第 12 章 e: 傅里叶级数

数学系 梁卓滨

2018-2019 学年 II





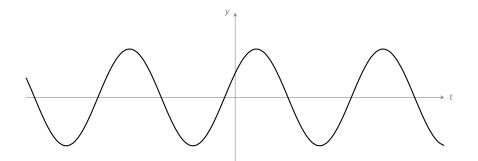
We are here now...

1. 傅里叶级数的概念

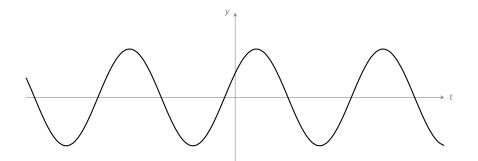
2. 周期为 2π 的周期函数的傅里叶级数

3. 一般周期函数的傅里叶级数

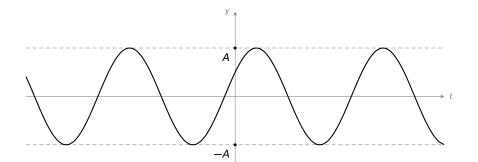
正弦函数 $y = A \sin(\omega t + \varphi)$



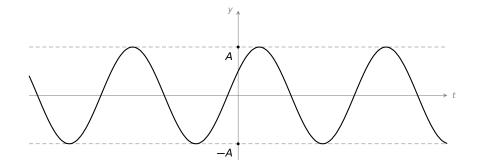
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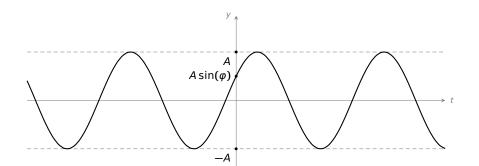
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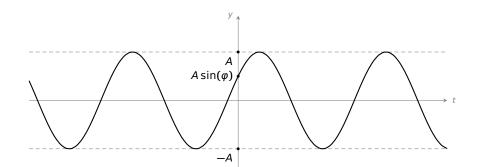
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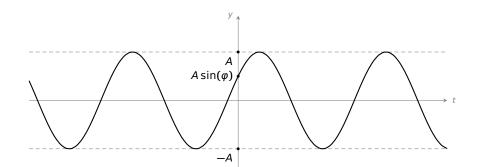


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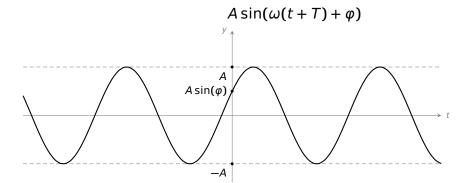
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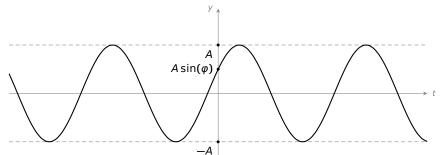
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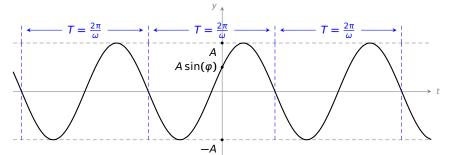
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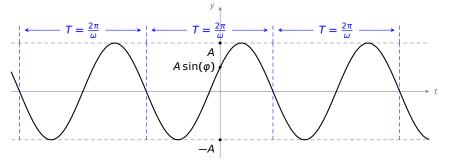
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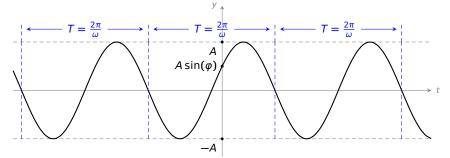
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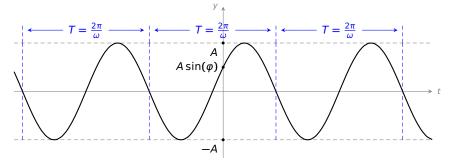
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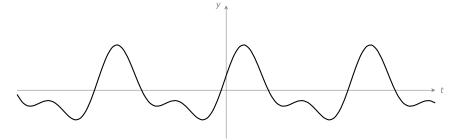
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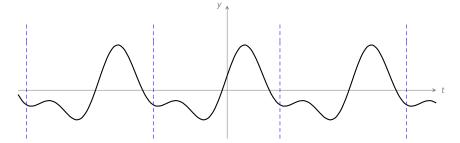


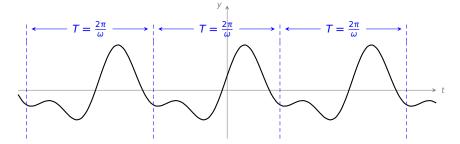
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然 $T = \frac{2\pi}{4}$ 也是周期

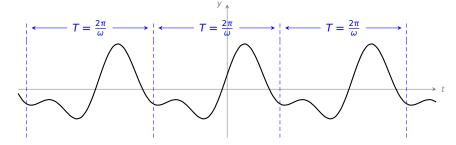








假设 f(t) 是定义域为 \mathbb{R} 的周期函数,周期也是 $T = \frac{2\pi}{\omega}$ 。

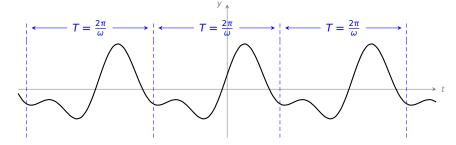


问题 是否有如下展开

$$f(t) = A_0 + \sum_{n=1}^{\infty} A_n \sin(n\omega t + \varphi_n)$$



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$$f(t) = A_0 + \sum_{n=1}^{\infty} A_n \sin(n\omega t + \varphi_n)$$

注 在电工学中,上述展开称为谐波分析; A_0 称为直流分量;

 $A_n \sin(n\omega t + \varphi_n)$ 称为 n 次谐波



设 $T = \frac{2\pi}{\omega} = 2l$,

注意到
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$$A_n \sin(n\omega t + \varphi_n) = A_n \sin(\frac{n\pi t}{l} + \varphi_n)$$

$$\sin \varphi_n \cos \frac{n\pi t}{l} + \cos \varphi_n \sin \frac{n\pi t}{l}$$

注意到
$$\omega = \frac{\pi}{l}$$
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$$A_n \sin(n\omega t + \varphi_n) = A_n \sin(\frac{n\pi t}{l} + \varphi_n)$$

$$= A_n \left[\sin \varphi_n \cos \frac{n\pi t}{l} + \cos \varphi_n \sin \frac{n\pi t}{l} \right]$$

设
$$T = \frac{2\pi}{\omega} = 2l$$
,故区间 $[-l, l]$ 是 $f(t)$ 的一个完整周期。

注意到
$$\omega = \frac{\pi}{l}$$
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这时
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设 $T = \frac{2\pi}{c_1} = 2l$,故区间[-l, l]是f(t)的一个完整周期。

注意到 $\omega = \frac{\pi}{7}$,所以

$$\Delta = \{i, i\}$$

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$$\Lambda \sin(n\omega t)$$

$$\Lambda \sin(n\alpha t)$$



- $= A_n \left[\sin \varphi_n \cos \frac{n\pi t}{l} + \cos \varphi_n \sin \frac{n\pi t}{l} \right]$

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的展开为

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性质 三角函数系

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$$\int_{-\pi}^{\pi} \cos nx dx = 0, \qquad \int_{-\pi}^{\pi} \sin nx dx = 0 \qquad (n = 1, 2, 3, \dots)$$

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另外

$$\int_{-\pi}^{\pi} \sin^2 nx dx = \int_{-\pi}^{\pi} \cos^2 nx dx = \pi \qquad (n = 1, 2, 3, \dots)$$



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$

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 $= \int_{-\pi}^{\pi} a_n \cos nx \cdot \cos nx dx = \pi a_n$



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$$n = 1, 2, 3, \cdots$$
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$$\int_{-\pi}^{\pi} f(x) \sin nx dx = \int_{-\pi}^{\pi} \left[\frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos kx + b_k \sin kx \right) \right] \sin nx dx$$
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$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$

则

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \qquad (n = 0, 1, 2, 3, \dots)$$

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"形式推导" (2) 当 $n = 1, 2, 3, \cdots$ 时,

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定义 f(x) 的傅里叶级数定义为

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$

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问题 何时成立
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$
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定理(收敛定理,狄利克雷充分条件)



- 1. 在一个周期内连续或只有有限个第一类间断点;
- 2. 在一个周期内至多只有有限个极值点,

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当 x 是 f(x) 的连续点时,

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• 当 $x \in f(x)$ 的间断点时,

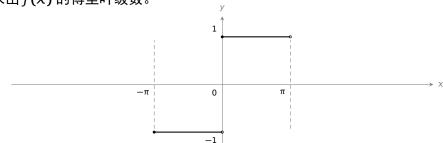
$$\frac{1}{2} \Big[f(x^{-}) + f(x^{+}) \Big] = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big(a_n \cos nx + b_n \sin nx \Big)$$

$$f(x) = \begin{cases} -1, & -\pi \le x < 0, \\ 1, & 0 \le x < \pi. \end{cases}$$

求出f(x)的傅里叶级数。

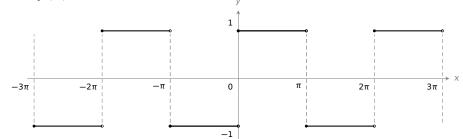
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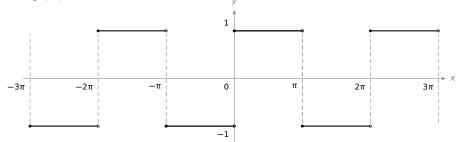
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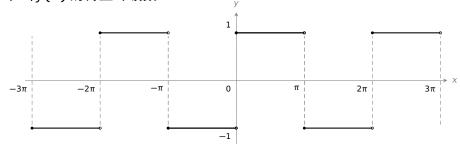
解 计算傅里叶系数如下:

 a_n



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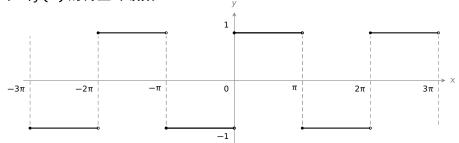
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求出f(x)的傅里叶级数。



解 计算傅里叶系数如下:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \xrightarrow{\frac{6}{10}} 0$$



$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \xrightarrow{\frac{6}{3}} 0,$$

 b_n

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \xrightarrow{\frac{6}{\pi} (4\pi)} 0,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \xrightarrow{\frac{6}{4}} 0,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin nx dx$$



$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \xrightarrow{\frac{6}{3}} 0,$$

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$$2 \cos nx \Big|_{0}^{\pi}$$

$$= \frac{2}{\pi} \cdot (-1) \cdot \frac{\cos nx}{n} \Big|_{0}^{\pi}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \stackrel{\text{fight}}{===} 0,$$

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$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \stackrel{\text{fight}}{=} 0,$$

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$$= \left\{ \begin{array}{c} n = 1, 3, 5, \cdots \\ n = 2, 4, 6, \cdots . \end{array} \right.$$



$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \stackrel{\text{fight}}{===} 0,$$

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$$= \begin{cases} \frac{4}{n\pi}, & n = 1, 3, 5, \cdots \\ 0, & n = 2, 4, 6, \cdots \end{cases}$$



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$$= \begin{cases} nn \\ 0, \end{cases}$$
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所以傅里叶级数为

$$\frac{a_0}{2} + \sum_{n=0}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$



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第 12 章 e: 傅里叶级数

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \xrightarrow{\frac{6}{6}} 0,$$

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$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{4}{\pi} \left[\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \cdots \right]$$

收敛定理分析可知:

当 x ≠ nπ 时,

当 x = nπ 是,

收敛定理分析可知:

• 当 $x \neq n\pi$ 时,是 f 的连续点,

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$$\frac{4}{\pi} \left[\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots \right] = \frac{1}{2} \left[f(x^{-}) + f(x^{+}) \right]$$

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(显然,可直接看出当 $x = n\pi$ 时傅里叶级数的值为 0)

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$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$$

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$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots = \frac{\pi}{4}$$

• $\exists x \neq n\pi$ 时,是 f 的连续点,此时

$$f(x) = \frac{4}{\pi} \left[\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \cdots \right]$$

• 当 $x = n\pi$ 是,是 f 的间断点,此时

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$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots = \frac{\pi}{4}$$



• 当 $x \neq n\pi$ 时,是 f 的连续点,此时

注 1f(x) 的傅里叶级数是 $\frac{4}{\pi} \left[\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \cdots \right]$, 利用

 $\frac{4}{\pi} \left[\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \cdots \right] = \frac{1}{2} \left[f(x^{-}) + f(x^{+}) \right] = 0$

$$f(x) = \frac{4}{\pi} \left[\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \cdots \right]$$

● 当 $x = n\pi$ 是,是 f 的间断点,此时

收敛定理分析可知:

(显然,可直接看出当
$$x = n\pi$$
 时傅里叶级数的值为 0) 注 2 取 $x = \frac{\pi}{2}$,可得到
$$1 - \frac{1}{2} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots = \frac{\pi}{4}$$

 $\frac{1}{2}$ 3 5 7 9 11 注 3 当 $x = \frac{\pi}{2}$,傅里叶级数仅仅是条件收敛

$$\frac{4}{\pi} \left[\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \cdots \right]$$

$$\frac{4}{\pi} \left[\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots \right] = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin[(2n-1)x]$$

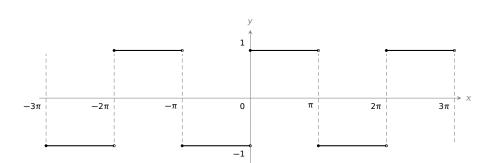
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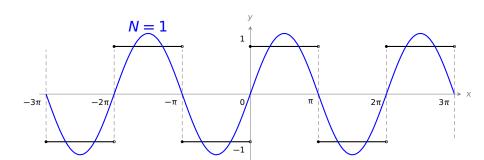
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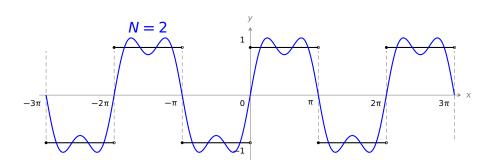
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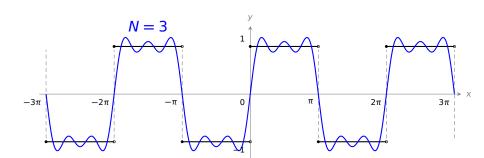
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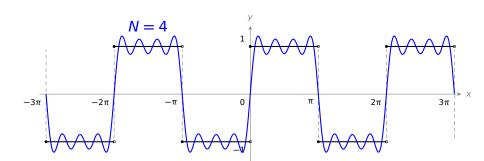
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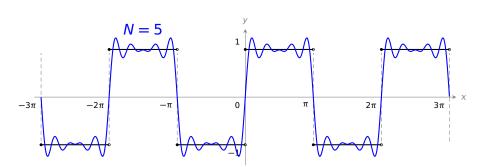
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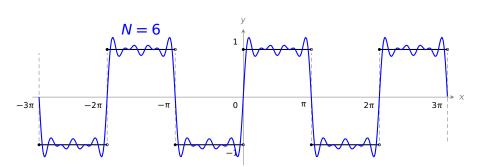
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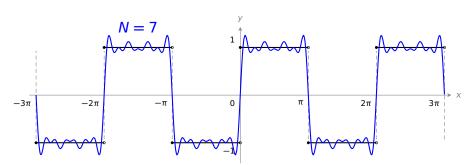
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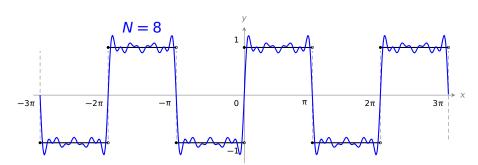
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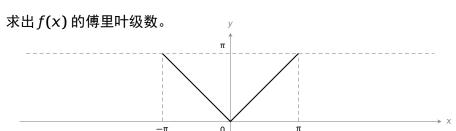
例 2 设 f(x) 是周期为 2π 的周期函数,在 $[-\pi, \pi)$ 上的表达式为

$$f(x) = |x|$$

求出f(x)的傅里叶级数。

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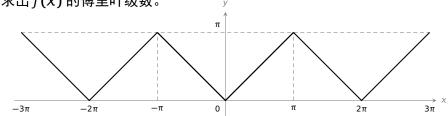
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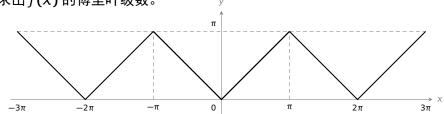
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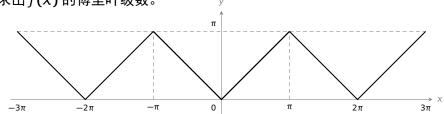
解 计算傅里叶系数如下:

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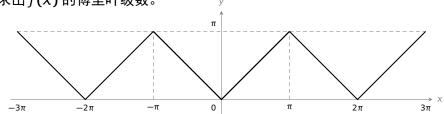
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解 计算傅里叶系数如下:

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$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \xrightarrow{\frac{\Phi(M)}{\pi}} 0,$$

$$a_n =$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \xrightarrow{\frac{6}{\pi} (4\pi)} 0,$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

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$$= \frac{2}{n\pi} \int_{0}^{\pi} x d \sin nx$$



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$$\pi \int_{-\pi}^{\pi} \pi \int_{0}^{\pi} x d \sin nx = \frac{2}{n\pi} \left[x \sin nx \right]_{0}^{\pi} - \int_{0}^{\pi} \sin nx dx$$

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$$2 \int_0^{\pi} 1 d\sin nx = \frac{2}{n\pi} \left[x \sin nx \Big|_0^{\pi} - \int_0^{\pi} \sin nx dx \Big|_0^{\pi} \right]$$

$$n\pi \int_{0}^{\pi} n\pi L \qquad |_{0}^{\pi} \int_{0}^{\pi} = \frac{2}{n\pi} \left[\frac{1}{n} \cos nx \Big|_{0}^{\pi} \right]$$

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$$n\pi \int_{0}^{\pi} n\pi \ln |n| = \frac{2}{n\pi} \left[\frac{1}{n} \cos nx \Big|_{0}^{\pi} \right] = \frac{2}{n^{2}\pi} \left[(-1)^{n} - 1 \right]$$

$$\frac{1}{n\pi} \left[-\cos nx \right]_0 = \frac{1}{n^2\pi} \left[(-1)^n - 1 \right]_0$$



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$$\frac{1}{n} \int_{0}^{\pi} \left[-\frac{2}{n^{2}\pi} \int_{0}^{\pi}$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{2}{\pi} \int_{0}^{\pi} f(x) dx = \frac{2}{\pi} \int_{0}^{\pi} x dx = \frac{2}{\pi} \cdot \frac{1}{2} x^2 \Big|_{0}^{\pi} = \pi.$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \xrightarrow{\text{fight}} 0,$$

$$a_{n} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos nx dx = \frac{2}{\pi} \int_{0}^{\pi} x \cos nx dx$$

 $= \frac{2}{n\pi} \left[\frac{1}{n} \cos nx \Big|_{0}^{n} \right] = \frac{2}{n^{2}\pi} \left[(-1)^{n} - 1 \right] = \begin{cases} -\frac{4}{n^{2}\pi}, & n = 1, 3, 5, \cdots \\ 0, & n = 2, 4, 6, \cdots \end{cases}$

$$a_n = -\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = -\frac{1}{\pi} \int_{0}^{\pi} f(x) \cos nx dx = -\frac{1}{\pi} \int_{0}^{\pi} x \cos nx dx$$
$$= -\frac{2}{n\pi} \int_{0}^{\pi} x d \sin nx = -\frac{2}{n\pi} \left[x \sin nx \right]_{0}^{\pi} - \int_{0}^{\pi} \sin nx dx$$

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所以傅里叶级数为 $\frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos nx$



第 12 章 e: 傅里叶级数

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \stackrel{\text{fight}}{===} 0,$$

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所以傅里叶级数为 $\frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos nx = \frac{\pi}{2} - \frac{4}{\pi} \left[\cos x + \frac{1}{3^2} \cos 3x + \frac{1}{5^2} \cos 5x + \cdots \right]$





$$\frac{\pi}{2} - \frac{4}{\pi} \left[\cos x + \frac{1}{3^2} \cos 3x + \frac{1}{5^2} \cos 5x + \cdots \right]$$

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又因为f(x)是连续函数,

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$$注 2$$
 取 $x = 0$,可得到

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$$\dot{r} 2$$
 取 $x = 0$. 可得到

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots = \frac{\pi^2}{8}$$

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注 3 偶函数 f(x) 的傅里叶级数是 $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$



$$\frac{\pi}{2} - \frac{4}{\pi} \left[\cos x + \frac{1}{3^2} \cos 3x + \frac{1}{5^2} \cos 5x + \cdots \right]$$

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考虑部分和

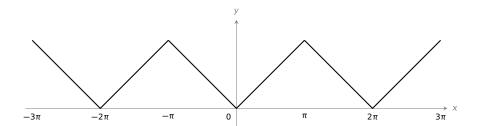
$$\frac{\pi}{2} - \frac{4}{\pi} \sum_{i=1}^{N} \frac{1}{(2n-1)^2} \cos[(2n-1)x]$$



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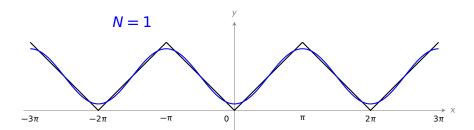
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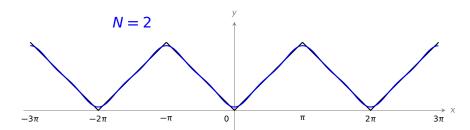
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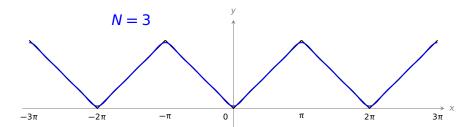
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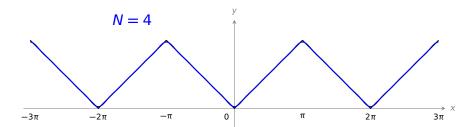
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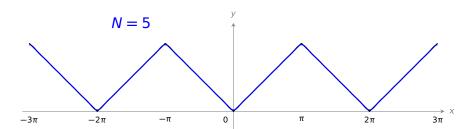
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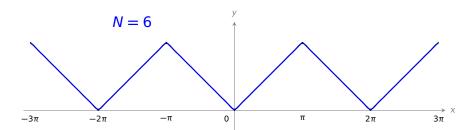
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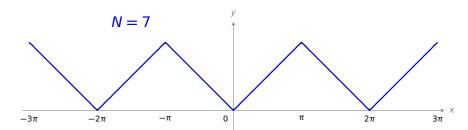
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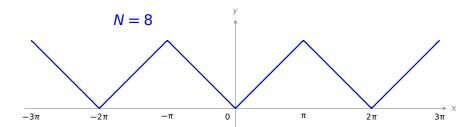
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证明(1)假设ƒ为奇函数,则

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \xrightarrow{\frac{4\pi}{3}} 0$$

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证明 (2) 假设 f 为偶函数,则

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \xrightarrow{\frac{6}{3}} 0$$

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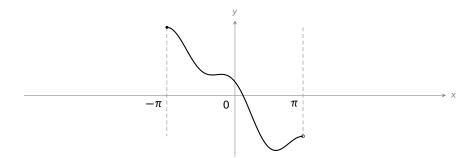
证明 (2) 假设 f 为偶函数,则

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \xrightarrow{\frac{\hat{\sigma}(\underline{M}\underline{M}\underline{M})}{\underline{\sigma}(\underline{M}\underline{M}\underline{M})}} 0$$

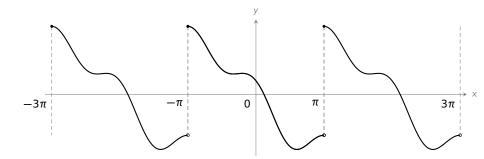
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设 f(x) 是定义在区间 $[-\pi, \pi)$ (或 $(-\pi, \pi]$)上的函数,可以对其进行周期延拓,从而得到定义在 \mathbb{R} 上的周期函数

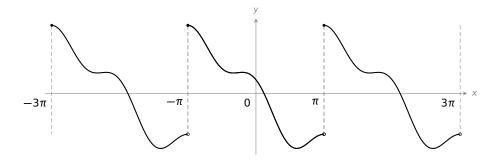
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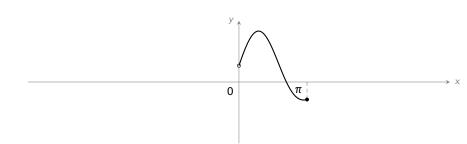


延拓后的周期函数任然记为f(x),此时可以进行傅里叶展开。

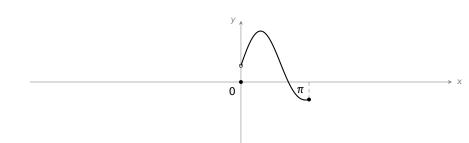


设 f(x) 是定义在区间 $(0, \pi]$ 上的函数,可以对其进行奇延拓,从而得到定义在 \mathbb{R} 上的周期奇函数。

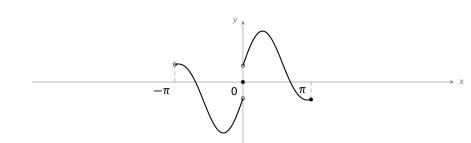
设 f(x) 是定义在区间 $(0, \pi]$ 上的函数,可以对其进行奇延拓,从而得到定义在 \mathbb{R} 上的周期奇函数。



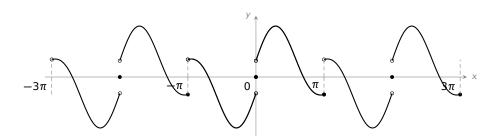
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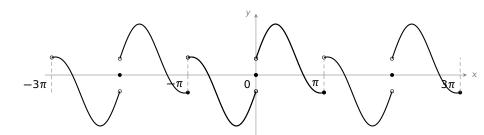




设 f(x) 是定义在区间 $(0, \pi]$ 上的函数,可以对其进行奇延拓,从而得到定义在 \mathbb{R} 上的周期奇函数。

奇延拓步骤:

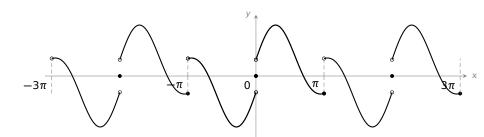
• 定义 f(0) = 0



设 f(x) 是定义在区间 $(0, \pi]$ 上的函数,可以对其进行奇延拓,从而得到定义在 \mathbb{R} 上的周期奇函数。

奇延拓步骤:

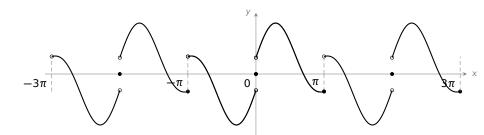
• $\mathbb{E} \times f(0) = 0$; $\mathbb{E} \times (-\pi, 0)$ 时, $\mathbb{E} \times f(x) = -f(-x)$;



设 f(x) 是定义在区间 $(0, \pi]$ 上的函数,可以对其进行奇延拓,从而得到定义在 \mathbb{R} 上的周期奇函数。

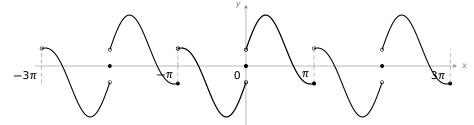
奇延拓步骤:

• 定义 f(0) = 0; 当 $x \in (-\pi, 0)$ 时,定义 f(x) = -f(-x); (此时 f 在 $(-\pi, \pi]$ 上有定义,且在 $(-\pi, \pi)$ 上为奇函数)



设 f(x) 是定义在区间 $(0, \pi]$ 上的函数,可以对其进行奇延拓,从而得到定义在 \mathbb{R} 上的周期奇函数。

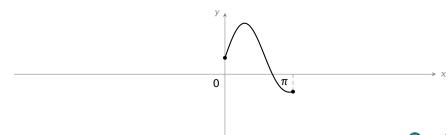
- 定义 f(0) = 0; 当 $x \in (-\pi, 0)$ 时,定义 f(x) = -f(-x); (此时 f 在 $(-\pi, \pi]$ 上有定义,且在 $(-\pi, \pi)$ 上为奇函数)
- 周期延拓 f 在 (-π, π] 上的取值。



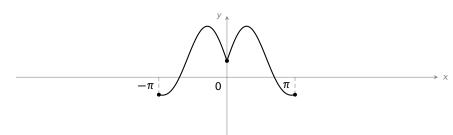


设 f(x) 是定义在区间 $[0, \pi]$ 上的函数,可以对其进行偶延拓,从而得到定义在 \mathbb{R} 上的周期偶函数。

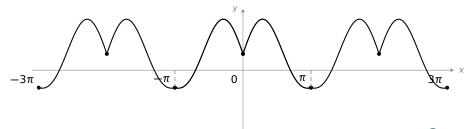
设 f(x) 是定义在区间 $[0, \pi]$ 上的函数,可以对其进行偶延拓,从而得到定义在 \mathbb{R} 上的周期偶函数。



设 f(x) 是定义在区间 $[0, \pi]$ 上的函数,可以对其进行偶延拓,从而得到定义在 \mathbb{R} 上的周期偶函数。



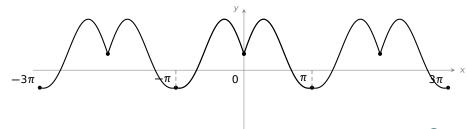
设 f(x) 是定义在区间 $[0, \pi]$ 上的函数,可以对其进行偶延拓,从而得到定义在 \mathbb{R} 上的周期偶函数。



设 f(x) 是定义在区间 $[0, \pi]$ 上的函数,可以对其进行偶延拓,从而得到定义在 \mathbb{R} 上的周期偶函数。

偶延拓步骤:

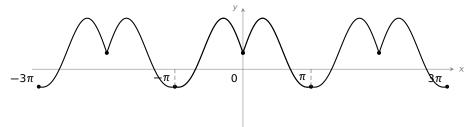
• $\exists x \in [-\pi, 0]$ 时,定义 f(x) = f(-x);



设 f(x) 是定义在区间 $[0, \pi]$ 上的函数,可以对其进行偶延拓,从而得到定义在 \mathbb{R} 上的周期偶函数。

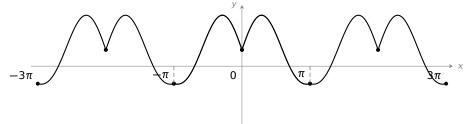
偶延拓步骤:

• 当 $x \in [-\pi, 0]$ 时,定义 f(x) = f(-x); (此时 f 成为定义在 $[-\pi, \pi]$ 上为偶函数)



设 f(x) 是定义在区间 $[0, \pi]$ 上的函数,可以对其进行偶延拓,从而得到定义在 \mathbb{R} 上的周期偶函数。

- 当 $x \in [-\pi, 0]$ 时,定义 f(x) = f(-x); (此时 f 成为定义在 $[-\pi, \pi]$ 上为偶函数)
- 周期延拓 f 在 $[-\pi, \pi]$ 上的取值。



We are here now...

1. 傅里叶级数的概念

2. 周期为 2π 的周期函数的傅里叶级数

3. 一般周期函数的傅里叶级数

假设 f(x) 是定义在 \mathbb{R} 上周期函数,周期为 T=2l,

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$$

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} dx$$
 $(n = 0, 1, 2, 3, \dots)$
 $1 \int_{-l}^{l} n\pi x$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} dx \qquad (n = 0, 1, 2, 3, \dots)$$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$$

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 $(n = 0, 1, 2, 3, \dots)$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} dx \qquad (n = 0, 1, 2, 3, \dots)$$

"推导"
$$\diamondsuit g(x) = f(\frac{l}{\pi}x),$$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$$

其中

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} dx \qquad (n = 0, 1, 2, 3, \dots)$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} dx \qquad (n = 0, 1, 2, 3, \dots)$$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$$

其中

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} dx \qquad (n = 0, 1, 2, 3, \dots)$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} dx \qquad (n = 0, 1, 2, 3, \dots)$$

$$a(x+2\pi)$$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$$

其中

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} dx \qquad (n = 0, 1, 2, 3, \dots)$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} dx \qquad (n = 0, 1, 2, 3, \dots)$$

$$g(x+2\pi) = f(\frac{l}{\pi}(x+2\pi))$$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$$

其中

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} dx \qquad (n = 0, 1, 2, 3, \dots)$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} dx \qquad (n = 0, 1, 2, 3, \dots)$$

$$g(x+2\pi) = f(\frac{l}{\pi}(x+2\pi)) = f(\frac{l}{\pi}x+2l)$$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$$

其中

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} dx \qquad (n = 0, 1, 2, 3, \dots)$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} dx \qquad (n = 0, 1, 2, 3, \dots)$$

$$g(x+2\pi) = f(\frac{l}{\pi}(x+2\pi)) = f(\frac{l}{\pi}x+2l) = f(\frac{l}{\pi}x)$$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$$

其中

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} dx \qquad (n = 0, 1, 2, 3, \dots)$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} dx \qquad (n = 0, 1, 2, 3, \dots)$$

$$g(x+2\pi) = f(\frac{l}{\pi}(x+2\pi)) = f(\frac{l}{\pi}x+2l) = f(\frac{l}{\pi}x) = g(x)$$



$$\frac{a_0}{2} + \sum_{l=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$$

其中

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} dx \qquad (n = 0, 1, 2, 3, \dots)$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} dx \qquad (n = 0, 1, 2, 3, \dots)$$

"推导" 令 $g(x) = f(\frac{l}{\pi}x)$, 则 g 是周期为 2π 的周期函数:

$$g(x+2\pi) = f(\frac{l}{\pi}(x+2\pi)) = f(\frac{l}{\pi}x+2l) = f(\frac{l}{\pi}x) = g(x)$$

所以

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$$g(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$



$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$$

其中 $a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} dx \qquad (n = 0, 1, 2, 3, \dots)$ $b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} dx \qquad (n = 0, 1, 2, 3, \dots)$

"推导" 令
$$g(x) = f(\frac{l}{\pi}x)$$
,则 g 是周期为 2π 的周期函数:
$$g(x+2\pi) = f(\frac{l}{\pi}(x+2\pi)) = f(\frac{l}{\pi}x+2l) = f(\frac{l}{\pi}x) = g(x)$$

所以 $f(\frac{l}{\pi}x) = g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right)$



既然
$$f(\frac{l}{\pi}x) = g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$



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$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$$



既然
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其中

 a_n

 b_n



既然 $f(\frac{l}{\pi}x) = g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$$

其中

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(z) \cos nz dz$$

 b_n



既然 $f(\frac{l}{\pi}x) = g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(z) \cos nz dz$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(z) \sin nz dz$$



死然 $f(\frac{l}{\pi}x) = g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$

所以
$$f(x) = \frac{a_0}{2} + \sum_{i=1}^{\infty} \left(a_i \cos \frac{n\pi x}{l} + b_i \sin \frac{n\pi x}{l} \right)$$

其中 $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(z) \cos nz dz = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\frac{l}{\pi}z) \cos nz dz$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(z) \sin nz dz$$



既然
$$f(\frac{l}{\pi}x) = g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(z) \cos nz dz = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\frac{l}{\pi}z) \cos nz dz$$

$$x = \frac{l}{\pi}z$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(z) \sin nz dz$$



既然
$$f(\frac{l}{\pi}x) = g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(z) \cos nz dz = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\frac{l}{\pi}z) \cos nz dz$$

$$\frac{x = \frac{l}{\pi}z}{\pi} \frac{1}{\pi} \int f(x) \cos \frac{n\pi x}{l}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(z) \sin nz dz$$



既然
$$f(\frac{l}{\pi}x) = g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(z) \cos nz dz = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\frac{l}{\pi}z) \cos nz dz$$

$$\xrightarrow{\frac{x = \frac{l}{\pi}z}{\pi}} \frac{1}{\pi} \int f(x) \cos \frac{n\pi x}{l} d(\frac{\pi}{l}x)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(z) \sin nz dz$$



既然
$$f(\frac{l}{\pi}x) = g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(z) \cos nz dz = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\frac{l}{\pi}z) \cos nz dz$$

$$= \frac{x = \frac{l}{\pi}z}{\pi} \frac{1}{\pi} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} d(\frac{\pi}{l}x)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(z) \sin nz dz$$



既然
$$f(\frac{l}{\pi}x) = g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$$

其中

$$\begin{aligned}
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} g(z) \cos nz dz = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\frac{l}{\pi}z) \cos nz dz \\
&= \frac{x = \frac{l}{\pi}z}{\pi} \frac{1}{\pi} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} d(\frac{\pi}{l}x) = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} dx, \\
b_n &= \frac{1}{l} \int_{-\pi}^{\pi} g(z) \sin nz dz
\end{aligned}$$

 $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(z) \sin nz dz$



既然
$$f(\frac{l}{\pi}x) = g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$$

$$a_{n} = \frac{1}{\pi} \int_{-\pi}^{\pi} g(z) \cos nz dz = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\frac{l}{\pi}z) \cos nz dz$$

$$\frac{x = \frac{l}{\pi}z}{\pi} \frac{1}{\pi} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} d(\frac{\pi}{l}x) = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} dx,$$

$$b_{n} = \frac{1}{\pi} \int_{-\pi}^{\pi} g(z) \sin nz dz = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\frac{l}{-z}) \sin nz dz$$



既然
$$f(\frac{l}{\pi}x) = g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(z) \cos nz dz = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\frac{l}{\pi}z) \cos nz dz$$

$$\frac{x = \frac{l}{\pi}z}{\pi} \frac{1}{\pi} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} d(\frac{\pi}{l}x) = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} dx,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(z) \sin nz dz = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\frac{l}{\pi}z) \sin nz dz$$



既然
$$f(\frac{l}{\pi}x) = g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right)$$

其中

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(z) \cos nz dz = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\frac{l}{\pi}z) \cos nz dz$$

$$\pi \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \pi \int_{-\pi}^{\pi} \pi \int_{-\pi}^{\pi} \frac{1}{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{1}{\pi} \int_{-\pi}^{\pi} \int_{-\pi$$

 $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(z) \sin nz dz = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\frac{l}{\pi}z) \sin nz dz$$

$$\frac{x = \frac{l}{\pi}z}{\pi} \frac{1}{\pi} \int f(x) \sin \frac{n\pi x}{l}$$



既然
$$f(\frac{l}{\pi}x) = g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(z) \cos nz dz = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\frac{l}{\pi}z) \cos nz dz$$

$$\frac{\pi}{1-\pi} \int_{-\pi}^{\pi} \frac{\pi}{1-\pi} \int_{-\pi}^{\pi} f(x) \cos \frac{n\pi x}{1} d(\frac{\pi}{1}x) = \frac{1}{1} \int_{-\pi}^{\pi} f(x) \cos \frac{n\pi x}{1} dx,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(z) \sin nz dz = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\frac{l}{\pi}z) \sin nz dz$$

$$\frac{x = \frac{l}{\pi}z}{\pi} \frac{1}{\pi} \int f(x) \sin \frac{n\pi x}{l} d(\frac{\pi}{l}x)$$



既然
$$f(\frac{l}{\pi}x) = g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(z) \cos nz dz = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\frac{l}{\pi}z) \cos nz dz$$

$$\int_{-\pi}^{\pi} \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{g(z) \cos nz dz}{\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{f(-z) \cos nz dz}{\pi}}$$

$$\frac{x = \frac{1}{\pi} z}{\pi} \int_{-\pi}^{\pi} \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f(x) \cos nz dz}{\int_{-\pi}^{\pi} \frac{f(-z) \cos nz dz}{\pi}} dx,$$

$$b_{n} = \frac{1}{\pi} \int_{-\pi}^{\pi} g(z) \sin nz dz = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\frac{l}{\pi}z) \sin nz dz$$

$$\frac{x = \frac{l}{\pi}z}{\pi} \frac{1}{\pi} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} d(\frac{\pi}{l}x)$$



既然
$$f(\frac{l}{\pi}x) = g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$$

其中

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(z) \cos nz dz = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\frac{l}{\pi}z) \cos nz dz$$

$$a_{n} = \frac{1}{\pi} \int_{-\pi}^{\pi} g(z) \cos nz dz = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\frac{1}{\pi}z) \cos nz dz$$

$$= \frac{x = \frac{1}{\pi}z}{\pi} \int_{-\pi}^{t} f(x) \cos \frac{n\pi x}{t} d(\frac{\pi}{t}x) = \frac{1}{t} \int_{-\pi}^{t} f(x) \cos \frac{n\pi x}{t} dx,$$

$$\pi \int_{-l}^{-l} f(z) dz = \frac{1}{\pi} \int_{-\pi}^{\pi} f(-z) \sin nz dz$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(z) \sin nz dz = \frac{1}{\pi} \int_{-\pi}^{\pi} f(-z) \sin nz dz$$

 $\frac{x=\frac{l}{\pi}z}{\pi} \frac{1}{\pi} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} d(\frac{\pi}{L}x) = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx.$

