§3.4 向量组的秩

数学系 梁卓滨

2017 - 2018 学年 I



m维 向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s$

m维 向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s \xrightarrow{\text{挑选}} \alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}$ (部分组)

$$m$$
维
向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s \xrightarrow{\text{挑选}} \alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}$ (部分组)

满足:

- α_{j1}, α_{j2}, ..., α_{jr} 线性无关;且
- 对 α_{i_1} , α_{i_2} , ..., α_{i_r} 再加入任一 α_i 后都是线性相关,

$$m$$
维
向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s \xrightarrow{\text{挑选}} \alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}$ (部分组)

满足:

- α_{j1}, α_{j2}, ..., α_{jr} 线性无关;且
- 对 α_{i_1} , α_{i_2} , ..., α_{i_r} 再加入任一 α_i 后都是线性相关,

("不可扩充"的 线性无关

$$m$$
维
向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s \xrightarrow{\text{挑选}} \alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}$ (部分组)

满足:

- α_{i1}, α_{i2}, ..., α_{ir} 线性无关;且
- 对 α_{i_1} , α_{i_2} , ..., α_{i_r} 再加入任一 α_i 后都是线性相关,

则称 α_{i_1} , α_{i_2} , ..., α_{i_r} 是 α_1 , α_2 , ..., α_s 的一个极大 (线性) 无关组。



("不可扩充"的 线性无关

$$m$$
维
向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s \xrightarrow{\text{挑选}} \alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}$ (部分组)

满足:

- α_{i1}, α_{i2}, ..., α_{ir} 线性无关;且
- 对 α_{j_1} , α_{j_2} , ..., α_{j_r} 再加入任一 α_i 后都是线性相关,

则称 α_{j_1} , α_{j_2} , . . . , α_{j_r} 是 α_1 , α_2 , . . . , α_s 的一个极大 (线性) 无关组。

 $注 r \leq s$



("不可扩充"的 线性无关

$$m$$
维
向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s \xrightarrow{\text{挑选}} \alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}$ (部分组)

满足:

- α_{i1}, α_{i2}, ..., α_{ir}, 线性无关; 且
 "个可扩充"的 线性无关
- 对 α_{j_1} , α_{j_2} , ..., α_{j_r} 再加入任一 α_i 后都是线性相关,

则称 $lpha_{j_1}$, $lpha_{j_2}$, . . . , $lpha_{j_r}$ 是 $lpha_1$, $lpha_2$, . . . , $lpha_s$ 的一个极大 (线性) 无关组。

 $注 r \leq s \perp r \leq m$ 。



$$m$$
维
向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s \xrightarrow{\text{挑选}} \alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}$ (部分组)

满足:

- α_{i1}, α_{i2}, ..., α_{ir}, 线性无关; 且
 "不可扩充"的 线性无关
- 对 α_{j_1} , α_{j_2} , ..., α_{j_r} 再加入任一 α_i 后都是线性相关,

则称 $lpha_{j_1}$, $lpha_{j_2}$, . . . , $lpha_{j_r}$ 是 $lpha_1$, $lpha_2$, . . . , $lpha_s$ 的一个极大 (线性) 无关组。

r < s 且 r < m。

例设
$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, 则极大无关组是:



$$m$$
维
向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s \xrightarrow{\text{挑选}} \alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}$ (部分组)

满足:

- $lpha_{j_1}, lpha_{j_2}, \ldots, lpha_{j_r}$ 线性无关;且 $\begin{pmatrix} \text{"不可扩充"的} \\ \text{线性无关} \end{pmatrix}$
- ullet 对 $lpha_{j_1}$, $lpha_{j_2}$, . . . , $lpha_{j_r}$ 再加入任一 $lpha_i$ 后都是线性相关,

则称 $lpha_{j_1}$, $lpha_{j_2}$, \ldots , $lpha_{j_r}$ 是 $lpha_1$, $lpha_2$, \ldots , $lpha_s$ 的一个极大 (线性) 无关组。

 $注 r \le s 且 r \le m$ 。

例设
$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, 则极大无关组是: α_1 , α_2 ;

$$m$$
维 向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s \stackrel{\text{挑选}}{\longrightarrow} \alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}$ (部分组)

满足:

•
$$\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}$$
线性无关;且 $\begin{pmatrix} \text{"不可扩充"的} \\ \text{线性无关} \end{pmatrix}$

ullet 对 $lpha_{j_1}$, $lpha_{j_2}$, \ldots , $lpha_{j_r}$ 再加入任一 $lpha_i$ 后都是线性相关,

则称 α_{j_1} , α_{j_2} , . . . , α_{j_r} 是 α_1 , α_2 , . . . , α_s 的一个极大 (线性) 无关组。

 $注 r \le s 且 r \le m$ 。

例 设
$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, 则极大无关组是:

 α_1 , α_2 ; α_1 , α_3 ;



$$m$$
维 向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s \stackrel{\text{挑选}}{\longrightarrow} \alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}$ (部分组)

满足:

- $lpha_{j_1}, lpha_{j_2}, \ldots, lpha_{j_r}$ 线性无关;且 $\begin{pmatrix} \text{"不可扩充"的} \\ \text{线性无关} \end{pmatrix}$
- ullet 对 $lpha_{j_1}$, $lpha_{j_2}$, . . . , $lpha_{j_r}$ 再加入任一 $lpha_i$ 后都是线性相关,

则称 $lpha_{j_1}$, $lpha_{j_2}$, . . . , $lpha_{j_r}$ 是 $lpha_1$, $lpha_2$, . . . , $lpha_s$ 的一个极大 (线性) 无关组。

 $注 r \le s 且 r \le m$ 。

例设
$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, 则极大无关组是:

 α_1 , α_2 ; α_1 , α_3 ; α_2 , α_3 ;



$$m$$
维 向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s \stackrel{\text{挑选}}{\longrightarrow} \alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}$ (部分组)

满足:

- $lpha_{j_1}, lpha_{j_2}, \ldots, lpha_{j_r}$ 线性无关;且 $\begin{pmatrix} \text{"不可扩充"的} \\ \text{线性无关} \end{pmatrix}$
- ullet 对 $lpha_{j_1}$, $lpha_{j_2}$, ..., $lpha_{j_r}$ 再加入任一 $lpha_i$ 后都是线性相关,

则称 $lpha_{j_1}$, $lpha_{j_2}$, . . . , $lpha_{j_r}$ 是 $lpha_1$, $lpha_2$, . . . , $lpha_s$ 的一个极大 (线性) 无关组。

 $注 r \leq s \perp r \leq m$ 。

例设
$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, 则极大无关组是:

 α_1 , α_2 ; α_1 , α_3 ; α_2 , α_3 ; α_2 , α_4 ;



$$m$$
维
向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s \xrightarrow{\text{挑选}} \alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}$ (部分组)

满足:

•
$$\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}$$
线性无关;且 $\begin{pmatrix} \text{"不可扩充"的} \\ \text{线性无关} \end{pmatrix}$

ullet 对 $lpha_{j_1}$, $lpha_{j_2}$, ..., $lpha_{j_r}$ 再加入任一 $lpha_i$ 后都是线性相关,

则称 α_{j_1} , α_{j_2} , . . . , α_{j_r} 是 α_1 , α_2 , . . . , α_s 的一个极大 (线性) 无关组。

 $注 r \leq s \perp r \leq m$ 。

例设
$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, 则极大无关组是:

 α_1 , α_2 ; α_1 , α_3 ; α_2 , α_3 ; α_2 , α_4 ; α_3 , α_4



定理 设 α_{i_1} , α_{i_2} , ..., α_{i_r} 是 α_1 , α_2 , ..., α_s 的线性无关部分组,则:

$$\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r} \iff$$
是极大无关组

定理 设 α_{i_1} , α_{i_2} , ..., α_{i_r} 是 α_1 , α_2 , ..., α_s 的线性无关部分组,则:

 $lpha_{j_1}, \, lpha_{j_2}, \, \ldots, \, lpha_{j_r} \qquad \qquad \qquad lpha_1, \, lpha_2, \, \ldots, \, lpha_s$ 中每个向量是极大无关组。 都可由 $lpha_{j_1}, \, lpha_{j_2}, \, \ldots, \, lpha_{j_r}$ 线性表示

定理 设 α_{j_1} , α_{j_2} , ..., α_{j_r} 是 α_1 , α_2 , ..., α_s 的线性无关部分组,则:

$$lpha_{j_1}, \, lpha_{j_2}, \, \ldots, \, lpha_{j_r} \qquad \Longleftrightarrow \qquad lpha_1, \, lpha_2, \, \ldots, \, lpha_s$$
中每个向量是极大无关组 都可由 $lpha_{j_1}, \, lpha_{j_2}, \, \ldots, \, lpha_{j_r}$ 线性表示

证明

定理 设 α_{j_1} , α_{j_2} , ..., α_{j_r} 是 α_1 , α_2 , ..., α_s 的线性无关部分组,则:

$$lpha_{j_1}, \, lpha_{j_2}, \, \ldots, \, lpha_{j_r} \qquad \Longleftrightarrow \qquad lpha_1, \, lpha_2, \, \ldots, \, lpha_s$$
中每个向量是极大无关组 都可由 $lpha_{j_1}, \, lpha_{j_2}, \, \ldots, \, lpha_{j_r}$ 线性表示

证明

"
$$\Rightarrow$$
 " 对任意 α_i 成立: α_{j_1} , α_{j_2} , ..., α_{j_r} , α_i 线性相关

" **_** '

定理 设 α_{j_1} , α_{j_2} , ..., α_{j_r} 是 α_1 , α_2 , ..., α_s 的线性无关部分组,则:

证明

"
$$\Rightarrow$$
 " 对任意 α_i 成立: α_{j_1} , α_{j_2} , ..., α_{j_r} , α_i 线性相关,所以 α_i 是 α_{j_1} , α_{j_2} , ..., α_{j_r} 线性组合

" **=** '



定理 设 α_{j_1} , α_{j_2} , ..., α_{j_r} 是 α_1 , α_2 , ..., α_s 的线性无关部分组,则:

证明

" \Rightarrow " 对任意 α_i 成立: α_{j_1} , α_{j_2} , . . . , α_{j_r} , α_i 线性相关,所以 α_i 是 α_{j_1} , α_{j_2} , . . . , α_{j_r} 线性组合

" ← " 对任意 α_i 成立: α_i 是 α_{i_1} , α_{i_2} , . . . , α_{i_r} 线性组合



定理 设 α_{j_1} , α_{j_2} , ..., α_{j_r} 是 α_1 , α_2 , ..., α_s 的线性无关部分组,则:

证明

" ⇒ " 对任意
$$\alpha_i$$
 成立: α_{j_1} , α_{j_2} , ..., α_{j_r} , α_i 线性相关,所以 α_i 是 α_{j_1} , α_{j_2} , ..., α_{j_r} 线性组合
" ⇐ " 对任意 α_i 成立: α_i 是 α_{j_1} , α_{j_2} , ..., α_{j_r} 线性组合,所以

 $lpha_{j_1},\,lpha_{j_2},\,\ldots,\,lpha_{j_r},\,lpha_i$ 线性相关



定理 极大无关组所包含向量的个数是唯一确定的。

定理 极大无关组所包含向量的个数是唯一确定的。即:若

$$\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}; \qquad \beta_{k_1}, \beta_{k_2}, \ldots, \beta_{k_t}$$

都是 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 的极大无关组,则

定理 极大无关组所包含向量的个数是唯一确定的。即: 若

$$\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}; \qquad \beta_{k_1}, \beta_{k_2}, \ldots, \beta_{k_t}$$

都是 α_1 , α_2 , ..., α_s 的极大无关组,则 r = t

定理 极大无关组所包含向量的个数是唯一确定的。即: 若

$$\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}; \qquad \beta_{k_1}, \beta_{k_2}, \ldots, \beta_{k_t}$$

都是 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 的极大无关组,则 r = t

证明 注意到

• α_{j_1} , α_{j_2} , ..., α_{j_r} 与 β_{k_1} , β_{k_2} , ..., β_{k_t} 等价(相互线性表示);且

定理 极大无关组所包含向量的个数是唯一确定的。即:若

$$\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}; \qquad \beta_{k_1}, \beta_{k_2}, \ldots, \beta_{k_t}$$

都是 α_1 , α_2 , . . . , α_s 的极大无关组,则 r = t

证明 注意到

- α_{j_1} , α_{j_2} , ..., α_{j_r} 与 β_{k_1} , β_{k_2} , ..., β_{k_t} 等价(相互线性表示);且
- α_{j_1} , α_{j_2} , ..., α_{j_r} 与 β_{k_1} , β_{k_2} , ..., β_{k_t} 都是线性无关,

定理 极大无关组所包含向量的个数是唯一确定的。即:若

$$\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}; \qquad \beta_{k_1}, \beta_{k_2}, \ldots, \beta_{k_t}$$

都是 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 的极大无关组,则 r = t

证明 注意到

- α_{j_1} , α_{j_2} , ..., α_{j_r} 与 β_{k_1} , β_{k_2} , ..., β_{k_t} 等价(相互线性表示);且
- α_{j_1} , α_{j_2} , ..., α_{j_r} 与 β_{k_1} , β_{k_2} , ..., β_{k_t} 都是线性无关,

所以r = t



定理 极大无关组所包含向量的个数是唯一确定的。即:若

$$\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}; \qquad \beta_{k_1}, \beta_{k_2}, \ldots, \beta_{k_t}$$

都是 α_1 , α_2 , . . . , α_s 的极大无关组,则 r=t

证明 注意到

- α_{j_1} , α_{j_2} , ..., α_{j_r} 与 β_{k_1} , β_{k_2} , ..., β_{k_t} 等价(相互线性表示);且
- α_{j_1} , α_{j_2} , ..., α_{j_r} 与 β_{k_1} , β_{k_2} , ..., β_{k_t} 都是线性无关,

所以r=t

定义 设向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 的极大无关组所包含向量的个数, 称向量组的秩, 记为:



定理 极大无关组所包含向量的个数是唯一确定的。即:若

$$\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}; \qquad \beta_{k_1}, \beta_{k_2}, \ldots, \beta_{k_t}$$

都是 α_1 , α_2 , . . . , α_s 的极大无关组,则 r=t

证明 注意到

- α_{j_1} , α_{j_2} , ..., α_{j_r} 与 β_{k_1} , β_{k_2} , ..., β_{k_t} 等价(相互线性表示);且
- α_{j_1} , α_{j_2} , ..., α_{j_r} 与 β_{k_1} , β_{k_2} , ..., β_{k_t} 都是线性无关,

所以r = t

定义 设向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 的极大无关组所包含向量的个数,称向量组的秩,记为:

$$r(\alpha_1, \alpha_2, \ldots, \alpha_s)$$



定理 极大无关组所包含向量的个数是唯一确定的。即:若

$$\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}; \qquad \beta_{k_1}, \beta_{k_2}, \ldots, \beta_{k_t}$$

都是 α_1 , α_2 , . . . , α_s 的极大无关组,则 r=t

证明 注意到

- α_{j_1} , α_{j_2} , ..., α_{j_r} 与 β_{k_1} , β_{k_2} , ..., β_{k_t} 等价(相互线性表示);且
- α_{j_1} , α_{j_2} , ..., α_{j_r} 与 β_{k_1} , β_{k_2} , ..., β_{k_t} 都是线性无关,

所以r=t

定义 设向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 的极大无关组所包含向量的个数,称向量组的秩,记为:

$$r(\alpha_1, \alpha_2, \ldots, \alpha_s)$$



例设
$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, 则 $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) =$

例设
$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, 则 $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = 2$



例设
$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, 则 $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \underline{2}$

这是:

例设
$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, 则 $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \underline{2}$

这是:

• α_1 , α_2 是极大无关组,所以极大无关组包含个 2 向量。



例设
$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, 则 $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \underline{2}$

这是:

α₁, α₂ 是极大无关组, 所以极大无关组包含个 2 向量。
 事实上, α₁, α₂ 线性无关,

例设
$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, 则 $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \underline{2}$

这是:

α₁, α₂ 是极大无关组, 所以极大无关组包含个 2 向量。
 事实上, α₁, α₂ 线性无关, 且 α₃, α₄ 均能由 α₁, α₂ 线性表示

例设
$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, 则 $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \underline{2}$

汶是:

 α₁, α₂ 是极大无关组, 所以极大无关组包含个 2 向量。 事实上, α_1 , α_2 线性无关, 且 α_3 , α_4 均能由 α_1 , α_2 线性表示 $\alpha_3 = \alpha_1 + \alpha_2$;

$$\alpha_3 = \alpha_1 + \alpha_2$$
;

例设
$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, 则 $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \underline{2}$

这是:

α₁, α₂ 是极大无关组,所以极大无关组包含个 2 向量。
 事实上,α₁,α₂ 线性无关,且α₃,α₄ 均能由α₁,α₂ 线性表示
 α₃ = α₁ + α₂; α₄ = 2α₂

例设
$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, 则 $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \underline{2}$

这是:

α₁, α₂ 是极大无关组, 所以极大无关组包含个 2 向量。
 事实上, α₁, α₂ 线性无关, 且 α₃, α₄ 均能由 α₁, α₂ 线性表示

或者说:

$$\alpha_3 = \alpha_1 + \alpha_2; \quad \alpha_4 = 2\alpha_2$$

• $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \leq 2$

例设
$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, 则 $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \underline{2}$

这是:

• α_1 , α_2 是极大无关组,所以极大无关组包含个 2 向量。 事实上, α_1 , α_2 线性无关,且 α_3 , α_4 均能由 α_1 , α_2 线性表示

 $\alpha_3 = \alpha_1 + \alpha_2$; $\alpha_4 = 2\alpha_2$

或者说:

•
$$r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \leq 2$$

• 有两个线性无关向量,

例设
$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, 则 $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \underline{2}$

这是:

• α_1 , α_2 是极大无关组,所以极大无关组包含个 2 向量。 事实上, α_1 , α_2 线性无关,且 α_3 , α_4 均能由 α_1 , α_2 线性表示

 $\alpha_3 = \alpha_1 + \alpha_2$; $\alpha_4 = 2\alpha_2$

或者说:

•
$$r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \leq 2$$

• 有两个线性无关向量,如 α_1 , α_2 ,



例设
$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, 则 $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \underline{2}$

这是:

α₁, α₂ 是极大无关组,所以极大无关组包含个 2 向量。
 事实上,α₁,α₂ 线性无关,且α₃,α₄ 均能由α₁,α₂ 线性表示

 $\alpha_3 = \alpha_1 + \alpha_2$; $\alpha_4 = 2\alpha_2$

- $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \leq 2$
- 有两个线性无关向量,如 α_1 , α_2 , 所以 $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \ge 2$



设

$$A_{m \times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

设
$$A_{m \times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

设
$$a_1 \quad a_2$$

$$A_{m \times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

$$A_{m \times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

设
$$A_{m\times n} = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_n \\ \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix} = (\alpha_1 \alpha_2 \cdots \alpha_n)$$

设
$$A_{m \times n} = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_n \\ \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix} = (\alpha_1 \, \alpha_2 \, \cdots \, \alpha_n)$$



设
$$A_{m \times n} = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_n \\ \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix} = (\alpha_1 \, \alpha_2 \, \cdots \, \alpha_n)$$

设
$$A_{m \times n} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix} = (\alpha_1 \alpha_2 \cdots \alpha_n)$$



设
$$A_{m\times n} = \begin{array}{cccc} \beta_1 & \alpha_1 & \alpha_2 & \alpha_n \\ \beta_2 & \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \beta_2 & \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_m & \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{array} \right) = (\alpha_1 \alpha_2 \cdots \alpha_n) = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{pmatrix}$$



设
$$A_{m\times n} = \begin{array}{cccc} \beta_1 & \alpha_1 & \alpha_2 & \alpha_n \\ \beta_2 & \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_m & \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{array} \right) = (\alpha_1 \alpha_2 \cdots \alpha_n) = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{pmatrix}$$

定义

r(α₁, α₂, ..., α_n) 称为 A 的列秩;



设
$$A_{m \times n} = \begin{array}{cccc} \beta_1 & \alpha_1 & \alpha_2 & \alpha_n \\ \beta_2 & \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_m & \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{array} \right) = (\alpha_1 \alpha_2 \cdots \alpha_n) = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{pmatrix}$$

定义

- r(α₁, α₂,..., α_n) 称为 A 的列秩;
- r(β₁, β₂, ..., β_m) 称为 A 的行秩;



设
$$A_{m \times n} = \begin{array}{cccc} \beta_1 & \alpha_1 & \alpha_2 & \alpha_n \\ \beta_2 & \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_m & \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{array} \right) = (\alpha_1 \alpha_2 \cdots \alpha_n) = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{pmatrix}$$

定义

- r(α₁, α₂,..., α_n) 称为 A 的列秩;
- r(β₁, β₂,...,β_m) 称为 A 的行秩;

定理
$$r(A) = r(\alpha_1, \alpha_2, \ldots, \alpha_n) = r(\beta_1, \beta_2, \ldots, \beta_m)$$



§3.4 向量组的秩

问题 给出 m 维的向量组 α_1 , α_2 , \cdots , α_n , 如何求出其一组极大无关组?

步骤

问题 给出 m 维的向量组 α_1 , α_2 , \cdots , α_n , 如何求出其一组极大无关组?

步骤
$$1. A_{m \times n} = \begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_n \\
\alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\
\alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn}
\end{pmatrix}$$

问题 给出 m 维的向量组 α_1 , α_2 , \cdots , α_n , 如何求出其一组极大无关组?

步骤
$$1. \ \, A_{m \times n} = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_n \\ \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix} \xrightarrow{\eta \$ \uparrow - \psi }$$
简化的阶梯型矩阵

问题 给出 m 维的向量组 α_1 , α_2 , \cdots , α_n , 如何求出其一组极大无关组?

步骤
$$1. \ A_{m \times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \xrightarrow{\eta \not s \not f \cdot g \not b}$$
简化的阶梯型矩阵

2. 通过简化的阶梯型矩阵 , 求出 r(A)。

问题 给出 m 维的向量组 $lpha_1$, $lpha_2$, \cdots , $lpha_n$, 如何求出其一组极大无关组 ?

步骤
$$1. \ \, A_{m \times n} = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_n \\ \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix} \xrightarrow{\eta \not \in \widehat{T}_{\mathfrak{D}} \not \to}$$
简化的阶梯型矩阵

2. 通过简化的阶梯型矩阵,求出 r(A)。

利用 $r(\alpha_1, \alpha_2, \dots, \alpha_n) = r(A)$, 得出极大无关组所包含向量的个数

问题 给出 m 维的向量组 $lpha_1$, $lpha_2$, \cdots , $lpha_n$, 如何求出其一组极大无关组 ?

- 3. 通过简化的阶梯型矩阵,容易看出线性无关的 r(A) 列,这就找到一组极大无关组



§3.4 向量组的秩

问题 给出 m 维的向量组 $lpha_1$, $lpha_2$, \cdots , $lpha_n$, 如何求出其一组极大无关组 ?

步骤
$$1. \ A_{m \times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \xrightarrow{\text{初等行变换}}$$
简化的阶梯型矩阵

- 2. 通过简化的阶梯型矩阵 , 求出 r(A)。 利用 $r(\alpha_1,\alpha_2,\ldots,\alpha_n)=r(A)$, 得出极大无关组所包含向量的个数
- 3. 通过简化的阶梯型矩阵,容易看出线性无关的 r(A) 列,这就找到一组极大无关组
- 4. 通过简化的阶梯型矩阵,容易看出其余列如何用极大无关组线性表示



例 求向量组 $\alpha_1 = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$ 的一个极大

例 求向量组
$$\alpha_1 = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$ 的一个极大

例 求向量组
$$\alpha_1 = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$ 的一个极大

$$\begin{pmatrix} 2 & 1 & 2 & 3 \\ 4 & 1 & 3 & 5 \\ 2 & 0 & 1 & 2 \end{pmatrix} \xrightarrow{r_2 - 2r_1} r_3 - r_1$$

例 求向量组
$$\alpha_1 = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$ 的一个极大

$$\begin{pmatrix}
2 & 1 & 2 & 3 \\
4 & 1 & 3 & 5 \\
2 & 0 & 1 & 2
\end{pmatrix}
\xrightarrow[r_3-r_1]{r_2-2r_1}
\begin{pmatrix}
2 & 1 & 2 & 3 \\
0 & -1 & -1 & -1 \\
0 & -1 & -1 & -1
\end{pmatrix}
\longrightarrow$$

例 求向量组
$$\alpha_1 = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$ 的一个极大

$$\begin{pmatrix} 2 & 1 & 2 & 3 \\ 4 & 1 & 3 & 5 \\ 2 & 0 & 1 & 2 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 2 & 1 & 2 & 3 \\ 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

例 求向量组
$$\alpha_1 = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$ 的一个极大

$$\begin{pmatrix}
2 & 1 & 2 & 3 \\
4 & 1 & 3 & 5 \\
2 & 0 & 1 & 2
\end{pmatrix}
\xrightarrow[r_3-r_1]{r_2-2r_1}
\begin{pmatrix}
2 & 1 & 2 & 3 \\
0 & -1 & -1 & -1 \\
0 & -1 & -1 & -1
\end{pmatrix}
\longrightarrow
\begin{pmatrix}
2 & 1 & 2 & 3 \\
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

$$r_1-r_2$$



例 求向量组
$$\alpha_1 = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$ 的一个极大



例 求向量组
$$\alpha_1 = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$ 的一个极大



例 求向量组
$$\alpha_1 = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$ 的一个极大



例 求向量组
$$\alpha_1 = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$ 的一个极大



例 求向量组
$$\alpha_1 = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$ 的一个极大

$$\begin{pmatrix}
2 & 1 & 2 & 3 \\
4 & 1 & 3 & 5 \\
2 & 0 & 1 & 2
\end{pmatrix}
\xrightarrow{r_2-2r_1}
\begin{pmatrix}
2 & 1 & 2 & 3 \\
0 & -1 & -1 & -1 \\
0 & -1 & -1 & -1
\end{pmatrix}
\longrightarrow
\begin{pmatrix}
2 & 1 & 2 & 3 \\
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

$$\xrightarrow{r_1-r_2}
\begin{pmatrix}
2 & 0 & 1 & 2 \\
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\xrightarrow{\frac{1}{2}\times r_1}
\begin{pmatrix}
1 & 0 & \frac{1}{2} & 1 \\
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

• $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2$;

B A ANN UNIVERSITY

例 求向量组
$$\alpha_1 = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$ 的一个极大

•
$$r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2$$
;

B A ANN UNIVERSITY

例 求向量组
$$\alpha_1 = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$ 的一个极大

$$\begin{pmatrix}
2 & 1 & 2 & 3 \\
4 & 1 & 3 & 5 \\
2 & 0 & 1 & 2
\end{pmatrix}
\xrightarrow[r_3-r_1]{r_2-2r_1}
\begin{pmatrix}
2 & 1 & 2 & 3 \\
0 & -1 & -1 & -1 \\
0 & -1 & -1 & -1
\end{pmatrix}
\longrightarrow
\begin{pmatrix}
2 & 1 & 2 & 3 \\
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

$$\xrightarrow{r_1-r_2} \left(\begin{array}{cccc} 2 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\frac{1}{2} \times r_1} \left(\begin{array}{cccc} 1 & 0 & \frac{1}{2} & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

所以

- $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2$;
- α₁, α₂ 是极大无关组;

例 求向量组
$$\alpha_1 = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$ 的一个极大

$$\begin{pmatrix}
2 & 1 & 2 & 3 \\
4 & 1 & 3 & 5 \\
2 & 0 & 1 & 2
\end{pmatrix}
\xrightarrow[r_3-r_1]{r_2-2r_1}
\begin{pmatrix}
2 & 1 & 2 & 3 \\
0 & -1 & -1 & -1 \\
0 & -1 & -1 & -1
\end{pmatrix}
\longrightarrow
\begin{pmatrix}
2 & 1 & 2 & 3 \\
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

$$\xrightarrow{r_1-r_2} \left(\begin{array}{cccc} 2 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\frac{1}{2} \times r_1} \left(\begin{array}{cccc} 1 & 0 & \frac{1}{2} & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

所以

- $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2;$
- α₁, α₂ 是极大无关组;
- $\alpha_3 = \frac{1}{2}\alpha_1 + \alpha_2$, $\alpha_4 = \alpha_1 + \alpha_2$

例 求向量组
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一个

例 求向量组
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一个

例 求向量组
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一个

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} r_3 - r_1$$

例 求向量组
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一个

$$\begin{pmatrix}
1 & 0 & 1 & 2 \\
2 & 1 & 1 & 4 \\
1 & 1 & 0 & 3 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow[r_3-r_1]{r_2-2r_1}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 1 & -1 & 1 \\
0 & 2 & -2 & 3
\end{pmatrix}$$

例 求向量组
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一个

$$\begin{pmatrix}
1 & 0 & 1 & 2 \\
2 & 1 & 1 & 4 \\
1 & 1 & 0 & 3 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow[r_3-r_1]{r_2-2r_1}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 1 & -1 & 1 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow[r_4-2r_2]{r_3-r_2}$$

例 求向量组
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一个

$$\begin{pmatrix}
1 & 0 & 1 & 2 \\
2 & 1 & 1 & 4 \\
1 & 1 & 0 & 3 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow[r_3-r_1]{r_2-2r_1}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 1 & -1 & 1 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow[r_4-2r_2]{r_3-r_2}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 3
\end{pmatrix}$$



例 求向量组
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一个

$$\begin{pmatrix}
1 & 0 & 1 & 2 \\
2 & 1 & 1 & 4 \\
1 & 1 & 0 & 3 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow[r_3-r_1]{r_2-2r_1}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 1 & -1 & 1 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow[r_4-2r_2]{r_3-r_2}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 3
\end{pmatrix}$$

$$-3r_{3}$$



例 求向量组
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一个

$$\begin{pmatrix}
1 & 0 & 1 & 2 \\
2 & 1 & 1 & 4 \\
1 & 1 & 0 & 3 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow[r_3-r_1]{r_2-2r_1}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 1 & -1 & 1 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow[r_4-2r_2]{r_3-r_2}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 3
\end{pmatrix}$$

$$\xrightarrow{r_4-3r_3} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

例 求向量组
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一个

$$\begin{pmatrix}
1 & 0 & 1 & 2 \\
2 & 1 & 1 & 4 \\
1 & 1 & 0 & 3 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow[r_3-r_1]{r_2-2r_1}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 1 & -1 & 1 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow[r_4-2r_2]{r_3-r_2}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 3
\end{pmatrix}$$

$$\xrightarrow{r_4-3r_3} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

例 求向量组
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一个

$$\begin{pmatrix}
1 & 0 & 1 & 2 \\
2 & 1 & 1 & 4 \\
1 & 1 & 0 & 3 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow{r_2-2r_1}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 1 & -1 & 1 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow{r_3-r_2}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 1 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow{r_4-2r_2}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 3
\end{pmatrix}$$

$$\xrightarrow{r_4 - 3r_3} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

所以



例 求向量组
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一个

$$\begin{pmatrix}
1 & 0 & 1 & 2 \\
2 & 1 & 1 & 4 \\
1 & 1 & 0 & 3 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow[r_3-r_1]{r_2-2r_1}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 1 & -1 & 1 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow[r_4-2r_2]{r_3-r_2}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 3
\end{pmatrix}$$

$$\xrightarrow{r_4 - 3r_3} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

所以
•
$$r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 3$$
;

例 求向量组
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一个

$$\begin{pmatrix}
1 & 0 & 1 & 2 \\
2 & 1 & 1 & 4 \\
1 & 1 & 0 & 3 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow[r_3-r_1]{r_2-2r_1}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 1 & -1 & 1 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow[r_4-2r_2]{r_3-r_2}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 3
\end{pmatrix}$$

$$\xrightarrow{r_4 - 3r_3} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

所以
•
$$r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 3$$
;

例 求向量组
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一个

$$\begin{pmatrix}
1 & 0 & 1 & 2 \\
2 & 1 & 1 & 4 \\
1 & 1 & 0 & 3 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow[r_3-r_1]{r_2-2r_1}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 1 & -1 & 1 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow[r_4-2r_2]{r_3-r_2}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 3
\end{pmatrix}$$

$$\xrightarrow{r_4-3r_3} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

所以

$$r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 3;$$

α₁, α₂, α₄ 是极大无关组;



例 求向量组
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一个极大无关组;并把其余向量用该极大无关组线性表示。

$$\begin{pmatrix}
1 & 0 & 1 & 2 \\
2 & 1 & 1 & 4 \\
1 & 1 & 0 & 3 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow{r_2-2r_1}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 1 & -1 & 1 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow{r_3-r_2}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}$$

$$\xrightarrow{r_4-3r_3}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

所以
•
$$r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 3$$
:

$$\bullet \ \alpha_3 = \alpha_1 - \alpha_2$$



例 求向量组
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个极

例 求向量组
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个极

$$\begin{pmatrix}
 1 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\
 1 & 2 & 3 & 4 \\
 2 & 3 & 4 & 5 \\
 3 & 4 & 5 & 6 \\
 4 & 5 & 6 & 7
\end{pmatrix}$$

例 求向量组
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个极

$$\begin{pmatrix}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 5 \\
3 & 4 & 5 & 6 \\
4 & 5 & 6 & 7
\end{pmatrix}
\xrightarrow[r_4-4r_1]{r_2-2r_1}
\xrightarrow[r_4-4r_1]{r_2-3r_1}$$

例 求向量组
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个极

$$\begin{pmatrix}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 5 \\
3 & 4 & 5 & 6 \\
4 & 5 & 6 & 7
\end{pmatrix}
\xrightarrow[r_4-4r_1]{r_2-2r_1}
\begin{pmatrix}
1 & 2 & 3 & 4 \\
0 & -1 & -2 & -3 \\
0 & -2 & -4 & -6 \\
0 & -3 & -6 & -9
\end{pmatrix}$$

例 求向量组
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个极

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \xrightarrow[r_4-4r_1]{r_2-2r_1} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{pmatrix}$$

$$r_3-2r_2$$

 r_4-3r_2

例 求向量组
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个极

$$\begin{pmatrix}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 5 \\
3 & 4 & 5 & 6 \\
4 & 5 & 6 & 7
\end{pmatrix}
\xrightarrow[r_4-3r_2]{r_2-2r_1}
\begin{pmatrix}
1 & 2 & 3 & 4 \\
0 & -1 & -2 & -3 \\
0 & -2 & -4 & -6 \\
0 & -3 & -6 & -9
\end{pmatrix}$$

$$\xrightarrow[r_4-3r_2]{r_4-3r_2}
\begin{pmatrix}
1 & 2 & 3 & 4 \\
0 & -1 & -2 & -3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

例 求向量组
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个极

$$\begin{pmatrix}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 5 \\
3 & 4 & 5 & 6 \\
4 & 5 & 6 & 7
\end{pmatrix}
\xrightarrow[r_4-3r_2]{r_2-2r_1}
\begin{pmatrix}
1 & 2 & 3 & 4 \\
0 & -1 & -2 & -3 \\
0 & -2 & -4 & -6 \\
0 & -3 & -6 & -9
\end{pmatrix}$$

$$\xrightarrow[r_3-2r_2]{r_4-3r_2}
\begin{pmatrix}
1 & 2 & 3 & 4 \\
0 & -1 & -2 & -3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\longrightarrow
\begin{pmatrix}
1 & 0 & -1 & -2 \\
0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

例 求向量组
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个极

$$\begin{pmatrix}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 5 \\
3 & 4 & 5 & 6 \\
4 & 5 & 6 & 7
\end{pmatrix}
\xrightarrow[r_{4}-3r_{2}]{r_{2}-2r_{1}}
\begin{pmatrix}
1 & 2 & 3 & 4 \\
0 & -1 & -2 & -3 \\
0 & -2 & -4 & -6 \\
0 & -3 & -6 & -9
\end{pmatrix}$$

$$\xrightarrow[r_{3}-2r_{2}]{r_{4}-3r_{2}}
\begin{pmatrix}
1 & 2 & 3 & 4 \\
0 & -1 & -2 & -3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\longrightarrow
\begin{pmatrix}
1 & 0 & -1 & -2 \\
0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}$$



例 求向量组
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个极

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \xrightarrow[r_4-4r_1]{r_2-2r_1} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{pmatrix}$$

$$\xrightarrow[r_4-3r_2]{r_4-3r_2} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

所以
•
$$r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2$$
;

例 求向量组
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个极

$$\begin{pmatrix}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 5 \\
3 & 4 & 5 & 6 \\
4 & 5 & 6 & 7
\end{pmatrix}
\xrightarrow[r_4-3r_2]{r_2-2r_1}
\begin{pmatrix}
1 & 2 & 3 & 4 \\
0 & -1 & -2 & -3 \\
0 & -2 & -4 & -6 \\
0 & -3 & -6 & -9
\end{pmatrix}$$

$$\xrightarrow[r_4-3r_2]{r_4-3r_2}
\begin{pmatrix}
1 & 2 & 3 & 4 \\
0 & -1 & -2 & -3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\longrightarrow
\begin{pmatrix}
1 & 0 & -1 & -2 \\
0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

所以
• $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2$;

例 求向量组
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个极

$$\begin{pmatrix}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 5 \\
3 & 4 & 5 & 6 \\
4 & 5 & 6 & 7
\end{pmatrix}
\xrightarrow[r_4-4r_1]{r_2-2r_1}
\begin{pmatrix}
1 & 2 & 3 & 4 \\
0 & -1 & -2 & -3 \\
0 & -2 & -4 & -6 \\
0 & -3 & -6 & -9
\end{pmatrix}$$

$$\xrightarrow[r_4-3r_2]{r_4-3r_2}
\begin{pmatrix}
1 & 2 & 3 & 4 \\
0 & -1 & -2 & -3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\longrightarrow
\begin{pmatrix}
1 & 0 & -1 & -2 \\
0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

- $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2;$
 - α₁, α₂ 是极大无关组;

例求向量组 $\alpha_1 = \begin{pmatrix} 1\\2\\3\\4 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2\\3\\4\\5 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3\\4\\5\\6 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 4\\5\\6\\7 \end{pmatrix}$ 的一个极

$$\begin{pmatrix}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 5 \\
3 & 4 & 5 & 6 \\
4 & 5 & 6 & 7
\end{pmatrix}
\xrightarrow[r_{3}-3r_{1}]{r_{3}-3r_{1}}
\begin{pmatrix}
1 & 2 & 3 & 4 \\
0 & -1 & -2 & -3 \\
0 & -2 & -4 & -6 \\
0 & -3 & -6 & -9
\end{pmatrix}$$

$$\xrightarrow[r_{3}-2r_{2}]{r_{4}-3r_{2}}
\begin{pmatrix}
1 & 2 & 3 & 4 \\
0 & -1 & -2 & -3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\longrightarrow
\begin{pmatrix}
1 & 0 & -1 & -2 \\
0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

所以
•
$$r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2$$
;

$$\alpha_3 = -\alpha_1 + 2\alpha_2$$



例 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个极

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \xrightarrow[r_4-4r_1]{r_2-2r_1} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{pmatrix}$$

$$r_{3-2r_{2}}$$
 $r_{4-3r_{2}}$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ \longrightarrow $\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ 所以

• $r(\alpha_{1}, \alpha_{2}, \alpha_{3}, \alpha_{4}) = r(A) = 2$;

•
$$\alpha_3 = -\alpha_1 + 2\alpha_2$$
, $\alpha_4 = -2\alpha_1 + 3\alpha_2$



证明设

$$r_1 = r(\alpha_1, \alpha_2, \dots, \alpha_s),$$

$$r_2 = r(\beta_1, \beta_2, \dots, \beta_t),$$

$$r_1 = r(\alpha_1, \alpha_2, \ldots, \alpha_s), \quad \alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_{r_1}}$$
 是极大无关组 $r_2 = r(\beta_1, \beta_2, \ldots, \beta_t),$

例 假设向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 可由 $\beta_1, \beta_2, \ldots, \beta_t$ 线性表示,则 $r(\alpha_1, \alpha_2, \ldots, \alpha_s) \leq r(\beta_1, \beta_2, \ldots, \beta_t).$

$$r_1 = r(\alpha_1, \alpha_2, \dots, \alpha_s), \quad \alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_{r_1}}$$
 是极大无关组 $r_2 = r(\beta_1, \beta_2, \dots, \beta_t), \quad \beta_{j_1}, \beta_{j_2}, \dots, \beta_{j_{r_2}}$ 是极大无关组

例 假设向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 可由 $\beta_1, \beta_2, \ldots, \beta_t$ 线性表示,则 $r(\alpha_1, \alpha_2, \ldots, \alpha_s) \leq r(\beta_1, \beta_2, \ldots, \beta_t).$

$$r_1 = r(\alpha_1, \alpha_2, \ldots, \alpha_s), \quad \alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_{r_1}}$$
 是极大无关组 $r_2 = r(\beta_1, \beta_2, \ldots, \beta_t), \quad \beta_{j_1}, \beta_{j_2}, \ldots, \beta_{j_{r_2}}$ 是极大无关组 注意到 $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_{r_1}}$ 能由 $\beta_{j_1}, \beta_{j_2}, \ldots, \beta_{j_{r_2}}$ 线性表示,

$$r_1 = r(\alpha_1, \alpha_2, ..., \alpha_s)$$
, $\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_{r_1}}$ 是极大无关组 $r_2 = r(\beta_1, \beta_2, ..., \beta_t)$, $\beta_{j_1}, \beta_{j_2}, ..., \beta_{j_{r_2}}$ 是极大无关组 注意到 $\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_{r_1}}$ 能由 $\beta_{j_1}, \beta_{j_2}, ..., \beta_{j_{r_2}}$ 线性表示,所以 $r_1 \leq r_2$ 。

例 假设向量组 α_1 , α_2 , ..., α_s 可由 β_1 , β_2 , ..., β_t 线性表示,则 $r(\alpha_1, \alpha_2, ..., \alpha_s) \leq r(\beta_1, \beta_2, ..., \beta_t).$

证明设

$$r_1 = r(\alpha_1, \alpha_2, ..., \alpha_s), \quad \alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_{r_1}}$$
 是极大无关组 $r_2 = r(\beta_1, \beta_2, ..., \beta_t), \quad \beta_{j_1}, \beta_{j_2}, ..., \beta_{j_{r_2}}$ 是极大无关组 注意到 $\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_{r_1}}$ 能由 $\beta_{j_1}, \beta_{j_2}, ..., \beta_{j_{r_2}}$ 线性表示,所以 $r_1 \leq r_2$ 。

定理 设有向量组 (A): $\alpha_1, \alpha_2, \ldots, \alpha_s$ (B): $\beta_1, \beta_2, \ldots, \beta_t$

若它们等价.

例 假设向量组 α_1 , α_2 , ..., α_s 可由 β_1 , β_2 , ..., β_t 线性表示,则 $r(\alpha_1, \alpha_2, ..., \alpha_s) \leq r(\beta_1, \beta_2, ..., \beta_t)$.

证明设

$$r_1 = r(\alpha_1, \alpha_2, ..., \alpha_s)$$
, $\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_{r_1}}$ 是极大无关组 $r_2 = r(\beta_1, \beta_2, ..., \beta_t)$, $\beta_{j_1}, \beta_{j_2}, ..., \beta_{j_{r_2}}$ 是极大无关组 注意到 $\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_{r_1}}$ 能由 $\beta_{j_1}, \beta_{j_2}, ..., \beta_{j_{r_2}}$ 线性表示,所以 $r_1 \leq r_2$ 。

定理 设有向量组
$$(A)$$
: $\alpha_1, \alpha_2, \ldots, \alpha_s$ (B) : $\beta_1, \beta_2, \ldots, \beta_t$

若它们等价,则 $r(\alpha_1, \alpha_2, \ldots, \alpha_s) = r(\beta_1, \beta_2, \ldots, \beta_t)$ 。



$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}$$

证明 设
$$AB = C_{m \times s}$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}$$

证明 设
$$AB = C_{m \times s}$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}$$

证明 设
$$AB = C_{m \times s}$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}$$

证明 设
$$AB = C_{m \times s}$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} \alpha_{11} & \alpha_{2} & \alpha_{n} \\ \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}$$

证明 设
$$AB = C_{m \times s}$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}$$

$$(\gamma_1 \ \gamma_2 \cdots \ \gamma_s) = (\alpha_1 \ \alpha_2 \cdots \ \alpha_n) \begin{pmatrix} b_{11} \ b_{12} \cdots b_{1s} \\ b_{21} \ b_{22} \cdots b_{2s} \\ \vdots \ \vdots \ \ddots \ \vdots \\ b_{n1} \ b_{n2} \cdots b_{ns} \end{pmatrix}$$



证明 设
$$AB = C_{m \times s}$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}$$

即

$$(\gamma_1 \ \gamma_2 \cdots \ \gamma_s) = (\alpha_1 \ \alpha_2 \cdots \ \alpha_n) \begin{pmatrix} b_{11} \ b_{12} \cdots b_{1s} \\ b_{21} \ b_{22} \cdots b_{2s} \\ \vdots \ \vdots \ \ddots \ \vdots \\ b_{n1} \ b_{n2} \cdots b_{ns} \end{pmatrix}$$

$$\Rightarrow \quad \gamma_1 = b_{11}\alpha_1 + b_{21}\alpha_2 + \dots + b_{n1}\alpha_n$$

证明 设
$$AB = C_{m \times s}$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}$$

即

$$(\gamma_1 \ \gamma_2 \cdots \ \gamma_s) = (\alpha_1 \ \alpha_2 \cdots \ \alpha_n) \begin{pmatrix} b_{11} \ b_{12} \cdots b_{1s} \\ b_{21} \ b_{22} \cdots b_{2s} \\ \vdots \ \vdots \ \ddots \ \vdots \\ b_{n1} \ b_{n2} \cdots b_{ns} \end{pmatrix}$$

$$\Rightarrow$$
 $\gamma_1 = b_{11}\alpha_1 + b_{21}\alpha_2 + \cdots + b_{n1}\alpha_n$ 等等

证明 设
$$AB = C_{m \times s}$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}$$

即

$$(\gamma_1 \ \gamma_2 \cdots \ \gamma_s) = (\alpha_1 \ \alpha_2 \cdots \ \alpha_n) \begin{pmatrix} b_{11} \ b_{12} \cdots b_{1s} \\ b_{21} \ b_{22} \cdots b_{2s} \\ \vdots \ \vdots \ \ddots \ \vdots \\ b_{n1} \ b_{n2} \cdots b_{ns} \end{pmatrix}$$

$$\Rightarrow \quad \gamma_1 = b_{11}\alpha_1 + b_{21}\alpha_2 + \dots + b_{n1}\alpha_n \quad 等等$$

可见 $\gamma_1, \ldots, \gamma_s$ 由 $\alpha_1, \ldots, \alpha_s$ 线性表示,

证明 设
$$AB = C_{m \times s}$$

$$\underbrace{\begin{pmatrix}
c_{11} & c_{12} & \cdots & c_{1s} \\
c_{21} & c_{22} & \cdots & c_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
c_{m1} & c_{m2} & \cdots & c_{ms}
\end{pmatrix}}_{C} = \underbrace{\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}}_{A} \begin{pmatrix}
b_{11} & b_{12} & \cdots & b_{1s} \\
b_{21} & b_{22} & \cdots & b_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
b_{n1} & b_{n2} & \cdots & b_{ns}
\end{pmatrix}}_{B}$$

即

$$(\gamma_1 \ \gamma_2 \cdots \gamma_s) = (\alpha_1 \ \alpha_2 \cdots \ \alpha_n) \begin{pmatrix} b_{11} \ b_{12} \cdots b_{1s} \\ b_{21} \ b_{22} \cdots b_{2s} \\ \vdots \ \vdots \ \ddots \ \vdots \\ b_{n1} \ b_{n2} \cdots b_{ns} \end{pmatrix}$$

可见
$$\gamma_1, \ldots, \gamma_s$$
 由 $\alpha_1, \ldots, \alpha_s$ 线性表示,所以

$$r(\gamma_1, \ldots, \gamma_s) \leq r(\alpha_1, \ldots, \alpha_n)$$

 $\gamma_1 = b_{11}\alpha_1 + b_{21}\alpha_2 + \cdots + b_{n1}\alpha_n$ 等等



证明 设
$$AB = C_{m \times s}$$

$$\underbrace{\begin{pmatrix}
c_{11} & c_{12} & \cdots & c_{1s} \\
c_{21} & c_{22} & \cdots & c_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
c_{m1} & c_{m2} & \cdots & c_{ms}
\end{pmatrix}}_{C} = \underbrace{\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}}_{A} \begin{pmatrix}
b_{11} & b_{12} & \cdots & b_{1s} \\
b_{21} & b_{22} & \cdots & b_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
b_{n1} & b_{n2} & \cdots & b_{ns}
\end{pmatrix}}_{B}$$

即

$$(\gamma_1 \ \gamma_2 \cdots \gamma_s) = (\alpha_1 \ \alpha_2 \cdots \alpha_n) \begin{pmatrix} b_{11} \ b_{12} \cdots b_{1s} \\ b_{21} \ b_{22} \cdots b_{2s} \\ \vdots \ \vdots \ \ddots \ \vdots \\ b_{n1} \ b_{n2} \cdots b_{ns} \end{pmatrix}$$

 $\gamma_1 = b_{11}\alpha_1 + b_{21}\alpha_2 + \cdots + b_{n1}\alpha_n$ 等等

可见
$$\gamma_1, \ldots, \gamma_s$$
 由 $\alpha_1, \ldots, \alpha_s$ 线性表示,所以

可见
$$\gamma_1,\ldots,\gamma_s$$
 由 $lpha_1,\ldots,lpha_s$ 线性表示,所以

$$r(\gamma_1, \ldots, \gamma_s) \le r(\alpha_1, \ldots, \alpha_n) = r(A)$$

证明 设
$$AB = C_{mxs}$$

$$\underbrace{\begin{pmatrix}
c_{11} & c_{12} & \cdots & c_{1s} \\
c_{21} & c_{22} & \cdots & c_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
c_{m1} & c_{m2} & \cdots & c_{ms}
\end{pmatrix}}_{C} = \underbrace{\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}}_{A} \begin{pmatrix}
b_{11} & b_{12} & \cdots & b_{1s} \\
b_{21} & b_{22} & \cdots & b_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
b_{n1} & b_{n2} & \cdots & b_{ns}
\end{pmatrix}}_{B}$$

即

$$(\gamma_1 \ \gamma_2 \cdots \gamma_s) = (\alpha_1 \ \alpha_2 \cdots \alpha_n) \begin{pmatrix} b_{11} \ b_{12} \cdots b_{1s} \\ b_{21} \ b_{22} \cdots b_{2s} \\ \vdots \ \vdots \ \ddots \ \vdots \\ b_{n1} \ b_{n2} \cdots b_{ns} \end{pmatrix}$$

$$\Rightarrow \gamma_1 = b_{11}\alpha_1 + b_{21}\alpha_2 + \cdots + b_{n1}\alpha_n \quad \text{\reff}$$

可见 $\gamma_1, \ldots, \gamma_s$ 由 $\alpha_1, \ldots, \alpha_s$ 线性表示,所以

$$r(AB) = r(\gamma_1, \ldots, \gamma_s) \le r(\alpha_1, \ldots, \alpha_n) = r(A)$$



$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}^{\beta_{1}}$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}^{\beta_{1}}$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{\beta_{n}}^{\beta_{1}}$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{\beta_{n}}^{\beta_{1}}$$

$$\underbrace{\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{bmatrix}}_{C} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{bmatrix}}_{B}^{\beta_{1}}$$

$$\underbrace{\begin{bmatrix}
c_{11} & c_{12} & \cdots & c_{1s} \\
c_{21} & c_{22} & \cdots & c_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
c_{m1} & c_{m2} & \cdots & c_{ms}
\end{bmatrix}}_{C} = \underbrace{\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}}_{A} \underbrace{\begin{bmatrix}
b_{11} & b_{12} & \cdots & b_{1s} \\
b_{21} & b_{22} & \cdots & b_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
b_{n1} & b_{n2} & \cdots & b_{ns}
\end{bmatrix}}_{\beta_{n}}^{\beta_{1}}$$

$$\frac{\delta_{1}}{\delta_{2}} \begin{pmatrix} c_{11} & c_{12} \cdots c_{1s} \\ c_{21} & c_{22} \cdots c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} \cdots c_{ms} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} \cdots a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \cdots b_{1s} \\ b_{21} & b_{22} \cdots b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} \cdots b_{ns} \end{pmatrix} \beta_{n}$$

$$\frac{\delta_{1}}{\delta_{2}} \left(\begin{array}{ccc} C_{11} & C_{12} & \cdots & C_{1s} \\ C_{21} & C_{22} & \cdots & C_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ C_{m1} & C_{m2} & \cdots & C_{ms} \end{array} \right) = \left(\begin{array}{ccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{array} \right) \left(\begin{array}{ccc} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{array} \right) \beta_{1}$$

$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$$

$$\Rightarrow \quad \delta_1 = a_{11}\beta_1 + a_{12}\beta_2 + \dots + a_{1n}\beta_n$$

证明 设 $AB = C_{mxs}$

$$\Rightarrow \delta_1 = a_{11}\beta_1 + a_{12}\beta_2 + \cdots + a_{1n}\beta_n \quad \text{(Signature)}$$

$$\Rightarrow \quad \delta_1 = a_{11}\beta_1 + a_{12}\beta_2 + \dots + a_{1n}\beta_n \quad 等等$$

证明 设 $AB = C_{m \times s}$

$$\frac{\delta_{1}}{\delta_{2}} \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix} \begin{pmatrix} \beta_{1} \\ \beta_{n} \\ \beta_$$

即

$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$$

$$\Rightarrow \quad \delta_1 = \alpha_{11}\beta_1 + \alpha_{12}\beta_2 + \dots + \alpha_{1n}\beta_n \quad \text{\mathfrak{F}}$$

可见 $\delta_1, \ldots, \delta_m$ 由 β_1, \ldots, β_s 线性表示,

证明 设 $AB = C_{m \times s}$

$$\underbrace{ \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{bmatrix}}_{C} = \underbrace{ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{A} \underbrace{ \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{bmatrix}}_{\beta_{n}} _{\beta_{n}}$$

即

$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$$

$$\Rightarrow \quad \delta_1 = a_{11}\beta_1 + a_{12}\beta_2 + \cdots + a_{1n}\beta_n \quad \text{\$}$$

可见 $\delta_1, \ldots, \delta_m$ 由 β_1, \ldots, β_s 线性表示,所以

$$r(\delta_1, \ldots, \delta_m) \leq r(\beta_1, \ldots, \beta_n)$$

证明 设 $AB = C_{m \times s}$

$$\underbrace{ \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{bmatrix}}_{C} = \underbrace{ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{A} \underbrace{ \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{bmatrix}}_{B}^{\beta_{1}}$$

即

$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$$

$$\Rightarrow \quad \delta_1 = a_{11}\beta_1 + a_{12}\beta_2 + \dots + a_{1n}\beta_n \quad$$
 等等

可见 $\delta_1, \ldots, \delta_m$ 由 β_1, \ldots, β_s 线性表示,所以

$$r(\delta_1, \ldots, \delta_m) \le r(\beta_1, \ldots, \beta_n) = r(B)$$

证明 设 $AB = C_{m \times s}$

$$\frac{\delta_{1}}{\delta_{2}} \underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{\beta_{n}}_{\beta_{n}}$$

即

$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$$

$$\Rightarrow \quad \delta_1 = a_{11}\beta_1 + a_{12}\beta_2 + \dots + a_{1n}\beta_n \quad$$
 等等

可见 $\delta_1, \ldots, \delta_m$ 由 β_1, \ldots, β_s 线性表示,所以

$$r(AB) = r(\delta_1, \ldots, \delta_m) \le r(\beta_1, \ldots, \beta_n) = r(B)$$