第 09 周作业解答

练习 1. 问 $\beta = \begin{pmatrix} 2 \\ 0 \\ 3 \\ -1 \\ 3 \end{pmatrix}$ 是否能由向量组 $\alpha_1 = \begin{pmatrix} 0 \\ 2 \\ 1 \\ 5 \\ -1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 4 \\ -1 \end{pmatrix}$ 线性表示？若能，写出其中一个线性组合的表达式。

解

$$\left(\alpha_1 \quad \alpha_2 \quad \alpha_3 \mid \beta \right) = \left(\begin{array}{ccc|c} 0 & 1 & 1 & 2 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 3 \\ 5 & 2 & 4 & -1 \\ -1 & 1 & -1 & 3 \end{array}\right) \xrightarrow{r_1 \leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 5 & 2 & 4 & -1 \\ -1 & 1 & -1 & 3 \end{array}\right) \xrightarrow[\substack{r_4-5r_1 \\ r_5+r_1}]{r_2-2r_1} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & -3 & 1 & -6 \\ 0 & 1 & 1 & 2 \\ 0 & -8 & 4 & -16 \\ 0 & 3 & -1 & 6 \end{array}\right)$$

$$\xrightarrow[\frac{1}{4}\times r_4]{r_2 \leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & -3 & 1 & -6 \\ 0 & -2 & 1 & -4 \\ 0 & 3 & -1 & 6 \end{array}\right) \xrightarrow[\substack{r_4+2r_2 \\ r_5-3r_2}]{r_3+3r_2} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right) \xrightarrow[\substack{\frac{1}{3}\times r_4 \\ -\frac{1}{4}\times r_5}]{\frac{1}{4}\times r_3} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[\substack{r_5-r_3 \\ r_2-r_3}]{r_4-r_3} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{r_1-2r_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

可见 $r(\alpha_1\alpha_2\alpha_3) = r(\alpha_1\alpha_2\alpha_3\beta)$，所以 β 能由 $\alpha_1, \alpha_2, \alpha_3$。并且从最后简化的阶梯型矩阵容易看出：

$$\beta = -\alpha_1 + 2\alpha_2 + 0\alpha_3 = -\alpha_1 + 2\alpha_2.$$

练习 2. 问向量组 $\alpha_1 = \begin{pmatrix} 3 \\ -1 \\ 2 \\ 3 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ 是否线性相关？若线性相关，写出它们的一个相关表达式。

解

$$\left(\alpha_1 \quad \alpha_2 \quad \alpha_3 \right) = \begin{pmatrix} 3 & 1 & 0 \\ -1 & 1 & 0 \\ 2 & 2 & 1 \\ 3 & 0 & 1 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{pmatrix} -1 & 1 & 0 \\ 3 & 1 & 0 \\ 2 & 2 & 1 \\ 3 & 0 & 1 \end{pmatrix} \xrightarrow[\substack{r_3+2r_1 \\ r_4+3r_1}]{r_2+3r_1} \begin{pmatrix} -1 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & 4 & 1 \\ 0 & 3 & 1 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{4}\times r_2} \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 3 & 1 \end{pmatrix} \xrightarrow[r_4-3r_2]{r_3-4r_2} \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{r_4-r_3} \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

可见 $r(\alpha_1\alpha_2\alpha_3) = 3 =$ 向量个数，所以 $\alpha_1, \alpha_2, \alpha_3$ 线性无关。

练习 3. 根据参数 a 的取值，讨论向量组 $\alpha_1 = \begin{pmatrix}3\\1\\a\end{pmatrix}$, $\alpha_2 = \begin{pmatrix}4\\a\\0\end{pmatrix}$, $\alpha_3 = \begin{pmatrix}1\\0\\a\end{pmatrix}$ 何时线性相关，何时线性无关。

解作矩阵

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix} = \begin{pmatrix} 3 & 4 & 1 \\ 1 & a & 0 \\ a & 0 & a \end{pmatrix},$$

则 $\alpha_1, \alpha_2, \alpha_3$ 线性相关当且仅当 $|A| = 0$，线性无关当且仅当 $|A| \neq 0$。计算行列式：

$$|A| = \begin{vmatrix} 3 & 4 & 1 \\ 1 & a & 0 \\ a & 0 & a \end{vmatrix} \xlongequal{c_1 - c_3} \begin{vmatrix} 2 & 4 & 1 \\ 1 & a & 0 \\ 0 & 0 & a \end{vmatrix} \xlongequal{\text{按第 3 行展开}} (-1)^{3+3} a \begin{vmatrix} 2 & 4 \\ 1 & a \end{vmatrix} = 2a(a-2).$$

所以

- $\alpha_1, \alpha_2, \alpha_3$ 线性相关 $\Leftrightarrow |A| = 0 \Leftrightarrow a = 0$ 或 $a = 2$
- $\alpha_1, \alpha_2, \alpha_3$ 线性无关 $\Leftrightarrow |A| \neq 0 \Leftrightarrow a \neq 0$ 且 $a \neq 2$

练习 4. 设 $\beta_1 = \alpha_1 + \alpha_2, \beta_2 = \alpha_2 + \alpha_3, \beta_3 = \alpha_3 + \alpha_4, \beta_4 = \alpha_4 + \alpha_1$。证明 $\beta_1, \beta_2, \beta_3, \beta_4$ 线性相关。

解设

$$\begin{aligned} 0 =& k_1\alpha + k_2(\alpha+\beta) + k_3(\alpha+\beta+\gamma) \\ =& (k_1+k_2+k_3)\alpha + (k_2+k_3)\beta + k_3\gamma \end{aligned}$$

因为 α, β, γ 线性无关，所以

$$\begin{cases} k_1 + k_2 + k_3 = 0 \\ \qquad k_2 + k_3 = 0 \\ \qquad\qquad k_3 = 0 \end{cases} \quad \Rightarrow \quad k_1 = k_2 = k_3 = 0$$

所以 $\alpha, \alpha+\beta, \alpha+\beta+\gamma$ 线性无关。

另证注意到

$$\underbrace{\begin{pmatrix} \alpha & \alpha+\beta & \alpha+\beta+\gamma \end{pmatrix}}_{\text{记为矩阵}Q} = \underbrace{\begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix}}_{\text{记为矩阵}P} \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{记为矩阵}A} \quad \Rightarrow \quad Q = PA$$

而 $|A| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$，所以 A 可逆，从而

$$r(Q) = r(PA) = r(P) = 3 = (\text{向量组}\alpha, \alpha+\beta, \alpha+\beta+\gamma\text{向量个数})$$

即：

$$r(\alpha, \alpha+\beta, \alpha+\beta+\gamma) = (\text{向量组}\alpha, \alpha+\beta, \alpha+\beta+\gamma\text{向量个数})$$

所以 $\alpha, \alpha+\beta, \alpha+\beta+\gamma$ 线性无关。