# 第 11 章 f: 高斯公式、斯托克斯公式

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# **Outline**

1. 高斯公式

2. 斯托克斯公式



# We are here now...

1. 高斯公式

2. 斯托克斯公式



定义 设 
$$F = (P, Q, R)$$
 是空间中向量场,定义

$$divF := \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

称为向量场 F 的 散度.

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例 1 计算向量场  $F = (x^2 + yz, y^2 + xz, z^2 + xy)$  的散度.

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 的散度.

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**例1** 计算向量场 
$$F = (x^2 + yz, y^2 + xz, z^2 + xy)$$
 的散度.

$$\operatorname{div} F = \frac{\partial}{\partial x}(x^2 + yz) + \frac{\partial}{\partial y}(y^2 + xz) + \frac{\partial}{\partial z}(z^2 + xy) = 2x + 2y + 2z.$$



**例 2** 计算梯度场  $\nabla_r^1$   $(r = \sqrt{x^2 + y^2 + z^2})$  的散度.

$$\nabla \frac{1}{r}$$

$$\operatorname{div}\nabla \frac{1}{r}$$



$$\nabla \frac{1}{r} = (\frac{\partial}{\partial x} r^{-1}, \frac{\partial}{\partial y} r^{-1}, \frac{\partial}{\partial z} r^{-1})$$

 $\operatorname{div}\nabla \frac{1}{r}$ 



$$\nabla \frac{1}{r} = (\frac{\partial}{\partial x} r^{-1}, \frac{\partial}{\partial y} r^{-1}, \frac{\partial}{\partial z} r^{-1})$$
$$-r^{-2} \cdot r_{x}$$
$$\operatorname{div} \nabla \frac{1}{r}$$



$$\nabla \frac{1}{r} = \left(\frac{\partial}{\partial x}r^{-1}, \frac{\partial}{\partial y}r^{-1}, \frac{\partial}{\partial z}r^{-1}\right)$$
$$= \left(-r^{-2} \cdot r_x, -r^{-2} \cdot r_y, -r^{-2} \cdot r_z\right)$$
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$$r_{x} = \frac{x}{r},$$

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**例 2** 计算梯度场  $\nabla_r^1$   $(r = \sqrt{x^2 + y^2 + z^2})$  的散度.

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$$(-\frac{1}{r^{3}} + \frac{3x^{2}}{r^{5}})$$

**例 2** 计算梯度场  $\nabla^1_r$   $(r = \sqrt{x^2 + y^2 + z^2})$  的散度.

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$$= (-\frac{1}{r^{3}} + \frac{3x^{2}}{r^{5}}) + (-\frac{1}{r^{3}} + \frac{3y^{2}}{r^{5}}) + (-\frac{1}{r^{3}} + \frac{3z^{2}}{r^{5}})$$

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$$= \left(-\frac{1}{r^3} + \frac{3x^2}{r^5}\right) + \left(-\frac{1}{r^3} + \frac{3y^2}{r^5}\right) + \left(-\frac{1}{r^3} + \frac{3z^2}{r^5}\right)$$
$$= -\frac{3}{r^3} + \frac{3(x^2 + y^2 + z^2)}{r^5}$$

$$r_X = \frac{x}{r}, \qquad r_y = \frac{y}{r}, \qquad r_z = \frac{z}{r},$$

$$\nabla \frac{1}{r} = (\frac{\partial}{\partial x} r^{-1}, \frac{\partial}{\partial y} r^{-1}, \frac{\partial}{\partial z} r^{-1})$$

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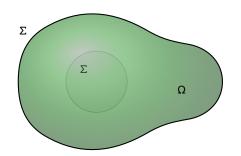
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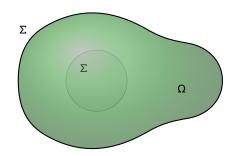
$$\operatorname{div} \nabla \frac{1}{r} = \frac{\partial}{\partial x}(-\frac{x}{r^3}) + \frac{\partial}{\partial y}(-\frac{y}{r^3}) + \frac{\partial}{\partial z}(-\frac{z}{r^3})$$

$$= \left(-\frac{1}{r^3} + \frac{3x^2}{r^5}\right) + \left(-\frac{1}{r^3} + \frac{3y^2}{r^5}\right) + \left(-\frac{1}{r^3} + \frac{3z^2}{r^5}\right)$$
$$= -\frac{3}{r^3} + \frac{3(x^2 + y^2 + z^2)}{r^5} = 0.$$

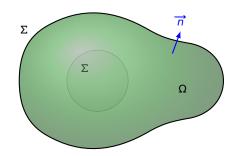
- 空间闭区域  $\Omega$  的边界是分片光滑的闭曲面  $\Sigma$ ,
- $\overrightarrow{n}$  是  $\Sigma$  的单位外法向量,



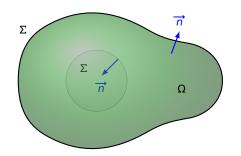
- 空间闭区域  $\Omega$  的边界是分片光滑的闭曲面  $\Sigma$ ,
- $\overrightarrow{n}$  是 Σ 的单位 外法向量,(指向 Ω 外部)



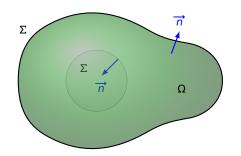
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- $\overrightarrow{n}$  是  $\Sigma$  的单位 外法向量,(指向  $\Omega$  外部)
- F = (P, Q, R) 是  $\Omega$  中向量场,且 P, Q, R 具有一阶连续的偏导数,

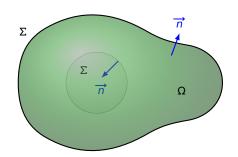


#### 定理(高斯公式) 假设

- 空间闭区域 Ω 的边界是分片光滑的闭曲面 Σ,
- $\overrightarrow{n}$  是  $\Sigma$  的单位 外法向量,(指向  $\Omega$  外部)
- F = (P, Q, R) 是  $\Omega$  中向量场,且 P, Q, R 具有一阶连续的偏导数,

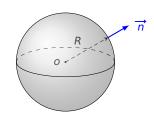
则

$$\iiint_{\Omega} \operatorname{div} F dv = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$



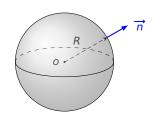
$$I = \iint_{\Sigma} 2x dy dz + y^2 dz dx + z^2 dx dy$$

其中定向曲面 Σ 是球面  $x^2 + y^2 + z^2 = R^2$ , 定向取外侧.



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$$I = \iiint_{\Sigma} F \cdot \overrightarrow{n} \, dS$$



$$I = \iint_{\Sigma} 2x dy dz + y^2 dz dx + z^2 dx dy$$

R n

其中定向曲面 Σ 是球面  $x^2 + y^2 + z^2 = R^2$ , 定向取外侧.

$$I = \iiint_{\Sigma} F \cdot \overrightarrow{n} dS = \frac{\overline{\overline{n}} \cdot \overline{\overline{n}}}{\int \int_{\Omega} div F dv}$$

$$I = \iint_{\Sigma} 2x dy dz + y^2 dz dx + z^2 dx dy$$

R

其中定向曲面  $\Sigma$  是球面  $x^2 + y^2 + z^2 = R^2$ , 定向取外侧.

$$I = \underbrace{F = (2x, y^2, z^2)}_{\Gamma} \iint_{\Sigma} F \cdot \overrightarrow{n} dS = \underbrace{\overline{\text{sycd}}}_{\Omega} \iiint_{\Omega} \text{div} F dv$$



$$I = \iint_{\Sigma} 2x dy dz + y^2 dz dx + z^2 dx dy$$

 $\overrightarrow{n}$ 

其中定向曲面 Σ 是球面  $x^2 + y^2 + z^2 = R^2$ , 定向取外侧.

$$I = \frac{F = (2x, y^2, z^2)}{\int \int_{\Sigma} F \cdot \overrightarrow{n} dS} = \frac{\overrightarrow{\text{sh}公式}}{\int \int_{\Omega} \text{div} F dv}$$
$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} (2x) + \frac{\partial}{\partial y} (y^2) + \frac{\partial}{\partial z} (z^2) \right] dv$$



$$I = \iint_{\Sigma} 2x dy dz + y^2 dz dx + z^2 dx dy$$

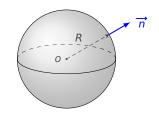
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其中定向曲面 Σ 是球面  $x^2 + y^2 + z^2 = R^2$ , 定向取外侧.

$$I = \frac{F = (2x, y^2, z^2)}{\iint_{\Sigma} F \cdot \overrightarrow{n} dS} = \frac{\overrightarrow{\text{sh}} \triangle \overrightarrow{\text{sh}}}{\iint_{\Omega} \text{div} F dv}$$
$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} (2x) + \frac{\partial}{\partial y} (y^2) + \frac{\partial}{\partial z} (z^2) \right] dv = \iiint_{\Omega} (2 + 2y + 2z) dv$$



$$I = \iint_{\Sigma} 2x dy dz + y^2 dz dx + z^2 dx dy$$



其中定向曲面 Σ 是球面  $x^2 + y^2 + z^2 = R^2$ , 定向取外侧.

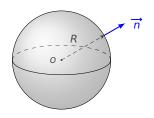
$$I \xrightarrow{F=(2x,y^2,z^2)} \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\overline{a} \underline{m} \triangle \underline{x}} \iiint_{\Omega} \operatorname{div} F \, dv$$

$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} (2x) + \frac{\partial}{\partial y} (y^2) + \frac{\partial}{\partial z} (z^2) \right] dv = \iiint_{\Omega} (2+2y+2z) \, dv$$

$$\xrightarrow{\underline{x} \underline{n} \underline{n}} \iiint_{\Omega} 2 \, dv$$



$$I = \iint_{\Sigma} 2x dy dz + y^2 dz dx + z^2 dx dy$$



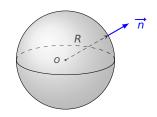
其中定向曲面  $\Sigma$  是球面  $x^2 + y^2 + z^2 = R^2$ , 定向取外侧.

$$I \xrightarrow{F=(2x,y^2,z^2)} \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\overline{a} \underline{m} \triangle \underline{x}} \iiint_{\Omega} \text{div} F dv$$

$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} (2x) + \frac{\partial}{\partial y} (y^2) + \frac{\partial}{\partial z} (z^2) \right] dv = \iiint_{\Omega} (2 + 2y + 2z) dv$$

$$\xrightarrow{\underline{\text{M}} \underline{m} \underline{m}} \iiint_{\Omega} 2 dv = 2 \text{Vol}(\Omega)$$

$$I = \iint_{\Sigma} 2x dy dz + y^2 dz dx + z^2 dx dy$$



其中定向曲面  $\Sigma$  是球面  $x^2 + y^2 + z^2 = R^2$ , 定向取外侧.

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$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} (2x) + \frac{\partial}{\partial y} (y^2) + \frac{\partial}{\partial z} (z^2) \right] \, dv = \iiint_{\Omega} (2+2y+2z) \, dv$$

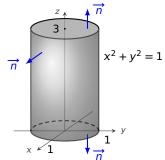
$$\xrightarrow{\underline{\text{MME}}} \iiint_{\Omega} 2 \, dv = 2 \operatorname{Vol}(\Omega) = \frac{8}{3} \pi R^3$$



#### 例2 计算

$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz \quad \overrightarrow{n}$$

其中定向曲面  $\Sigma$  是右图柱体的边界曲面.

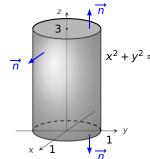


### 例2计算

$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz \quad \overrightarrow{n}$$

其中定向曲面  $\Sigma$  是右图柱体的边界曲面.

$$I = \iiint_{\Sigma} F \cdot \overrightarrow{n} \, dS$$



### 例2计算

$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz \quad \overrightarrow{n}$$



$$I = F = ((y-z)x, 0, x-y)$$
 
$$\iint_{\Sigma} F \cdot \overrightarrow{n} dS = \frac{\overrightarrow{\text{sh}公式}}{\iint_{\Omega}} \iint_{\Omega} div F dv$$



$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz \quad \overrightarrow{n}$$

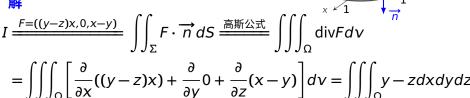
其中定向曲面  $\Sigma$  是右图柱体的边界曲面.

$$I = \frac{F = ((y-z)x, 0, x-y)}{\int \int_{\Sigma} F \cdot \overrightarrow{n} dS} = \frac{\overline{\text{syst}}}{\int \int_{\Omega} \text{div} F dv}$$
$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv$$



$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz \quad \overrightarrow{n}$$

其中定向曲面  $\Sigma$  是右图柱体的边界曲面.



$$=\iiint_{\Omega} \left[ \frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dz$$

例 
$$Z$$
 计算 
$$I = \iint_{\Sigma} (x-y) dx dy + (y-z)x dy dz \quad \vec{n}$$
 其中定向曲面  $\Sigma$  是右图柱体的边界曲面.

# 解

 $I \xrightarrow{F = ((y-z)x, 0, x-y)} \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{a}\underline{m} \subseteq \underline{m}} \iiint_{\Omega} div F dv$   $= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dz$ 



$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz \quad \overrightarrow{n}$$

其中定向曲面 Σ 是右图柱体的边界曲面.

 $= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dz$ 

$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz \quad \overrightarrow{n}$$

$$I \xrightarrow{F = ((y-z)x, 0, x-y)} \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{a}\underline{n} \subseteq \Sigma} \iiint_{\Omega} div F dv$$
$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dz$$



$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz \quad \overrightarrow{n}$$

其中定向曲面 Σ 是右图柱体的边界曲面.

I = F = ((y-z)x, 0, x-y)  $\iint_{\Sigma} F \cdot \overrightarrow{n} dS = \frac{\overrightarrow{\text{sh}} \triangle \overrightarrow{\text{sh}}}{} \iiint_{\Sigma} \overrightarrow{\text{div}} F dv$  $= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dz$ 



$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz \quad \overrightarrow{n}$$

$$I = \frac{F = ((y - z)x, 0, x - y)}{\int \int_{\Sigma} F \cdot \overrightarrow{n} dS} = \frac{\overrightarrow{\text{sh}} \triangle \overrightarrow{\text{sh}}}{\int \int_{\Omega} \text{div} F dv}$$

$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} ((y - z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x - y) \right] dv = \iiint_{\Omega} y - z dx dy dz$$



$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz \quad \overrightarrow{n}$$

其中定向曲面  $\Sigma$  是右图柱体的边界曲面.

$$I \xrightarrow{F = ((y-z)x, 0, x-y)} \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\underline{a}\underline{m} \underline{\Delta}\underline{d}} \iiint_{\Omega} div F dv$$

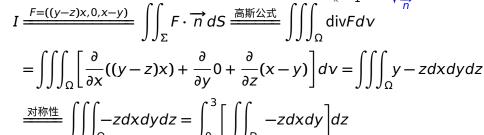
$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dz$$

$$\xrightarrow{\underline{M}\underline{M}\underline{d}} \iiint_{\Omega} -z dx dy dz = \int_{0}^{3} \left[ \iint_{\Omega} -z dx dy \right] dz$$



$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz \quad \overrightarrow{n}$$

# 解



● 整角大型

$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz \quad \overrightarrow{n}$$

$$I \xrightarrow{F = ((y-z)x, 0, x-y)} \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{a}\underline{m} \underline{C}\underline{C}\underline{C}} \iiint_{\Omega} div F dv$$

$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dz$$

$$\xrightarrow{\underline{M}\underline{M}\underline{C}\underline{C}} \iiint_{\Omega} -z dx dy dz = \int_{\Omega} \left[ \iint_{\Omega} -z dx dy \right] dz = -z |D_{z}|$$



$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz \quad \overrightarrow{n}$$

$$I \xrightarrow{F = ((y-z)x, 0, x-y)} \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{a}\underline{m}\underline{C}\underline{C}\underline{C}} \iiint_{\Omega} div F dv$$

$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dz$$

$$\xrightarrow{\underline{M}\underline{M}\underline{C}\underline{C}} \iiint_{\Omega} -z dx dy dz = \int_{\Omega} \left[ \iint_{\Omega} -z dx dy \right] dz = -z |D_z|$$



$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz \quad \overrightarrow{n}$$

$$I \xrightarrow{F = ((y-z)x, 0, x-y)} \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\underline{a}\underline{m} \leq \underline{x}} \iiint_{\Omega} \operatorname{div} F \, dv$$

$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z \, dx \, dy \, dz$$

$$\xrightarrow{\underline{y}\underline{m}\underline{t}} \iiint_{\Omega} -z \, dx \, dy \, dz = \int_{0}^{3} \left[ \iint_{\Omega} -z \, dx \, dy \right] dz = \int_{0}^{3} \left[ -z |D_{z}| \right] dz$$



$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz \quad \overrightarrow{n}$$

其中定向曲面 Σ 是右图柱体的边界曲面.

$$I \xrightarrow{F = ((y-z)x, 0, x-y)} \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{a}\underline{m} \underline{\triangle}} \iiint_{\Omega} div F dv$$

$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dz$$

 $= \int_{-1}^{3} \left[ -z\pi \right] dz$ 



$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz \quad \overrightarrow{n}$$

其中定向曲面 
$$\Sigma$$
 是右图柱体的边界曲面.

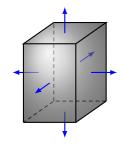
I = F = ((y-z)x, 0, x-y)  $\iint_{\Sigma} F \cdot \overrightarrow{n} dS = \frac{\overrightarrow{\text{sh}} \triangle \overrightarrow{\text{sh}}}{\overrightarrow{\text{oliv}}} \iiint_{\Sigma} A \cdot \overrightarrow{\text{oliv}} F dv$ 

 $= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dz$ 

$$\int J J J_{\Omega}$$

$$= \int_{0}^{3} \left[ -z\pi \right] dz = -\frac{9}{2}\pi$$

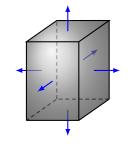




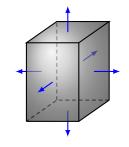
**例3** 计算流体  $F = (x - y^2, y, z^3)$  流向长方体

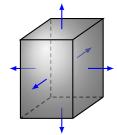


$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS$$



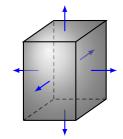
$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{a}\underline{m}\underline{c}\underline{m}} \iiint_{\Omega} div F dv$$





$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\underline{\hat{n}} \underline{\hat{n}} \underline{\hat{n}}} \iiint_{\Omega} \operatorname{div} F \, dv$$
$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} (x - y^2) + \frac{\partial}{\partial y} (y) + \frac{\partial}{\partial z} (z^3) \right] \, dv$$



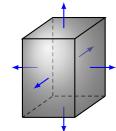


$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\underline{n} \underline{n} \underline{n}} \iiint_{\Omega} \operatorname{div} F \, dv$$

$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} (x - y^2) + \frac{\partial}{\partial y} (y) + \frac{\partial}{\partial z} (z^3) \right] \, dv$$

$$= \iiint_{\Omega} (2 + 3z^2) \, dx \, dy \, dz$$



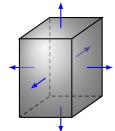


$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{a}\underline{m} \triangle \underline{x}} \iiint_{\Omega} \operatorname{div} F dv$$

$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} (x - y^{2}) + \frac{\partial}{\partial y} (y) + \frac{\partial}{\partial z} (z^{3}) \right] dv$$

$$= \iiint_{\Omega} (2 + 3z^{2}) dx dy dz = \int_{\Omega} \left[ \int_{\Omega} \left[ \int_{\Omega} (2 + 3z^{2}) dz \right] dy \right] dx$$



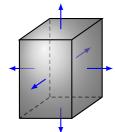


$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\underline{a}\underline{m} \triangle \underline{x}} \iiint_{\Omega} \operatorname{div} F \, dv$$

$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} (x - y^2) + \frac{\partial}{\partial y} (y) + \frac{\partial}{\partial z} (z^3) \right] \, dv$$

$$= \iiint_{\Omega} (2 + 3z^2) \, dx \, dy \, dz = \int_{0}^{1} \left[ \int_{\Omega} \left[ \int_{\Omega} (2 + 3z^2) \, dz \right] \, dy \right] \, dx$$



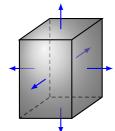


$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\underline{a}\underline{m} \triangle \underline{x}} \iiint_{\Omega} \operatorname{div} F \, dv$$

$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} (x - y^2) + \frac{\partial}{\partial y} (y) + \frac{\partial}{\partial z} (z^3) \right] \, dv$$

$$= \iiint_{\Omega} (2 + 3z^2) \, dx \, dy \, dz = \int_{0}^{1} \left[ \int_{1}^{2} \left[ \int (2 + 3z^2) \, dz \right] \, dy \right] \, dx$$



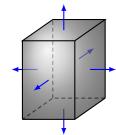


$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{a}\underline{m} \triangle \underline{x}} \iiint_{\Omega} \operatorname{div} F dv$$

$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} (x - y^2) + \frac{\partial}{\partial y} (y) + \frac{\partial}{\partial z} (z^3) \right] dv$$

$$= \iiint_{\Omega} (2 + 3z^2) dx dy dz = \int_{0}^{1} \left[ \int_{1}^{2} \left[ \int_{1}^{4} (2 + 3z^2) dz \right] dy \right] dx$$





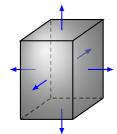
$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\underline{a}\underline{m}\underline{\omega}\underline{\omega}} \iiint_{\Omega} \operatorname{div} F \, dv$$

$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} (x - y^2) + \frac{\partial}{\partial y} (y) + \frac{\partial}{\partial z} (z^3) \right] \, dv$$

$$= \iiint_{\Omega} (2 + 3z^2) \, dx \, dy \, dz = \int_{0}^{1} \left[ \int_{1}^{2} \left[ \int_{1}^{4} (2 + 3z^2) \, dz \right] \, dy \right] \, dx$$

$$= \int_{0}^{1} 1 \, dx \cdot \int_{1}^{2} 1 \, dy \cdot \int_{1}^{4} (2 + 3z^2) \, dz$$





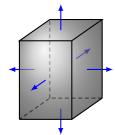
$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\underline{a}\underline{m}\underline{\square}} \iiint_{\Omega} \operatorname{div} F \, dv$$

$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} (x - y^2) + \frac{\partial}{\partial y} (y) + \frac{\partial}{\partial z} (z^3) \right] \, dv$$

$$= \iiint_{\Omega} (2 + 3z^2) \, dx \, dy \, dz = \int_{0}^{1} \left[ \int_{1}^{2} \left[ \int_{1}^{4} (2 + 3z^2) \, dz \right] \, dy \right] \, dx$$

$$= \int_{0}^{1} 1 \, dx \cdot \int_{1}^{2} 1 \, dy \cdot \int_{1}^{4} (2 + 3z^2) \, dz = 1 \cdot 1 \cdot (2z + z^3) \Big|_{1}^{4}$$





$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\underline{\underline{a}}\underline{\underline{m}}\underline{\underline{m}}\underline{\underline{m}}} \iiint_{\Omega} \operatorname{div} F \, dv$$

$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} (x - y^2) + \frac{\partial}{\partial y} (y) + \frac{\partial}{\partial z} (z^3) \right] \, dv$$

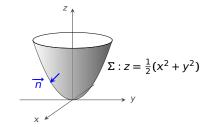
$$= \iiint_{\Omega} (2 + 3z^2) \, dx \, dy \, dz = \int_{0}^{1} \left[ \int_{1}^{2} \left[ \int_{1}^{4} (2 + 3z^2) \, dz \right] \, dy \right] \, dx$$

$$= \int_{0}^{1} 1 \, dx \cdot \int_{1}^{2} 1 \, dy \cdot \int_{1}^{4} (2 + 3z^2) \, dz = 1 \cdot 1 \cdot (2z + z^3) \Big|_{1}^{4} = 69$$



$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

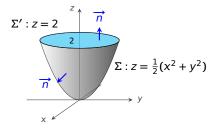
其中定向曲面 Σ 是抛物面的一部分, 取单位外法向量,如右图:



$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

其中定向曲面  $\Sigma$  是抛物面的一部分,

取单位外法向量,如右图:

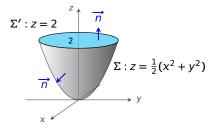


 $\mathbf{m}$  如图补充平面  $\Sigma'$ ,则  $\Sigma \cup \Sigma'$  构成 3 维区域  $\Omega$  边界,应用高斯公式:

$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

其中定向曲面  $\Sigma$  是抛物面的一部分,

取单位外法向量,如右图:



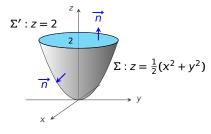
 $\mathbf{p}$  如图补充平面  $\Sigma'$ ,则  $\Sigma \cup \Sigma'$  构成 3 维区域  $\Omega$  边界,应用高斯公式:

原式 = 
$$\iint_{\Sigma \cup \Sigma'} F \cdot \overrightarrow{n} \, dS - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$

$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

其中定向曲面  $\Sigma$  是抛物面的一部分,

取单位外法向量,如右图:



 $\mathbf{m}$  如图补充平面  $\Sigma'$ ,则  $\Sigma \cup \Sigma'$  构成 3 维区域  $\Omega$  边界,应用高斯公式:

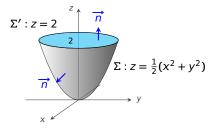
原式 = 
$$\iint_{\Sigma \cup \Sigma'} F \cdot \overrightarrow{n} dS - \iint_{\Sigma'} F \cdot \overrightarrow{n} dS$$
$$= \iiint_{\Omega} \operatorname{div} F dV$$

### 例4计算

$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

其中定向曲面  $\Sigma$  是抛物面的一部分,

取单位外法向量,如右图:



 $\mathbf{p}$  如图补充平面  $\Sigma'$ ,则  $\Sigma \cup \Sigma'$  构成 3 维区域  $\Omega$  边界,应用高斯公式:

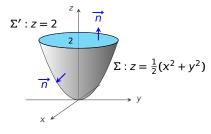
原式 = 
$$\iint_{\Sigma \cup \Sigma'} F \cdot \overrightarrow{n} \, dS - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$
$$= \iiint_{\Omega} \operatorname{div} F \, dv - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$

### 例4计算

$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

其中定向曲面  $\Sigma$  是抛物面的一部分,

取单位外法向量,如右图:



 $\mathbf{m}$  如图补充平面  $\Sigma'$ ,则  $\Sigma \cup \Sigma'$  构成 3 维区域  $\Omega$  边界,应用高斯公式:

原式 = 
$$\iint_{\Sigma \cup \Sigma'} F \cdot \overrightarrow{n} \, dS - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$
$$= \iiint_{\Omega} \operatorname{div} F \, dv - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$
$$\underline{F = (z^2 + x, 0, -z)}$$

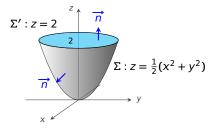


### 例4计算

$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

其中定向曲面  $\Sigma$  是抛物面的一部分,

取单位外法向量,如右图:



 $\mu$  如图补充平面  $\Sigma'$ ,则  $\Sigma \cup \Sigma'$  构成 3 维区域  $\Omega$  边界,应用高斯公式:

原式 = 
$$\iint_{\Sigma \cup \Sigma'} F \cdot \overrightarrow{n} \, dS - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$
$$= \iiint_{\Omega} \operatorname{div} F \, dv - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$
$$\underbrace{\underbrace{F = (z^2 + x, 0, -z)}_{\operatorname{div} F = 0}}$$



例4计算

$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

其中定向曲面  $\Sigma$  是抛物面的一部分,

 $\Sigma': z = 2$   $\Sigma: z = \frac{1}{2}(x^2 + y^2)$  X = 2 X = 2 X = 2 X = 2 Y = 2

取单位外法向量,如右图:

 $\mathbf{p}$  如图补充平面  $\Sigma'$ ,则  $\Sigma \cup \Sigma'$  构成 3 维区域  $\Omega$  边界,应用高斯公式:

原式 = 
$$\iint_{\Sigma \cup \Sigma'} F \cdot \overrightarrow{n} \, dS - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$
= 
$$\iiint_{\Omega} \operatorname{div} F \, dv - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$

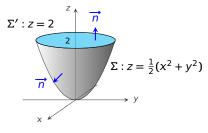
$$\frac{F = (z^2 + x, 0, -z)}{\operatorname{div} F = 0} - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$



例4计算

$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

其中定向曲面 Σ 是抛物面的一部分, 取单位外法向量,如右图:



 $\mathbf{p}$  如图补充平面  $\Sigma'$ ,则  $\Sigma \cup \Sigma'$  构成 3 维区域  $\Omega$  边界,应用高斯公式:

原式 = 
$$\iint_{\Sigma \cup \Sigma'} F \cdot \overrightarrow{n} \, dS - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$
= 
$$\iiint_{\Omega} \operatorname{div} F \, dv - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$

$$\frac{F = (z^2 + x, 0, -z)}{\operatorname{div} F = 0} - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$

$$\overrightarrow{n} = (0, 0, 1)$$

例4 计算

$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

其中定向曲面 Σ 是抛物面的一部分, 取单位外法向量,如右图:

$$\Sigma': z = 2$$

$$\Sigma: z = \frac{1}{2}(x^2 + y^2)$$

 $\mathbf{m}$  如图补充平面  $\Sigma'$ ,则  $\Sigma \cup \Sigma'$  构成 3 维区域  $\Omega$  边界,应用高斯公式:

原式 = 
$$\iint_{\Sigma \cup \Sigma'} F \cdot \overrightarrow{n} \, dS - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$
= 
$$\iiint_{\Omega} \operatorname{div} F \, dv - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$

$$\frac{F = (z^2 + x, 0, -z)}{\operatorname{div} F = 0} - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$

$$\frac{\overrightarrow{n} = (0, 0, 1)}{\operatorname{div} F = 0}$$

例 4 计算

其中定向曲面 Σ 是抛物面的一部分, 取单位外法向量,如右图:

$$\Sigma': z = 2$$

$$\Sigma: z = \frac{1}{2}(x^2 + y^2)$$

$$x = 2$$

 $\mathbf{p}$  如图补充平面  $\Sigma'$ ,则  $\Sigma \cup \Sigma'$  构成 3 维区域  $\Omega$  边界,应用高斯公式:

原式 = 
$$\iint_{\Sigma \cup \Sigma'} F \cdot \overrightarrow{n} \, dS - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$
= 
$$\iiint_{\Omega} \operatorname{div} F \, dv - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$

$$\frac{F = (z^2 + x, 0, -z)}{\operatorname{div} F = 0} - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$

$$\frac{\overrightarrow{n} = (0, 0, 1)}{F \cdot \overrightarrow{n} = -z} - \iint_{\Sigma'} -2 \, dS$$



例 4 计算

$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$
其中定向曲面  $\Sigma$  是抛物面的一部分,

 $\Sigma': z = 2$   $\Sigma: z = \frac{1}{2}(x^2 + y^2)$ 

取单位外法向量,如右图:

 $\mathbf{M}$  如图补充平面  $\Sigma'$ ,则  $\Sigma \cup \Sigma'$  构成 3 维区域  $\Omega$  边界,应用高斯公式:

原式 = 
$$\iint_{\Sigma \cup \Sigma'} F \cdot \overrightarrow{n} \, dS - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$
= 
$$\iiint_{\Omega} \operatorname{div} F \, dv - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$

$$\frac{F = (z^2 + x, 0, -z)}{\operatorname{div} F = 0} - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$

$$\frac{\overrightarrow{n} = (0, 0, 1)}{F \cdot \overrightarrow{n} = -z} - \iint_{\Sigma'} -2 \, dS = 2\operatorname{Area}(\Sigma')$$



**例 4** 计算

$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$
  
中面  $\Sigma$  旱坳物面的一部分

其中定向曲面  $\Sigma$  是抛物面的一部分, 取单位外法向量,如右图:

$$\Sigma': z = 2$$

$$\Sigma: z = \frac{1}{2}(x^2 + y^2)$$

$$x$$

 $\mathbf{K}$  如图补充平面  $\Sigma'$ ,则  $\Sigma \cup \Sigma'$  构成 3 维区域  $\Omega$  边界,应用高斯公式:

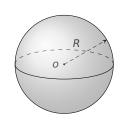
原式 = 
$$\iint_{\Sigma \cup \Sigma'} F \cdot \overrightarrow{n} \, dS - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$
= 
$$\iiint_{\Omega} \operatorname{div} F \, dv - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$

$$\frac{F = (z^2 + x, 0, -z)}{\operatorname{div} F = 0} - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$

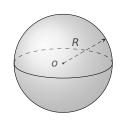
$$\frac{\overrightarrow{n} = (0, 0, 1)}{F \cdot \overrightarrow{n} = -z} - \iint_{\Sigma'} -2 \, dS = 2 \operatorname{Area}(\Sigma') = 8\pi$$

## 

$$V = \frac{R}{3}S.$$



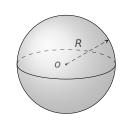
$$V = \frac{R}{3}S.$$



$$V = \iiint_{\Omega} 1 dv$$

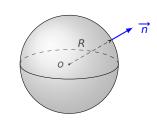


$$V = \frac{R}{3}S.$$



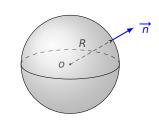
$$V = \iiint_{\Omega} 1 dv = \iiint_{\Omega} \operatorname{div} F dv$$

$$V = \frac{R}{3}S.$$



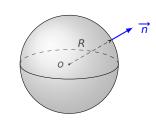
$$V = \iiint_{\Omega} 1 dv$$
 \_\_\_\_\_\_  $\iint_{\Omega} \text{div} F dv = \frac{\vec{\text{s}} \vec{\text{m}} \vec{\text{c}} \vec{\text{s}}}{\int_{\Sigma} F \cdot \vec{n} dS}$ 

$$V = \frac{R}{3}S.$$



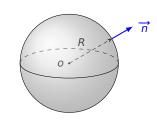
$$V = \iiint_{\Omega} 1 dv \frac{F = (x, y, z)}{\iiint_{\Omega}} \qquad \iiint_{\Omega} \text{div} F dv = \frac{\bar{\text{s}} \underline{\text{m}} \underline{\text{d}} \underline{\text{s}}}{\iiint_{\Sigma}} F \cdot \overrightarrow{n} dS$$

$$V = \frac{R}{3}S.$$



$$V = \iiint_{\Omega} 1 dv \frac{F = (x, y, z)}{\text{div} F = 3}$$
 
$$\iiint_{\Omega} \text{div} F dv \stackrel{\underline{=}\underline{\text{mid}}\underline{\text{mid}}}{=} \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$

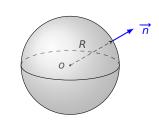
$$V = \frac{R}{3}S.$$



$$V = \iiint_{\Omega} 1 dv \frac{F = (x, y, z)}{\text{div} F = 3} \frac{1}{3} \iiint_{\Omega} \text{div} F dv \frac{\overline{\text{s}} \underline{\text{m}} \underline{\text{c}} \underline{\text{d}}}{\int_{\Sigma} F \cdot \overrightarrow{n} dS}$$

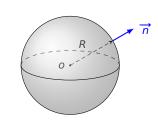


$$V = \frac{R}{3}S.$$



$$V = \iiint_{\Omega} 1 dv \frac{F = (x, y, z)}{\text{div} F = 3} \frac{1}{3} \iiint_{\Omega} \text{div} F dv \stackrel{\underline{a} \underline{\text{mid}} \underline{\text{mid}}}{\underline{\text{div}}} \frac{1}{3} \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$

$$V = \frac{R}{3}S.$$

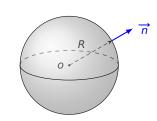


$$V = \iiint_{\Omega} 1 dv \frac{F = (x, y, z)}{\text{div}F = 3} \frac{1}{3} \iiint_{\Omega} \text{div}F dv \frac{\overline{\text{s}} \underline{\text{m}} \underline{\text{d}} \underline{\text{s}}}{3} \frac{1}{3} \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$

$$\overrightarrow{n} = \frac{1}{8}(x, y, z)$$



$$V = \frac{R}{3}S.$$

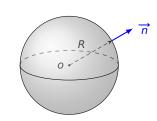


$$V = \iiint_{\Omega} 1 dv \frac{F = (x, y, z)}{\text{div}F = 3} \frac{1}{3} \iiint_{\Omega} \text{div}F dv \stackrel{\overline{\text{sh} \triangle I}}{=} \frac{1}{3} \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$

$$\overrightarrow{n} = \frac{1}{8}(x, y, z)$$

$$F \cdot \overrightarrow{n} = \frac{1}{R} (x^2 + y^2 + z^2)$$

$$V = \frac{R}{3}S.$$

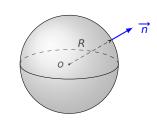


$$V = \iiint_{\Omega} 1 dv \frac{F = (x, y, z)}{\text{div}F = 3} \frac{1}{3} \iiint_{\Omega} \text{div}F dv \frac{\overline{a} \text{斯公式}}{\overline{a}} \frac{1}{3} \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$

$$\overrightarrow{n} = \frac{1}{B}(x, y, z)$$

$$F \cdot \overrightarrow{n} = \frac{1}{R} (x^2 + y^2 + z^2) = R$$

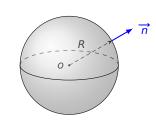
$$V = \frac{R}{3}S.$$



$$V = \iiint_{\Omega} 1 dv \frac{F = (x, y, z)}{\text{div}F = 3} \frac{1}{3} \iiint_{\Omega} \text{div}F dv \stackrel{\underline{=} \underline{\text{mid}}}{\underline{=} \frac{1}{3}} \frac{1}{3} \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$

$$\frac{\overrightarrow{n} = \frac{1}{R}(x, y, z)}{\overrightarrow{F \cdot \overrightarrow{n}} = \frac{1}{R}(x^2 + y^2 + z^2) = R} \frac{1}{3} \iint_{\Sigma} R dS$$

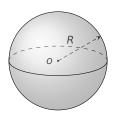
$$V = \frac{R}{3}S.$$



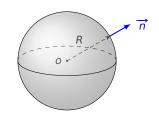
$$V = \iiint_{\Omega} 1 dv \frac{F = (x, y, z)}{\text{div}F = 3} \frac{1}{3} \iiint_{\Omega} \text{div}F dv = \frac{\bar{n} + \bar{n} + \bar{n} + \bar{n} + \bar{n}}{3} \iint_{\Sigma} F \cdot \vec{n} dS$$

$$\frac{\vec{n} = \frac{1}{R}(x, y, z)}{\vec{F} \cdot \vec{n} = \frac{1}{R}(x^2 + y^2 + z^2) = R} \frac{1}{3} \iint_{\Sigma} R dS = \frac{1}{3} RS$$

$$I = \iint_{\Sigma} (x^2 + y + z) dS$$



$$I = \iint_{\Sigma} (x^2 + y + z) dS$$



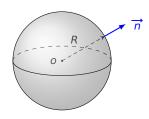
其中曲面 Σ 是球面  $x^2 + y^2 + z^2 = R^2$ .

$$\iint_{\Sigma} (x^2 + y + z) dS$$

$$= \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\overline{\text{a}} \underline{\text{m}} \triangle \exists} \iiint_{\Omega} \text{div} F dv$$



$$I = \iint_{\Sigma} (x^2 + y + z) dS$$



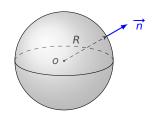
其中曲面 Σ 是球面  $x^2 + y^2 + z^2 = R^2$ .

解球面单位外法向量  $\overrightarrow{n} = \frac{1}{R}(x, y, z)$ ,所以

$$\iint_{\Sigma} (x^2 + y + z) dS$$

$$=\iint_{\Sigma} F \cdot \overrightarrow{n} dS \stackrel{\overline{a} \underline{m} \triangle \underline{\exists}}{===} \iiint_{\Omega} \operatorname{div} F dv$$

$$I = \iint_{\Sigma} (x^2 + y + z) dS$$



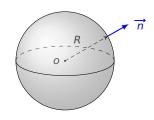
其中曲面 Σ 是球面  $x^2 + y^2 + z^2 = R^2$ .

**解**球面单位外法向量  $\overrightarrow{n} = \frac{1}{R}(x, y, z)$ ,所以

$$\iint_{\Sigma} (x^{2} + y + z) dS \qquad ( , , ) \cdot \frac{1}{R}(x, y, z)$$

$$= \iiint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{n} \text{ in } \Delta X} \iiint_{\Omega} \text{div } F dv$$

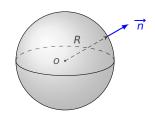
$$I = \iint_{\Sigma} (x^2 + y + z) dS$$



解球面单位外法向量 
$$\overrightarrow{n} = \frac{1}{R}(x, y, z)$$
,所以

$$\iint_{\Sigma} (x^2 + y + z) dS \qquad R(x, 1, 1) \cdot \frac{1}{R}(x, y, z)$$
$$= \iint_{\Sigma} F \cdot \overrightarrow{n} dS \stackrel{\overline{\text{ahcd}}}{====} \iiint_{\Omega} \text{div} F dv$$

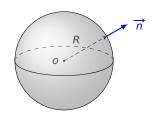
$$I = \iint_{\Sigma} (x^2 + y + z) dS$$



解球面单位外法向量 
$$\overrightarrow{n} = \frac{1}{R}(x, y, z)$$
,所以

$$\iint_{\Sigma} (x^{2} + y + z) dS = \iint_{\Sigma} R(x, 1, 1) \cdot \frac{1}{R}(x, y, z) dS$$
$$= \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{\mathrm{sh} \triangle \mathbb{I}}} \iiint_{\Omega} \mathrm{div} F dv$$

$$I = \iint_{\Sigma} (x^2 + y + z) dS$$



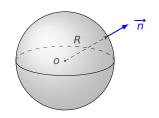
$$\mathbf{m}$$
 球面单位外法向量  $\overrightarrow{n} = \frac{1}{R}(x, y, z)$ ,所以

$$\iint_{\Sigma} (x^{2} + y + z) dS = \iint_{\Sigma} R(x, 1, 1) \cdot \frac{1}{R}(x, y, z) dS$$

$$= \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{a}\underline{m} \triangle \underline{x}} \iiint_{\Omega} div F dv$$

$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} (Rx) + \frac{\partial}{\partial y} (R) + \frac{\partial}{\partial z} (R) \right] dv$$

$$I = \iint_{\Sigma} (x^2 + y + z) dS$$



$$\mathbf{m}$$
 球面单位外法向量  $\overrightarrow{n} = \frac{1}{R}(x, y, z)$ ,所以

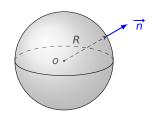
$$\iint_{\Sigma} (x^{2} + y + z) dS = \iint_{\Sigma} R(x, 1, 1) \cdot \frac{1}{R}(x, y, z) dS$$

$$= \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{a}\underline{m} \underline{\Delta}\underline{\Xi}} \iiint_{\Omega} \operatorname{div} F dv$$

$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} (Rx) + \frac{\partial}{\partial y} (R) + \frac{\partial}{\partial z} (R) \right] dv$$

$$= \iiint_{\Omega} R dx dy dz$$

$$I = \iint_{\Sigma} (x^2 + y + z) dS$$



$$\mathbf{m}$$
 球面单位外法向量  $\overrightarrow{n} = \frac{1}{R}(x, y, z)$ ,所以

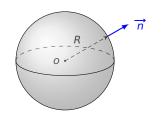
$$\iint_{\Sigma} (x^{2} + y + z) dS = \iint_{\Sigma} R(x, 1, 1) \cdot \frac{1}{R}(x, y, z) dS$$

$$= \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{a}\underline{\mathsf{M}} \triangle \underline{\mathsf{M}}} \iiint_{\Omega} \mathrm{div} F dv$$

$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} (Rx) + \frac{\partial}{\partial y} (R) + \frac{\partial}{\partial z} (R) \right] dv$$

$$= \iiint_{\Omega} R dx dy dz = R \text{Vol}(\Omega)$$

$$I = \iint_{\Sigma} (x^2 + y + z) dS$$



其中曲面 Σ 是球面  $x^2 + y^2 + z^2 = R^2$ .

**解**球面单位外法向量 
$$\overrightarrow{n} = \frac{1}{R}(x, y, z)$$
,所以

$$\iint_{\Sigma} (x^{2} + y + z) dS = \iint_{\Sigma} R(x, 1, 1) \cdot \frac{1}{R}(x, y, z) dS$$

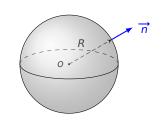
$$= \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{a} \underline{m} \underline{\triangle} \underline{\square}} \iiint_{\Omega} div F dv$$

$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} (Rx) + \frac{\partial}{\partial y} (R) + \frac{\partial}{\partial z} (R) \right] dv$$

 $= \iiint_{\Omega} R dx dy dz = R Vol(\Omega) = \frac{4}{3} \pi R^4$ 

$$I = \iint_{\Sigma} (x^2 + y + z) dS$$

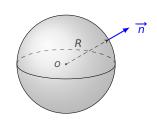
其中曲面 Σ 是球面  $x^2 + y^2 + z^2 = R^2$ .



### 例 6 计算

$$I = \iint_{\Sigma} (x^2 + y + z) dS$$

其中曲面 Σ 是球面  $x^2 + y^2 + z^2 = R^2$ .



$$\iint_{\Sigma} (x^2 + y + z) dS \xrightarrow{\text{phie}} \iint_{\Sigma} x^2 dS$$



$$I = \iint_{\Sigma} (x^2 + y + z) dS$$

R

其中曲面 Σ 是球面  $x^2 + y^2 + z^2 = R^2$ .

$$\iint_{\Sigma} (x^2 + y + z) dS \xrightarrow{\text{spate}} \iint_{\Sigma} x^2 dS = \frac{1}{3} \iint_{\Sigma} (x^2 + x^2 + x^2) dS$$

$$I = \iint_{\Sigma} (x^2 + y + z) dS$$

R

其中曲面 Σ 是球面  $x^2 + y^2 + z^2 = R^2$ .

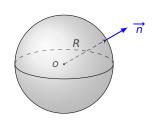
$$\iint_{\Sigma} (x^2 + y + z) dS \xrightarrow{\text{span}} \iint_{\Sigma} x^2 dS = \frac{1}{3} \iint_{\Sigma} (x^2 + x^2 + x^2) dS$$

$$\xrightarrow{\text{span}} \frac{1}{3} \iint_{\Sigma} (x^2 + y^2 + z^2) dS$$

#### 例 6 计算

$$I = \iint_{\Sigma} (x^2 + y + z) dS$$

 $JJ_{\Sigma}$ 其中曲面  $\Sigma$  是球面  $x^2 + y^2 + z^2 = R^2$ .



$$\iint_{\Sigma} (x^2 + y + z) dS \xrightarrow{\frac{\sqrt{3}}{2}} \iint_{\Sigma} x^2 dS = \frac{1}{3} \iint_{\Sigma} (x^2 + x^2 + x^2) dS$$

$$\xrightarrow{\frac{\sqrt{3}}{2}} \frac{1}{3} \iint_{\Sigma} (x^2 + y^2 + z^2) dS$$

$$= \frac{1}{3} \iint_{\Sigma} R^2 dS$$

#### 例 6 计算

$$I = \iint_{\Sigma} (x^2 + y + z) dS$$

R

其中曲面 Σ 是球面  $x^2 + y^2 + z^2 = R^2$ .

$$\iint_{\Sigma} (x^2 + y + z) dS \xrightarrow{\text{对称性}} \iint_{\Sigma} x^2 dS = \frac{1}{3} \iint_{\Sigma} (x^2 + x^2 + x^2) dS$$

$$\xrightarrow{\text{对称性}} \frac{1}{3} \iint_{\Sigma} (x^2 + y^2 + z^2) dS$$

$$= \frac{1}{3} \iint_{\Sigma} R^2 dS = \frac{1}{3} R^2 \text{Area}(\Sigma)$$

$$I = \iint_{\Sigma} (x^2 + y + z) dS$$

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其中曲面 Σ 是球面  $x^2 + y^2 + z^2 = R^2$ .

$$\iint_{\Sigma} (x^2 + y + z) dS \xrightarrow{\text{sparse}} \iint_{\Sigma} x^2 dS = \frac{1}{3} \iint_{\Sigma} (x^2 + x^2 + x^2) dS$$

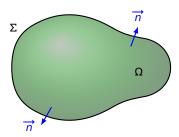
$$\xrightarrow{\text{sparse}} \frac{1}{3} \iint_{\Sigma} (x^2 + y^2 + z^2) dS$$

$$= \frac{1}{3} \iint_{\Sigma} R^2 dS = \frac{1}{3} R^2 \text{Area}(\Sigma) = \frac{4}{3} \pi R^4$$

高斯公式 
$$\iiint_{\Omega} \operatorname{div} F dv = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$



• 假设 F = (P, Q, R) 是流体的速度向量场,



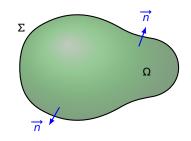
高斯公式 
$$\iiint_{\Omega} \operatorname{div} F dv = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$



• 假设 F = (P, Q, R) 是流体的速度向量场,则

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS$$

表示单位时间流向  $\Sigma$  外侧的通量.



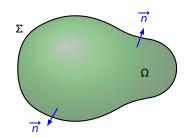
高斯公式 
$$\iiint_{\Omega} \operatorname{div} F dv = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$



• 假设 F = (P, Q, R) 是流体的速度向量场,则

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS$$

表示单位时间流向 Σ 外侧的通量.



$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS > 0$$

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS < 0$$

高斯公式 
$$\iiint_{\Omega} \operatorname{div} F dv = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$



• 假设 F = (P, Q, R) 是流体的速度向量场,则

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS$$

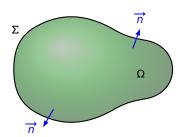
表示单位时间流向  $\Sigma$  外侧的通量.

● 进一步假设流体是不可压,则

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS > 0$$

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS < 0$$

高斯公式 
$$\iiint_{\Omega} \operatorname{div} F dv = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$

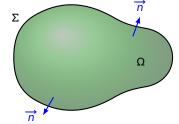


• 假设 F = (P, Q, R) 是流体的速度向量场,则

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS$$

表示单位时间流向  $\Sigma$  外侧的通量.

● 进一步假设流体是不可压,则



$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS > 0 \Rightarrow \Omega \, \text{内有 "source"}$$

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS < 0$$

高斯公式  $\iiint_{\Omega} \operatorname{div} F dv = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$ 

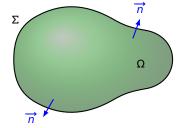


• 假设 F = (P, Q, R) 是流体的速度向量场,则

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS$$

表示单位时间流向  $\Sigma$  外侧的通量.

• 进一步假设流体是不可压,则



$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS > 0 \Rightarrow \Omega \, \text{内有 "source"}$$

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS < 0 \Rightarrow \Omega \, \text{内有 "sink"}$$

高斯公式 
$$\iiint_{\Omega} \operatorname{div} F dv = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$



• 假设 F = (P, Q, R) 是流体的速度向量场,则

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS$$

表示单位时间流向  $\Sigma$  外侧的通量.

● 进一步假设流体是不可压,则

$$\Sigma$$
 $\Omega$ 

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS > 0 \Rightarrow \Omega \, \text{内有 "source"}$$

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS < 0 \Rightarrow \Omega \, \text{内有 "sink"}$$

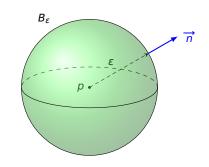
 $\mathbf{\dot{z}}$  高斯公式  $\iiint_{\mathbf{O}} \operatorname{div} F dv = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$  表明:  $\operatorname{div} F$  反映这种

"source"和 "sink"的强度.



p •

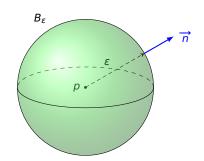




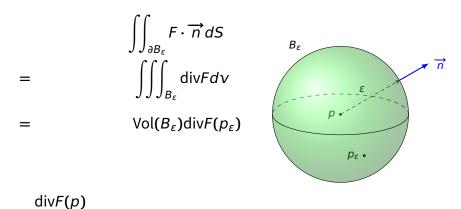


$$\iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \iiint_{B_{\varepsilon}} \operatorname{div} F \, dv$$









$$\frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \iiint_{B_{\varepsilon}} \operatorname{div} F \, dv$$

$$= \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \operatorname{Vol}(B_{\varepsilon}) \operatorname{div} F(p_{\varepsilon})$$

$$p_{\varepsilon} \cdot e^{-\beta \varepsilon}$$



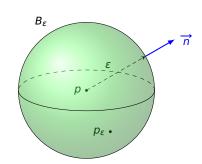
$$\frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \iiint_{B_{\varepsilon}} \text{div} F \, dV$$

$$= \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \text{Vol}(B_{\varepsilon}) \text{div} F(p_{\varepsilon})$$

$$= \text{div} F(p_{\varepsilon})$$

$$\text{div} F(p)$$



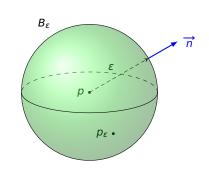
$$\lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \iiint_{B_{\varepsilon}} \text{div} F dv$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \text{Vol}(B_{\varepsilon}) \text{div} F(p_{\varepsilon})$$

$$= \lim_{\varepsilon \to 0^{+}} \text{div} F(p_{\varepsilon})$$

$$\text{div} F(p)$$



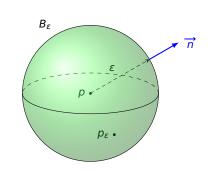
$$\lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \iiint_{B_{\varepsilon}} \text{div} F \, dv$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \text{Vol}(B_{\varepsilon}) \text{div} F(p_{\varepsilon})$$

$$= \lim_{\varepsilon \to 0^{+}} \text{div} F(p_{\varepsilon})$$

$$= \text{div} F(p)$$



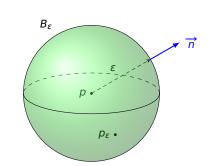
$$\lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \iiint_{B_{\varepsilon}} \text{div} F \, dv$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \text{Vol}(B_{\varepsilon}) \text{div} F(p_{\varepsilon})$$

$$= \lim_{\varepsilon \to 0^{+}} \text{div} F(p_{\varepsilon})$$

$$= \text{div} F(p)$$



- divF(p)>0时,
- divF(p)<0 时,

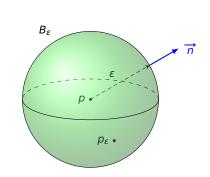
$$\lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \iiint_{B_{\varepsilon}} \text{div} F \, dv$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \text{Vol}(B_{\varepsilon}) \text{div} F(p_{\varepsilon})$$

$$= \lim_{\varepsilon \to 0^{+}} \text{div} F(p_{\varepsilon})$$

$$= \text{div} F(p)$$



- $\operatorname{div} F(p) > 0$  时, $\iint_{\partial B_{\epsilon}} F \cdot \overrightarrow{n} \, dS > 0$ ( $\epsilon$  充分小),
- divF(p)<0 时,

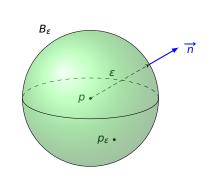


$$\lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \iiint_{B_{\varepsilon}} \text{div} F \, dv$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \text{Vol}(B_{\varepsilon}) \text{div} F(p_{\varepsilon})$$

$$= \lim_{\varepsilon \to 0^{+}} \text{div} F(p_{\varepsilon})$$



- $\operatorname{div} F(p) > 0$  时, $\iint_{\partial B_{\epsilon}} F \cdot \overrightarrow{n} dS > 0$ ( $\epsilon$  充分小),说明 p 点是 source
- divF(p)<0 时,

 $= \operatorname{div} F(p)$ 

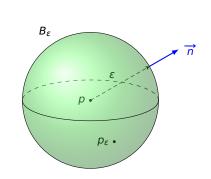


$$\lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \iiint_{B_{\varepsilon}} \text{div} F \, dv$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \text{Vol}(B_{\varepsilon}) \text{div} F(p_{\varepsilon})$$

$$= \lim_{\varepsilon \to 0^{+}} \text{div} F(p_{\varepsilon})$$



- $\operatorname{div} F(p) > 0$  时, $\iint_{\partial B_{\epsilon}} F \cdot \overrightarrow{n} \, dS > 0$ ( $\epsilon$  充分小),说明 p 点是 source
- $\operatorname{div} F(p) < 0$  时, $\iint_{\partial B_{\epsilon}} F \cdot \overrightarrow{n} \, dS < 0$ ( $\epsilon$  充分小),



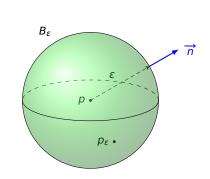
 $= \operatorname{div} F(p)$ 

$$\lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \iiint_{B_{\varepsilon}} \text{div} F \, dV$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \text{Vol}(B_{\varepsilon}) \text{div} F(p_{\varepsilon})$$

$$= \lim_{\varepsilon \to 0^{+}} \text{div} F(p_{\varepsilon})$$



- $\operatorname{div} F(p) > 0$  时, $\iint_{\partial B_{\epsilon}} F \cdot \overrightarrow{n} \, dS > 0$ ( $\epsilon$  充分小),说明 p 点是 source
- $\operatorname{div} F(p) < 0$  时,  $\iint_{\partial B_{\epsilon}} F \cdot \overrightarrow{n} \, dS < 0$  ( $\epsilon$  充分小),说明 p 点是  $\operatorname{sink}$



 $= \operatorname{div} F(p)$ 

### We are here now...

1. 高斯公式

2. 斯托克斯公式



定义 设 F = (P, Q, R) 是空间中向量场,定义

$$\operatorname{rot} F := \left| \begin{array}{ccc} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{array} \right|$$

称为向量场 F 的 旋度.

定义 设 F = (P, Q, R) 是空间中向量场,定义

$$\operatorname{rot} F := \left| \begin{array}{ccc} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{array} \right| = \left( \left| \begin{array}{ccc} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{array} \right|,$$



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称为向量场 F 的 旋 g.



定义 设 F = (P, Q, R) 是空间中向量场,定义

$$\operatorname{rot} F := \left| \begin{array}{ccc} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{array} \right| = \left( \left| \begin{array}{ccc} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{array} \right|, \ - \left| \begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{array} \right|, \ \left| \begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{array} \right| \right)$$

定义 设 F = (P, Q, R) 是空间中向量场,定义

$$\operatorname{rot} F := \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left( \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} \right)$$
$$= (R_{y} - Q_{z}, \qquad , \qquad )$$



定义 设 F = (P, Q, R) 是空间中向量场,定义

$$\operatorname{rot} F := \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left( \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} \right)$$
$$= (R_{y} - Q_{z}, P_{z} - R_{x}, )$$



定义 设 F = (P, Q, R) 是空间中向量场,定义

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$$= (R_{y} - Q_{z}, P_{z} - R_{x}, Q_{x} - P_{y})$$

称为向量场 F 的 旋 g.

<mark>例</mark> 计算向量场  $F = (y, -x, e^{xz})$  的旋度.

定义 设 F = (P, Q, R) 是空间中向量场,定义

$$\operatorname{rot} F := \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left( \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} \right)$$
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定义 设 F = (P, Q, R) 是空间中向量场,定义

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例 计算向量场  $F = (y, -x, e^{xz})$  的旋度.

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定义 设 F = (P, Q, R) 是空间中向量场,定义

$$\operatorname{rot} F := \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left( \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} \right)$$
$$= (R_{y} - Q_{z}, P_{z} - R_{x}, Q_{x} - P_{y})$$

称为向量场 F 的 旋 g .

例 计算向量场  $F = (y, -x, e^{xz})$  的旋度.

#### 解

$$\operatorname{rot} F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & e^{xz} \end{vmatrix} = \left( \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -x & e^{xz} \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ y & e^{xz} \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ y & -x \end{vmatrix} \right)$$

 $= (0, -ze^{xz}, )$ 



定义 设 F = (P, Q, R) 是空间中向量场,定义

$$\operatorname{rot} F := \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left( \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} \right)$$
$$= (R_{y} - Q_{z}, P_{z} - R_{x}, Q_{x} - P_{y})$$

称为向量场 F 的 旋 g.

例 计算向量场  $F = (y, -x, e^{xz})$  的旋度.

$$\operatorname{rot} F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & e^{XZ} \end{vmatrix} = \left( \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -x & e^{XZ} \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ y & e^{XZ} \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ y & -x \end{vmatrix} \right)$$

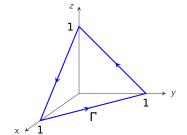
$$=(0, -ze^{xz}, -2)$$



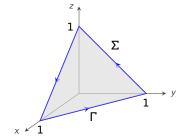
#### 例 1 试利用斯托克斯公式计算

$$I = \int_{\Gamma} z dx + x dy + y dz$$

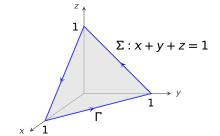
其中有向曲线 Γ 如右图.



$$I = \int_{\Gamma} z dx + x dy + y dz$$

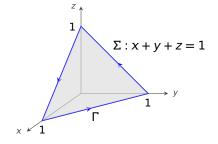


$$I = \int_{\Gamma} z dx + x dy + y dz$$



$$I = \int_{\Gamma} z dx + x dy + y dz$$

$$\mathbf{H}$$
设 $F = (z, x, y)$ ,则

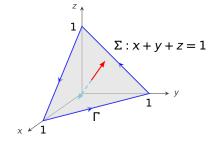


$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} dS$$



$$I = \int_{\Gamma} z dx + x dy + y dz$$

解设F = (z, x, y),则

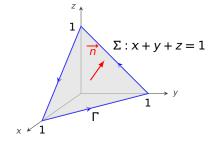


$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} dS$$



$$I = \int_{\Gamma} z dx + x dy + y dz$$

$$\mathbf{H}$$
设 $F = (z, x, y)$ ,则

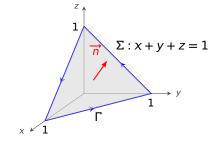


$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} dS$$



$$I = \int_{\Gamma} z dx + x dy + y dz$$
  
其中有向曲线  $\Gamma$  如右图.

 $\mathbf{H}$ 设F = (z, x, y),则



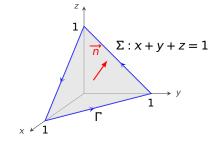
$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1, 1, 1)}$$



$$I = \int_{\Gamma} z dx + x dy + y dz$$

$$\mathbf{H}$$
设 $F = (z, x, y)$ ,则

$$rot F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix}$$



$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1, 1, 1)}$$



$$I = \int_{\Gamma} z dx + x dy + y dz$$

$$\mathbf{H}$$
设 $F = (z, x, y)$ ,则

$$\operatorname{rot} F = \begin{pmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{pmatrix} = \begin{pmatrix} \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ z & y \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ z & x \end{vmatrix} \end{pmatrix}$$

$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)}$$



$$I = \int_{\Gamma} z dx + x dy + y dz$$

$$\mathbf{H}$$
设 $F = (z, x, y)$ ,则

$$\operatorname{rot} F = (z, x, y), \quad \mathbb{I}$$

$$\operatorname{rot} F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \left( \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ z & y \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ z & x \end{vmatrix} \right)$$

$$= (1, , , )$$

$$=(1, , )$$

$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)}$$

$$I = \int_{\Gamma} z dx + x dy + y dz$$

$$\mathbf{H}$$
设 $F = (z, x, y)$ ,则

Fig. (2, x, y), 则
$$rot F = \begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
z & x & y
\end{vmatrix} = \left(\begin{vmatrix}
\frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
x & y
\end{vmatrix}, -\begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\
z & y
\end{vmatrix}, \begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\
z & x
\end{vmatrix}\right)$$

$$= (1, 1, )$$

$$=(1, 1, )$$

$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)}$$

$$I = \int_{\Gamma} z dx + x dy + y dz$$

$$\mathbf{H}$$
设 $F = (z, x, y)$ ,则

$$\operatorname{rot} F = \begin{pmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y \end{pmatrix}, - \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ z & y \end{pmatrix}, \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ z & x \end{pmatrix} \end{pmatrix}$$

$$=(1, 1, 1)$$

$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Gamma} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)}$$

$$I = \int_{\Gamma} z dx + x dy + y dz$$

解设
$$F = (z, x, y)$$
,则

$$\operatorname{rot} F = \begin{pmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{pmatrix} = \begin{pmatrix} \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ z & y \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ z & x \end{vmatrix} \end{pmatrix}$$

$$= (1, 1, 1)$$

$$=(1, 1, 1)$$

$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)} \iint_{\Sigma} \sqrt{3} dS$$

$$I = \int_{\Gamma} z dx + x dy + y dz$$

解设
$$F=(z,x,y)$$
,则

$$\operatorname{rot} F = \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y \end{pmatrix}, - \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ z & y \end{pmatrix}, \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ z & x \end{pmatrix} \end{pmatrix}$$

$$=(1, 1, 1)$$

$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)} \iint_{\Sigma} \sqrt{3} dS$$
$$= \sqrt{3} \operatorname{Area}(\Sigma)$$

$$I = \int_{\Gamma} z dx + x dy + y dz$$

其中有向曲线 Γ 如右图.

$$\mathbf{H}$$
设 $F = (z, x, y)$ ,则

$$\operatorname{rot} F = \begin{pmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{pmatrix} = \begin{pmatrix} \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ z & y \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ z & x \end{vmatrix} \end{pmatrix}$$

$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)} \iint_{\Sigma} \sqrt{3} dS$$
$$= \sqrt{3} \operatorname{Area}(\Sigma) = \sqrt{3} \cdot \frac{1}{2} \cdot \sqrt{2} \cdot \sqrt{2} \cdot \sin \frac{\pi}{3}$$

=(1, 1, 1)



$$I = \int_{\Gamma} z dx + x dy + y dz$$

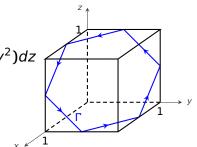
其中有向曲线 Γ 如右图.

 $\mathbf{H}$  设 F = (z, x, y),则

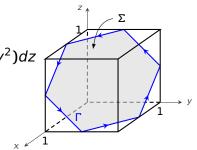
rot 
$$F = \begin{pmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y \end{pmatrix}, - \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ z & y \end{pmatrix}, \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ z & x \end{pmatrix} \end{pmatrix}$$

$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1, 1, 1)} \iint_{\Sigma} \sqrt{3} dS$$
$$= \sqrt{3} \operatorname{Area}(\Sigma) = \sqrt{3} \cdot \frac{1}{2} \cdot \sqrt{2} \cdot \sqrt{2} \cdot \sin \frac{\pi}{3} = \frac{3}{2}$$

$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$



$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$



例 2 试利用斯托克斯公式计算 
$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$
 其中有向曲线  $\Gamma$  如右图.



$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$
  
其中有向曲线  $\Gamma$  如右图.  
解设  $F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$ ,则

所以

$$I = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} \, dS$$



 $\Sigma: x+y+z=\tfrac{3}{2}$ 

$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$
  
其中有向曲线  $\Gamma$  如右图.  
解设  $F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$ ,则

所以

$$I = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} \, dS$$



 $\Sigma: x+y+z=\tfrac{3}{2}$ 

$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$
  
其中有向曲线  $\Gamma$  如右图.  
解设  $F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$ ,则

所以

$$I = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} \, dS$$



 $\Sigma: x + y + z = \frac{3}{2}$ 

$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$
  
其中有向曲线  $\Gamma$  如右图.  
解设  $F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$ ,则

所以

$$I = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} \, dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)}$$



 $\Sigma : x + y + z = \frac{3}{2}$ 

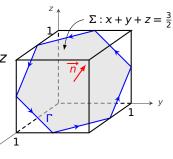
$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$

其中有向曲线 Γ 如右图.

解设
$$F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$
,则
$$rot F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix}$$

所以

$$I = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} \, dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1, 1, 1)}$$



$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$

其中有向曲线 Γ 如右图.

解设
$$F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$
,则
$$rot F = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix} = (-2y - 2z, , ,$$

所以

$$I = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} \, dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)}$$



 $\Sigma: x+y+z=\tfrac{3}{2}$ 

$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$

其中有向曲线 Γ 如右图.

解设
$$F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$
,则
$$rot F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix} = (-2y - 2z, -2z - 2x,$$

所以

$$I = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} \, dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)}$$



 $\Sigma: x+y+z=\tfrac{3}{2}$ 

$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$

其中有向曲线 Γ 如右图.

解设
$$F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$
,则
$$rot F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix} = (-2y - 2z, -2z - 2x, -2x - 2y)$$

所以

$$I = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)}$$



 $\Sigma : x + y + z = \frac{3}{2}$ 

$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$

其中有向曲线 Γ 如右图.

解设
$$F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$
,则
$$rot F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix} = (-2y - 2z, -2z - 2x, -2x - 2y)$$

所以

$$I = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)} \frac{-4}{\sqrt{3}} \iint_{\Sigma} (x+y+z) dS$$



 $\Sigma: x+y+z=\tfrac{3}{2}$ 

$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$
  
其中有向曲线  $\Gamma$  如右图.

解设
$$F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$
,则
$$rot F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix} = (-2y - 2z, -2z - 2x, -2x - 2y)$$

$$I = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)} \frac{-4}{\sqrt{3}} \iint_{\Sigma} (x+y+z) dS = \frac{-4}{\sqrt{3}} \iint_{\Sigma} \frac{3}{2} dS$$



$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$
其中有向曲线  $\Gamma$  如右图.

解设  $F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$ ,则
$$rot F = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix} = (-2y - 2z, -2z - 2x, -2x - 2y)$$
所以

$$I = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} \, dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)} \frac{-4}{\sqrt{3}} \iint_{\Sigma} (x+y+z) dS = \frac{-4}{\sqrt{3}} \iint_{\Sigma} \frac{3}{2} dS$$

$$=-2\sqrt{3}$$
Area( $\Sigma$ )



$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$

其中有向曲线 [ 如右图.

解设
$$F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$
,则
$$rot F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix} = (-2y - 2z, -2z - 2x, -2x - 2y)$$

$$I = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} \, dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)} \frac{-4}{\sqrt{3}} \iint_{\Sigma} (x+y+z) dS = \frac{-4}{\sqrt{3}} \iint_{\Sigma} \frac{3}{2} dS$$

$$=-2\sqrt{3}$$
Area( $\Sigma$ )



 $\Sigma : x + y + z = \frac{3}{2}$ 

$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$

解设
$$F = (v^2 - z^2 \ z^2 - z^2)$$

其中有向曲线 Γ 如右图.

**解**设 $F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$ ,则

$$x^2 - x^2, x^2$$

$$\overrightarrow{k}$$

$$\frac{k}{\frac{\partial}{\partial z}}$$

$$\frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial z}$$

 $\Sigma: x+y+z=\tfrac{3}{2}$ 

 $\operatorname{rot} F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix} = (-2y - 2z, -2z - 2x, -2x - 2y)$ 



$$I = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} \, dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)} \frac{-4}{\sqrt{3}} \iint_{\Sigma} (x+y+z) dS = \frac{-4}{\sqrt{3}} \iint_{\Sigma} \frac{3}{2} dS$$

$$= -2\sqrt{3}\operatorname{Area}(\Sigma) = -2\sqrt{3}\cdot 6\cdot \frac{1}{2}\cdot \sqrt{\frac{1}{2}}\cdot \sqrt{\frac{1}{2}}\cdot \sin\frac{\pi}{3}$$



$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$

解设
$$F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$
,则

$$32F = (v^2 - z^2 + z^2 - z^2)$$

$$x^2 - x^2, x$$

$$\frac{\partial}{\partial y} x^2 x^2 - y$$

$$\frac{K}{\frac{\partial}{\partial z}}$$

 $\operatorname{rot} F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix} = (-2y - 2z, -2z - 2x, -2x - 2y)$ 

$$-2x-2y$$

 $\Sigma: x+y+z=\tfrac{3}{2}$ 

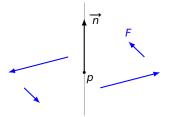
所以
$$I = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} \, dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)} \frac{-4}{\sqrt{3}} \iint_{\Sigma} (x+y+z) \, dS = \frac{-4}{\sqrt{3}} \iint_{\Sigma} \frac{3}{2} \, dS$$

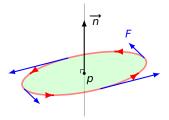
 $= -2\sqrt{3}\operatorname{Area}(\Sigma) = -2\sqrt{3} \cdot 6 \cdot \frac{1}{2} \cdot \sqrt{\frac{1}{2} \cdot \sqrt{\frac{1}{2} \cdot \sin \frac{\pi}{3}}} = -\frac{9}{2}$ 



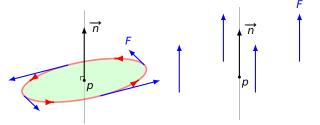




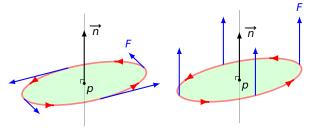




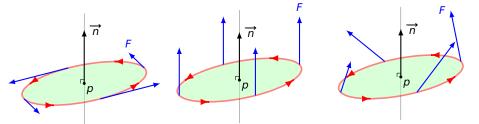




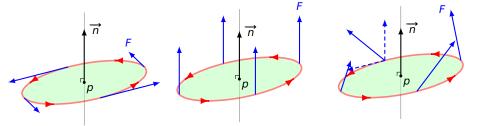




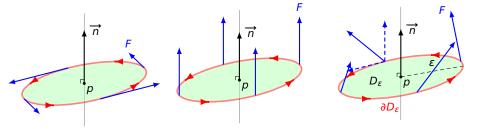




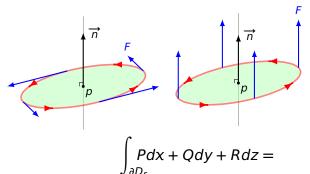


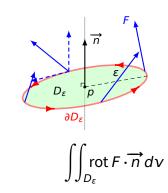


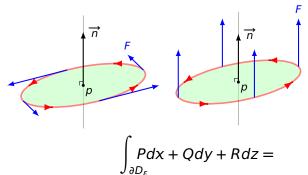






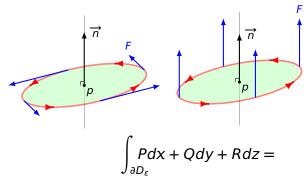






$$\int\int_{D_{\varepsilon}} \operatorname{rot} F \cdot \overrightarrow{n} \, dv$$



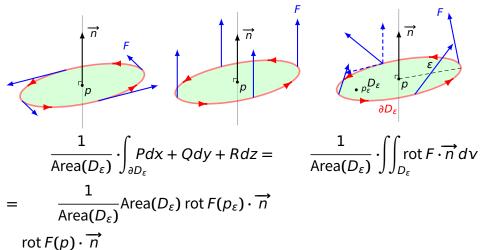


$$\int \int \operatorname{rot} F \cdot \overrightarrow{n} \, dv$$

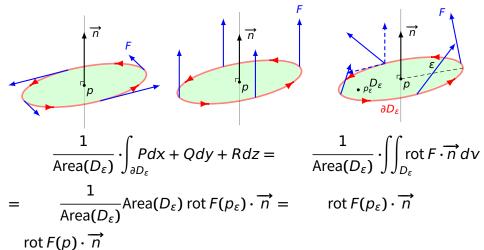
$$= \operatorname{Area}(D_{\varepsilon}) \operatorname{rot} F(p_{\varepsilon}) \cdot \overrightarrow{n}$$

$$rot F(p) \cdot \overrightarrow{n}$$

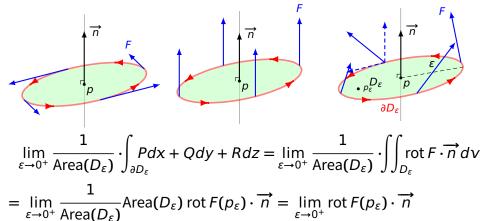






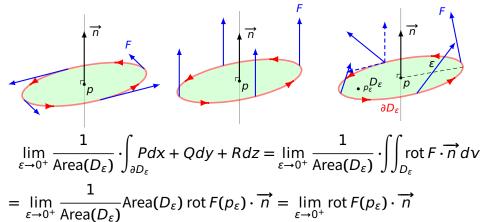






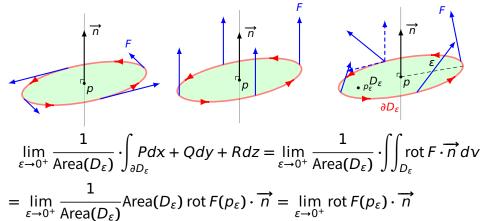
$$\varepsilon \to 0^+$$
 Area(*E* rot  $F(p) \cdot \overrightarrow{n}$ 





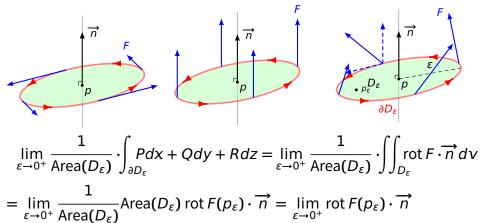
$$= \operatorname{rot} F(p) \cdot \overrightarrow{n}$$





$$= \operatorname{rot} F(p) \cdot \overrightarrow{n}$$



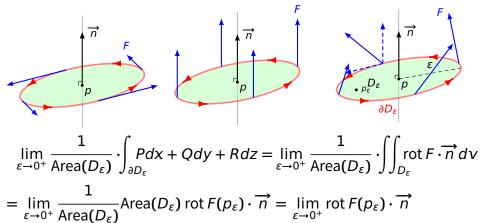


$$= \lim_{\varepsilon \to 0^+} \frac{1}{\operatorname{Area}(D_{\varepsilon})} \operatorname{Area}(D_{\varepsilon}) \operatorname{rot} F(p_{\varepsilon}) \cdot \Pi = \lim_{\varepsilon \to 0^+} \operatorname{rot} F(p_{\varepsilon}) \cdot \Pi$$

$$= \operatorname{rot} F(p) \cdot \overrightarrow{n}$$

 $\mathbf{\dot{z}}$  cot  $F \neq 0$  说明有旋,此时可认为 F 在  $\mathbf{p}$  点附近绕轴  $\overrightarrow{\mathbf{n}} = \frac{\text{rot } F}{\text{trot } F}$  旋转;





$$= \operatorname{rot} F(p) \cdot \overrightarrow{n}$$

 $\mathbf{\dot{z}}$  cot  $F \neq 0$  说明有旋,此时可认为 F 在  $\mathbf{p}$  点附近绕轴  $\overrightarrow{\mathbf{n}} = \frac{\text{rot } F}{\text{trot } F}$  旋转;  $\cot F = 0$  说明无旋.

