第 3 章 d：向量组的秩

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2019-2020 学年 I

向量组的极大无关组

$\alpha_1, \alpha_2, \ldots, \alpha_s$

向量组的极大无关组

$$\alpha_1, \alpha_2, \ldots, \alpha_s \xrightarrow[\text{能被其余向量线性表示的向量}]{\text{逐个剔除}}$$

向量组的极大无关组

$$\alpha_1, \alpha_2, \ldots, \alpha_s \xrightarrow[\text{直到不能再剔除为止}]{\substack{\text{逐个剔除}\\ \text{能被其余向量线性表示的向量}}}$$

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例 求 $\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ 的一个极大无关组。

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解

$\alpha_1, \alpha_2, \alpha_3, \alpha_4$

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解

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_4 = 2\alpha_1 + 0\alpha_2 + 0\alpha_3}$$

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$$\alpha_1,\alpha_2,\alpha_3,\alpha_4 \xrightarrow[\text{剔除}\alpha_1]{\alpha_1=-\alpha_2+\alpha_3+0\alpha_4} \alpha_2,\alpha_3,\alpha_4 \xrightarrow[\text{剔除}\alpha_2]{\alpha_2=\alpha_3-\frac{1}{2}\alpha_4} \alpha_3,\alpha_4 \ \text{极大无关组}$$

$$\alpha_1,\alpha_2,\alpha_3,\alpha_4 \xrightarrow[\text{剔除}\alpha_2]{\alpha_2=0\alpha_1+\alpha_3-\frac{1}{2}\alpha_4} \alpha_1,\alpha_3,\alpha_4 \xrightarrow{\alpha_4=2\alpha_1+0\alpha_3}$$

向量组的极大无关组

$$\alpha_1, \alpha_2, \ldots, \alpha_s \xrightarrow[\text{直到不能再剔除为止}]{\substack{\text{逐个剔除}\\ \text{能被其余向量线性表示的向量}}} \alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}$$ **极大无关组**

例 求 $\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ 的一个极大无关组。

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$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow[\text{剔除}\alpha_1]{\alpha_1 = -\alpha_2 + \alpha_3 + 0\alpha_4} \alpha_2, \alpha_3, \alpha_4 \xrightarrow[\text{剔除}\alpha_2]{\alpha_2 = \alpha_3 - \frac{1}{2}\alpha_4} \alpha_3, \alpha_4$$ 极大无关组

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还有其他极大无关组吗?

向量组的极大无关组

$$\alpha_1,\alpha_2,\ldots,\alpha_s \xrightarrow[\text{直到不能再剔除为止}]{\substack{\text{逐个剔除}\\ \text{能被其余向量线性表示的向量}}} \alpha_{j_1},\alpha_{j_2},\ldots,\alpha_{j_r}$$ **极大无关组**

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注 极大无关组不唯一!

极大无关组的性质

定理 $\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}$ 是 $\alpha_1, \alpha_2, \cdots, \alpha_s$ 的极大无关组，当且仅当

- $\alpha_1, \alpha_2, \cdots, \alpha_s$ 中每个向量都可由 $\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}$ 线性表示
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$$\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}; \qquad \beta_{k_1}, \beta_{k_2}, \ldots, \beta_{k_t}$$

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$$\alpha_1, \alpha_2; \quad \alpha_1, \alpha_3; \quad \alpha_2, \alpha_3; \quad \alpha_2, \alpha_4; \quad \alpha_3, \alpha_4$$

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可见，每个极大无关组都由 2 个向量构成。

向量组的秩

定义 向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 的极大无关组所包含向量的个数，称为向量组的**秩**，记为:

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注 $r(\alpha_1, \alpha_2, \ldots, \alpha_s) \leq s$ 且 $\leq m$ (维数)。

例 设 $A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$

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$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 3 & 5 \\ 0 & -2 & -4 \end{pmatrix}$$

- $r(\alpha_1, \alpha_2, \alpha_3) = 2$:

$$\alpha_1, \alpha_2, \alpha_3 \xrightarrow[\text{剔除}\alpha_3]{\alpha_3 = -\alpha_1 + 2\alpha_2} \alpha_1, \alpha_2 \xrightarrow{\alpha_1, \alpha_2\text{线性无关}} \alpha_1, \alpha_2\text{为极大无关组}$$

- $r(\beta_1, \beta_2) = \quad$:

$$\beta_1, \beta_2 \xrightarrow{\beta_1, \beta_2\text{线性无关}}$$

例 设 $A = \begin{array}{cc} \begin{matrix} \alpha_1 & \alpha_2 & \alpha_3 \end{matrix} & \\ \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} & \begin{matrix} \beta_1 \\ \beta_2 \end{matrix} \end{array}$

- $r(A) = 2$:

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 3 & 5 \\ 0 & -2 & -4 \end{pmatrix}$$

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- $r(\beta_1, \beta_2) = \quad$:

$$\beta_1, \beta_2 \xrightarrow{\beta_1, \beta_2\text{线性无关}} \beta_1, \beta_2\text{为极大无关组}$$

例 设 $A = \begin{array}{cc} & \begin{array}{ccc} \alpha_1 & \alpha_2 & \alpha_3 \end{array} \\ & \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} \begin{array}{c} \beta_1 \\ \beta_2 \end{array} \end{array}$

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例 设 $A = \begin{array}{cc} & \begin{array}{ccc} \alpha_1 & \alpha_2 & \alpha_3 \end{array} \\ & \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} \begin{array}{c} \beta_1 \\ \beta_2 \end{array} \end{array}$

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- $r(\beta_1, \beta_2) = 2$:

$$\beta_1, \beta_2 \xrightarrow{\beta_1, \beta_2\text{线性无关}} \beta_1, \beta_2\text{为极大无关组}$$

可见，以上上个秩均相等，即 $r(A) = r(\alpha_1, \alpha_2, \alpha_3) = r(\beta_1, \beta_2)$。

例 设 $A = \begin{array}{cc} \begin{array}{ccc} \alpha_1 & \alpha_2 & \alpha_3 \end{array} & \\ \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} & \begin{array}{c} \beta_1 \\ \beta_2 \end{array} \end{array}$

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- $r(\beta_1, \beta_2) = 2$:

$$\beta_1, \beta_2 \xrightarrow{\beta_1, \beta_2\text{线性无关}} \beta_1, \beta_2\text{为极大无关组}$$

可见，以上上个秩均相等，即 $r(A) = r(\alpha_1, \alpha_2, \alpha_3) = r(\beta_1, \beta_2)$。

这不是巧合，而是恒成立！

秩

设

$$\begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & & \alpha_n \end{array} \\ A_{m\times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array}$$

秩

设

$$
A_{m\times n}=\begin{array}{cc}
\begin{array}{cccc} \alpha_1 & \alpha_2 & & \alpha_n \end{array} & \\
\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} & \begin{array}{c} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{array}
\end{array}
$$

秩

设

$$
A_{m\times n} = \begin{array}{cc}
\begin{array}{cccc} \alpha_1 & \alpha_2 & & \alpha_n \end{array} & \\
\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} & \begin{array}{c} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{array}
\end{array}
$$

定义

- $r(\alpha_1, \alpha_2, \ldots, \alpha_n)$ 称为 A 的**列秩**；
- $r(\beta_1, \beta_2, \ldots, \beta_m)$ 称为 A 的**行秩**；

秩

设

$$
A_{m\times n}=\begin{array}{c}
\begin{array}{cccc} \alpha_1 & \alpha_2 & & \alpha_n \end{array} \\
\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}
\end{array}
\begin{array}{c} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{array}
$$

定义

- $r(\alpha_1, \alpha_2, \ldots, \alpha_n)$ 称为 A 的**列秩**；
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定理 $r(A) = r(\alpha_1, \alpha_2, \ldots, \alpha_n) = r(\beta_1, \beta_2, \ldots, \beta_m)$

秩

设

$$
\begin{array}{cc}
 & \begin{array}{cccc} \alpha_1 & \alpha_2 & & \alpha_n \end{array} \\
A_{m\times n} = & \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{array}{c} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{array}
\end{array}
$$

定义

- $r(\alpha_1, \alpha_2, \ldots, \alpha_n)$ 称为 A 的**列秩**；
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定理 $r(A) = r(\alpha_1, \alpha_2, \ldots, \alpha_n) = r(\beta_1, \beta_2, \ldots, \beta_m)$

应用 计算向量组的秩可转化为计算矩阵的秩。

初等变换求极大无关组

问题 给出 m 维的向量组 $\alpha_1, \alpha_2, \cdots, \alpha_n$，如何求出其一组极大无关组?

步骤

初等变换求极大无关组

问题 给出 m 维的向量组 $\alpha_1, \alpha_2, \cdots, \alpha_n$，如何求出其一组极大无关组?

步骤

1.

$$A_{m\times n} = \begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & & \alpha_n \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array}$$

初等变换求极大无关组

问题 给出 m 维的向量组 $\alpha_1, \alpha_2, \cdots, \alpha_n$，如何求出其一组极大无关组?

步骤

1. $$\begin{array}{cc} & \begin{array}{cccc} \alpha_1 & \alpha_2 & & \alpha_n \end{array} \\ A_{m\times n} = & \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \xrightarrow{\text{初等行变换}} \text{简化的阶梯型矩阵} \end{array}$$

初等变换求极大无关组

问题 给出 m 维的向量组 $\alpha_1, \alpha_2, \cdots, \alpha_n$，如何求出其一组极大无关组?

步骤

1. $$A_{m\times n} = \begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & & \alpha_n \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array} \xrightarrow{\text{初等行变换}} \text{简化的阶梯型矩阵}$$

2. 通过简化的阶梯型矩阵，求出 $r(A)$。

初等变换求极大无关组

问题 给出 m 维的向量组 $\alpha_1, \alpha_2, \cdots, \alpha_n$，如何求出其一组极大无关组?

步骤

1.
$$\begin{array}{cc} & \begin{array}{cccc} \alpha_1 & \alpha_2 & & \alpha_n \end{array} \\ A_{m\times n} = & \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array} \xrightarrow{\text{初等行变换}} \text{简化的阶梯型矩阵}$$

2. 通过简化的阶梯型矩阵，求出 $r(A)$。
   利用 $r(\alpha_1, \alpha_2, \ldots, \alpha_n) = r(A)$，得出极大无关组所包含向量的个数

初等变换求极大无关组

问题 给出 m 维的向量组 $\alpha_1, \alpha_2, \cdots, \alpha_n$，如何求出其一组极大无关组?

步骤

1. $$A_{m\times n} = \begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & & \alpha_n \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array} \xrightarrow{\text{初等行变换}} \text{简化的阶梯型矩阵}$$

2. 通过简化的阶梯型矩阵，求出 $r(A)$。
   利用 $r(\alpha_1,\alpha_2,\ldots,\alpha_n) = r(A)$，得出极大无关组所包含向量的个数
3. 通过简化的阶梯型矩阵，容易看出线性无关的 $r(A)$ 列，这就找到一组极大无关组

初等变换求极大无关组

问题 给出 m 维的向量组 $\alpha_1, \alpha_2, \cdots, \alpha_n$，如何求出其一组极大无关组?

步骤

1. $$A_{m\times n} = \begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & & \alpha_n \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array} \xrightarrow{\text{初等行变换}} \text{简化的阶梯型矩阵}$$
2. 通过简化的阶梯型矩阵，求出 $r(A)$。
   利用 $r(\alpha_1,\alpha_2,\ldots,\alpha_n) = r(A)$，得出极大无关组所包含向量的个数
3. 通过简化的阶梯型矩阵，容易看出线性无关的 $r(A)$ 列，这就找到一组极大无关组
4. 通过简化的阶梯型矩阵，容易看出其余列如何用该选定极大无关组线性表示

例 1 求向量组 $\alpha_1 = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \alpha_4 = \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

例 1 求向量组 $\alpha_1 = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \alpha_4 = \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{c} \begin{matrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{matrix} \\ \begin{pmatrix} 2 & 1 & 2 & 3 \\ 4 & 1 & 3 & 5 \\ 2 & 0 & 1 & 2 \end{pmatrix} \end{array}$$

例 1 求向量组 $\alpha_1=\begin{pmatrix}2\\4\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}1\\1\\0\end{pmatrix}$, $\alpha_3=\begin{pmatrix}2\\3\\1\end{pmatrix}$, $\alpha_4=\begin{pmatrix}3\\5\\2\end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{array}$$

$$\begin{pmatrix}2 & 1 & 2 & 3\\4 & 1 & 3 & 5\\2 & 0 & 1 & 2\end{pmatrix}\xrightarrow[r_3-r_1]{r_2-2r_1}$$

例 1 求向量组 $\alpha_1=\begin{pmatrix}2\\4\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}1\\1\\0\end{pmatrix}$, $\alpha_3=\begin{pmatrix}2\\3\\1\end{pmatrix}$, $\alpha_4=\begin{pmatrix}3\\5\\2\end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{array}$$

$$\begin{pmatrix}2&1&2&3\\4&1&3&5\\2&0&1&2\end{pmatrix}\xrightarrow[r_3-r_1]{r_2-2r_1}\begin{pmatrix}2&1&2&3\\0&-1&-1&-1\\0&-1&-1&-1\end{pmatrix}\longrightarrow$$

例 1 求向量组 $\alpha_1=\begin{pmatrix}2\\4\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}1\\1\\0\end{pmatrix}$, $\alpha_3=\begin{pmatrix}2\\3\\1\end{pmatrix}$, $\alpha_4=\begin{pmatrix}3\\5\\2\end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{array}$$

$$\begin{pmatrix} 2 & 1 & 2 & 3 \\ 4 & 1 & 3 & 5 \\ 2 & 0 & 1 & 2 \end{pmatrix} \xrightarrow[r_3-r_1]{r_2-2r_1} \begin{pmatrix} 2 & 1 & 2 & 3 \\ 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

例 1 求向量组 $\alpha_1=\begin{pmatrix}2\\4\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}1\\1\\0\end{pmatrix}$, $\alpha_3=\begin{pmatrix}2\\3\\1\end{pmatrix}$, $\alpha_4=\begin{pmatrix}3\\5\\2\end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{array}$$

$$\begin{pmatrix}2&1&2&3\\4&1&3&5\\2&0&1&2\end{pmatrix}\xrightarrow[r_3-r_1]{r_2-2r_1}\begin{pmatrix}2&1&2&3\\0&-1&-1&-1\\0&-1&-1&-1\end{pmatrix}\longrightarrow\begin{pmatrix}2&1&2&3\\0&1&1&1\\0&0&0&0\end{pmatrix}$$

$$\xrightarrow{r_1-r_2}$$

例 1 求向量组 $\alpha_1=\begin{pmatrix}2\\4\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}1\\1\\0\end{pmatrix}$, $\alpha_3=\begin{pmatrix}2\\3\\1\end{pmatrix}$, $\alpha_4=\begin{pmatrix}3\\5\\2\end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{array}$$

$$\begin{pmatrix}2&1&2&3\\4&1&3&5\\2&0&1&2\end{pmatrix}\xrightarrow[r_3-r_1]{r_2-2r_1}\begin{pmatrix}2&1&2&3\\0&-1&-1&-1\\0&-1&-1&-1\end{pmatrix}\longrightarrow\begin{pmatrix}2&1&2&3\\0&1&1&1\\0&0&0&0\end{pmatrix}$$

$$\xrightarrow{r_1-r_2}\begin{pmatrix}2&0&1&2\\0&1&1&1\\0&0&0&0\end{pmatrix}$$

例 1 求向量组 $\alpha_1 = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \alpha_4 = \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{array}$$

$$\begin{pmatrix} 2 & 1 & 2 & 3 \\ 4 & 1 & 3 & 5 \\ 2 & 0 & 1 & 2 \end{pmatrix} \xrightarrow[r_3 - r_1]{r_2 - 2r_1} \begin{pmatrix} 2 & 1 & 2 & 3 \\ 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{r_1 - r_2} \begin{pmatrix} 2 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\frac{1}{2} \times r_1}$$

例 1 求向量组 $\alpha_1=\begin{pmatrix}2\\4\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}1\\1\\0\end{pmatrix}$, $\alpha_3=\begin{pmatrix}2\\3\\1\end{pmatrix}$, $\alpha_4=\begin{pmatrix}3\\5\\2\end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{array}$$

$$\begin{pmatrix}2&1&2&3\\4&1&3&5\\2&0&1&2\end{pmatrix}\xrightarrow[r_3-r_1]{r_2-2r_1}\begin{pmatrix}2&1&2&3\\0&-1&-1&-1\\0&-1&-1&-1\end{pmatrix}\longrightarrow\begin{pmatrix}2&1&2&3\\0&1&1&1\\0&0&0&0\end{pmatrix}$$

$$\xrightarrow{r_1-r_2}\begin{pmatrix}2&0&1&2\\0&1&1&1\\0&0&0&0\end{pmatrix}\xrightarrow{\frac{1}{2}\times r_1}\begin{pmatrix}1&0&\frac{1}{2}&1\\0&1&1&1\\0&0&0&0\end{pmatrix}$$

例 1 求向量组 $\alpha_1 = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \alpha_4 = \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{array} \\ \begin{pmatrix} 2 & 1 & 2 & 3 \\ 4 & 1 & 3 & 5 \\ 2 & 0 & 1 & 2 \end{pmatrix} \end{array} \xrightarrow[r_3 - r_1]{r_2 - 2r_1} \begin{pmatrix} 2 & 1 & 2 & 3 \\ 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{r_1 - r_2} \begin{pmatrix} 2 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\frac{1}{2} \times r_1} \begin{pmatrix} 1 & 0 & \frac{1}{2} & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

例 1 求向量组 $\alpha_1=\begin{pmatrix}2\\4\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}1\\1\\0\end{pmatrix}$, $\alpha_3=\begin{pmatrix}2\\3\\1\end{pmatrix}$, $\alpha_4=\begin{pmatrix}3\\5\\2\end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{c}\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{array}\\ \begin{pmatrix}2&1&2&3\\4&1&3&5\\2&0&1&2\end{pmatrix}\end{array}\xrightarrow[r_3-r_1]{r_2-2r_1}\begin{pmatrix}2&1&2&3\\0&-1&-1&-1\\0&-1&-1&-1\end{pmatrix}\longrightarrow\begin{pmatrix}2&1&2&3\\0&1&1&1\\0&0&0&0\end{pmatrix}$$

$$\xrightarrow{r_1-r_2}\begin{pmatrix}2&0&1&2\\0&1&1&1\\0&0&0&0\end{pmatrix}\xrightarrow{\frac{1}{2}\times r_1}\begin{pmatrix}1&0&\frac{1}{2}&1\\0&1&1&1\\0&0&0&0\end{pmatrix}$$

所以

- $r(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=r(A)=2$;

例 1 求向量组 $\alpha_1=\begin{pmatrix}2\\4\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}1\\1\\0\end{pmatrix}$, $\alpha_3=\begin{pmatrix}2\\3\\1\end{pmatrix}$, $\alpha_4=\begin{pmatrix}3\\5\\2\end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{array}$$

$$\begin{pmatrix}2&1&2&3\\4&1&3&5\\2&0&1&2\end{pmatrix}\xrightarrow[r_3-r_1]{r_2-2r_1}\begin{pmatrix}2&1&2&3\\0&-1&-1&-1\\0&-1&-1&-1\end{pmatrix}\longrightarrow\begin{pmatrix}2&1&2&3\\0&1&1&1\\0&0&0&0\end{pmatrix}$$

$$\xrightarrow{r_1-r_2}\begin{pmatrix}2&0&1&2\\0&1&1&1\\0&0&0&0\end{pmatrix}\xrightarrow{\frac{1}{2}\times r_1}\begin{pmatrix}1&0&\frac{1}{2}&1\\0&1&1&1\\0&0&0&0\end{pmatrix}$$

所以

- $r(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=r(A)=2$;

例 1 求向量组 $\alpha_1=\begin{pmatrix}2\\4\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}1\\1\\0\end{pmatrix}$, $\alpha_3=\begin{pmatrix}2\\3\\1\end{pmatrix}$, $\alpha_4=\begin{pmatrix}3\\5\\2\end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解 $\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4$

$$\begin{pmatrix}2&1&2&3\\4&1&3&5\\2&0&1&2\end{pmatrix}\xrightarrow[r_3-r_1]{r_2-2r_1}\begin{pmatrix}2&1&2&3\\0&-1&-1&-1\\0&-1&-1&-1\end{pmatrix}\longrightarrow\begin{pmatrix}2&1&2&3\\0&1&1&1\\0&0&0&0\end{pmatrix}$$

$$\xrightarrow{r_1-r_2}\begin{pmatrix}2&0&1&2\\0&1&1&1\\0&0&0&0\end{pmatrix}\xrightarrow{\frac{1}{2}\times r_1}\begin{pmatrix}1&0&\frac{1}{2}&1\\0&1&1&1\\0&0&0&0\end{pmatrix}$$

所以

- $r(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=r(A)=2$;
- α_1,α_2 是极大无关组;

例 1 求向量组 $\alpha_1=\begin{pmatrix}2\\4\\2\end{pmatrix}, \alpha_2=\begin{pmatrix}1\\1\\0\end{pmatrix}, \alpha_3=\begin{pmatrix}2\\3\\1\end{pmatrix}, \alpha_4=\begin{pmatrix}3\\5\\2\end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{array}$$

$$\begin{pmatrix}2&1&2&3\\4&1&3&5\\2&0&1&2\end{pmatrix}\xrightarrow[r_3-r_1]{r_2-2r_1}\begin{pmatrix}2&1&2&3\\0&-1&-1&-1\\0&-1&-1&-1\end{pmatrix}\longrightarrow\begin{pmatrix}2&1&2&3\\0&1&1&1\\0&0&0&0\end{pmatrix}$$

$$\xrightarrow{r_1-r_2}\begin{pmatrix}2&0&1&2\\0&1&1&1\\0&0&0&0\end{pmatrix}\xrightarrow{\frac{1}{2}\times r_1}\begin{pmatrix}1&0&\frac{1}{2}&1\\0&1&1&1\\0&0&0&0\end{pmatrix}$$

所以

- $r(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=r(A)=2$;
- α_1,α_2 是极大无关组;
- $\alpha_3=\frac{1}{2}\alpha_1+\alpha_2,\quad \alpha_4=\alpha_1+\alpha_2$

例 2 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

例 2 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}, \alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 0 & 2 & -2 & 3 \end{pmatrix} \end{array}$$

例 2 求向量组 $\alpha_1=\begin{pmatrix}1\\2\\1\\0\end{pmatrix}$, $\alpha_2=\begin{pmatrix}0\\1\\1\\2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}1\\1\\0\\-2\end{pmatrix}$, $\alpha_4=\begin{pmatrix}2\\4\\3\\3\end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{c} \quad\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \\ \begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 0 & 2 & -2 & 3 \end{pmatrix} \end{array} \xrightarrow[r_3-r_1]{r_2-2r_1}$$

例 2 求向量组 $\alpha_1=\begin{pmatrix}1\\2\\1\\0\end{pmatrix}$, $\alpha_2=\begin{pmatrix}0\\1\\1\\2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}1\\1\\0\\-2\end{pmatrix}$, $\alpha_4=\begin{pmatrix}2\\4\\3\\3\end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{c} \begin{matrix}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{matrix} \\ \begin{pmatrix}1 & 0 & 1 & 2\\ 2 & 1 & 1 & 4\\ 1 & 1 & 0 & 3\\ 0 & 2 & -2 & 3\end{pmatrix}\end{array} \xrightarrow[r_3-r_1]{r_2-2r_1} \begin{pmatrix}1 & 0 & 1 & 2\\ 0 & 1 & -1 & 0\\ 0 & 1 & -1 & 1\\ 0 & 2 & -2 & 3\end{pmatrix}$$

例 2 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}, \alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{array}$$

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow[r_3-r_1]{r_2-2r_1} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow[r_4-2r_2]{r_3-r_2}$$

例 2 求向量组 $\alpha_1=\begin{pmatrix}1\\2\\1\\0\end{pmatrix}$, $\alpha_2=\begin{pmatrix}0\\1\\1\\2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}1\\1\\0\\-2\end{pmatrix}$, $\alpha_4=\begin{pmatrix}2\\4\\3\\3\end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{c}\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{array}\\ \begin{pmatrix}1&0&1&2\\2&1&1&4\\1&1&0&3\\0&2&-2&3\end{pmatrix}\end{array}\xrightarrow[r_3-r_1]{r_2-2r_1}\begin{pmatrix}1&0&1&2\\0&1&-1&0\\0&1&-1&1\\0&2&-2&3\end{pmatrix}\xrightarrow[r_4-2r_2]{r_3-r_2}\begin{pmatrix}1&0&1&2\\0&1&-1&0\\0&0&0&1\\0&0&0&3\end{pmatrix}$$

例 2 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}, \alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 0 & 2 & -2 & 3 \end{pmatrix} \end{array} \xrightarrow[r_3-r_1]{r_2-2r_1} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow[r_4-2r_2]{r_3-r_2} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$\xrightarrow[r_1-2r_3]{r_4-3r_3}$$

例 2 求向量组 $\alpha_1=\begin{pmatrix}1\\2\\1\\0\end{pmatrix}$, $\alpha_2=\begin{pmatrix}0\\1\\1\\2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}1\\1\\0\\-2\end{pmatrix}$, $\alpha_4=\begin{pmatrix}2\\4\\3\\3\end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{c}\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{array}\\ \begin{pmatrix}1&0&1&2\\2&1&1&4\\1&1&0&3\\0&2&-2&3\end{pmatrix}\end{array}\xrightarrow[r_3-r_1]{r_2-2r_1}\begin{pmatrix}1&0&1&2\\0&1&-1&0\\0&1&-1&1\\0&2&-2&3\end{pmatrix}\xrightarrow[r_4-2r_2]{r_3-r_2}\begin{pmatrix}1&0&1&2\\0&1&-1&0\\0&0&0&1\\0&0&0&3\end{pmatrix}$$

$$\xrightarrow[r_1-2r_3]{r_4-3r_3}\begin{pmatrix}1&0&1&0\\0&1&-1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}$$

例 2 求向量组 $\alpha_1=\begin{pmatrix}1\\2\\1\\0\end{pmatrix}$, $\alpha_2=\begin{pmatrix}0\\1\\1\\2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}1\\1\\0\\-2\end{pmatrix}$, $\alpha_4=\begin{pmatrix}2\\4\\3\\3\end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{c}\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{array}\\ \begin{pmatrix}1&0&1&2\\2&1&1&4\\1&1&0&3\\0&2&-2&3\end{pmatrix}\end{array}\xrightarrow[r_3-r_1]{r_2-2r_1}\begin{pmatrix}1&0&1&2\\0&1&-1&0\\0&1&-1&1\\0&2&-2&3\end{pmatrix}\xrightarrow[r_4-2r_2]{r_3-r_2}\begin{pmatrix}1&0&1&2\\0&1&-1&0\\0&0&0&1\\0&0&0&3\end{pmatrix}$$

$$\xrightarrow[r_1-2r_3]{r_4-3r_3}\begin{pmatrix}1&0&1&0\\0&1&-1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}$$

例 2 求向量组 $\alpha_1=\begin{pmatrix}1\\2\\1\\0\end{pmatrix}$, $\alpha_2=\begin{pmatrix}0\\1\\1\\2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}1\\1\\0\\-2\end{pmatrix}$, $\alpha_4=\begin{pmatrix}2\\4\\3\\3\end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{c}\alpha_1\ \ \alpha_2\ \ \alpha_3\ \ \alpha_4\\ \begin{pmatrix}1&0&1&2\\2&1&1&4\\1&1&0&3\\0&2&-2&3\end{pmatrix}\end{array}\xrightarrow[r_3-r_1]{r_2-2r_1}\begin{pmatrix}1&0&1&2\\0&1&-1&0\\0&1&-1&1\\0&2&-2&3\end{pmatrix}\xrightarrow[r_4-2r_2]{r_3-r_2}\begin{pmatrix}1&0&1&2\\0&1&-1&0\\0&0&0&1\\0&0&0&3\end{pmatrix}$$

$$\xrightarrow[r_1-2r_3]{r_4-3r_3}\begin{pmatrix}1&0&1&0\\0&1&-1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}$$

所以

例 2 求向量组 $\alpha_1 = \begin{pmatrix}1\\2\\1\\0\end{pmatrix}$, $\alpha_2 = \begin{pmatrix}0\\1\\1\\2\end{pmatrix}$, $\alpha_3 = \begin{pmatrix}1\\1\\0\\-2\end{pmatrix}$, $\alpha_4 = \begin{pmatrix}2\\4\\3\\3\end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{c} \begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{array} \\ \begin{pmatrix}1&0&1&2\\2&1&1&4\\1&1&0&3\\0&2&-2&3\end{pmatrix}\end{array} \xrightarrow[r_3-r_1]{r_2-2r_1} \begin{pmatrix}1&0&1&2\\0&1&-1&0\\0&1&-1&1\\0&2&-2&3\end{pmatrix} \xrightarrow[r_4-2r_2]{r_3-r_2} \begin{pmatrix}1&0&1&2\\0&1&-1&0\\0&0&0&1\\0&0&0&3\end{pmatrix}$$

$$\xrightarrow[r_1-2r_3]{r_4-3r_3} \begin{pmatrix}1&0&1&0\\0&1&-1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}$$

所以

- $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 3$;

例 2 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 0 & 2 & -2 & 3 \end{pmatrix} \end{array} \xrightarrow[r_3-r_1]{r_2-2r_1} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow[r_4-2r_2]{r_3-r_2} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$\xrightarrow[r_1-2r_3]{r_4-3r_3} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

所以

- $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 3$;

例 2 求向量组 $\alpha_1=\begin{pmatrix}1\\2\\1\\0\end{pmatrix}$, $\alpha_2=\begin{pmatrix}0\\1\\1\\2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}1\\1\\0\\-2\end{pmatrix}$, $\alpha_4=\begin{pmatrix}2\\4\\3\\3\end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{c}\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{array}\\ \begin{pmatrix}1&0&1&2\\2&1&1&4\\1&1&0&3\\0&2&-2&3\end{pmatrix}\end{array}\xrightarrow[r_3-r_1]{r_2-2r_1}\begin{pmatrix}1&0&1&2\\0&1&-1&0\\0&1&-1&1\\0&2&-2&3\end{pmatrix}\xrightarrow[r_4-2r_2]{r_3-r_2}\begin{pmatrix}1&0&1&2\\0&1&-1&0\\0&0&0&1\\0&0&0&3\end{pmatrix}$$

$$\xrightarrow[r_1-2r_3]{r_4-3r_3}\begin{pmatrix}1&0&1&0\\0&1&-1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}$$

所以

- $r(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=r(A)=3$;
- $\alpha_1,\alpha_2,\alpha_4$ 是极大无关组;

例 2 求向量组 $\alpha_1=\begin{pmatrix}1\\2\\1\\0\end{pmatrix}, \alpha_2=\begin{pmatrix}0\\1\\1\\2\end{pmatrix}, \alpha_3=\begin{pmatrix}1\\1\\0\\-2\end{pmatrix}, \alpha_4=\begin{pmatrix}2\\4\\3\\3\end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{c}\quad\alpha_1\quad\alpha_2\quad\alpha_3\quad\alpha_4\\ \begin{pmatrix}1&0&1&2\\2&1&1&4\\1&1&0&3\\0&2&-2&3\end{pmatrix}\end{array} \xrightarrow[r_3-r_1]{r_2-2r_1} \begin{pmatrix}1&0&1&2\\0&1&-1&0\\0&1&-1&1\\0&2&-2&3\end{pmatrix} \xrightarrow[r_4-2r_2]{r_3-r_2} \begin{pmatrix}1&0&1&2\\0&1&-1&0\\0&0&0&1\\0&0&0&3\end{pmatrix}$$

$$\xrightarrow[r_1-2r_3]{r_4-3r_3} \begin{pmatrix}1&0&1&0\\0&1&-1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}$$

所以

- $r(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=r(A)=3$;
- $\alpha_1,\alpha_2,\alpha_4$ 是极大无关组;
- $\alpha_3=\alpha_1-\alpha_2$

例 3 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}, \alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

例 3 求向量组 $\alpha_1=\begin{pmatrix}1\\2\\3\\4\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\3\\4\\5\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\4\\5\\6\end{pmatrix}$, $\alpha_4=\begin{pmatrix}4\\5\\6\\7\end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{c} \begin{matrix}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{matrix} \\ \begin{pmatrix}1 & 2 & 3 & 4\\2 & 3 & 4 & 5\\3 & 4 & 5 & 6\\4 & 5 & 6 & 7\end{pmatrix}\end{array}$$

例 3 求向量组 $\alpha_1=\begin{pmatrix}1\\2\\3\\4\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\3\\4\\5\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\4\\5\\6\end{pmatrix}$, $\alpha_4=\begin{pmatrix}4\\5\\6\\7\end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{array}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \xrightarrow[\substack{r_3-3r_1\\ r_4-4r_1}]{r_2-2r_1}$$

例 3 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{c} \begin{matrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{matrix} \\ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \end{array} \xrightarrow[\substack{r_3-3r_1 \\ r_4-4r_1}]{r_2-2r_1} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{pmatrix}$$

例 3 求向量组 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{array}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \xrightarrow[\substack{r_3-3r_1 \\ r_4-4r_1}]{r_2-2r_1} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{pmatrix}$$

$$\xrightarrow[r_4-3r_2]{r_3-2r_2}$$

例 3 求向量组 $\alpha_1=\begin{pmatrix}1\\2\\3\\4\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\3\\4\\5\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\4\\5\\6\end{pmatrix}$, $\alpha_4=\begin{pmatrix}4\\5\\6\\7\end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{c} \begin{matrix}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{matrix} \\ \begin{pmatrix}1&2&3&4\\2&3&4&5\\3&4&5&6\\4&5&6&7\end{pmatrix}\end{array} \xrightarrow[\substack{r_3-3r_1\\r_4-4r_1}]{r_2-2r_1} \begin{pmatrix}1&2&3&4\\0&-1&-2&-3\\0&-2&-4&-6\\0&-3&-6&-9\end{pmatrix}$$

$$\xrightarrow[r_4-3r_2]{r_3-2r_2} \begin{pmatrix}1&2&3&4\\0&-1&-2&-3\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

例 3 求向量组 $\alpha_1=\begin{pmatrix}1\\2\\3\\4\end{pmatrix}, \alpha_2=\begin{pmatrix}2\\3\\4\\5\end{pmatrix}, \alpha_3=\begin{pmatrix}3\\4\\5\\6\end{pmatrix}, \alpha_4=\begin{pmatrix}4\\5\\6\\7\end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解

$$\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{array}$$

$$\begin{pmatrix}1&2&3&4\\2&3&4&5\\3&4&5&6\\4&5&6&7\end{pmatrix}\xrightarrow[\substack{r_3-3r_1\\r_4-4r_1}]{r_2-2r_1}\begin{pmatrix}1&2&3&4\\0&-1&-2&-3\\0&-2&-4&-6\\0&-3&-6&-9\end{pmatrix}$$

$$\xrightarrow[r_4-3r_2]{r_3-2r_2}\begin{pmatrix}1&2&3&4\\0&-1&-2&-3\\0&0&0&0\\0&0&0&0\end{pmatrix}\longrightarrow\begin{pmatrix}1&0&-1&-2\\0&1&2&3\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

例 3 求向量组 $\alpha_1=\begin{pmatrix}1\\2\\3\\4\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\3\\4\\5\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\4\\5\\6\end{pmatrix}$, $\alpha_4=\begin{pmatrix}4\\5\\6\\7\end{pmatrix}$ 的一个极大无关组；并把其余向量用该极大无关组线性表示。

解 $\alpha_1\ \ \alpha_2\ \ \alpha_3\ \ \alpha_4$

$$\begin{pmatrix}1&2&3&4\\2&3&4&5\\3&4&5&6\\4&5&6&7\end{pmatrix}\xrightarrow[\substack{r_3-3r_1\\r_4-4r_1}]{r_2-2r_1}\begin{pmatrix}1&2&3&4\\0&-1&-2&-3\\0&-2&-4&-6\\0&-3&-6&-9\end{pmatrix}$$

$$\xrightarrow[r_4-3r_2]{r_3-2r_2}\begin{pmatrix}1&2&3&4\\0&-1&-2&-3\\0&0&0&0\\0&0&0&0\end{pmatrix}\longrightarrow\begin{pmatrix}1&0&-1&-2\\0&1&2&3\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

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所以

- $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2$;

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即

$$(\gamma_1\ \gamma_2\ \cdots\ \gamma_s) = (\alpha_1\ \alpha_2\ \cdots\ \alpha_n) \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}$$

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$$\underbrace{\begin{array}{cccc} \gamma_1 & \gamma_2 & & \gamma_s \\ \multicolumn{4}{c}{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}} \end{array}}_{C} = \underbrace{\begin{array}{cccc} \alpha_1 & \alpha_2 & & \alpha_n \\ \multicolumn{4}{c}{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}} \end{array}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}$$

即

$$(\gamma_1\ \gamma_2\ \cdots\ \gamma_s) = (\alpha_1\ \alpha_2\ \cdots\ \alpha_n)\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}$$

$$\Rightarrow \quad \gamma_1 = b_{11}\alpha_1 + b_{21}\alpha_2 + \cdots + b_{n1}\alpha_n$$

例 设 $A_{m\times n}$, $B_{n\times s}$ 为矩阵，则 $r(AB) \le \min\{r(A), r(B)\}$。

证明 设 $AB = C_{m\times s}$

$$\underbrace{\begin{array}{c} \begin{array}{cccc} \gamma_1 & \gamma_2 & & \gamma_s \end{array} \\ \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix} \end{array}}_{C} = \underbrace{\begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & & \alpha_n \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}$$

即

$$(\gamma_1\ \gamma_2\ \cdots\ \gamma_s) = (\alpha_1\ \alpha_2\ \cdots\ \alpha_n) \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}$$

$\Rightarrow \quad \gamma_1 = b_{11}\alpha_1 + b_{21}\alpha_2 + \cdots + b_{n1}\alpha_n$ 等等

例 设 $A_{m\times n}$, $B_{n\times s}$ 为矩阵，则 $r(AB) \le \min\{r(A), r(B)\}$。

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即

$$(\gamma_1 \ \gamma_2 \ \cdots \ \gamma_s) = (\alpha_1 \ \alpha_2 \ \cdots \ \alpha_n) \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}$$

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$$\underbrace{\begin{array}{c} \begin{array}{cccc} \gamma_1 & \gamma_2 & & \gamma_s \end{array} \\ \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix} \end{array}}_{C} = \underbrace{\begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & & \alpha_n \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}$$

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$$r(\gamma_1, \ldots, \gamma_s) \le r(\alpha_1, \ldots, \alpha_n) = r(A)$$

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证明 设 $AB = C_{m\times s}$

$$\underbrace{\begin{array}{c}\begin{array}{cccc}\gamma_1 & \gamma_2 & & \gamma_s\end{array}\\ \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}\end{array}}_{C} = \underbrace{\begin{array}{c}\begin{array}{cccc}\alpha_1 & \alpha_2 & & \alpha_n\end{array}\\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}\end{array}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}$$

即

$$(\gamma_1 \ \gamma_2 \ \cdots \ \gamma_s) = (\alpha_1 \ \alpha_2 \ \cdots \ \alpha_n) \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}$$

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$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{array}{c} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix} \end{array}}_{B} \begin{array}{l} \beta_1 \\ \beta_2 \\ \\ \\ \end{array}$$

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$$\begin{array}{c} \delta_1 \\ \\ \\ \\ \end{array}\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}\begin{array}{c} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{array}$$

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证明 设 $AB = C_{m\times s}$

$$
\underbrace{\begin{array}{c} \delta_1 \\ \delta_2 \\ \\ \\ \end{array}\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}\begin{array}{c} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{array}
$$

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证明 设 $AB=C_{m\times s}$

$$\begin{array}{c}\delta_1\\ \delta_2\\ \vdots\\ \\ \end{array}\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s}\\ c_{21} & c_{22} & \cdots & c_{2s}\\ \vdots & \vdots & \ddots & \vdots\\ c_{m1} & c_{m2} & \cdots & c_{ms}\end{pmatrix}}_{C}=\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{pmatrix}}_{A}\underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s}\\ b_{21} & b_{22} & \cdots & b_{2s}\\ \vdots & \vdots & \ddots & \vdots\\ b_{n1} & b_{n2} & \cdots & b_{ns}\end{pmatrix}}_{B}\begin{array}{c}\beta_1\\ \beta_2\\ \vdots\\ \beta_n\end{array}$$

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$$\begin{array}{c}\delta_1\\ \delta_2\\ \vdots\\ \delta_m\end{array}\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s}\\ c_{21} & c_{22} & \cdots & c_{2s}\\ \vdots & \vdots & \ddots & \vdots\\ c_{m1} & c_{m2} & \cdots & c_{ms}\end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{pmatrix}}_{A}\underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s}\\ b_{21} & b_{22} & \cdots & b_{2s}\\ \vdots & \vdots & \ddots & \vdots\\ b_{n1} & b_{n2} & \cdots & b_{ns}\end{pmatrix}}_{B}\begin{array}{c}\beta_1\\ \beta_2\\ \vdots\\ \beta_n\end{array}$$

即

$$\begin{pmatrix}\delta_1\\ \delta_2\\ \vdots\\ \delta_m\end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{pmatrix}\begin{pmatrix}\beta_1\\ \beta_2\\ \vdots\\ \beta_n\end{pmatrix}$$

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即
$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$$

$$\Rightarrow \quad \delta_1 = a_{11}\beta_1 + a_{12}\beta_2 + \cdots + a_{1n}\beta_n$$

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$$\begin{array}{c} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{array} \underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B} \begin{array}{c} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{array}$$

即
$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$$

$$\Rightarrow \quad \delta_1 = a_{11}\beta_1 + a_{12}\beta_2 + \cdots + a_{1n}\beta_n \quad \text{等等}$$

例 设 $A_{m\times n}$, $B_{n\times s}$ 为矩阵，则 $r(AB) \le \min\{r(A), r(B)\}$。

证明 设 $AB = C_{m\times s}$

$$\begin{array}{c} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{array} \underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B} \begin{array}{c} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{array}$$

即
$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$$

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可见 $\delta_1, \ldots, \delta_m$ 由 $\beta_1, \ldots, \beta_n$ 线性表示，

例 设 $A_{m\times n}$, $B_{n\times s}$ 为矩阵，则 $r(AB)\le\min\{r(A),r(B)\}$。

证明 设 $AB=C_{m\times s}$

$$
\underbrace{\begin{array}{c} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{array}\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}\begin{array}{c} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{array}}_{B}
$$

即
$$
\begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}
$$

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\Rightarrow \quad \delta_1 = a_{11}\beta_1 + a_{12}\beta_2 + \cdots + a_{1n}\beta_n \quad \text{等等}
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可见 $\delta_1,\ldots,\delta_m$ 由 $\beta_1,\ldots,\beta_n$ 线性表示，所以

$$
r(\delta_1,\ldots,\delta_m) \le r(\beta_1,\ldots,\beta_n)
$$

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可见 $\delta_1, \ldots, \delta_m$ 由 $\beta_1, \ldots, \beta_n$ 线性表示，所以

$$r(\delta_1, \ldots, \delta_m) \le r(\beta_1, \ldots, \beta_n) = r(B)$$

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$$r(AB) = r(\delta_1, \ldots, \delta_m) \le r(\beta_1, \ldots, \beta_n) = r(B)$$