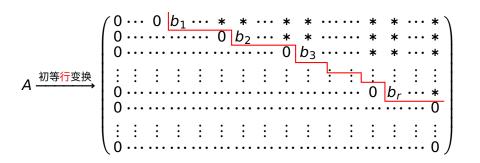
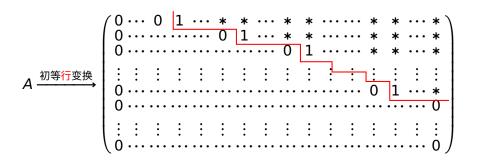
## 第3章a:线性方程组的消元解法

数学系 梁卓滨

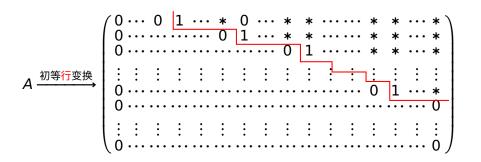
2019-2020 学年 I



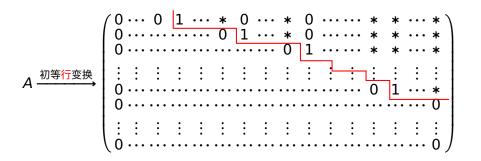
消元法 1/17 < ▶ △ ▼



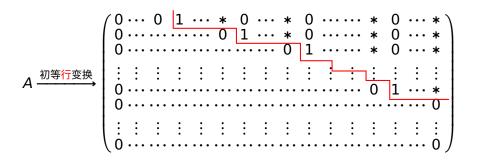
消元法 1/17 < ▶ △ ▽



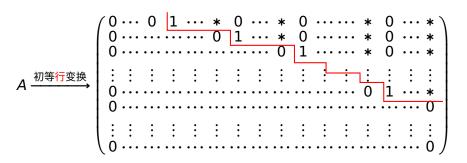
消元法 1/17 ▼ ▷ △ ▼



消元法 1/17 ▼ ▷ △ ▼



消元法 1/17 < ▶ △ ▼



后者称为简化的阶梯型矩阵。

考虑n个未知量m个方程的线性方程组:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

考虑 n 个未知量 m 个方程的线性方程组:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m \times n} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

考虑 n 个未知量 m 个方程的线性方程组:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m \times n}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}_{b}$$

考虑 n 个未知量 m 个方程的线性方程组:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m \times n}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}_{b} \quad \Rightarrow \quad Ax = b$$

考虑 n 个未知量 m 个方程的线性方程组:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

可以,等价地,改写成矩阵形式

$$\underbrace{\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}_{m \times n}}_{A}
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix} = \underbrace{\begin{pmatrix}
b_1 \\
b_2 \\
\vdots \\
b_m
\end{pmatrix}}_{b}$$

$$\Rightarrow Ax = b$$

整个方程组的信息包含在:  $(A:b) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix}$ 

考虑 n 个未知量 m 个方程的线性方程组:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m \times n}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}_{b} \quad \Rightarrow \quad Ax = b$$

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ 3x_1 + 4x_2 - 2x_3 = 3 \end{cases}$$

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ 3x_1 + 4x_2 - 2x_3 = 3 \end{cases}$$

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 & (2)-(1) \\ 4x_1 + 7x_2 - x_3 = -1 & (3)-4\times(1) \\ 3x_1 + 4x_2 - 2x_3 = 3 & (4)-3\times(1) \end{cases}$$

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ 3x_1 + 4x_2 - 2x_3 = 3 \end{cases} \xrightarrow{(2)-(1)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \\ 3x_2 + 3x_3 = -9 \\ x_2 + x_3 = -3 \end{cases}$$

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ 3x_1 + 4x_2 - 2x_3 = 3 \end{cases} \xrightarrow{(2)-(1)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \\ 3x_2 + 3x_3 = -9 \\ x_2 + x_3 = -3 \end{cases}$$

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ 3x_1 + 4x_2 - 2x_3 = 3 \end{cases} \xrightarrow{(2)-(1)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \\ 3x_2 + 3x_3 = -9 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(1)-(2)}$$

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ 3x_1 + 4x_2 - 2x_3 = 3 \end{cases} \xrightarrow{(3)-4\times(1)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \\ 3x_2 + 3x_3 = -9 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(1)-(2)} \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \\ 0 = 0 \\ 0 = 0 \end{cases}$$

例 解方程组

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ 3x_1 + 4x_2 - 2x_3 = 3 \end{cases} \xrightarrow{(3)-4\times(1)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \\ 3x_2 + 3x_3 = -9 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(1)-(2)} \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \\ 0 = 0 \\ 0 = 0 \end{cases}$$

$$\begin{cases} x_1 & -2x_3 = 5 \\ x_2 + & x_3 = -3 \\ 0 = 0 \\ 0 = 0 \end{cases} \Rightarrow$$

例 解方程组

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ 3x_1 + 4x_2 - 2x_3 = 3 \end{cases} \xrightarrow{(3)-4\times(1)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \\ 3x_2 + 3x_3 = -9 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(1)-(2)} \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \\ 0 = 0 \\ 0 = 0 \end{cases}$$

$$\begin{cases} x_1 & -2x_3 = 5 \\ x_2 + & x_3 = -3 \\ 0 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 5 + 2x_3 \\ x_2 = -3 - x_3 \end{cases}$$

例 解方程组

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ 3x_1 + 4x_2 - 2x_3 = 3 \end{cases} \xrightarrow{(2)-(1)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \\ 3x_2 + 3x_3 = -9 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(1)-(2)} \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \\ 0 = 0 \\ 0 = 0 \end{cases}$$

1

$$(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ 3 & 4 & 2 & 3 \end{pmatrix}$$

$$\begin{cases} x_1 & -2x_3 = 5 \\ x_2 + & x_3 = -3 \\ 0 = 0 & \Rightarrow \begin{cases} x_1 = 5 + 2x_3 \\ x_2 = -3 - x_3 \end{cases} \\ 0 = 0 & \end{cases}$$

例 解方程组

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ 3x_1 + 4x_2 - 2x_3 = 3 \end{cases} \xrightarrow{(2)-(1)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \\ 3x_2 + 3x_3 = -9 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(1)-(2)} \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \\ 0 = 0 \\ 0 = 0 \end{cases}$$

$$(A : b) = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ 3 & 4 & 2 & 3 \end{pmatrix}$$

$$\begin{cases} x_1 & -2x_3 = 5 \\ x_2 + & x_3 = -3 \\ 0 = 0 & \Rightarrow \begin{cases} x_1 = 5 + 2x_3 \\ x_2 = -3 - x_3 \end{cases} \\ 0 = 0 & \end{cases}$$

例 解方程组

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ 3x_1 + 4x_2 - 2x_3 = 3 \end{cases} \xrightarrow{(2)-(1)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \\ 3x_2 + 3x_3 = -9 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(1)-(2)} \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \\ 0 = 0 \\ 0 = 0 \end{cases}$$

1

$$(A:b) = \begin{pmatrix} \boxed{1} & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ 3 & 4 & 2 & 3 \end{pmatrix} \quad \frac{r_2 - r_1}{r_3 - 4r_1}$$

$$\begin{cases} x_1 & -2x_3 = 5 \\ x_2 + & x_3 = -3 \\ 0 = 0 & \Rightarrow \begin{cases} x_1 = 5 + 2x_3 \\ x_2 = -3 - x_3 \end{cases} \\ 0 = 0 & \end{cases}$$

例 解方程组

$$\begin{cases} x_{1} + x_{2} - x_{3} = 2 \\ x_{1} + 2x_{2} = -1 \\ 4x_{1} + 7x_{2} - x_{3} = -1 \\ 3x_{1} + 4x_{2} - 2x_{3} = 3 \end{cases} \xrightarrow{(2)-(1)} \begin{cases} x_{1} + x_{2} - x_{3} = 2 \\ x_{2} + x_{3} = -3 \\ 3x_{2} + 3x_{3} = -9 \\ x_{2} + x_{3} = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_{1} - 2x_{3} = 5 \\ x_{2} + x_{3} = -3 \\ 0 = 0 \\ 0 = 0 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ 3 & 4 & 2 & 3 \end{pmatrix} \xrightarrow[r_4-3r_1]{r_2-r_1} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 3 & -9 \\ 0 & 1 & 1 & -3 \end{pmatrix}$$

$$\begin{cases} x_1 & -2x_3 = 5 \\ x_2 + & x_3 = -3 \\ 0 = 0 & \Rightarrow \begin{cases} x_1 = 5 + 2x_3 \\ x_2 = -3 - x_3 \end{cases} \\ 0 = 0 & \end{cases}$$

例 解方程组

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ 3x_1 + 4x_2 - 2x_3 = 3 \end{cases} \xrightarrow{(3)-4\times(1)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \\ 3x_2 + 3x_3 = -9 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \\ 0 = 0 \\ 0 = 0 \end{cases}$$

$$(A:b) = \begin{pmatrix} \boxed{1} & \boxed{1} & -1 & \boxed{2} \\ \boxed{1} & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ 3 & 4 & 2 & \boxed{3} \end{pmatrix} \xrightarrow[r_4 - 3r_1]{r_2 - r_1} \begin{pmatrix} \boxed{1} & \boxed{1} & -1 & \boxed{2} \\ \boxed{0} & \boxed{1} & \boxed{1} & -3 \\ \boxed{0} & 3 & \boxed{3} & -9 \\ \boxed{0} & 1 & \boxed{1} & -3 \end{pmatrix}$$

$$\begin{cases} x_1 & -2x_3 = 5 \\ x_2 + & x_3 = -3 \\ & 0 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 5 + 2x_3 \\ x_2 = -3 - x_3 \end{cases}$$

例 解方程组

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ 3x_1 + 4x_2 - 2x_3 = 3 \end{cases} \xrightarrow{(3)-4x(1)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \\ 3x_2 + 3x_3 = -9 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(1)-(2)} \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \\ 0 = 0 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 1 & 2 \\ 12 & 0 & -1 \\ 47 - 1 & -1 & \frac{r_2 - r_1}{r_3 - 4r_1} \\ 34 & 2 & 3 \end{pmatrix} \xrightarrow{r_3 - 4r_1} \begin{cases} 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 3 & -9 \\ 0 & 1 & 1 & -3 \end{cases} \xrightarrow{r_1 - r_2} \xrightarrow{r_3 - 3r_2} \begin{cases} 1 & -2x_3 = 5 \\ x_2 + x_3 = -3 \end{cases}$$

$$\begin{cases} x_1 & -2x_3 = 5 \\ x_2 + & x_3 = -3 \\ 0 = 0 & \Rightarrow \begin{cases} x_1 = 5 + 2x_3 \\ x_2 = -3 - x_3 \end{cases} \\ 0 = 0 & \end{cases}$$

#### 例 解方程组

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ 3x_1 + 4x_2 - 2x_3 = 3 \end{cases} \xrightarrow{(3)-4x(1)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \\ 3x_2 + 3x_3 = -9 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3x(2)} \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \\ 0 = 0 \\ 0 = 0 \end{cases}$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$\begin{cases} x_1 & -2x_3 = 5 \\ x_2 + & x_3 = -3 \\ 0 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 5 + 2x_3 \\ x_2 = -3 - x_3 \end{cases}$$

例 解方程组

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ 3x_1 + 4x_2 - 2x_3 = 3 \end{cases} \xrightarrow{(2)-(1)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \\ 3x_2 + 3x_3 = -9 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3x(2)} \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \\ 0 = 0 \\ 0 = 0 \end{cases}$$

$$\downarrow \qquad \qquad \downarrow \qquad$$

$$\begin{cases} x_1 & -2x_3 = 5 \\ x_2 + & x_3 = -3 \\ 0 = 0 & \Rightarrow \begin{cases} x_1 = 5 + 2x_3 \\ x_2 = -3 - x_3 \end{cases} \end{cases}$$

#### 例 解方程组

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ 3x_1 + 4x_2 - 2x_3 = 3 \end{cases} \xrightarrow{(2)-(1)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \\ 3x_2 + 3x_3 = -9 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(0)-(2)} \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(0)-(2)} \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(0)-(2)} \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(0)-(2)} \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(0)-(2)} \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(0)-(2)} \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(0)-(2)} \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(0)-(2)} \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(0)-(2)} \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(0)-(2)} \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(0)-(2)} \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(0)-(2)} \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(0)-(2)} \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(0)-(2)} \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(0)-(2)} \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(0)-(2)} \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(0)-(2)} \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(0)-(2)} \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(0)-(2)} \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(0)-(2)} \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(0)-(2)} \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(0)-(2)} \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(0)-(2)} \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(0)-(2)} \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(0)-(2)} \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(0)-(2)} \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(0)-(2)} \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(0)-(2)} \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(0)-(2)} \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(0)-(2)} \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(0)-(2)} \begin{cases} x_1 - 2x_3 = 5 \end{cases} \xrightarrow{(0)-(2)} \begin{cases} x_$$

所以

$$\begin{cases} x_1 & -2x_3 = 5 \\ x_2 + & x_3 = -3 \\ 0 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 5 + 2x_3 \\ x_2 = -3 - x_3 \end{cases}$$

主元

#### 例 解方程组

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ 3x_1 + 4x_2 - 2x_3 = 3 \end{cases} \xrightarrow{(2)-(1)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \\ 3x_2 + 3x_3 = -9 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(1)-(2)} \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(3)-3\times(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(4)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(4)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(4)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(4)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(4)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(4)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(4)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(4)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(4)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(4)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(4)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(4)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(4)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(4)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(4)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \end{cases} \xrightarrow{(4)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_3 + x_3 = -3 \end{cases} \xrightarrow{(4)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_3 + x_3 = -3 \end{cases} \xrightarrow{(4)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_3 + x_3 = -3 \end{cases} \xrightarrow{(4)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_3 + x_3 = -3 \end{cases} \xrightarrow{(4)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_3 + x_3 = -3 \end{cases} \xrightarrow{(4)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_3 + x_3 = -3 \end{cases} \xrightarrow{(4)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_3 + x_3 = -3 \end{cases} \xrightarrow{(4)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_3 + x_3 = -3 \end{cases} \xrightarrow{(4)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_3 + x_3 = -3 \end{cases} \xrightarrow{(4)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_3 + x_3 = -3 \end{cases} \xrightarrow{(4)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_3 + x_3 = -3 \end{cases} \xrightarrow{(4)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_3 + x_3 = -3 \end{cases} \xrightarrow{(4)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_3 + x_3 = -3 \end{cases} \xrightarrow{(4)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_3 + x_3 = -3 \end{cases} \xrightarrow{(4)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \end{cases} \xrightarrow{(4)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \end{cases} \xrightarrow{(4)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \end{cases} \xrightarrow{(4)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \end{cases} \xrightarrow{(4)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \end{cases} \xrightarrow{(4)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \end{cases} \xrightarrow{(4)-(2)} \begin{cases} x_1 + x_2 - x_3 = 2 \end{cases} \xrightarrow{(4)-(2)$$

自由变量

$$\vec{x}_{1} - 2\vec{x}_{3} \\
\vec{x}_{2} + \vec{x}_{3} \\
0 \\
0$$

$$\begin{pmatrix}
01 & 1 & -3 \\
03 & 3 & -9 \\
01 & 1 & -3
\end{pmatrix}$$

$$\begin{cases} x_1 & -2x_3 = 5 \\ x_2 + & x_3 = -3 \\ 0 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 5 + 2x_3 \\ x_2 = -3 - x_3 \end{cases}$$

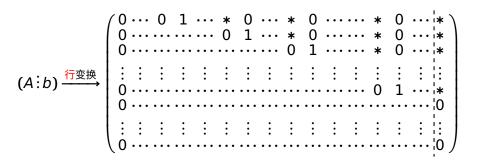
步骤:

- 1. Ax = b  $\Longrightarrow$   $(A : b) \xrightarrow{ij}$  简化的阶梯型矩阵
- 2. 确定主元、自由变量

步骤:

- 1. Ax = b  $\Longrightarrow$   $(A : b) \xrightarrow{ij}$  简化的阶梯型矩阵
- 2. 确定主元、自由变量

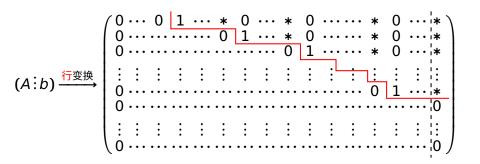
例如



步骤:

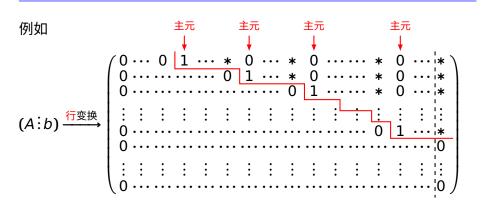
- 1. Ax = b  $\Longrightarrow$   $(A : b) \xrightarrow{\eta = \eta = 0}$  简化的阶梯型矩阵
- 2. 确定主元、自由变量

例如



步骤:

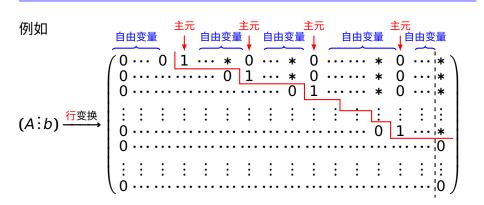
- 1. Ax = b  $\Longrightarrow$   $(A : b) \xrightarrow{\eta \in f \circ h}$  简化的阶梯型矩阵
- 2. 确定主元、自由变量



## 初等行变换求解线性方程组

步骤:

- 1. Ax = b  $\Longrightarrow$   $(A : b) \xrightarrow{\eta = \eta = 0}$  简化的阶梯型矩阵
- 2. 确定主元、自由变量



**例1** 解方程组:  $\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$ 

例1 解方程组: 
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$

$$(A \vdots b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{pmatrix}$$

例 1 解方程组: 
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{pmatrix}$$

**例1** 解方程组: 
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$
**解** (A:b) = 
$$\begin{pmatrix} 1 & 1 & -1 & 2 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{pmatrix} \xrightarrow[r_3 - 2r_1]{r_3 - 2r_1} r_{4+2r_1}$$

$$x_1 + x_2 - x_3 = x_1 + 2x_2 = x_1 + 2x_2 = x_1 + x_2 + x_3 = x_1 + x_2 + x_2 + x_3 = x_1 + x_2 + x_2 + x_3 = x_1 + x_$$

$$\begin{array}{c}
2x_1 + 5x_2 + x_3 = -5 \\
-2x_1 - 3x_2 + x_3 = -1
\end{array}$$

**例1** 解方程组: 
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$
**解** 
$$(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{pmatrix} \xrightarrow[r_3 - 2r_1]{r_3 - 2r_1} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 5 & 1 & -1 & r_4 + 2r_1 \end{pmatrix}$$

$$x_1 + x_2 - x_3 = 2$$
  
 $x_1 + 2x_2 = -$ 

$$\begin{cases} 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$

**例1** 解方程组: 
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$
**解** 
$$(A:b) = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 5 & 1 \\ -2 & -3 & 1 & -1 \end{pmatrix} \xrightarrow[r_3 - 2r_1]{r_3 - 2r_1} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 & -3 \\ -2 & -3 & 1 & -1 \end{pmatrix}$$

$$x_1 + x_2 - x_3 = 2$$
  
**例1** 解方程组:  $x_1 + 2x_2 = -$ 

1 解方程组: 
$$\begin{cases} 2x_1 + 5x_2 + x_3 = -1 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$

**例1**解方程组: 
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$
**解** (A:b) = 
$$\begin{pmatrix} 1 & 1 & -1 \\ 2 & 5 & 1 \\ -5 & 1 & -1 \end{pmatrix} \xrightarrow[r_3 - 2r_1]{r_3 - 2r_1} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 3 & 3 & -9 \end{pmatrix}$$

**列1** 解方程组: 
$$\begin{cases} x_1 + 2x_2 & = - \\ 2x_1 + 5x_2 + x_3 & = - \end{cases}$$

$$2x_{1} + 5x_{2} + x_{3} = -2x_{1} - 3x_{2} + x$$

**例1** 解方程组: 
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$
**解** 
$$(A:b) = \begin{pmatrix} 1 & 1 & -1 & | & 2 \\ 2 & 5 & 1 & | & -5 \\ 2 & 5 & 1 & | & -1 \end{pmatrix} \xrightarrow[r_3 - 2r_1]{r_3 - 2r_1} \begin{pmatrix} 1 & 1 & -1 & | & -3 \\ 0 & 1 & 3 & 3 & | & -9 \\ 0 & -1 & -1 & | & 3 \end{pmatrix}$$

**例1** 解方程组: 
$$\begin{cases} x_1 + 2x_2 & = - \\ 2x_1 + 5x_2 + x_3 & = - \end{cases}$$

例 I 解方程组: 
$$\begin{cases} 2x_1 + 5x_2 + x_3 = -4x_1 - 3x_2 + x_3 = -4x_1 - 3x_1 - 3x_2 + x_2 + x_3 = -4x_1 - 3x_1 - 3x_2 + x_2 + x_3 = -4x_1 - 3x_1 - 3x_1 - 3x_2 + x_2 + x_3 = -4x_1 - 3x_1 - 3x_1 - 3x_2 + x_2 + x_3 = -4x_1 - 3x_1 - 3x_1$$

**例1** 解方程组: 
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$
**解** 
$$(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{pmatrix} \xrightarrow[r_3 - 2r_1]{r_3 - 2r_1} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & -1 & -1 & 3 \end{pmatrix}$$

**例1** 解方程组: 
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$
**解** (A:b) = 
$$\begin{pmatrix} 1 & 1 & -1 & 2 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{pmatrix} \xrightarrow[r_3 - 2r_1]{r_3 - 2r_1} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 3 & 3 & 3 \\ 0 & -1 & -1 & 3 \end{pmatrix}$$

$$r_1-r_2 \rightarrow r_3-3r_2$$

**例 1** 解方程组: 
$$\begin{cases} x_1 + 2x_2 & = - \\ 2x_1 + 5x_2 + x_3 & = - \end{cases}$$

$$2x_1 + 5x_2 + x_3 = -$$

$$-2x_1 - 3x_2 + x_3 = -$$

**例1** 解方程组: 
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$
**解** 
$$(A:b) = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{pmatrix} \xrightarrow[r_3 - 2r_1]{r_3 - 2r_1} \begin{pmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & -3 \\ 0 & -1 & -1 & -3 \\ 0 & -1 & -1 & -3 \end{pmatrix}$$

$$\xrightarrow[r_1+r_2]{r_1-r_2} \left( \begin{array}{ccc|c} 0 & 1 & 1 & -3 \end{array} \right)$$

**列1** 解方程组: 
$$\begin{cases} x_1 + 2x_2 & = -2x_1 + 5x_2 + x_3 = -2x_1 - 3x_2 + x_3 = -2x_1 - x_1 - x_1 - x_1 - x_2 + x_2 - x_2 - x_1 - x_1 - x_2 - x_2 - x_2 - x_1 - x_1 - x_2 - x_2$$

**例1** 解方程组: 
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$
**解** 
$$(A:b) = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{pmatrix} \xrightarrow[r_4+2r_1]{r_3-2r_1} \begin{pmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & -3 \\ 0 & -1 & -1 & -3$$

$$\frac{r_1 - r_2}{r_3 - 3r_2} \left( \begin{array}{ccc} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -3 \\ \end{array} \right)$$

列1 解方程组: 
$$\begin{cases} x_1 + 2x_2 & = - \\ 2x_1 + 5x_2 + x_3 & = - \\ -2x_1 - 3x_2 + x_3 & = - \end{cases}$$

**例1** 解方程组: 
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$
**解** 
$$(A:b) = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 5 & 1 \\ -5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{pmatrix} \xrightarrow[r_3 - 2r_1]{r_3 - 2r_1} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & -1 & -1 & 3 \end{pmatrix}$$

**列1** 解方程组: 
$$\begin{cases} x_1 + 2x_2 & = -2x_1 + 5x_2 + x_3 & = -2x_1 - 3x_2 + x_2 & = -2x_1 - 3x_2 + x_3 & = -2x_1 - 3x_2 + x_2 + x_3 & = -2x_1 - 3x_2 + x_2 + x_3 & = -2x_1 - 3x_2 + x_2 + x_3 & = -2x_1 - 3x_2 + x_2 + x_3 + x_3 + x_3 + x_2 + x_3 + x_$$

**例1** 解方程组:  $\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$  **解**  $(A:b) = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 5 & 1 \\ -5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{pmatrix} \xrightarrow[r_3 - 2r_1]{r_3 - 2r_1} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & -1 & -1 & 3 \end{pmatrix}$ 

$$\begin{array}{c|cccc}
 & r_1 - r_2 \\
\hline
 & r_3 - 3r_2 \\
 & r_4 + r_2
\end{array}
\left(\begin{array}{ccccc}
1 & 0 & -2 & 5 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right)$$

**例1** 解方程组: 
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$
**解** 
$$(A:b) = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 5 & 1 \\ -5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{pmatrix} \xrightarrow[r_3 - 2r_1]{r_3 - 2r_1} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & -1 & -1 & 3 \end{pmatrix}$$

$$\frac{r_1 - r_2}{r_3 - 3r_2} \begin{pmatrix}
\begin{vmatrix}
1 & 0 & -2 & | & 5 \\
0 & 1 & 1 & | & -3 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{vmatrix}$$

例 1 解方程组: 
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{pmatrix} \xrightarrow[r_4+2r_1]{r_2-r_1} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 3 & 3 & -9 \\ 0 & -1 & -1 & 3 \end{pmatrix}$$

$$\xrightarrow[r_4+r_2]{r_1-r_2} \begin{pmatrix} \boxed{1} & 0 & -2 & | & 5 \\ \hline 0 & \boxed{1} & \boxed{1} & | & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

5/17 ⊲ ⊳ Δ ⊽

例 1 解方程组: 
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{pmatrix} \xrightarrow[r_4+2r_1]{r_2-r_1} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 3 & 3 & -9 \\ 0 & -1 & -1 & 3 \end{pmatrix}$$

$$\xrightarrow[r_4+r_2]{r_1-r_2} \begin{pmatrix} 1 & 0 & -2 & 5 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

 $x_1, x_2$  为主元, $x_3$  为自由变量。

例 1 解方程组: 
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$

$$\begin{cases} 1 & 1 - 1 \\ 2 & 0 - 1 \\ 1 & 2 - 1 \end{cases} \xrightarrow{r_2 - r_1} \begin{cases} 1 & 1 - 1 \\ 2 & 0 - 1 \\ 1 & 2 - 1 \end{cases}$$

$$\mathbf{A}:b) = \begin{pmatrix}
1 & 1 & -1 & 2 \\
1 & 2 & 0 & -1 \\
2 & 5 & 1 & -5 \\
-2 & -3 & 1 & -1
\end{pmatrix} \xrightarrow[r_4+2r_1]{r_2-r_1} \begin{pmatrix}
1 & 1 & -1 & 2 \\
0 & 3 & 3 & -9 \\
0 & -1 & -1 & 3
\end{pmatrix}$$

$$\xrightarrow[r_3-3r_2]{r_3-3r_2} \begin{pmatrix}
1 & 0 & -2 & 5 \\
0 & 1 & 1 & -3 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

 $x_1, x_2$  为主元, $x_3$  为自由变量。所以原方程组等价于

$$\begin{cases} x_1 + & -2x_3 = 5 \\ & x_2 + & x_3 = -3 \end{cases}$$

例1 解方程组: 
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{pmatrix} \xrightarrow[r_4+2r_1]{r_2-r_1} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 3 & 3 & -9 \\ 0 & -1 & -1 & 3 \end{pmatrix}$$

$$\xrightarrow[r_3-3r_2]{r_3-3r_2} \begin{pmatrix} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

 $x_1, x_2$  为主元, $x_3$  为自由变量。所以原方程组等价于

$$\begin{cases} x_1 + & -2x_3 = 5 \\ & x_2 + & x_3 = -3 \end{cases} \iff \begin{cases} x_1 + & = 5 + 2x_3 \\ & x_2 = -3 - x_3 \end{cases}$$

例1 解方程组:  $\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$  $(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{pmatrix} \xrightarrow[r_1+2r_3]{r_2-r_1} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -2 \\ 0 & 3 & 3 & -9 \\ 0 & -1 & -1 & 3 \end{pmatrix}$  $\frac{r_1 - r_2}{r_3 - 3r_2} \begin{pmatrix} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$  $x_1, x_2$  为主元, $x_3$  为自由变量。所以原方程组等价干  $\begin{cases} x_1 + & -2x_3 = 5 \\ & x_2 + & x_3 = -3 \end{cases} \iff \begin{cases} x_1 + & = 5 + 2x_3 \\ & x_2 = -3 - x_3 \end{cases}$ 所以通解是:  $(c_1$ 为任意常数) 消元法 5/17 ⊲ ⊳ Δ ⊽

例1 解方程组: 
$$\begin{cases} x_{1} + x_{2} - x_{3} = 2 \\ x_{1} + 2x_{2} = -1 \\ 2x_{1} + 5x_{2} + x_{3} = -5 \\ -2x_{1} - 3x_{2} + x_{3} = -1 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -1 \end{pmatrix} \xrightarrow{r_{2} - r_{1}} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 3 & 3 & -9 \\ 0 & -1 & -1 & -3 \end{pmatrix}$$

$$\xrightarrow{r_{1} - r_{2}} \begin{pmatrix} 1 & 0 & -2 & 5 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$x_{1}, x_{2} \to 2x_{3}, x_{3} \to 3x_{2} \to 3x_{3}$$

$$x_{1} + x_{2} \to 2x_{3} \to 3x_{2} \to 3x_{3}$$

$$x_{2} + x_{3} = -3 \to 3x_{3}$$

$$x_{1} + x_{2} \to 2x_{3} \to 3x_{3}$$

$$x_{2} + x_{3} = -3 \to 3x_{3}$$

$$x_{3} = c_{1}$$

$$x_{1} \to 3x_{2} \to 3x_{3}$$

$$x_{1} \to 3x_{2} \to 3x_{3}$$

$$x_{2} \to 3x_{3} \to 3x_{3}$$

$$x_{2} \to 3x_{3} \to 3x_{3}$$

$$x_{3} \to 3x_{3} \to 3x_{3}$$

$$x_{1} \to 3x_{3} \to 3x_{3}$$

$$x_{2} \to 3x_{3} \to 3x_{3}$$

$$x_{2} \to 3x_{3} \to 3x_{3}$$

$$x_{3} \to 3x_{3} \to 3x_{3}$$

$$x_{4} \to 3x_{4} \to 3x_{4} \to 3x_{4}$$

$$x_{5} \to 3x_{4} \to 3x_{4} \to 3x_{4}$$

$$x_{1} \to 3x_{4} \to 3x_{4} \to 3x_{4}$$

$$x_{2} \to 3x_{4} \to 3x_{4} \to 3x_{4}$$

$$x_{2} \to 3x_{4} \to 3x_{4} \to 3x_{4}$$

$$x_{3} \to 3x_{4} \to 3x_{4} \to 3x_{4} \to 3x_{4}$$

$$x_{4} \to 3x_{4} \to 3x_{4} \to 3x_{4} \to 3x_{4}$$

$$x_{5} \to 3x_{4} \to 3x_{4} \to 3x_{4}$$

$$x_{5} \to 3x_{4} \to 3x_{4} \to 3x_{4}$$

$$x_{5} \to 3x_{4} \to 3x_{4} \to 3x_{4} \to 3x_{4}$$

$$x_{5} \to 3x_{4} \to 3x_{4} \to 3x_{4} \to 3x_{4}$$

$$x_{5} \to 3x_{4} \to 3x_{4} \to 3x_{4} \to 3x_{4}$$

$$x_{5} \to 3x_{4} \to 3x_{4} \to 3x_{4} \to 3x_{4} \to 3x_{4}$$

$$x_{5} \to 3x_{4} \to 3x_{4} \to 3x_{4} \to 3x_{4} \to 3x_{4}$$

$$x_{5} \to 3x_{4} \to 3x_{4}$$

$$m{R}$$
  $(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \end{pmatrix} \frac{r_2 - r_1}{r_3 - 2r_1} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & -1 & -1 & -3 \end{pmatrix}$   $\begin{pmatrix} r_1 - r_2 \\ r_3 - 3r_2 \\ r_4 + r_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$   $x_1, x_2$  为主元, $x_3$  为自由变量。所以原方程组等价于 
$$\begin{cases} x_1 + & -2x_3 = 5 \\ x_2 + & x_3 = -3 \end{cases} \iff \begin{cases} x_1 + & = 5 + 2x_3 \\ x_2 = -3 - x_3 \end{cases}$$
 所以通解是: 
$$\begin{cases} x_1 = 5 + 2c_1 \\ x_2 = -3 - c_1 \end{cases} \begin{pmatrix} x_1 + x_2 + x_3 + x_3 + x_3 + x_4 + x_5 + x_5 + x_5 \\ x_2 = -3 - c_1 \end{pmatrix} \begin{pmatrix} x_1 + x_2 + x_3 + x_3 + x_5 + x_5 \\ x_2 = -3 - c_1 \end{pmatrix}$$
  $\begin{cases} x_1 = 5 + 2c_1 \\ x_2 = -3 - c_1 \end{pmatrix} \begin{pmatrix} x_1 + x_2 + x_3 + x_5 \\ x_2 = -3 - c_1 \end{pmatrix} \begin{pmatrix} x_1 + x_2 + x_3 + x_5 \\ x_2 = -3 - c_1 \end{pmatrix}$   $\begin{cases} x_1 + x_2 + x_3 + x_5 \\ x_2 = -3 - c_1 \end{pmatrix} \begin{pmatrix} x_1 + x_2 + x_3 + x_5 \\ x_2 = -3 - c_1 \end{pmatrix} \begin{pmatrix} x_1 + x_2 + x_3 + x_5 \\ x_2 = -3 - c_1 \end{pmatrix} \begin{pmatrix} x_1 + x_2 + x_3 + x_5 \\ x_2 = -3 - c_1 \end{pmatrix} \begin{pmatrix} x_1 + x_2 + x_3 + x_5 \\ x_2 = -3 - c_1 \end{pmatrix} \begin{pmatrix} x_1 + x_2 + x_3 + x_5 \\ x_2 = -3 - c_1 \end{pmatrix} \begin{pmatrix} x_1 + x_2 + x_3 + x_5 \\ x_2 = -3 - c_1 \end{pmatrix} \begin{pmatrix} x_1 + x_2 + x_3 + x_5 \\ x_2 = -3 - c_1 \end{pmatrix} \begin{pmatrix} x_1 + x_2 + x_3 + x_5 \\ x_2 = -3 - c_1 \end{pmatrix} \begin{pmatrix} x_1 + x_2 + x_3 + x_5 \\ x_2 = -3 - c_1 \end{pmatrix} \begin{pmatrix} x_1 + x_2 + x_3 + x_5 \\ x_2 = -3 - c_1 \end{pmatrix} \begin{pmatrix} x_1 + x_2 + x_3 + x_5 \\ x_2 = -3 - c_1 \end{pmatrix} \begin{pmatrix} x_1 + x_2 + x_3 + x_5 \\ x_3 = c_1 \end{pmatrix}$ 

例1 解方程组:  $\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$ 

**例2** 解方程组:  $\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$ 

例2 解方程组: 
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix}$$

例2 解方程组: 
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix}$$

例2 解方程组: 
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow[r_4-5r_1]{r_2+2r_1}$$

例2 解方程组: 
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$3x_1 + 6x_2 + 13x_3 = 88$$
  
 $5x_1 + 9x_2 + 22x_3 = 141$ 

$$(A:b) = \begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow[r_4-5r_1]{r_2+2r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ r_3-3r_1 & r_4-5r_1 & r_4-5r_1 & r_4-5r_1 \end{pmatrix}$$

例2 解方程组: 
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$3x_1 + 6x_2 + 13x_3 = 88$$
  
 $5x_1 + 9x_2 + 22x_3 = 141$ 

$$(A:b) = \begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow{r_2 + 2r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \end{pmatrix}$$

消元法 6/17 ⊲ ⊳ ∆ ⊽

例2 解方程组: 
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$3x_1 + 6x_2 + 13x_3 = 88$$
  
 $5x_1 + 9x_2 + 22x_3 = 141$ 

$$(A:b) = \begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow[r_3 - 5r_1]{r_2 + 2r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

**例 2** 解方程组: 
$$\begin{cases} -2x_1 - 3x_2 - 9x_3 = -5 \\ 2x_1 + 6x_2 + 12x_3 = -5 \end{cases}$$

例2 解方程组: 
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow{r_2 + 2r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

消元法 6/17 < ▷ △ ▽

例2 解方程组: 
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$3x_1 + 6x_2 + 13x_3 = 88$$
  
 $5x_1 + 9x_2 + 22x_3 = 141$ 

$$(A:b) = \begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow{r_2 + 2r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

消元法 6/17 < ▷ △ ▽

例2 解方程组: 
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$3x_1 + 6x_2 + 13x_3 = 88$$
$$5x_1 + 9x_2 + 22x_3 = 141$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow[r_3-5r_1]{r_2+2r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

例2 解方程组: 
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$5x_1 + 6x_2 + 13x_3 = 88$$
  
 $5x_1 + 9x_2 + 22x_3 = 141$ 

$$(A:b) = \begin{pmatrix} \begin{vmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow[r_4-5r_1]{r_2+2r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

$$\xrightarrow[r_4+r_2]{r_1-2r_2} \left( \begin{array}{cc|c} 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \end{array} \right)$$

例2 解方程组: 
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

解

$$(A:b) = \begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow[r_3-3r_1]{r_2+2r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

$$\xrightarrow[r_4+r_2]{r_1-2r_2} \left( \begin{array}{ccc|c} 1 & 0 & 6 & 22 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \end{array} \right)$$

例2 解方程组: 
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

解

$$(A:b) = \begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow[r_2 - 5r_1]{r_2 + 2r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

$$\xrightarrow[r_4+r_2]{r_1-2r_2} \left( \begin{array}{ccc|c} 1 & 0 & 6 & 22 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{array} \right)$$

例2 解方程组: 
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

解

$$(A:b) = \begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow[r_3-3r_1]{r_2+2r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

$$\xrightarrow[r_4+r_2]{r_1-2r_2} \left( \begin{array}{ccc|c} 1 & 0 & 6 & 22 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{array} \right)$$

例2 解方程组: 
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

解

$$(A:b) = \begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow[r_3 - 3r_1]{r_2 + 2r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

$$\xrightarrow[r_4+r_2]{r_1-2r_2} \left(\begin{array}{ccc} 1 & 0 & 6 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{array}\right) \xrightarrow[r_4-r_3]{r_1-6r_3} \left(\begin{array}{ccc} 1 & 0 & 6 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{array}\right) \xrightarrow[r_4-r_3]{r_1-6r_3} \left(\begin{array}{ccc} 1 & 0 & 6 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{array}\right) \xrightarrow[r_4-r_3]{r_1-6r_3} \left(\begin{array}{ccc} 1 & 0 & 6 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{array}\right) \xrightarrow[r_4-r_3]{r_4-r_3} \left(\begin{array}{ccc} 1 & 0 & 6 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{array}\right) \xrightarrow[r_4-r_3]{r_4-r_3} \left(\begin{array}{ccc} 1 & 0 & 6 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{array}\right) \xrightarrow[r_4-r_3]{r_4-r_3} \left(\begin{array}{ccc} 1 & 0 & 6 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{array}\right) \xrightarrow[r_4-r_3]{r_4-r_3} \left(\begin{array}{ccc} 1 & 0 & 6 \\ 0 & 1 & -1 \\ 0 & 0 & 1$$

例2 解方程组: 
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow{r_2 + 2r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

$$\xrightarrow[r_4+r_2]{r_1-2r_2} \left( \begin{array}{ccc} 1 & 0 & 6 & 22 \\ 0 & 1 & -1 & 23 \\ 0 & 0 & 1 & 4 \end{array} \right) \xrightarrow[r_4-r_3]{r_1-6r_3} \left( \begin{array}{ccc} 0 & 0 & 1 & 4 \end{array} \right)$$

例 2 解方程组: 
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow{r_2 + 2r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

$$\xrightarrow{r_1 - 2r_2} \begin{pmatrix} 1 & 0 & 6 & 22 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \end{pmatrix} \xrightarrow{r_2 + r_3} \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

例2 解方程组: 
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow{r_2 + 2r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 4 & 3 \\ r_3 - 3r_1 & 0 & 0 & 1 & 4 \end{pmatrix}$$

$$\xrightarrow{r_1 - 2r_2} \begin{pmatrix} 1 & 0 & 6 & 22 \\ 0 & 1 & -1 & 3 \\ 0 & 1 & 1 & 3 \end{pmatrix} \xrightarrow{r_1 - 6r_3} \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 1 & 1 & 7 \end{pmatrix}$$

$$\xrightarrow[r_4+r_2]{r_4+r_2} \begin{pmatrix} 1 & 0 & 6 & 22 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{pmatrix} \xrightarrow[r_2+r_3]{r_1-6r_3} \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

例2 解方程组: 
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow{r_2 + 2r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

$$\xrightarrow[r_4+r_2]{r_4+r_2} \begin{pmatrix} 1 & 0 & 6 & 22 \\ 0 & 1 & -1 & 23 \\ 0 & 0 & 1 & 4 \end{pmatrix} \xrightarrow[r_4-r_3]{r_1-6r_3} \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

例2 解方程组: 
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow{r_2 + 2r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

$$\xrightarrow{r_1 - 2r_2} \begin{pmatrix} 1 & 0 & 6 & 22 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \end{pmatrix} \xrightarrow{r_2 + r_3} \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \end{pmatrix} \xrightarrow{r_2 + r_3} \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

例2 解方程组: 
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow{r_2 + 2r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

$$\xrightarrow{r_1 - 2r_2} \begin{pmatrix} 1 & 0 & 6 & 22 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{pmatrix} \xrightarrow{r_2 + r_3} \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

消元法 6/17 ⊲ ⊳ ∆ ⊽

例 2 解方程组: 
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow{r_2 + 2r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

$$\frac{r_1 - 2r_2}{r_1 - 2r_2} \begin{pmatrix} 1 & 0 & 6 & 22 \\ 0 & 1 & -1 & 2 & 3 \end{pmatrix} \xrightarrow{r_1 - 6r_3} \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -7 \\ 0 & 1 & 0 & 7 \end{pmatrix}$$

$$\frac{r_1 - 2r_2}{r_4 + r_2} \leftarrow \begin{pmatrix}
1 & 0 & 6 & 22 \\
0 & 1 & -1 & 3 \\
0 & 0 & 1 & 4
\end{pmatrix}
\frac{r_1 - 6r_3}{r_2 + r_3} \leftarrow \begin{pmatrix}
1 & 0 & 0 & -2 \\
0 & 1 & 0 & 7 \\
0 & 0 & 1 & 4
\end{pmatrix}$$

 $X_1, X_2, X_3$  为主元,没有自由变量。

例2 解方程组: 
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow{r_2 + 2r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

$$\xrightarrow{r_1 - 2r_2} \begin{pmatrix} 1 & 0 & 6 & 22 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \end{pmatrix} \xrightarrow{r_2 + r_3} \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

 $X_1, X_2, X_3$  为主元,没有自由变量。所以原方程组等价于

$$\begin{cases} x_1 & =-2 \\ x_2 & =7 \\ x_3 & =4 \end{cases}$$

例3 解方程组:  $\begin{cases} 4x_1 + 2x_2 - 7x_3 = -3\\ 2x_1 + x_2 - 4x_3 = -1\\ 5x_1 + 3x_2 - 11x_3 = 2\\ x_1 + x_2 - 4x_3 = 2 \end{cases}$ 

例3 解方程组: 
$$\begin{cases} 4x_1 + 2x_2 - 7x_3 = -3\\ 2x_1 + x_2 - 4x_3 = -1\\ 5x_1 + 3x_2 - 11x_3 = 2\\ x_1 + x_2 - 4x_3 = 2 \end{cases}$$

$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix}$$

例3 解方程组: 
$$\begin{cases} 4x_1 + 2x_2 - 7x_3 = -3\\ 2x_1 + x_2 - 4x_3 = -1\\ 5x_1 + 3x_2 - 11x_3 = 2\\ x_1 + x_2 - 4x_3 = 2 \end{cases}$$

$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4}$$

例3 解方程组: 
$$\begin{cases} 4x_1 + 2x_2 - 7x_3 = -3\\ 2x_1 + x_2 - 4x_3 = -1\\ 5x_1 + 3x_2 - 11x_3 = 2\\ x_1 + x_2 - 4x_3 = 2 \end{cases}$$

$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & -3 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 1 & 1 & -4 & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & 2 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 4 & 2 & -7 & -3 \end{pmatrix}$$

例3 解方程组: 
$$\begin{cases} 4x_1 + 2x_2 - 7x_3 = -3\\ 2x_1 + x_2 - 4x_3 = -1\\ 5x_1 + 3x_2 - 11x_3 = 2\\ x_1 + x_2 - 4x_3 = 2 \end{cases}$$

$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 - 5r_1} \xrightarrow{r_4 - 4r_1}$$

例3 解方程组: 
$$\begin{cases} 4x_1 + 2x_2 - 7x_3 = -3\\ 2x_1 + x_2 - 4x_3 = -1\\ 5x_1 + 3x_2 - 11x_3 = 2\\ x_1 + x_2 - 4x_3 = 2 \end{cases}$$

$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 - 5r_1} \xrightarrow{r_4 - 4r_1}$$

$$\begin{pmatrix} 1 & 1-4 & 2 \\ & & \end{pmatrix}$$

例3 解方程组: 
$$\begin{cases} 4x_1 + 2x_2 - 7x_3 = -3\\ 2x_1 + x_2 - 4x_3 = -1\\ 5x_1 + 3x_2 - 11x_3 = 2\\ x_1 + x_2 - 4x_3 = 2 \end{cases}$$

$$(A : b) = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 - 5r_1} \xrightarrow{r_4 - 4r_1}$$

$$\begin{pmatrix} 1 & 1 & -4 \\ 0 & -1 & 4 \\ \end{pmatrix} \begin{pmatrix} -5 \\ \end{pmatrix}$$

例3 解方程组: 
$$\begin{cases} 4x_1 + 2x_2 - 7x_3 = -3\\ 2x_1 + x_2 - 4x_3 = -1\\ 5x_1 + 3x_2 - 11x_3 = 2\\ x_1 + x_2 - 4x_3 = 2 \end{cases}$$

$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 - 5r_1}$$

$$\begin{pmatrix}
1 & 1 - 4 \\
0 - 1 & 4 \\
0 - 2 & 9
\end{pmatrix} - 5$$

例3 解方程组: 
$$\begin{cases} 4x_1 + 2x_2 - 7x_3 = -3\\ 2x_1 + x_2 - 4x_3 = -1\\ 5x_1 + 3x_2 - 11x_3 = 2\\ x_1 + x_2 - 4x_3 = 2 \end{cases}$$

$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -1 & 1 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -1 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 - 5r_1} \xrightarrow{r_4 - 4r_1}$$

$$\begin{pmatrix}
1 & 1 - 4 & 2 \\
0 - 1 & 4 & -5 \\
0 - 2 & 9 & -8 \\
0 - 2 & 9 & -11
\end{pmatrix}$$

例3 解方程组: 
$$\begin{cases} 4x_1 + 2x_2 - 7x_3 = -3\\ 2x_1 + x_2 - 4x_3 = -1\\ 5x_1 + 3x_2 - 11x_3 = 2\\ x_1 + x_2 - 4x_3 = 2 \end{cases}$$

$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 - 5r_1}$$

$$\begin{pmatrix}
1 & 1 - 4 & 2 \\
0 - 1 & 4 & -5 \\
0 - 2 & 9 & -8 \\
0 - 2 & 9 - 11
\end{pmatrix}
\xrightarrow[r_4 - 2r_2]{r_3 - 2r_2}
\xrightarrow[r_4 - 2r_2]{r_4 - 2r_2}$$

例3 解方程组: 
$$\begin{cases} 4x_1 + 2x_2 - 7x_3 = -3\\ 2x_1 + x_2 - 4x_3 = -1\\ 5x_1 + 3x_2 - 11x_3 = 2\\ x_1 + x_2 - 4x_3 = 2 \end{cases}$$

$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 - 5r_1} \xrightarrow{r_4 - 4r_1}$$

$$\begin{pmatrix} 1 & 1-4 & 2 \\ 0-1 & 4 & -5 \\ 0-2 & 9 & -8 \\ 0-2 & 9 & -11 \end{pmatrix} \xrightarrow{r_3-2r_2} \begin{pmatrix} 1 & 1-4 & 2 \\ 0-1 & 4 & -5 \\ r_4-2r_2 \end{pmatrix}$$

例3 解方程组: 
$$\begin{cases} 4x_1 + 2x_2 - 7x_3 = -3\\ 2x_1 + x_2 - 4x_3 = -1\\ 5x_1 + 3x_2 - 11x_3 = 2\\ x_1 + x_2 - 4x_3 = 2 \end{cases}$$

$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 - 5r_1} \xrightarrow{r_4 - 4r_1}$$

$$\begin{pmatrix} 1 & 1-4 & 2 \\ 0-1 & 4 & -5 \\ 0-2 & 9 & -8 \\ 0-2 & 9 & -11 \end{pmatrix} \xrightarrow{r_3-2r_2} \begin{pmatrix} 1 & 1-4 & 2 \\ 0-1 & 4 & -5 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

例3 解方程组: 
$$\begin{cases} 4x_1 + 2x_2 - 7x_3 = -3\\ 2x_1 + x_2 - 4x_3 = -1\\ 5x_1 + 3x_2 - 11x_3 = 2\\ x_1 + x_2 - 4x_3 = 2 \end{cases}$$

$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 - 5r_1} \xrightarrow{r_4 - 4r_1}$$

$$\begin{pmatrix} 1 & 1-4 & 2 \\ 0-1 & 4 & -5 \\ 0-2 & 9 & -8 \\ 0-2 & 9 & -11 \end{pmatrix} \xrightarrow{r_3-2r_2} \begin{pmatrix} 1 & 1-4 & 2 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

例3 解方程组: 
$$\begin{cases} 4x_1 + 2x_2 - 7x_3 = -3\\ 2x_1 + x_2 - 4x_3 = -1\\ 5x_1 + 3x_2 - 11x_3 = 2\\ x_1 + x_2 - 4x_3 = 2 \end{cases}$$

$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 - 5r_1} \xrightarrow{r_4 - 4r_1}$$

$$\begin{pmatrix} 1 & 1 - 4 \\ 0 & -1 & 4 \\ 0 & -2 & 9 \\ 0 & -2 & 9 \\ -11 \end{pmatrix} \xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 & 1 - 4 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 \end{pmatrix} \xrightarrow{r_4 - r_3}$$

例3 解方程组: 
$$\begin{cases} 4x_1 + 2x_2 - 7x_3 = -3\\ 2x_1 + x_2 - 4x_3 = -1\\ 5x_1 + 3x_2 - 11x_3 = 2\\ x_1 + x_2 - 4x_3 = 2 \end{cases}$$

$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & -3 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 1 & 1 & -4 & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & 2 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 4 & 2 & -7 & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} r_2 - 2r_1 & r_3 - 5r_1 & r_4 - 4r_1 & r_4 &$$

例3 解方程组: 
$$\begin{cases} 4x_1 + 2x_2 - 7x_3 = -3\\ 2x_1 + x_2 - 4x_3 = -1\\ 5x_1 + 3x_2 - 11x_3 = 2\\ x_1 + x_2 - 4x_3 = 2 \end{cases}$$

$$\begin{array}{l}
(A : b) = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \xrightarrow{r_3 - 5r_1} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 0 & -1 & 4 & | & -5 \\ 0 & 0 & 1 & | & -1 \end{pmatrix} \xrightarrow{r_4 - r_3} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 0 & 0 & 1 & | & 2 \\ 0 & 0 & 0 & | & -3 \end{pmatrix}$$

所以原方程组等价干

例3 解方程组: 
$$\begin{cases} 4x_1 + 2x_2 - 7x_3 = -3\\ 2x_1 + x_2 - 4x_3 = -1\\ 5x_1 + 3x_2 - 11x_3 = 2\\ x_1 + x_2 - 4x_3 = 2 \end{cases}$$

$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \frac{r_2 - 2r_1}{r_3 - 5r_1}$$

$$\begin{pmatrix} 1 & 1 - 4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & -2 & 9 & -8 \\ 0 & -2 & 9 & -11 \end{pmatrix} \xrightarrow[r_4 - 2r_2]{r_3 - 2r_2} \begin{pmatrix} 1 & 1 - 4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{pmatrix} \xrightarrow[r_4 - r_3]{r_4 - r_3} \begin{pmatrix} 1 & 1 - 4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

所以原方程组等价于

$$\begin{cases} x_1 + & x_2 - 4x_3 = 2 \\ -x_2 + 4x_3 = -5 \\ & x_3 = 2 \\ & 0 = -3 \end{cases}$$

例3 解方程组: 
$$\begin{cases} 4x_1 + 2x_2 - 7x_3 = -3\\ 2x_1 + x_2 - 4x_3 = -1\\ 5x_1 + 3x_2 - 11x_3 = 2\\ x_1 + x_2 - 4x_3 = 2 \end{cases}$$

$$(A : b) = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 - 5r_1}$$

$$\begin{pmatrix} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & -2 & 9 & -11 \end{pmatrix} \xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{pmatrix} \xrightarrow{r_4 - r_3} \begin{pmatrix} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

所以原方程组等价于

$$\begin{cases} x_1 + & x_2 - 4x_3 = 2 \\ -x_2 + 4x_3 = -5 \\ & x_3 = 2 \\ & 0 = -3 \end{cases} \Rightarrow \mathcal{E}_{\mathbf{M}}^{\mathbf{M}}!$$

例3 解方程组: 
$$\begin{cases} 4x_1 + 2x_2 - 7x_3 = -3\\ 2x_1 + x_2 - 4x_3 = -1\\ 5x_1 + 3x_2 - 11x_3 = 2\\ x_1 + x_2 - 4x_3 = 2 \end{cases}$$

$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & -3 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 1 & 1 & -4 & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & 2 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 4 & 2 & -7 & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 1 & -4 & 2 \\ 5 & 3 & -11 & 2 \\ 4 & 2 & -7 & -3 \end{pmatrix} \xrightarrow{r_3 - 5r_1} \begin{pmatrix} 1 & 1 & -4 & 2 \\ r_3 - 5r_1 & r_4 - 4r_1 & r_3 - 5r_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 & -7 & -3 \end{pmatrix} \xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 & -7 & -7 & -7 \end{pmatrix} \xrightarrow{r_4 - r_3} \begin{pmatrix} 1 & 1 & -4 & -7 & -7 & -7 \\ 0 & -1 & 4 & -7 & -7 & -7 & -7 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -4 & -7 & -7 & -7 \\ 0 & -1 & 4 & -7 & -7 & -7 & -7 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -4 & -7 & -7 & -7 \\ 0 & -1 & 4 & -7 & -7 & -7 & -7 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -4 & -7 & -7 & -7 \\ 0 & -1 & 4 & -7 & -7 & -7 & -7 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -4 & -7 & -7 & -7 \\ 0 & -1 & 4 & -7 & -7 & -7 & -7 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -4 & -7 & -7 & -7 \\ 0 & -1 & 4 & -7 & -7 & -7 & -7 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -4 & -7 & -7 & -7 \\ 0 & -1 & 4 & -7 & -7 & -7 & -7 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -4 & -7 & -7 & -7 \\ 0 & -1 & 4 & -7 & -7 & -7 & -7 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -4 & -7 & -7 & -7 \\ 0 & -1 & 4 & -7 & -7 & -7 & -7 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -4 & -7 & -7 & -7 \\ 0 & -1 & 4 & -7 & -7 & -7 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -4 & -7 & -7 & -7 \\ 0 & -1 & 4 & -7 & -7 & -7 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -7 & -7 & -7 & -7 \\ 0 & -1 & 4 & -7 & -7 & -7 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -7 & -7 & -7 & -7 \\ 0 & -1 & 4 & -7 & -7 & -7 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -7 & -7 & -7 & -7 \\ 0 & -1 & 4 & -7 & -7 & -7 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -7 & -7 & -7 & -7 \\ 0 & -1 & 4 & -7 & -7 & -7 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -7 & -7 & -7 & -7 \\ 0 & -1 & 4 & -7 & -7 & -7 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -7 & -7 & -7 & -7 \\ 0 & -1 & 4 & -7 & -7 & -7 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -7 & -7 & -7 & -7 \\ 0 & -1 & 4 & -7 & -7 & -7 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -7 & -7 & -7 & -7 \\ 0 & -1 & 4 & -7 & -7 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -7 & -7 & -7 & -7 \\ 0 & -1 & 4 & -7 & -7 & -7 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -7 & -7 & -7 \\ 0 & -1 & 4 & -7 & -7 & -7 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -7 & -7 & -7 & -7 \\ 0 & -1 & 4 & -7 & -7 & -7 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -7 & -7 & -7 & -7 \\ 0 & -1 & 4 & -7 & -7 & -7 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -7 & -7 & -7 & -7 \\ 0 & -1 & 4 & -7 & -7 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -7 & -7 & -7 & -7 \\ 0 & -1 & 4 & -7 & -7 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -7 & -7 & -7 & -7 \\ 0 & -1 & 4 & -7 & -7 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -7 & -7 & -7 & -7$$

$$\begin{pmatrix} 1 & 1 - 4 & 2 \\ 0 - 1 & 4 & -5 \\ 0 - 2 & 9 & -11 \end{pmatrix} \xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 & 1 - 4 & 2 \\ 0 - 1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{pmatrix} \xrightarrow{r_4 - r_3} \begin{pmatrix} 1 & 1 - 4 & 2 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

所以原方程组等价于

$$\begin{cases} x_1 + & x_2 - 4x_3 = 2 \\ -x_2 + 4x_3 = -5 \\ & x_3 = 2 \\ & 0 = -3 \end{cases} \Rightarrow \mathcal{E}_{\mathbf{M}}^{\mathbf{M}}!$$

$$\begin{cases} x_{1} + & x_{2} - & x_{3} = 2 \\ x_{1} + & 2x_{2} & = -1 \\ 2x_{1} + & 5x_{2} + & x_{3} = -5 \\ -2x_{1} - & 3x_{2} + & x_{3} = -1 \end{cases} \qquad \begin{cases} x_{1} + & 2x_{2} + & 4x_{3} = & 28 \\ -2x_{1} - & 3x_{2} - & 9x_{3} = & -53 \\ 3x_{1} + & 6x_{2} + & 13x_{3} = & 88 \\ 5x_{1} + & 9x_{2} + & 22x_{3} = & 141 \end{cases} \qquad \begin{cases} 4x_{1} + & 2x_{2} - & 7x_{3} = & -3 \\ 2x_{1} + & x_{2} - & 4x_{3} = & -1 \\ 5x_{1} + & 3x_{2} - & 11x_{3} = & 2 \\ x_{1} + & x_{2} - & 4x_{3} = & 2 \end{cases}$$

$$\begin{cases} x_{1} + & x_{2} - & x_{3} = 2 \\ x_{1} + & 2x_{2} & = -1 \\ 2x_{1} + & 5x_{2} + & x_{3} = -5 \\ -2x_{1} - & 3x_{2} + & x_{3} = -1 \end{cases} \qquad \begin{cases} x_{1} + & 2x_{2} + & 4x_{3} = & 28 \\ -2x_{1} - & 3x_{2} - & 9x_{3} = & -53 \\ 3x_{1} + & 6x_{2} + & 13x_{3} = & 88 \\ 5x_{1} + & 9x_{2} + & 22x_{3} = & 141 \end{cases} \qquad \begin{cases} 4x_{1} + & 2x_{2} - & 7x_{3} = & -3 \\ 2x_{1} + & x_{2} - & 4x_{3} = & -1 \\ 5x_{1} + & 3x_{2} - & 11x_{3} = & 2 \\ x_{1} + & x_{2} - & 4x_{3} = & 2 \end{cases}$$

$$\downarrow \qquad \qquad \downarrow \qquad$$

r(A) r(A:b)

r(A) r(A:b)

 $r(A) \neq r(A : b)$ 

r(A) = r(A : b)

r(A) = r(A : b)

 $r(A) \neq r(A : b)$ 

r(A) = r(A : b)

r(A) = r(A : b)

r(A) < r(A : b)

r(A) = r(A : b)

r(A) = r(A : b) < n

r(A) < r(A : b)

r(A) = r(A : b) < n r(A) = r(A : b) = n

r(A) < r(A : b)

总结  
定理 方程组 
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

总结  
定理 方程组 
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

解有如下情形:

$$1. r(A : b) = r(A)$$

 $r(A) \neq r(A : b)$ 

总结  
定理 方程组 
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

1. 
$$r(A : b) = r(A)$$

2. 
$$r(A) \neq r(A : b) \Leftrightarrow r(A) < r(A : b)$$

总结  
定理 方程组 
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

1. 
$$r(A : b) = r(A)$$
$$r(A) = r(A : b) < n$$

$$r(A) = r(A : b) = n$$

2. 
$$r(A) \neq r(A : b) \Leftrightarrow r(A) < r(A : b)$$

总结  
定理 方程组 
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

1. 
$$r(A : b) = r(A)$$

$$r(A) = r(A : b) < n$$

• 只有唯一解 
$$\Leftrightarrow r(A) = r(A : b) = n$$

2. 
$$r(A) \neq r(A : b) \Leftrightarrow r(A) < r(A : b)$$

总结  
定理 方程组 
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

1. 
$$r(A : b) = r(A)$$

- 有无穷多解  $\Leftrightarrow r(A) = r(A : b) < n$
- 只有唯一解  $\Leftrightarrow$  r(A) = r(A : b) = n
- $r(A) \neq r(A : b) \Leftrightarrow r(A) < r(A : b)$

总结  
定理 方程组 
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

- 1. 有解  $\Leftrightarrow r(A:b) = r(A)$ 
  - 有无穷多解  $\Leftrightarrow r(A) = r(A : b) < n$
  - 只有唯一解  $\Leftrightarrow r(A) = r(A : b) = n$
- $r(A) \neq r(A : b) \Leftrightarrow r(A) < r(A : b)$

总结  
定理 方程组 
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_n$$

### 解有如下情形:

- 1. 有解  $\Leftrightarrow r(A:b) = r(A)$ 
  - 有无穷多解  $\Leftrightarrow r(A) = r(A : b) < n$
  - 只有唯一解  $\Leftrightarrow r(A) = r(A : b) = n$
- 2. 无解  $\Leftrightarrow$   $r(A) \neq r(A : b) \Leftrightarrow r(A) < r(A : b)$

⇔ Ax = b 的

总结  
定理 方程组 
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

解有如下情形:

- 1. 有解  $\Leftrightarrow r(A:b) = r(A)$ 
  - 有无穷多解  $\Leftrightarrow r(A) = r(A : b) < n$
  - 只有唯一解  $\Leftrightarrow r(A) = r(A : b) = n$
- $\Xi$ 解  $\Leftrightarrow$   $r(A) \neq r(A : b) \Leftrightarrow r(A) < r(A : b)$

### 注

• r(A:b) = r(A) 的值,相当于方程组中"独立"方程个数;此时

消元法 9/17 ⊲ ⊳ ∆ ⊽

总结 定理 方程组  $\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$ 

⇔ Ax = b 的

解有如下情形:

- 1. 有解  $\Leftrightarrow r(A:b) = r(A)$ 
  - 有无穷多解  $\Leftrightarrow r(A) = r(A : b) < n$
  - 只有唯一解  $\Leftrightarrow r(A) = r(A : b) = n$
- $\Xi$ 解  $\Leftrightarrow$   $r(A) \neq r(A : b) \Leftrightarrow r(A) < r(A : b)$

### 注

- r(A:b) = r(A) 的值,相当于方程组中"独立"方程个数;此时
- n − r(A) 为自由变量的个数

练习 1 求解  $\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1 \\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3 \\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7 \\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$ 

练习1 求解 
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1\\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3\\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7\\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$

练习1 求解 
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1\\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3\\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7\\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 + r_1}$$

练习 1 求解 
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1 \\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3 \\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7 \\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$

练习 1 求解 
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1 \\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3 \\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7 \\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$

$$\frac{r_3-2r_2}{r_4-2r_2}$$

练习 1 求解 
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1\\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3\\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7\\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 6 & 0 & 6 \\ 0 & 0 & 0 & 2 & 7 & 9 & 7 \end{pmatrix}$$

$$\frac{r_3 - 2r_2}{r_4 - 2r_2} \left( \begin{array}{ccccc} 1 & 2 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 6 & 0 & 6 \\ 0 & 0 & 0 & 7 & 0 & 7 \end{array} \right)$$

练习 1 求解 
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1\\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3\\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7\\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 6 & 0 & 6 \\ 0 & 0 & 0 & 6 & 0 & 6 \end{pmatrix} \xrightarrow{\frac{1}{6} \times r_3} \xrightarrow{\frac{1}{7} \times r_4}$$

消元法

10/17 ⊲ ⊳ ∆ ⊽

练习1 求解 
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1\\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3\\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7\\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$

$$r_{3-2r_2} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{pmatrix} \xrightarrow{\frac{1}{6} \times r_3} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{pmatrix}$$

练习1 求解 
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1\\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3\\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7\\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 6 & 0 & 6 \\ 0 & 0 & 0 & 7 & 0 & 7 \end{pmatrix} \xrightarrow{\frac{1}{7} \times r_4} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$$r_4-r_3$$

消元法

练习1 求解 
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1\\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3\\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7\\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 6 & 0 & 6 \\ 0 & 0 & 0 & 7 & 0 & 7 \end{pmatrix} \xrightarrow{\frac{1}{6} \times r_3} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{r_4 - r_3} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

消元法

练习1 求解 
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1\\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3\\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7\\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$
 
$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 6 & 0 & 6 \\ 0 & 0 & 0 & 7 & 0 & 7 \end{pmatrix} \xrightarrow{\frac{1}{6} \times r_3} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$
 
$$\xrightarrow{r_4 - r_3} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \xrightarrow{r_2 + r_3}$$
 
$$\xrightarrow{r_1 - r_3}$$

消元法 10/17 < ▶ △ ▽

练习 1 求解 
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1 \\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3 \\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7 \\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 6 & 0 & 6 \\ 0 & 0 & 0 & 7 & 0 & 7 \end{pmatrix} \xrightarrow{\frac{1}{5} \times r_3} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{r_4 - r_3} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \xrightarrow{r_2 + r_3} \begin{pmatrix} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

消元法 10/17 < ▶ △ ▽

练习1 求解 
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1\\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3\\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7\\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 6 & 0 & 6 \\ 0 & 0 & 0 & 7 & 0 & 7 \end{pmatrix} \xrightarrow{\frac{1}{6} \times r_3} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{r_4 - r_3} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{r_2 + r_3} \begin{pmatrix} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$r_1-r_2$$

消元法

练习 1 求解 
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1\\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3\\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7\\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 6 & 0 & 6 \\ 0 & 0 & 0 & 7 & 0 & 7 \end{pmatrix} \xrightarrow{\frac{1}{7} \times r_4} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{r_4 - r_3} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{r_2 + r_3} \begin{pmatrix} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{r_1 - r_2} \begin{pmatrix} 1 & 2 & 0 & 0 & 2 & | -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & | 0 \end{pmatrix}$$

消元法

练习1 求解 
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1\\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3\\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7\\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 6 & 0 & 6 \\ 0 & 0 & 0 & 7 & 0 & 7 \end{pmatrix} \xrightarrow{\frac{1}{7} \times r_4} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{r_4 - r_3} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{r_2 + r_3} \begin{pmatrix} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{r_1 - r_2} \begin{pmatrix} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

练习 2 求解 
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1\\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3\\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7\\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

练习 2 求解 
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1\\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3\\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7\\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

● 可见 r(A) = r(A : b) = 3 < 5,有无穷多的解,

练习 2 求解 
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1\\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3\\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7\\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

● 可见 r(A) = r(A : b) = 3 < 5,有无穷多的解,含 5 — 3 = 2 个自由变量

练习 2 求解 
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1 \\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3 \\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7 \\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- 可见 r(A) = r(A:b) = 3 < 5, 有无穷多的解,含5-3=2个自由变量
- 由既约阶梯形矩阵可知,原方程组等价于

$$\begin{cases} x_1 + 2x_2 & + 2x_5 = -2 \\ x_3 & - x_5 = 2 \\ x_4 & = 1 \end{cases}$$

练习 2 求解 
$$\begin{cases} x_1 + \ 2x_2 + \ x_3 + \ x_4 + \ x_5 = \ 1 \\ 2x_1 + \ 4x_2 + \ 3x_3 + \ x_4 + \ x_5 = \ 3 \\ -x_1 - \ 2x_2 + \ x_3 + \ 3x_4 - \ 3x_5 = \ 7 \\ 2x_3 + \ 5x_4 - \ 2x_5 = \ 9 \end{cases}$$

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- 可见 r(A) = r(A : b) = 3 < 5,有无穷多的解,含 5 3 = 2 个自由变量
- 由既约阶梯形矩阵可知,原方程组等价于

$$\begin{cases} x_1 + 2x_2 & + 2x_5 = -2 \\ x_3 & - x_5 = 2 \\ x_4 & = 1 \end{cases} \Rightarrow \begin{cases} x_1 = -2 - 2x_2 - 2x_5 \\ x_3 = 2 + x_5 \\ x_4 = 1 \end{cases}$$

练习 2 求解 
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1\\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3\\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7\\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- 可见 r(A) = r(A:b) = 3 < 5,有无穷多的解,含 5 3 = 2 个自由变量
- 由既约阶梯形矩阵可知,原方程组等价于

$$\begin{cases} x_1 + 2x_2 & + 2x_5 = -2 \\ x_3 & - x_5 = 2 \\ x_4 & = 1 \end{cases} \Rightarrow \begin{cases} x_1 = -2 - 2x_2 - 2x_5 \\ x_3 = 2 + x_5 \\ x_4 = 1 \end{cases}$$

所以通解是

$$\begin{cases} x_1 = \\ x_2 = \\ x_3 = \\ x_4 = \\ x_5 = \end{cases}$$

练习 2 求解 
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1 \\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3 \\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7 \\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- 可见 r(A) = r(A : b) = 3 < 5,有无穷多的解,含 5 3 = 2 个自由变量
- 由既约阶梯形矩阵可知,原方程组等价于

$$\begin{cases} x_1 + 2x_2 & + 2x_5 = -2 \\ x_3 & - x_5 = 2 \\ x_4 & = 1 \end{cases} \Rightarrow \begin{cases} x_1 = -2 - 2x_2 - 2x_5 \\ x_3 = 2 + x_5 \\ x_4 = 1 \end{cases}$$

$$\begin{cases} x_1 = \\ x_2 = c_1 \\ x_3 = \\ x_4 = \\ x_5 = c_2 \end{cases}$$
  $(c_1, c_2$ 为任意常数)

练习 2 求解 
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1 \\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3 \\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7 \\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- 可见 r(A) = r(A : b) = 3 < 5,有无穷多的解,含 5 − 3 = 2 个自由变量
- 由既约阶梯形矩阵可知,原方程组等价于

$$\begin{cases} x_1 + 2x_2 & + 2x_5 = -2 \\ x_3 & - x_5 = 2 \\ x_4 & = 1 \end{cases} \Rightarrow \begin{cases} x_1 = -2 - 2x_2 - 2x_5 \\ x_3 = 2 + x_5 \\ x_4 = 1 \end{cases}$$

定 
$$\begin{cases} x_1 = -2 - 2c_1 - 2c_2 \\ x_2 = c_1 \\ x_3 = \\ x_4 = \\ x_5 = c_2 \end{cases} (c_1, c_2 为任意常数)$$

练习 2 求解 
$$\begin{cases} x_1 + \ 2x_2 + \ x_3 + \ x_4 + \ x_5 = \ 1 \\ 2x_1 + \ 4x_2 + \ 3x_3 + \ x_4 + \ x_5 = \ 3 \\ -x_1 - \ 2x_2 + \ x_3 + \ 3x_4 - \ 3x_5 = \ 7 \\ 2x_3 + \ 5x_4 - \ 2x_5 = \ 9 \end{cases}$$

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- 可见 r(A) = r(A : b) = 3 < 5,有无穷多的解,含 5 − 3 = 2 个自由变量
- 由既约阶梯形矩阵可知,原方程组等价于

$$\begin{cases} x_1 + 2x_2 & + 2x_5 = -2 \\ x_3 & - x_5 = 2 \\ x_4 & = 1 \end{cases} \Rightarrow \begin{cases} x_1 = -2 - 2x_2 - 2x_5 \\ x_3 = 2 + x_5 \\ x_4 = 1 \end{cases}$$

定 
$$\begin{cases} X_1 = -2 - 2c_1 - 2c_2 \\ X_2 = c_1 \\ X_3 = 2 + c_2 \\ X_4 = \\ X_5 = c_2 \end{cases}$$
  $(c_1, c_2$ 为任意常数)

练习 2 求解 
$$\begin{cases} x_1 + \ 2x_2 + \ x_3 + \ x_4 + \ x_5 = \ 1 \\ 2x_1 + \ 4x_2 + \ 3x_3 + \ x_4 + \ x_5 = \ 3 \\ -x_1 - \ 2x_2 + \ x_3 + \ 3x_4 - \ 3x_5 = \ 7 \\ 2x_3 + \ 5x_4 - \ 2x_5 = \ 9 \end{cases}$$

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- 可见 r(A) = r(A : b) = 3 < 5,有无穷多的解,含 5 3 = 2 个自由变量
- 由既约阶梯形矩阵可知,原方程组等价于

$$\begin{cases} x_1 + 2x_2 & + 2x_5 = -2 \\ x_3 & - x_5 = 2 \\ x_4 & = 1 \end{cases} \Rightarrow \begin{cases} x_1 = -2 - 2x_2 - 2x_5 \\ x_3 = 2 + x_5 \\ x_4 = 1 \end{cases}$$

定 
$$\begin{cases} X_1 = -2 - 2c_1 - 2c_2 \\ X_2 = c_1 \\ X_3 = 2 + c_2 \\ X_4 = 1 \\ X_5 = c_2 \end{cases}$$
  $(c_1, c_2$ 为任意常数)

$$\begin{cases} x_{1}+ & x_{2}+ & x_{3}+ & x_{4}=0\\ & x_{2}+ & 2x_{3}+ & 2x_{4}=1\\ & -x_{2}+ & (\alpha-3)x_{3}- & 2x_{4}=b\\ 3x_{1}+ & 2x_{2}+ & x_{3}+ & \alpha x_{4}=-1 \end{cases}$$
 有无穷解、唯一解,及无解?

$$\begin{cases} x_{1}+ & x_{2}+ & x_{3}+ & x_{4}=0\\ & x_{2}+ & 2x_{3}+ & 2x_{4}=1\\ & -x_{2}+ & (\alpha-3)x_{3}- & 2x_{4}=b\\ 3x_{1}+ & 2x_{2}+ & x_{3}+ & ax_{4}=-1 \end{cases}$$
 有无穷解、唯一解,及无解?

$$(A \vdots b) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & a - 2 & 2 & 1 \\ 0 & -1 & a - 3 & -2 & b \\ 3 & 2 & 1 & a - 1 \end{pmatrix}$$

$$\begin{cases} x_{1}+ & x_{2}+ & x_{3}+ & x_{4}=0\\ & x_{2}+ & 2x_{3}+ & 2x_{4}=1\\ & -x_{2}+ & (\alpha-3)x_{3}- & 2x_{4}=b\\ 3x_{1}+ & 2x_{2}+ & x_{3}+ & ax_{4}=-1 \end{cases}$$
 有无穷解、唯一解,及无解?

解

$$(A \vdots b) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & a - 1 & 2 & 2 & 1 \\ 0 & -1 & a - 1 & a & -1 \end{pmatrix} \xrightarrow{r_4 - 3r_1}$$

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 2x_4 = 1 \\ -x_2 + (a-3)x_3 - 2x_4 = b \\ 3x_1 + 2x_2 + x_3 + ax_4 = -1 \end{cases}$$
 有无穷解、唯一解,及无解?

解

$$(A \vdots b) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & a - 3 & -2 & b \\ 0 & -1 & a - 3 & -2 & b \\ 3 & 2 & a - 3 & -1 \end{pmatrix} \xrightarrow{r_4 - 3r_1} \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 2 & 1 \\ 0 & -1 & a - 3 & -2 & b \\ 0 & -1 & a - 3 & -2 & a - 3 & -1 \end{pmatrix}$$

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 2x_4 = 1 \\ -x_2 + (a-3)x_3 - 2x_4 = b \\ 3x_1 + 2x_2 + x_3 + ax_4 = -1 \end{cases}$$
 有无穷解、唯一解,及无解?

解

$$(A \vdots b) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & a - \frac{2}{3} & -\frac{2}{3} & | & 1 \\ 0 & -1 & a & | & -1 \end{pmatrix} \xrightarrow{r_4 - 3r_1} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & 2 \\ 0 & -1 & a - \frac{3}{3} & -2 & | & b \\ 0 & -1 & -2 & a - 3 & | & -1 \end{pmatrix}$$

$$r_3+r_2$$

$$\begin{cases} x_{1}+ & x_{2}+ & x_{3}+ & x_{4}=0\\ & x_{2}+ & 2x_{3}+ & 2x_{4}=1\\ & -x_{2}+ & (\alpha-3)x_{3}- & 2x_{4}=b\\ 3x_{1}+ & 2x_{2}+ & x_{3}+ & ax_{4}=-1 \end{cases}$$
 有无穷解、唯一解,及无解?

解

$$(A \vdots b) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & a - 3 & -2 & b \\ 3 & 2 & a - 3 & -2 & b \end{pmatrix} \xrightarrow{r_4 - 3r_1} \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & a - 3 & -2 & b \\ 0 & -1 & a - 3 & -2 & b \\ 0 & -1 & a - 3 & -2 & a - 3 & -1 \end{pmatrix}$$

$$\xrightarrow{r_3 + r_2} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & b \\ 0 & 0 & a - 1 & 0 & b + 1 \\ 0 & 0 & 0 & a - 1 & 0 \end{pmatrix} b + 1$$

 $M_2$  讨论 a, b 取何值时,方程组

$$\begin{cases} x_1 + & x_2 + & x_3 + & x_4 = 0 \\ & x_2 + & 2x_3 + & 2x_4 = 1 \\ & -x_2 + & (\alpha - 3)x_3 - & 2x_4 = b \end{cases}$$
  $f(x_1 + x_2 + x_3 + x_4 = 0)$   $f(x_2 + x_4 + x_4$ 

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & a - 1 & 0 & b + 1 \\ 0 & 0 & 0 & a - 1 \end{pmatrix}$$

$$\begin{cases} x_1 + & x_2 + & x_3 + & x_4 = 0 \\ & x_2 + & 2x_3 + & 2x_4 = 1 \\ & -x_2 + & (\alpha - 3)x_3 - & 2x_4 = b \end{cases}$$
  $fatar = 0$   $fatar =$ 

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & a - 1 & 2 & 2 & 1 \\ 0 & 0 & a - 1 & 0 & b + 1 \\ 0 & 0 & a - 1 & 0 \end{pmatrix}$$

- 当 a ≠ 1 时
- 当 a = 1 时

$$\begin{cases} x_{1}+&x_{2}+&x_{3}+&x_{4}=&0\\ &x_{2}+&2x_{3}+&2x_{4}=&1\\ &-x_{2}+&(\alpha-3)x_{3}-&2x_{4}=&b\\ 3x_{1}+&2x_{2}+&x_{3}+&\alpha x_{4}=&-1 \end{cases}$$
 有无穷解、唯一解,及无解?

$$(A : b) \longrightarrow \begin{pmatrix} \begin{vmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

- 当 a ≠ 1 时
- 当 a = 1 时

**例 2** 讨论  $\alpha$ , b 取何值时,方程组

$$\begin{cases} x_{1}+&x_{2}+&x_{3}+&x_{4}=&0\\ &x_{2}+&2x_{3}+&2x_{4}=&1\\ &-x_{2}+&(\alpha-3)x_{3}-&2x_{4}=&b\\ 3x_{1}+&2x_{2}+&x_{3}+&ax_{4}=&-1 \end{cases}$$
 有无穷解、唯一解,及无解?

解

$$(A \vdots b) \longrightarrow \begin{pmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & A - 1 & 0 \\ 0 & 0 & A - 1 \end{pmatrix} b + 1 \\ b & 0 & 0 & A - 1 \end{pmatrix}$$

- 当 α ≠ 1 时(b 为任意数), r(A) = r(A · b) = 4,
- 当 a = 1 时

**例 2** 讨论  $\alpha$ , b 取何值时,方程组

$$\begin{cases} x_{1}+&x_{2}+&x_{3}+&x_{4}=&0\\ &x_{2}+&2x_{3}+&2x_{4}=&1\\ &-x_{2}+&(\alpha-3)x_{3}-&2x_{4}=&b\\ 3x_{1}+&2x_{2}+&x_{3}+&ax_{4}=&-1 \end{cases}$$
 有无穷解、唯一解,及无解?

解

$$(A:b) \longrightarrow \begin{pmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix} b + 1 \\ b & 0 & 0 & 0 & 0 \end{pmatrix}$$

- 当 $\alpha \neq 1$ 时(b为任意数), r(A) = r(A : b) = 4, 有唯一解;
- 当 a = 1 时

**M 2** 讨论 <math> a, b 取何值时,方程组

$$\begin{cases} x_{1}+&x_{2}+&x_{3}+&x_{4}=&0\\ &x_{2}+&2x_{3}+&2x_{4}=&1\\ &-x_{2}+&(\alpha-3)x_{3}-&2x_{4}=&b\\ 3x_{1}+&2x_{2}+&x_{3}+&ax_{4}=&-1 \end{cases}$$
 有无穷解、唯一解,及无解?

解

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & a-1 & 0 & b+1 \\ 0 & 0 & a-1 & 0 \end{pmatrix}$$

- 当 α ≠ 1 时(b 为任意数), r(A) = r(A:b) = 4, 有唯一解;
- 当 a = 1 时

$$\begin{cases} x_1 + & x_2 + & x_3 + & x_4 = 0 \\ & x_2 + & 2x_3 + & 2x_4 = 1 \\ & -x_2 + & (\alpha - 3)x_3 - & 2x_4 = b \end{cases}$$
 有无穷解、唯一解,及无解? 
$$\begin{cases} 3x_1 + & 2x_2 + & x_3 + & ax_4 = -1 \end{cases}$$

解

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 2 & 1 \\ 0 & 0 & a - 1 & 0 & b + 1 \\ 0 & 0 & 0 & a - 1 & 0 \end{pmatrix}$$

- 当 $\alpha \neq 1$ 时(b为任意数), r(A) = r(A : b) = 4, 有唯一解;
- 当 a = 1 时

$$(A \vdots b) \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & b+1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} b + \frac{1}{0}$$

$$\begin{cases} x_1 + & x_2 + & x_3 + & x_4 = 0 \\ & x_2 + & 2x_3 + & 2x_4 = 1 \\ & -x_2 + & (\alpha - 3)x_3 - & 2x_4 = b \end{cases}$$
 有无穷解、唯一解,及无解? 
$$\begin{cases} 3x_1 + & 2x_2 + & x_3 + & ax_4 = -1 \end{cases}$$

$$(A \vdots b) \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 2 & 2 \\ 0 & 0 & a - 1 & 0 & a - 1 & b + 1 \\ 0 & 0 & a - 1 & a - 1 & b \end{pmatrix}$$

- 当 α ≠ 1 时(b 为任意数), r(A) = r(A:b) = 4, 有唯一解;
- 当 a = 1 时

$$(A:b) \longrightarrow \begin{pmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & b+1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$-a=1$$
.  $b=-1$  时

- 
$$a = 1, b \neq -1$$
 时

$$\begin{cases} x_1 + & x_2 + & x_3 + & x_4 = 0 \\ & x_2 + & 2x_3 + & 2x_4 = 1 \\ & -x_2 + & (\alpha - 3)x_3 - & 2x_4 = b \end{cases}$$
 有无穷解、唯一解,及无解? 
$$\begin{cases} 3x_1 + & 2x_2 + & x_3 + & ax_4 = -1 \end{cases}$$

$$(A \vdots b) \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 2 & 2 \\ 0 & 0 & a - 1 & 0 & a - 1 & b + 1 \\ 0 & 0 & a - 1 & a - 1 & b \end{pmatrix}$$

- 当 α ≠ 1 时(b 为任意数), r(A) = r(A:b) = 4, 有唯一解;
- 当 a = 1 时

$$(A:b) \longrightarrow \begin{pmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & b+1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$-a=1$$
.  $b=-1$  时

- 
$$a = 1, b \neq -1$$
 时

$$\begin{cases} x_{1}+&x_{2}+&x_{3}+&x_{4}=&0\\ &x_{2}+&2x_{3}+&2x_{4}=&1\\ &-x_{2}+&(\alpha-3)x_{3}-&2x_{4}=&b\\ 3x_{1}+&2x_{2}+&x_{3}+&ax_{4}=&-1 \end{cases}$$
 有无穷解、唯一解,及无解?

$$(A \vdots b) \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 2 & 2 \\ 0 & 0 & a - 1 & 0 & a - 1 \\ 0 & 0 & a - 1 & b + 1 \\ 0 & 0 & a - 1 & 0 \end{pmatrix}$$

- 当 α ≠ 1 时(b 为任意数), r(A) = r(A · b) = 4, 有唯一解;
- 当 a = 1 时

$$(A:b) \longrightarrow \begin{pmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & b+1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- 
$$a = 1$$
,  $b = -1$  时,  $r(A) = r(A : b) = 2 < 4$ ,

- 
$$a = 1, b \neq -1$$
 时

$$\begin{cases} x_1 + & x_2 + & x_3 + & x_4 = 0 \\ & x_2 + & 2x_3 + & 2x_4 = 1 \\ & -x_2 + & (\alpha - 3)x_3 - & 2x_4 = b \end{cases}$$
 有无穷解、唯一解,及无解? 
$$\begin{cases} 3x_1 + & 2x_2 + & x_3 + & ax_4 = -1 \end{cases}$$

$$(A \vdots b) \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 2 & 2 \\ 0 & 0 & a - 1 & 0 & a - 1 \\ 0 & 0 & a - 1 & b + 1 \\ 0 & 0 & a - 1 & 0 \end{pmatrix}$$

- 当 a ≠ 1 时(b 为任意数), r(A) = r(A:b) = 4, 有唯一解;
- 当 a = 1 时

$$(A:b) \longrightarrow \begin{pmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & b+1 \\ 0 & 0 & 0 & 0 & b+1 \end{pmatrix}$$

- 
$$a = 1$$
,  $b = -1$  时, $r(A) = r(A : b) = 2 < 4$ ,有无穷多解

- 
$$a = 1, b \neq -1$$
 时

$$\begin{cases} x_1 + & x_2 + & x_3 + & x_4 = 0 \\ & x_2 + & 2x_3 + & 2x_4 = 1 \\ & -x_2 + & (a-3)x_3 - & 2x_4 = b \end{cases}$$
 有无穷解、唯一解,及无解? 
$$\begin{cases} 3x_1 + & 2x_2 + & x_3 + & ax_4 = -1 \end{cases}$$

$$(A \vdots b) \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & a - 1 & 0 & 0 & a - 1 \\ 0 & 0 & a - 1 & a - 1 & b + 1 \\ 0 & 0 & a - 1 & a - 1 & 0 \end{pmatrix}$$

- 当 α ≠ 1 时(b 为任意数), r(A) = r(A:b) = 4, 有唯一解;
- 当 a = 1 时

$$(A \vdots b) \longrightarrow \begin{pmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & b+1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$-a = 1, b = -1$$
 时, $r(A) = r(A : b) = 2 < 4$ ,有无穷多解

- 
$$a = 1, b \neq -1$$
 时

$$\begin{cases} x_1 + & x_2 + & x_3 + & x_4 = 0 \\ & x_2 + & 2x_3 + & 2x_4 = 1 \\ & -x_2 + & (\alpha - 3)x_3 - & 2x_4 = b \end{cases}$$
 有无穷解、唯一解,及无解? 
$$\begin{cases} 3x_1 + & 2x_2 + & x_3 + & ax_4 = -1 \end{cases}$$

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & a - 1 & 2 & 2 \\ 0 & 0 & a - 1 & 0 & a - 1 \\ 0 & 0 & a - 1 & 0 \end{pmatrix} b + \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & a - 1 & 0 \end{pmatrix}$$

- 当 $\alpha \neq 1$ 时(b为任意数), r(A) = r(A:b) = 4, 有唯一解;
- 当 a = 1 时

$$(A \vdots b) \longrightarrow \begin{pmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & b+1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$-a = 1, b = -1$$
 时, $r(A) = r(A : b) = 2 < 4$ ,有无穷多解

- 
$$a = 1$$
,  $b \neq -1$  时,  $r(A) = 2 < 3 = r(A : b)$ ,

$$\begin{cases} x_1 + & x_2 + & x_3 + & x_4 = 0 \\ & x_2 + & 2x_3 + & 2x_4 = 1 \\ & -x_2 + & (\alpha - 3)x_3 - & 2x_4 = b \end{cases}$$
 有无穷解、唯一解,及无解? 
$$\begin{cases} 3x_1 + & 2x_2 + & x_3 + & ax_4 = -1 \end{cases}$$

- 当 α ≠ 1 时(b 为任意数), r(A) = r(A:b) = 4, 有唯一解;
- 当 a = 1 时

$$(A:b) \longrightarrow \begin{pmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & b+1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- 
$$a = 1$$
,  $b = -1$  时, $r(A) = r(A : b) = 2 < 4$ ,有无穷多解

- 
$$a = 1, b \neq -1$$
 时,  $r(A) = 2 < 3 = r(A : b)$ , 无解

例 3 讨论 a, b 取何值时,方程组  $\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \\ 3x_1 + 4x_2 + ax_3 = b \end{cases}$ 

穷解、唯一解,及无解?

例 3 讨论 
$$a$$
,  $b$  取何值时,方程组 
$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \\ 3x_1 + 4x_2 + ax_3 = b \end{cases}$$

$$(A \vdots b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \frac{1}{b}$$

例 3 讨论 
$$a$$
,  $b$  取何值时,方程组 
$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \\ 3x_1 + 4x_2 + ax_3 = b \end{cases}$$

$$(A \vdots b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & \alpha \end{pmatrix} - \frac{1}{b} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 - 3r_1}$$

例 3 讨论 
$$a$$
,  $b$  取何值时,方程组 
$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \\ 3x_1 + 4x_2 + ax_3 = b \end{cases}$$

解

$$(A:b) = \begin{pmatrix} 1 & 2 & 3 & | & 1 \\ 2 & 3 & 5 & | & -1 \\ 3 & 4 & a & | & -b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 3 & | & 1 \\ & & & & & | & & 1 \end{pmatrix}$$

例 3 讨论 
$$a$$
,  $b$  取何值时,方程组 
$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \end{cases}$$
有无 
$$3x_1 + 4x_2 + ax_3 = b$$

解

$$(A:b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$$

例 3 讨论 
$$a$$
,  $b$  取何值时,方程组 
$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \\ 3x_1 + 4x_2 + ax_3 = b \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -1 \\ 0 & -2 & a - 9 \end{pmatrix} \begin{pmatrix} 1 \\ b - 3 \end{pmatrix}$$

例 3 讨论 
$$a$$
,  $b$  取何值时,方程组 
$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \end{cases}$$
有无 
$$3x_1 + 4x_2 + ax_3 = b$$

解

$$(A:b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -1 \\ 0 & -2 & a - 9 \end{pmatrix} \begin{pmatrix} 1 \\ b - 3 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2}$$

消元法 14/17 < ▶ △ ▼

例 3 讨论 
$$a$$
,  $b$  取何值时,方程组 
$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \end{cases}$$
有无 
$$3x_1 + 4x_2 + ax_3 = b$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 \\ 0 & -1 \\ 0 & -2 & a - 9 \end{pmatrix} \begin{vmatrix} -1 \\ b - 3 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 & -1 \\ 0 & 0 & a - 7 \end{vmatrix} \begin{pmatrix} -1 \\ b + 3 \end{pmatrix}$$

例 3 讨论 
$$a$$
,  $b$  取何值时,方程组 
$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \\ 3x_1 + 4x_2 + ax_3 = b \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 \\ 0 \\ -1 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 3r_1} \begin{pmatrix} 1 \\ 0 \\ -2 \\ a - 9 \end{pmatrix} \begin{vmatrix} -1 \\ b - 3 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{r_3 - 2r_3} \begin{vmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{r_3 - 3r_3} \xrightarrow{r_3 - 3r_3} \begin{pmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- 当 α ≠ 7 时
- 当 a = 7 时

例 3 讨论 
$$a$$
,  $b$  取何值时,方程组 
$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \\ 3x_1 + 4x_2 + ax_3 = b \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \begin{pmatrix} 1 \\ b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \xrightarrow{r_3 - 3r_1} \begin{pmatrix} 1 \\ 0 \\ -2 \\ a - 9 \end{pmatrix} \begin{pmatrix} 3 \\ b - 3 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ a - 7 \end{pmatrix} \begin{pmatrix} 3 \\ b + 3 \end{pmatrix}$$

- 当 α ≠ 7 时
- 当 a = 7 时

例 3 讨论 
$$a$$
,  $b$  取何值时,方程组 
$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \\ 3x_1 + 4x_2 + ax_3 = b \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 \\ 0 & -1 \\ 0 & -2 & a - 9 \end{pmatrix} \begin{vmatrix} -1 \\ b - 3 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 & a - 7 \end{vmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \\ a - 7 \end{vmatrix} \begin{pmatrix} 1 \\ -3 \\ b + 3 \end{pmatrix}$$

- 当 α ≠ 7 时 (b 为任意数), r(A:b) = r(A) = 3,
- 当 a = 7 时

例 3 讨论 
$$a$$
,  $b$  取何值时,方程组 
$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \\ 3x_1 + 4x_2 + ax_3 = b \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 \\ 0 & -1 \\ 0 & -2 & a - 9 \end{pmatrix} \begin{vmatrix} -1 \\ b - 3 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 & a - 7 \end{vmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \\ a - 7 \end{vmatrix} \begin{pmatrix} 1 \\ -3 \\ b + 3 \end{pmatrix}$$

- 当 $\alpha \neq 7$ 时(b为任意数), r(A:b) = r(A) = 3, 有唯一解;
- 当α=7时

例 3 讨论 
$$a$$
,  $b$  取何值时,方程组 
$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \end{cases}$$
有无 
$$3x_1 + 4x_2 + ax_3 = b$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 \\ 0 \\ -1 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 3r_1} \begin{pmatrix} 1 \\ 0 \\ -2 \\ a - 9 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \\ b - 3 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 \\ 0 \\ -1 \\ -3 \\ b + 3 \end{pmatrix}$$

- 当 $\alpha \neq 7$ 时(b为任意数), r(A:b) = r(A) = 3, 有唯一解;
- 当α=7时

例 3 讨论 
$$a$$
,  $b$  取何值时,方程组 
$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \end{cases}$$
有无 
$$3x_1 + 4x_2 + ax_3 = b$$

解

$$(A:b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 \\ 0 \\ -1 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 3r_1} \begin{pmatrix} 1 \\ 0 \\ -2 \\ a - 9 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \\ b - 3 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 \\ 0 \\ -1 \\ -3 \\ b + 3 \end{pmatrix}$$

- 当 $\alpha \neq 7$ 时(b为任意数), r(A:b) = r(A) = 3, 有唯一解;
- 当 a = 7 时

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & b+3 \end{pmatrix}$$

消元法

例 3 讨论 
$$a$$
,  $b$  取何值时,方程组 
$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \\ 3x_1 + 4x_2 + ax_3 = b \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 \\ 0 & -1 \\ 0 & -2 & a - 9 \end{pmatrix} \begin{vmatrix} -3 \\ b - 3 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 & -1 \\ 0 & a - 7 \end{vmatrix} \begin{pmatrix} 3 \\ b + 3 \end{pmatrix}$$

- 当  $\alpha \neq 7$  时(b 为任意数),r(A : b) = r(A) = 3,有唯一解;
- 当 a = 7 时

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & b+3 \end{pmatrix}$$

- a = 7, b = -3 时
- $a = 7, b \neq -3$  时

例 3 讨论 
$$a$$
,  $b$  取何值时,方程组 
$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \\ 3x_1 + 4x_2 + ax_3 = b \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 3 & | & -1 \\ 2 & 3 & 5 & | & -1 \\ 3 & 4 & a & | & b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 3 & | & -1 \\ 0 & -1 & -1 & | & -3 \\ 0 & -2 & a - 9 & | & b - 3 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 & 2 & 3 & | & -1 \\ 0 & 0 & a - 7 & | & b + 3 \end{pmatrix}$$

- 当  $\alpha \neq 7$  时(b 为任意数),r(A : b) = r(A) = 3,有唯一解;
- 当 a = 7 时

$$(A:b) \longrightarrow \begin{pmatrix} \begin{vmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & b+3 \end{pmatrix}$$

$$-a = 7$$
.  $b = -3$  时

- 
$$a = 7$$
,  $b \neq -3$  时

例 3 讨论 
$$a$$
,  $b$  取何值时,方程组 
$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \end{cases}$$
有无 
$$3x_1 + 4x_2 + ax_3 = b$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 \\ 0 \\ -1 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 3r_1} \begin{pmatrix} 1 \\ 0 \\ -2 \\ a - 9 \end{pmatrix} \begin{pmatrix} -1 \\ b - 3 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_3} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_3} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_3} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_3} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_3} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_3} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_3} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_3} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_3} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_3} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_3} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_3} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_3} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_3} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_3} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_3} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_3} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_3} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_3} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_3} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_3} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_3} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_3} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_3} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_3} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_3} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_3} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_3} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_3} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_3} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 2r_3$$

- 当 α ≠ 7 时(b 为任意数), r(A:b) = r(A) = 3, 有唯一解;
- 当 a = 7 时

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & b+3 \end{pmatrix}$$

- 
$$a = 7$$
,  $b = -3$  时,  $r(A : b) = r(A) = 2 < 3$ ,

- 
$$a = 7, b \neq -3$$
 时

例 3 讨论 
$$a$$
,  $b$  取何值时,方程组 
$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \end{cases}$$
有无 
$$3x_1 + 4x_2 + ax_3 = b$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 \\ 0 \\ -1 \\ -2 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 0 -2 \end{pmatrix} - \begin{pmatrix} 1 \\ -3 \\ b -3 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 -2 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ b + 3 \end{pmatrix}$$

- 当 $\alpha \neq 7$ 时(b为任意数),r(A:b) = r(A) = 3,有唯一解;
- 当 a = 7 时

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & b+3 \end{pmatrix}$$

$$-a = 7$$
,  $b = -3$  时, $r(A : b) = r(A) = 2 < 3$ ,有无穷多解

- 
$$a = 7, b \neq -3$$
 时

例 3 讨论 
$$a$$
,  $b$  取何值时,方程组 
$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \\ 3x_1 + 4x_2 + ax_3 = b \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \frac{1}{b} \begin{pmatrix} \frac{r_2 - 2r_1}{r_3 - 3r_1} & \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -1 \\ 0 & -2 & a - 9 \end{pmatrix} \begin{vmatrix} -1 \\ b - 3 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -1 \\ 0 & a - 7 \end{vmatrix} \begin{pmatrix} 1 \\ b + 3 \end{pmatrix}$$

- 当  $\alpha \neq 7$  时(b 为任意数),r(A : b) = r(A) = 3,有唯一解;
- 当 a = 7 时

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & b+3 \end{pmatrix}$$

- a = 7, b = -3 时,r(A : b) = r(A) = 2 < 3,有无穷多解
- $a = 7, b \neq -3$  时

例 3 讨论 
$$a$$
,  $b$  取何值时,方程组 
$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \end{cases}$$
有无 
$$3x_1 + 4x_2 + ax_3 = b$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 \\ 0 \\ -1 \\ -2 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 0 -2 \end{pmatrix} \begin{pmatrix} -1 \\ -3 \\ b -3 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ b + 3 \end{pmatrix}$$

- 当 α ≠ 7 时(b 为任意数), r(A:b) = r(A) = 3, 有唯一解;
- 当 α = 7 时

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & b+3 \end{pmatrix}$$

$$-a = 7$$
,  $b = -3$  时, $r(A : b) = r(A) = 2 < 3$ ,有无穷多解

- 
$$a = 7, b \neq -3$$
 时

例 3 讨论 
$$a$$
,  $b$  取何值时,方程组 
$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \\ 3x_1 + 4x_2 + ax_3 = b \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 \\ 0 \\ -1 \\ -2 \\ a - 9 \end{pmatrix} \begin{pmatrix} 3 \\ b - 3 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \xrightarrow{r_3 - 2r_1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{r_3 - 3r_1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- 当 $\alpha \neq 7$ 时(b为任意数),r(A:b) = r(A) = 3,有唯一解;
- 当 a = 7 时

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & b+3 \end{pmatrix}$$

$$-a = 7$$
,  $b = -3$  时, $r(A : b) = r(A) = 2 < 3$ ,有无穷多解

- 
$$a = 7$$
,  $b \neq -3$  时,  $r(A : b) = 3 \neq 2 = r(A)$ ,

例 3 讨论 
$$a$$
,  $b$  取何值时,方程组 
$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \\ 3x_1 + 4x_2 + ax_3 = b \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 3 & | & -1 \\ 2 & 3 & 5 & | & -1 \\ 3 & 4 & a & | & b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 3 & | & -1 \\ 0 & -1 & -1 & | & -3 \\ 0 & -2 & a - 9 & | & b - 3 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 & 2 & 3 & | & 1 \\ 0 & -1 & -1 & | & -3 \\ 0 & 0 & a - 7 & | & b + 3 \end{pmatrix}$$

- 当  $\alpha \neq 7$  时(b 为任意数),r(A : b) = r(A) = 3,有唯一解;
- 当 a = 7 时

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & b+3 \end{pmatrix}$$

$$-a = 7$$
,  $b = -3$  时, $r(A : b) = r(A) = 2 < 3$ ,有无穷多解

- 
$$\alpha$$
 = 7,  $b$  ≠ -3 时,  $r(A:b)$  = 3 ≠ 2 =  $r(A)$ , 无解

• 一般线性方程组  $A_{m \times n} x = b$  (m 个方程, n 个未知量)

Ax = b	有无穷解	有唯一解	无解
	r(A) = r(A : b) < n	r(A) = r(A : b) = n	r(A) < r(A : b)

• 一般线性方程组  $A_{m \times n} x = b$  (m 个方程, n 个未知量)

Ax = b	有无穷解	有唯一解	无解
	r(A) = r(A : b) < n	$r(A) = r(A \vdots b) = n$	r(A) < r(A : b)

• 一般线性方程组  $A_{m \times n} x = b$  (m 个方程, n 个未知量)

Ax = b	有无穷解	有唯一解	无解
	r(A) = r(A : b) < n	r(A) = r(A : b) = n	r(A) < r(A : b)

Ax = 0	有无穷解	有唯一解(零解)

• 一般线性方程组  $A_{m \times n} x = b$  (m 个方程, n 个未知量)

Ax = b	有无穷解	有唯一解	无解
	r(A) = r(A : b) < n	r(A) = r(A : b) = n	r(A) < r(A : b)

Ax = 0	有无穷解	有唯一解(零解)
	r(A) < n	

• 一般线性方程组  $A_{m \times n} x = b$  (m 个方程, n 个未知量)

Ax = b	有无穷解	有唯一解	无解
	r(A) = r(A : b) < n	r(A) = r(A : b) = n	r(A) < r(A : b)

Ax = 0	有无穷解	有唯一解(零解)
	r(A) < n	r(A) = n

例 解齐次线性方程组  $\begin{cases} x_1-&x_2+&5x_3-&x_4&=0\\ x_1+&x_2-&2x_3+&3x_4&=0\\ 3x_1-&x_2+&8x_3+&x_4&=0\\ x_1+&3x_2-&9x_3+&7x_4&=0 \end{cases}$ 

$$(A:b) = \begin{pmatrix} 1 & -1 & 5 & -1 & 0 \\ 1 & 1 & -2 & 3 & 0 \\ 3 & -1 & 8 & 1 & 0 \\ 1 & 3 & -9 & 7 & 0 \end{pmatrix}$$

$$(A:b) = \begin{pmatrix} 1 & -1 & 5 & -1 & 0 \\ 1 & 1 & -2 & 3 & 0 \\ 3 & -1 & 8 & 1 & 0 \\ 1 & 3 & -9 & 7 & 0 \end{pmatrix} \xrightarrow[r_3 - 3r_1]{r_3 - 3r_1} \xrightarrow[r_4 - r_1]{r_4 - r_1}$$

$$(A:b) = \begin{pmatrix} 1 & -1 & 5 & -1 & 0 \\ 1 & 1 & -2 & 3 & 0 \\ 3 & -1 & 8 & 1 & 0 \\ 1 & 3 & -9 & 7 & 0 \end{pmatrix} \xrightarrow[r_3 - 3r_1]{r_3 - 3r_1} \begin{pmatrix} 1 & -1 & 5 & -1 & 0 \\ r_3 - 3r_1 & r_4 - r_1 & r_4 - r_1 & r_4 - r_1 \end{pmatrix}$$

例 解齐次线性方程组 
$$\begin{cases} x_1- & x_2+ 5x_3- x_4=0 \\ x_1+ & x_2- 2x_3+ 3x_4=0 \\ 3x_1- & x_2+ 8x_3+ x_4=0 \\ x_1+ 3x_2- 9x_3+ 7x_4=0 \end{cases}$$

$$(A \vdots b) = \begin{pmatrix} 1 & -1 & 5 & -1 & 0 \\ 1 & 1 & -2 & 3 & 0 \\ 3 & -1 & 8 & 1 & 0 \\ 1 & 3 & -9 & 7 & 0 \end{pmatrix} \xrightarrow[r_3 - 3r_1]{r_3 - 3r_1} \begin{pmatrix} 1 & -1 & 5 & -1 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 2 & -7 & 4 & 0 \end{pmatrix}$$

例 解齐次线性方程组 
$$\begin{cases} x_1- & x_2+ 5x_3- x_4=0\\ x_1+ & x_2- 2x_3+ 3x_4=0\\ 3x_1- & x_2+ 8x_3+ x_4=0\\ x_1+ 3x_2- 9x_3+ 7x_4=0 \end{cases}$$

$$(A \vdots b) = \begin{pmatrix} 1 & -1 & 5 & -1 & 0 \\ 1 & 1 & -2 & 3 & 0 \\ 3 & -1 & 8 & 1 & 0 \\ 1 & 3 & -9 & 7 & 0 \end{pmatrix} \xrightarrow[r_3 - 3r_1]{r_3 - 3r_1} \begin{pmatrix} 1 & -1 & 5 & -1 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 4 & -14 & 8 & 0 \end{pmatrix}$$

解

$$(A \vdots b) = \begin{pmatrix} 1 & -1 & 5 & -1 & 0 \\ 1 & 1 & -2 & 3 & 0 \\ 3 & -1 & 8 & 1 & 0 \\ 1 & 3 & -9 & 7 & 0 \end{pmatrix} \xrightarrow[r_3 - r_1]{r_3 - 3r_1} \begin{pmatrix} 1 & -1 & 5 & -1 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 4 & -14 & 8 & 0 \end{pmatrix}$$

$$r_3-r_2$$

消元法

例 解齐次线性方程组 
$$\begin{cases} x_1- & x_2+ 5x_3- x_4=0\\ x_1+ & x_2- 2x_3+ 3x_4=0\\ 3x_1- & x_2+ 8x_3+ x_4=0\\ x_1+ 3x_2- 9x_3+ 7x_4=0 \end{cases}$$

$$\frac{1}{2} \times r_2$$

例 解齐次线性方程组 
$$\begin{cases} x_1- & x_2+ 5x_3- x_4=0\\ x_1+ & x_2- 2x_3+ 3x_4=0\\ 3x_1- & x_2+ 8x_3+ x_4=0\\ x_1+ 3x_2- 9x_3+ 7x_4=0 \end{cases}$$

例 解齐次线性方程组 
$$\begin{cases} x_1- & x_2+ 5x_3- x_4=0\\ x_1+ & x_2- 2x_3+ 3x_4=0\\ 3x_1- & x_2+ 8x_3+ x_4=0\\ x_1+ & 3x_2- 9x_3+ 7x_4=0 \end{cases}$$

例 解齐次线性方程组 
$$\begin{cases} x_1- & x_2+ 5x_3- x_4=0\\ x_1+ & x_2- 2x_3+ 3x_4=0\\ 3x_1- & x_2+ 8x_3+ x_4=0\\ x_1+ 3x_2- 9x_3+ 7x_4=0 \end{cases}$$

解

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 0 & 3/2 & 1 & | & 0 \\ 0 & 1 & -7/2 & 2 & | & 0 \\ 0 & 0 & & 0 & 0 & | & 0 \\ 0 & 0 & & 0 & 0 & | & 0 \end{pmatrix}$$

所以原方程组等价于

例 解齐次线性方程组 
$$\begin{cases} x_1- & x_2+ 5x_3- x_4=0\\ x_1+ & x_2- 2x_3+ 3x_4=0\\ 3x_1- & x_2+ 8x_3+ x_4=0\\ x_1+ 3x_2- 9x_3+ 7x_4=0 \end{cases}$$

解

所以原方程组等价干

$$\begin{cases} x_1 + & \frac{3}{2}x_3 + x_4 = 0 \\ x_2 - \frac{7}{2}x_3 + 2x_4 = 0 \end{cases}$$

例 解齐次线性方程组 
$$\begin{cases} x_1- & x_2+ 5x_3- & x_4=0\\ x_1+ & x_2- 2x_3+ 3x_4=0\\ 3x_1- & x_2+ 8x_3+ x_4=0\\ x_1+ 3x_2- 9x_3+ 7x_4=0 \end{cases}$$

解

所以原方程组等价干

$$\begin{cases} x_1 + \frac{3}{2}x_3 + x_4 = 0 \\ x_2 - \frac{7}{2}x_3 + 2x_4 = 0 \end{cases} \iff \begin{cases} x_1 + \frac{3}{2}x_3 - x_4 \\ x_2 = \frac{7}{2}x_3 - 2x_4 \end{cases}$$

消元法

例 解齐次线性方程组  $\begin{cases} x_1 - x_2 + 5x_3 - x_4 &= 0 \\ x_1 + x_2 - 2x_3 + 3x_4 &= 0 \\ 3x_1 - x_2 + 8x_3 + x_4 &= 0 \\ x_1 + 3x_2 - 9x_3 + 7x_4 &= 0 \end{cases}$ 

解

$$\begin{cases} x_1 + \frac{3}{2}x_3 + x_4 = 0 \\ x_2 - \frac{7}{2}x_3 + 2x_4 = 0 \end{cases} \iff \begin{cases} x_1 + \frac{3}{2}x_3 - x_4 \\ x_2 = \frac{7}{2}x_3 - 2x_4 \end{cases}$$

所以  $\begin{cases} x_3 = c_1 \\ x_4 = c_2 \end{cases}$ 

例 解齐次线性方程组  $\begin{cases} x_1 - x_2 + 5x_3 - x_4 &= 0 \\ x_1 + x_2 - 2x_3 + 3x_4 &= 0 \\ 3x_1 - x_2 + 8x_3 + x_4 &= 0 \\ x_1 + 3x_2 - 9x_3 + 7x_4 &= 0 \end{cases}$ 

所以原方程组等价干

$$\begin{cases} x_{1} + \frac{3}{2}x_{3} + x_{4} = 0 \\ x_{2} - \frac{7}{2}x_{3} + 2x_{4} = 0 \end{cases} \iff \begin{cases} x_{1} + \frac{3}{2}x_{3} - x_{4} \\ x_{2} = \frac{7}{2}x_{3} - 2x_{4} \end{cases}$$

$$\begin{cases} x_{1} = -\frac{3}{2}c_{1} - c_{2} \\ x_{3} = c_{1} \\ x_{4} = c_{2} \end{cases}$$

例 解齐次线性方程组  $\begin{cases} x_1 - x_2 + 5x_3 - x_4 &= 0 \\ x_1 + x_2 - 2x_3 + 3x_4 &= 0 \\ 3x_1 - x_2 + 8x_3 + x_4 &= 0 \\ x_1 + 3x_2 - 9x_3 + 7x_4 &= 0 \end{cases}$ 

所以原方程组等价干

$$\begin{cases} x_1 + \frac{3}{2}x_3 + x_4 = 0 \\ x_2 - \frac{7}{2}x_3 + 2x_4 = 0 \end{cases} \iff \begin{cases} x_1 + \frac{3}{2}x_3 - x_4 \\ x_2 = \frac{7}{2}x_3 - 2x_4 \end{cases}$$

所以  $\begin{cases} x_1 = -\frac{3}{2}c_1 - c_2 \\ x_2 = \frac{7}{2}c_1 - 2c_2 \\ x_3 = c_1 \\ x_4 = c_2 \end{cases}$ 

例 解齐次线性方程组  $\begin{cases} x_1 - x_2 + 5x_3 - x_4 &= 0 \\ x_1 + x_2 - 2x_3 + 3x_4 &= 0 \\ 3x_1 - x_2 + 8x_3 + x_4 &= 0 \\ x_1 + 3x_2 - 9x_3 + 7x_4 &= 0 \end{cases}$ 解

所以原方程组等价干

$$\begin{cases} x_1 + \frac{3}{2}x_3 + x_4 = 0 \\ x_2 - \frac{7}{2}x_3 + 2x_4 = 0 \end{cases} \iff \begin{cases} x_1 + \frac{3}{2}x_3 - x_4 \\ x_2 = \frac{7}{2}x_3 - 2x_4 \end{cases}$$

所以  $\begin{cases} x_1 = -\frac{3}{2}c_1 - c_2 \\ x_2 = \frac{7}{2}c_1 - 2c_2 \\ x_3 = c_1 \\ x_4 = c_2 \end{cases}$ 

(注 自由变量个数 = 2 = 4 - r(A))

消元法 17/17 ⊲ ⊳ ∆ ⊽