# §1.4 克莱姆法则

数学系 梁卓滨

2017 - 2018 学年 I



# Outline of §1.4

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_2 + a_{22}y_2 = b_2 \end{cases}$$



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$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} , \quad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

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$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_2 + a_{22}y_2 = b_2 \end{cases}$$

的解是

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{D_1}{D}, \quad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{D_2}{D}$$

注

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$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$
 称为系数行列式

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•  $D_i$ : 将 D 的第 i 列换成常数项  $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ 



三元线性方程组  $\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases}$  的解是

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$$x_{1} = \frac{\begin{vmatrix} b_{1} & a_{12} & a_{13} \\ b_{2} & a_{22} & a_{23} \\ b_{3} & a_{32} & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} = \frac{D_{1}}{D}, \quad x_{2} = \frac{\begin{vmatrix} a_{11} & b_{1} & a_{13} \\ a_{21} & b_{2} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & b_{3} \end{vmatrix}} = \frac{D_{2}}{D}$$

$$x_{3} = \frac{\begin{vmatrix} a_{11} & a_{12} & b_{1} \\ a_{21} & a_{22} & b_{2} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}$$
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$$x_{3} = \frac{A_{11} + A_{12} + A_{13} + A_{13}}{A_{21} + A_{22} + A_{23} + A_{23}}$$

$$a_{21} = \frac{A_{22} + A_{23}}{A_{23} + A_{23}}$$

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$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

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$$\begin{vmatrix} a_{11} & \cdots & a_{1j-1} & a_{1,j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2,j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{n,j} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}$$

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### 定理(克莱姆法则) 线性方程组

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当系数行列式  $D \neq 0$  时,方程具有唯一解:



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注 两个前提: (1) 未知元个数 = 方程个数; (2) 系数行列式  $D \neq 0$ 



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注 两个前提: (1) 未知元个数 = 方程个数; (2) 系数行列式  $D \neq 0$ 

(3) 若 D=0,则方程有无穷多解或无解(以后详说)



克莱姆法则证明 (仅验证  $x_j = \frac{D_j}{D}$  是解,唯一性的证明要用到矩阵知识,略去。)

$$x_j = \frac{D_j}{D}$$

验证第 
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$$x_{j} = \frac{D_{j}}{D} = \frac{\begin{vmatrix} a_{11} & \cdots & a_{1j-1} & b_{1} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & b_{2} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & b_{n} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}}$$

验证第 k 条方程成立  $(k = 1, 2, \dots, n)$ :

$$a_{k1}x_1 + \cdots + a_{kn}x_n =$$

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$$x_{j} = \frac{D_{j}}{D} = \frac{\begin{vmatrix} a_{21} & a_{2j+1} & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & b_{n} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}}$$

 $a_{k1}x_1 + \cdots + a_{kn}x_n =$ 



D

 $x_{j} = \frac{D_{j}}{D} = \frac{\begin{vmatrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & b_{n} & a_{nj+1} & \cdots & a_{nn} \\ a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}} = \frac{b_{1} + b_{2} + \cdots + b_{n}}{D}$ 



 $x_{j} = \frac{D_{j}}{D} = \frac{\begin{vmatrix} \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & b_{n} & a_{nj+1} & \cdots & a_{nn} \\ a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}} = \frac{b_{1}A_{1j} + b_{2} + \cdots + b_{n}}{D}$ 

验证第 k 条方程成立  $(k = 1, 2, \dots, n)$ :



 $x_{j} = \frac{D_{j}}{D} = \frac{\begin{vmatrix} \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & b_{n} & a_{nj+1} & \cdots & a_{nn} \\ a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}} = \frac{b_{1}A_{1j} + b_{2}A_{2j} + \cdots + b_{n}}{D}$ 





 $x_{j} = \frac{D_{j}}{D} = \frac{\begin{vmatrix} \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & b_{n} & a_{nj+1} & \cdots & a_{nn} \\ a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}} = \frac{b_{1}A_{1j} + b_{2}A_{2j} + \cdots + b_{n}}{D}$ 





 $x_{j} = \frac{D_{j}}{D} = \frac{\begin{vmatrix} \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & b_{n} & a_{nj+1} & \cdots & a_{nn} \\ a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}} = \frac{b_{1}A_{1j} + b_{2}A_{2j} + \cdots + b_{n}A_{nj}}{D}$ 

验证第 
$$K$$
 条万桯成立( $K=1,2,\cdots,n$ )

$$x_{j} = \frac{D_{j}}{D} = \frac{\begin{vmatrix} a_{21} & \cdots & a_{2j-1} & b_{2} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj-1} & b_{n} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}} = \frac{b_{1}A_{1j} + b_{2}A_{2j} + \cdots + b_{n}A_{nj}}{D}$$

$$\sum_{i=1} b_i A_{ij}$$

验证第 k 条方程成立  $(k = 1, 2, \dots, n)$ :

$$a_{k1}x_1 + \cdots + a_{kn}x_n =$$



克莱姆法则证明 (仅验证  $x_j = \frac{D_j}{D}$  是解,唯一性的证明要用到矩阵知识,略去。)  $|a_{11} \cdots a_{1i-1}|_{D} |a_{1i+1} \cdots a_{1i}|_{D}$ 

$$x_{j} = \frac{D_{j}}{D} = \frac{\begin{vmatrix} a_{11} & \cdots & a_{1j-1} & b_{1} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & b_{2} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & b_{n} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}} = \frac{b_{1}A_{1j} + b_{2}A_{2j} + \cdots + b_{n}A_{nj}}{D}$$

$$= \frac{1}{D} \sum_{i=1}^{n} b_{i}A_{ij}$$

验证第 
$$k$$
 条方程成立( $k = 1, 2, \dots, n$ ):

 $a_{k1}x_1 + \cdots + a_{kn}x_n =$ 





克莱姆法则证明 (仅验证  $X_i = \frac{D_i}{D}$  是解,唯一性的证明要用到矩阵知识,略去。)  $\begin{vmatrix} a_{11} & \cdots & a_{1j-1} & b_1 & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & b_2 & a_{2j+1} & \cdots & a_{2n} \end{vmatrix}$ 

$$x_{j} = \frac{D_{j}}{D} = \frac{\begin{vmatrix} \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & b_{n} & a_{nj+1} & \cdots & a_{nn} \\ a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}} = \frac{b_{1}A_{1j} + b_{2}A_{2j} + \cdots + b_{n}A_{nj}}{D}$$

验证第 k 条方程成立  $(k = 1, 2, \dots, n)$ :

 $a_{k1}x_1 + \dots + a_{kn}x_n = \sum_{i=1}^n a_{kj}x_j$ 

 $=\frac{1}{D}\sum_{i}^{n}b_{i}A_{ij}$ 

克莱姆法则证明 (仅验证  $X_i = \frac{D_i}{D}$  是解,唯一性的证明要用到矩阵知识,略去。)  $a_{11} \cdots a_{1j-1} \xrightarrow{b_1} a_{1j+1} \cdots a_{1n}$  $a_{21} \cdots a_{2j-1} \xrightarrow{b_2} a_{2j+1} \cdots a_{2n}$ 

$$x_{j} = \frac{D_{j}}{D} = \frac{\begin{vmatrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & b_{n} & a_{nj+1} & \cdots & a_{nn} \\ a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}} = \frac{b_{1}A_{1j} + b_{2}A_{2j} + \cdots + b_{n}A_{nj}}{D}$$

$$= \frac{1}{D} \sum_{i=1}^{n} b_{i}A_{ij}$$
验证第  $k$  条方程成立  $(k = 1, 2, \cdots, n)$ :

 $a_{k1}x_1 + \dots + a_{kn}x_n = \sum_{i=1}^n a_{kj}x_j = \sum_{i=1}^n a_{kj} \left(\frac{1}{D}\sum_{i=1}^n b_i A_{ij}\right)$ 





克莱姆法则证明 (仅验证  $X_i = \frac{D_i}{D}$  是解,唯一性的证明要用到矩阵知识,略去。)  $a_{11} \cdots a_{1j-1} \xrightarrow{b_1} a_{1j+1} \cdots a_{1n}$  $a_{21} \cdots a_{2j-1} \xrightarrow{b_2} a_{2j+1} \cdots a_{2n}$ 

$$x_{j} = \frac{D_{j}}{D} = \frac{\begin{vmatrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & b_{n} & a_{nj+1} & \cdots & a_{nn} \\ a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}} = \frac{b_{1}A_{1j} + b_{2}A_{2j} + \cdots + b_{n}A_{nj}}{D}$$

$$G_j = \frac{D_j}{D} = \frac{|a_{ij}|}{|a_{11} \cdots a_{1j-1} a_{1j} a_{1j+1} \cdots a_{1j-1} a_{1j+1} a_{1j+1} \cdots a_{1j-1} a_{1j+1} a_{1j+1} \cdots a_{1j-1} a_{1j+1} \cdots a_{1j+1} \cdots a_{1j+1} a_{1j+1} \cdots a_{1j+1}$$

 $=\frac{1}{D}\sum_{i}^{n}b_{i}A_{ij}$ 

验证第 k 条方程成立  $(k = 1, 2, \dots, n)$ :

 $a_{k1}x_1 + \dots + a_{kn}x_n = \sum_{i=1}^n a_{kj}x_j = \sum_{i=1}^n a_{kj} \left(\frac{1}{D}\sum_{i=1}^n b_i A_{ij}\right) = \frac{1}{D}\sum_{i=1}^n \sum_{i=1}^n a_{kj}b_i A_{ij}$ 

 $a_{11} \cdots a_{1j-1} \xrightarrow{b_1} a_{1j+1} \cdots a_{1n}$  $a_{21} \cdots a_{2j-1} \xrightarrow{b_2} a_{2j+1} \cdots a_{2n}$ 

克莱姆法则证明 (仅验证  $X_i = \frac{D_i}{D}$  是解,唯一性的证明要用到矩阵知识,略去。)

$$x_{j} = \frac{D_{j}}{D} = \frac{\begin{vmatrix} \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & b_{n} & a_{nj+1} & \cdots & a_{nn} \\ a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}} = \frac{b_{1}A_{1j} + b_{2}A_{2j} + \cdots + b_{n}A_{nj}}{D}$$

 $=\frac{1}{D}\sum_{i}^{n}b_{i}A_{ij}$ 

$$D \stackrel{\frown}{\underset{i=1}{\sum}} D(n)$$
  
验证第  $k$  条方程成立( $k = 1, 2, \dots, n$ ):

 $a_{k1}x_1 + \dots + a_{kn}x_n = \sum_{i=1}^n a_{kj}x_j = \sum_{i=1}^n a_{kj} \left(\frac{1}{D}\sum_{i=1}^n b_i A_{ij}\right) = \frac{1}{D}\sum_{i=1}^n \sum_{i=1}^n a_{kj}b_i A_{ij}$ 

 $=\frac{1}{D}\sum_{i=1}\sum_{i=1}a_{kj}b_iA_{ij}$ 

 $b_k$ 

 $a_{11} \cdots a_{1j-1} \xrightarrow{b_1} a_{1j+1} \cdots a_{1n}$  $a_{21} \cdots a_{2j-1} \xrightarrow{b_2} a_{2j+1} \cdots a_{2n}$ 

克莱姆法则证明 (仅验证  $X_i = \frac{D_i}{D}$  是解,唯一性的证明要用到矩阵知识,略去。)

$$x_{j} = \frac{D_{j}}{D} = \frac{\begin{vmatrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & b_{n} & a_{nj+1} & \cdots & a_{nn} \\ a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}} = \frac{b_{1}A_{1j} + b_{2}A_{2j} + \cdots + b_{n}A_{nj}}{D}$$

$$= \frac{1}{D} \sum_{i=1}^{n} b_{i}A_{ij}$$

$$=\frac{1}{D}\sum_{i=1}^{N}b_{i}A_{ij}$$
验证第  $k$  条方程成立( $k=1,2,\cdots,n$ ):

$$a_{k1}x_1 + \dots + a_{kn}x_n = \sum_{i=1}^n a_{kj}x_j = \sum_{i=1}^n a_{kj} \left(\frac{1}{D}\sum_{i=1}^n b_i A_{ij}\right) = \frac{1}{D}\sum_{i=1}^n \sum_{i=1}^n a_{kj}b_i A_{ij}$$

$$\sum_{j=1}^{n} a_{i,j} x_{i,j} = \sum_{j=1}^{n} a_{i,j} \left( D \sum_{i=1}^{n} a_{i,j} x_{i,j} \right) = D \sum_{j=1}^{n} \sum_{i=1}^{n} a_{i,j} x_{i,j} x_{i,j}$$

br

 $a_{11} \cdots a_{1j-1} \xrightarrow{b_1} a_{1j+1} \cdots a_{1n}$  $a_{21} \cdots a_{2j-1} \xrightarrow{b_2} a_{2j+1} \cdots a_{2n}$  $x_{j} = \frac{D_{j}}{D} = \frac{\begin{vmatrix} \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & b_{n} & a_{nj+1} & \cdots & a_{nn} \\ a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}} = \frac{b_{1}A_{1j} + b_{2}A_{2j} + \cdots + b_{n}A_{nj}}{D}$ 

克莱姆法则证明 (仅验证  $X_i = \frac{D_i}{D}$  是解,唯一性的证明要用到矩阵知识,略去。)

$$\begin{vmatrix} a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}$$

$$= \frac{1}{D} \sum_{i=1}^{n} b_i A_{ij}$$

验证第 k 条方程成立 ( $k = 1, 2, \dots, n$ ):

 $a_{k1}x_1 + \dots + a_{kn}x_n = \sum_{i=1}^n a_{kj}x_j = \sum_{i=1}^n a_{kj} \left(\frac{1}{D}\sum_{i=1}^n b_i A_{ij}\right) = \frac{1}{D}\sum_{i=1}^n \sum_{i=1}^n a_{kj}b_i A_{ij}$ 

 $\sum_{i=1}^{n} a_{kj} b_i A_{ij} = b_i \sum_{i=1}^{n} a_{kj} A_{ij}$ 

 $a_{11} \cdots a_{1j-1} \xrightarrow{b_1} a_{1j+1} \cdots a_{1n}$  $a_{21} \cdots a_{2j-1} \xrightarrow{b_2} a_{2j+1} \cdots a_{2n}$ 

克莱姆法则证明 (仅验证  $x_j = \frac{D_j}{D}$  是解,唯一性的证明要用到矩阵知识,略去。)

$$x_{j} = \frac{D_{j}}{D} = \frac{\begin{vmatrix} \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & b_{n} & a_{nj+1} & \cdots & a_{nn} \\ a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}} = \frac{b_{1}A_{1j} + b_{2}A_{2j} + \cdots + b_{n}A_{nj}}{D}$$

$$= \frac{1}{D} \sum_{i=1}^{n} b_{i}A_{ij}$$

$$D \underset{i=1}{\overset{r}{\smile}}$$
 验证第  $k$  条方程成立( $k=1,2,\cdots,n$ ):

验证第 
$$k$$
 条方程成立( $k = 1, 2, \dots, n$ ):
$$a_{k1}x_1 + \dots + a_{kn}x_n = \sum_{i=1}^n a_{kj}x_j = \sum_{i=1}^n a_{kj} \left(\frac{1}{D}\sum_{i=1}^n b_i A_{ij}\right) = \frac{1}{D}\sum_{i=1}^n \sum_{i=1}^n a_{kj}b_i A_{ij}$$

$$a_{k1}x_1 + \dots + a_{kn}x_n = \sum_{j=1}^{n} a_{kj}x_j = \sum_{j=1}^{n} a_{kj} \left(\frac{1}{D}\sum_{i=1}^{n} b_i A_{ij}\right) = \frac{1}{D}\sum_{j=1}^{n} \sum_{i=1}^{n} a_{kj}b_i A_{ij}$$

 $= \frac{1}{D} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{kj} b_i A_{ij} = b_i \sum_{j=1}^{n} a_{kj} A_{ij}$ 

 $a_{11} \cdots a_{1j-1} \xrightarrow{b_1} a_{1j+1} \cdots a_{1n}$  $a_{21} \cdots a_{2j-1} \xrightarrow{b_2} a_{2j+1} \cdots a_{2n}$  $x_{j} = \frac{D_{j}}{D} = \frac{\begin{vmatrix} \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & b_{n} & a_{nj+1} & \cdots & a_{nn} \\ a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}} = \frac{b_{1}A_{1j} + b_{2}A_{2j} + \cdots + b_{n}A_{nj}}{D}$ 

克莱姆法则证明 (仅验证  $X_i = \frac{D_i}{D}$  是解,唯一性的证明要用到矩阵知识,略去。)

$$\begin{vmatrix} \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}$$

$$= \frac{1}{D} \sum_{i=1}^{n} b_i A_{ij}$$
验证第  $k$  条方程成立( $k = 1, 2, \cdots, n$ ):

 $a_{k1}x_1 + \dots + a_{kn}x_n = \sum_{i=1}^n a_{kj}x_j = \sum_{i=1}^n a_{kj} \left(\frac{1}{D}\sum_{i=1}^n b_i A_{ij}\right) = \frac{1}{D}\sum_{i=1}^n \sum_{i=1}^n a_{kj}b_i A_{ij}$ 

 $= \frac{1}{D} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{kj} b_i A_{ij} = \frac{1}{D} \sum_{i=1}^{n} b_i \sum_{j=1}^{n} a_{kj} A_{ij}$ 

 $a_{11} \cdots a_{1j-1} \xrightarrow{b_1} a_{1j+1} \cdots a_{1n}$  $a_{21} \cdots a_{2j-1} \xrightarrow{b_2} a_{2j+1} \cdots a_{2n}$  $x_{j} = \frac{D_{j}}{D} = \frac{\begin{vmatrix} \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & b_{n} & a_{nj+1} & \cdots & a_{nn} \\ a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}} = \frac{b_{1}A_{1j} + b_{2}A_{2j} + \cdots + b_{n}A_{nj}}{D}$ 

克莱姆法则证明 (仅验证  $x_j = \frac{D_j}{D}$  是解,唯一性的证明要用到矩阵知识,略去。)

$$=rac{1}{D}\sum_{i=1}^n b_i A_{ij}$$
  
验证第  $k$  条方程成立( $k=1,2,\cdots,n$ ):

 $a_{k1}x_1 + \dots + a_{kn}x_n = \sum_{i=1}^n a_{kj}x_j = \sum_{i=1}^n a_{kj} \left(\frac{1}{D}\sum_{i=1}^n b_i A_{ij}\right) = \frac{1}{D}\sum_{i=1}^n \sum_{i=1}^n a_{kj}b_i A_{ij}$ 

 $= \frac{1}{D} \sum_{i=1}^{n} \sum_{i=1}^{n} a_{kj} b_i A_{ij} = \frac{1}{D} \sum_{i=1}^{n} b_i \sum_{i=1}^{n} a_{kj} A_{ij} \qquad b_k \sum_{i=1}^{n} a_{kj} A_{kj}$ 

 $\begin{vmatrix} a_{11} & \cdots & a_{1j-1} & b_1 & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & b_2 & a_{2j+1} & \cdots & a_{2n} \end{vmatrix}$  $x_{j} = \frac{D_{j}}{D} = \frac{\begin{vmatrix} \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & b_{n} & a_{nj+1} & \cdots & a_{nn} \\ a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}} = \frac{b_{1}A_{1j} + b_{2}A_{2j} + \cdots + b_{n}A_{nj}}{D}$ 

克莱姆法则证明 (仅验证  $x_j = \frac{D_j}{D}$  是解,唯一性的证明要用到矩阵知识,略去。)

$$\begin{vmatrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}$$

$$= \frac{1}{D} \sum_{i=1}^{n} b_i A_{ij}$$

验证第 k 条方程成立 ( $k = 1, 2, \dots, n$ ):

 $a_{k1}x_1 + \dots + a_{kn}x_n = \sum_{i=1}^n a_{kj}x_j = \sum_{i=1}^n a_{kj} \left(\frac{1}{D}\sum_{i=1}^n b_i A_{ij}\right) = \frac{1}{D}\sum_{i=1}^n \sum_{i=1}^n a_{kj}b_i A_{ij}$ 

 $= \frac{1}{D} \sum_{i=1}^{n} \sum_{k=1}^{n} a_{kj} b_i A_{ij} = \frac{1}{D} \sum_{i=1}^{n} b_i \sum_{k=1}^{n} a_{kj} A_{ij} = \frac{1}{D} \cdot b_k \sum_{k=1}^{n} a_{kj} A_{kj}$ 

克莱姆法则证明 (仅验证  $x_j = \frac{D_j}{D}$  是解,唯一性的证明要用到矩阵知识,略去。)  $a_{11} \cdots a_{1j-1} \xrightarrow{b_1} a_{1j+1} \cdots a_{1n}$  $a_{21} \cdots a_{2j-1} \xrightarrow{b_2} a_{2j+1} \cdots a_{2n}$  $x_{j} = \frac{D_{j}}{D} = \frac{\begin{vmatrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & b_{n} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}} = \frac{b_{1}A_{1j} + b_{2}A_{2j} + \cdots + b_{n}A_{nj}}{D}$  $=\frac{1}{D}\sum_{i}^{n}b_{i}A_{ij}$ 

$$\begin{vmatrix} a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}$$

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验证第  $k$  条方程成立( $k = 1, 2, \cdots, n$ ):

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$$\bullet \begin{cases} x+y=1 \\ x+y=0 \end{cases}, D=\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}=0$$



• 
$$\begin{cases} x+y=1 \\ x+y=1 \end{cases}, D = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0, 实质上只有一条方程 x+y=1,$$
 显然有无穷多解。

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• 
$$\begin{cases} x+y=1\\ x+y=0 \end{cases}$$
 ,  $D=\begin{vmatrix} 1 & 1\\ 1 & 1 \end{vmatrix}=0$  , 方程组包含矛盾方程,显然无解。



#### 例 解线性方程组

$$\begin{cases} 2x_1 + x_2 - x_3 = 1\\ 3x_1 - x_2 - x_3 = -2\\ -x_1 + 2x_2 + x_3 = 6 \end{cases}$$

### 练习 解线性方程组

$$\begin{cases} x_1 + x_2 = 90 \\ x_2 + x_3 = 86 \\ x_1 + x_3 = 80 \end{cases}$$

#### 例 解线性方程组

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提示 
$$D = -5$$
,  $D_1 = -5$ ,  $D_2 = -10$ ,  $D_3 = -15$ 

### 练习 解线性方程组

$$\begin{cases} x_1 + x_2 = 90 \\ x_2 + x_3 = 86 \\ x_1 + x_3 = 80 \end{cases}$$

#### 例 解线性方程组

$$\begin{cases} 2x_1 + x_2 - x_3 = 1\\ 3x_1 - x_2 - x_3 = -2\\ -x_1 + 2x_2 + x_3 = 6 \end{cases}$$

提示 
$$D = -5$$
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### 练习 解线性方程组

$$\begin{cases} x_1 + x_2 = 90 \\ x_2 + x_3 = 86 \\ x_1 + x_3 = 80 \end{cases}$$

提示 
$$D = 2$$
,  $D_1 = 84$ ,  $D_2 = 96$ ,  $D_3 = 76$ 



#### 定理 齐次线性方程组

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = 0 \end{cases}$$

当系数行列式 
$$D \neq 0$$
 时,仅有零解( $x_1 = x_2 = \cdots = x_n = 0$ )

#### 定理 齐次线性方程组

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 当系数行列式  $D \neq 0$  时,仅有零解( $x_1 = x_2 = \cdots = x_n = 0$ )证明  $x_1 = x_2 = \cdots = x_n = 0$  显然是方程组的解

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另一方面,因为 $D \neq 0$ ,所以方程组有唯一解(克莱姆法则)

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§1.4 克莱姆法则

#### 定理 齐次线性方程组

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = 0 \end{cases}$$

当系数行列式 
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### 注

其实是充分必要条件:仅有零解的充分必要条件是 D ≠ 0



### 定理 齐次线性方程组

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当系数行列式 
$$D \neq 0$$
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另一方面,因为 $D \neq 0$ ,所以方程组有唯一解(克莱姆法则)

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### 注

其实是充分必要条件: 仅有零解的充分必要条件是 D ≠ 0



例 齐次方程组 
$$\begin{cases} x_1 - 2x_2 = 0 \\ 2x_1 - 4x_2 = 0 \end{cases}$$
 的系数矩阵  $D = \begin{vmatrix} 1 & -2 \\ 2 & -4 \end{vmatrix}$ 



例 齐次方程组 
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例 判断线性方程组 
$$\begin{cases} 2x_1 + 3x_2 + 4x_3 + 5x_4 = 0 \\ 3x_1 + 4x_2 + 5x_3 + 5x_4 = 0 \\ 4x_1 + 5x_2 + 6x_3 + 6x_4 = 0 \\ 5x_1 + 6x_2 + 8x_3 + 9x_4 = 0 \end{cases}$$
 是否只有零解

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2 3 4 5 3 4 5 5 4 5 6 6 5 6 8 9

$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 5 & 6 & 8 & 9 \end{vmatrix} \underline{r_4 - r_3}$$

$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 5 & 6 & 8 & 9 \end{vmatrix} \xrightarrow{\underline{r_4 - r_3}} \begin{vmatrix} \\ \\ \\ 1 & 1 & 2 & 3 \end{vmatrix}$$

2	3	4	5		2	3	4	5
3	4	5	5	$r_4-r_3$	3	4	5	5
4	5	6	6		4	5	6	6
5	6	8	9		1	1	2	3

$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 5 & 6 & 8 & 9 \end{vmatrix} \xrightarrow{\underline{r_4 - r_3}} \begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 1 & 1 & 2 & 3 \end{vmatrix} \xrightarrow{\underline{c_2 - c_1}}_{\substack{c_3 - 2c_1 \\ c_4 - 3c_1}}$$

2	3	4	5		2	3	4	5		2
3	4	5	5	$r_4-r_3$	3	4	5	5	$c_2 - c_1$	3
4	5	6	6		4	5	6	6	$c_3 - 2c_1$	4
5	6	8	9		1	1	2	3		1

-	2	3	4	5		2	3	4	5		2	1
	3	4	5	5	$r_4-r_3$	3	4	5	5	$c_2 - c_1$	3	1
1	4	5	6	6		4	5	6	6	$c_3 - 2c_1$	4	1
	5	6	8	9		1	1	2	3	$c_4 - 3c_1$	1	0
	4 5	5 6	6 8	6 9	<u>r_4-r_3</u>	4	5 1	6 2	6	$c_2-c_1$ $c_3-2c$ $c_4-3c$	1 1	$= \begin{vmatrix} 3 \\ 4 \\ 1 \end{vmatrix}$



2 3 4 5    2 3 4 5    2 1 0	
$\begin{vmatrix} 3 & 4 & 5 & 5 \end{vmatrix} \begin{vmatrix} r_4-r_3 \end{vmatrix} \begin{vmatrix} 3 & 4 & 5 & 5 \end{vmatrix} \begin{vmatrix} c_2-c_1 \end{vmatrix} \begin{vmatrix} 3 & 1 & -1 \end{vmatrix}$	L
$\begin{vmatrix} 4 & 5 & 6 & 6 \end{vmatrix} = \begin{vmatrix} 4 & 5 & 6 & 6 \end{vmatrix} = \begin{vmatrix} \frac{1}{c_3 - 2c_1} & 4 & 1 & -1 \end{vmatrix}$	2
$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 5 & 6 & 8 & 9 \end{vmatrix} \xrightarrow{\underline{r_4 - r_3}} \begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 1 & 1 & 2 & 3 \end{vmatrix} \xrightarrow{\underline{c_2 - c_1}} \begin{vmatrix} 2 & 1 & 0 \\ 3 & 1 & -1 \\ 4 & 1 & -1 \\ 1 & 0 & 0 \end{vmatrix}$	



• •													
2	3	4	5		2	3	4	5		2	1	0	-1
3	4	5	5	$r_4-r_3$	3	4	5	5	$c_2 - c_1$	3	1	-1	<b>- 4</b>
4	5	6	6		4	5	6	6	$c_3-2c_1$	4	1	<b>-</b> 2	- 6
5	6	8	9		1	1	2	3	$\frac{c_2 - c_1}{c_3 - 2c_1}$ $c_4 - 3c_1$	1	0	0	0
			- 1		1			- 1		1			



$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 5 & 6 & 8 & 9 \end{vmatrix} \xrightarrow{r_4 - r_3} \begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 1 & 1 & 2 & 3 \end{vmatrix} \xrightarrow{c_2 - c_1} \begin{vmatrix} 2 & 1 & 0 & -1 \\ 3 & 1 & -1 & -4 \\ 4 & 1 & -2 & -6 \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

$$= 1 \times (-1)^{4+1} \times \begin{vmatrix} 1 & 0 & -1 \\ 1 & -1 & -4 \\ 1 & -2 & -6 \end{vmatrix}$$



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$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 5 & 6 & 8 & 9 \end{vmatrix} \xrightarrow{\underline{r_4 - r_3}} \begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 1 & 1 & 2 & 3 \end{vmatrix} \xrightarrow{\underline{c_2 - c_1}} \begin{vmatrix} 2 & 1 & 0 & -1 \\ 3 & 1 & -1 & -4 \\ 4 & 1 & -2 & -6 \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

$$= 1 \times (-1)^{4+1} \times \begin{vmatrix} 1 & 0 & -1 \\ 1 & -1 & -4 \\ 1 & -2 & -6 \end{vmatrix} \xrightarrow{C_3 + C_1} - \begin{vmatrix} 1 & 0 & 0 \\ 1 & -1 & -3 \\ 1 & -2 & -5 \end{vmatrix}$$



$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 5 & 6 & 8 & 9 \end{vmatrix} \xrightarrow{\underline{r_4 - r_3}} \begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 1 & 1 & 2 & 3 \end{vmatrix} \xrightarrow{\underline{c_2 - c_1}} \begin{vmatrix} 2 & 1 & 0 & -1 \\ 3 & 1 & -1 & -4 \\ 4 & 1 & -2 & -6 \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

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$$= - \begin{vmatrix} -1 & -3 \\ -2 & -5 \end{vmatrix} = 1 \neq 0$$



$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 5 & 6 & 8 & 9 \end{vmatrix} \xrightarrow{\underline{r_4 - r_3}} \begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 1 & 1 & 2 & 3 \end{vmatrix} \xrightarrow{\underline{c_2 - c_1}} \begin{vmatrix} 2 & 1 & 0 & -1 \\ 3 & 1 & -1 & -4 \\ 4 & 1 & -2 & -6 \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

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$$= - \begin{vmatrix} -1 & -3 \\ -2 & -5 \end{vmatrix} = 1 \neq 0$$

所以齐次线性方程组有唯一解



练习 齐次线性方程组  $\begin{cases} kx_1 & + x_4 = 0 \\ x_1 + 2x_2 & - x_4 = 0 \\ (k+2)x_1 - x_2 & + 4x_4 = 0 \\ 2x_1 + x_2 + 3x_3 + kx_4 = 0 \end{cases}$ 有非零解

的充分必要条件是 k 满足  $_{\_\_\_}$ 



练习 齐次线性方程组 
$$\begin{cases} kx_1 + x_4 = 0 \\ x_1 + 2x_2 - x_4 = 0 \\ (k+2)x_1 - x_2 + 4x_4 = 0 \\ 2x_1 + x_2 + 3x_3 + kx_4 = 0 \end{cases}$$

有非零解

的充分必要条件是 k 满足 \_\_\_\_

$$D = \begin{vmatrix} k & 0 & 0 & 1 \\ 1 & 2 & 0 & -1 \\ k+2 & -1 & 0 & 4 \\ 2 & 1 & 3 & k \end{vmatrix}$$



练习 齐次线性方程组 
$$\begin{cases} kx_1 & + x_4 = 0 \\ x_1 + 2x_2 & - x_4 = 0 \\ (k+2)x_1 - x_2 & + 4x_4 = 0 \\ 2x_1 + x_2 + 3x_3 + kx_4 = 0 \end{cases}$$

的充分必要条件是 k 满足  $_{\_\_\_}$ 

$$D = \begin{vmatrix} k & 0 & 0 & 1 \\ 1 & 2 & 0 & -1 \\ k+2 & -1 & 0 & 4 \\ 2 & 1 & 3 & k \end{vmatrix} = 3.$$



有非零解

练习 齐次线性方程组 
$$\begin{cases} kx_1 + x_4 = 0 \\ x_1 + 2x_2 - x_4 = 0 \\ (k+2)x_1 - x_2 + 4x_4 = 0 \\ 2x_1 + x_2 + 3x_3 + kx_4 = 0 \end{cases}$$
 有非零解

的充分必要条件是 k 满足 \_\_\_\_

解

$$D = \begin{vmatrix} k & 0 & 0 & 1 \\ 1 & 2 & 0 & -1 \\ k+2 & -1 & 0 & 4 \\ 2 & 1 & 3 & k \end{vmatrix} = 3 \cdot (-1)^{3+4} \begin{vmatrix} k & 0 & 1 \\ 1 & 2 & -1 \\ k+2 & -1 & 4 \end{vmatrix}$$

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$$_{2}+r_{1}$$



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$$\frac{r_2 + r_1}{k} (-3) \cdot \begin{vmatrix} k & 0 & 1 \\ k & 0 & 1 \end{vmatrix}$$



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有非零解

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§1.4 克莱姆法则

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=-3(5k-5)

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有非零解当且仅当 D=0.

=-3(5k-5)



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有非零解当且仅当 D=0,当且仅当 k=1

=-3(5k-5)

