ 分别展开成 (x + 4) 的幂级数:

解 1. 注意到
$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$
.

2. 利用
$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \dots + t^n + \dots$$
, $t \in (-1, 1)$

将
$$\frac{1}{x+1}$$
, $\frac{1}{x+2}$ 分别展开成 $(x+4)$ 的幂级数: 令 $t=x+4$, 则

*
$$\frac{1}{x+1} = \frac{1}{t-3}$$

解 1. 注意到
$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$
.

2. 利用
$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \dots + t^n + \dots$$
, $t \in (-1, 1)$

将
$$\frac{1}{x+1}$$
, $\frac{1}{x+2}$ 分别展开成 $(x+4)$ 的幂级数: 令 $t=x+4$, 则

*
$$\frac{1}{x+1} = \frac{1}{t-3} = \frac{1}{-3} \cdot \frac{1}{1-\frac{t}{3}}$$

解 1. 注意到
$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$
.

2. 利用
$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \dots + t^n + \dots$$
, $t \in (-1, 1)$

将
$$\frac{1}{x+1}$$
, $\frac{1}{x+2}$ 分别展开成 $(x+4)$ 的幂级数: 令 $t=x+4$, 则

*
$$\frac{1}{x+1} = \frac{1}{t-3} = \frac{1}{-3} \cdot \frac{1}{1-\frac{t}{3}} = -\frac{1}{3} \sum_{n=0}^{\infty} \frac{t^n}{3^n}$$

解 1. 注意到
$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$
.

2. 利用
$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots + t^n + \cdots$$
, $t \in (-1, 1)$

将
$$\frac{1}{x+1}$$
, $\frac{1}{x+2}$ 分别展开成 $(x+4)$ 的幂级数: 令 $t=x+4$, 则

*
$$\frac{1}{x+1} = \frac{1}{t-3} = \frac{1}{-3} \cdot \frac{1}{1-\frac{t}{2}} = -\frac{1}{3} \sum_{n=0}^{\infty} \frac{t^n}{3^n} = -\sum_{n=0}^{\infty} \frac{(x+4)^n}{3^{n+1}}$$

解 1. 注意到
$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$
.

2. 利用
$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \dots + t^n + \dots$$
, $t \in (-1, 1)$

将
$$\frac{1}{x+1}$$
, $\frac{1}{x+2}$ 分别展开成 $(x+4)$ 的幂级数: 令 $t=x+4$, 则 1 1 1 1 $\stackrel{\triangle}{=}$ t^n $\stackrel{\triangle}{=}$ $(x+4)^n$

*
$$\frac{1}{x+1} = \frac{1}{t-3} = \frac{1}{-3} \cdot \frac{1}{1-\frac{t}{3}} = -\frac{1}{3} \sum_{n=0}^{\infty} \frac{t^n}{3^n} = -\sum_{n=0}^{\infty} \frac{(x+4)^n}{3^{n+1}}$$

其中
$$|\frac{t}{3}| < 1$$

解 1. 注意到
$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$
.

2. 利用
$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \dots + t^n + \dots$$
, $t \in (-1, 1)$

*
$$\frac{1}{x+1} = \frac{1}{t-3} = \frac{1}{-3} \cdot \frac{1}{1-\frac{t}{3}} = -\frac{1}{3} \sum_{n=0}^{\infty} \frac{t^n}{3^n} = -\sum_{n=0}^{\infty} \frac{(x+4)^n}{3^{n+1}}$$

其中
$$\left| \frac{x+4}{3} \right| = \left| \frac{t}{3} \right| < 1$$

解 1. 注意到
$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$
.

2. 利用 $\frac{1}{1-t} = 1 + t + t^2 + t^3 + \dots + t^n + \dots$, $t \in (-1, 1)$

将
$$\frac{1}{x+1}$$
, $\frac{1}{x+2}$ 分别展开成 $(x+4)$ 的幂级数: 令 $t=x+4$, 则 1 1 1 1 $\stackrel{\frown}{=}$ t^n $\stackrel{\frown}{=}$ $(x+4)^n$

*
$$\frac{1}{x+1} = \frac{1}{t-3} = \frac{1}{-3} \cdot \frac{1}{1-\frac{t}{3}} = -\frac{1}{3} \sum_{n=0}^{\infty} \frac{t^n}{3^n} = -\sum_{n=0}^{\infty} \frac{(x+4)^n}{3^{n+1}}$$

其中 $\left|\frac{x+4}{3}\right| = \left|\frac{t}{3}\right| < 1$,即 -7 < x < -1。

解 1. 注意到
$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$
.

2. 利用 $\frac{1}{1-t} = 1 + t + t^2 + t^3 + \dots + t^n + \dots$, $t \in (-1, 1)$

将
$$\frac{1}{x+1}$$
, $\frac{1}{x+2}$ 分别展开成 $(x+4)$ 的幂级数: 令 $t=x+4$, 则 1 1 1 1 $\stackrel{\frown}{=}$ t^n $\stackrel{\frown}{=}$ $(x+4)^n$

* $\frac{1}{x+1} = \frac{1}{t-3} = \frac{1}{-3} \cdot \frac{1}{1-\frac{t}{3}} = -\frac{1}{3} \sum_{n=0}^{\infty} \frac{t^n}{3^n} = -\sum_{n=0}^{\infty} \frac{(x+4)^n}{3^{n+1}}$

其中
$$\left|\frac{x+4}{3}\right| = \left|\frac{t}{3}\right| < 1$$
,即 $-7 < x < -1$ 。

*
$$\frac{1}{x+2} = \frac{1}{t-2}$$

解 1. 注意到
$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$
.

2. 利用 $\frac{1}{1-t} = 1 + t + t^2 + t^3 + \dots + t^n + \dots$, $t \in (-1, 1)$

将
$$\frac{1}{x+1}$$
, $\frac{1}{x+2}$ 分别展开成 $(x+4)$ 的幂级数: 令 $t=x+4$, 则 1 1 1 1 $\sum_{n=0}^{\infty} t^n$ $\sum_{n=0}^{\infty} (x+4)^n$

*
$$\frac{1}{x+1} = \frac{1}{t-3} = \frac{1}{-3} \cdot \frac{1}{1-\frac{t}{3}} = -\frac{1}{3} \sum_{n=0}^{\infty} \frac{t^n}{3^n} = -\sum_{n=0}^{\infty} \frac{(x+4)^n}{3^{n+1}}$$

其中
$$\left|\frac{x+4}{3}\right| = \left|\frac{t}{3}\right| < 1$$
,即 $-7 < x < -1$ 。

*
$$\frac{1}{x+2} = \frac{1}{t-2} = \frac{1}{-2} \cdot \frac{1}{1-\frac{t}{2}}$$

解 1. 注意到
$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$
.

2. 利用 $\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots + t^n + \cdots$, $t \in (-1, 1)$

将
$$\frac{1}{x+1}$$
, $\frac{1}{x+2}$ 分别展开成 $(x+4)$ 的幂级数: 令 $t=x+4$, 则 1 1 1 1 $\frac{1}{x+1}$ $\frac{1}{x+2}$ $\frac{1}{x+$

*
$$\frac{1}{x+1} = \frac{1}{t-3} = \frac{1}{-3} \cdot \frac{1}{1-\frac{t}{3}} = -\frac{1}{3} \sum_{n=0}^{\infty} \frac{t^n}{3^n} = -\sum_{n=0}^{\infty} \frac{(x+4)^n}{3^{n+1}}$$

其中
$$\left|\frac{x+4}{3}\right| = \left|\frac{t}{3}\right| < 1$$
,即 $-7 < x < -1$ 。

*
$$\frac{1}{x+2} = \frac{1}{t-2} = \frac{1}{-2} \cdot \frac{1}{1-\frac{t}{2}} = -\frac{1}{2} \sum_{n=0}^{\infty} \frac{t^n}{2^n}$$

解 1. 注意到
$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$
.

2. 利用 $\frac{1}{1-t} = 1 + t + t^2 + t^3 + \dots + t^n + \dots$, $t \in (-1, 1)$

*
$$\frac{1}{x+1} = \frac{1}{t-3} = \frac{1}{-3} \cdot \frac{1}{1-\frac{t}{3}} = -\frac{1}{3} \sum_{n=0}^{\infty} \frac{t^n}{3^n} = -\sum_{n=0}^{\infty} \frac{(x+4)^n}{3^{n+1}}$$

其中
$$\left|\frac{x+4}{3}\right| = \left|\frac{t}{3}\right| < 1$$
,即 $-7 < x < -1$ 。

*
$$\frac{1}{x+2} = \frac{1}{t-2} = \frac{1}{-2} \cdot \frac{1}{1-\frac{t}{2}} = -\frac{1}{2} \sum_{n=0}^{\infty} \frac{t^n}{2^n} = -\sum_{n=0}^{\infty} \frac{(x+4)^n}{2^{n+1}}$$



解 1. 注意到
$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$
.

2. 利用 $\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots + t^n + \cdots$, $t \in (-1, 1)$

将
$$\frac{1}{x+1}$$
, $\frac{1}{x+2}$ 分别展开成 $(x+4)$ 的幂级数: 令 $t=x+4$, 则

*
$$\frac{1}{x+1} = \frac{1}{t-3} = \frac{1}{-3} \cdot \frac{1}{1-\frac{t}{3}} = -\frac{1}{3} \sum_{n=0}^{\infty} \frac{t^n}{3^n} = -\sum_{n=0}^{\infty} \frac{(x+4)^n}{3^{n+1}}$$

其中
$$\left|\frac{x+4}{3}\right| = \left|\frac{t}{3}\right| < 1$$
,即 $-7 < x < -1$ 。

*
$$\frac{1}{x+2} = \frac{1}{t-2} = \frac{1}{-2} \cdot \frac{1}{1-\frac{t}{2}} = -\frac{1}{2} \sum_{n=0}^{\infty} \frac{t^n}{2^n} = -\sum_{n=0}^{\infty} \frac{(x+4)^n}{2^{n+1}}$$

其中
$$|\frac{t}{2}| < 1$$

解 1. 注意到
$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$
.

2. 利用 $\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots + t^n + \cdots$, $t \in (-1, 1)$

将
$$\frac{1}{x+1}$$
, $\frac{1}{x+2}$ 分别展开成 $(x+4)$ 的幂级数: 令 $t=x+4$, 则

*
$$\frac{1}{x+1} = \frac{1}{t-3} = \frac{1}{-3} \cdot \frac{1}{1-\frac{t}{3}} = -\frac{1}{3} \sum_{n=0}^{\infty} \frac{t^n}{3^n} = -\sum_{n=0}^{\infty} \frac{(x+4)^n}{3^{n+1}}$$

其中
$$\left|\frac{x+4}{3}\right| = \left|\frac{t}{3}\right| < 1$$
,即 $-7 < x < -1$ 。

*
$$\frac{1}{x+2} = \frac{1}{t-2} = \frac{1}{-2} \cdot \frac{1}{1-\frac{t}{2}} = -\frac{1}{2} \sum_{n=0}^{\infty} \frac{t^n}{2^n} = -\sum_{n=0}^{\infty} \frac{(x+4)^n}{2^{n+1}}$$

其中 $\left|\frac{x+4}{2}\right| = \left|\frac{t}{2}\right| < 1$

解 1. 注意到
$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$
.

2. 利用 $\frac{1}{1-t} = 1 + t + t^2 + t^3 + \dots + t^n + \dots$, $t \in (-1, 1)$

*
$$\frac{1}{x+1} = \frac{1}{t-3} = \frac{1}{-3} \cdot \frac{1}{1-\frac{t}{3}} = -\frac{1}{3} \sum_{n=0}^{\infty} \frac{t^n}{3^n} = -\sum_{n=0}^{\infty} \frac{(x+4)^n}{3^{n+1}}$$

其中
$$\left|\frac{x+4}{3}\right| = \left|\frac{t}{3}\right| < 1$$
,即 $-7 < x < -1$ 。

*
$$\frac{1}{x+2} = \frac{1}{t-2} = \frac{1}{-2} \cdot \frac{1}{1-\frac{t}{2}} = -\frac{1}{2} \sum_{n=0}^{\infty} \frac{t^n}{2^n} = -\sum_{n=0}^{\infty} \frac{(x+4)^n}{2^{n+1}}$$

其中 $\left|\frac{x+4}{2}\right| = \left|\frac{t}{2}\right| < 1$,即-6 < x < -2。

解 1. 注意到
$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$
.

2. 利用 $\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots + t^n + \cdots$, $t \in (-1, 1)$

*
$$\frac{1}{x+1} = \frac{1}{t-3} = \frac{1}{-3} \cdot \frac{1}{1-\frac{t}{3}} = -\frac{1}{3} \sum_{n=0}^{\infty} \frac{t^n}{3^n} = -\sum_{n=0}^{\infty} \frac{(x+4)^n}{3^{n+1}}$$

其中
$$\left|\frac{x+4}{3}\right| = \left|\frac{t}{3}\right| < 1$$
,即 $-7 < x < -1$ 。

*
$$\frac{1}{x+2} = \frac{1}{t-2} = \frac{1}{-2} \cdot \frac{1}{1-\frac{t}{2}} = -\frac{1}{2} \sum_{n=0}^{\infty} \frac{t^n}{2^n} = -\sum_{n=0}^{\infty} \frac{(x+4)^n}{2^{n+1}}$$

其中
$$\left|\frac{x+4}{2}\right| = \left|\frac{t}{2}\right| < 1$$
,即 $-6 < x < -2$ 。

3. 所以 -6 < x < -2 时

例 3 把函数 $f(x) = \frac{1}{x^2 + 3x + 2}$ 展开成 (x + 4) 的幂级数。 解 1. 注意到 $\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$.

解 1. 注意到
$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+1}$$

将
$$\frac{1}{x+1}$$
, $\frac{1}{x+2}$ 分别展开成 $(x+4)$ 的幂级数: 令 $t=x+4$, 则

* $\frac{1}{x+1} = \frac{1}{t-3} = \frac{1}{-3} \cdot \frac{1}{1-\frac{t}{3}} = -\frac{1}{3} \sum_{n=0}^{\infty} \frac{t^n}{3^n} = -\sum_{n=0}^{\infty} \frac{(x+4)^n}{3^{n+1}}$

其中
$$\left|\frac{x+4}{3}\right| = \left|\frac{t}{3}\right| < 1$$
,即 $-7 < x < -1$ 。

1 1 1 1 1 $\sum_{n=0}^{\infty} t^n$ $\sum_{n=0}^{\infty} (x+4)^n$

其中
$$\left|\frac{x+3}{3}\right| = \left|\frac{1}{3}\right| < 1$$
,即 $-7 < x < -1$ 。

* $\frac{1}{x+2} = \frac{1}{t-2} = \frac{1}{-2} \cdot \frac{1}{1-\frac{t}{2}} = -\frac{1}{2} \sum_{n=0}^{\infty} \frac{t^n}{2^n} = -\sum_{n=0}^{\infty} \frac{(x+4)^n}{2^{n+1}}$

其中 $\left|\frac{x+4}{2}\right| = \left|\frac{t}{2}\right| < 1$,即 -6 < x < -2。

3. 所以-6 < x < -2时

 $\frac{1}{x^2 + 3x + 2} = \frac{1}{x + 1} - \frac{1}{x + 2} = \sum_{n=0}^{\infty} \left(\frac{1}{2^{n+1}} - \frac{1}{3^{n+1}} \right) (x + 4)_{0}^{n}$