§1.4　克莱姆法则

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2018 - 2019 学年上学期

二元线性方程组

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_2 + a_{22}y_2 = b_2 \end{cases}$$

的解是

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$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \qquad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

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- $D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ 称为系数行列式

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- $D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ 称为系数行列式，上述公式隐含要求 $D \neq 0$
- D_i：将 D 的第 i 列换成常数项 $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

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$$\begin{vmatrix} a_{11} & \cdots & a_{1j-1} & a_{1,j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2,j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{n,j} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}$$

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令 $D_j = \begin{vmatrix} a_{11} & \cdots & a_{1\,j-1} & b_1 & a_{1\,j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2\,j-1} & b_2 & a_{2\,j+1} & \cdots & a_{2n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n\,j-1} & b_n & a_{n\,j+1} & \cdots & a_{nn} \end{vmatrix}$

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定理（克莱姆法则） 线性方程组

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注 1 两个前提：(1) 未知元个数 = 方程个数；(2) 系数行列式 $D \neq 0$

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注 1 两个前提：(1) 未知元个数 = 方程个数；(2) 系数行列式 $D \neq 0$

注 2 若 $D = 0$，则方程或者无解、或者有无穷多解（以后详说）

克莱姆法则证明（仅验证 $x_j = \frac{D_j}{D}$ 是解，唯一性的证明要用到矩阵知识，略去。）

$$x_j = \frac{D_j}{D}$$

验证第 k 条方程成立（$k = 1, 2, \cdots, n$）：

$$a_{k1}x_1 + \cdots + a_{kn}x_n =$$

$$b_k$$

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克莱姆法则证明（仅验证 $x_j = \frac{D_j}{D}$ 是解，唯一性的证明要用到矩阵知识，略去。）

$$x_j = \frac{D_j}{D} = \frac{\begin{vmatrix} a_{11} & \cdots & a_{1j-1} & b_1 & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & b_2 & a_{2j+1} & \cdots & a_{2n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj-1} & b_n & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}} = \frac{b_1 A_{1j} + b_2 \qquad + \cdots + b_n}{D}$$

验证第 k 条方程成立（$k = 1, 2, \cdots, n$）：

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$$a_{k1}x_1 + \cdots + a_{kn}x_n = \sum_{j=1}^{n} a_{kj}x_j$$

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$$=\frac{1}{D}\sum_{i=1}^{n}b_iA_{ij}$$

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$$a_{k1}x_1+\cdots+a_{kn}x_n=\sum_{j=1}^{n}a_{kj}x_j=\sum_{j=1}^{n}a_{kj}\left(\frac{1}{D}\sum_{i=1}^{n}b_iA_{ij}\right)$$

$$b_k$$

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$$= \quad \sum_{j=1}^{n} a_{kj} b_i A_{ij} = \quad b_i \sum_{j=1}^{n} a_{kj} A_{ij} \quad\quad b_k$$

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b_k

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$$=\frac{1}{D}\sum_{i=1}^{n}\sum_{j=1}^{n}a_{kj}b_iA_{ij}=\frac{1}{D}\sum_{i=1}^{n}b_i\sum_{j=1}^{n}a_{kj}A_{ij} \qquad b_k\sum_{j=1}^{n}a_{kj}A_{kj} \qquad b_k$$

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克莱姆法则证明（仅验证 $x_j = \frac{D_j}{D}$ 是解，唯一性的证明要用到矩阵知识，略去。）

$$x_j = \frac{D_j}{D} = \frac{\begin{vmatrix} a_{11} & \cdots & a_{1j-1} & b_1 & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & b_2 & a_{2j+1} & \cdots & a_{2n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj-1} & b_n & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}} = \frac{b_1A_{1j} + b_2A_{2j} + \cdots + b_nA_{nj}}{D}$$

$$= \frac{1}{D}\sum_{i=1}^{n} b_i A_{ij}$$

验证第 k 条方程成立（$k = 1, 2, \cdots, n$）：

$$a_{k1}x_1 + \cdots + a_{kn}x_n = \sum_{j=1}^{n} a_{kj}x_j = \sum_{j=1}^{n} a_{kj}\left(\frac{1}{D}\sum_{i=1}^{n} b_iA_{ij}\right) = \frac{1}{D}\sum_{j=1}^{n}\sum_{i=1}^{n} a_{kj}b_iA_{ij}$$

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- $\begin{cases} x+y=1 \\ x+y=1 \end{cases}$，$D=\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}=0$

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- $\begin{cases} x+y=1 \\ x+y=0 \end{cases}$，$D=\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}=0$，方程组包含矛盾方程，显然无解。

例 解线性方程组

$$\begin{cases} 2x_1 + \quad x_2 - x_3 = 1 \\ 3x_1 - \quad x_2 - x_3 = -2 \\ -x_1 + 2x_2 + x_3 = 6 \end{cases}$$

练习 解线性方程组

$$\begin{cases} x_1 + x_2 = 90 \\ x_2 + x_3 = 86 \\ x_1 + x_3 = 80 \end{cases}$$

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提示 $D = -5$, $D_1 = -5$, $D_2 = -10$, $D_3 = -15$

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练习 解线性方程组

$$\begin{cases} x_1 + x_2 = 90 \\ x_2 + x_3 = 86 \\ x_1 + x_3 = 80 \end{cases}$$

提示 $D = 2,\ D_1 = 84,\ D_2 = 96,\ D_3 = 76$

齐次线性方程组

定理 齐次线性方程组

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = 0 \end{cases}$$

当系数行列式 $D \neq 0$ 时，仅有零解（$x_1 = x_2 = \cdots = x_n = 0$）

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注

- 实际上，$D \neq 0 \quad \Rightarrow \quad$ 只有零解 $x_1 = x_2 = \cdots = x_n = 0$

齐次线性方程组

定理 齐次线性方程组

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所以方程组除 $x_1 = x_2 = \cdots = x_n = 0$ 外，没有其他解

注

- 实际上，$D \neq 0 \iff$ 只有零解 $x_1 = x_2 = \cdots = x_n = 0$
- 若 $D = 0$，方程有无穷多的解

例子

例 齐次方程组 $\begin{cases} x_1 - 2x_2 = 0 \\ 2x_1 - 4x_2 = 0 \end{cases}$ 的系数矩阵 $D = \begin{vmatrix} 1 & -2 \\ 2 & -4 \end{vmatrix}$

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例 判断线性方程组 $\begin{cases} 2x_1 + 3x_2 + 4x_3 + 5x_4 = 0 \\ 3x_1 + 4x_2 + 5x_3 + 5x_4 = 0 \\ 4x_1 + 5x_2 + 6x_3 + 6x_4 = 0 \\ 5x_1 + 6x_2 + 8x_3 + 9x_4 = 0 \end{cases}$ 是否只有零解

解

$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 5 & 6 & 8 & 9 \end{vmatrix}$$

解

$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 5 & 6 & 8 & 9 \end{vmatrix} \xlongequal{r_4 - r_3}$$

解

$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 5 & 6 & 8 & 9 \end{vmatrix} \xlongequal{r_4 - r_3} \begin{vmatrix} & & & \\ & & & \\ & & & \\ 1 & 1 & 2 & 3 \end{vmatrix}$$

解

$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 5 & 6 & 8 & 9 \end{vmatrix} \xlongequal{r_4-r_3} \begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 1 & 1 & 2 & 3 \end{vmatrix}$$

解

$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 5 & 6 & 8 & 9 \end{vmatrix} \xlongequal{r_4-r_3} \begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 1 & 1 & 2 & 3 \end{vmatrix} \xlongequal[\substack{c_3-2c_1 \\ c_4-3c_1}]{c_2-c_1}$$

解

$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 5 & 6 & 8 & 9 \end{vmatrix} \xlongequal{r_4 - r_3} \begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 1 & 1 & 2 & 3 \end{vmatrix} \xlongequal[\substack{c_3-2c_1 \\ c_4-3c_1}]{c_2-c_1} \begin{vmatrix} 2 & & & \\ 3 & & & \\ 4 & & & \\ 1 & & & \end{vmatrix}$$

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解

$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 5 & 6 & 8 & 9 \end{vmatrix} \xlongequal{r_4-r_3} \begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 1 & 1 & 2 & 3 \end{vmatrix} \xlongequal[\substack{c_3-2c_1 \\ c_4-3c_1}]{c_2-c_1} \begin{vmatrix} 2 & 1 & 0 & \\ 3 & 1 & -1 & \\ 4 & 1 & -2 & \\ 1 & 0 & 0 & \end{vmatrix}$$

解

$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 5 & 6 & 8 & 9 \end{vmatrix} \xlongequal{r_4-r_3} \begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 1 & 1 & 2 & 3 \end{vmatrix} \xlongequal[\substack{c_3-2c_1 \\ c_4-3c_1}]{c_2-c_1} \begin{vmatrix} 2 & 1 & 0 & -1 \\ 3 & 1 & -1 & -4 \\ 4 & 1 & -2 & -6 \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

解

$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 5 & 6 & 8 & 9 \end{vmatrix} \xlongequal{r_4 - r_3} \begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 1 & 1 & 2 & 3 \end{vmatrix} \xlongequal[\substack{c_3 - 2c_1 \\ c_4 - 3c_1}]{c_2 - c_1} \begin{vmatrix} 2 & 1 & 0 & -1 \\ 3 & 1 & -1 & -4 \\ 4 & 1 & -2 & -6 \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

$$= 1 \times (-1)^{4+1} \times \begin{vmatrix} 1 & 0 & -1 \\ 1 & -1 & -4 \\ 1 & -2 & -6 \end{vmatrix}$$

解

$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 5 & 6 & 8 & 9 \end{vmatrix} \xlongequal{r_4 - r_3} \begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 1 & 1 & 2 & 3 \end{vmatrix} \xlongequal[\substack{c_3-2c_1 \\ c_4-3c_1}]{c_2-c_1} \begin{vmatrix} 2 & 1 & 0 & -1 \\ 3 & 1 & -1 & -4 \\ 4 & 1 & -2 & -6 \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

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解

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$$= 1 \times (-1)^{4+1} \times \begin{vmatrix} 1 & 0 & -1 \\ 1 & -1 & -4 \\ 1 & -2 & -6 \end{vmatrix} \xlongequal{c_3 + c_1} - \begin{vmatrix} 1 & 0 & 0 \\ 1 & -1 & -3 \\ 1 & -2 & -5 \end{vmatrix}$$

解

$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 5 & 6 & 8 & 9 \end{vmatrix} \overset{r_4-r_3}{=\!=\!=} \begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 1 & 1 & 2 & 3 \end{vmatrix} \xlongequal[\substack{c_3-2c_1 \\ c_4-3c_1}]{c_2-c_1} \begin{vmatrix} 2 & 1 & 0 & -1 \\ 3 & 1 & -1 & -4 \\ 4 & 1 & -2 & -6 \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

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$$= -\begin{vmatrix} -1 & -3 \\ -2 & -5 \end{vmatrix}$$

解

$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 5 & 6 & 8 & 9 \end{vmatrix} \xlongequal{r_4-r_3} \begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 1 & 1 & 2 & 3 \end{vmatrix} \xlongequal[\substack{c_3-2c_1 \\ c_4-3c_1}]{c_2-c_1} \begin{vmatrix} 2 & 1 & 0 & -1 \\ 3 & 1 & -1 & -4 \\ 4 & 1 & -2 & -6 \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

$$= 1 \times (-1)^{4+1} \times \begin{vmatrix} 1 & 0 & -1 \\ 1 & -1 & -4 \\ 1 & -2 & -6 \end{vmatrix} \xlongequal{c_3+c_1} -\begin{vmatrix} 1 & 0 & 0 \\ 1 & -1 & -3 \\ 1 & -2 & -5 \end{vmatrix}$$

$$= -\begin{vmatrix} -1 & -3 \\ -2 & -5 \end{vmatrix} = 1 \neq 0$$

解

$$\begin{vmatrix}2&3&4&5\\3&4&5&5\\4&5&6&6\\5&6&8&9\end{vmatrix} \xlongequal{r_4-r_3} \begin{vmatrix}2&3&4&5\\3&4&5&5\\4&5&6&6\\1&1&2&3\end{vmatrix} \xlongequal[c_3-2c_1 \atop c_4-3c_1]{c_2-c_1} \begin{vmatrix}2&1&0&-1\\3&1&-1&-4\\4&1&-2&-6\\1&0&0&0\end{vmatrix}$$

$$= 1\times(-1)^{4+1}\times\begin{vmatrix}1&0&-1\\1&-1&-4\\1&-2&-6\end{vmatrix} \xlongequal{c_3+c_1} -\begin{vmatrix}1&0&0\\1&-1&-3\\1&-2&-5\end{vmatrix}$$

$$= -\begin{vmatrix}-1&-3\\-2&-5\end{vmatrix} = 1 \neq 0$$

所以齐次线性方程组有唯一解

练习 齐次线性方程组 $\begin{cases} kx_1 + x_4 = 0 \\ x_1 + 2x_2 - x_4 = 0 \\ (k+2)x_1 - x_2 + 4x_4 = 0 \\ 2x_1 + x_2 + 3x_3 + kx_4 = 0 \end{cases}$ 有非零解

的充分必要条件是 k 满足 ____

练习 齐次线性方程组 $\begin{cases} kx_1 + x_4 = 0 \\ x_1 + 2x_2 - x_4 = 0 \\ (k+2)x_1 - x_2 + 4x_4 = 0 \\ 2x_1 + x_2 + 3x_3 + kx_4 = 0 \end{cases}$ 有非零解

的充分必要条件是 k 满足 ____

解

$$D = \begin{vmatrix} k & 0 & 0 & 1 \\ 1 & 2 & 0 & -1 \\ k+2 & -1 & 0 & 4 \\ 2 & 1 & 3 & k \end{vmatrix}$$

练习 齐次线性方程组 $\begin{cases} kx_1 + x_4 = 0 \\ x_1 + 2x_2 - x_4 = 0 \\ (k+2)x_1 - x_2 + 4x_4 = 0 \\ 2x_1 + x_2 + 3x_3 + kx_4 = 0 \end{cases}$ 有非零解的充分必要条件是 k 满足 ____

解

$$D = \begin{vmatrix} k & 0 & 0 & 1 \\ 1 & 2 & 0 & -1 \\ k+2 & -1 & 0 & 4 \\ 2 & 1 & 3 & k \end{vmatrix} = 3 \cdot$$

练习 齐次线性方程组 $\begin{cases} kx_1 \quad\quad\quad\quad + \quad x_4 = 0 \\ x_1 + 2x_2 \quad\quad - \quad x_4 = 0 \\ (k+2)x_1 - \quad x_2 \quad\quad + 4x_4 = 0 \\ 2x_1 + \quad x_2 + 3x_3 + kx_4 = 0 \end{cases}$ 有非零解

的充分必要条件是 k 满足 ____

解

$$D = \begin{vmatrix} k & 0 & 0 & 1 \\ 1 & 2 & 0 & -1 \\ k+2 & -1 & 0 & 4 \\ 2 & 1 & 3 & k \end{vmatrix} = 3\cdot(-1)^{3+4}\begin{vmatrix} k & 0 & 1 \\ 1 & 2 & -1 \\ k+2 & -1 & 4 \end{vmatrix}$$

练习 齐次线性方程组 $\begin{cases} kx_1 \qquad\qquad\qquad + \ x_4 = 0 \\ x_1 + 2x_2 \qquad\quad - \ x_4 = 0 \\ (k+2)x_1 - \ x_2 \qquad + 4x_4 = 0 \\ 2x_1 + \ x_2 + 3x_3 + kx_4 = 0 \end{cases}$ 有非零解

的充分必要条件是 k 满足 ____

解

$$D = \begin{vmatrix} k & 0 & 0 & 1 \\ 1 & 2 & 0 & -1 \\ k+2 & -1 & 0 & 4 \\ 2 & 1 & 3 & k \end{vmatrix} = 3 \cdot (-1)^{3+4} \begin{vmatrix} k & 0 & 1 \\ 1 & 2 & -1 \\ k+2 & -1 & 4 \end{vmatrix}$$

$$\underline{\underline{r_2 + r_1}}$$

练习 齐次线性方程组 $\begin{cases} kx_1 \qquad\qquad\qquad + \ x_4 = 0 \\ x_1 + 2x_2 \qquad\quad - \ x_4 = 0 \\ (k+2)x_1 - \ x_2 \qquad + 4x_4 = 0 \\ 2x_1 + \ x_2 + 3x_3 + kx_4 = 0 \end{cases}$ 有非零解

的充分必要条件是 k 满足 ____

解

$$D = \begin{vmatrix} k & 0 & 0 & 1 \\ 1 & 2 & 0 & -1 \\ k+2 & -1 & 0 & 4 \\ 2 & 1 & 3 & k \end{vmatrix} = 3 \cdot (-1)^{3+4} \begin{vmatrix} k & 0 & 1 \\ 1 & 2 & -1 \\ k+2 & -1 & 4 \end{vmatrix}$$

$$\xlongequal{r_2+r_1} (-3) \cdot \begin{vmatrix} k & 0 & 1 \\ & & \\ & & \end{vmatrix}$$

练习 齐次线性方程组 $\begin{cases} kx_1 & & & + & x_4 = 0 \\ x_1 + 2x_2 & & & - & x_4 = 0 \\ (k+2)x_1 - & x_2 & & + & 4x_4 = 0 \\ 2x_1 + & x_2 + & 3x_3 & + & kx_4 = 0 \end{cases}$ 有非零解

的充分必要条件是 k 满足 ____

解

$$D = \begin{vmatrix} k & 0 & 0 & 1 \\ 1 & 2 & 0 & -1 \\ k+2 & -1 & 0 & 4 \\ 2 & 1 & 3 & k \end{vmatrix} = 3 \cdot (-1)^{3+4} \begin{vmatrix} k & 0 & 1 \\ 1 & 2 & -1 \\ k+2 & -1 & 4 \end{vmatrix}$$

$$\xlongequal{r_2+r_1} (-3) \cdot \begin{vmatrix} k & 0 & 1 \\ k+1 & 2 & 0 \\ & & \end{vmatrix}$$

练习 齐次线性方程组 $\begin{cases} kx_1 + x_4 = 0 \\ x_1 + 2x_2 - x_4 = 0 \\ (k+2)x_1 - x_2 + 4x_4 = 0 \\ 2x_1 + x_2 + 3x_3 + kx_4 = 0 \end{cases}$ 有非零解的充分必要条件是 k 满足 ____

解

$$D = \begin{vmatrix} k & 0 & 0 & 1 \\ 1 & 2 & 0 & -1 \\ k+2 & -1 & 0 & 4 \\ 2 & 1 & 3 & k \end{vmatrix} = 3\cdot(-1)^{3+4}\begin{vmatrix} k & 0 & 1 \\ 1 & 2 & -1 \\ k+2 & -1 & 4 \end{vmatrix}$$

$$\xlongequal[r_3-4r_1]{r_2+r_1} (-3)\cdot\begin{vmatrix} k & 0 & 1 \\ k+1 & 2 & 0 \\ & & \end{vmatrix}$$

练习 齐次线性方程组 $\begin{cases} kx_1 \qquad\qquad\qquad + \quad x_4 = 0 \\ x_1 + 2x_2 \qquad\quad - \quad x_4 = 0 \\ (k+2)x_1 - \quad x_2 \qquad + 4x_4 = 0 \\ 2x_1 + \quad x_2 + 3x_3 + kx_4 = 0 \end{cases}$ 有非零解

的充分必要条件是 k 满足 ____

解

$$D = \begin{vmatrix} k & 0 & 0 & 1 \\ 1 & 2 & 0 & -1 \\ k+2 & -1 & 0 & 4 \\ 2 & 1 & 3 & k \end{vmatrix} = 3\cdot(-1)^{3+4}\begin{vmatrix} k & 0 & 1 \\ 1 & 2 & -1 \\ k+2 & -1 & 4 \end{vmatrix}$$

$$\overset{r_2+r_1}{\underset{r_3-4r_1}{=\!=\!=}} (-3)\cdot\begin{vmatrix} k & 0 & 1 \\ k+1 & 2 & 0 \\ -3k+2 & -1 & 0 \end{vmatrix}$$

练习 齐次线性方程组 $\begin{cases} kx_1 + x_4 = 0 \\ x_1 + 2x_2 - x_4 = 0 \\ (k+2)x_1 - x_2 + 4x_4 = 0 \\ 2x_1 + x_2 + 3x_3 + kx_4 = 0 \end{cases}$ 有非零解

的充分必要条件是 k 满足 ____

解

$$D = \begin{vmatrix} k & 0 & 0 & 1 \\ 1 & 2 & 0 & -1 \\ k+2 & -1 & 0 & 4 \\ 2 & 1 & 3 & k \end{vmatrix} = 3\cdot(-1)^{3+4}\begin{vmatrix} k & 0 & 1 \\ 1 & 2 & -1 \\ k+2 & -1 & 4 \end{vmatrix}$$

$$\xlongequal[r_3-4r_1]{r_2+r_1} (-3)\cdot\begin{vmatrix} k & 0 & 1 \\ k+1 & 2 & 0 \\ -3k+2 & -1 & 0 \end{vmatrix} = (-3)\cdot(-1)^{1+3}\cdot\begin{vmatrix} k+1 & 2 \\ -3k+2 & -1 \end{vmatrix}$$

练习 齐次线性方程组 $\begin{cases} kx_1 \qquad\qquad\qquad + \ x_4 = 0 \\ x_1 + 2x_2 \qquad\quad - \ x_4 = 0 \\ (k+2)x_1 - \ x_2 \qquad + 4x_4 = 0 \\ 2x_1 + \ x_2 + 3x_3 + kx_4 = 0 \end{cases}$ 有非零解

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$$D = \begin{vmatrix} k & 0 & 0 & 1 \\ 1 & 2 & 0 & -1 \\ k+2 & -1 & 0 & 4 \\ 2 & 1 & 3 & k \end{vmatrix} = 3\cdot(-1)^{3+4}\begin{vmatrix} k & 0 & 1 \\ 1 & 2 & -1 \\ k+2 & -1 & 4 \end{vmatrix}$$

$$\xlongequal[r_3-4r_1]{r_2+r_1} (-3)\cdot\begin{vmatrix} k & 0 & 1 \\ k+1 & 2 & 0 \\ -3k+2 & -1 & 0 \end{vmatrix} = (-3)\cdot(-1)^{1+3}\cdot\begin{vmatrix} k+1 & 2 \\ -3k+2 & -1 \end{vmatrix}$$

$$= -3(5k-5)$$

练习 齐次线性方程组 $\begin{cases} kx_1 + x_4 = 0 \\ x_1 + 2x_2 - x_4 = 0 \\ (k+2)x_1 - x_2 + 4x_4 = 0 \\ 2x_1 + x_2 + 3x_3 + kx_4 = 0 \end{cases}$ 有非零解

的充分必要条件是 k 满足 ____

解

$$D = \begin{vmatrix} k & 0 & 0 & 1 \\ 1 & 2 & 0 & -1 \\ k+2 & -1 & 0 & 4 \\ 2 & 1 & 3 & k \end{vmatrix} = 3\cdot(-1)^{3+4}\begin{vmatrix} k & 0 & 1 \\ 1 & 2 & -1 \\ k+2 & -1 & 4 \end{vmatrix}$$

$$\xlongequal[r_3-4r_1]{r_2+r_1}(-3)\cdot\begin{vmatrix} k & 0 & 1 \\ k+1 & 2 & 0 \\ -3k+2 & -1 & 0 \end{vmatrix} = (-3)\cdot(-1)^{1+3}\cdot\begin{vmatrix} k+1 & 2 \\ -3k+2 & -1 \end{vmatrix}$$

$$= -3(5k-5)$$

有非零解当且仅当 $D=0$,

练习 齐次线性方程组 $\begin{cases} kx_1 \qquad\qquad\quad + \ x_4 = 0 \\ x_1 + 2x_2 \qquad - \ x_4 = 0 \\ (k+2)x_1 - \ x_2 \qquad + 4x_4 = 0 \\ 2x_1 + \ x_2 + 3x_3 + kx_4 = 0 \end{cases}$ 有非零解

的充分必要条件是 k 满足 ____

解

$$D = \begin{vmatrix} k & 0 & 0 & 1 \\ 1 & 2 & 0 & -1 \\ k+2 & -1 & 0 & 4 \\ 2 & 1 & 3 & k \end{vmatrix} = 3\cdot(-1)^{3+4}\begin{vmatrix} k & 0 & 1 \\ 1 & 2 & -1 \\ k+2 & -1 & 4 \end{vmatrix}$$

$$\xlongequal[r_3-4r_1]{r_2+r_1}(-3)\cdot\begin{vmatrix} k & 0 & 1 \\ k+1 & 2 & 0 \\ -3k+2 & -1 & 0 \end{vmatrix} = (-3)\cdot(-1)^{1+3}\cdot\begin{vmatrix} k+1 & 2 \\ -3k+2 & -1 \end{vmatrix}$$

$$= -3(5k-5)$$

有非零解当且仅当 $D=0$，当且仅当 $k=1$