§4.3 实对称矩阵的特征值和特征向量

数学系 梁卓滨

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本节内容

- ◇ 向量的内积
- ♣ 正交向量组,施密特正交化方法
- ♡ 正交矩阵
- ♠ 对称矩阵可对角化

定义
$$\mathbb{R}^n$$
 中两个向量 $\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$ 和 $\beta = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ 的内积定义为:

$$\alpha^T \beta =$$

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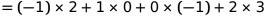
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$$= (-1) \times 2 + 1 \times 0 + 0 \times (-1) + 2 \times 3 = 4$$



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$$\alpha^T \beta = \beta^T \alpha$$

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$$(k\alpha)^T\beta = k\alpha^T\beta$$
, $(k$ 是实数)

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$$(\alpha + \beta)^T \gamma = \alpha^T \gamma + \beta^T \gamma$$

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定义

$$||\alpha|| := \sqrt{\alpha^T \alpha} = \sqrt{\alpha_1^2 + \alpha_2^2 + \dots + \alpha_n^2}$$

称为向量的长度或范数。

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$$\alpha = \begin{pmatrix} -4 \\ -5 \\ 6 \end{pmatrix}$$
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$$|\alpha^T \beta| \le ||\alpha|| \cdot ||\beta||$$

即

$$|a_1b_1 + \dots + a_nb_n| \le \sqrt{a_1^2 + \dots + a_n^2} \cdot \sqrt{b_1^2 + \dots + b_n^2}$$



● 定义 长度为 1 的向量称为单位向量。

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主单位向量

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• 设 $\alpha \neq 0$,则 $||\alpha|| \neq 0$,向量 $\frac{1}{||\alpha||} \alpha$ 是单位向量:



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$$\left\| \frac{1}{||\alpha||} \alpha \right\| = \frac{1}{||\alpha||} ||\alpha|| = 1$$

 $\pi \frac{1}{||\alpha||} \alpha$ 为 α 的单位化



$$\alpha = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \beta = \begin{pmatrix} 2 \\ 2 \\ 4 \\ 5 \end{pmatrix}$$

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解

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$$||\alpha|| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$
,

$$\alpha = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \beta = \begin{pmatrix} 2 \\ 2 \\ 4 \\ 5 \end{pmatrix}$$

解

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$$||\alpha|| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$
, 所以的 α 单位化为:
$$\frac{1}{||\alpha||} \alpha = \frac{1}{\sqrt{14}} \begin{pmatrix} 1\\2\\3 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{14}\\2/\sqrt{14}\\3/\sqrt{14} \end{pmatrix}$$

$$\alpha = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \beta = \begin{pmatrix} 2 \\ 2 \\ 4 \\ 5 \end{pmatrix}$$

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2.
$$||\beta|| = \sqrt{2^2 + 2^2 + 4^2 + 5^2} = \sqrt{49} = 7$$
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2. $||\beta|| = \sqrt{2^2 + 2^2 + 4^2 + 5^2} = \sqrt{49} = 7$, 所以的 β 单位化为:

$$\frac{1}{||\beta||}\beta = \frac{1}{7} \begin{pmatrix} 2\\2\\4\\5 \end{pmatrix} = \begin{pmatrix} 2/7\\2/7\\4/7\\5/7 \end{pmatrix}$$

定义 若 $\alpha^T \beta = 0$, 则称 α , β 正交(或垂直)

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例 零向量与任意向量正交:

 $0^T \alpha$

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$$0^T \alpha = 0 \cdot a_1 + 0 \cdot a_2 + \dots + 0 \cdot a_n = 0$$

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例 向量组
$$\varepsilon_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
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$$\varepsilon_1^T \varepsilon_2 = 0$$
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定义 若 \mathbb{R}^n 中向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 满足

- 1. 每个向量非零: $\alpha_i \neq 0$, i = 1, 2, ..., s
- 2. 两两正交: $\alpha_i^T \alpha_j = 0$, $i \neq j$

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证明设

$$k_1\alpha_1 + k_2\alpha_2 + \cdots + k_s\alpha_s = 0$$

$$k_1 = k_2 = \cdots = k_s = 0$$

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$$0 = \alpha_i^T (k_1 \alpha_1 + k_2 \alpha_2 + \dots + k_i \alpha_i + \dots + k_s \alpha_s)$$

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$$k_1\alpha_1 + k_2\alpha_2 + \cdots + k_s\alpha_s = 0$$

$$0 = \alpha_i^T (k_1 \alpha_1 + k_2 \alpha_2 + \dots + k_i \alpha_i + \dots + k_s \alpha_s) = \frac{\alpha_i^T \alpha_j = 0 \text{ for } i \neq j}{m_i + m_i + m_i}$$

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所以 $k_i = 0$ 。

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所以 $k_i = 0$ 。由 i 的任意性

$$k_1 = k_2 = \cdots = k_s = 0$$

正交化

 $\alpha_1, \alpha_2, \ldots, \alpha_s$ (线性无关) $\longrightarrow \beta_1, \beta_2, \ldots, \beta_s$ (等价, 两两正交)



 $\alpha_1, \alpha_2, \ldots, \alpha_s$ (线性无关) $\xrightarrow{\mathbb{E}^{\Sigma(k)}} \beta_1, \beta_2, \ldots, \beta_s$ (等价, 两两正交)



$$\alpha_1, \alpha_2, \ldots, \alpha_s$$
(线性无关) $\xrightarrow{\text{正交化}} \beta_1, \beta_2, \ldots, \beta_s$ (等价,两两正交) 实现正交化步骤(施密特正交化方法):

$$\beta_1 =$$

$$\beta_2 =$$

$$\beta_3 =$$

$$\beta_s =$$

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(线性无关) $\xrightarrow{\text{E文}\ell}$ $\beta_1, \beta_2, \ldots, \beta_s$ (等价,两两正交) 实现正交化步骤(施密特正交化方法):

$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \cdots - \beta_1$$

$$\beta_3 =$$

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 (线性无关) $\xrightarrow{\text{EX}(k)} \beta_1, \beta_2, \ldots, \beta_s$ (等价, 两两正交)

实现正交化步骤(施密特正交化方法):

$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \frac{1}{||\beta_1||^2} \beta_1$$

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正交化.

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例 1 将线性无关组 $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 3 \\ 3 \\ -1 \\ -1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix}$ 正交化

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$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2^T \beta_1}{||\beta_1||^2} \beta_1$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3^T \beta_1}{||\beta_1||^2} \beta_1 - \frac{\alpha_3^T \beta_2}{||\beta_2||^2} \beta_2$$

例 1 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1\\1\\1\\1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 3\\3\\-1\\-1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} -2\\0\\6\\8 \end{pmatrix}$ 正交化

$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \cdots - \beta_1$$

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$$\beta_1 = \alpha_1 = \begin{pmatrix} \frac{1}{1} \\ \frac{1}{1} \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \cdots - \beta_1$$

$$\beta_3 = \alpha_3 - \cdots - \beta_1 - \cdots - \beta_2$$

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$$\beta_1 = \alpha_1 = \begin{pmatrix} \frac{1}{1} \\ \frac{1}{1} \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\beta_1}{-1} = \begin{pmatrix} \frac{3}{3} \\ -\frac{1}{-1} \end{pmatrix} - -\begin{pmatrix} \frac{1}{1} \\ \frac{1}{1} \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \dots - \beta_1 - \dots - \beta_2$$

例 1 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 3 \\ 3 \\ -1 \\ -1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix}$ 正交化

$$\beta_{1} = \alpha_{1} = \begin{pmatrix} \frac{1}{1} \\ \frac{1}{1} \end{pmatrix}$$

$$\beta_{2} = \alpha_{2} - \frac{\beta_{1}}{\beta_{1}} = \begin{pmatrix} \frac{3}{3} \\ -\frac{1}{-1} \end{pmatrix} - \frac{1}{4} \begin{pmatrix} \frac{1}{1} \\ \frac{1}{1} \end{pmatrix}$$

$$\beta_{3} = \alpha_{3} - \frac{\beta_{1}}{\beta_{2}} - \frac{\beta_{2}}{\beta_{2}}$$



例 1 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 3 \\ 3 \\ -1 \\ -1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix}$ 正交化

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1\\1\\1\\1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \dots - \beta_1 = \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} - \frac{4}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

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例 1 将线性无关组
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$$\beta_2 = \alpha_2 - \dots - \beta_1 = \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} - \frac{4}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -2 \\ -2 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \cdots - \beta_1 - \cdots - \beta_2$$



例 1 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$
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$$\beta_2 = \alpha_2 - \frac{\beta_1}{-1} - \frac{4}{4} \begin{pmatrix} \frac{1}{1} \\ \frac{1}{1} \end{pmatrix} = \begin{pmatrix} \frac{2}{2} \\ -\frac{2}{2} \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\beta_1 - \beta_2}{\beta_1 - \beta_2}$$

$$= \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix} - \frac{1}{1} - \frac{2}{1} - \frac{2$$



例 1 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$
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$$= \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix} - \frac{1}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{2}{2} \begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix}$$



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$$= \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix} - \frac{12}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \dots - \begin{pmatrix} 2 \\ 2 \\ -2 \\ -2 \end{pmatrix}$$

 $\beta_3 = \alpha_3 - \cdots - \beta_1 - \cdots - \beta_2$



例 1 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$
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$$= \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix} - \frac{12}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{2}{16} \begin{pmatrix} 2 \\ 2 \\ -2 \\ -2 \end{pmatrix}$$

 $\beta_3 = \alpha_3 - \cdots - \beta_1 - \cdots - \beta_2$



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$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$
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$$\beta_{3} = \alpha_{3} - \frac{\beta_{1} - \beta_{2}}{\beta_{0}}$$

$$= \begin{pmatrix} -2 \\ 0 \\ 6 \end{pmatrix} - \frac{12}{4} \begin{pmatrix} \frac{1}{1} \\ \frac{1}{1} \end{pmatrix} - \frac{-32}{16} \begin{pmatrix} \frac{2}{2} \\ -\frac{2}{3} \end{pmatrix}$$



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$$\beta_3 = \alpha_3 - \frac{\beta_1 - \beta_2}{\beta_1 - \beta_2}$$

$$= \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix} - \frac{12}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{-32}{16} \begin{pmatrix} 2 \\ 2 \\ -2 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$



例 2 将线性无关组 $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ 正交化

例 2 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ 正交化

$$\beta_1 =$$

$$\beta_2 =$$

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$$\beta_1 = \alpha_1 = \begin{pmatrix} \frac{1}{1} \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \dots - \beta_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} - - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \cdots - \beta_1 - \cdots - \beta_2$$

例 2 将线性无关组
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$$\beta_2 = \alpha_2 - \frac{3}{2} - \frac{3}{2} - \frac{1}{3} - \frac{1}{3} \begin{pmatrix} \frac{1}{1} \\ 0 \\ 1 \end{pmatrix}$$

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$$\beta_2 = \alpha_2 - \dots - \beta_1 = \begin{pmatrix} \frac{3}{2} \\ \frac{1}{1} \end{pmatrix} - \frac{6}{3} \begin{pmatrix} \frac{1}{1} \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \cdots - \beta_1 - \cdots - \beta_2$$

例 2 将线性无关组
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$$\beta_3 = \alpha_3 - \cdots - \beta_1 - \cdots - \beta_2$$

例 2 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$
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$$\beta_1 = \alpha_1 = \begin{pmatrix} 1\\1\\0\\1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \dots - \beta_1 = \begin{pmatrix} \frac{3}{2} \\ \frac{1}{1} \end{pmatrix} - \frac{6}{3} \begin{pmatrix} \frac{1}{1} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{0} \\ 1 \\ -1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\beta_1 - \beta_2}{\beta_1 - \beta_2}$$

$$= \begin{pmatrix} \frac{1}{1} \\ \frac{1}{3} \end{pmatrix} - - \begin{pmatrix} \frac{1}{1} \\ \frac{1}{0} \\ \frac{1}{1} \end{pmatrix} - - \begin{pmatrix} \frac{1}{0} \\ \frac{1}{1} \\ -1 \end{pmatrix}$$



例 2 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ 正交化

$$\beta_1 = \alpha_1 = \begin{pmatrix} \frac{1}{1} \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \dots - \beta_1 = \begin{pmatrix} \frac{3}{2} \\ \frac{1}{1} \end{pmatrix} - \frac{6}{3} \begin{pmatrix} \frac{1}{1} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{0} \\ 1 \\ -1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\beta_1 - \beta_2}{\beta_1 - \beta_2}$$

$$= \begin{pmatrix} \frac{2}{1} \\ \frac{1}{3} \end{pmatrix} - \frac{1}{3} \begin{pmatrix} \frac{1}{1} \\ \frac{1}{1} \end{pmatrix} - - \begin{pmatrix} \frac{1}{0} \\ \frac{1}{-1} \end{pmatrix}$$



例 2 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ 正交化

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1\\1\\0\\1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \dots - \beta_1 = \begin{pmatrix} \frac{3}{2} \\ \frac{1}{1} \end{pmatrix} - \frac{6}{3} \begin{pmatrix} \frac{1}{1} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{0} \\ 1 \\ -1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\beta_1 - \beta_2}{\beta_1 - \beta_2}$$

$$= \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix} - \frac{6}{3} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} - - \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$



例 2 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ 正交化

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1\\1\\0\\1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\beta_1}{\beta_1} = \begin{pmatrix} 3\\2\\1\\1 \end{pmatrix} - \frac{6}{3} \begin{pmatrix} 1\\1\\0\\1 \end{pmatrix} = \begin{pmatrix} 1\\0\\1\\-1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\beta_1}{\beta_1} - \frac{\beta_2}{\beta_1}$$

$$= \begin{pmatrix} \frac{2}{1} \\ \frac{1}{3} \end{pmatrix} - \frac{6}{3} \begin{pmatrix} \frac{1}{1} \\ 0 \\ 1 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} \frac{1}{0} \\ 1 \\ -1 \end{pmatrix}$$



例 2 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ 正交化

$$eta_1=lpha_1=\left(egin{array}{c} 1\ 1\ 0\ 1 \end{array}
ight)$$

$$\beta_2 = \alpha_2 - \frac{3}{2} - \frac{6}{3} \begin{pmatrix} \frac{1}{1} \\ \frac{1}{1} \end{pmatrix} - \frac{6}{3} \begin{pmatrix} \frac{1}{1} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{0} \\ 1 \\ -1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\beta_1 - \beta_2}{\beta_1 - \beta_2}$$

$$= \begin{pmatrix} \frac{2}{1} \\ \frac{1}{2} \end{pmatrix} - \frac{6}{3} \begin{pmatrix} \frac{1}{1} \\ \frac{1}{2} \end{pmatrix} - \frac{0}{3} \begin{pmatrix} \frac{1}{0} \\ \frac{1}{2} \end{pmatrix}$$



例 2 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ 正交化

$$eta_1=lpha_1=\left(egin{array}{c} rac{1}{1} \ 0 \end{array}
ight)$$

$$\beta_2 = \alpha_2 - \dots - \beta_1 = \begin{pmatrix} \frac{3}{2} \\ \frac{1}{1} \end{pmatrix} - \frac{6}{3} \begin{pmatrix} \frac{1}{1} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{0} \\ 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2\\1\\1\\3 \end{pmatrix} - \frac{6}{3} \begin{pmatrix} 1\\1\\0\\1 \end{pmatrix} - \frac{0}{3} \begin{pmatrix} 1\\0\\1\\-1 \end{pmatrix} = \begin{pmatrix} 0\\-1\\1\\1 \end{pmatrix}$$

 $\beta_3 = \alpha_3 - \cdots - \beta_1 - \cdots - \beta_2$



例 3 将线性无关组 $\alpha_1 = \begin{pmatrix} 1\\1\\1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 0\\1\\2\\1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} -1\\0\\1\\1 \end{pmatrix}$ 正交化

例 3 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ 正交化

$$\beta_1 =$$

$$\beta_2 =$$

$$\beta_3 =$$

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$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$
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$$\beta_2 = \alpha_2 - \cdots - \beta_1$$

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例 3 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1\\1\\1 \end{pmatrix}$$
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$$\beta_1 = \alpha_1 = \begin{pmatrix} \frac{1}{1} \\ \frac{1}{1} \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \cdots - \beta_1$$

$$\beta_3 = \alpha_3 - \cdots - \beta_1 - \cdots - \beta_2$$

例 3 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$
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$$\beta_1 = \alpha_1 = \begin{pmatrix} \frac{1}{1} \\ \frac{1}{1} \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \dots - \beta_1 = \begin{pmatrix} 0 \\ \frac{1}{2} \\ 1 \end{pmatrix} - - \begin{pmatrix} \frac{1}{1} \\ \frac{1}{1} \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \cdots - \beta_1 - \cdots - \beta_2$$



例 3 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ 正交化

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1\\1\\1\\1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \dots - \beta_1 = \begin{pmatrix} 0 \\ \frac{1}{2} \\ 1 \end{pmatrix} - \frac{1}{4} \begin{pmatrix} \frac{1}{1} \\ \frac{1}{1} \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \cdots - \beta_1 - \cdots - \beta_2$$

例 3 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ 正交化

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$$= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - - \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

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$$\beta_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad 4 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\beta_1 - \beta_2}{\beta_1 - \beta_2}$$

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正交矩阵

定义 设 n 阶矩阵 Q 满足 $Q^TQ = I_n$,则称 Q 是正交矩阵。

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证明 设
$$Q = (\alpha_1 \, \alpha_2 \, \dots \, \alpha_n)$$
,则 $Q^T Q = \begin{pmatrix} \alpha_1^T \\ \alpha_2^T \\ \vdots \\ \alpha_n^T \end{pmatrix} (\alpha_1 \, \alpha_2 \, \dots \, \alpha_n)$

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$$Q^{T}Q = \begin{pmatrix} \alpha_{1}^{T} \\ \alpha_{2}^{T} \\ \vdots \\ \alpha_{n}^{T} \end{pmatrix} (\alpha_{1} \alpha_{2} \dots \alpha_{n}) = \begin{pmatrix} \alpha_{1}^{T} \alpha_{1} & \alpha_{1}^{T} \alpha_{2} & \cdots & \alpha_{1}^{T} \alpha_{n} \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{pmatrix}$$

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定理 n 阶矩阵 Q 是正交矩阵的充分必要条件是: Q 的列(行)向量组是单位正交向量组。

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所以

$$Q^TQ = I$$



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所以

$$Q^{T}Q = I \quad \Leftrightarrow \quad \begin{cases} \alpha_{i}^{T}\alpha_{i} = 1, \\ \alpha_{i}^{T}\alpha_{j} = 0, \end{cases}$$



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所以

$$Q^{T}Q = I \iff \begin{cases} \alpha_{i}^{T}\alpha_{i} = 1, & (i = 1, 2, ..., n) \\ \alpha_{i}^{T}\alpha_{j} = 0, & (i \neq j; i, j = 1, 2, ..., n) \end{cases}$$



$$A_1 = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{pmatrix}, \quad A_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{pmatrix},$$

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提示 验证: 列向量组是单位正交向量组



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答案 A1 是正交矩阵

$$A_1 = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{pmatrix}, \quad A_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{pmatrix},$$

提示验证:列向量组是单位正交向量组

答案 A_1 是正交矩阵, A_2 不是正交矩阵

- 对任意 n 阶方阵:
 - 1. 一定有n个特征值(计算重数,复数域内),可能有非实数特征值
 - 2. 不一定能对角化

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$$MA = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}$$
 的特征值方程是

$$0 = |\lambda I - A| =$$

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例
$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
 的特征值方程是

$$0 = |\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ 1 & \lambda \end{vmatrix} =$$

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所以特征值是 $\lambda_1 = -\sqrt{-1}$, $\lambda_2 = \sqrt{-1}$ 。



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- 对实对称矩阵,总成立:
 - 1. 定理 实对称矩阵的特征值都是实数。
 - 2. 定理 实对称矩阵一定可以对角化。



也就是:设A为实对称矩阵,则一定存在可逆矩阵P,使得

$$P^{-1}AP = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ & \ddots \\ & & \lambda_n \end{pmatrix}$$

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事实上,还可以进一步要求 P 是正交矩阵:

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注 由于正交矩阵满足 $Q^{-1} = Q^T$,

也就是:设A为实对称矩阵,则一定存在可逆矩阵P,使得

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注 由于正交矩阵满足 $Q^{-1} = Q^T$,上述等价于 $Q^T A Q = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \ddots \\ \ddots \end{pmatrix}$

$$\alpha_2^T \alpha_1 = 0$$



$$A\alpha_1 = \lambda_1\alpha_1$$

$$A\alpha_2 = \lambda_2\alpha_2$$

$$\alpha_2^T \alpha_1 = 0$$

$$A\alpha_1 = \lambda_1 \alpha_1 \quad \Rightarrow \quad \alpha_2^T A \alpha_1 = \lambda_1 \ \alpha_2^T \alpha_1$$

 $A\alpha_2 = \lambda_2 \alpha_2$

$$\alpha_2^T \alpha_1 = 0$$

$$A\alpha_1 = \lambda_1 \alpha_1 \implies \alpha_2^T A \alpha_1 = \lambda_1 \alpha_2^T \alpha_1$$

 $A\alpha_2 = \lambda_2 \alpha_2 \implies \alpha_1^T A \alpha_2 = \lambda_2 \alpha_1^T \alpha_2$

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$$A\alpha_1 = \lambda_1 \alpha_1 \quad \Rightarrow \quad \alpha_2^T A \alpha_1 = \lambda_1 \alpha_2^T \alpha_1$$

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$$\alpha_2^T \alpha_1 = 0$$



$$A\alpha_{1} = \lambda_{1}\alpha_{1} \Rightarrow \alpha_{2}^{T}A\alpha_{1} = \lambda_{1}\alpha_{2}^{T}\alpha_{1}$$

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注意
$$\alpha_2^T A \alpha_1 = (\alpha_2^T A \alpha_1)^T =$$

$$\alpha_2^T \alpha_1 = 0$$

$$A\alpha_{1} = \lambda_{1}\alpha_{1} \implies \boxed{\alpha_{2}^{T}A\alpha_{1}} = \lambda_{1} \boxed{\alpha_{2}^{T}\alpha_{1}}$$

$$A\alpha_{2} = \lambda_{2}\alpha_{2} \implies \boxed{\alpha_{1}^{T}A\alpha_{2}} = \lambda_{2} \boxed{\alpha_{1}^{T}\alpha_{2}}$$

注意
$$\alpha_2^T A \alpha_1 = \left(\alpha_2^T A \alpha_1\right)^T = \alpha_1^T A^T \left(\alpha_2^T\right)^T =$$

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$$A\alpha_{1} = \lambda_{1}\alpha_{1} \quad \Rightarrow \quad \alpha_{2}^{T}A\alpha_{1} = \lambda_{1}\alpha_{2}^{T}\alpha_{1}$$

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$$\alpha_2^T A \alpha_1 = (\alpha_2^T A \alpha_1)^T = \alpha_1^T A^T (\alpha_2^T)^T = \alpha_1^T A \alpha_2$$

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注意
$$\alpha_2^T A \alpha_1 = \left(\alpha_2^T A \alpha_1\right)^T = \alpha_1^T A^T \left(\alpha_2^T\right)^T = \alpha_1^T A \alpha_2$$
,两式相减得
$$0 = (\lambda_1 - \lambda_2) \alpha_2^T \alpha_1$$

$$\alpha_2^T \alpha_1 = 0$$



证明 设 A 为实对称矩阵, $\lambda_1 \neq \lambda_2$ 为两特征值, α_1 , α_2 为相应特征向量,则

$$A\alpha_{1} = \lambda_{1}\alpha_{1} \quad \Rightarrow \quad \boxed{\alpha_{2}^{T}A\alpha_{1}} = \lambda_{1} \boxed{\alpha_{2}^{T}\alpha_{1}}$$

$$A\alpha_{2} = \lambda_{2}\alpha_{2} \quad \Rightarrow \quad \boxed{\alpha_{1}^{T}A\alpha_{2}} = \lambda_{2} \boxed{\alpha_{1}^{T}\alpha_{2}}$$

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,两式相减得
$$0 = (\lambda_1 - \lambda_2) \alpha_2^T \alpha_1$$

由于 $\lambda_1 \neq \lambda_2$, 所以

$$\alpha_2^T \alpha_1 = 0$$



不同 特征值	重 数		正交化	单位化		
λ_1	n_1					
λ_2	n ₂					
:	:					
λ_{s}	ns					
	共n					
$ \lambda I - A = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_s)^{n_s}$						

不同 特征值	重 数	$(\lambda_i I - A)x = 0$ 基础解系	正交化	单位化		
λ_1	n_1					
λ_2	n_2					
÷	÷					
λ_{s}	ns					
	共 n					
$ \lambda I - A = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_s)^{n_s}$						

不同 特征值	重 数	$(\lambda_i I - A)x = 0$ 基础解系	正交化	单位化		
λ_1	n ₁	$\alpha_1^{(1)}, \cdots, \alpha_{n_1}^{(1)}$				
λ_2	n_2					
÷	÷					
λ_{s}	ns					
	共 n					
$ \lambda I - A = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_s)^{n_s}$						

不同 特征值	重 数	$(\lambda_i I - A)x = 0$ 基础解系	正交化	单位化			
λ_1	n_1	$\alpha_1^{(1)}, \cdots, \alpha_{n_1}^{(1)}$					
λ_2	n_2	$\alpha_1^{(2)}, \cdots, \alpha_{n_2}^{(2)}$					
:	:						
$\lambda_{\scriptscriptstyle S}$	ns						
	共n						
$ \lambda I - A = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_s)^{n_s}$							

	不同 特征值	重 数	$(\lambda_i I - A)x = 0$ 基础解系	正交化	单位化
	λ_1	n_1	$\alpha_1^{(1)}, \cdots, \alpha_{n_1}^{(1)}$		
	λ_2	n_2	$\alpha_1^{(2)}, \cdots, \alpha_{n_2}^{(2)}$		
	:	i	÷.		
	λ_{s}	ns	$\alpha_1^{(s)}, \cdots, \alpha_{n_s}^{(s)}$		
		共n			
$ \lambda I - A = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_s)^{n_s}$					



	不同 特征值	重 数	$(\lambda_i I - A)x = 0$ 基础解系	正交化	单位化	
	λ_1	n_1	$\alpha_1^{(1)}, \cdots, \alpha_{n_1}^{(1)}$			
	λ_2	n_2	$\alpha_1^{(2)}, \cdots, \alpha_{n_2}^{(2)}$			
	:	÷	:			
	λ_s	ns	$\alpha_1^{(s)}, \cdots, \alpha_{n_s}^{(s)}$			
		共n	共n个无关特征向量			
$ \lambda I - A = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_s)^{n_s}$						

解释示意图

不同	重	$(\lambda_i I - A)x = 0$	正交化	单位化
特征值_	数	基础解系		
λ_1	n_1	$\alpha_1^{(1)}, \cdots, \alpha_{n_1}^{(1)}$		
λ_2	n ₂	$\alpha_1^{(2)}, \cdots, \alpha_{n_2}^{(2)}$		
÷	÷	:		
λ_s	ns	$\alpha_1^{(s)}, \cdots, \alpha_{n_s}^{(s)}$		
	共 n	共 n 个无关特征向量		

•
$$\Leftrightarrow P = (\alpha_1^{(1)}, \dots, \alpha_{n_c}^{(n_s)}), \ \text{MI} \ P^{-1}AP = \Lambda_o$$



解释示意图

不同 特征值	重 数	$(\lambda_i I - A)x = 0$ 基础解系	正交化	单位化
λ_1	n_1	$\alpha_1^{(1)}, \cdots, \alpha_{n_1}^{(1)}$		
λ_2	n_2	$\alpha_1^{(2)}, \cdots, \alpha_{n_2}^{(2)}$		
÷	÷	i:		
λ_s	ns	$\alpha_1^{(s)}, \cdots, \alpha_{n_s}^{(s)}$		
	共 n	共 <i>n</i> 个无关特征向量		

• 令 $P = (\alpha_1^{(1)}, \dots, \alpha_{n_s}^{(n_s)})$,则 $P^{-1}AP = \Lambda$ 。但一般地,P 不是正交矩阵。



解释示意图

不同 特征值	重 数	$(\lambda_i I - A)x = 0$ 基础解系	正交化	单位化
λ_1	n ₁	$\alpha_1^{(1)}, \cdots, \alpha_{n_1}^{(1)}$	$\Rightarrow \beta_1^{(1)}, \cdots, \beta_{n_1}^{(1)}$	
λ_2	n_2	$\alpha_1^{(2)}, \cdots, \alpha_{n_2}^{(2)}$		
÷	:	i i		
λ_s	ns	$\alpha_1^{(s)}, \cdots, \alpha_{n_s}^{(s)}$		
	共 n	共n个无关特征向量		

• 令 $P = (\alpha_1^{(1)}, \dots, \alpha_{n_s}^{(n_s)})$,则 $P^{-1}AP = \Lambda$ 。但一般地,P 不是正交矩阵。



解释示意图

不同 特征值	重 数	$(\lambda_i I - A)x = 0$ 基础解系		正交化		单位化	
λ_1	n_1	$\alpha_1^{(1)}, \cdots, \alpha_{n_1}^{(1)}$	⇒	$\beta_1^{(1)}, \cdots, \beta_{n_1}^{(1)}$	⇒	$\gamma_1^{(1)}, \cdots, \gamma_{n_1}^{(1)}$	
λ_2	n ₂	$\alpha_1^{(2)}, \cdots, \alpha_{n_2}^{(2)}$					
÷	÷	i:					
λ_s	ns	$\alpha_1^{(s)}, \cdots, \alpha_{n_s}^{(s)}$					
	共 n	共 n 个无关特征向量					

$$|\lambda I - A| = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_s)^{n_s}$$

• 令 $P = (\alpha_1^{(1)}, \dots, \alpha_{p_s}^{(n_s)})$,则 $P^{-1}AP = \Lambda$ 。但一般地,P 不是正交矩阵。



解释示意图

不同 特征值	重 数	$(\lambda_i I - A)x = 0$ 基础解系		正交化		单位化
λ_1	n_1	$\alpha_1^{(1)}, \cdots, \alpha_{n_1}^{(1)}$	⇒	$\beta_1^{(1)}, \cdots, \beta_{n_1}^{(1)}$	⇒	$\gamma_1^{(1)}, \cdots, \gamma_{n_1}^{(1)}$
λ_2	n ₂	$\alpha_1^{(2)}, \cdots, \alpha_{n_2}^{(2)}$	⇒	$\beta_1^{(2)}, \cdots, \beta_{n_2}^{(2)}$		
÷	:	÷				
λ_{s}	ns	$\alpha_1^{(s)}, \cdots, \alpha_{n_s}^{(s)}$				
	共 n	共 n 个无关特征向量				

• 令 $P = (\alpha_1^{(1)}, \dots, \alpha_{n_s}^{(n_s)})$,则 $P^{-1}AP = \Lambda$ 。但一般地,P 不是正交矩阵。



解释示意图

不同 特征值	重 数	$(\lambda_i I - A)x = 0$ 基础解系		正交化		单位化
λ_1	n ₁	$\alpha_1^{(1)}, \cdots, \alpha_{n_1}^{(1)}$	⇒	$\beta_1^{(1)}, \cdots, \beta_{n_1}^{(1)}$	⇒	$\gamma_1^{(1)}, \cdots, \gamma_{n_1}^{(1)}$
λ_2	n ₂	$\alpha_1^{(2)}, \cdots, \alpha_{n_2}^{(2)}$	⇒	$\beta_1^{(2)},\cdots,\beta_{n_2}^{(2)}$	\Rightarrow	$\gamma_1^{(2)}, \cdots, \gamma_{n_2}^{(2)}$
÷	÷	:				:
λ_s	ns	$\alpha_1^{(s)}, \cdots, \alpha_{n_s}^{(s)}$				
	共 n	共 n 个无关特征向量	ţ			

• 令
$$P = (\alpha_1^{(1)}, \dots, \alpha_{n_s}^{(n_s)})$$
,则 $P^{-1}AP = \Lambda$ 。但一般地, P 不是正交矩阵。

 $|\lambda I - A| = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_s)^{n_s}$



解释示意图

不同 特征值	重 数	$(\lambda_i I - A)x = 0$ 基础解系		正交化		单位化
λ_1	n_1	$\alpha_1^{(1)}, \cdots, \alpha_{n_1}^{(1)}$	⇒	$\beta_1^{(1)}, \cdots, \beta_{n_1}^{(1)}$	⇒	$\gamma_1^{(1)}, \cdots, \gamma_{n_1}^{(1)}$
λ_2	n ₂	$\alpha_1^{(2)}, \cdots, \alpha_{n_2}^{(2)}$	\Rightarrow	$\beta_1^{(2)},\cdots,\beta_{n_2}^{(2)}$	\Rightarrow	$\gamma_1^{(2)}, \cdots, \gamma_{n_2}^{(2)}$
:	÷	:		:		:
λ_{s}	ns	$\alpha_1^{(s)}, \cdots, \alpha_{n_s}^{(s)}$	\Rightarrow	$\beta_1^{(s)}, \cdots, \beta_{n_s}^{(s)}$		
	共n	共 n 个无关特征向	量			

• 令 $P = (\alpha_1^{(1)}, \dots, \alpha_{n_s}^{(n_s)})$,则 $P^{-1}AP = \Lambda$ 。但一般地,P 不是正交矩阵。

 $|\lambda I - A| = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_s)^{n_s}$



解释示意图

不同 特征值	重 数	$(\lambda_i I - A)x = 0$ 基础解系		正交化		单位化	
λ_1	n_1	$\alpha_1^{(1)}, \cdots, \alpha_{n_1}^{(1)}$	⇒	$\beta_1^{(1)}, \cdots, \beta_{n_1}^{(1)}$	⇒	$\gamma_1^{(1)}, \cdots, \gamma_{n_1}^{(1)}$	
λ_2	n_2	$\alpha_1^{(2)}, \cdots, \alpha_{n_2}^{(2)}$	⇒	$\beta_1^{(2)},\cdots,\beta_{n_2}^{(2)}$	⇒	$\gamma_1^{(2)}, \cdots, \gamma_{n_2}^{(2)}$	
÷	:	:		:		:	
λ_s	ns	$\alpha_1^{(s)}, \cdots, \alpha_{n_s}^{(s)}$	\Rightarrow	$\beta_1^{(s)}, \cdots, \beta_{n_s}^{(s)}$	\Rightarrow	$\gamma_1^{(s)}, \cdots, \gamma_{n_s}^{(s)}$	
	共n	共 n 个无关特征向	量				

$$|\lambda I - A| = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_s)^{n_s}$$

• 令 $P = (\alpha_1^{(1)}, \dots, \alpha_{p_c}^{(n_s)})$,则 $P^{-1}AP = \Lambda$ 。但一般地,P 不是正交矩阵。



解释示意图

不同 特征值	重 数	$(\lambda_i I - A)x = 0$ 基础解系		正交化		单位化
λ_1	n_1	$\alpha_1^{(1)}, \cdots, \alpha_{n_1}^{(1)}$	⇒	$\beta_1^{(1)}, \cdots, \beta_{n_1}^{(1)}$	⇒	$\gamma_1^{(1)}, \cdots, \gamma_{n_1}^{(1)}$
λ_2	n ₂	$\alpha_1^{(2)}, \cdots, \alpha_{n_2}^{(2)}$	⇒	$\beta_1^{(2)},\cdots,\beta_{n_2}^{(2)}$	\Rightarrow	$\gamma_1^{(2)}, \cdots, \gamma_{n_2}^{(2)}$
:	÷	:		:		:
λ_s	ns	$\alpha_1^{(s)}, \cdots, \alpha_{n_s}^{(s)}$	⇒	$\beta_1^{(s)}, \cdots, \beta_{n_s}^{(s)}$	\Rightarrow	$\gamma_1^{(s)}, \cdots, \gamma_{n_s}^{(s)}$
	共n	共 n 个无关特征向				构成单位正交特 征向量

$$a riangle P - (\alpha^{(1)} \dots \alpha^{(n_s)})$$
 刚 $P^{-1}\Delta P - \Lambda$ 但一般地 P 不早正交矩阵

• 令 $P = (\alpha_1^{(1)}, \dots, \alpha_{n_s}^{(n_s)})$,则 $P^{-1}AP = \Lambda$ 。但一般地,P 不是正交矩阵。



 $|\lambda I - A| = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_s)^{n_s}$

 $(\lambda_i I - A)x = 0$

基础解系

解释示意图

不同

特征值

 λ_1

数

 n_1

λ_2	n_2	$\alpha_1^{(2)}, \cdots, \alpha_{n_2}^{(2)}$	\Rightarrow	$\beta_1^{(2)}, \cdots, \beta_{n_2}^{(2)}$	⇒	$\gamma_1^{(2)}, \cdots, \gamma_{n_2}^{(2)}$
÷	÷	:		:		:
λ_s	ns	$\alpha_1^{(s)}, \cdots, \alpha_{n_s}^{(s)}$	⇒	$\beta_1^{(s)}, \cdots, \beta_{n_s}^{(s)}$	⇒	$\gamma_1^{(s)}, \cdots, \gamma_{n_s}^{(s)}$
	共 n	共 n 个无关特征向	 量			构成单位正交特

正交化.

 $\alpha_1^{(1)}, \dots, \alpha_{n_1}^{(1)} \Rightarrow \beta_1^{(1)}, \dots, \beta_{n_1}^{(1)} \Rightarrow \gamma_1^{(1)}, \dots, \gamma_{n_1}^{(1)}$

• 令 $P = (\alpha_1^{(1)}, \dots, \alpha_{n_s}^{(n_s)})$,则 $P^{-1}AP = \Lambda$ 。但一般地,P 不是正交矩阵。
• 令 $Q = (\gamma_1^{(1)}, \dots, \gamma_n^{(n_s)})$,

 $|\lambda I - A| = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_s)^{n_s}$

单位化

征向量

 $(\lambda_i I - A)x = 0$

基础解系

 $|\lambda I - A| = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_s)^{n_s}$

正交化.

 $\alpha_1^{(1)}, \cdots, \alpha_{n_1}^{(1)} \Rightarrow \beta_1^{(1)}, \cdots, \beta_{n_1}^{(1)} \Rightarrow \gamma_1^{(1)}, \cdots, \gamma_{n_1}^{(1)}$

 $\alpha_1^{(2)}, \cdots, \alpha_{n_2}^{(2)} \Rightarrow \beta_1^{(2)}, \cdots, \beta_{n_2}^{(2)} \Rightarrow \gamma_1^{(2)}, \cdots, \gamma_{n_2}^{(2)}$

解释示意图

不同

特征值

 λ_1

 λ_2

数

 n_1

 n_2

	÷	÷	:		:		:
	λ_{s}	ns	$\alpha_1^{(s)}, \cdots, \alpha_{n_s}^{(s)}$	⇒	$\beta_1^{(s)}, \cdots, \beta_{n_s}^{(s)}$	\Rightarrow	$\gamma_1^{(s)}, \cdots, \gamma_{n_s}^{(s)}$
_		共 n	共 n 个无关特征向	量			构成单位正交特 征向量

• 令 $P = (\alpha_1^{(1)}, \dots, \alpha_n^{(n_s)})$,则 $P^{-1}AP = \Lambda$ 。但一般地,P 不是正交矩阵。

• 令 $Q=(\gamma_1^{(1)},\cdots,\gamma_{n_S}^{(n_S)})$,则 $Q^{-1}AQ=\Lambda$,
§4.3 实对称矩阵的特征值和特征向量

单位化

解释示意图

特征值	数	基础解系				
λ_1	n_1	$\alpha_1^{(1)}, \cdots, \alpha_{n_1}^{(1)}$	⇒	$\beta_1^{(1)}, \cdots, \beta_{n_1}^{(1)}$	⇒	$\gamma_1^{(1)}, \cdots, \gamma_{n_1}^{(1)}$
λ_2	n ₂	$\alpha_1^{(2)}, \cdots, \alpha_{n_2}^{(2)}$	\Rightarrow	$\beta_1^{(2)}, \cdots, \beta_{n_2}^{(2)}$	⇒	$\gamma_1^{(2)}, \cdots, \gamma_{n_2}^{(2)}$

正交化.

$$\lambda_s$$
 n_s $\alpha_1^{(s)}, \cdots, \alpha_{n_s}^{(s)}$ \Rightarrow $\beta_1^{(s)}, \cdots, \beta_{n_s}^{(s)}$ \Rightarrow $\gamma_1^{(s)}, \cdots, \gamma_{n_s}^{(s)}$
共 n 共 n 个无关特征向量

 $|\lambda I - A| = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_s)^{n_s}$

 $(\lambda_i I - A)x = 0$

构成单位正交特

征向量

单位化

• 今
$$P = (\alpha_s^{(1)}, \dots, \alpha_s^{(n_s)})$$
,则 $P^{-1}AP = \Lambda$ 。但一般地, P 不是正交矩阵。

• 令 $P = (\alpha_1^{(1)}, \dots, \alpha_{n_s}^{(n_s)})$,则 $P^{-1}AP = \Lambda$ 。但一般地,P 不是正交矩阵。

• 令 $Q = (\gamma_1^{(1)}, \dots, \gamma_n^{(n_s)})$,则 $Q^{-1}AQ = \Lambda$,并且 Q 是正交矩阵。 实对称矩阵的特征值和特征向量

$$Q^{-1}AQ = \begin{pmatrix} * \\ * \end{pmatrix}$$

$$Q^{-1}AQ = \begin{pmatrix} * \\ * \end{pmatrix}$$

例 1 求矩阵 $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ 的特征值与特征向量。 $\mathbf{H} \bullet \mathbf{H}$ 特征方程: $\mathbf{0} = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = \mathbf{H}$

$$Q^{-1}AQ = \begin{pmatrix} * \\ * \end{pmatrix}$$



解 ● 特征方程:
$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^2 - 1$$

$$Q^{-1}AQ = \left(\begin{array}{c} * \\ * \end{array}\right)$$

解 ● 特征方程:
$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$$

$$Q^{-1}AQ = \left(\begin{array}{c} * \\ * \end{array} \right)$$

解・特征方程:
$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$$
• $\lambda_1 = 1$

•
$$\lambda_2 = 3$$

$$Q^{-1}AQ = \begin{pmatrix} * \\ * \end{pmatrix}$$



解・特征方程:
$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$$
• $\lambda_1 = 1$

$$\lambda_2 = 3$$

$$Q^{-1}AQ = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

解 ● 特征方程:
$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$$

• $\lambda_1 = 1$, $\pi K (\lambda_1 I - A) x = 0$:

•
$$\lambda_2 = 3$$
, \overline{x} $\mathbf{R}(\lambda_2 I - A)\mathbf{x} = 0$:

$$Q^{-1}AQ = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$



解・特征方程:
$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$$

• $\lambda_1 = 1$. 求解 $(\lambda_1 I - A)x = 0$:

$$(1I - A \vdots 0) = \begin{pmatrix} -1 - 1 & 0 \\ -1 - 1 & 0 \end{pmatrix}$$

$$Q^{-1}AQ = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

例 1 求矩阵 $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ 的特征值与特征向量。 解 ● 特征方程: $\hat{0} = |\hat{\lambda I} - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$

解 ● 特征方程:
$$0 = |\lambda I - A| = |A - 1| = |A - 2| = (\lambda - 1)(\lambda - 3)$$

• $\lambda_1 = 1$. 求解 $(\lambda_1 I - A)x = 0$:

$$(1I - A \vdots 0) = \begin{pmatrix} -1 - 1 & | & 0 \\ -1 - 1 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix}$$

$$Q^{-1}AQ = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

例 1 求矩阵 $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ 的特征值与特征向量。 解 ● 特征方程: $\hat{0} = |\hat{\lambda I} - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$

$$\mathbf{H} \bullet$$
 特征方程: $0 = |\lambda I - A| = \begin{vmatrix} 1 & 1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$

• $\lambda_1 = 1$. 求解 $(\lambda_1 I - A)x = 0$:

$$(1I - A \vdots 0) = \begin{pmatrix} -1 - 1 & 0 \\ -1 - 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{array}{c} x_1 + x_2 = 0 \\ \end{array}$$

$$Q^{-1}AQ = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

例 1 求矩阵 $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ 的特征值与特征向量。 解 ● 特征方程: $\hat{0} = |\hat{\lambda I} - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$

$$\mathbf{H}$$
 ● 特征方程: $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$

$$(1I - A : 0) = \begin{pmatrix} -1 - 1 & | & 0 \\ -1 - 1 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix} \qquad \begin{matrix} x_1 + x_2 & = & 0 \\ x_1 & = & -x_2 \end{matrix}$$

$$Q^{-1}AQ = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$



例 1 求矩阵 $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ 的特征值与特征向量。 解 ● 特征方程: $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$

$$\mathbf{M} \bullet$$
 特征方程: $0 = |\lambda I - A| = \begin{vmatrix} 1 & 1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$

• $\lambda_1 = 1$, $\pi M (\lambda_1 I - A) x = 0$:

$$(1I - A : 0) = \begin{pmatrix} -1 - 1 & 0 \\ -1 - 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{array}{c} x_1 + x_2 = 0 \\ \vdots \\ x_1 = -x_2 \end{array}$$

基础解系: $\alpha_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$Q^{-1}AQ = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

例 1 求矩阵 $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ 的特征值与特征向量。 解 ● 特征方程: $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$

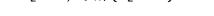
*● 特征方程:
$$0 = |AI - A| = |-1| \lambda - 2| = (\lambda - 1)(\lambda - 3)$$

• $\lambda_1 = 1$, \overline{x} $\mathbf{R}(\lambda_1 I - A)x = 0$:

$$(1I - A : 0) = \begin{pmatrix} -1 - 1 & | & 0 \\ -1 - 1 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix} \qquad \begin{array}{c} x_1 + x_2 = 0 \\ x_1 = -x_2 \end{array}$$

基础解系: $\alpha_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

•
$$\lambda_2 = 3$$
, \overline{x} $\mathbf{R}(\lambda_2 I - A)\mathbf{X} = 0$:



$$Q^{-1}AQ = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

解 ● 特征方程:
$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$$

• $\lambda_1 = 1$. 求解 $(\lambda_1 I - A)x = 0$:

$$(1I - A : 0) = \begin{pmatrix} -1 - 1 & 0 \\ -1 - 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{cases} x_1 + x_2 = 0 \\ x_1 = -x_2 \end{cases}$$

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$$\alpha_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$(3I - A : 0) = \begin{pmatrix} 1 & -1 & | & 0 \\ -1 & 1 & | & 0 \end{pmatrix}$$

$$Q^{-1}AQ = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$



解 • 特征方程:
$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$$

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解 ● 特征方程:
$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$$

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解 • 特征方程:
$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$$

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$$(1I - A : 0) = \begin{pmatrix} -1 - 1 & 0 \\ -1 - 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{cases} x_1 + x_2 = 0 \\ x_1 = -x_2 \end{cases}$$

基础解系:
$$\alpha_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$(3I - A : 0) = \begin{pmatrix} 1 & -1 & | & 0 \\ -1 & 1 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix} \qquad \begin{array}{c} x_1 - x_2 & = & 0 \\ x_1 & = & x_2 \end{array}$$

$$Q^{-1}AQ = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$



例 1 求矩阵 $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ 的特征值与特征向量。 解 ● 特征方程: $\hat{0} = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$

• $\lambda_1 = 1$. 求解 $(\lambda_1 I - A)x = 0$:

$$(1I - A \vdots 0) = \begin{pmatrix} -1 - 1 & 0 \\ -1 - 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{array}{c} x_1 + x_2 = 0 \\ x_1 = -x_2 \end{array}$$

基础解系: $\alpha_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

•
$$\lambda_2 = 3$$
, $\forall x \in A$

$$(3I - A : 0) = \begin{pmatrix} 1 & -1 & | & 0 \\ -1 & 1 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix} \qquad \begin{array}{c} x_1 - x_2 & = & 0 \\ x_1 & = & x_2 \end{array}$$

基础解系: $\alpha_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$Q^{-1}AQ = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$



例 1 求矩阵 $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ 的特征值与特征向量。 解 ● 特征方程: $\hat{0} = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$

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$$(1I - A \vdots 0) = \begin{pmatrix} -1 - 1 & | & 0 \\ -1 - 1 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix} \qquad \begin{matrix} x_1 + x_2 & = & 0 \\ x_1 & = & -x_2 \end{matrix}$$

基础解系: $\alpha_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

•
$$\lambda_2 = 3$$
, \overline{x} M ($\lambda_2 I - A$) $X = 0$:

$$(3I - A : 0) = \begin{pmatrix} 1 & -1 & | & 0 \\ -1 & 1 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix} \qquad \begin{array}{c} x_1 - x_2 = 0 \\ y \\ x_1 = x_2 \end{array}$$

基础解系:
$$\alpha_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

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解 ● 特征方程: $0 = |\lambda I - A| = |\lambda - 2| = (\lambda - 1)(\lambda - 3)$ • $\lambda_1 = 1$, $\pi M(\lambda_1 I - A) X = 0$:

$$(1I - A:0) = \begin{pmatrix} -1 - 1 & 0 \\ -1 - 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{matrix} x_1 + x_2 = 0 \\ y \\ x_1 = -x_2 \end{matrix}$$
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 γ_1 所以取 $Q = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$,则 $Q^{-1}AQ = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$$Q^{-1}AQ = \begin{pmatrix} * \\ * \end{pmatrix}$$



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例 2 求矩阵 $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ 的特征值与特征向量。 $\mathbf{H} \bullet \mathbf{H}$ 特征方程: $\mathbf{0} = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = \mathbf{H}$

$$Q^{-1}AQ = \left(\begin{array}{c} * \\ * \end{array} \right)$$

解 ● 特征方程:
$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2 - (-2)^2$$

$$Q^{-1}AQ = \left(\begin{array}{c} * \\ * \end{array} \right)$$

解 ● 特征方程:
$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = (\lambda + 1)(\lambda - 3)$$

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例 2 求矩阵 $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ 的特征值与特征向量。 解 ● 特征方程: $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ 2 & 1 \end{vmatrix}$

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解 ● 特征方程:
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• $\lambda_1 = -1$, \overline{x} $\mathbb{R}(\lambda_1 I - A)x = 0$:

•
$$\lambda_2 = 3$$
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解 ● 特征方程:
$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = (\lambda + 1)(\lambda - 3)$$

• $\lambda_1 = -1$, \vec{x} \vec{x}

$$(1I - A \vdots 0) = \begin{pmatrix} -2 - 2 & 0 \\ -2 - 2 & 0 \end{pmatrix}$$

• $\lambda_2 = 3$, $\Re (\lambda_2 I - A)x = 0$:

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解 ● 特征方程:
$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = (\lambda + 1)(\lambda - 3)$$

・ 特征方程: $0 = |\lambda_I - A| = |-2|\lambda - 1| = |\lambda + 1|(\lambda - 3)$ • $\lambda_1 = -1$,求解 $(\lambda_1 I - A)x = 0$:

$$(1I - A \vdots 0) = \begin{pmatrix} -2 - 2 & | & 0 \\ -2 - 2 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix}$$

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解 ● 特征方程:
$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = (\lambda + 1)(\lambda - 3)$$

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$$(1I - A \vdots 0) = \begin{pmatrix} -2 - 2 & | & 0 \\ -2 - 2 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix} \qquad \begin{matrix} x_1 + x_2 & = & 0 \\ & \downarrow & \\ x_1 & = & -x_2 \end{matrix}$$

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•
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 γ_1 所以取 $Q = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$,则 $Q^{-1}AQ = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

例
$$1A = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 3 \end{pmatrix}$$

$$Q^{-1}AQ = \begin{pmatrix} * & * \\ & * \end{pmatrix}$$



例
$$1A = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 3 \end{pmatrix}$$
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•
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,

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, 特征向量 $\alpha_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$

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•
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例
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•
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$$Q^{-1}AQ = \begin{pmatrix} -1 & 2 \\ & 5 \end{pmatrix}$$



例
$$1A = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 3 \end{pmatrix}$$
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•
$$\lambda_1 = -1$$
,特征向量 $\alpha_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ 单位化 $\gamma_1 = \begin{pmatrix} 2/3 \\ 2/3 \\ 1/3 \end{pmatrix}$

•
$$\lambda_2 = 2$$
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例
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•
$$\lambda_3 = 5$$
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$$Q^{-1}AQ = \begin{pmatrix} -1 & 2 \\ 5 & 5 \end{pmatrix}$$



例
$$1A = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 3 \end{pmatrix}$$
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•
$$\lambda_2 = 2$$
,特征向量 $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$ 单位化 $\gamma_2 = \begin{pmatrix} 2/3 \\ -1/3 \\ -2/3 \end{pmatrix}$

•
$$\lambda_3 = 5$$
,特征向量 $\alpha_3 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ 单位化 $\gamma_3 = \begin{pmatrix} 1/3 \\ -2/3 \\ 2/3 \end{pmatrix}$

$$Q^{-1}AQ = \begin{pmatrix} -1 & 2 \\ 5 & 5 \end{pmatrix}$$



例
$$1A = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 3 \end{pmatrix}$$
, 特征方程: $0 = |\lambda I - A| = (\lambda + 1)(\lambda - 2)(\lambda - 5)$

•
$$\lambda_1 = -1$$
,特征向量 $\alpha_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_1 = \begin{pmatrix} 2/3 \\ 2/3 \\ 1/3 \end{pmatrix}$

•
$$\lambda_2 = 2$$
, 特征向量 $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$ 单位化 $\gamma_2 = \begin{pmatrix} 2/3 \\ -1/3 \\ -2/3 \end{pmatrix}$

•
$$\lambda_3 = 5$$
,特征向量 $\alpha_3 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ 单位化 $\gamma_3 = \begin{pmatrix} 1/3 \\ -2/3 \\ 2/3 \end{pmatrix}$

所以取
$$Q = \underbrace{\begin{pmatrix} 2/3 & 2/3 & 1/3 \\ 2/3 & -1/3 & -2/3 \\ 1/3 & -2/3 & 2/3 \end{pmatrix}}_{Q: F 交降}, 则 $Q^{-1}AQ = \begin{pmatrix} -1 & 2 & 1 \\ 2 & 5 & 1 \end{pmatrix}$$$



例
$$2A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{pmatrix}$$

$$Q^{-1}AQ = \left(\begin{array}{c} * \\ * \\ \end{array}\right)$$



例
$$2A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{pmatrix}$$
,特征方程: $0 = |\lambda I - A| = (\lambda - 1)^2(\lambda - 10)$

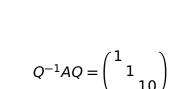
$$Q^{-1}AQ = \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix}$$

$$Q^{-1}AQ = \begin{pmatrix} * \\ * \\ * \end{pmatrix}$$

•
$$\lambda_3 = 10$$

• $\lambda_1 = 1$ (二重)

例 $2A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{pmatrix}$, 特征方程: $0 = |\lambda I - A| = (\lambda - 1)^2 (\lambda - 10)$



例 $2A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{pmatrix}$, 特征方程: $0 = |\lambda I - A| = (\lambda - 1)^2(\lambda - 10)$

• $\lambda_1 = 1$ (二重)

•
$$\lambda_3 = 10$$

• $\lambda_3 = 10$

λ₁ = 1 (二重). 特征向量

 $\begin{cases} \alpha_1 = \begin{pmatrix} -2\\1\\0 \end{pmatrix} \\ \alpha_2 = \begin{pmatrix} 2\\0\\1 \end{pmatrix} \end{cases}$

 $Q^{-1}AQ = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

例 $2A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{pmatrix}$, 特征方程: $0 = |\lambda I - A| = (\lambda - 1)^2 (\lambda - 10)$

•
$$\lambda_1 = 1$$
(二重),特征向量

例 $2A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ 2 & -4 & 5 \end{pmatrix}$, 特征方程: $0 = |\lambda I - A| = (\lambda - 1)^2(\lambda - 10)$

$$\begin{cases}
\alpha_1 = \begin{pmatrix} -2\\1\\0 \end{pmatrix} \\
\alpha_2 = \begin{pmatrix} 2\\0\\1 \end{pmatrix}
\end{cases}$$

•
$$\lambda_3 = 10$$
, 特征向量 $\alpha_3 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$



 $Q^{-1}AQ = \begin{pmatrix} 1 & 1 & 1 \\ & 1 & 1 \end{pmatrix}$

例
$$2A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{pmatrix}$$
, 特征方程: $0 = |\lambda I - A| = (\lambda - 1)^2 (\lambda - 10)$

λ₁ = 1 (二重). 特征向量

$$\begin{cases}
\alpha_1 = \begin{pmatrix} -2\\1\\0 \end{pmatrix} \xrightarrow{\mathbb{E}^{2}(\mathbb{N})} \begin{cases}
\beta_1 = \begin{pmatrix} -2\\1\\0 \end{pmatrix} \\
\beta_2 = \begin{pmatrix} 2/5\\4/5\\1 \end{pmatrix}
\end{cases}$$

• $\lambda_3 = 10$,特征向量 $\alpha_3 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$

$$Q^{-1}AQ = \begin{pmatrix} 1 & 1 \\ & 10 \end{pmatrix}$$



例
$$2A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{pmatrix}$$
, 特征方程: $0 = |\lambda I - A| = (\lambda - 1)^2 (\lambda - 10)$

λ₁ = 1 (二重). 特征向量

$$\begin{cases}
\alpha_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \xrightarrow{\text{EXC}}
\begin{cases}
\beta_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \xrightarrow{\text{$\frac{1}{\sqrt{5}}$}} \begin{pmatrix} \gamma_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \\
\alpha_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}
\end{cases}$$

$$\begin{cases}
\beta_2 = \begin{pmatrix} 2/5 \\ 4/5 \\ 1 \end{pmatrix}
\end{cases}$$

$$\begin{cases}
\gamma_2 = \frac{5}{3\sqrt{5}} \begin{pmatrix} 2/5 \\ 4/5 \\ 1 \end{pmatrix}
\end{cases}$$

$$\begin{cases}
\alpha_3 = 10, \quad 450 \text{ for } 360 \text{ fo$$

• $\lambda_3 = 10$,特征向量 $\alpha_3 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$

$$Q^{-1}AQ = \begin{pmatrix} 1 & 1 & 1 \\ & 10 & 1 \end{pmatrix}$$



例 $2A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{pmatrix}$, 特征方程: $0 = |\lambda I - A| = (\lambda - 1)^2 (\lambda - 10)$

λ₁ = 1 (二重). 特征向量

 $Q^{-1}AQ = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$



例 $2A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ 2 & 5 & 5 \end{pmatrix}$, 特征方程: $0 = |\lambda I - A| = (\lambda - 1)^2(\lambda - 10)$ λ₁ = 1 (二重), 特征向量

$$\alpha_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{cases} \alpha_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \xrightarrow{\text{正交化}} \begin{cases} \beta_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \xrightarrow{\text{单位化}} \begin{cases} \gamma_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \end{cases} \\ \alpha_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} & \begin{cases} \beta_2 = \begin{pmatrix} 2/5 \\ 4/5 \\ 1 \end{cases} \end{cases} \end{cases} \begin{cases} \gamma_2 = \frac{5}{3\sqrt{5}} \begin{pmatrix} 2/5 \\ 4/5 \\ 1 \end{cases} \end{cases}$$

$$\bullet \ \lambda_3 = 10, \ \text{特征向量} \alpha_3 = \begin{pmatrix} 1 \\ 2 \\ -2 \\ \gamma_3 \end{cases} \xrightarrow{\text{单位化}} \ \gamma_3 = \begin{pmatrix} 1/3 \\ 2/3 \\ -2/3 \end{pmatrix}$$

$$\text{所以取} \ Q = \begin{pmatrix} -2/\sqrt{5} \ 2/3 \sqrt{5} \ 1/3 \\ 1/\sqrt{5} \ 4/3 \sqrt{5} \ 2/3 \\ 0 \ \sqrt{5}/3 \ -2/3 \end{pmatrix}, \ \text{例} \ Q^{-1}AQ = \begin{pmatrix} 1 \\ 1 \\ 10 \end{pmatrix}$$

例
$$3A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

 $Q^{-1}AQ = \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix}$

例 3
$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$
,特征方程: $0 = |\lambda I - A| =$

$$Q^{-1}AQ = \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix}$$

例 3
$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$
,特征方程: $0 = |\lambda I - A| = (\lambda + 1)^2 (\lambda - 5)$ **Det**

$$Q^{-1}AQ = \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix}$$

例 3 $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$,特征方程: $0 = |\lambda I - A| = (\lambda + 1)^2 (\lambda - 5)$ Del

•
$$\lambda_2 = 5$$

• $\lambda_1 = -1$ (二重)

 $Q^{-1}AQ = \begin{pmatrix} * \\ * \\ * \end{pmatrix}$

• $\lambda_1 = -1$ (二重)

$$Q^{-1}AQ = \begin{pmatrix} -1 \\ -1 \\ 5 \end{pmatrix}$$

例 3 $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$,特征方程: $0 = |\lambda I - A| = (\lambda + 1)^2 (\lambda - 5)$ Det

•
$$\lambda_2 = 5$$

§4.3 实对称矩阵的特征值和特征向量



例 3 $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$, 特征方程: $0 = |\lambda I - A| = (\lambda + 1)^2(\lambda - 5)$ **Det**

•
$$\lambda_2 = 5$$

$$\bullet \ \lambda_2 = 5$$

 $Q^{-1}AQ = \begin{pmatrix} -1 & \\ & -1 & \\ & 5 \end{pmatrix}$

 $\begin{cases} \alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \\ \alpha_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \end{cases}$





• $\lambda_2 = 5$,特征向量: • Det $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

λ₁ = −1 (二重), 特征向量: ▶ Detail

 $Q^{-1}AQ = \begin{pmatrix} -1 & \\ & -1 & \\ & 5 \end{pmatrix}$

例 3 $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$,特征方程: $0 = |\lambda I - A| = (\lambda + 1)^2 (\lambda - 5)$ Del

• $\lambda_2 = 5$,特征向量: $\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

例 3 $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$, 特征方程: $0 = |\lambda I - A| = (\lambda + 1)^2(\lambda - 5)$ **Det**

$$\begin{cases}
\alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \\
\alpha_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}
\end{cases}
\xrightarrow{\mathbb{E}^{\frac{1}{2}}\mathbb{E}^{\frac{1}{2}}}$$

$$2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \xrightarrow{\text{lex}(\mathcal{H})}$$

λ₁ = −1 (二重), 特征向量: ▶ Detail

 $Q^{-1}AQ = \begin{pmatrix} -1 & \\ & -1 & \\ & & 5 \end{pmatrix}$

例 3 $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$,特征方程: $0 = |\lambda I - A| = (\lambda + 1)^2 (\lambda - 5)$ **Det**

$$\begin{cases}
\alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \xrightarrow{\text{if } \text{ if } \text{$$

•
$$\lambda_2 = 5$$
,特征向量: $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$



 $Q^{-1}AQ = \begin{pmatrix} -1 \\ -1 \\ \xi \end{pmatrix}$



例 3
$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$
, 特征方程: $0 = |\lambda I - A| = (\lambda + 1)^2 (\lambda - 5)$ Detail $\begin{pmatrix} \lambda_1 = -1 & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) & (-1) &$

•
$$\lambda_2 = 5$$
,特征向量: • Det $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$



 $Q^{-1}AQ = \begin{pmatrix} -1 & \\ & -1 & \\ & 5 \end{pmatrix}$

例 3
$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$
, 特征方程: $0 = |\lambda I - A| = (\lambda + 1)^2 (\lambda - 5)$ Det $\lambda_1 = -1$ (二重), 特征向量: Detail

$$\begin{cases} \alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \xrightarrow{\text{EX}} \begin{cases} \beta_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \xrightarrow{\text{A}} \end{cases}$$

$$\lambda_{1} = -1 \quad (三重) \quad , \quad \text{特征向量:} \quad \text{Detail}$$

$$\begin{cases} \alpha_{1} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} & \text{ } \\ \alpha_{2} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} & \text{ } \end{cases} \quad \begin{cases} \beta_{1} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} & \text{ } \\ \beta_{2} = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix} & \text{ } \end{cases} \quad \begin{cases} \gamma_{1} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} & \text{ } \end{cases}$$

$$\gamma_{2} = \begin{pmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix}$$

• $\lambda_2 = 5$,特征向量: ①Det $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_3 = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$

 $Q^{-1}AQ = \begin{pmatrix} -1 & \\ & -1 & \\ & 5 \end{pmatrix}$

例 3
$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$
, 特征方程: $0 = |\lambda I - A| = (\lambda + 1)^2 (\lambda - 5)$ Det
• $\lambda_1 = -1$ (二重) ,特征向量: Detail
$$\begin{pmatrix} \alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} & \begin{pmatrix} \beta_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} & \begin{pmatrix} \gamma_1 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \end{pmatrix}$$

$$\lambda_{1} = -1 \ (三重) \ , \ \text{特征向量:} \ \ \overset{\text{Detail}}{\bullet} \ \begin{pmatrix} \alpha_{1} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \\ \alpha_{2} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \ \ \overset{\text{正交化}}{\bullet} \ \begin{pmatrix} \beta_{1} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \\ \beta_{2} = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix} \ \ & \begin{cases} \gamma_{1} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \\ \gamma_{2} = \begin{pmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix} \end{pmatrix}$$

• $\lambda_2 = 5$,特征向量: Obt $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_3 = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$

取 $Q = \begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}$, 则 $Q^{-1}AQ = \begin{pmatrix} -1 \\ -1 \\ 5 \end{pmatrix}$

———The End———

$$0 = |\lambda I - A| =$$

$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ -2 & -2 & \lambda - 1 \end{vmatrix}$$

$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ -2 & -2 & \lambda - 1 \end{vmatrix}$$

$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ -2 & -2 & \lambda - 1 \end{vmatrix}$$

$$\frac{r_3 - r_2}{2} \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -\lambda - 1 & \lambda + 1 \end{vmatrix}$$



$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ -2 & -2 & \lambda - 1 \end{vmatrix}$$
$$\frac{r_3 - r_2}{2} \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -\lambda - 1 & \lambda + 1 \end{vmatrix}$$
$$= (\lambda + 1) \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -1 & 1 \end{vmatrix}$$

$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ -2 & -2 & \lambda - 1 \end{vmatrix}$$

$$\frac{r_3 - r_2}{2} \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -\lambda - 1 & \lambda + 1 \end{vmatrix}$$

$$= (\lambda + 1) \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -1 & 1 \end{vmatrix} \frac{c_2 + c_3}{2}$$

$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ -2 & -2 & \lambda - 1 \end{vmatrix}$$

$$\frac{r_3 - r_2}{2} \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -\lambda - 1 & \lambda + 1 \end{vmatrix}$$

$$= (\lambda + 1) \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -1 & 1 \end{vmatrix} \xrightarrow{c_2 + c_3} (\lambda + 1) \begin{vmatrix} \lambda - 1 & -4 & -2 \\ -2 & \lambda - 3 & -2 \\ 0 & 0 & 1 \end{vmatrix}$$





$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ -2 & -2 & \lambda - 1 \end{vmatrix}$$

$$\frac{r_3 - r_2}{2} \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -\lambda - 1 & \lambda + 1 \end{vmatrix}$$

$$= (\lambda + 1) \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -1 & 1 \end{vmatrix} \stackrel{c_2 + c_3}{=} (\lambda + 1) \begin{vmatrix} \lambda - 1 & -4 & -2 \\ -2 & \lambda - 3 & -2 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (\lambda + 1) \begin{vmatrix} \lambda - 1 & -4 \\ -2 & \lambda - 3 \end{vmatrix}$$



$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ -2 & -2 & \lambda - 1 \end{vmatrix}$$

$$\frac{r_3 - r_2}{2} \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -\lambda - 1 & \lambda + 1 \end{vmatrix}$$

$$= (\lambda + 1) \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -1 & 1 \end{vmatrix} \frac{c_2 + c_3}{2} (\lambda + 1) \begin{vmatrix} \lambda - 1 & -4 & -2 \\ -2 & \lambda - 3 & -2 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (\lambda + 1) \begin{vmatrix} \lambda - 1 & -4 \\ -2 & \lambda - 3 \end{vmatrix}$$





 $=(\lambda+1)(\lambda^2-4\lambda-5)$

$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix}$$

$$\frac{r_3 - r_2}{2} \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -\lambda - 1 & \lambda + 1 \end{vmatrix}$$

$$= (\lambda + 1) \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -1 & 1 \end{vmatrix} \frac{c_2 + c_3}{2} (\lambda + 1) \begin{vmatrix} \lambda - 1 & -4 & -2 \\ -2 & \lambda - 3 & -2 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (\lambda + 1) \begin{vmatrix} \lambda - 1 & -4 \\ -2 & \lambda - 3 \end{vmatrix}$$

$$= (\lambda + 1)(\lambda^2 - 4\lambda - 5)$$



 $=(\lambda+1)^2(\lambda-5)$

•
$$\exists \lambda_1 = -1$$
, $\forall M (\lambda_1 I - A) x = 0$:

$$(-I - A : 0) =$$



$$(-I-A:0) = \begin{pmatrix} -2 & -2 & -2 & 0 \\ -2 & -2 & -2 & 0 \\ -2 & -2 & -2 & 0 \end{pmatrix} \rightarrow$$





$$(-I-A:0) = \begin{pmatrix} -2 & -2 & -2 & | & 0 \\ -2 & -2 & -2 & | & 0 \\ -2 & -2 & -2 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$





$$(-I - A \vdots 0) = \begin{pmatrix} -2 & -2 & -2 & | & 0 \\ -2 & -2 & -2 & | & 0 \\ -2 & -2 & -2 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$x_1 + x_2 + x_3 = 0$$





$$(-I-A:0) = \begin{pmatrix} -2 & -2 & -2 & | & 0 \\ -2 & -2 & -2 & | & 0 \\ -2 & -2 & -2 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$x_1 + x_2 + x_3 = 0 \Rightarrow x_1 = -x_2 - x_3$$





$$(-I-A:0) = \begin{pmatrix} -2 & -2 & -2 & | & 0 \\ -2 & -2 & -2 & | & 0 \\ -2 & -2 & -2 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$x_1 + x_2 + x_3 = 0$$
 \Rightarrow $x_1 = -x_2 - x_3$ 基础解系: $\alpha_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



$$(-I-A:0) = \begin{pmatrix} -2 & -2 & -2 & | & 0 \\ -2 & -2 & -2 & | & 0 \\ -2 & -2 & -2 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$x_1 + x_2 + x_3 = 0 \Rightarrow x_1 = -x_2 - x_3$$

基础解系: $\alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



$$(-I-A:0) = \begin{pmatrix} -2 & -2 & -2 & | & 0 \\ -2 & -2 & -2 & | & 0 \\ -2 & -2 & -2 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$x_1 + x_2 + x_3 = 0 \Rightarrow x_1 = -x_2 - x_3$$

基础解系: $\alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$



$$(5I - A : 0) =$$

$$(5I - A \vdots 0) = \begin{pmatrix} 4 & -2 & -2 & | & 0 \\ -2 & 4 & -2 & | & 0 \\ -2 & -2 & 4 & | & 0 \end{pmatrix}$$



$$(5I - A \vdots 0) = \begin{pmatrix} 4 & -2 & -2 & 0 \\ -2 & 4 & -2 & 0 \\ -2 & -2 & 4 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & -1 & -1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{pmatrix}$$



$$(5I - A \vdots 0) = \begin{pmatrix} 4 & -2 & -2 & | & 0 \\ -2 & 4 & -2 & | & 0 \\ -2 & -2 & 4 & | & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & -1 & -1 & | & 0 \\ 1 & -2 & 1 & | & 0 \\ 1 & 1 & -2 & | & 0 \end{pmatrix}$$

$$r_1 \leftrightarrow r_3$$





$$(5I - A \vdots 0) = \begin{pmatrix} 4 & -2 & -2 & | & 0 \\ -2 & 4 & -2 & | & 0 \\ -2 & -2 & 4 & | & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & -1 & -1 & | & 0 \\ 1 & -2 & 1 & | & 0 \\ 1 & 1 & -2 & | & 0 \end{pmatrix}$$

$$\xrightarrow{r_1 \leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 1 & -2 & 1 & 0 \\ 2 & -1 & -1 & 0 \end{array} \right)$$



$$(5I - A \vdots 0) = \begin{pmatrix} 4 & -2 & -2 & | & 0 \\ -2 & 4 & -2 & | & 0 \\ -2 & -2 & 4 & | & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & -1 & -1 & | & 0 \\ 1 & -2 & 1 & | & 0 \\ 1 & 1 & -2 & | & 0 \end{pmatrix}$$

$$\xrightarrow{r_1 \leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 1 & -2 & 1 & 0 \\ 2 & -1 & -1 & 0 \end{array} \right) \xrightarrow[r_3 - 2r_1]{r_2 - r_1} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & -3 & 3 & 0 \end{array} \right)$$



$$(5I - A \vdots 0) = \begin{pmatrix} 4 & -2 & -2 & | & 0 \\ -2 & 4 & -2 & | & 0 \\ -2 & -2 & 4 & | & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & -1 & -1 & | & 0 \\ 1 & -2 & 1 & | & 0 \\ 1 & 1 & -2 & | & 0 \end{pmatrix}$$

$$\xrightarrow{r_1 \leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 1 & -2 & 1 & 0 \\ 2 & -1 & -1 & 0 \end{array}\right) \xrightarrow[r_3 - 2r_1]{r_2 - r_1} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$



• 当 $\lambda_2 = 5$, 求解 $(\lambda_2 I - A)x = 0$:

$$(5I - A \vdots 0) = \begin{pmatrix} 4 & -2 & -2 & | & 0 \\ -2 & 4 & -2 & | & 0 \\ -2 & -2 & 4 & | & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & -1 & -1 & | & 0 \\ 1 & -2 & 1 & | & 0 \\ 1 & 1 & -2 & | & 0 \end{pmatrix}$$

$$\xrightarrow{r_1 \leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 1 & -2 & 1 & 0 \\ 2 & -1 & -1 & 0 \end{array}\right) \xrightarrow[r_3 - 2r_1]{r_2 - r_1} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{r_1 - r_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$



• $\exists \lambda_2 = 5$, \vec{x} \vec{x} $(\lambda_2 I - A)x = 0$:

$$(5I - A \vdots 0) = \begin{pmatrix} 4 & -2 & -2 & | & 0 \\ -2 & 4 & -2 & | & 0 \\ -2 & -2 & 4 & | & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & -1 & -1 & | & 0 \\ 1 & -2 & 1 & | & 0 \\ 1 & 1 & -2 & | & 0 \end{pmatrix}$$

$$\xrightarrow{r_1 \leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 1 & -2 & 1 & 0 \\ 2 & -1 & -1 & 0 \end{array}\right) \xrightarrow[r_3 - 2r_1]{r_2 - r_1} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right)$$

$$\longrightarrow \begin{pmatrix} 1 & 1 & -2 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \xrightarrow{r_1 - r_2} \begin{pmatrix} 1 & 0 & -1 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

所以
$$\begin{cases} x_1 & -x_3 = 0 \end{cases}$$





• 当 $\lambda_2 = 5$, 求解 $(\lambda_2 I - A)x = 0$:

$$(5I - A \vdots 0) = \begin{pmatrix} 4 & -2 & -2 & | & 0 \\ -2 & 4 & -2 & | & 0 \\ -2 & -2 & 4 & | & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & -1 & -1 & | & 0 \\ 1 & -2 & 1 & | & 0 \\ 1 & 1 & -2 & | & 0 \end{pmatrix}$$

$$\xrightarrow{r_1 \leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 1 & -2 & 1 & 0 \\ 2 & -1 & -1 & 0 \end{array} \right) \xrightarrow[r_3 - 2r_1]{r_2 - r_1} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & -3 & 3 & 0 \end{array} \right)$$

$$\longrightarrow \begin{pmatrix} 1 & 1 & -2 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \xrightarrow{r_1 - r_2} \begin{pmatrix} 1 & 0 & -1 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$\begin{cases} x_1 & -x_3 = 0 \end{cases}$$

所以
$$\begin{cases} x_1 & -x_3 = 0 \\ x_2 - x_3 = 0 \end{cases}$$



• 当 $\lambda_2 = 5$,求解 $(\lambda_2 I - A)x = 0$:

$$(5I - A \vdots 0) = \begin{pmatrix} 4 & -2 & -2 & | & 0 \\ -2 & 4 & -2 & | & 0 \\ -2 & -2 & 4 & | & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & -1 & -1 & | & 0 \\ 1 & -2 & 1 & | & 0 \\ 1 & 1 & -2 & | & 0 \end{pmatrix}$$

$$\xrightarrow{r_1 \leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 1 & -2 & 1 & 0 \\ 2 & -1 & -1 & 0 \end{array} \right) \xrightarrow[r_3 - 2r_1]{r_2 - r_1} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & -3 & 3 & 0 \end{array} \right)$$

$$\longrightarrow \left(\begin{array}{cc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \xrightarrow{r_1 - r_2} \left(\begin{array}{cc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

所以
$$\begin{cases} x_1 & -x_3 = 0 \\ x_2 - x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = x_3 \\ x_2 = x_3 \end{cases}$$





• 当 $\lambda_2 = 5$,求解 $(\lambda_2 I - A)x = 0$:

$$(5I - A \vdots 0) = \begin{pmatrix} 4 & -2 & -2 & | & 0 \\ -2 & 4 & -2 & | & 0 \\ -2 & -2 & 4 & | & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & -1 & -1 & | & 0 \\ 1 & -2 & 1 & | & 0 \\ 1 & 1 & -2 & | & 0 \end{pmatrix}$$

$$\xrightarrow{r_1 \leftrightarrow r_3} \begin{pmatrix} 1 & 1 & -2 & | & 0 \\ 1 & -2 & 1 & | & 0 \\ 2 & -1 & -1 & | & 0 \end{pmatrix} \xrightarrow{r_2 - r_1} \begin{pmatrix} 1 & 1 & -2 & | & 0 \\ 0 & -3 & 3 & | & 0 \\ 0 & -3 & 3 & | & 0 \end{pmatrix}$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{r_1 - r_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

所以 $\begin{cases} x_1 & -x_3 = 0 \\ x_2 - x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = x_3 \\ x_2 = x_3 \end{cases}$

基础解系:
$$\alpha_3 = \begin{pmatrix} \\ 1 \end{pmatrix}$$





• 当 $\lambda_2 = 5$, 求解 $(\lambda_2 I - A)x = 0$:

$$(5I - A \vdots 0) = \begin{pmatrix} 4 & -2 & -2 & | & 0 \\ -2 & 4 & -2 & | & 0 \\ -2 & -2 & 4 & | & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & -1 & -1 & | & 0 \\ 1 & -2 & 1 & | & 0 \\ 1 & 1 & -2 & | & 0 \end{pmatrix}$$

$$\xrightarrow{r_1 \leftrightarrow r_3} \begin{pmatrix} 1 & 1 & -2 & 0 \\ 1 & -2 & 1 & 0 \\ 2 & -1 & -1 & 0 \end{pmatrix} \xrightarrow{r_2 - r_1} \begin{pmatrix} 1 & 1 & -2 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & -3 & 3 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -2 & 0 \\ 1 & 1 & -2 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & -3 & 3 & 0 \end{pmatrix}$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{r_1 - r_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

所以 $\begin{cases} x_1 & -x_3 = 0 \\ x_2 - x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = x_3 \\ x_2 = x_3 \end{cases}$

基础解系:
$$\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$



将线性无关组
$$\alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ 正交化:





将线性无关组
$$\alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ 正交化:

$$\beta_1 =$$

$$\beta_2 =$$





将线性无关组
$$\alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ 正交化:

$$\beta_1 = \alpha_1$$

$$\beta_2 =$$



将线性无关组
$$\alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ 正交化:

$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \cdots - \beta_1$$

将线性无关组
$$\alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ 正交化:

$$\beta_1 = \alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \cdots - \beta_1$$



将线性无关组
$$\alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ 正交化:

$$\beta_1 = \alpha_1 = \left(\begin{array}{c} -1\\1\\0 \end{array}\right)$$

$$\beta_2 = \alpha_2 - \dots - \beta_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - - \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$





将线性无关组
$$\alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ 正交化:

$$\beta_1 = \alpha_1 = \left(\begin{array}{c} -1\\1\\0\end{array}\right)$$

$$\beta_2 = \alpha_2 - \dots - \beta_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{-1}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

▶ Back



将线性无关组
$$\alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ 正交化:

$$\beta_1 = \alpha_1 = \left(\begin{array}{c} -1\\1\\0\end{array}\right)$$

$$\beta_2 = \alpha_2 - \dots - \beta_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$





将线性无关组
$$\alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ 正交化:

$$\beta_1 = \alpha_1 = \left(\begin{array}{c} -1\\1\\0\end{array}\right)$$

$$\beta_2 = \alpha_2 - \dots - \beta_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}$$

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