# 第 11 章 f: 高斯公式、斯托克斯公式

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2016-2017 **学年** II



### Outline

1. 高斯公式

2. 斯托克斯公式

We are here now...

1. 高斯公式

2. 斯托克斯公式

定义 设 
$$F = (P, Q, R)$$
 是空间中向量场,定义

$$\mathrm{div} F := \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

称为向量场 F 的散度。

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$$\operatorname{div} F = \frac{\partial}{\partial x}(x^2 + yz) + \frac{\partial}{\partial y}(y^2 + xz) + \frac{\partial}{\partial z}(z^2 + xy) = 2x + 2y + 2z.$$



$$\nabla \frac{1}{r}$$

$$\operatorname{div} \nabla \frac{1}{r}$$

$$\nabla \frac{1}{r} = (\frac{\partial}{\partial x} r^{-1}, \frac{\partial}{\partial y} r^{-1}, \frac{\partial}{\partial y} r^{-1})$$

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$$-r^{-2} \cdot r_{x}$$
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$$\operatorname{div} \nabla \frac{1}{r}$$

例 计算梯度场 
$$\nabla \frac{1}{r}$$
  $(r = \sqrt{x^2 + y^2 + z^2})$  的散度。

$$r_{x} = \frac{x}{r},$$

$$\nabla \frac{1}{r} = (\frac{\partial}{\partial x}r^{-1}, \frac{\partial}{\partial y}r^{-1}, \frac{\partial}{\partial y}r^{-1})$$

$$= (-r^{-2} \cdot r_{x}, -r^{-2} \cdot r_{y}, -r^{-2} \cdot r_{y})$$

$$\operatorname{div} \nabla \frac{1}{r}$$

$$r_{x} = \frac{x}{r}, \qquad r_{y} = \frac{y}{r}, \qquad r_{z} = \frac{z}{r},$$

$$\nabla \frac{1}{r} = (\frac{\partial}{\partial x}r^{-1}, \frac{\partial}{\partial y}r^{-1}, \frac{\partial}{\partial y}r^{-1})$$

$$= (-r^{-2} \cdot r_{x}, -r^{-2} \cdot r_{y}, -r^{-2} \cdot r_{y})$$

$$\operatorname{div} \nabla \frac{1}{r}$$

$$\begin{aligned} r_x &= \frac{x}{r}, & r_y &= \frac{y}{r}, & r_z &= \frac{z}{r}, \\ \nabla \frac{1}{r} &= (\frac{\partial}{\partial x} r^{-1}, \frac{\partial}{\partial y} r^{-1}, \frac{\partial}{\partial y} r^{-1}) \\ &= (-r^{-2} \cdot r_x, -r^{-2} \cdot r_y, -r^{-2} \cdot r_y) = (-\frac{x}{r^3}, -\frac{y}{r^3}, -\frac{z}{r^3}), \\ \operatorname{div} \nabla \frac{1}{r} &= (\frac{1}{r^3} r^{-1}, \frac{1}{r^3} r^{-1}, \frac{1}{r^3}, -\frac{z}{r^3}), \end{aligned}$$

$$r_{x} = \frac{x}{r}, \qquad r_{y} = \frac{y}{r}, \qquad r_{z} = \frac{z}{r},$$

$$\nabla \frac{1}{r} = (\frac{\partial}{\partial x} r^{-1}, \frac{\partial}{\partial y} r^{-1}, \frac{\partial}{\partial y} r^{-1})$$

$$= (-r^{-2} \cdot r_{x}, -r^{-2} \cdot r_{y}, -r^{-2} \cdot r_{y}) = (-\frac{x}{r^{3}}, -\frac{y}{r^{3}}, -\frac{z}{r^{3}}),$$

$$\operatorname{div} \nabla \frac{1}{r} = \frac{\partial}{\partial x} (-\frac{x}{r^{3}}) + \frac{\partial}{\partial y} (-\frac{y}{r^{3}}) + \frac{\partial}{\partial z} (-\frac{z}{r^{3}})$$



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$$(-\frac{1}{r^{3}} + \frac{3x^{2}}{r^{5}})$$

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$$= (-\frac{1}{r^{3}} + \frac{3x^{2}}{r^{5}}) + (-\frac{1}{r^{3}} + \frac{3y^{2}}{r^{5}}) + (-\frac{1}{r^{3}} + \frac{3z^{2}}{r^{5}})$$



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$$= (-\frac{1}{r^{3}} + \frac{3x^{2}}{r^{5}}) + (-\frac{1}{r^{3}} + \frac{3y^{2}}{r^{5}}) + (-\frac{1}{r^{3}} + \frac{3z^{2}}{r^{5}})$$

$$= -\frac{3}{r^{3}} + \frac{3(x^{2} + y^{2} + z^{2})}{r^{5}}$$

解

$$r_{x} = \frac{x}{r}, \qquad r_{y} = \frac{y}{r}, \qquad r_{z} = \frac{z}{r},$$

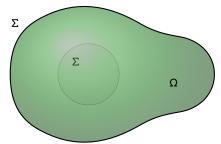
$$\nabla \frac{1}{r} = (\frac{\partial}{\partial x} r^{-1}, \frac{\partial}{\partial y} r^{-1}, \frac{\partial}{\partial y} r^{-1})$$

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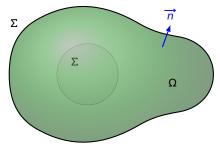
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 $= \left(-\frac{1}{r^3} + \frac{3x^2}{r^5}\right) + \left(-\frac{1}{r^3} + \frac{3y^2}{r^5}\right) + \left(-\frac{1}{r^3} + \frac{3z^2}{r^5}\right)$  $= -\frac{3}{r^3} + \frac{3(x^2 + y^2 + z^2)}{r^5} = 0.$ 

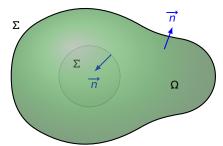
- 空间闭区域  $\Omega$  的边界是分片光滑的闭曲面  $\Sigma$ ,
- $\overrightarrow{n}$  是  $\Sigma$  的单位外法向量,



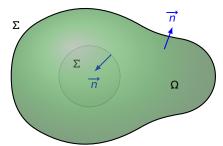
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- F = (P, Q, R) 是  $\Omega$  中向量场,且 P, Q, R 具有一阶连续的偏导数,



#### 定理(高斯公式) 假设

- 空间闭区域  $\Omega$  的边界是分片光滑的闭曲面  $\Sigma$ ,
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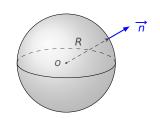
则

$$\iiint_{\Omega} \operatorname{div} F dv = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$

$$\sum_{\overrightarrow{n}} \int_{\Omega} \int_{\Omega} \operatorname{div} F dv = \int_{\Sigma} F \cdot \overrightarrow{n} dS$$

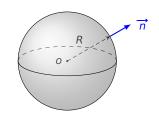
$$I = \iint_{\Sigma} 2x \, dy \, dz + y^2 \, dz \, dx + z^2 \, dx \, dy$$

其中定向曲面  $\Sigma$  是球面  $x^2 + y^2 + z^2 = R^2$ , 定向取外侧



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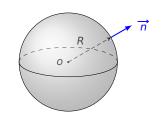


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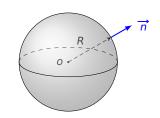
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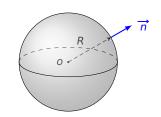


$$I = \underbrace{F = (2x, y^2, z^2)}_{\Gamma} = \iint_{\Sigma} F \cdot \overrightarrow{n} dS = \underbrace{\overline{\text{sinGR}}}_{\Omega} \iint_{\Omega} \text{div} F dv$$



$$I = \iint_{\Sigma} 2x dy dz + y^2 dz dx + z^2 dx dy$$

其中定向曲面 Σ 是球面  $x^2 + y^2 + z^2 = R^2$ , 定向取外侧

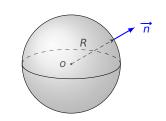


$$I = \frac{F = (2x, y^2, z^2)}{\int \int_{\Sigma} F \cdot \overrightarrow{n} \, dS} = \frac{\overline{\text{syndx}}}{\int \int \int_{\Omega} \operatorname{div} F \, dv}$$
$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} (2x) + \frac{\partial}{\partial y} (y^2) + \frac{\partial}{\partial z} (z^2) \right] dv$$



$$I = \iint_{\Sigma} 2x dy dz + y^2 dz dx + z^2 dx dy$$

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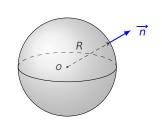


$$I = \frac{F = (2x, y^2, z^2)}{\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS} = \frac{\overrightarrow{\text{sin} \Delta x}}{\iint_{\Omega} \text{div} F \, dv}$$
$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} (2x) + \frac{\partial}{\partial y} (y^2) + \frac{\partial}{\partial z} (z^2) \right] dv = \iiint_{\Omega} (2 + y + z) \, dx \, dy \, dz$$



$$I = \iint_{\Sigma} 2x dy dz + y^2 dz dx + z^2 dx dy$$

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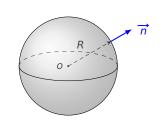


$$I = \frac{F = (2x, y^2, z^2)}{\int \int_{\Sigma} F \cdot \overrightarrow{n} \, dS} = \frac{\overrightarrow{\text{sh} \triangle x}}{\int \int_{\Omega} \text{div} F \, dv}$$
$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} (2x) + \frac{\partial}{\partial y} (y^2) + \frac{\partial}{\partial z} (z^2) \right] dv = \iiint_{\Omega} (2 + y + z) \, dx \, dy \, dz$$
$$= \frac{\overrightarrow{\text{sh} \triangle x}}{\int \int_{\Omega} 2 \, dy \, dz}$$



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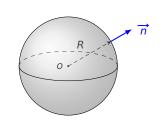


$$I = \frac{F = (2x, y^2, z^2)}{\int \int_{\Sigma} F \cdot \overrightarrow{n} \, dS} = \frac{\overrightarrow{\text{short}}}{\int \int_{\Omega} \text{div} F \, dv}$$
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$$= \frac{\overrightarrow{\text{short}}}{\int \int_{\Omega} 2 \, dy \, dz} = 2 \text{Vol}(\Omega)$$



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$$I = \frac{F = (2x, y^{2}, z^{2})}{\int \int_{\Sigma} F \cdot \overrightarrow{n} dS} = \frac{\overrightarrow{\text{short}}}{\int \int_{\Omega} \text{div} F dv}$$

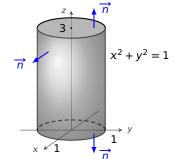
$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} (2x) + \frac{\partial}{\partial y} (y^{2}) + \frac{\partial}{\partial z} (z^{2}) \right] dv = \iiint_{\Omega} (2 + y + z) dx dy dz$$

$$= \frac{\overrightarrow{\text{short}}}{\int \int_{\Omega} 2 dy dz} = 2 \text{Vol}(\Omega) = \frac{8}{3} \pi R^{3}$$



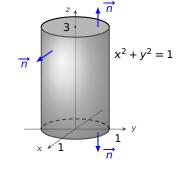
$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$

其中定向曲面  $\Sigma$  是右图柱体的边界曲面



$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$

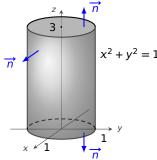
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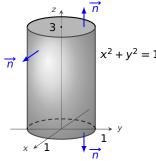
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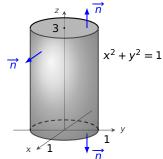
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$$I = F = ((y-z)x, 0, x-y)$$
  $\iint_{\Sigma} F \cdot \overrightarrow{n} dS = \overline{\text{S斯公式}}$   $\iint_{\Omega} \text{div} F dv$ 



$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$
  
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$$I = \frac{F = ((y-z)x, 0, x-y)}{\int \int_{\Sigma} F \cdot \overrightarrow{n} dS} = \frac{\overline{\text{synch}}}{\int \int_{\Omega} \text{div} F dv}$$
$$= \int \left[ \int_{\Omega} \left[ \frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv$$



例 计算 
$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$
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$$x^{2} + y^{2} = 1$$

$$I = \frac{F = ((y-z)x, 0, x-y)}{\int \int_{\Sigma} F \cdot \overrightarrow{n} dS} = \frac{\overrightarrow{\text{sin} \triangle x}}{\int \int_{\Omega} \text{div} F dv}$$
$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dz$$



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$$\overrightarrow{n}$$

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$$= \frac{\overrightarrow{\text{sin}} \cdot \overrightarrow{\text{con}}}{\int \int_{\Omega} -z dx dy dz} = \int \left[ \iint_{\Omega} -z dx dy \right] dz$$



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$$x^{2} + y^{2} = 1$$

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$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z \, dx \, dy \, dz$$

$$\xrightarrow{\underline{x}\underline{n}\underline{n}\underline{n}} \iiint_{\Omega} -z \, dx \, dy \, dz = \int_{\Omega} \left[ \int_{\Omega} -z \, dx \, dy \, dz \right] dz$$



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$$I \xrightarrow{F = ((y-z)x, 0, x-y)} \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{a}\underline{m} \triangle \underline{x}} \iiint_{\Omega} \operatorname{div} F dv$$

$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dz$$

$$\xrightarrow{\underline{x}\underline{n}\underline{n}\underline{n}} \iiint_{\Omega} -z dx dy dz = \int_{\Omega} \left[ \int_{\Omega} -z dx dy dz \right] dz$$



別 り 昇 
$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$
 其中定向曲面  $\Sigma$  是右图柱体的边界曲面

$$x^{2} + y^{2} = 1$$

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$$x^{2} + y^{2} = 1$$

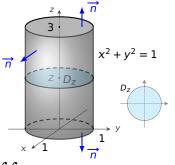
$$I = \frac{F = ((y-z)x, 0, x-y)}{\int \int_{\Sigma} F \cdot \overrightarrow{n} dS} = \frac{\overrightarrow{\text{a}} \cdot \overrightarrow{\text{m}} \cdot \overrightarrow{\text{m}}}{\int \int_{\Omega} \text{div} F dv}$$

$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dz$$

$$= \frac{\overrightarrow{\text{m}} \cdot \overrightarrow{\text{m}}}{\int \int_{\Omega} -z dx dy dz} = \int \left[ \iint_{\Omega} -z dx dy \right] dz$$



例 订昇 
$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$
 其中定向曲面  $\Sigma$  是右图柱体的边界曲面



$$I = \frac{F = ((y-z)x, 0, x-y)}{\int \int_{\Sigma} F \cdot \overrightarrow{n} dS} = \frac{\overline{\text{Sh} \triangle x}}{\int \int_{\Omega} \text{div} F dv}$$

$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dz$$

$$= \frac{\overline{\text{Sh} \triangle x}}{\int \int_{\Omega} -z dx dy dz} = \int_{0}^{3} \left[ \int_{\Omega} -z dx dy \right] dz$$



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$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dz$$

$$= \frac{\overline{\text{sh} \triangle x}}{\int \int_{\Omega} -z dx dy dz} = \int_{0}^{3} \left[ \iint_{D_{z}} -z dx dy \right] dz$$



例 计算 
$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$
 其中定向曲面  $\Sigma$  是右图柱体的边界曲面

$$x^{2} + y^{2} = 1$$

$$y$$

$$x^{2} + y^{2} = 1$$

$$I = \frac{F = ((y-z)x, 0, x-y)}{\int \int_{\Sigma} F \cdot \overrightarrow{n} \, dS} = \frac{\overrightarrow{\text{sin}} \triangle \overrightarrow{x}}{\int \int_{\Omega} \text{div} F \, dv}$$
$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z \, dx \, dy \, dz$$

$$= \iiint_{\Omega} \left[ \frac{1}{\partial x} ((y-z)x) + \frac{1}{\partial y} + \frac{1}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dx$$

$$\frac{\forall k \neq 0}{k \neq 0} \iiint_{\Omega} -z dx dy dz = \int_{0}^{3} \left[ \iint_{\Omega} -z dx dy \right] dz -z |D_{z}|$$



例 计算 
$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$
 其中定向曲面  $\Sigma$  是右图柱体的边界曲面

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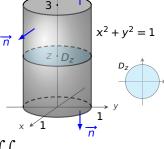
$$I = \frac{F = ((y-z)x, 0, x-y)}{\int \int_{\Sigma} F \cdot \overrightarrow{n} dS} = \frac{\overline{\sin}\Delta x}{\int \int_{\Omega} \operatorname{div} F dv}$$

$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dz$$

$$= \frac{\overline{\sin}\Delta x}{\int \int_{\Omega} -z dx dy dz} = \int_{0}^{3} \left[ \iint_{\Omega} -z dx dy \right] dz = \int_{0}^{3} \left[ -z |D_{z}| \right] dz$$



 $I = \iint_{-\infty} (x - y) dx dy + (y - z) x dy dz$ 其中定向曲面 Σ 是右图柱体的边界曲面



I = F = ((y-z)x, 0, x-y)  $\iint_{\mathbb{R}} F \cdot \overrightarrow{n} \, dS = \overline{\text{sh} \Delta t} \iiint_{\mathbb{R}} \text{div} F dv$ 

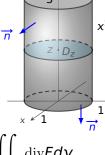
$$=\iiint_{\Omega} \left[ \frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dz$$

$$\xrightarrow{\text{print}} \iiint_{\Omega} -z dx dy dz = \int_{0}^{3} \left[ \iint_{\Omega} -z dx dy \right] dz = \int_{0}^{3} \left[ -z |D_{z}| \right] dz$$

 $= \int_{a}^{3} \left[ -z\pi \right] dz$ 

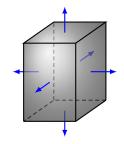


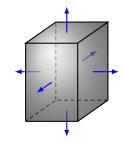
 $I = \iint_{-\infty} (x - y) dx dy + (y - z) x dy dz$ 其中定向曲面 Σ 是右图柱体的边界曲面



I = F = ((y-z)x, 0, x-y)  $\iint_{\mathbb{R}} F \cdot \overrightarrow{n} \, dS = \overline{\text{sh} \Delta t} \iiint_{\mathbb{R}} \text{div} F dv$  $= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dz$ 

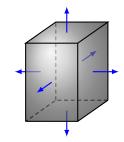
 $= \int_{0}^{3} \left[ -z\pi \right] dz = -\frac{9}{2}\pi$ 





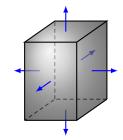
$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS$$





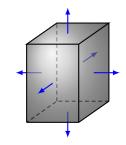
$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\underline{a}\underline{\mathsf{m}} \underline{\mathsf{m}}\underline{\mathsf{m}}} \iiint_{\Omega} \mathrm{div} F d\nu$$

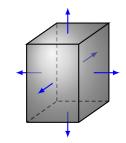




$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\underline{=}\underline{\text{midst}}} \iiint_{\Omega} \operatorname{div} F \, dv$$
$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} (x - y^2) + \frac{\partial}{\partial y} (y) + \frac{\partial}{\partial z} (z^3) \right] \, dv$$





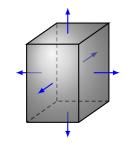


$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\underline{a}\underline{m}\underline{\omega}\underline{\omega}} \iiint_{\Omega} \operatorname{div} F \, dv$$

$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} (x - y^2) + \frac{\partial}{\partial y} (y) + \frac{\partial}{\partial z} (z^3) \right] dv$$

$$= \iiint_{\Omega} (2 + 3z^2) \, dx \, dy \, dz = \int_{\Omega} \left[ \int_{\Omega} \left[ \int_{\Omega} (2 + 3z^2) \, dz \, dy \, dx \right] dx$$



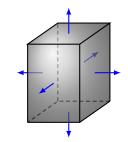


$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\underline{a}\underline{h}\underline{G}\underline{S}} \iiint_{\Omega} \operatorname{div} F \, dV$$

$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} (x - y^2) + \frac{\partial}{\partial y} (y) + \frac{\partial}{\partial z} (z^3) \right] dV$$

$$= \iiint_{\Omega} (2 + 3z^2) \, dx \, dy \, dz = \int_{0}^{1} \left[ \int_{\Omega} \left[ \left[ (2 + 3z^2) \, dz \right] \, dy \right] \, dx$$



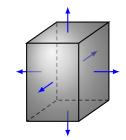


$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\underline{\underline{a}}\underline{\underline{m}}\underline{\underline{M}}\underline{\underline{M}}} \iiint_{\Omega} \operatorname{div} F \, dv$$

$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} (x - y^2) + \frac{\partial}{\partial y} (y) + \frac{\partial}{\partial z} (z^3) \right] \, dv$$

$$= \iiint_{\Omega} (2 + 3z^2) \, dx \, dy \, dz = \int_{\Omega}^{1} \left[ \int_{1}^{2} \left[ \int_{1}^{2} (2 + 3z^2) \, dz \right] \, dy \right] \, dx$$



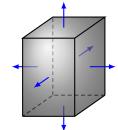


$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\underline{a}\underline{m}\underline{\omega}\underline{\omega}} \iiint_{\Omega} \operatorname{div} F \, dV$$

$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} (x - y^2) + \frac{\partial}{\partial y} (y) + \frac{\partial}{\partial z} (z^3) \right] \, dV$$

$$= \iiint_{\Omega} (2 + 3z^2) \, dx \, dy \, dz = \int_{0}^{1} \left[ \int_{1}^{2} \left[ \int_{1}^{4} (2 + 3z^2) \, dz \right] \, dy \right] \, dx$$





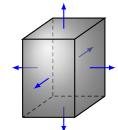
$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{a}\underline{m}\underline{\omega}\underline{\omega}} \iiint_{\Omega} \operatorname{div} F dV$$

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$$= \iiint_{\Omega} (2 + 3z^2) dx dy dz = \int_{0}^{1} \left[ \int_{1}^{2} \left[ \int_{1}^{4} (2 + 3z^2) dz \right] dy \right] dx$$

$$= \int_{0}^{1} 1 dx \cdot \int_{1}^{2} 1 dy \cdot \int_{1}^{4} (2 + 3z^2) dz$$



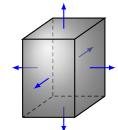


$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{a}\underline{m} \underline{\wedge}\underline{\wedge}} \iiint_{\Omega} \operatorname{div} F dV$$

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$$= \iiint_{\Omega} (2 + 3z^2) dx dy dz = \int_{0}^{1} \left[ \int_{1}^{2} \left[ \int_{1}^{4} (2 + 3z^2) dz \right] dy \right] dx$$

$$= \int_{0}^{1} 1 dx \cdot \int_{1}^{2} 1 dy \cdot \int_{1}^{4} (2 + 3z^2) dz = 1 \cdot 1 \cdot (2z + z^3) \Big|_{1}^{4}$$



$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\underline{a}\underline{y}\underline{y}} \iiint_{\Omega} \operatorname{div} F \, dv$$

$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} (x - y^2) + \frac{\partial}{\partial y} (y) + \frac{\partial}{\partial z} (z^3) \right] \, dv$$

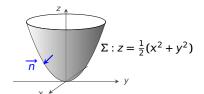
$$= \iiint_{\Omega} (2 + 3z^2) \, dx \, dy \, dz = \int_{0}^{1} \left[ \int_{1}^{2} \left[ \int_{1}^{4} (2 + 3z^2) \, dz \right] \, dy \right] \, dx$$

$$= \int_{0}^{1} 1 \, dx \cdot \int_{1}^{2} 1 \, dy \cdot \int_{1}^{4} (2 + 3z^2) \, dz = 1 \cdot 1 \cdot (2z + z^3) \Big|_{1}^{4} = 69$$

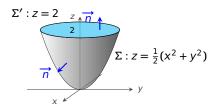


$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

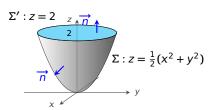
其中定向曲面 Σ 是抛物面的一部分, 取单位外法向量,如图:



$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$



$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

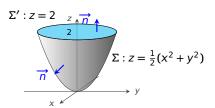


$$\iint_{\Sigma} (z^{2} + x) dy dz - z dx dy = \iint_{\Sigma} F \cdot \overrightarrow{n} dS,$$

$$\iint_{\Sigma'} (z^{2} + x) dy dz - z dx dy = \iint_{\Sigma'} F \cdot \overrightarrow{n} dS$$

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS + \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$

$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$



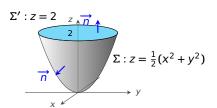
$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy = \iint_{\Sigma} F \cdot \overrightarrow{n} dS,$$

$$\iint_{\Sigma'} (z^2 + x) dy dz - z dx dy = \iint_{\Sigma'} F \cdot \overrightarrow{n} dS$$

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS + \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS = \iint_{\Sigma \cup \Sigma'} F \cdot \overrightarrow{n} \, dS$$



$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$



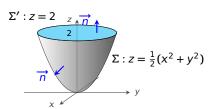
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$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS + \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS = \iint_{\Sigma \cup \Sigma'} \operatorname{div} F \, dV$$



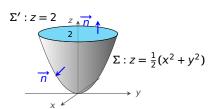
$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$



$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS + \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS = \iint_{\Sigma \cup \Sigma'} \operatorname{div} F \, dv \xrightarrow{\operatorname{div} F = 0}$$



$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$



$$\iint_{\Sigma} (z^{2} + x) dy dz - z dx dy = \iint_{\Sigma} F \cdot \overrightarrow{n} dS,$$

$$\iint_{\Sigma'} (z^{2} + x) dy dz - z dx dy = \iint_{\Sigma'} F \cdot \overrightarrow{n} dS$$

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS + \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS = \iint_{\Sigma \cup \Sigma'} \operatorname{div} F \, dv \xrightarrow{\operatorname{div} F = 0} 0.$$



$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

$$\Sigma': z = 2$$

$$Z : z = \frac{1}{2}(x^2 + y^2)$$

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$$\iint_{\Sigma} (z^{2} + x) dy dz - z dx dy = \iint_{\Sigma} F \cdot \overrightarrow{n} dS,$$

$$\iint_{\Sigma'} (z^{2} + x) dy dz - z dx dy = \iint_{\Sigma'} F \cdot \overrightarrow{n} dS \xrightarrow{F = (z^{2} + x, 0, -z)}$$

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS + \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS = \iint_{\Sigma \cup \Sigma'} \operatorname{div} F \, dv \xrightarrow{\operatorname{div} F = 0} 0.$$



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$$\iint_{\Sigma'} (z^2 + x) dy dz - z dx dy = \iint_{\Sigma'} F \cdot \overrightarrow{n} dS \xrightarrow{F = (z^2 + x, 0, -z)}{\overrightarrow{n} = (0, 0, 1)}$$

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS + \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS = \iint_{\Sigma} \operatorname{div} F \, dv \xrightarrow{\operatorname{div} F = 0} 0.$$



$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

其中定向曲面 Σ 是抛物面的一部分, 取单位外法向量,如图:

$$\Sigma': z = 2$$

$$\Sigma: z = \frac{1}{2}(x^2 + y^2)$$

$$Y$$

 $\mathbf{m}$  如图补充平面  $\Sigma'$ ,则  $\Sigma \cup \Sigma'$  构成 3 维区域  $\Omega$  边界,应用高斯公式:

$$\iint_{\Sigma} (z^{2} + x) dy dz - z dx dy = \iint_{\Sigma} F \cdot \overrightarrow{n} dS,$$

$$\iint_{\Sigma'} (z^{2} + x) dy dz - z dx dy = \iint_{\Sigma'} F \cdot \overrightarrow{n} dS \xrightarrow{F = (z^{2} + x, 0, -z)}{\overrightarrow{n} = (0, 0, 1)} \iint_{\Sigma'} -z dS$$

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS + \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS = \iint_{\Sigma \cup \Sigma'} \operatorname{div} F \, dv \xrightarrow{\operatorname{div} F = 0} 0.$$



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解 如图称允十個 
$$Z$$
 ,则  $Z$   $O$   $Z$  和成  $S$  维区域  $\Omega$  边齐,应用高利公式:
$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$
$$\iint_{\Sigma'} (z^2 + x) dy dz - z dx dy = \iint_{\Sigma'} F \cdot \overrightarrow{n} dS \frac{F = (z^2 + x, 0, -z)}{\overrightarrow{n} = (0, 0, 1)} \iint_{\Sigma'} -z dS$$
$$= \iint_{\Sigma'} -2 dS$$

$$= \iiint_{\Sigma'} -2dS$$

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS + \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS = \iint_{\Sigma \cup \Sigma'} \operatorname{div} F \, dv \xrightarrow{\operatorname{div} F = 0} 0.$$



$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

其中定向曲面 Σ 是抛物面的一部分, 取单位外法向量,如图:

 $\mathbf{m}$  如图补充平面  $\Sigma'$ ,则  $\Sigma \cup \Sigma'$  构成 3 维区域  $\Omega$  边界,应用高斯公式:

解 如图称充平面 
$$Z$$
 ,则  $Z$   $O$   $Z$  构成  $S$  维区域  $\Omega$  边齐,应用高利公式:
$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy = \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{F = (z^2 + x, 0, -z)} \iint_{\Sigma'} -z dS$$
$$= \iint_{\Sigma'} -2 dS = -2 \operatorname{Area}(\Sigma')$$

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS + \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS = \iint_{\Sigma \cup \Sigma'} \operatorname{div} F \, dv \xrightarrow{\operatorname{div} F = 0} 0.$$



$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

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$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy = \iint_{\Sigma} F \cdot \overrightarrow{n} dS,$$

$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy = \iint_{\Sigma'} F \cdot \overrightarrow{n} dS \xrightarrow{F = (z^2 + x, 0, -z)} \iint_{\Sigma'} -z dS$$

$$= \iint_{\Sigma'} -2 dS = -2 \operatorname{Area}(\Sigma') = -8\pi,$$

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS + \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS = \iint_{\Sigma \cup \Sigma'} \operatorname{div} F \, dv \xrightarrow{\operatorname{div} F = 0} 0.$$



其中定向曲面 Σ 是抛物面的一部分, 取单位外法向量,如图:

$$\Sigma': z = 2$$

$$\sum_{x \in \Pi} \sum_{x \in \Pi} \sum_{y \in \Pi} \sum_{x \in \Pi} \sum_$$

 $\mathbf{m}$  如图补充平面  $\Sigma'$ ,则  $\Sigma \cup \Sigma'$  构成 3 维区域  $\Omega$  边界,应用高斯公式:

$$\iint_{\Sigma} (z^{2} + x) dy dz - z dx dy = \iint_{\Sigma} F \cdot \overrightarrow{n} dS,$$

$$\iint_{\Sigma'} (z^{2} + x) dy dz - z dx dy = \iint_{\Sigma'} F \cdot \overrightarrow{n} dS \xrightarrow{F = (z^{2} + x, 0, -z)} \iint_{\Sigma'} -z dS$$

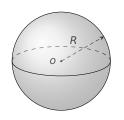
$$= \iint_{\Sigma'} -2 dS = -2 \operatorname{Area}(\Sigma') = -8\pi,$$

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS + \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS = \iint_{\Sigma \cup \Sigma'} \operatorname{div} F \, dv \xrightarrow{\operatorname{div} F = 0} 0.$$

所以原积分等于 8π。

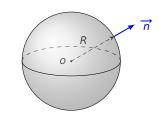


$$I = \iint_{\Sigma} (x^2 + y + z) dS$$



$$I = \iint_{\Sigma} (x^2 + y + z) dS$$

其中曲面 Σ 是球面  $x^2 + y^2 + z^2 = R^2$ 



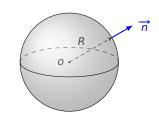
解

$$\iint_{\Sigma} (x^2 + y + z) dS$$

$$= \iint_{\Sigma} F \cdot \overrightarrow{n} dS \stackrel{\overline{\text{sh}} \triangle \overrightarrow{x}}{=} \iiint_{\Omega} \operatorname{div} F dv$$



$$I = \iint_{\Sigma} (x^2 + y + z) dS$$

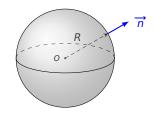


解 球面单位外法向量 
$$\overrightarrow{n} = \frac{1}{R}(x, y, z)$$
,所以

$$\iint_{\Sigma} (x^2 + y + z) dS$$

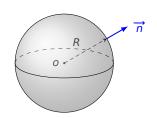
$$= \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{a}\underline{m}\underline{\wedge}\underline{\wedge}\underline{\wedge}} \iiint_{\Omega} \mathrm{div} F dv$$

$$I = \iint_{\Sigma} (x^2 + y + z) dS$$



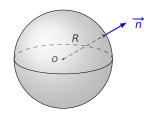
解 球面单位外法向量 
$$\overrightarrow{n} = \frac{1}{R}(x, y, z)$$
,所以 
$$\iint_{\Sigma} (x^2 + y + z) dS \qquad ( , , ) \cdot \frac{1}{R}(x, y, z)$$
 
$$= \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\overline{\text{sh}\Delta t}} \iiint_{\Omega} \text{div} F dv$$

$$I = \iint_{\Sigma} (x^2 + y + z) dS$$



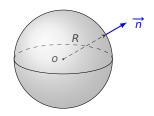
解 球面单位外法向量 
$$\overrightarrow{n} = \frac{1}{R}(x, y, z)$$
,所以 
$$\iint_{\Sigma} (x^2 + y + z) dS \qquad R(x, 1, 1) \cdot \frac{1}{R}(x, y, z)$$
 
$$= \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\overline{\text{SH}} \triangle T} \iiint_{\Omega} \text{div} F dv$$

$$I = \iint_{\Sigma} (x^2 + y + z) dS$$



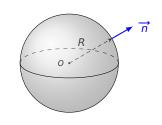
解 球面单位外法向量 
$$\overrightarrow{n} = \frac{1}{R}(x, y, z)$$
,所以 
$$\iint_{\Sigma} (x^2 + y + z) dS = \iint_{\Sigma} R(x, 1, 1) \cdot \frac{1}{R}(x, y, z) dS$$
 
$$= \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\overline{\text{sh}\Delta t}} \iiint_{\Omega} \text{div} F dV$$

$$I = \iint_{\Sigma} (x^2 + y + z) dS$$



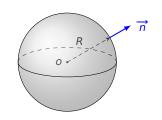
解 球面单位外法向量 
$$\overrightarrow{n} = \frac{1}{R}(x, y, z)$$
,所以 
$$\iint_{\Sigma} (x^2 + y + z) dS = \iint_{\Sigma} R(x, 1, 1) \cdot \frac{1}{R}(x, y, z) dS$$
 
$$= \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{n} \times \underline{x}} \iiint_{\Omega} \operatorname{div} F dv$$
 
$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} (Rx) + \frac{\partial}{\partial y} (R) + \frac{\partial}{\partial z} (R) \right] dv$$

$$I = \iint_{\Sigma} (x^2 + y + z) dS$$



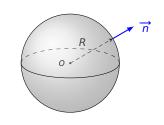
解 球面单位外法向量 
$$\overrightarrow{n} = \frac{1}{R}(x, y, z)$$
,所以 
$$\iint_{\Sigma} (x^2 + y + z) dS = \iint_{\Sigma} R(x, 1, 1) \cdot \frac{1}{R}(x, y, z) dS$$
 
$$= \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{n} \underline{n} \triangle \exists} \iiint_{\Omega} \operatorname{div} F dv$$
 
$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} (Rx) + \frac{\partial}{\partial y} (R) + \frac{\partial}{\partial z} (R) \right] dv$$
 
$$= \iiint_{\Omega} R dx dy dz$$

$$I = \iint_{\Sigma} (x^2 + y + z) dS$$



解 球面单位外法向量 
$$\overrightarrow{n} = \frac{1}{R}(x, y, z)$$
,所以 
$$\iint_{\Sigma} (x^2 + y + z) dS = \iint_{\Sigma} R(x, 1, 1) \cdot \frac{1}{R}(x, y, z) dS$$
 
$$= \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{n} \underline{n} \triangle \exists} \iiint_{\Omega} \operatorname{div} F dv$$
 
$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} (Rx) + \frac{\partial}{\partial y} (R) + \frac{\partial}{\partial z} (R) \right] dv$$
 
$$= \iiint_{\Omega} R dx dy dz = R \operatorname{Vol}(\Omega)$$

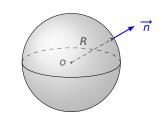
$$I = \iint_{\Sigma} (x^2 + y + z) dS$$



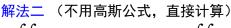
解 球面单位外法向量 
$$\overrightarrow{n} = \frac{1}{R}(x, y, z)$$
,所以 
$$\iint_{\Sigma} (x^2 + y + z) dS = \iint_{\Sigma} R(x, 1, 1) \cdot \frac{1}{R}(x, y, z) dS$$
 
$$= \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{n} \underline{n} \underline{n} \underline{n}} \iiint_{\Omega} \operatorname{div} F dV$$
 
$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} (Rx) + \frac{\partial}{\partial y} (R) + \frac{\partial}{\partial z} (R) \right] dV$$
 
$$= \iiint_{\Omega} R dx dy dz = R \operatorname{Vol}(\Omega) = \frac{4}{3} \pi R^4$$

$$I = \iint_{\Sigma} (x^2 + y + z) dS$$

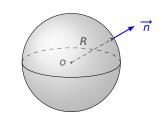
其中曲面 Σ 是球面  $x^2 + y^2 + z^2 = R^2$ 



$$I = \iint_{\Sigma} (x^2 + y + z) dS$$

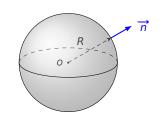


$$\iint_{\Sigma} (x^2 + y + z) dS \xrightarrow{\underline{\text{yh}}\underline{\text{th}}} \iint_{\Sigma} x^2 dS$$



$$I = \iint_{\Sigma} (x^2 + y + z) dS$$

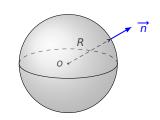
其中曲面 Σ 是球面  $x^2 + y^2 + z^2 = R^2$ 



$$\iint_{\Sigma} (x^2 + y + z) dS \xrightarrow{\text{span}} \iint_{\Sigma} x^2 dS = \frac{1}{3} \iint_{\Sigma} (x^2 + x^2 + x^2) dS$$

$$I = \iint_{\Sigma} (x^2 + y + z) dS$$

其中曲面 Σ 是球面  $x^2 + y^2 + z^2 = R^2$ 

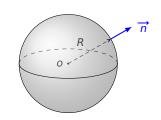


$$\iint_{\Sigma} (x^2 + y + z) dS \xrightarrow{\text{spate}} \iint_{\Sigma} x^2 dS = \frac{1}{3} \iint_{\Sigma} (x^2 + x^2 + x^2) dS$$

$$\xrightarrow{\text{spate}} \frac{1}{3} \iint_{\Sigma} (x^2 + y^2 + z^2) dS$$

$$I = \iint_{\Sigma} (x^2 + y + z) dS$$

其中曲面 Σ 是球面  $x^2 + y^2 + z^2 = R^2$ 



$$\iint_{\Sigma} (x^2 + y + z)dS \xrightarrow{\text{print}} \iint_{\Sigma} x^2 dS = \frac{1}{3} \iint_{\Sigma} (x^2 + x^2 + x^2)dS$$

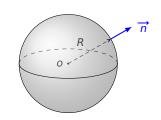
$$\xrightarrow{\text{print}} \frac{1}{3} \iint_{\Sigma} (x^2 + y^2 + z^2)dS$$

$$= \frac{1}{3} \iint_{\Sigma} R^2 dS$$



$$I = \iint_{\Sigma} (x^2 + y + z) dS$$

其中曲面 Σ 是球面  $x^2 + y^2 + z^2 = R^2$ 



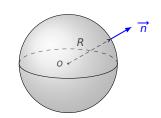
$$\iint_{\Sigma} (x^2 + y + z) dS \xrightarrow{\text{print}} \iint_{\Sigma} x^2 dS = \frac{1}{3} \iint_{\Sigma} (x^2 + x^2 + x^2) dS$$

$$\xrightarrow{\text{print}} \frac{1}{3} \iint_{\Sigma} (x^2 + y^2 + z^2) dS$$

$$= \frac{1}{3} \iint_{\Sigma} R^2 dS = \frac{1}{3} R^2 \text{Area}(\Sigma)$$

$$I = \iint_{\Sigma} (x^2 + y + z) dS$$

其中曲面 Σ 是球面  $x^2 + y^2 + z^2 = R^2$ 



$$\iint_{\Sigma} (x^{2} + y + z)dS \xrightarrow{\frac{\sqrt{2}}{2}} \iint_{\Sigma} x^{2}dS = \frac{1}{3} \iint_{\Sigma} (x^{2} + x^{2} + x^{2})dS$$

$$\xrightarrow{\frac{\sqrt{2}}{2}} \frac{1}{3} \iint_{\Sigma} (x^{2} + y^{2} + z^{2})dS$$

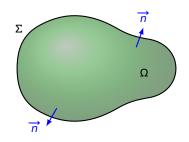
$$= \frac{1}{3} \iint_{\Sigma} R^{2}dS = \frac{1}{3}R^{2}Area(\Sigma) = \frac{4}{3}\pi R^{4}$$



高斯公式 
$$\iiint_{\Omega} \operatorname{div} F dv = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$



• 假设 F = (P, Q, R) 是流体的速度向量场,



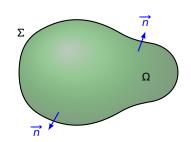
高斯公式 
$$\iiint_{\Omega} \operatorname{div} F dv = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$



假设 F = (P, Q, R) 是流体的速度向量场,则

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS$$

表示单位时间流向  $\Sigma$  外侧的通量。



高斯公式 
$$\iiint_{\Omega} \operatorname{div} F dv = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$



假设 F = (P, Q, R) 是流体的速度向量场,则

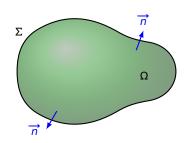
$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS$$

表示单位时间流向  $\Sigma$  外侧的通量。

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS > 0$$

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS < 0$$

高斯公式 
$$\iiint_{\Omega} \operatorname{div} F dv = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$



 假设 F = (P, Q, R) 是流体的速度向 量场,则

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS$$

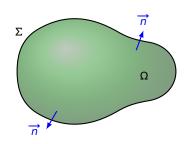
表示单位时间流向 Σ 外侧的通量。

• 进一步假设流体是不可压,则

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS > 0$$

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS < 0$$

高斯公式 
$$\iiint_{\Omega} \operatorname{div} F dv = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$



 假设 F = (P, Q, R) 是流体的速度向 量场,则

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS$$

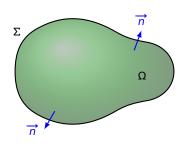
表示单位时间流向 Σ 外侧的通量。

• 进一步假设流体是不可压,则

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS > 0 \Rightarrow \Omega \text{ 内有 "source"}$$

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS < 0$$

高斯公式 
$$\iiint_{\Omega} \operatorname{div} F dv = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$



 假设 F = (P, Q, R) 是流体的速度向 量场,则

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS$$

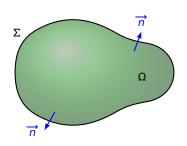
表示单位时间流向  $\Sigma$  外侧的通量。

• 进一步假设流体是不可压,则

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS > 0 \Rightarrow \Omega \text{ 内有 "source"}$$

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS < 0 \Rightarrow \Omega \text{ 内有 "sink"}$$

高斯公式 
$$\iiint_{\Omega} \operatorname{div} F dv = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$



假设 F = (P, Q, R) 是流体的速度向量场,则

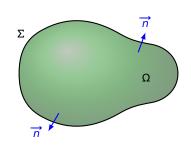
$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS$$

表示单位时间流向 Σ 外侧的通量。

• 进一步假设流体是不可压,则

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS > 0 \Rightarrow \Omega \text{ 内有 "source"}$$

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS < 0 \Rightarrow \Omega \text{ 内有 "sink"}$$



注 高斯公式  $\iiint_{\Omega} \operatorname{div} F dv = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$  表明:  $\operatorname{div} F$  反映这种

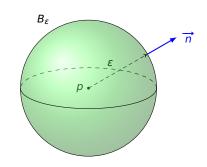
"source"和 "sink"的强度。



p •

 $\operatorname{div} F(p)$ 



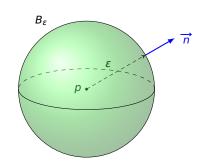


 $\operatorname{div} F(p)$ 



$$\iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \iiint_{B_{\varepsilon}} \operatorname{div} F \, dv$$



divF(p)



$$\iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$\iiint_{B_{\varepsilon}} \operatorname{div} F \, dV$$

$$= \operatorname{Vol}(B_{\varepsilon}) \operatorname{div} F(p_{\varepsilon})$$





$$\frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \iiint_{B_{\varepsilon}} \operatorname{div} F \, dv$$

$$= \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \operatorname{Vol}(B_{\varepsilon}) \operatorname{div} F(p_{\varepsilon})$$

 $\operatorname{div} F(p)$ 



$$\frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \iiint_{B_{\varepsilon}} \operatorname{div} F dv$$

$$= \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \operatorname{Vol}(B_{\varepsilon}) \operatorname{div} F(p_{\varepsilon})$$

$$= \operatorname{div} F(p_{\varepsilon})$$

## 散度 $\operatorname{div} F$ 的物理解释 (2)

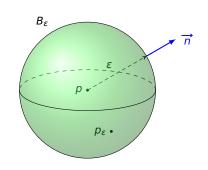
$$\lim_{\varepsilon \to 0^{+}} \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \iiint_{B_{\varepsilon}} \operatorname{div} F \, dv$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \operatorname{Vol}(B_{\varepsilon}) \operatorname{div} F(p_{\varepsilon})$$

$$= \lim_{\varepsilon \to 0^{+}} \operatorname{div} F(p_{\varepsilon})$$

$$\operatorname{div} F(p)$$



# 散度 div F 的物理解释 (2)

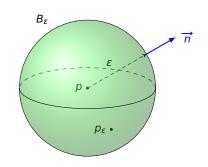
$$\lim_{\varepsilon \to 0^{+}} \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \iiint_{B_{\varepsilon}} \operatorname{div} F \, dv$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \operatorname{Vol}(B_{\varepsilon}) \operatorname{div} F(p_{\varepsilon})$$

$$= \lim_{\varepsilon \to 0^{+}} \operatorname{div} F(p_{\varepsilon})$$

$$= \operatorname{div} F(p)$$



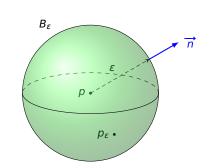
$$\lim_{\varepsilon \to 0^{+}} \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \iiint_{B_{\varepsilon}} \operatorname{div} F \, dv$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \operatorname{Vol}(B_{\varepsilon}) \operatorname{div} F(p_{\varepsilon})$$

$$= \lim_{\varepsilon \to 0^{+}} \operatorname{div} F(p_{\varepsilon})$$

$$= \operatorname{div} F(p)$$



- div*F*(p)>0 时,
- div*F*(*p*)<0 时,

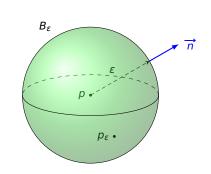
$$\lim_{\varepsilon \to 0^{+}} \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \iiint_{B_{\varepsilon}} \operatorname{div} F \, dv$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \operatorname{Vol}(B_{\varepsilon}) \operatorname{div} F(p_{\varepsilon})$$

$$= \lim_{\varepsilon \to 0^{+}} \operatorname{div} F(p_{\varepsilon})$$

$$= \operatorname{div} F(p)$$



- div*F*(*p*)>0 时, ∫∫∂B, *F* · *n* dS > 0 (ε 充分小),
- div*F*(*p*)<0 时,



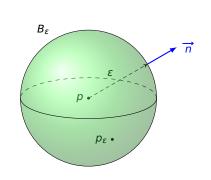
$$\lim_{\varepsilon \to 0^{+}} \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \iiint_{B_{\varepsilon}} \operatorname{div} F \, dv$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \operatorname{Vol}(B_{\varepsilon}) \operatorname{div} F(p_{\varepsilon})$$

$$= \lim_{\varepsilon \to 0^{+}} \operatorname{div} F(p_{\varepsilon})$$

$$= \operatorname{div} F(p)$$



- $\operatorname{div} F(p) > 0$  时,  $\iint_{\partial B_{\epsilon}} F \cdot \overrightarrow{n} dS > 0$  ( $\epsilon$  充分小),说明 p 点是 source
- div*F*(p)<0 时,



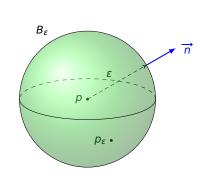
$$\lim_{\varepsilon \to 0^{+}} \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \iiint_{B_{\varepsilon}} \operatorname{div} F \, dv$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \operatorname{Vol}(B_{\varepsilon}) \operatorname{div} F(p_{\varepsilon})$$

$$= \lim_{\varepsilon \to 0^{+}} \operatorname{div} F(p_{\varepsilon})$$

$$= \operatorname{div} F(p)$$



- $\operatorname{div} F(p) > 0$  时,  $\iint_{\partial B_{\epsilon}} F \cdot \overrightarrow{n} dS > 0$  ( $\epsilon$  充分小),说明 p 点是 source
- div*F*(*p*)<0 时,∫∫∂Bε *F* · *n* dS <0(ε 充分小),



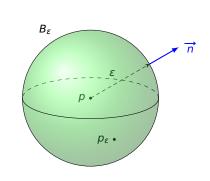
$$\lim_{\varepsilon \to 0^{+}} \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \iiint_{B_{\varepsilon}} \operatorname{div} F \, dv$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \operatorname{Vol}(B_{\varepsilon}) \operatorname{div} F(p_{\varepsilon})$$

$$= \lim_{\varepsilon \to 0^{+}} \operatorname{div} F(p_{\varepsilon})$$

$$= \operatorname{div} F(p)$$



- $\operatorname{div} F(p) > 0$  时, $\iint_{\partial B_{\epsilon}} F \cdot \overrightarrow{n} \, dS > 0$  ( $\epsilon$  充分小),说明 p 点是 source
- $\operatorname{div} F(p) < 0$  时,  $\iint_{\partial B_{\epsilon}} F \cdot \overrightarrow{n} \, dS < 0$  ( $\epsilon$  充分小),说明 p 点是  $\operatorname{sink}$



We are here now...

1. 高斯公式

2. 斯托克斯公式

定义 设 
$$F = (P, Q, R)$$
 是空间中向量场, 定义

$$\operatorname{rot} F := \left| \begin{array}{ccc} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{array} \right|$$

定义 设 
$$F = (P, Q, R)$$
 是空间中向量场,定义

$$\operatorname{rot} F := \left| \begin{array}{ccc} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{array} \right| = \left( \left| \begin{array}{ccc} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{array} \right|,$$



定义 设 F = (P, Q, R) 是空间中向量场, 定义

$$\operatorname{rot} F := \left| \begin{array}{ccc} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & O & R \end{array} \right| = \left( \left| \begin{array}{ccc} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{array} \right|, - \left| \begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{array} \right|,$$



定义 设 F = (P, Q, R) 是空间中向量场, 定义

$$\operatorname{rot} F := \left| \begin{array}{ccc} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{array} \right| = \left( \left| \begin{array}{ccc} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{array} \right|, \ - \left| \begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{array} \right|, \ \left| \begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{array} \right| \right)$$



定义 设 F = (P, Q, R) 是空间中向量场, 定义

$$\operatorname{rot} F := \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left( \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} \right)$$
$$= (R_{y} - Q_{z}, \qquad , \qquad )$$



定义 设 F = (P, Q, R) 是空间中向量场, 定义

$$\operatorname{rot} F := \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \begin{pmatrix} \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} \end{pmatrix}$$
$$= (R_{y} - Q_{z}, P_{z} - R_{x}, )$$



定义 设 F = (P, Q, R) 是空间中向量场, 定义

$$\operatorname{rot} F := \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left( \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} \right)$$
$$= (R_{y} - Q_{z}, P_{z} - R_{x}, Q_{x} - P_{y})$$



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$$= (R_{y} - Q_{z}, P_{z} - R_{x}, Q_{x} - P_{y})$$

称为向量场 F 的旋度。

例 计算向量场  $F = (y, -x, e^{xz})$  的旋度。



定义 设 F = (P, Q, R) 是空间中向量场, 定义

$$\operatorname{rot} F := \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left( \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} \right)$$
$$= (R_{V} - Q_{Z}, P_{Z} - R_{X}, Q_{X} - P_{Y})$$

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例 计算向量场  $F = (y, -x, e^{xz})$  的旋度。

$$\operatorname{rot} F = \begin{vmatrix}
\overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
y & -x & e^{xz}
\end{vmatrix}$$

定义 设 F = (P, Q, R) 是空间中向量场,定义

$$\operatorname{rot} F := \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left( \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} \right)$$
$$= (R_{V} - Q_{Z}, P_{Z} - R_{X}, Q_{X} - P_{Y})$$

称为向量场 F 的旋度。

例 计算向量场  $F = (v_1 - x_2)$  的旋度。

$$rot F = \begin{vmatrix}
\overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
y & -x & e^{XZ}
\end{vmatrix} = \left(\begin{vmatrix}
\frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
-x & e^{XZ}
\end{vmatrix}, -\begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\
y & e^{XZ}
\end{vmatrix}, \begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\
y & -x
\end{vmatrix}\right)$$



定义 设 
$$F = (P, Q, R)$$
 是空间中向量场, 定义

$$\operatorname{rot} F := \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left( \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} \right)$$
$$= (R_{V} - Q_{Z}, P_{Z} - R_{X}, Q_{X} - P_{Y})$$

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定义 设 
$$F = (P, Q, R)$$
 是空间中向量场, 定义

$$\operatorname{rot} F := \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left( \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} \right)$$
$$= \left( R_{V} - Q_{Z}, P_{Z} - R_{X}, Q_{X} - P_{V} \right)$$

称为向量场 F 的旋度。

例 计算向量场  $F = (v_1 - x_2)$  的旋度。

 $\operatorname{rot} F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & e^{XZ} \end{vmatrix} = \left( \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -x & e^{XZ} \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ y & e^{XZ} \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ y & -x \end{vmatrix} \right)$ 



定义 设 
$$F = (P, Q, R)$$
 是空间中向量场, 定义

$$\operatorname{rot} F := \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left( \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} \right)$$
$$= \left( R_{V} - Q_{Z}, P_{Z} - R_{X}, Q_{X} - P_{V} \right)$$

称为向量场 F 的旋度。

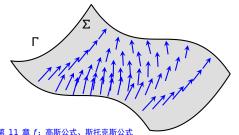
例 计算向量场  $F = (v_1 - x_2)$  的旋度。

 $\operatorname{rot} F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & e^{Xz} \end{vmatrix} = \left( \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -x & e^{Xz} \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ y & e^{Xz} \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ y & -x \end{vmatrix} \right)$ 

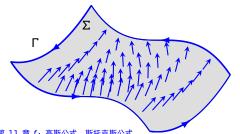
$$|y| - x \in |$$
  
=  $(0, -ze^{xz}, -2)$ 



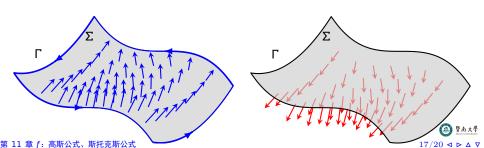
- $\Sigma$  是空间中分片光滑的定向曲面,选定单位外法向量  $\overrightarrow{n}$  ,
- $\Gamma$  是  $\Sigma$  的边界, 且赋予 "边界定向",



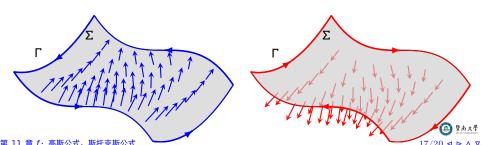
- $\Sigma$  是空间中分片光滑的定向曲面,选定单位外法向量  $\overrightarrow{n}$  ,
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- $\Gamma$  是  $\Sigma$  的边界, 且赋予 "边界定向",
- F = (P, Q, R) 是空间向量场, 且 P, Q, R 具有一阶连续偏导数,

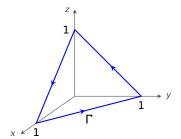


#### 定理(斯托克斯公式) 假设

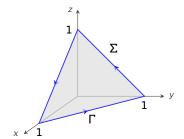
- $\Sigma$  是空间中分片光滑的定向曲面,选定单位外法向量  $\overrightarrow{n}$ ,
- $\Gamma$  是  $\Sigma$  的边界, 且赋予"边界定向",
- F = (P, Q, R) 是空间向量场, 且 P, Q, R 具有一阶连续偏导数,

则成立:  $\iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} \, dS = \int_{\Gamma} P dx + Q dy + R dz.$ 

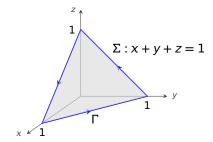
$$I = \int_{\Gamma} z dx + x dy + y dz$$



$$I = \int_{\Gamma} z dx + x dy + y dz$$

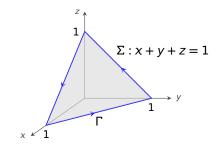


$$I = \int_{\Gamma} z dx + x dy + y dz$$



$$I = \int_{\Gamma} z dx + x dy + y dz$$

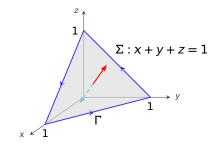
解设
$$F=(z,x,y)$$
,则



所以
$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} dS$$

$$I = \int_{\Gamma} z dx + x dy + y dz$$

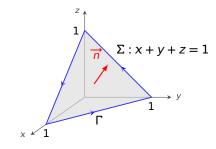
解设
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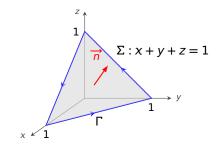
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所以
$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} dS$$

$$I = \int_{\Gamma} z dx + x dy + y dz$$

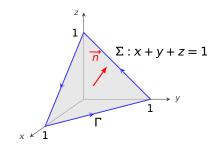
解设
$$F=(z,x,y)$$
,则



所以
$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)}$$

$$I = \int_{\Gamma} z dx + x dy + y dz$$

解设 
$$F = (z, x, y)$$
,则 
$$\operatorname{rot} F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix}$$

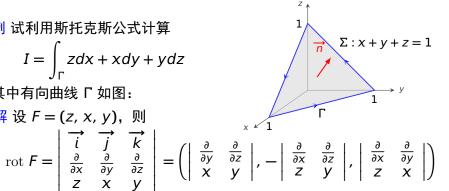


所以
$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)}$$

$$I = \int_{\Gamma} z dx + x dy + y dz$$

解设
$$F = (z, x, y)$$
,则

$$\operatorname{rot} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix}$$



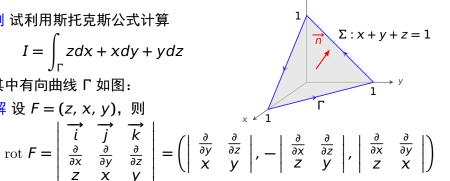
所以
$$\int_{\mathbb{R}} z dx + x dy + y dz = \iint_{\mathbb{R}} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)}$$



$$I = \int_{\Gamma} z dx + x dy + y dz$$

解设
$$F = (z, x, y)$$
,则

$$\operatorname{rot} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix}$$



$$=(1, , ]$$

所以
$$\int_{\mathbb{R}} z dx + x dy + y dz = \iint_{\mathbb{R}} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)}$$

$$I = \int_{\Gamma} z dx + x dy + y dz$$

解设
$$F = (z, x, y)$$
,则

$$\operatorname{rot} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix}$$

$$=(1, 1, )$$

所以
$$\int_{\mathbb{R}} z dx + x dy + y dz = \iint_{\mathbb{R}} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)}$$

$$I = \int_{\Gamma} z dx + x dy + y dz$$

解设
$$F = (z, x, y)$$
,则

$$\operatorname{rot} F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix}$$

以試利用斯托克斯公式计算 
$$I = \int_{\Gamma} z dx + x dy + y dz$$
 中有向曲线  $\Gamma$  如图: 
$$\mathbf{F} = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y \end{bmatrix}, - \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ z & y \end{bmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ z & x \end{vmatrix}$$

$$=(1, 1, 1)$$

所以
$$\int_{\mathbb{R}} z dx + x dy + y dz = \iint_{\mathbb{R}} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)}$$

$$I = \int_{\Gamma} z dx + x dy + y dz$$

解设
$$F = (z, x, y)$$
,则

$$\operatorname{rot} F = \left| \begin{array}{ccc} l & J & K \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Z & X & Y \end{array} \right|$$

以 试利用斯托克斯公式计算 
$$I = \int_{\Gamma} z dx + x dy + y dz$$
 中有向曲线  $\Gamma$  如图: 
$$\mathbf{F} = \begin{bmatrix} \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \mathbf{F} & \mathbf{F} & \mathbf$$

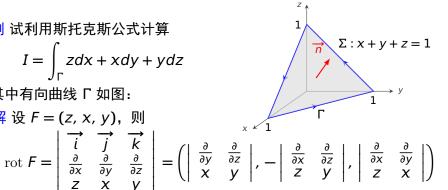
$$=(1, 1, 1)$$

所以
$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)} \iint_{\Sigma} \sqrt{3} dS$$

$$I = \int_{\Gamma} z dx + x dy + y dz$$

解设
$$F = (z, x, y)$$
,则

$$\operatorname{rot} F = \left| \begin{array}{ccc} l & J & K \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Z & X & Y \end{array} \right|$$



$$=(1, 1, 1)$$

所以
$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)} \iint_{\Sigma} \sqrt{3} dS$$

$$=\sqrt{3}\operatorname{Area}(\Sigma)$$

$$I = \int_{\Gamma} z dx + x dy + y dz$$

其中有向曲线 [如图:

解设
$$F = (z, x, y)$$
,则

$$\operatorname{rot} F = \begin{bmatrix} l & J & K \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Z & X & Y \end{bmatrix}$$

以 试利用斯托克斯公式计算 
$$I = \int_{\Gamma} z dx + x dy + y dz$$
 中有向曲线  $\Gamma$  如图: 
$$\mathbf{F} = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y \end{bmatrix}, - \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ z & y \end{bmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ z & x \end{vmatrix}$$

$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1, 1, 1)} \iint_{\Sigma} \sqrt{3} dS$$

=(1, 1, 1)

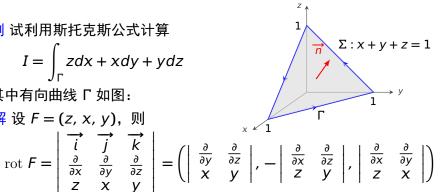
$$= \sqrt{3} \operatorname{Area}(\Sigma) = \sqrt{3} \cdot \frac{1}{2} \cdot \sqrt{2} \cdot \sqrt{2} \cdot \sin \frac{\pi}{3}$$



$$I = \int_{\Gamma} z dx + x dy + y dz$$

解设
$$F = (z, x, y)$$
,则

$$\cot F = \begin{bmatrix} 1 & J & K \\ \frac{\partial}{\partial X} & \frac{\partial}{\partial Y} & \frac{\partial}{\partial Z} \\ Z & X & Y \end{bmatrix}$$

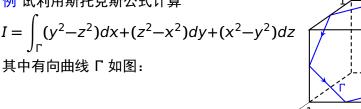


$$=(1, 1, 1)$$

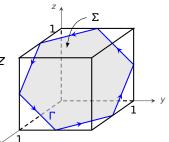
所以
$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)} \iint_{\Sigma} \sqrt{3} dS$$

$$= \sqrt{3}\operatorname{Area}(\Sigma) = \sqrt{3} \cdot \frac{1}{2} \cdot \sqrt{2} \cdot \sqrt{2} \cdot \sin \frac{\pi}{3} = \frac{3}{2}$$

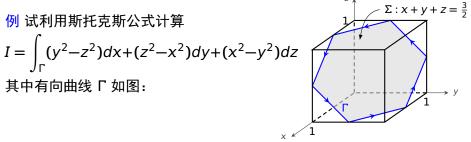
$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$



$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$



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解设 
$$F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$
,则

所以
$$I = \iint_{-\infty} \mathbf{rot} \ F \cdot \overrightarrow{n} \ dS$$



$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$

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 $\Sigma: x + y + z = \frac{3}{2}$ 

$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$

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$$I = \iint_{\Gamma} \operatorname{rot} F \cdot \overrightarrow{n} \, dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)}$$



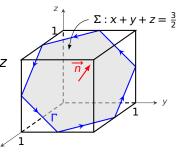
$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$

其中有向曲线 Γ 如图:

解设 
$$F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$
, 则
$$\operatorname{rot} F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix}$$

所以

$$I = \iint_{-\infty} \mathbf{rot} \ F \cdot \overrightarrow{n} \ dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)}$$



$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$

解设 
$$F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$
, 则  $\overrightarrow{i}$   $\overrightarrow{j}$   $\overrightarrow{k}$ 

$$\operatorname{rot} F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix} = (-2y - 2z, \qquad , \qquad )$$

所以

$$I = \iint \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)}$$



$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$

解设 
$$F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$
, 则  $\overrightarrow{i}$   $\overrightarrow{j}$   $\overrightarrow{k}$ 

$$\operatorname{rot} F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix} = (-2y - 2z, -2z - 2x,$$

所以

$$I = \iint_{\Gamma} \operatorname{rot} F \cdot \overrightarrow{n} \, dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1, 1, 1)}$$



$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$

其中有向曲线 Γ如图:

解设 
$$F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$
, 则
$$rot F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix} = (-2y - 2z, -2z - 2x, -2x - 2y)$$

所以

$$I = \iint_{\Gamma} \operatorname{rot} F \cdot \overrightarrow{n} \, dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1, 1, 1)}$$



$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$

解设 
$$F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$
, 则
$$rot F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix} = (-2y - 2z, -2z - 2x, -2x - 2y)$$

$$I = \iint_{\Gamma} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)} \frac{-4}{\sqrt{3}} \iint_{\Gamma} (x+y+z) dS$$



例 试利用斯托克斯公式计算
$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$

解设  $F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$ ,则

rot 
$$F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix} = (-2y - 2z, -2z - 2x, -2x - 2y)$$
所以

所以
$$I = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} \, dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)} \frac{-4}{\sqrt{3}} \iint_{\Sigma} (x+y+z) \, dS = \frac{-4}{\sqrt{3}} \iint_{\Sigma} \frac{3}{2} \, dS$$



 $I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$ 

$$\int_{\Gamma}$$

解设 
$$F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$
,则

$$\operatorname{rot} F = \begin{vmatrix} \frac{i}{\partial x} & \frac{\partial}{\partial x} & \frac{\partial$$

$$\operatorname{rot} F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix} = (-2y - 2z, -2z - 2x, -2x - 2y)$$
所以
$$I = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)} \frac{-4}{\sqrt{3}} \iint_{\Sigma} (x + y + z) dS = \frac{-4}{\sqrt{3}} \iint_{\Sigma} \frac{3}{2} dS$$

$$=-2\sqrt{3}\operatorname{Area}(\Sigma)$$



 $I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$ 

解设
$$F = (v^2 - z^2, z^2 -$$

解设
$$F = (y^2 - z^2)$$

其中有向曲线 Γ 如图:

解设 
$$F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$
,则

$$\operatorname{rot} F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix} = (-2y - 2z, -2z - 2x, -2x - 2y)$$

$$\cot F = \begin{vmatrix} \frac{\partial}{\partial x} \\ y^2 - \end{vmatrix}$$

$$\int_{\partial x} F = \int_{\partial x} \frac{1}{\partial x} y^2 - z^2 z^2$$

所以
$$I = \iint_{\Sigma} \cot F \cdot \overrightarrow{n} \, dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)} \frac{-4}{\sqrt{3}} \iint_{\Sigma} (x+y+z) dS = \frac{-4}{\sqrt{3}} \iint_{\Sigma} \frac{3}{2} dS$$

 $=-2\sqrt{3}\mathrm{Area}(\Sigma)$ 



 $\Sigma : x + y + z = \frac{3}{2}$ 

$$I = \int_{\mathbb{R}} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$

解设 
$$F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$
,则

$$Z$$
 设  $F = (y^2 - z^2, z^2 - x^2, x^2)$ 

rot 
$$F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix} = (-2y - 2z, -2z - 2x, -2x - 2y)$$
所以

 $I = \iint_{\Sigma} \cot F \cdot \overrightarrow{n} \, dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)} \frac{-4}{\sqrt{3}} \iint_{\Sigma} (x+y+z) dS = \frac{-4}{\sqrt{3}} \iint_{\Sigma} \frac{3}{2} dS$  $= -2\sqrt{3}\operatorname{Area}(\Sigma) = -2\sqrt{3}\cdot 6\cdot \frac{1}{2}\cdot \sqrt{\frac{1}{2}\cdot \sqrt{\frac{1}{2}\cdot \sin\frac{\pi}{3}}}$ 



 $\Sigma: x + y + z = \frac{3}{2}$ 

 $\operatorname{rot} F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - v^2 \end{vmatrix} = (-2y - 2z, -2z - 2x, -2x - 2y)$ 

 $=-2\sqrt{3}\operatorname{Area}(\Sigma)=-2\sqrt{3}\cdot 6\cdot \frac{1}{2}\cdot \sqrt{\frac{1}{2}\cdot \sqrt{\frac{1}{2}\cdot \sin\frac{\pi}{3}}}=-\frac{9}{2}$ 

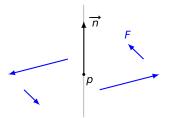
 $I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$ 

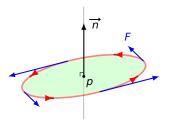
解设  $F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$ ,则

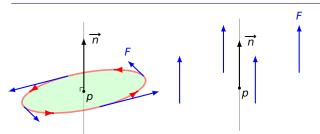
 $\Sigma : x + y + z = \frac{3}{2}$ 

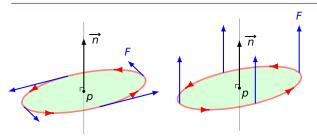
 $I = \iint_{\Sigma} \cot F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)} \frac{-4}{\sqrt{3}} \iint_{\Sigma} (x+y+z) dS = \frac{-4}{\sqrt{3}} \iint_{\Sigma} \frac{3}{2} dS$ 

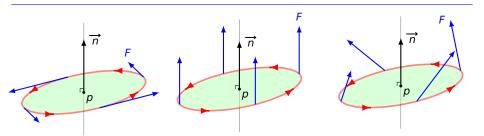


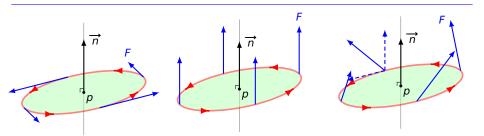


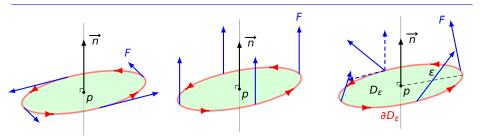


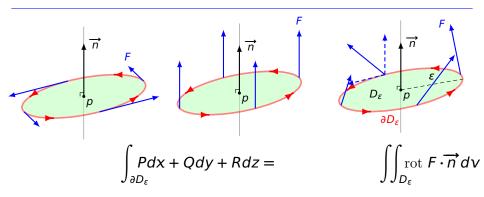


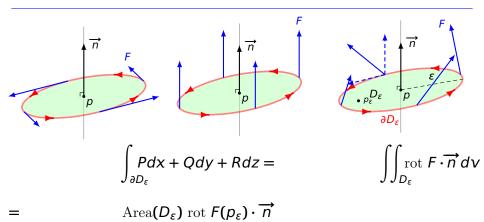


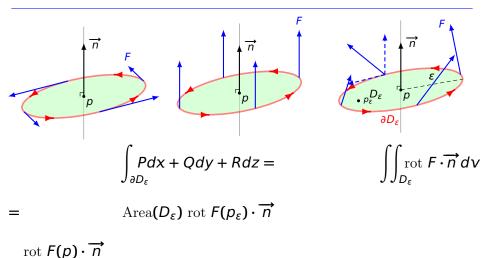


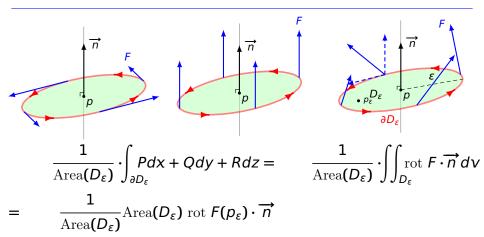




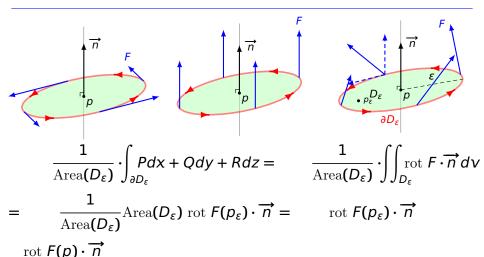




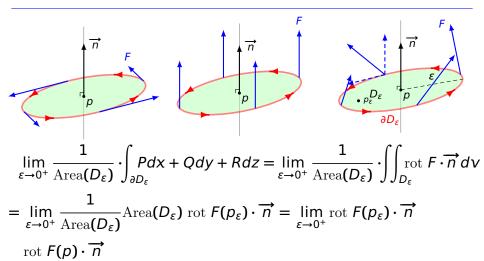




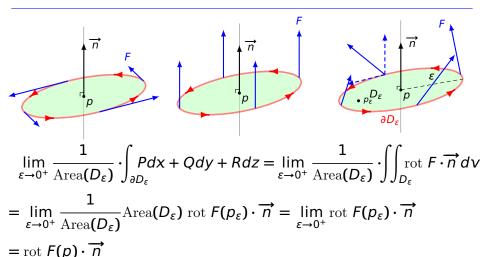




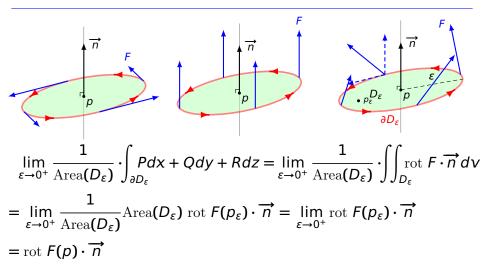


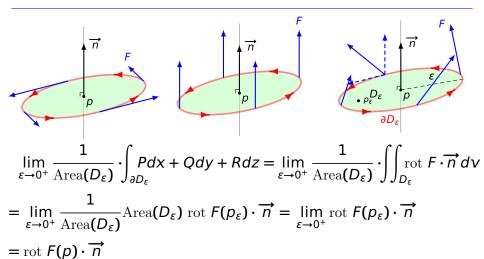




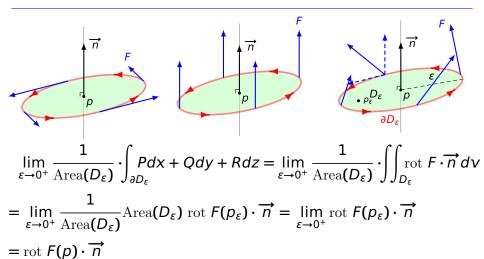








注  $\cot F \neq 0$  说明有旋,此时可认为 F 在 p 点附近绕轴  $\overrightarrow{n} = \frac{\cot F}{|\cot F|}$  旋



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转;  $\cot F = 0$  说明无旋。

20/20 ◀