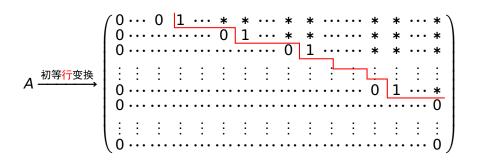
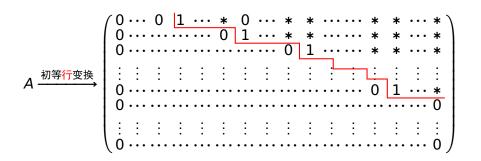
§3.1 线性方程组的消元解法

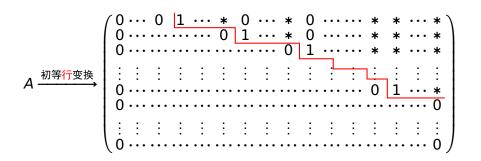
数学系 梁卓滨

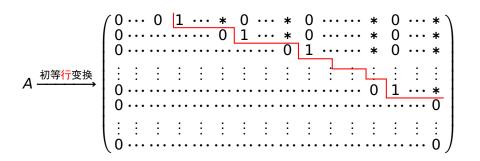
2018 - 2019 学年上学期

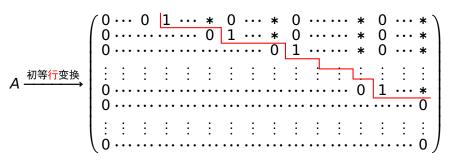












后者称为简化的阶梯型矩阵。



记号

考虑 n 个未知量 m 个方程的线性方程组:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

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可以,等价地,改写成矩阵形式

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m \times n} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

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整个方程组的信息包含在:

$$(A : b) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix}$$

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整个方程组的信息包含在:

增广矩阵
$$(A:b) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix}$$



$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ 3x_1 + 4x_2 - 2x_3 = 3 \end{cases}$$

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$$\begin{cases} x_1 & -2x_3 = 5 \\ x_2 + & x_3 = -3 \\ 0 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 5 + 2x_3 \\ x_2 = -3 - x_3 \end{cases}$$

例 解方程组

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例 解方程组

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$$(A:b) = \begin{pmatrix} \boxed{1} & 1 & & & & \\ \boxed{1} & 1 & & & & \\ \boxed{1} & 1 & & & \\ \boxed{1} & 2 & 0 & -1 \\ 47 & -1 & -1 \\ 34 & 2 & 3 \end{pmatrix} \xrightarrow[\substack{r_2 - r_1 \\ r_3 - 3r_1}} \begin{pmatrix} \boxed{1} & 1 - 1 & 2 \\ 0 \boxed{1} & 1 & -3 \\ 0 & 3 & 3 & -9 \\ 0 & 1 & 1 & -3 \end{pmatrix} \xrightarrow[\substack{r_1 - r_2 \\ r_3 - 3r_2 \\ r_4 - r_2}} \begin{pmatrix} \boxed{1} & 0 - 2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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$$\begin{cases} x_1 & -2x_3 = 5 \\ x_2 + & x_3 = -3 \\ & 0 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 5 + 2x_3 \\ x_2 = -3 - x_3 \end{cases}$$

例 解方程组

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ 3x_1 + 4x_2 - 2x_3 = 3 \end{cases} \xrightarrow{(2)-(1)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \\ 3x_2 + 3x_3 = -9 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(1)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 +$$

$$(3x_1 + 4x_2 - 2x_3 = 3) \xrightarrow{(r) \ 3x(2)} (x_2 + x_3 = -3) \xrightarrow{(r) \ 4} (0 = 0)$$

$$\downarrow \qquad \qquad \downarrow \qquad$$

$$\begin{cases} x_1 & -2x_3 = 5 \\ x_2 + & x_3 = -3 \\ 0 = 0 & \Rightarrow \begin{cases} x_1 = 5 + 2x_3 \\ x_2 = -3 - x_3 \end{cases} \end{cases}$$

例 解方程组

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ 3x_1 + 4x_2 - 2x_3 = 3 \end{cases} \xrightarrow{(3)-4\times(1)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \\ 3x_2 + 3x_3 = -9 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_3 + x_3 = -3 \end{cases} \xrightarrow{(3)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_3 + x_3 = -3 \end{cases} \xrightarrow{(3)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \end{cases} \xrightarrow{(3)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \end{cases} \xrightarrow{(3)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \end{cases} \xrightarrow{(3)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \end{cases} \xrightarrow{(3)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \end{cases} \xrightarrow{(3)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \end{cases} \xrightarrow{(3)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \end{cases} \xrightarrow{(3)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \end{cases} \xrightarrow{(3)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \end{cases} \xrightarrow{(3)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \end{cases} \xrightarrow{($$

$$\begin{cases} 4x_1 + /x_2 - x_3 = -1 & (3) - 4 \times (1) \\ 3x_1 + 4x_2 - 2x_3 = 3 & (4) - 3 \times (1) \end{cases}$$

$$(A:b) = \begin{pmatrix} \boxed{1} & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ 3 & 4 & 2 & 3 \end{pmatrix} \xrightarrow[r_3 - 3r_1]{r_2 - r_1} \begin{pmatrix} \boxed{1} & 1 & -1 & 2 \\ 0 \boxed{1} & 1 & -3 \\ 0 & 3 & 3 & -9 \\ 0 & 1 & 1 & -3 \end{pmatrix} \xrightarrow[r_4 - r_2]{r_1 - r_2} \begin{pmatrix} \boxed{1} & 0 - 2 & 5 \\ 0 \boxed{1} & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} x_1 & -2x_3 = 5 \\ x_2 + & x_3 = -3 \\ 0 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 5 + 2x_3 \\ x_2 = -3 - x_3 \end{cases}$$

自由变量

$$\begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \\ 0 = 0 \\ 0 = 0 \end{cases}$$



例 解方程组

$$x_1 + x_2 - x_3 = 2$$
 $(x_1 + x_2 - x_3 = 2)$

$$x_1 + 2x_2 = -1$$
 (2)-(1)
 $4x_1 + 7x_2 - x_3 = -1$ (3)-4×(1)

$$- x_3 = -1 _{(3)-4\times(1)}$$

$$-2x_3 = 3 ^{(4)-3\times(1)}$$

$$(A : b) = \begin{pmatrix} \boxed{1} \boxed{1-1} & 2 \\ 12 & 0 & -1 \\ 47 - 1 & -1 \\ 34 & 2 & 3 \end{pmatrix} \xrightarrow[r_4 - 3r_1]{r_2 - r_1} \begin{pmatrix} \boxed{1} \boxed{1-1} & 2 \\ 0 \boxed{1} \boxed{1} & -3 \\ 03 & 3 & -9 \\ 01 & 1 & -3 \end{pmatrix} \xrightarrow[r_4 - r_2]{r_1 - r_2} \begin{pmatrix} \boxed{10} - 2 & 5 \\ 0 \boxed{1} \boxed{1} & -3 \\ 00 & 0 & 0 \\ 00 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} x_1 & -2x_3 = 5 \\ x_2 + & x_3 = -3 \\ & 0 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 5 + 2x_3 \\ x_2 = -3 - x_3 \end{cases}$$

解方程,等同于:增广矩阵 ^{初等行变换} 简化的阶梯型矩阵

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ 3x_1 + 4x_2 - 2x_3 = 3 \end{cases} \xrightarrow{(3)-4\times(1)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \\ 3x_2 + 3x_3 = -9 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(1)-(2)} \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \\ 0 = 0 \end{cases}$$



例 解方程组

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ 3x_1 + 4x_2 - 2x_3 = 3 \end{cases} \xrightarrow{(3)-4\times(1)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \\ 3x_2 + 3x_3 = -9 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x$$

$$(A : b) = \begin{pmatrix} \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} \\ \boxed{12 \ 0} & -1 \\ 47 - 1 - 1 \\ \boxed{34 \ 2} & \boxed{3} \end{pmatrix} \xrightarrow[r_{4} - 3r_{1}]{r_{2} - 4r_{1}} \begin{pmatrix} \boxed{11 - 1} & 2 \\ 0\boxed{1} & 1 & -3 \\ 0 & 3 & 3 & -9 \\ 0 & 1 & 1 & -3 \end{pmatrix} \xrightarrow[r_{4} - 2]{r_{1} - r_{2}} \begin{pmatrix} \boxed{10 - 2} & 5 \\ 0\boxed{1} & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} x_1 & -2x_3 = 5 \\ x_2 + & x_3 = -3 \\ & 0 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 5 + 2x_3 \\ x_2 = -3 - x_3 \end{cases}$$

- 解方程,等同于: 增广矩阵 ^{初等行变换} 简化的阶梯型矩阵
- 独立方程个数 = 阶梯型矩阵的非零行的行数

自由变量

$$\begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \\ 0 = 0 \\ 0 = 0 \end{cases}$$

例 解方程组

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \end{cases} \xrightarrow{(2)-(1)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -1 \\ 3x_2 + 3x_3 = -1 \end{cases}$$

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ 3x_1 + 4x_2 - 2x_3 = 3 \end{cases} \xrightarrow{(2)-(1)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \\ (4) - 3 \times (1) \end{cases}$$

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \\ x_2 + 3x_3 = -9 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3 \times (2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3 \times (2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3 \times (2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3 \times (2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3 \times (2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3 \times (2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3 \times (2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3 \times (2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3 \times (2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3 \times (2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3 \times (2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3 \times (2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3 \times (2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3 \times (2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3 \times (2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3 \times (2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3 \times (2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3 \times (2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3 \times (2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3 \times (2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3 \times (2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3 \times (2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3 \times (2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3 \times (2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3 \times (2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3 \times (2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3 \times (2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3 \times (2)} \begin{cases} x_1 + x_2 - x_3 = 2 \end{cases} \xrightarrow{(3)-3 \times (2)} \begin{cases} x_1 + x_2 - x_3 = 2 \end{cases} \xrightarrow{(3)-3 \times (2)} \begin{cases} x_1 + x_2 - x_3 = 2 \end{cases} \xrightarrow{(3)-3 \times (2)} \begin{cases} x_1 + x_2 - x_3 = 2 \end{cases} \xrightarrow{(3)-3 \times (2)} \begin{cases} x_1 + x_2 - x_3 = 2 \end{cases} \xrightarrow{(3)-3 \times (2)} \begin{cases} x_1 + x_2 - x_3 = 2 \end{cases} \xrightarrow{(3)-3 \times (2)} \begin{cases} x_1 + x_2 - x_3 = 2 \end{cases} \xrightarrow{(3)-3 \times (2)} \begin{cases} x$$

$$(A:b) = \begin{pmatrix} \boxed{1} \boxed{1-1} & 2 \\ 12 & 0 & -1 \\ 47 & -1 & -1 \\ 34 & 2 & 3 \end{pmatrix} \xrightarrow[r_4-3r_1]{r_2-r_1} \begin{pmatrix} \boxed{0} \boxed{1} & 1 & -3 \\ 0 \boxed{1} & 1 & -3 \\ 0 & 3 & 3 & -9 \\ 0 & 1 & 1 & -3 \end{pmatrix} \xrightarrow[r_4-r_2]{r_1-r_2} \begin{pmatrix} \boxed{10} & -2 & 5 \\ \boxed{0} \boxed{1} & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix}
0.3 & 3 & -3 \\
0.1 & 1 & -3
\end{pmatrix} \xrightarrow{r_3 - 3r_4} \xrightarrow{r_4 - r_2}$$

$$r_3 = 5$$

$$r_3 = -3$$

$$r_4 = 5 + 2x_3$$

$$r_4 = 5 + 2x_3$$

$$\begin{cases} x_1 & -2x_3 = 5 \\ x_2 + & x_3 = -3 \\ & 0 = 0 \\ & 0 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 5 + 2x_3 \\ x_2 = -3 - x_3 \end{cases}$$

- 解方程,等同于: 增广矩阵 ^{初等行变换} 简化的阶梯型矩阵
- 独立方程个数 = 阶梯型矩阵的非零行的行数
- 通解中, 主元由自由变量表示, 自由变量取任意常数

自由变量

$$\begin{cases} x_{2} - 2x_{3} = 5 \\ x_{2} + x_{3} = -3 \\ 0 = 0 \\ 0 = 0 \end{cases}$$





初等行变换求解线性方程组

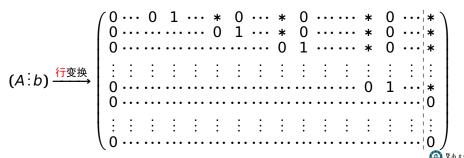
步骤:

- 1. $Ax = b \implies (A : b) \xrightarrow{instance high parts and instance high parts and insta$
- 2. 确定主元、自由变量
- 3. 通解中, 主元由自由变量表示, 自由变量取任意常数

步骤:

- 1. $Ax = b \implies (A : b) \xrightarrow{instance high parts and instance high parts and insta$
- 2. 确定主元、自由变量
- 3. 通解中, 主元由自由变量表示, 自由变量取任意常数

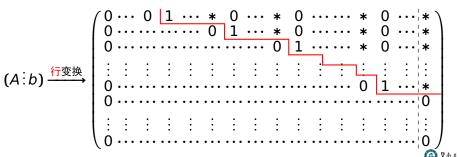
例如



步骤:

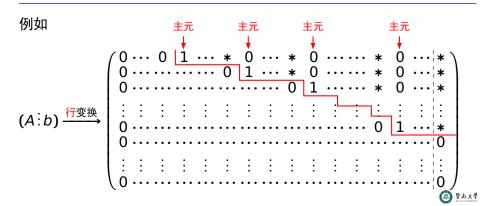
- 1. $Ax = b \implies (A \dot{\cdot} b) \xrightarrow{\eta + \eta + \eta}$ 简化的阶梯型矩阵
- 2. 确定主元、自由变量
- 3. 通解中, 主元由自由变量表示, 自由变量取任意常数

例如



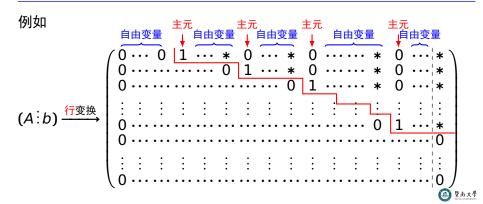
步骤:

- 1. $Ax = b \implies (A \dot{\cdot} b) \xrightarrow{\eta + \eta + \eta}$ 简化的阶梯型矩阵
- 2. 确定主元、自由变量
- 3. 通解中, 主元由自由变量表示, 自由变量取任意常数



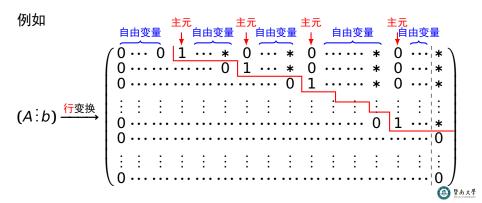
步骤:

- 1. $Ax = b \implies (A \dot{\cdot} b) \xrightarrow{\eta + \eta + \eta}$ 简化的阶梯型矩阵
- 2. 确定主元、自由变量
- 3. 通解中, 主元由自由变量表示, 自由变量取任意常数



步骤:

- 1. $Ax = b \implies (A : b) \xrightarrow{\eta + \eta + \eta}$ 简化的阶梯型矩阵
- 2. 确定主元、自由变量(自由变量个数 = 变量个数 主元个数)
- 3. 通解中,主元由自由变量表示,自由变量取任意常数



例 1 解方程组:
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$

$$\begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ -2 & -3 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{pmatrix}$$

例 1 解方程组:
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{pmatrix}$$

$$\mathbf{H}(A:b) = \begin{pmatrix} -2x_1 - 3x_2 + 3x_2 + 3x_3 + 3x_4 +$$

例 1 解方程组: $\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$ $\begin{pmatrix} 1 & 1 & -1 & 2 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{pmatrix} \xrightarrow[r_3 - 2r_1]{r_3 - 2r_1} \xrightarrow[r_4 + 2r_1]{r_4 + 2r_1}$

$$\mathbf{M} (A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{pmatrix} \frac{r_2 - r_3 - r_4}{r_4 + r_5}$$

例 1 解方程组:
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{pmatrix} \xrightarrow[r_3 - 2r_1]{r_3 - 2r_1} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -1 \\ -2 & -3 & 1 & -1 \end{pmatrix}$$

例 1 解方程组:
$$\begin{cases} x_1 + x_2 - x_3 = x_3 \\ x_1 + 2x_2 = -x_3 \\ 2x_1 + 5x_2 + x_3 = -x_3 \end{cases}$$

例 1 解方程组:
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$

$$\begin{pmatrix} 1 & 1 & -1 & 2 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{pmatrix} \xrightarrow[r_3 - 2r_1]{r_3 - 2r_1} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 1 & 1 & 1 & -3 \end{pmatrix}$$

例 1 解方程组:
$$\begin{cases} x_{1} + x_{2} - x_{3} = 2 \\ x_{1} + 2x_{2} = -1 \\ 2x_{1} + 5x_{2} + x_{3} = -5 \\ -2x_{1} - 3x_{2} + x_{3} = -1 \end{cases}$$

$$\begin{pmatrix} 1 & 1 & -1 & 2 \\ \frac{1}{2} & 2 & 0 & -\frac{1}{2} \\ \frac{1}{2} & 5 & 1 & -\frac{1}{2} \\ -2 & -3 & 1 & -1 \end{pmatrix} \xrightarrow[r_{3}-2r_{1}\\ r_{4}+2r_{1} \end{cases} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 3 & 3 & -9 \end{pmatrix}$$



$$2x_{1} + 5x_{2} + x_{3} = -5$$

$$-2x_{1} - 3x_{2} + x_{3} = -1$$

$$(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -\frac{1}{5} & \frac{r_{2}-r_{1}}{5} & 0 \end{pmatrix}$$

例 1 解方程组:
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{pmatrix} \xrightarrow[r_3 - 2r_1]{r_3 - 2r_1} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & -1 & -$$



例 1 解方程组:
$$\begin{cases} x_{1} + x_{2} - x_{3} = 2 \\ x_{1} + 2x_{2} = -1 \\ 2x_{1} + 5x_{2} + x_{3} = -5 \\ -2x_{1} - 3x_{2} + x_{3} = -1 \end{cases}$$

$$\begin{pmatrix} 1 & 1 & -1 & 2 \\ \frac{1}{2} & 2 & 0 & -\frac{1}{2} \\ \frac{1}{2} & 5 & 1 & -\frac{1}{2} \\ -2 & -3 & 1 & -1 \end{pmatrix} \xrightarrow[r_{3}-2r_{1}\\ r_{4}+2r_{1} \end{cases} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 3 & 3 & -\frac{2}{3} \\ 0 & -1 & -1 & 3 \end{pmatrix}$$

例 1 解方程组:
$$\begin{cases} x_{1} + x_{2} - x_{3} = 2 \\ x_{1} + 2x_{2} = -1 \\ 2x_{1} + 5x_{2} + x_{3} = -5 \\ -2x_{1} - 3x_{2} + x_{3} = -1 \end{cases}$$

$$\begin{pmatrix} 1 & 1 & -1 & 2 \\ \frac{1}{2} & 2 & 0 & -\frac{1}{2} \\ \frac{1}{2} & 5 & 1 & -\frac{1}{2} \\ -2 & -3 & 1 & -1 \end{pmatrix} \xrightarrow[r_{3}-2r_{1}\\ r_{4}+2r_{1} \end{cases} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 3 & 3 & -\frac{2}{3} \\ 0 & -1 & -1 & 3 \end{pmatrix}$$

$$\frac{r_1-r_2}{r_3-3r_2}$$



$$\frac{r_1-r_2}{r_3-3r_2} \left(\begin{array}{ccc} 0 & 1 & 1 \\ \end{array} \right)$$





$$\begin{array}{c|cccc}
(A \cdot b) - \begin{pmatrix} 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{pmatrix} \xrightarrow{r_3} \\
\xrightarrow{r_1 - r_2} \begin{pmatrix} 1 & 0 & -2 & -5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\left. \begin{array}{c|cc} 1 & 0 & -2 & -5 \\ 0 & 1 & 0 & 0 & 0 \end{array} \right| \begin{array}{c|cc} -5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

$$2x_{1} + 5x_{2} + x_{3} = -5$$

$$-2x_{1} - 3x_{2} + x_{3} = -1$$

$$(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{pmatrix} \xrightarrow[r_{4}+2r_{1}]{r_{3}-2r_{1}} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 3 & 3 & -9 \\ 0 & -1 & -1 & 3 \end{pmatrix}$$

$$\begin{pmatrix}
-2 & 5 & 1 & -5 \\
-2 & -3 & 1 & -1
\end{pmatrix} \xrightarrow{r_3} \begin{pmatrix}
1 & 0 & -2 & 5 \\
0 & 1 & 1 & -3 \\
0 & 0 & 0 & 0
\end{pmatrix}$$



$$\begin{pmatrix}
-2 & -3 & 1 & -1 \\
-2 & -3 & 1 & -1
\end{pmatrix} \xrightarrow{r_3} \xrightarrow{r_4}$$

$$\xrightarrow{r_1 - r_2} \xrightarrow{r_3 - 3r_2} \begin{pmatrix}
1 & 0 & -2 & 5 \\
0 & 1 & 1 & -3 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

例 1 解方程组:
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$

$$\begin{pmatrix} 1 & 1 & -1 & 2 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{pmatrix} \xrightarrow[r_3 - 2r_1]{r_3 - 2r_1} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 3 & 3 & -9 \\ 0 & -1 & -1 & 3 \end{pmatrix}$$

$$\begin{array}{c|cccc}
 & -2 & -3 & 1 & -5 & r_3 \\
 & -2 & -3 & 1 & -1 & r_4 \\
\hline
 & r_{1}-r_{2} & 0 & 0 & 0 & 0 \\
\hline
 & r_{3}-3r_{2} & 0 & 0 & 0 & 0
\end{array}$$

例 1 解方程组: $\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$ $\mathbf{H} (A:b) = \begin{pmatrix} \begin{vmatrix} 1 & 1 & -1 & | & -2 \\ 1 & 2 & 0 & | & -1 \\ 2 & 5 & 1 & | & -5 \\ -2 & -3 & 1 & | & -1 \end{pmatrix} \xrightarrow[r_1 \to 2r_1]{r_2 - r_1} \begin{pmatrix} 1 & 1 & -1 & | & -2 \\ 0 & 1 & 1 & | & -3 \\ 0 & -1 & -1 & | & -3 \\ 0 & -1 & -1 & | & 3 \end{pmatrix}$ $r_{1}-r_{2}$ $r_{3}-3r_{2}$ r_{2} $r_{3}-3r_{2}$ $r_{3}-3r_{2}$

 x_1, x_2 为主元, x_3 为自由变量。

例 1 解方程组: $\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$ $\mathbf{H}(A:b) = \begin{pmatrix} \begin{vmatrix} 1 & 1 & -1 & | & 2 \\ 1 & 2 & 0 & | & -1 & | & -1 \\ 2 & 5 & 1 & | & -5 & | & r_3 - 2r_1 \\ -2 & -3 & 1 & | & -1 & | & -1 & | & -3 \\ -2 & -3 & 1 & | & -1 & | & -1 & | & -3 \\ -2 & -3 & 1 & | & -1 & | & -1 & | & -3 \\ -2 & -3 & 1 & | & -1 & | & -3 & | & -3 \\ -2 & -3 & 1 & | & -1 & | & -3 & | & -3 \\ -2 & -3 & 1 & | & -1 & | & -3 & | & -3 \\ -2 & -3 & 1 & | & -1 & | & -3 & | & -3 & | & -3 \\ -2 & -3 & 1 & | & -1 & | & -3 & | & -3 & | & -3 \\ -2 & -3 & 1 & | & -1 & | & -3 & | & -3 & | & -3 \\ -2 & -3 & 1 & | & -1 & | & -3 & | & -3 & | & -3 \\ -2 & -3 & 1 & | & -1 & | & -3 & | & -3 & | & -3 & | & -3 \\ -2 & -3 & 1 & | & -1 & | & -3 & | & -3 & | & -3 \\ -2 & -3 & 1 & | & -1 & | & -3 & | & -3 & | & -3 & | & -3 \\ -2 & -3 & 1 & | & -1 & | & -3 & | & -3 & | & -3 & | & -3 \\ -2 & -3 & 1 & | & -1 & | & -3 & | & -3 & | & -3 & | & -3 \\ -2 & -3 & 1 & | & -1 & | & -3 & | & -3 & | & -3 & | & -3 \\ -2 & -3 & 1 & | & -1 & | & -3 & | & -3 & | & -3 & | & -3 \\ -2 & -3 & 1 & | & -1 & | & -1 & | & -3 & | & -3 & | & -3 \\ -2 & -3 & 1 & | & -1 & | & -3 & | & -3 & | & -3 & | & -3 \\ -2 & -3 & 1 & | & -1 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 \\ -2 & -3 & 1 & | & -1 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 \\ -2 & -3 & 1 & | & -1 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 \\ -2 & -3 & 1 & | & -1 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 \\ -2 & -3 & 1 & | & -1 & | & -1 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 &$ $\xrightarrow[r_3-3r_2]{r_1-r_2} \left(\begin{array}{c|c} 1 & 0 & -2 & -5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array} \right)$ x_1, x_2 为主元, x_3 为自由变量。所以原方程组等价于

$$\begin{cases} x_1, x_2 \ \text{为主儿}, \ x_3 \ \text{为自由支重。所以}. \\ x_1 + -2x_3 = 5 \\ x_2 + x_3 = -3 \end{cases}$$



例 1 解方程组:
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$

$$\mathbf{H} (A:b) = \begin{pmatrix} \begin{vmatrix} 1 & 1 & -1 & | & 2 \\ 1 & 2 & 0 & | & -1 \\ 2 & 5 & 1 & | & -5 \\ -2 & -3 & 1 & | & -1 \end{pmatrix} \xrightarrow[r_4+2r_1]{r_3-2r_1} \begin{pmatrix} 1 & 1 & -1 & | & -2 \\ 0 & 3 & 3 & | & -9 \\ 0 & -1 & -1 & | & 3 \end{pmatrix}$$

$$\xrightarrow[r_4+r_2]{r_1-r_2} \begin{pmatrix} \boxed{1} & 0 & -2 & 5 \\ 0 & \boxed{1} & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

 x_1, x_2 为主元, x_3 为自由变量。所以原方程组等价于

$$\begin{cases} x_{1} + -2x_{3} = 5 \\ x_{2} + x_{3} = -3 \end{cases} \Leftrightarrow \begin{cases} x_{1} + = 5 + 2x_{3} \\ x_{2} = -3 - x_{3} \end{cases}$$

例1解方程组: $\begin{cases} x_1 + x_2 - x_3 = 2\\ x_1 + 2x_2 = -1\\ 2x_1 + 5x_2 + x_3 = -5\\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$ $\mathbf{H}(A:b) = \begin{pmatrix} \begin{vmatrix} 1 & 1 & -1 & | & 2 \\ 1 & 2 & 0 & | & -1 \\ 2 & 5 & 1 & | & -5 \\ -2 & -3 & 1 & | & -1 \end{pmatrix} \xrightarrow[r_3-2r_1]{r_3-2r_1} \begin{pmatrix} 1 & 1 & -1 & | & -3 \\ 0 & 3 & 3 & | & -9 \\ 0 & -1 & -1 & | & -3 \end{pmatrix}$ $\xrightarrow[r_3-3r_2]{r_1-r_2} \left(\begin{array}{c|c} 1 & 0 & -2 & -5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array} \right)$ x_1, x_2 为主元, x_3 为自由变量。所以原方程组等价于 $\begin{cases} x_{1} + -2x_{3} = 5 \\ x_{2} + x_{3} = -3 \end{cases} \Leftrightarrow \begin{cases} x_{1} + = 5 + 2x_{3} \\ x_{2} = -3 - x_{3} \end{cases}$ 所以通解是: $(c_1$ 为任意常数)

例1解方程组: $\begin{cases} x_1 + x_2 - x_3 = 2\\ x_1 + 2x_2 = -1\\ 2x_1 + 5x_2 + x_3 = -5\\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$ $\mathbf{H}(A:b) = \begin{pmatrix} \begin{vmatrix} 1 & 1 & -1 & | & 2 \\ 1 & 2 & 0 & | & -1 \\ 2 & 5 & 1 & | & -5 \\ -2 & -3 & 1 & | & -1 \end{pmatrix} \xrightarrow[r_3-2r_1]{r_3-2r_1} \begin{pmatrix} 1 & 1 & -1 & | & -3 \\ 0 & 3 & 3 & | & -9 \\ 0 & -1 & -1 & | & -3 \end{pmatrix}$ $\xrightarrow[r_3-3r_2]{r_1-r_2} \left(\begin{array}{c|c} 1 & -2 & -5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array} \right)$ x_1, x_2 为主元, x_3 为自由变量。所以原方程组等价于 $\begin{cases} x_{1} + -2x_{3} = 5 \\ x_{2} + x_{3} = -3 \end{cases} \Leftrightarrow \begin{cases} x_{1} + = 5 + 2x_{3} \\ x_{2} = -3 - x_{3} \end{cases}$ 所以通解是: $\begin{cases} x_1 = 5 + 2c_1 \\ (c_1$ 为任意常数) $x_2 = c_1$

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 $x_1 + x_2 - x_3 = 2$

$$\frac{r_{1}-r_{2}}{r_{3}-3r_{2}} \xrightarrow{\begin{pmatrix} 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}} \xrightarrow{\text{Mad } \hat{r} \neq \hat{r} \neq \hat{r}} \frac{x_{1}+x_{2}}{x_{1}+x_{2}}$$

$$\begin{array}{cccc}
x_{1}, x_{2} & \hat{r} \neq \hat{$$

所以通解是: $\begin{cases} x_1 = 5 + 2c_1 \\ x_2 = -3 - c_1 \end{cases}$ (c_1) (c_1) 任意常数)

例1解方程组: $\begin{cases} x_1 + 2x_2 & = -1 \\ 2x_1 + 5x_2 + x_3 & = -5 \\ -2x_1 - 3x_2 + x_3 & = -1 \end{cases}$ $\mathbf{H} (A:b) = \begin{pmatrix} \begin{vmatrix} 1 & 1 & -1 & | & 2 \\ 1 & 2 & 0 & | & -1 & | & -1 \\ 2 & 5 & 1 & | & -5 & | & r_{3}-2r_{1} \\ 2 & 7 & 1 & | & -1 & | & -1 & | & -3 \\ 3 & 7 & 1 & | & -1 & | & -3 & | & -3 \\ 3 & 7 & 1 & | & -1 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 & | & -3 \\ 0 & -1 & -1 & | & -3 \\ 0 & -1 & -1 & | & -3 \\ 0$ $r_{1}-r_{2}$ $r_{3}-3r_{2}$ $\begin{pmatrix} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ 独立方程数 = 主元数 \Leftrightarrow 存在解

 $x_1 + x_2 - x_3 = 2$

$$x_1, x_2$$
 为主元, x_3 为自由变量。所以原方程组等价于
$$\begin{cases} x_1 + & -2x_3 = 5 \\ & x_2 + & x_3 = -3 \end{cases} \iff \begin{cases} x_1 + & = 5 + 2x_3 \\ & x_2 = -3 - x_3 \end{cases}$$

所以通解是: $\begin{cases} x_2 + x_3 = -3 & (x_2 = x_3) \\ x_1 = 5 + 2c_1 \\ x_2 = -3 - c_1 & (c_1 为任意常数) \\ x_3 = c_1 \end{cases}$

例 1 解方程组: $\begin{cases} x_1 + 2x_2 &= -1\\ 2x_1 + 5x_2 + x_3 &= -5\\ -2x_1 - 3x_2 + x_3 &= -1 \end{cases}$ $(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2\\ 1 & 2 & 0 & -1\\ 2 & 5 & 1 & -5\\ -2 & -3 & 1 & -1 \end{pmatrix} \xrightarrow[r_3 - 2r_1]{r_3 - 2r_1} \begin{pmatrix} 1 & 1 & -1 & 2\\ 0 & 3 & 3 & -9\\ 0 & -1 & -1 & 3 \end{pmatrix}$ $\xrightarrow{r_1 - r_2} \begin{pmatrix} 1 & 0 & -2 & 5\\ 0 & 1 & 1 & -3\\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[r_3 - 3r_2]{\frac{4\pi}{3} - 3r_2}} \xrightarrow{\frac{\pi}{3} - 3r_2} \begin{pmatrix} 1 & 0 & -2 & 5\\ 0 & 1 & 1 & -3\\ 0 & 0 & 0 & 0 \end{pmatrix}$

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例 1 解方程组: $\begin{cases} x_1 + x_2 - x_3 &= 2 \\ x_1 + 2x_2 &= -1 \\ 2x_1 + 5x_2 + x_3 &= -5 \\ -2x_1 - 3x_2 + x_3 &= -1 \end{cases}$ $\begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{pmatrix} \xrightarrow[r_3 - 2r_1]{r_3 - 2r_1} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 3 & 3 & -9 \\ 0 & -1 & -1 & 3 \end{pmatrix}$

人1,人2 万工儿,人3 万日田支星。 所以原乃任垣寺川 1
(Y1+ - 2Y2 = 5 (Y1+ = 5+2Y2)

$$\begin{cases} x_{1} + -2x_{3} = 5 \\ x_{2} + x_{3} = -3 \end{cases} \iff \begin{cases} x_{1} + = 5 + 2x_{3} \\ x_{2} = -3 - x_{3} \end{cases}$$
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例 2 解方程组:
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

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$$(A:b) = \begin{pmatrix} -\frac{1}{2} & -\frac{2}{3} & -\frac{4}{9} & -\frac{28}{53} \\ \frac{3}{5} & \frac{6}{9} & \frac{13}{22} & \frac{88}{141} \end{pmatrix}$$



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$$(A:b) = \begin{pmatrix} \begin{vmatrix} 1 \\ -2 \\ 3 \\ 6 \\ 13 \\ 6 \end{vmatrix} \begin{vmatrix} 28\\ -53\\ 88\\ 141 \end{pmatrix} \frac{r_2 + 2r_1}{r_3 - 3r_1}$$



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$$(A:b) = \begin{pmatrix} \boxed{1} & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow[r_4-5r_1]{r_2+2r_1} \begin{pmatrix} \boxed{1} & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \end{pmatrix}$$



例 2 解方程组:
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

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$$(A:b) = \begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow[r_4-5r_1]{r_2+2r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$



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$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$(A:b) = \begin{pmatrix} \begin{vmatrix} 1 \\ -2 \\ -3 \\ 3 \\ 6 \\ 13 \\ 5 \end{vmatrix} \begin{vmatrix} 28\\ -53\\ 88\\ 141 \end{pmatrix} \xrightarrow[r_2+2r_1]{r_2+2r_1} \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ -1 \end{vmatrix} \xrightarrow[2]{28} \begin{pmatrix} 1 \\ 3 \\ 4 \\ 1 \end{pmatrix}$$



例 2 解方程组:
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$(A:b) = \begin{pmatrix} \begin{vmatrix} 1 \\ -2 \\ 3 \\ 6 \\ 13 \\ 6 \end{vmatrix} \begin{vmatrix} 28\\ -53\\ 88\\ 141 \end{pmatrix} \xrightarrow[r_3-3r_1]{r_2+2r_1} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ -1 \end{vmatrix} = \begin{pmatrix} 24\\ 28\\ 3\\ 4\\ 1 \end{pmatrix}$$

$$r_1-2r_2$$



例 2 解方程组:
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow[r_4-5r_1]{r_2+2r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

$$\xrightarrow[r_4+r_2]{r_1-2r_2} \left(\begin{array}{ccc|c} 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \end{array} \right)$$

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$$(A:b) = \begin{pmatrix} \begin{vmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow{r_2 + 2r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

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例 2 解方程组:
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$(A:b) = \begin{pmatrix} \boxed{1} & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow[r_4-5r_1]{r_2+2r_1} \begin{pmatrix} \boxed{1} & 2 & 4 & 28 \\ 0 & \boxed{1} & -1 & 3 \\ 0 & 0 & \boxed{1} & 4 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

$$\xrightarrow[r_4+r_2]{r_1-2r_2} \left(\begin{array}{ccc} 1 & 0 & 6 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{array} \right| \left. \begin{array}{c} 22 \\ 3 \\ 4 \\ 4 \end{array} \right)$$

例 2 解方程组:
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$$(A:b) = \begin{pmatrix} \boxed{1} & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow[r_3 - 3r_1]{r_2 + 2r_1} \begin{pmatrix} \boxed{1} & 2 & 4 & 28 \\ 0 & \boxed{1} & -1 & 3 \\ 0 & 0 & \boxed{1} & 4 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

$$\xrightarrow[r_4+r_2]{r_1-2r_2} \left(\begin{array}{ccc} 1 & 0 & 6 & 22 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \end{array} \right) \xrightarrow[r_4-r_3]{r_1-6r_3}$$

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$$(A:b) = \begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow{r_2 + 2r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

$$\frac{r_1 - 2r_2}{r_1 - 2r_2} \begin{pmatrix} 1 & 0 & 6 & 22 \\ 0 & 1 & -1 & 3 \end{pmatrix} \xrightarrow{r_1 - 6r_3} \begin{pmatrix} 1 & 0 & 6 & 22 \\ 0 & 1 & -1 & 3 & 3 \end{pmatrix}$$

$$\xrightarrow[r_4+r_2]{r_4+r_2} \begin{pmatrix} 1 & 0 & 6 & 22 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{pmatrix} \xrightarrow[r_4-r_3]{r_1-6r_3} \begin{pmatrix} 0 & 0 & 1 & 4 \end{pmatrix}$$

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$$\xrightarrow{r_1 - 2r_2} \begin{pmatrix} \boxed{1} & 0 & 6 & 22 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & \boxed{1} & 4 \end{pmatrix} \xrightarrow{r_2 + r_3} \begin{pmatrix} \boxed{1} & 0 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$



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$$(A:b) = \begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow{r_2 + 2r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

$$r_{1-2r_2} \begin{pmatrix} 1 & 0 & 6 & 22 \\ 0 & 1 & -1 & 3 \end{pmatrix} \xrightarrow{r_1 - 6r_3} \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \end{pmatrix}$$

$$\xrightarrow[r_4+r_2]{r_1-2r_2} \left(\begin{array}{ccc|c} 1 & 0 & 6 & 22 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{array} \right) \xrightarrow[r_4+r_3]{r_1-6r_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \end{array} \right)$$

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$$(A:b) = \begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow{r_2 + 2r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ r_4 - 5r_1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

$$\xrightarrow{r_1 - 2r_2} \begin{pmatrix} 1 & 0 & 6 & 22 \\ 0 & 0 & -1 & 2 & 3 \\ 0 & 0 & 1 & 4 \end{pmatrix} \xrightarrow{r_1 - 6r_3} \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

$$\frac{r_1 - 2r_2}{r_4 + r_2} \leftarrow
\begin{pmatrix}
1 & 0 & 6 & 22 \\
0 & 1 & -1 & 3 \\
0 & 0 & 1 & 4 \\
0 & 0 & 1 & 4
\end{pmatrix}
\frac{r_1 - 6r_3}{r_2 + r_3} \leftarrow
\begin{pmatrix}
1 & 0 & 0 & -2 \\
0 & 1 & 0 & 7 \\
0 & 0 & 1 & 4 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

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$$\xrightarrow[r_4+r_2]{\begin{array}{c}1&0&6\\0&1&-1\\0&0&1\\0&1&4\end{array}}\xrightarrow[r_4+r_3]{\begin{array}{c}1&0&0\\0&1&-2\\0&0&1\\0&0&1\end{array}}$$

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解

 x_1, x_2, x_3 为主元,没有自由变量。

解

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$$\xrightarrow{r_1 - 2r_2} \begin{pmatrix} \boxed{1} & 0 & 6 & 22 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & \boxed{1} & 4 \end{pmatrix} \xrightarrow{r_2 + r_3} \begin{pmatrix} \boxed{1} & 0 & 0 & -2 \\ 0 & \boxed{1} & 0 & 7 \\ 0 & 0 & \boxed{1} & 4 \end{pmatrix}$$

 x_1, x_2, x_3 为主元,没有自由变量。所以原方程组等价于

$$\begin{cases} x_1 & = -2 \\ x_2 & = 7 \\ x_3 & = 4 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow{r_2 + 2r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

$$\frac{r_1 - 2r_2}{r_4 + r_2} \begin{pmatrix}
1 & 0 & 6 & 22 \\
0 & 1 & -1 & 3 \\
0 & 0 & 1 & 4
\end{pmatrix} \xrightarrow[r_2 + r_3]{r_1 - 6r_3} \begin{pmatrix}
1 & 0 & 0 & -2 \\
0 & 1 & 0 & 7 \\
0 & 0 & 1 & 4 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

 x_1, x_2, x_3 为主元,没有自由变量。所以原方程组等价于

$$\begin{cases} x_1 & = -2 \\ x_2 & = 7 \\ x_3 & = 4 \end{cases}$$
 独立方程数 = 主元数

解

$$(A:b) = \begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow{r_2 + 2r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

$$\frac{r_1 - 2r_2}{r_4 + r_2} \leftarrow \begin{pmatrix} 1 & 0 & 6 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 22 \\ 3 \\ 4 \\ 4 \end{pmatrix} \xrightarrow[r_2 + r_3]{r_1 - 6r_3} \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

 X_1, X_2, X_3 为主元,没有自由变量。所以原方程组等价于

$$\begin{cases} x_1 & = -2 \\ x_2 & = 7 \\ x_3 & = 4 \end{cases}$$

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解

$$(A:b) = \begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow{r_2 + 2r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

$$\frac{r_1 - 2r_2}{r_4 + r_2} \leftarrow \begin{pmatrix}
1 & 0 & 6 & 22 \\
0 & 1 & -1 & 3 \\
0 & 0 & 1 & 4
\end{pmatrix} \xrightarrow[r_4 - r_3]{r_1 - 6r_3} \leftarrow \begin{pmatrix}
1 & 0 & 0 & -2 \\
0 & 1 & 0 & 7 \\
0 & 0 & 1 & 4 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

 X_1, X_2, X_3 为主元,没有自由变量。所以原方程组等价于

$$\begin{cases} x_1 & = -2 \\ x_2 & = 7 \\ x_3 & = 4 \end{cases} \xrightarrow{\underbrace{\underline{\text{$\underline{4}$}$}\underline{\text{$\underline{\alpha}$}}}\underline{\text{$\underline{\beta}$}}\underline{\text{$\underline{$$



$$(A:b) = \begin{pmatrix} \boxed{1} & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow[r_4-5r_1]{r_2+2r_1} \begin{pmatrix} \boxed{1} & 2 & 4 & 28 \\ 0 & \boxed{1} & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

$$\xrightarrow[r_4+r_2]{\begin{array}{c} 1 & 0 & 6 & 22 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \\ \end{array}} \xrightarrow[r_4+r_3]{\begin{array}{c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ \end{array}}$$

 x_1, x_2, x_3 为主元,没有自由变量。所以原方程组等价于

$$\begin{cases} x_1 & = -2 \\ x_2 & = 7 \\ x_3 & = 4 \end{cases} \xrightarrow{\underline{\text{独立方程数}} = \pm \overline{\text{元数}}} \Leftrightarrow \overline{\text{存在解}}$$

$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & | -3 \\ 2 & 1 & -4 & | -1 \\ 5 & 3 & -11 & 2 \\ 1 & 1 & -4 & 2 \end{pmatrix}$$

$$(A:b) = \begin{pmatrix}
4 & 2 & -7 & | & -3 \\
2 & 1 & -4 & | & -1 \\
5 & 3 & -11 & | & 2 \\
1 & 1 & -4 & | & 2
\end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4}$$

$$\widehat{\mathbf{H}}(A:b) = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix}$$



$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & -3 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 1 & 1 & -4 & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} \boxed{1} & -4 & 2 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 4 & 2 & -7 & -3 \end{pmatrix}$$



$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 - 5r_1} \xrightarrow{r_4 - 4r_1}$$



$$\begin{array}{c}
(A : b) = \begin{pmatrix} 4 & 2 & -7 & | -3 \\ 2 & 1 & -4 & | -1 \\ 5 & 3 & -11 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \\
\begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix}$$



$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & | -3 \\ 2 & 1 & -4 & | -1 \\ 5 & 3 & -11 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} \boxed{1} & 1 & -4 & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} \boxed{1} & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | & -1 & | &$$



$$\begin{array}{c}
\stackrel{\text{pr}}{\text{(A:b)}} = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 - 5r_1} \\
\begin{pmatrix} 1 & 1 & -4 & | & -2 \\ 0 & -1 & 4 & | & -5 \\ 0 & -2 & 9 & | & -8 \end{pmatrix}$$



$$\mathbf{(A:b)} = \begin{pmatrix}
42 & -7 & -3 \\
21 & -4 & -1 \\
53 & -11 & 2 \\
11 & -4 & 2
\end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix}
11 & -4 & 2 \\
21 & -4 & -1 \\
53 & -11 & 2 \\
42 & -7 & -3
\end{pmatrix} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 - 5r_1} \begin{pmatrix}
1 & 1 & -4 & 2 \\
6 & -1 & 4 & -5 \\
0 & -2 & 9 & -8 \\
0 & -2 & 9 & -8
\end{pmatrix}$$

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$$\mathbf{\hat{R}} (A:b) = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 - 5r_1} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 0 & -1 & 4 & | & -5 \\ 0 & -2 & 9 & | & -11 \end{pmatrix}$$



$$\mathbf{(A:b)} = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 - 5r_1} \xrightarrow{r_4 - 4r_1}$$

$$\begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 0 & -1 & 4 & | & -5 \\ 0 & -2 & 9 & | & -11 \end{pmatrix} \xrightarrow{r_3 + 2r_2} \xrightarrow{r_4 + 2r_2}$$



$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 - 5r_1} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \xrightarrow{r_3 + 2r_2} \begin{pmatrix} 0 & -1 & 4 & | & -5 \\ 0 & -2 & 9 & | & -11 \end{pmatrix} \xrightarrow{r_3 + 2r_2} \begin{pmatrix} 0 & -1 & 4 & | & -5 \\ 0 & -2 & 9 & | & -11 \end{pmatrix}$$



$$\begin{array}{c}
(A:b) = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & 3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 - 5r_1} \xrightarrow{r_4 - 4r_1} \\
\begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 0 & -1 & 4 & | & -5 \\ 0 & -2 & 9 & | & -11 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{pmatrix} 1 & 0 & 0 & | & -3 \\ 0 & -1 & 4 & | & -5 \\ 0 & -2 & 9 & | & -11 \end{pmatrix}$$



$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 - 5r_1} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 1 & 1 & -4 & | & 2 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \xrightarrow{r_3 - 5r_1} \xrightarrow{r_4 - 4r_1} \begin{pmatrix} 1 & 1 & 0 & 0 & | & -3 \\ 0 & -1 & 4 & | & -5 \\ 0 & -2 & 9 & | & -11 \end{pmatrix} \xrightarrow{r_3 + 2r_2} \begin{pmatrix} 1 & 0 & 0 & | & -3 \\ 0 & -1 & 4 & | & -5 \\ 0 & 0 & 1 & | & 2 \end{pmatrix}$$



$$\mathbf{\widetilde{H}} (A:b) = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -1 & 1 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} \boxed{1} & 1 & -4 & | & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -1 & 1 & | & 2 \\ 5 & 3 & -1 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 - 5r_1} \xrightarrow{r_4 - 4r_1}$$

$$\begin{pmatrix} \boxed{1} & \boxed{1} - 4 & | & 2 \\ 0 & \boxed{1} & \boxed{1} & 4 & | & -5 \\ 0 & -2 & 9 & | & -11 \end{pmatrix} \xrightarrow{r_1 + r_2} \begin{pmatrix} \boxed{1} & 0 & 0 & | & -3 \\ 0 & 0 & 1 & | & -5 \\ 0 & 0 & 1 & | & -1 \end{pmatrix}$$



$$\mathbf{\widetilde{H}} (A:b) = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -1 & 1 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} \boxed{1} & 1 & -4 & | & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -1 & 1 & | & 2 \\ 5 & 3 & -1 & 1 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 - 5r_1} \begin{pmatrix} \boxed{1} & \boxed{1} & -4 & | & 2 \\ 5 & 3 & -1 & 1 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \xrightarrow{r_3 - 5r_1} \xrightarrow{r_4 - 4r_1} \begin{pmatrix} \boxed{1} & \boxed{1} & -4 & | & 2 \\ 5 & 3 & -1 & 1 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \xrightarrow{r_3 - 5r_1} \xrightarrow{r_4 - 4r_1} \begin{pmatrix} \boxed{1} & \boxed{1$$

$$\begin{array}{l}
(A:b) = \begin{pmatrix} 4 & 2 & -7 & | -3 \\ 2 & 1 & -4 & | -1 \\ 5 & 3 & -11 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & 3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \\
\begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & 3 \end{pmatrix} \xrightarrow{r_3 - 5r_1} \\
\begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 0 & -1 & 4 & | & -5 \\ 0 & -2 & 9 & | & -11 \end{pmatrix} \xrightarrow{r_1 + r_2} \begin{pmatrix} 1 & 0 & 0 & | & -3 \\ 0 & 0 & 1 & | & -5 \\ 0 & 0 & 1 & | & -1 \end{pmatrix} \xrightarrow{r_3 - 5r_1} \begin{pmatrix} 1 & 0 & 0 & | & -3 \\ 0 & 1 & 0 & | & 13 \\ 0 & 0 & 1 & | & 2 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

(3) 15 A

$$\begin{array}{l}
(A:b) = \begin{pmatrix} 4 & 2 & -7 & | -3 \\ 2 & 1 & -4 & | -1 \\ 5 & 3 & -11 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & 3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \\
\begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & 3 \end{pmatrix} \xrightarrow{r_3 - 5r_1} \\
\begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 0 & -1 & 4 & | & -5 \\ 0 & -2 & 9 & | & -11 \end{pmatrix} \xrightarrow{r_1 + r_2} \begin{pmatrix} 1 & 0 & 0 & | & -3 \\ 0 & 0 & 1 & | & 2 \\ 0 & 0 & 1 & | & -1 \end{pmatrix} \xrightarrow{r_1 + r_2} \begin{pmatrix} 1 & 0 & 0 & | & -3 \\ 0 & 0 & 1 & | & 2 \\ 0 & 0 & 1 & | & 2 \\ 0 & 0 & 0 & | & -3 \end{pmatrix}$$



例 3 解方程组: $\begin{cases} 4x_1 + 2x_2 - 7x_3 = -3\\ 2x_1 + x_2 - 4x_3 = -1\\ 5x_1 + 3x_2 - 11x_3 = 2\\ x_1 + x_2 - 4x_3 = 2 \end{cases}$ $\mathbf{\widetilde{R}} (A:b) = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -1 & 1 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -1 & 1 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 - 5r_1} \xrightarrow{r_4 - 4r_1}$ $\begin{pmatrix} 1 & 1 & -4 \\ 0 & -1 & 4 \\ 0 & -2 & 9 \\ 0 & -2 & 9 \\ -11 \end{pmatrix} \xrightarrow[r_{1}+r_{2}]{r_{1}+r_{2}} \begin{pmatrix} 1 & 0 & 0 & | & -3 \\ 0 & -1 & 4 & | & -5 \\ 0 & 0 & 1 & | & 2 \\ 0 & 0 & 1 & | & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & | & -3 \\ 0 & 1 & 0 & | & 13 \\ 0 & 0 & 1 & | & 2 \\ 0 & 0 & 0 & | & -3 \end{pmatrix}$ 所以原方程组等价于 $\begin{cases} x_1 & = -3 \\ x_2 & = 13 \\ x_3 & = 2 \\ 0 & = -3 \end{cases}$

例 3 解方程组: $\begin{cases} 4x_1 + 2x_2 - 7x_3 = -3\\ 2x_1 + x_2 - 4x_3 = -1\\ 5x_1 + 3x_2 - 11x_3 = 2\\ x_1 + x_2 - 4x_3 = 2 \end{cases}$ $\mathbf{\widetilde{R}} (A:b) = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -1 & 1 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -1 & 1 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 - 5r_1} \xrightarrow{r_4 - 4r_1}$ $\begin{pmatrix} 1 & 1 & -4 \\ 0 & -1 & 4 \\ 0 & -2 & 9 \\ 0 & -2 & 9 \\ -11 \end{pmatrix} \xrightarrow[r_3+2r_2]{} \begin{pmatrix} 1 & 0 & 0 & | & -3 \\ 0 & -1 & 4 & | & -5 \\ 0 & 0 & 1 & | & 2 \\ 0 & 0 & 1 & | & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & | & -3 \\ 0 & 1 & 0 & | & 13 \\ 0 & 0 & 1 & | & 2 \\ 0 & 0 & 0 & | & -3 \end{pmatrix}$ 所以原方程组等价于 $\begin{cases} x_1 & = -3 \\ x_2 & = 13 \\ x_3 & = 2 \\ 0 = -3 \end{cases} \Rightarrow \mathcal{E}_{\mathbf{M}}^{\mathbf{M}}$

例 3 解方程组: $\begin{cases} 4x_1 + 2x_2 - 7x_3 = -3\\ 2x_1 + x_2 - 4x_3 = -1\\ 5x_1 + 3x_2 - 11x_3 = 2\\ x_1 + x_2 - 4x_3 = 2 \end{cases}$ $\mathbf{(A:b)} = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -1 & 1 & 2 \\ 1 & 1 & -A & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & 2 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -1 & 2 \\ 4 & 2 & -7 & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 - 5r_1} \xrightarrow{r_4 - 4r_1}$ $\begin{pmatrix} 1 & 1 & -4 \\ 0 & -1 & 4 \\ 0 & -2 & 9 \\ 0 & -2 & 9 \\ -11 \end{pmatrix} \xrightarrow[r_1+r_2]{r_1+r_2} \begin{pmatrix} 1 & 0 & 0 & | & -3 \\ 0 & -1 & 4 & | & -5 \\ 0 & 0 & 1 & | & 2 \\ 0 & 0 & 1 & | & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & | & -3 \\ 0 & 1 & 0 & | & 13 \\ 0 & 0 & 1 & | & 2 \\ 0 & 0 & 0 & | & -3 \end{pmatrix}$ 所以原方程组等价于 $\begin{cases} x_1 & = -3 & \text{独立方程数 > 主元数} \\ x_2 & = 13 & \\ x_3 & = 2 & \Rightarrow & \text{无解!} \\ 0 = -3 & & & & \end{cases}$

例 3 解方程组: $\begin{cases} 4x_1 + 2x_2 - 7x_3 = -3\\ 2x_1 + x_2 - 4x_3 = -1\\ 5x_1 + 3x_2 - 11x_3 = 2\\ x_1 + x_2 - 4x_3 = 2 \end{cases}$ $\mathbf{(A:b)} = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -1 & 1 & 2 \\ 1 & 1 & -A & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -1 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 - 5r_1} \xrightarrow{r_4 - 4r_1}$ $\begin{pmatrix} 1 & 1 & -4 \\ 0 & -1 & 4 \\ 0 & -2 & 9 \\ 0 & -2 & 9 \\ -11 \end{pmatrix} \xrightarrow[r_1+r_2]{r_1+r_2} \begin{pmatrix} 1 & 0 & 0 & | & -3 \\ 0 & -1 & 4 & | & -5 \\ 0 & 0 & 1 & | & 2 \\ 0 & 0 & 1 & | & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & | & -3 \\ 0 & 1 & 0 & | & 13 \\ 0 & 0 & 1 & | & 2 \\ 0 & 0 & 0 & | & -3 \end{pmatrix}$ 所以原方程组等价于 $\begin{cases} x_1 & = -3 \\ x_2 & = 13 \\ x_3 & = 2 \end{cases} \xrightarrow{\text{独立方程数}} \Rightarrow \text{ 无解!}$

$$\begin{cases} x_{1}+ & x_{2}- & x_{3} = 2 \\ x_{1}+ & 2x_{2} & = -1 \\ 2x_{1}+ & 5x_{2}+ & x_{3} = -5 \\ -2x_{1}- & 3x_{2}+ & x_{3} = -1 \end{cases} \qquad \begin{cases} x_{1}+ & 2x_{2}+ & 4x_{3} = & 28 \\ -2x_{1}- & 3x_{2}- & 9x_{3} = & -53 \\ 3x_{1}+ & 6x_{2}+ & 13x_{3} = & 88 \\ 5x_{1}+ & 9x_{2}+ & 22x_{3} = & 141 \end{cases} \qquad \begin{cases} 4x_{1}+ & 2x_{2}- & 7x_{3} = & -3 \\ 2x_{1}+ & x_{2}- & 4x_{3} = & -1 \\ 5x_{1}+ & 3x_{2}- & 11x_{3} = & 2 \\ x_{1}+ & x_{2}- & 4x_{3} = & 2 \end{cases}$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$











主元数 = 独立方程数



主元数 = 独立方程数 < n



主元数 = 独立方程数 < nr(A) = r(A : b) < n



r(A) = r(A : b) < n

主元数 = 独立方程数 < n 主元数 = 独立方程数

主元数 = 独立方程数 < n 主元数 = 独立方程数 = n



r(A) = r(A : b) < n

主元数 = 独立方程数 = n

r(A) = r(A : b) = n



主元数 = 独立方程数 < n

r(A) = r(A : b) < n

$$\begin{pmatrix}
1 & 0 & -2 & 5 \\
0 & 1 & 1 & -3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

无穷多解

主元数 = 独立方程数
$$< n$$
 $r(A) = r(A : b) < n$

$$\begin{pmatrix} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{pmatrix} \begin{pmatrix} 4x_1 + 2x_2 - 7x_3 = -3 \\ 2x_1 + x_2 - 4x_3 = -1 \\ 5x_1 + 3x_2 - 11x_3 = 2 \\ x_1 + x_2 - 4x_3 = 2 \end{pmatrix}$$

$$\begin{cases} 5x_1 + 6x_2 + 15x_3 = 66 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

初等↓行变换

$$\begin{pmatrix} 1 & 0 & 0 & | & -2 \\ 0 & 1 & 0 & | & 7 \\ 0 & 0 & 1 & | & 4 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

唯一解

主元数 = 独立方程数=
$$n$$

 $r(A) = r(A : b) = n$

(A:b)

初等 ↓ 行变换

$$\begin{pmatrix} 1 & 0 & 0 & | & -3 \\ 0 & 1 & 0 & | & 13 \\ 0 & 0 & 1 & | & 2 \\ 0 & 0 & 0 & | & -3 \end{pmatrix}$$

无解

主元数 < 独立方程数



$$\begin{pmatrix} 1 & 0 & 0 & | & -2 \\ 0 & 1 & 0 & | & 7 \\ 0 & 0 & 1 & | & 4 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$\begin{pmatrix}
0 & 1 & 1 & -3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

初等↓行变换

无穷多解

主元数 = 独立方程数
$$< n$$

 $r(A) = r(A : b) < n$

唯一解

主元数 = 独立方程数=
$$n$$

 $r(A) = r(A : b) = n$

$$\begin{cases} 4x_1 + 2x_2 - 7x_3 = -3\\ 2x_1 + x_2 - 4x_3 = -1\\ 5x_1 + 3x_2 - 11x_3 = 2\\ x_1 + x_2 - 4x_3 = 2 \end{cases}$$

(A:b)

初等 **↓ 行**变换

$$\begin{pmatrix} 1 & 0 & 0 & | & -3 \\ 0 & 1 & 0 & 13 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & | & -3 \end{pmatrix}$$

无解

主元数 < 独立方程数 r(A) < r(A : b)



总结 定理方程组 $\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$

$$⇔Ax = b$$
 的

解有如下情形:



总结 定理方程组 $\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1\\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2\\ \vdots\\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$

$$a_{1n}x_n = b_1$$

 $\Leftrightarrow Ax = b$ 的

解有如下情形:

- 1. 有解 ⇔
 - 有无穷多解 ⇔
 - 只有唯一解 ⇔
- 2. 无解 ⇔

总结
定理方程组
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1\\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2\\ \vdots\\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

$$⇔ Ax = b$$
 的

解有如下情形:

- 1. 有解 \Leftrightarrow r(A) = r(A : b)
 - 有无穷多解 ⇔
 - 只有唯一解 ⇔
- 2. 无解 \Leftrightarrow $r(A) \neq r(A : b)$

解有如下情形:

- 1. 有解 \Leftrightarrow r(A) = r(A : b)
 - 有无穷多解 ⇔
 - 只有唯一解 ⇔
- 2. 无解 \Leftrightarrow $r(A) \neq r(A : b)$

注

● r(A) = 主元数; r(A:b) = 独立方程数



解有如下情形:

- 1. 有解 \Leftrightarrow r(A) = r(A : b) (主元数 = 独立方程数)
 - 有无穷多解 ⇔
 - 只有唯一解 ⇔
- 2. 无解 \Leftrightarrow $r(A) \neq r(A : b)$

(主元数 ≠ 独立方程数)

注

• r(A) = 主元数; r(A:b) = 独立方程数



解有如下情形:

- 1. 有解 \Leftrightarrow r(A) = r(A : b) (主元数 = 独立方程数)
 - 有无穷多解 ⇔
 - 只有唯一解 ⇔
- 2. 无解 \Leftrightarrow $r(A) \neq r(A : b) \Leftrightarrow r(A) < r(A : b)$ (主元数 \neq 独立方程数)

注

r(A) = 主元数; r(A:b) = 独立方程数



 $\Leftrightarrow Ax = b$ 的

解有如下情形:

- 1. 有解 \Leftrightarrow r(A) = r(A : b) (主元数 = 独立方程数)
 - 有无穷多解 \Leftrightarrow r(A) = r(A : b) < n
 - 只有唯一解 \Leftrightarrow r(A) = r(A : b) = n
- 2. 无解 \Leftrightarrow $r(A) \neq r(A : b) \Leftrightarrow r(A) < r(A : b)$ (主元数 \neq 独立方程数)

注

r(A) = 主元数; r(A:b) = 独立方程数



 $\Leftrightarrow Ax = b$ 的

解有如下情形:

- 1. 有解 \Leftrightarrow r(A) = r(A : b) (主元数 = 独立方程数)
 - 有无穷多解 ⇔ r(A) = r(A:b) < n
 - 只有唯一解 \Leftrightarrow r(A) = r(A : b) = n
- 2. 无解 \Leftrightarrow $r(A) \neq r(A : b) \Leftrightarrow r(A) < r(A : b)$ (主元数 \neq 独立方程数)

注

- r(A) = 主元数; r(A:b) = 独立方程数
 - n − r(A) 为自由变量的个数



解有如下情形:

- 1. 有解 \Leftrightarrow r(A) = r(A : b) (主元数 = 独立方程数)
 - 有无穷多解 \Leftrightarrow r(A) = r(A : b) < n (自由变量数 ≥ 1)
 - 只有唯一解 \Leftrightarrow r(A) = r(A : b) = n (自由变量数 = 0)
- 2. 无解 \Leftrightarrow $r(A) \neq r(A : b) \Leftrightarrow r(A) < r(A : b)$ (主元数 \neq 独立方程数)

注

- r(A) = 主元数; r(A:b) = 独立方程数
 - n − r(A) 为自由变量的个数



练习 1 求解 $\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1\\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3\\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7\\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$

练习 1 求解
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1\\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3\\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7\\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$
 解

$$(A:b) = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$

练习 1 求解
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1\\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3\\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7\\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$
 解

$$(A:b) = \begin{pmatrix} \boxed{1} & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$

练习1求解
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1\\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3\\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7\\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$
 解

$$(A:b) = \begin{pmatrix} \boxed{1} & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 + r_1}$$

练习1求解
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1 \\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3 \\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7 \\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$

$$(A : b) = \begin{pmatrix} \boxed{1} & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} \boxed{1} & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$



练习 1 求解
$$\begin{cases} x_{1} + 2x_{2} + x_{3} + x_{4} + x_{5} = 1\\ 2x_{1} + 4x_{2} + 3x_{3} + x_{4} + x_{5} = 3\\ -x_{1} - 2x_{2} + x_{3} + 3x_{4} - 3x_{5} = 7\\ 2x_{3} + 5x_{4} - 2x_{5} = 9 \end{cases}$$

$$(A:b) = \begin{pmatrix} \boxed{1} & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} \boxed{1} & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & \boxed{1} & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$



练习 1 求解
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1 \\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3 \\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7 \\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$

脌

$$(A \vdots b) = \begin{pmatrix} \boxed{1} & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} \boxed{1} & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & \boxed{1} & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$

$$\frac{r_3-2r_2}{r_4-2r_3}$$



练习 1 求解
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1 \\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3 \\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7 \\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$

胖

$$(A \vdots b) = \begin{pmatrix} \boxed{1} & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & \boxed{1} & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$

$$\frac{r_3 - 2r_2}{r_4 - 2r_2} \left(\begin{array}{ccccc}
1 & 2 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & -1 & -1 & 1 \\
0 & 0 & 0 & 6 & 0 & 6 \\
0 & 0 & 0 & 7 & 0 & 7
\end{array} \right)$$



练习 1 求解
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1 \\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3 \\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7 \\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$

$$(A:b) = \begin{pmatrix} \boxed{1} & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} \boxed{1} & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & \boxed{1} & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$

$$\frac{r_3 - 2r_2}{r_4 - 2r_2} \left(\begin{array}{ccccc}
1 & 2 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & -1 & -1 & 1 \\
0 & 0 & 0 & \boxed{6} & 0 & 6 \\
0 & 0 & 0 & 7 & 0 & 7
\end{array} \right)$$

练习1求解
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1\\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3\\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7\\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$

$$(A:b) = \begin{pmatrix} \boxed{1} & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} \boxed{1} & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & \boxed{1} & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$



练习 1 求解
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1 \\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3 \\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7 \\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$

脌

$$(A:b) = \begin{pmatrix} \boxed{1} & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} \boxed{1} & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & \boxed{1} & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$

$$\frac{r_{3}-2r_{2}}{r_{4}-2r_{2}} \left(\begin{array}{cccccc} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 6 & 0 & 6 \\ 0 & 0 & 0 & 7 & 0 & 7 \end{array} \right) \xrightarrow{\frac{1}{6}\times r_{3}} \left(\begin{array}{ccccccc} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array} \right) \\
\xrightarrow{r_{4}-r_{3}} \left(\begin{array}{cccccccc} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$



练习1求解
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1\\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3\\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7\\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$

$$(A:b) = \begin{pmatrix} \boxed{1} & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} \boxed{1} & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & \boxed{1} & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$

$$\frac{r_{3}-2r_{2}}{r_{4}-2r_{2}} \left(\begin{array}{ccccccc}
1 & 2 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & -1 & -1 & 1 \\
0 & 0 & 0 & 6 & 0 & 6 \\
0 & 0 & 0 & 7 & 0 & 7
\end{array}\right) \frac{\frac{1}{6} \times r_{3}}{\frac{1}{7} \times r_{4}} \left(\begin{array}{ccccccc}
1 & 2 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & -1 & -1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1
\end{array}\right)$$

$$\frac{r_{4}-r_{3}}{r_{4}-r_{3}} \left(\begin{array}{ccccccc}
1 & 2 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & -1 & -1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1
\end{array}\right)$$

$$\frac{r_{4}-r_{3}}{r_{4}-r_{3}} \left(\begin{array}{cccccc}
1 & 2 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & -1 & 2 \\
0 & 0 & 0 & 1 & 0 & 1
\end{array}\right)$$



练习1求解
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1\\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3\\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7\\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$

$$(A:b) = \begin{pmatrix} \boxed{1} & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} \boxed{1} & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & \boxed{1} & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$

$$\frac{r_{3}-2r_{2}}{r_{4}-2r_{2}} \left(\begin{array}{ccccccc}
1 & 2 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & -1 & -1 & 1 \\
0 & 0 & 0 & 6 & 0 & 6 \\
0 & 0 & 0 & 7 & 0 & 7
\end{array}\right) \frac{\frac{1}{6} \times r_{3}}{\frac{1}{7} \times r_{4}} \left(\begin{array}{ccccccc}
1 & 2 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & -1 & -1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1
\end{array}\right)$$

$$\frac{r_{4}-r_{3}}{r_{4}-r_{3}} \left(\begin{array}{ccccccc}
1 & 2 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & -1 & -1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1
\end{array}\right)$$

$$\frac{r_{4}-r_{3}}{r_{4}-r_{3}} \left(\begin{array}{cccccc}
1 & 2 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & -1 & 2 \\
0 & 0 & 0 & 1 & 0 & 1
\end{array}\right)$$



练习 1 求解
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1\\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3\\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7\\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$
解

$$\frac{r_{4}-r_{2}}{r_{4}-2r_{2}} \begin{pmatrix}
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 6 & 0 & 6 & 0 & \frac{1}{7} \\
0 & 0 & 0 & 7 & 0 & 7
\end{pmatrix}
\frac{1}{7} \times r_{4} \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & -1 & -1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$\frac{r_{4}-r_{3}}{r_{4}-r_{3}} \begin{pmatrix}
1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & -1 & -1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

$$\frac{r_{1}-r_{2}}{r_{1}-r_{2}} \begin{pmatrix}
1 & 2 & 0 & 0 & 2 & | -2 & 2 \\
0 & 0 & 1 & 0 & -1 & | 2 & 2 \\
0 & 0 & 0 & 1 & 0 & | 1 & 0 & |
\end{pmatrix}$$



练习 1 求解
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1\\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3\\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7\\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$

$$\tag{A:b)} = \begin{pmatrix} \boxed{1} & 2 & 1 & 1 & 1 & 1\\ 2 & 4 & 3 & 1 & 1 & 3\\ -1 & -2 & 1 & 3 & -3 & 7\\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1\\ 0 & 0 & \boxed{1} & -1 & -1 & 1\\ 0 & 0 & 2 & 4 & -2 & 8\\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$

$$\frac{r_{3}-2r_{2}}{r_{4}-2r_{2}} \left(\begin{array}{cccccc} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & \boxed{6} & 0 & 6 \\ 0 & 0 & 0 & 7 & 0 & 7 \end{array} \right) \frac{\frac{1}{6} \times r_{3}}{\frac{1}{7} \times r_{4}} \left(\begin{array}{ccccccc} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array} \right) \\
\xrightarrow{r_{4}-r_{3}} \left(\begin{array}{ccccccc} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & 0 & \boxed{1} & 0 & 1 \end{array} \right) \xrightarrow{r_{2}+r_{3}} \left(\begin{array}{cccccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & \boxed{1} & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

$$\xrightarrow{r_{1}-r_{2}} \left(\begin{array}{ccccccc} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array} \right)$$



$$(A:b) \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$



$$(A:b) \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$



$$(A:b) \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 0 & 2 & | & -2 \\ 0 & 0 & 1 & 0 & -1 & | & 2 \\ 0 & 0 & 0 & 1 & 0 & | & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

● 主元: X₁, X₃, X₄; 自由变量: X₂, X₅。



$$(A:b) \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 0 & 2 & | & -2 \\ 0 & 0 & 1 & 0 & -1 & | & 2 \\ 0 & 0 & 0 & 1 & 0 & | & 1 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$$

• 主元: x_1, x_3, x_4 ; 自由变量: x_2, x_5 。 (r(A) = r(A : b) = 3 < 5,无穷多解)

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- 主元: x_1, x_3, x_4 ; 自由变量: x_2, x_5 。 (r(A) = r(A : b) = 3 < 5, 无穷多解)
- 由既约阶梯形矩阵可知,原方程组等价于

$$\begin{cases} x_1 + 2x_2 & +2x_5 = -2 \\ x_3 & -x_5 = 2 \\ x_4 & = 1 \end{cases}$$

$$(A:b) \longrightarrow \begin{pmatrix} \boxed{1} & 2 & 0 & 0 & 2 & | & -2 \\ 0 & 0 & \boxed{1} & 0 & -1 & | & 2 \\ 0 & 0 & 0 & \boxed{1} & 0 & | & 1 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$$

- 主元: x_1, x_3, x_4 ; 自由变量: x_2, x_5 。 (r(A) = r(A : b) = 3 < 5, 无穷多解)
- 由既约阶梯形矩阵可知,原方程组等价于

$$\begin{cases} x_1 + 2x_2 & +2x_5 = -2 \\ x_3 & -x_5 = 2 \\ x_4 & = 1 \end{cases} \Rightarrow \begin{cases} x_1 = -2 - 2x_2 - 2x_5 \\ x_3 = 2 + x_5 \\ x_4 = 1 \end{cases}$$

$$(A:b) \longrightarrow \begin{pmatrix} \boxed{1} & 2 & 0 & 0 & 2 & | & -2 \\ 0 & 0 & \boxed{1} & 0 & -1 & | & 2 \\ 0 & 0 & 0 & \boxed{1} & 0 & | & 1 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$$

- 主元: x_1, x_3, x_4 ; 自由变量: x_2, x_5 。 (r(A) = r(A : b) = 3 < 5, 无穷多解)
- 由既约阶梯形矩阵可知,原方程组等价于

$$\begin{cases} x_1 + 2x_2 & +2x_5 = -2 \\ x_3 & -x_5 = 2 \\ x_4 & = 1 \end{cases} \Rightarrow \begin{cases} x_1 = -2 - 2x_2 - 2x_5 \\ x_3 = 2 + x_5 \\ x_4 = 1 \end{cases}$$

所以通解是
$$\begin{cases} x_1 = \\ x_2 = c_1 \\ x_3 = \\ x_4 = \\ x_5 = c_2 \end{cases}$$
 $(C_1, C_2$ 为任意常数)



$$(A:b) \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 0 & 2 & | & -2 \\ 0 & 0 & 1 & 0 & -1 & | & 2 \\ 0 & 0 & 0 & 1 & 0 & | & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$$

- 主元: x_1, x_3, x_4 ; 自由变量: x_2, x_5 。 (r(A) = r(A : b) = 3 < 5, 无穷多解)
- 由既约阶梯形矩阵可知,原方程组等价于

$$\begin{cases} x_1 + 2x_2 & +2x_5 = -2 \\ x_3 & -x_5 = 2 \\ x_4 & = 1 \end{cases} \Rightarrow \begin{cases} x_1 = -2 - 2x_2 - 2x_5 \\ x_3 = 2 + x_5 \\ x_4 = 1 \end{cases}$$

所以通解是
$$\begin{cases} x_1 = -2 - 2c_1 - 2c_2 \\ x_2 = c_1 \\ x_3 = \\ x_4 = \\ x_5 = c_2 \end{cases}$$
 $(c_1, c_2$ 为任意常数)



$$(A:b) \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 0 & 2 & | & -2 \\ 0 & 0 & 1 & 0 & -1 & | & 2 \\ 0 & 0 & 0 & 1 & 0 & | & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$$

- 主元: x_1, x_3, x_4 ; 自由变量: x_2, x_5 。 (r(A) = r(A : b) = 3 < 5, 无穷多解)
- 由既约阶梯形矩阵可知,原方程组等价于

$$\begin{cases} x_1 + 2x_2 & +2x_5 = -2 \\ x_3 & -x_5 = 2 \\ x_4 & = 1 \end{cases} \Rightarrow \begin{cases} x_1 = -2 - 2x_2 - 2x_5 \\ x_3 = 2 + x_5 \\ x_4 = 1 \end{cases}$$

所以通解是
$$\begin{cases} x_1 = -2 - 2c_1 - 2c_2 \\ x_2 = c_1 \\ x_3 = 2 + c_2 \\ x_4 = \\ x_5 = c_2 \end{cases}$$
 (c_1 , c_2 为任意常数)

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- 主元: x_1, x_3, x_4 ; 自由变量: x_2, x_5 。 (r(A) = r(A : b) = 3 < 5, 无穷多解)
- 由既约阶梯形矩阵可知,原方程组等价于

$$\begin{cases} x_1 + 2x_2 & +2x_5 = -2 \\ x_3 & -x_5 = 2 \\ x_4 & = 1 \end{cases} \Rightarrow \begin{cases} x_1 = -2 - 2x_2 - 2x_5 \\ x_3 = 2 + x_5 \\ x_4 = 1 \end{cases}$$

所以通解是
$$\begin{cases} x_1 = -2 - 2c_1 - 2c_2 \\ x_2 = c_1 \\ x_3 = 2 + c_2 \\ x_4 = 1 \end{cases}$$
 $(c_1, c_2$ 为任意常数)

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 2x_4 = 1 \\ -x_2 + (a-3)x_3 - 2x_4 = b \\ 3x_1 + 2x_2 + x_3 + ax_4 = -1 \end{cases}$$
 有无穷解、唯一解,及无解?

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 2x_4 = 1 \\ -x_2 + (a-3)x_3 - 2x_4 = b \\ 3x_1 + 2x_2 + x_3 + ax_4 = -1 \end{cases}$$
有无穷解、唯一解,及无解?

$$(A \vdots b) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & a - 2 & 2 & 1 \\ 0 & -1 & a - 3 & -2 & b \\ 3 & 2 & 1 & a - 1 \end{pmatrix}$$

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 2x_4 = 1 \\ -x_2 + (a-3)x_3 - 2x_4 = b \\ 3x_1 + 2x_2 + x_3 + ax_4 = -1 \end{cases}$$
 有无穷解、唯一解,及无解?

$$(A : b) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & a - 3 & -2 & b \\ 0 & -1 & a & -1 \end{pmatrix} \xrightarrow{r_4 - 3r_1}$$

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 2x_4 = 1 \\ -x_2 + (a-3)x_3 - 2x_4 = b \\ 3x_1 + 2x_2 + x_3 + ax_4 = -1 \end{cases}$$
 有无穷解、唯一解,及无解?

$$(A : b) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & a - \frac{2}{3} & -\frac{2}{3} & b \\ 0 & -1 & a - \frac{2}{3} & -\frac{2}{3} & b \end{pmatrix} \xrightarrow{r_4 - 3r_1} \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & -1 & a - \frac{3}{3} & -\frac{2}{3} & b \\ 0 & -1 & -2 & a - 3 & -1 \end{pmatrix}$$



$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 2x_4 = 1 \\ -x_2 + (a-3)x_3 - 2x_4 = b \\ 3x_1 + 2x_2 + x_3 + ax_4 = -1 \end{cases}$$
 有无穷解、唯一解,及无解?

$$(A \vdots b) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & a - \frac{2}{3} & -\frac{2}{3} & | & 1 \\ 0 & -1 & a & | & -1 \end{pmatrix} \xrightarrow{r_4 - 3r_1} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & | & 1 \\ 0 & -1 & a - \frac{3}{3} & -\frac{2}{3} & | & 1 \\ 0 & -1 & -2 & a - 3 & | & -1 \end{pmatrix}$$

$$\frac{r_3+r_2}{r_3+r_4}$$



$$(A \vdots b) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ 0 & -1 & a & -2 & b & 0 \\ 3 & 2 & a & 1 & a & -1 \end{pmatrix} \xrightarrow{r_4 - 3r_1} \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 2 & 1 \\ 0 & -1 & a & -3 & -2 & b \\ 0 & -1 & a & -3 & -2 & a & -3 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

$$\frac{r_3+r_2}{r_4+r_2} \leftarrow \begin{pmatrix} \frac{1}{0} & \frac{1}{1} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & a-1 & 0 \\ 0 & 0 & a-1 \end{pmatrix} b + \frac{1}{0}$$

例 2 讨论 α , b 取何值时, 方程组

$$+ x_2 + x_3 + x_4 =$$

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 2x_4 = 1 \\ -x_2 + (a-3)x_3 - 2x_4 = b \\ 3x_1 + 2x_2 + x_3 + ax_4 = -1 \end{cases}$$
 有无穷解、唯一解,及无解?

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & a - 1 & 0 & b + 1 \\ 0 & 0 & a - 1 & b + 1 \end{pmatrix}$$

$$\begin{cases} x_1 + & x_2 + & x_3 + x_4 = 0 \\ & x_2 + & 2x_3 + 2x_4 = 1 \\ & -x_2 + & (\alpha - 3)x_3 - 2x_4 = b \\ 3x_1 + & 2x_2 + & x_3 + \alpha x_4 = -1 \end{cases}$$
 $A = 0$

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & a - 1 & 0 & a - 1 \\ 0 & 0 & a - 1 & b + 1 \\ 0 & 0 & a - 1 & 0 \end{pmatrix}$$

- 当 a ≠ 1 时
- 当 a = 1 时

$$\begin{cases} x_1 + & x_2 + & x_3 + x_4 = 0 \\ & x_2 + & 2x_2 + 2x_4 = 0 \end{cases}$$

$$\begin{cases} x_1 + & x_2 + & x_3 + x_4 = 0 \\ & x_2 + & 2x_3 + 2x_4 = 1 \\ & -x_2 + & (\alpha - 3)x_3 - 2x_4 = b \\ 3x_1 + & 2x_2 + & x_3 + \alpha x_4 = -1 \end{cases}$$
 有无穷解、唯一解,及无解?

$$(A:b) \longrightarrow \begin{pmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & \alpha - 1 & 0 & b + 1 \\ 0 & 0 & 0 & \alpha - 1 & 0 \end{pmatrix}$$

- 当 α ≠ 1 时
- 当 a = 1 时

 Θ 2 讨论 α , b 取何值时,方程组

$$(A:b) \longrightarrow \begin{pmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & a-1 & 0 & b+1 \\ 0 & 0 & 0 & a-1 & 0 \end{pmatrix}$$

- 当 α ≠ 1 时(b 为任意数), r(A) = r(A:b) = 4,
- 当 a = 1 时

$$\begin{cases} x_1 + & x_2 + & x_3 + x_4 = 0 \\ & x_2 + & 2x_3 + 2x_4 = 1 \\ & -x_2 + & (\alpha - 3)x_3 - 2x_4 = b \\ 3x_1 + & 2x_2 + & x_3 + ax_4 = -1 \end{cases}$$
 5π

$$(A:b) \longrightarrow \begin{pmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 2 & 1 \\ 0 & 0 & \alpha - 1 & 0 & b + 1 \\ 0 & 0 & 0 & \alpha - 1 & 0 \end{pmatrix}$$

- 当 $\alpha \neq 1$ 时 (b 为任意数), r(A) = r(A : b) = 4, 有唯一解;
- 当 a = 1 时

 Θ 2 讨论 α , b 取何值时,方程组

$$\begin{cases} x_1 + & x_2 + & x_3 + x_4 = 0 \\ & x_2 + & 2x_3 + 2x_4 = 1 \\ & -x_2 + & (\alpha - 3)x_3 - 2x_4 = b \\ 3x_1 + & 2x_2 + & x_3 + ax_4 = -1 \end{cases}$$
 5π

$$(A:b) \longrightarrow \begin{pmatrix} \frac{1}{0} & \frac{1}{1} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & a - \frac{1}{0} & a - 1 & 0 \\ 0 & 0 & a - 1 & 0 \end{pmatrix} b + \frac{1}{0}$$

- 当 α ≠ 1 时(b 为任意数), r(A) = r(A : b) = 4, 有唯一解;
- 当 a = 1 时

<mark>例 2</mark> 讨论 α, b 取何值时,方程组 (

$$\begin{cases} x_1 + & x_2 + & x_3 + x_4 = 0 \\ & x_2 + & 2x_3 + 2x_4 = 1 \\ & -x_2 + & (a-3)x_3 - 2x_4 = b \\ 3x_1 + & 2x_2 + & x_3 + ax_4 = -1 \end{cases}$$
 有无穷解、唯一解,及无解?

- 当 $\alpha \neq 1$ 时(b为任意数), $r(A) = r(A \cdot b) = 4$, 有唯一解;
- 当 a = 1 时

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & b+1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$x_4 = 0$$

$$2x_4 = 1$$

 $(A:b) \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & a-1 & 0 & b+1 \\ 0 & 0 & a-1 & 0 & a \end{pmatrix}$

 $\begin{cases} x_1 + & x_2 + & x_3 + x_4 = 0 \\ & x_2 + & 2x_3 + 2x_4 = 1 \\ & -x_2 + & (\alpha - 3)x_3 - 2x_4 = b \\ 3x_1 + & 2x_2 + & x_3 + \alpha x_4 = -1 \end{cases}$ 有无穷解、唯一解,及无解?

解

• 当
$$a \neq 1$$
 时 (b 为任意数), $r(A) = r(A : b) = 4$, 有唯一解;
• 当 $a = 1$ 时
$$(A : b) \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & b + 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

-a=1, b=-1 时

-
$$a = 1, b \neq -1$$
 时

 $\begin{cases} x_1 + & x_2 + & x_3 + x_4 = 0 \\ & x_2 + & 2x_3 + 2x_4 = 1 \\ & -x_2 + & (\alpha - 3)x_3 - 2x_4 = b \\ 3x_1 + & 2x_2 + & x_3 + ax_4 = -1 \end{cases}$ $f(x_1 + x_2)$ $f(x_2 + x_3)$ $f(x_3 + x_4)$ $f(x_4 + x_4)$ $f(x_$

 $(A:b) \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & a-1 & 0 & b+1 \\ 0 & 0 & a-1 & 0 & a \end{pmatrix}$

解

• 当
$$a \neq 1$$
 时 (b 为任意数), $r(A) = r(A : b) = 4$, 有唯一解;
• 当 $a = 1$ 时 ($A : b$) — \rightarrow $\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & b+1 \end{pmatrix}$

-a=1, b=-1 时

-
$$a = 1, b \neq -1$$
 时



$$x_4 + x_4 = 0$$

 $(A:b) \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & a & 1 & 1 & 1 \\ 0 & 0 & a & 1 & 0 & b+1 \end{pmatrix}$

$$\begin{cases} x_1 + & x_2 + & x_3 + x_4 = 0 \\ & x_2 + & 2x_3 + 2x_4 = 1 \\ & -x_2 + & (\alpha - 3)x_3 - 2x_4 = b \\ 3x_1 + & 2x_2 + & x_3 + ax_4 = -1 \end{cases}$$
 $f(x_1 + x_2)$ $f(x_2 + x_3)$ $f(x_3 + x_4)$ $f(x_4 + x_4)$ $f(x_$

• 当
$$\alpha \neq 1$$
时(b 为任意数), $r(A) = r(A : b) = 4$,有唯一解;
• 当 $\alpha = 1$ 时

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = 0$$

•
$$\stackrel{\text{def}}{=} \alpha = 1 \text{ FT}$$

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & b+1 \end{pmatrix}$$

-
$$a = 1$$
, $b = -1$ 时, $r(A) = r(A : b) = 2 < 4$,

-
$$a = 1, b \neq -1$$
 时



$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 2x_4 = 1 \\ -x_2 + (\alpha - 3)x_3 - 2x_4 = b \end{cases}$$
 $father a father a fath$

- 当 a = 1 时

$$(A:b) \longrightarrow \begin{pmatrix} \begin{vmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & b+1 \\ 0 & 0 & 0 & 0 & b+1 \end{pmatrix}$$

- -a = 1, b = -1 时, r(A) = r(A : b) = 2 < 4, 有无穷多解
- $a = 1, b \neq -1$ 时



$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 2x_4 = 1 \\ -x_2 + (\alpha - 3)x_3 - 2x_4 = b \end{cases}$$
 $father a father a fath$

- 当 a = 1 时

- -a = 1, b = -1 时, r(A) = r(A : b) = 2 < 4, 有无穷多解
- $a = 1, b \neq -1$ 时



$$\begin{cases} x_1 + & x_2 + & x_3 + x_4 = 0 \\ & x_2 + & 2x_3 + 2x_4 = 1 \\ & -x_2 + & (a-3)x_3 - 2x_4 = b \\ 3x_1 + & 2x_2 + & x_3 + ax_4 = -1 \end{cases}$$
 有无穷解、唯一解,及无解?

$$(A : b) \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & a - 1 & 0 & b + 1 \\ 0 & 0 & 0 & a - 1 & 0 \end{pmatrix}$$

- 当 α ≠ 1 时(b 为任意数), r(A) = r(A : b) = 4, 有唯一解;
- 当 a = 1 时

$$(A : b) \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & b+1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- -a = 1, b = -1 时, r(A) = r(A : b) = 2 < 4, 有无穷多解
- $a = 1, b \neq -1$ 时



例 2 讨论 a, b 取何值时,方程组 $\begin{cases} x_1 + & x_2 + & x_3 + x_4 = 0 \\ & x_2 + & 2x_3 + 2x_4 = 1 \\ & -x_2 + & (\alpha - 3)x_3 - 2x_4 = b \\ 3x_1 + & 2x_2 + & x_3 + \alpha x_4 = -1 \end{cases}$ 有无穷解、唯一解,及无解?

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 2 & 2 \\ 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & a - 1 & 0 \\ 0 & 0 & a - 1 & b + 1 \end{pmatrix}$$

$$\bullet \, \, \exists \, a \neq 1 \, \text{时} \, (b \, \text{为任意数}), \, r(A) = r(A:b) = 4, \, \, \text{有唯一解};$$

- 当 a = 1 时

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & b+1 \\ 0 & 0 & 0 & 0 & b \end{pmatrix}$$

- $-\alpha = 1, b = -1$ 时, r(A) = r(A : b) = 2 < 4, 有无穷多解
- a = 1, $b \neq -1$ 时, r(A) = 2 < 3 = r(A : b),



例 2 讨论 a, b 取何值时,方程组 $\begin{cases} x_1 + & x_2 + & x_3 + x_4 = 0 \\ & x_2 + & 2x_3 + 2x_4 = 1 \\ & -x_2 + & (\alpha - 3)x_3 - 2x_4 = b \\ 3x_1 + & 2x_2 + & x_3 + \alpha x_4 = -1 \end{cases}$ 有无穷解、唯一解,及无解?

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 2 & 2 \\ 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & a - 1 & 0 \\ 0 & 0 & a - 1 & b + 1 \end{pmatrix} b + 1$$
• 当 $a \ne 1$ 时(b 为任意数), $r(A) = r(A:b) = 4$,有唯一解;

- 当 a = 1 时

•
$$\exists a = 1 \text{ F}$$

$$(A : b) \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & b+1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- -a = 1, b = -1 时, r(A) = r(A : b) = 2 < 4, 有无穷多解
- $\alpha = 1$, $b \neq -1$ 时, r(A) = 2 < 3 = r(A : b), 无解



$$(A:b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \frac{1}{b}$$



例 3 讨论 a, b 取何值时,方程组 $\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \end{cases}$ 有无 $3x_1 + 4x_2 + ax_3 = b$

$$(A:b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \frac{1}{b} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 - 3r_1}$$



$$(A \vdots b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \begin{pmatrix} 1 \\ b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 \end{pmatrix}$$



$$(A:b) = \begin{pmatrix} 1 & 2 & 3 & -1 \\ 2 & 3 & 5 & -1 \\ 3 & 4 & a & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & -2 & -3 & -1 \\ 0 & -1 & -3 & -3 \end{pmatrix}$$



$$(A:b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \begin{pmatrix} 1 \\ b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \xrightarrow{r_3 - 3r_1} \begin{pmatrix} 1 \\ 0 \\ -2 \\ a - 9 \end{pmatrix} \begin{pmatrix} 3 \\ b - 3 \end{pmatrix}$$



$$(A:b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 \\ 0 \\ -1 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 3r_1} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \begin{pmatrix} -1 \\ a - 9 \\ b - 3 \end{pmatrix}$$



例 3 讨论
$$a$$
, b 取何值时,方程组
$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \\ 3x_1 + 4x_2 + ax_3 = b \end{cases}$$

$$(A:b) = \begin{pmatrix} \frac{1}{2} & \frac{2}{3} & \frac{3}{5} & -\frac{1}{1} \\ \frac{2}{3} & \frac{3}{4} & a & -\frac{1}{1} \\ \frac{7}{3} - \frac{3}{3}r_{1} & \frac{1}{3} & -\frac{2}{3} & -\frac{3}{1} \\ \frac{r_{3} - 2r_{2}}{0} & \frac{1}{0} & -\frac{2}{1} & -\frac{3}{1} \\ 0 & 0 & a - 7 & b + 3 \end{pmatrix}$$

例 3 讨论
$$a$$
, b 取何值时,方程组
$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \\ 3x_1 + 4x_2 + ax_3 = b \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \begin{pmatrix} 1 \\ b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \xrightarrow{-1} \begin{pmatrix} 3 \\ 0 - 2 \end{pmatrix} \begin{pmatrix} -1 \\ 0 - 3 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \xrightarrow{-1} \begin{pmatrix} 2 \\ 0 \\ 0 - 7 \end{pmatrix} \begin{pmatrix} 3 \\ b + 3 \end{pmatrix}$$

- 当 a ≠ 7 时
- 当 a = 7 时

例 3 讨论
$$a$$
, b 取何值时,方程组
$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \end{cases}$$
 有无 $3x_1 + 4x_2 + ax_3 = b$

$$(A : b) = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 3 & 5 & -1 \\ 3 & 4 & a & b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & -2 & -3 & -3 \\ 0 & -2 & a - 9 & b - 3 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & a - 7 & b + 3 \end{pmatrix}$$

- 当 a ≠ 7 时
- 当 α = 7 时

例 3 讨论
$$a$$
, b 取何值时,方程组
$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \\ 3x_1 + 4x_2 + ax_3 = b \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 3 & 5 & -1 \\ 2 & 3 & 5 & -1 & b & \frac{r_2 - 2r_1}{r_3 - 3r_1} & \begin{pmatrix} 1 & -2 & -3 & -3 \\ 0 & -2 & a - 9 & b - 3 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & a - 7 & b + 3 \end{pmatrix}$$

- 当 α ≠ 7 时 (b 为任意数), r(A · b) = r(A) = 3,
- 当 a = 7 时

例 3 讨论
$$a$$
, b 取何值时,方程组
$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \\ 3x_1 + 4x_2 + ax_3 = b \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \begin{pmatrix} 1 \\ b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \xrightarrow{r_3 - 3r_1} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{a - 9} \begin{pmatrix} -3 \\ b - 3 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \xrightarrow{a - 7} \begin{pmatrix} 1 \\ b + 3 \end{pmatrix}$$

- 当 $\alpha \neq 7$ 时 (b 为任意数), r(A : b) = r(A) = 3, 有唯一解;
- 当 α = 7 时

例 3 讨论
$$a$$
, b 取何值时,方程组
$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \\ 3x_1 + 4x_2 + ax_3 = b \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 3 & 5 & -1 \\ 3 & 4 & a & b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & -1 \\ 0 & -2 & a - 9 & b - 3 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 & 2 & 3 & -1 \\ 0 & 0 & a - 7 & b + 3 \end{pmatrix}$$

- 当 $\alpha \neq 7$ 时 (b 为任意数), r(A : b) = r(A) = 3, 有唯一解;
- 当 α = 7 时



例 3 讨论
$$a$$
, b 取何值时,方程组
$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \\ 3x_1 + 4x_2 + ax_3 = b \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 3 & 5 & -1 \\ 3 & 4 & a & b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & -1 & -1 \\ 0 & -1 & -1 & -3 \\ 0 & -2 & a - 9 & b - 3 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 & 2 & 3 & -1 \\ 0 & 0 & a - 7 & b + 3 \end{pmatrix}$$

- 当 $\alpha \neq 7$ 时 (b 为任意数), r(A : b) = r(A) = 3, 有唯一解;
- $\stackrel{\text{def}}{=} a = 7 \text{ pl}$ $(A:b) \longrightarrow \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & b+3 \end{pmatrix}$



 $(A:b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & 6 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 \\ 0 \\ -1 \\ -2 \\ -2 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \\ -3 \end{pmatrix}$

$$\frac{r_{3}-2r_{2}}{0} \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & a-7 & b+3 \end{pmatrix}$$
• 当 $a \neq 7$ 时 $(b$ 为任意数), $r(A:b) = r(A) = 3$, 有唯一解;

- $\exists a = 7 \text{ fi}$ $(A : b) \longrightarrow \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
- (71.5)
 - a = 7, b = -3 时
 - $a = 7, b \neq -3$ 时



例 3 讨论 a, b 取何值时,方程组 $\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \end{cases}$ 有无 $3x_1 + 4x_2 + ax_3 = b$

 $(A:b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & 6 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 \\ 0 \\ -1 \\ -2 \\ -2 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \\ -3 \end{pmatrix}$

$$\frac{r_3-2r_2}{0} \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & a-7 & b+3 \end{pmatrix}$$
• 当 $a \neq 7$ 时(b 为任意数), $r(A:b) = r(A) = 3$,有唯一解;

- 当 α = 7 时
- $(A:b) \longrightarrow \begin{pmatrix} 1 & 2 & 3 & -\frac{1}{3} \\ 0 & -\frac{1}{3} & -\frac{1}{3} \end{pmatrix}$
 - -a=7. b=-3 时
 - a = 7, $b \neq -3$ 时



$$(A:b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 \\ 0 \\ -1 \\ r_3 - 3r_1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -2 \\ a - 9 \end{pmatrix} \begin{pmatrix} 3 \\ b - 3 \end{pmatrix} \\
 \xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ b + 3 \end{pmatrix}$$

- 当 $\alpha \neq 7$ 时(b为任意数), r(A : b) = r(A) = 3, 有唯一解;
- - a = 7, b = -3 时, r(A : b) = r(A) = 2 < 3,
 - $a = 7, b \neq -3$ 时



- 当 $\alpha \neq 7$ 时 (b 为任意数), r(A : b) = r(A) = 3, 有唯一解;
- - -a = 7, b = -3 时, r(A : b) = r(A) = 2 < 3, 有无穷多解
 - a = 7, $b \neq -3$ 时



- 当 $\alpha \neq 7$ 时(b为任意数), r(A : b) = r(A) = 3, 有唯一解;
- $\stackrel{\text{def}}{=} a = 7 \text{ pri}$ $(A:b) \longrightarrow \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & b+3 \end{pmatrix}$
 - -a = 7, b = -3 时, r(A : b) = r(A) = 2 < 3, 有无穷多解
 - $a = 7, b \neq -3$ 时



$$(A:b) = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 3 & 5 & -1 \\ 3 & 4 & a & -1 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & -2 & -3 & -3 \\ 0 & -2 & a - 9 & b - 3 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 & -2 & -3 & -3 \\ 0 & 0 & a - 7 & b + 3 \end{pmatrix}$$

- 当 $\alpha \neq 7$ 时 (b 为任意数), r(A : b) = r(A) = 3, 有唯一解;
- - -a = 7, b = -3 时, r(A : b) = r(A) = 2 < 3, 有无穷多解
 - $a = 7, b \neq -3$ 时



例 3 讨论 a, b 取何值时,方程组 $\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \end{cases}$ 有无 $3x_1 + 4x_2 + ax_3 = b$ 穷解、唯一解,及无解?

$$(A:b) = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 3 & 5 & -1 \\ 3 & 4 & a & -1 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & -2 & -3 & -3 \\ 0 & -2 & a - 9 & b - 3 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 & 2 & 3 & -1 \\ 0 & 0 & a - 7 & b + 3 \end{pmatrix}$$

- 当 $\alpha \neq 7$ 时(b为任意数), r(A : b) = r(A) = 3, 有唯一解;
- - $-\alpha = 7, b = -3$ 时, r(A : b) = r(A) = 2 < 3, 有无穷多解
 - a = 7, $b \neq -3$ 时, $r(A : b) = 3 \neq 2 = r(A)$,



$$\mathbf{(A:b)} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \frac{1}{b} \begin{pmatrix} \frac{r_2 - 2r_1}{r_3 - 3r_1} \end{pmatrix} \begin{pmatrix} 1 & -\frac{2}{a} & -\frac{3}{b} \\ 0 & -\frac{2}{a} & a - 9 \end{pmatrix} \begin{pmatrix} \frac{1}{b} - \frac{3}{3} \\ 0 & -\frac{3}{a} \end{pmatrix} \\
\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & a - 7 \end{pmatrix} \begin{pmatrix} \frac{1}{b} + \frac{3}{3} \end{pmatrix}$$

- 当 $\alpha \neq 7$ 时(b为任意数), r(A:b) = r(A) = 3, 有唯一解;
- - $-\alpha = 7, b = -3$ 时, r(A : b) = r(A) = 2 < 3, 有无穷多解
 - a = 7, $b \neq -3$ 时, $r(A : b) = 3 \neq 2 = r(A)$, 无解



• 一般线性方程组 $A_{m \times n} x = b$ (m 个方程, n 个未知量)

Ax = b	有无穷解	有唯一解	无解
	r(A) = r(A : b) < n	r(A) = r(A : b) = n	r(A) < r(A : b)

• 一般线性方程组 $A_{m \times n} x = b$ (m 个方程, n 个未知量)

Ax = b	有无穷解	有唯一解	无解
	r(A) = r(A : b) < n	r(A) = r(A : b) = n	r(A) < r(A : b)

• 齐次线性方程组 $A_{m\times n}x=0$,一定有解(至少有零解), $r(A)=r(A \stackrel{!}{\cdot} 0)$

• 一般线性方程组 $A_{m \times n} x = b$ (m 个方程, n 个未知量)

Ax = b	有无穷解	有唯一解	无解
	r(A) = r(A : b) < n	r(A) = r(A : b) = n	r(A) < r(A : b)

• 齐次线性方程组 $A_{m\times n}x = 0$,一定有解(至少有零解), $r(A) = r(A \stackrel{!}{\cdot} 0)$

Ax = 0	有无穷解	有唯一解(零解)

• 一般线性方程组 $A_{m \times n} x = b$ (m 个方程, n 个未知量)

Ax = b	有无穷解	有唯一解	无解
	r(A) = r(A : b) < n	r(A) = r(A : b) = n	r(A) < r(A : b)

• 齐次线性方程组 $A_{m\times n}x = 0$,一定有解(至少有零解), $r(A) = r(A \stackrel{!}{\cdot} 0)$

Ax = 0	有无穷解	有唯一解(零解)
	r(A) < n	

• 一般线性方程组 $A_{m \times n} x = b$ (m 个方程, n 个未知量)

Ax = b	有无穷解	有唯一解	无解
	r(A) = r(A : b) < n	r(A) = r(A : b) = n	r(A) < r(A : b)

• 齐次线性方程组 $A_{m\times n}x = 0$,一定有解(至少有零解), $r(A) = r(A \stackrel{!}{\cdot} 0)$

Ax = 0	有无穷解	有唯一解(零解)
	r(A) < n	r(A) = n

例解齐次线性方程组
$$\begin{cases} x_1-&x_2+&5x_3-&x_4&=0\\ x_1+&x_2-&2x_3+&3x_4&=0\\ 3x_1-&x_2+&8x_3+&x_4&=0\\ x_1+&3x_2-&9x_3+&7x_4&=0 \end{cases}$$

例解齐次线性方程组 $\begin{cases} x_1- & x_2+ 5x_3- & x_4=0\\ x_1+ & x_2- 2x_3+ 3x_4=0\\ 3x_1- & x_2+ 8x_3+ x_4=0\\ x_1+ 3x_2- 9x_3+ 7x_4=0 \end{cases}$

$$(A \vdots b) = \begin{pmatrix} 1 & -1 & 5 & -1 & 0 \\ 1 & 1 & -2 & 3 & 0 \\ 3 & -1 & 8 & 1 & 0 \\ 1 & 3 & -9 & 7 & 0 \end{pmatrix}$$

例解齐次线性方程组 $\begin{cases} x_1- & x_2+ 5x_3- & x_4=0\\ x_1+ & x_2- 2x_3+ 3x_4=0\\ 3x_1- & x_2+ 8x_3+ x_4=0\\ x_1+ 3x_2- 9x_3+ 7x_4=0 \end{cases}$

$$(A \vdots b) = \begin{pmatrix} 1 & -1 & 5 & -1 & 0 \\ 1 & 1 & -2 & 3 & 0 \\ 3 & -1 & 8 & 1 & 0 \\ 1 & 3 & -9 & 7 & 0 \end{pmatrix}$$



$$(A:b) = \begin{pmatrix} 1 & -1 & 5 & -1 & 0 \\ 1 & 1 & -2 & 3 & 0 \\ 3 & -1 & 8 & 1 & 0 \\ 1 & 3 & -9 & 7 & 0 \end{pmatrix} \xrightarrow[r_4-r_1]{r_2-r_1} \xrightarrow[r_4-r_1]{r_3-3r_1}$$

例解齐次线性方程组
$$\begin{cases} x_1- & x_2+ 5x_3- & x_4=0\\ x_1+ & x_2- 2x_3+ 3x_4=0\\ 3x_1- & x_2+ 8x_3+ x_4=0\\ x_1+ 3x_2- 9x_3+ 7x_4=0 \end{cases}$$

$$(A \vdots b) = \begin{pmatrix} \boxed{1} & -1 & 5 & -1 & 0 \\ 1 & 1 & -2 & 3 & 0 \\ 3 & -1 & 8 & 1 & 0 \\ 1 & 3 & -9 & 7 & 0 \end{pmatrix} \xrightarrow[r_4 - r_1]{r_2 - r_1} \begin{pmatrix} 1 & -1 & 5 & -1 & 0 \\ \\ r_3 - 3r_1 & \\ r_4 - r_1 & \end{pmatrix}$$



$$(A : b) = \begin{pmatrix} \boxed{1} & -1 & 5 & -1 & 0 \\ 1 & 1 & -2 & 3 & 0 \\ 3 & -1 & 8 & 1 & 0 \\ 1 & 3 & -9 & 7 & 0 \end{pmatrix} \xrightarrow[r_4 - r_1]{r_2 - r_1} \begin{pmatrix} 1 & -1 & 5 & -1 & 0 \\ 0 & 2 & -7 & 4 & 0 \end{pmatrix}$$



$$(A:b) = \begin{pmatrix} 1 & -1 & 5 & -1 & 0 \\ 1 & 1 & -2 & 3 & 0 \\ 3 & -1 & 8 & 1 & 0 \\ 1 & 3 & -9 & 7 & 0 \end{pmatrix} \xrightarrow[r_4-r_1]{r_2-r_1} \begin{pmatrix} 1 & -1 & 5 & -1 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 2 & -7 & 4 & 0 \end{pmatrix}$$



$$(A \vdots b) = \begin{pmatrix} \boxed{1} & -1 & 5 & -1 & 0 \\ 1 & 1 & -2 & 3 & 0 \\ 3 & -1 & 8 & 1 & 0 \\ 1 & 3 & -9 & 7 & 0 \end{pmatrix} \xrightarrow[r_3 - 3r_1]{r_3 - 3r_1} \begin{pmatrix} 1 & -1 & 5 & -1 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 4 & -14 & 8 & 0 \end{pmatrix}$$



$$(A \vdots b) = \begin{pmatrix} \boxed{1} & -1 & 5 & -1 & 0 \\ 1 & 1 & -2 & 3 & 0 \\ 3 & -1 & 8 & 1 & 0 \\ 1 & 3 & -9 & 7 & 0 \end{pmatrix} \xrightarrow[r_3 - 3r_1]{r_2 - r_1} \begin{pmatrix} 1 & -1 & 5 & -1 & 0 \\ 0 & \boxed{2} & -7 & 4 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 4 & -14 & 8 & 0 \end{pmatrix}$$

例解齐次线性方程组
$$\begin{cases} x_1- & x_2+ 5x_3- & x_4=0\\ x_1+ & x_2- 2x_3+ 3x_4=0\\ 3x_1- & x_2+ 8x_3+ x_4=0\\ x_1+ 3x_2- 9x_3+ 7x_4=0 \end{cases}$$

$$(A \vdots b) = \begin{pmatrix} \boxed{1} & -1 & 5 & -1 & 0 \\ 1 & 1 & -2 & 3 & 0 \\ 3 & -1 & 8 & 1 & 0 \\ 1 & 3 & -9 & 7 & 0 \end{pmatrix} \xrightarrow[r_4 - r_1]{r_2 - r_1} \begin{pmatrix} 1 & -1 & 5 & -1 & 0 \\ 0 & \boxed{2} & -7 & 4 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 4 & -14 & 8 & 0 \end{pmatrix}$$

$$r_3 - r_2$$

例解齐次线性方程组
$$\begin{cases} x_1- & x_2+ 5x_3- & x_4=0\\ x_1+ & x_2- 2x_3+ 3x_4=0\\ 3x_1- & x_2+ 8x_3+ & x_4=0\\ x_1+ 3x_2- 9x_3+ 7x_4=0 \end{cases}$$





$$(A:b) = \begin{pmatrix} \boxed{1} & -1 & 5 & -1 & | & 0 \\ 1 & 1 & -2 & 3 & | & 0 \\ 3 & -1 & 8 & 1 & | & 0 \\ 1 & 3 & -9 & 7 & | & 0 \end{pmatrix} \xrightarrow[r_2-r_1]{r_2-r_1} \begin{pmatrix} \boxed{1} & -1 & 5 & -1 & | & 0 \\ 0 & \boxed{2} & -7 & 4 & | & 0 \\ 0 & 2 & -7 & 4 & | & 0 \\ 0 & 4 & -14 & 8 & | & 0 \end{pmatrix}$$

$$\xrightarrow[r_4-2r_2]{r_4-2r_2}
\begin{pmatrix}
1 & -1 & 5 & -1 & 0 \\
0 & 2 & -7 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$\xrightarrow{\frac{1}{2} \times r_2} \begin{pmatrix}
1 & -1 & 5 & -1 & 0 \\
0 & 1 & -7/2 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix} \xrightarrow{r_1 + r_2} \begin{pmatrix}
1 & 0 & 3/2 & 1 & 0 \\
0 & 1 & -7/2 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$(A:b) = \begin{pmatrix} \boxed{1} & -1 & 5 & -1 & | & 0 \\ 1 & 1 & -2 & 3 & | & 0 \\ 3 & -1 & 8 & 1 & | & 0 \\ 1 & 3 & -9 & 7 & | & 0 \end{pmatrix} \xrightarrow[r_2-r_1]{r_2-r_1} \begin{pmatrix} \boxed{1} & -1 & 5 & -1 & | & 0 \\ 0 & \boxed{2} & -7 & 4 & | & 0 \\ 0 & 2 & -7 & 4 & | & 0 \\ 0 & 4 & -14 & 8 & | & 0 \end{pmatrix}$$





解

主元: X_1, X_2 ; 自由变量: X_3, X_4 。

主元:
$$x_1, x_2$$
; 自由变量: x_3, x_4 。原方程组等价于
$$\begin{cases} x_1 + & \frac{3}{2}x_3 + x_4 = 0 \\ & x_2 - \frac{7}{2}x_3 + 2x_4 = 0 \end{cases}$$

解

主元: X_1, X_2 ;自由变量: X_3, X_4 。原方程组等价于

$$\begin{cases} x_1 + \frac{3}{2}x_3 + x_4 = 0 \\ x_2 - \frac{7}{2}x_3 + 2x_4 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 + \frac{3}{2}x_3 - x_4 \\ x_2 = \frac{7}{2}x_3 - 2x_4 \end{cases}$$

主元: X_1, X_2 ; 自由变量: X_3, X_4 。原方程组等价于 $\begin{cases} x_1 + \frac{3}{2}x_3 + x_4 = 0 \\ x_2 - \frac{7}{2}x_3 + 2x_4 = 0 \end{cases} \iff \begin{cases} x_1 + \frac{3}{2}x_3 - x_4 \\ x_2 = \frac{7}{2}x_3 - 2x_4 \end{cases}$

所以
$$\begin{cases} x_3 = c_1 \\ x_4 = c_2 \end{cases}$$
 (其中 c_1 , c_2 为任意常数)

主元:
$$x_1, x_2$$
; 自由变量: x_3, x_4 。原方程组等价于

主元:
$$x_1, x_2$$
; 自由变量: x_3, x_4 。原方程组等价于
$$\begin{cases} x_1 + & \frac{3}{2}x_3 + x_4 = 0 \\ & x_2 - \frac{7}{2}x_3 + 2x_4 = 0 \end{cases} \iff \begin{cases} x_1 + & = -\frac{3}{2}x_3 - x_4 \\ & x_2 = \frac{7}{2}x_3 - 2x_4 \end{cases}$$

$$X_2 - \frac{1}{2}X_3 + 2X_4 = 0$$
 $X_2 =$ $X_1 = -\frac{3}{2}C_1 - C_2$ $X_3 = C_1$ $X_4 = C_2$ $X_4 = C_2$ $X_5 = C_1$ $X_6 = C_2$ $X_6 = C_2$ $X_6 = C_2$ $X_6 = C_2$ $X_6 = C_2$

主元: X_1, X_2 ; 自由变量: X_3, X_4 。原方程组等价于

$$\begin{cases} x_{1} + & \frac{3}{2}x_{3} + x_{4} = 0 \\ x_{2} - \frac{7}{2}x_{3} + 2x_{4} = 0 \end{cases} \iff \begin{cases} x_{1} + & = -\frac{3}{2}x_{3} - x_{4} \\ x_{2} = \frac{7}{2}x_{3} - 2x_{4} \end{cases}$$

$$\begin{cases} x_{1} = -\frac{3}{2}c_{1} - c_{2} \\ x_{2} = \frac{7}{2}c_{1} - 2c_{2} \\ x_{3} = c_{1} \\ x_{4} = c_{2} \end{cases}$$
(其中 c_{1} , c_{2} 为任意常数)