第 9 章 d：隐函数的求导公式

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隐函数的求导法 I

公式 设 $y = y(x)$ 满足 $F(x, y) = 0$，即 $F(x, y(x)) = 0$，则

$$\frac{dy}{dx} = -\frac{F_x}{F_y} \qquad (F_y \neq 0)$$

证明

$$\because \quad F(x, y(x)) = 0$$

$$\therefore \quad 0 = \frac{d}{dx} F(x, y(x)) = F_x + F_y \cdot \frac{dy}{dx}$$

$$\therefore \quad \frac{dy}{dx} = -\frac{F_x}{F_y}$$

$$\begin{tikzcd}
 & x & \\
F \arrow[ur, "F_x"] \arrow[dr, "F_y"'] & & \\
 & y \arrow[r, "\frac{dy}{dx}"] & x
\end{tikzcd}$$

例 设 $y=f(x)$ 满足 $\sin y+e^x=xy^2$，求 $\frac{dy}{dx}$

方法一 注意 $\sin y+e^x-xy^2=0$，令 $F(x,y)=\sin y+e^x-xy^2$，则 $F(x,y)=0$，所以

$$\frac{dy}{dx}=-\frac{F_x}{F_y}=-\frac{(\sin y+e^x-xy^2)'_x}{(\sin y+e^x-xy^2)'_y}=-\frac{e^x-y^2}{\cos y-2xy}$$

方法二 注意 $\sin y(x)+e^x-xy(x)^2=0$，所以

$$\begin{aligned}0&=\left(\sin y(x)+e^x-xy(x)^2\right)'_x\\&=(\sin y(x))'_x+(e^x)'_x-(xy(x)^2)'_x\\&=\cos y\cdot y'+e^x-y^2-2xy\cdot y'\\&=e^x-y^2+(\cos y-2xy)y'\end{aligned}$$

所以 $y'=-\frac{e^x-y^2}{\cos y-2xy}$

例 设 $y=f(x)$ 满足 $\ln(x^2+y^2)+3xy=4$，求 $\frac{dy}{dx}$

解 注意 $\ln(x^2+y^2)+3xy-4=0$，令

$$F(x,y)=\ln(x^2+y^2)+3xy-4$$

则 $F(x,y)=0$，所以

$$\frac{dy}{dx}=-\frac{F_x}{F_y}=-\frac{(\ln(x^2+y^2)+3xy-4)'_x}{(\ln(x^2+y^2)+3xy-4)'_y}$$

$$=-\frac{\frac{2x}{x^2+y^2}+3y}{\frac{2y}{x^2+y^2}+3x}$$

$$=-\frac{2x+3x^2y+3y^3}{2y+3xy^2+3x^3}$$

隐函数的求导法 II

公式 设 $z = f(x, y)$ 满足 $F(x, y, z) = 0$，即 $F(x, y, z(x, y)) = 0$，则

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} \qquad (F_z \neq 0)$$

证明

$$\because \quad F(x, y, z(x, y)) = 0$$

$$\therefore \quad 0 = \frac{\partial}{\partial x} F(x, y, z(x, y)) = F_x + F_z \cdot \frac{\partial z}{\partial x}$$

$$\therefore \quad \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \qquad \text{同理} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

$$\begin{tikzcd}
 & x & \\
F \arrow[ur, "F_x"] \arrow[r, "F_y"] \arrow[dr, "F_z"'] & y & x \\
 & z \arrow[ur, "\frac{\partial z}{\partial x}"] \arrow[r, "\frac{\partial z}{\partial y}"'] & y
\end{tikzcd}$$

例 设 $z=f(x,y)$ 满足 $x+y+xz=e^z-1$，求 $\frac{\partial z}{\partial x}$ 和 $\frac{\partial z}{\partial y}$

解 令 $F(x,y,z)=x+y+xz-e^z+1$，则 $F(x,y,z)=0$，所以

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{(x+y+xz-e^z+1)'_x}{(x+y+xz-e^z+1)'_z}$$

$$=-\frac{1+0+z-0+0}{0+0+x-e^z+0}=-\frac{1+z}{x-e^z}$$

$$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-\frac{(x+y+xz-e^z+1)'_y}{(x+y+xz-e^z+1)'_z}$$

$$=-\frac{0+1+0-0+0}{0+0+x-e^z+0}=-\frac{1}{x-e^z}$$

例 设 $z = f(x, y)$ 满足 $2\sin(x+2y-3z) = x+2y-3z$，求 $\frac{\partial z}{\partial x}$ 和 $\frac{\partial z}{\partial y}$

解 令 $F(x, y, z) = 2\sin(x+2y-3z) - x - 2y + 3z$，则

$F(x, y, z) = 0$，所以

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{(2\sin(x+2y-3z)-x-2y+3z)'_x}{(2\sin(x+2y-3z)-x-2y+3z)'_z}$$

$$= -\frac{2\cos(x+2y-3z)-1}{-6\cos(x+2y-3z)+3} = \frac{1}{3}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{(2\sin(x+2y-3z)-x-2y+3z)'_y}{(2\sin(x+2y-3z)-x-2y+3z)'_z}$$

$$= -\frac{4\cos(x+2y-3z)-2}{-6\cos(x+2y-3z)+3}$$

例 设 $z=f(x,y)$ 满足 $z-y-x+xe^{z-y-x}=0$，求 dz

解 令 $F(x,y,z)=z-y-x+xe^{z-y-x}$，则 $F(x,y,z)=0$，所以

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{(z-y-x+xe^{z-y-x})'_x}{(z-y-x+xe^{z-y-x})'_z}$$

$$=-\frac{-1+e^{z-y-x}-xe^{z-y-x}}{1+xe^{z-y-x}}=\frac{1+(x-1)e^{z-y-x}}{1+xe^{z-y-x}}$$

$$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-\frac{(z-y-x+xe^{z-y-x})'_y}{(z-y-x+xe^{z-y-x})'_z}=-\frac{-1-xe^{z-y-x}}{1+xe^{z-y-x}}=1$$

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=\frac{1+(x-1)e^{z-y-x}}{1+xe^{z-y-x}}dx+dy$$

例 设 $\Phi(u,v)$ 具有连续偏导数，函数 $z=z(x,y)$ 满足 $\Phi(cx-az,cy-bz)=0$，证明：

$$a\frac{\partial z}{\partial x}+b\frac{\partial z}{\partial y}=c.$$

解 令 $F(x,y,z)=\Phi(cx-az,cy-bz)$，则

$$F_x=\Phi_u\cdot u_x+\Phi_v\cdot v_x=c\Phi_u$$

$$F_y=\Phi_u\cdot u_y+\Phi_v\cdot v_y=c\Phi_v$$

$$F_z=\Phi_u\cdot u_z+\Phi_v\cdot v_z=-a\Phi_u-b\Phi_v$$

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=\frac{c\Phi_u}{a\Phi_u+b\Phi_v}$$

$$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=\frac{c\Phi_v}{a\Phi_u+b\Phi_v}$$

$$a\frac{\partial z}{\partial x}+b\frac{\partial z}{\partial y}=\frac{ac\Phi_u}{a\Phi_u+b\Phi_v}+\frac{bc\Phi_v}{a\Phi_u+b\Phi_v}=c$$

例 设 $z=f(x,y)$ 满足 $z=x+ye^z$，求 $\frac{\partial^2 z}{\partial x\partial y}$

解 $F(x,y,z)=x+ye^z-z$，则 $F(x,y,z)=0$ ，所以

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{(x+ye^z-z)_x}{(x+ye^z-z)_z}=-\frac{1}{ye^z-1}$$

$$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-\frac{(x+ye^z-z)_y}{(x+ye^z-z)_z}=-\frac{e^z}{ye^z-1}$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\left(-\frac{1}{ye^z-1}\right)'_y=\frac{(ye^z-1)'_y}{(ye^z-1)^2}$$

$$=\frac{e^z+y(e^z)'_y}{(ye^z-1)^2}=\frac{e^z+ye^z\cdot\frac{\partial z}{\partial y}}{(ye^z-1)^2}=\frac{e^z+ye^z\cdot\left(-\frac{e^z}{ye^z-1}\right)}{(ye^z-1)^2}$$

$$=\frac{-e^z}{(ye^z-1)^3}=\frac{e^z}{(1+x-z)^3}$$

回顾：二元线性方程组的求解

二元线性方程组

$$\begin{cases} a_{11}\,a_{22}\,a_{21}x + a_{12}\,a_{22}\,a_{21}y = a_{22}\,a_{21}b_1 & (1)\times a_{22}\times a_{21} \\ a_{21}\,a_{12}\,a_{11}x + a_{22}\,a_{12}\,a_{11}y = a_{12}\,a_{11}b_2 & (2)\times a_{12}\times a_{11} \end{cases}$$

用消元法解：

$(1)\times a_{22}-(2)\times a_{12}$，消去 y，得：

$$x=\frac{b_1a_{22}-a_{12}b_2}{a_{11}a_{22}-a_{12}a_{21}}=\frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

$(2)\times a_{11}-(1)\times a_{21}$，消去 x，得：

$$y=\frac{a_{11}b_2-b_1a_{21}}{a_{11}a_{22}-a_{12}a_{21}}=\frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

公式：

$$\begin{cases} a_{11}x + a_{12}y = b_1 \\ a_{21}x + a_{22}y = b_2 \end{cases} \Rightarrow \quad x = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

练习 利用二阶行列式求解下面二元线性方程组

1. $\begin{cases} 2x + 5y = 0 \\ 3x + 8y = 4 \end{cases}$ $\quad x = \frac{\begin{vmatrix} 0 & 5 \\ 4 & 8 \end{vmatrix}}{\begin{vmatrix} 2 & 5 \\ 3 & 8 \end{vmatrix}} = \frac{-20}{1} = -20, \quad y = \frac{\begin{vmatrix} 2 & 0 \\ 3 & 4 \end{vmatrix}}{\begin{vmatrix} 2 & 5 \\ 3 & 8 \end{vmatrix}} = \frac{8}{1} = 8$

2. $\begin{cases} 7x + 16y = 1 \\ 2x + 5y = -1 \end{cases}$ $\quad x = \frac{\begin{vmatrix} 1 & 16 \\ -1 & 5 \end{vmatrix}}{\begin{vmatrix} 7 & 16 \\ 2 & 5 \end{vmatrix}} = \frac{21}{3} = 7, \quad y = \frac{\begin{vmatrix} 7 & 1 \\ 2 & -1 \end{vmatrix}}{\begin{vmatrix} 7 & 16 \\ 2 & 5 \end{vmatrix}} = \frac{-9}{3} = -3$

方程组的隐函数求导公式

假设函数 $u = u(x, y)$, $v = v(x, y)$ 满足方程组

$$\begin{cases} F(x, y, u, v) = 0, \\ G(x, y, u, v) = 0. \end{cases}$$

问题：如何计算 $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ ？

求解如下：

$$\begin{cases} F(x, y, u, v) = 0 \\ G(x, y, u, v) = 0 \end{cases} \xRightarrow{\frac{\partial}{\partial x}} \begin{cases} F_u \cdot u_x + F_v \cdot v_x = -F_x \\ G_u \cdot u_x + G_v \cdot v_x = -G_x \end{cases}$$

$$\Rightarrow u_x = -\frac{\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}, \quad v_x = -\frac{\begin{vmatrix} F_u & F_x \\ G_u & G_x \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}$$

$$= -\frac{1}{J}\frac{\partial(F, G)}{\partial(x, v)} \qquad = -\frac{1}{J}\frac{\partial(F, G)}{\partial(u, x)}$$

$$\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases} \xRightarrow{\frac{\partial}{\partial y}} \begin{cases} F_u \cdot u_y + F_v \cdot v_y = -F_y \\ G_u \cdot u_y + G_v \cdot v_y = -G_y \end{cases}$$

$$\Rightarrow u_y = -\frac{\begin{vmatrix} F_y & F_v \\ G_y & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}, \quad v_y = -\frac{\begin{vmatrix} F_u & F_y \\ G_u & G_y \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}$$

$$= -\frac{1}{J}\frac{\partial(F,G)}{\partial(y,v)} \qquad = -\frac{1}{J}\frac{\partial(F,G)}{\partial(u,y)}$$

总结 设 $u = u(x, y),\ v = v(x, y)$ 满足方程组

$$\begin{cases} F(x, y, u, v) = 0 \\ G(x, y, u, v) = 0 \end{cases} \xRightarrow{\frac{\partial}{\partial x}} \begin{cases} F_x + F_u \cdot u_x + F_v \cdot v_x = 0 \\ G_x + G_u \cdot u_x + G_v \cdot v_x = 0 \end{cases}$$

$$\xRightarrow{\frac{\partial}{\partial y}} \begin{cases} F_y + F_u \cdot u_y + F_v \cdot v_y = 0 \\ G_y + G_u \cdot u_y + G_v \cdot v_y = 0 \end{cases}$$

所以

$$u_x = -\frac{\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{1}{J}\frac{\partial(F, G)}{\partial(x, v)}, \quad v_x = -\frac{\begin{vmatrix} F_u & F_x \\ G_u & G_x \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{1}{J}\frac{\partial(F, G)}{\partial(u, x)}$$

$$u_y = -\frac{\begin{vmatrix} F_y & F_v \\ G_y & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{1}{J}\frac{\partial(F, G)}{\partial(y, v)}, \quad v_y = -\frac{\begin{vmatrix} F_u & F_y \\ G_u & G_y \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{1}{J}\frac{\partial(F, G)}{\partial(u, y)}$$

例 设 $\begin{cases} x = e^u + u\sin v \\ y = e^u - u\cos v \end{cases}$，求 $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$

解

$$\begin{cases} e^u + u\sin v = x \\ e^u - u\cos v = y \end{cases} \xRightarrow{\frac{\partial}{\partial x}} \begin{cases} (e^u + \sin v)u_x + u\cos v \cdot v_x = 1 \\ (e^u - \cos v)u_x + u\sin v \cdot v_x = 0 \end{cases}$$

$$\xRightarrow{\frac{\partial}{\partial y}} \begin{cases} (e^u + \sin v)u_y + u\cos v \cdot v_y = 0 \\ (e^u - \cos v)u_y + u\sin v \cdot v_y = 1 \end{cases}$$

所以 $J = \begin{vmatrix} e^u + \sin v & u\cos v \\ e^u - \cos v & u\sin v \end{vmatrix} = ue^u(\sin v - \cos v) + u$

$$u_x = \frac{\begin{vmatrix} 1 & u\cos v \\ 0 & u\sin v \end{vmatrix}}{J} = \frac{\sin v}{e^u(\sin v - \cos v) + 1}, \quad v_x = \frac{\begin{vmatrix} e^u + \sin v & 1 \\ e^u - \cos v & 0 \end{vmatrix}}{J} = \frac{-e^u + \cos v}{ue^u(\sin v - \cos v) + u}$$

$$u_y = \frac{\begin{vmatrix} 0 & u\cos v \\ 1 & u\sin v \end{vmatrix}}{J} = \frac{-\cos v}{e^u(\sin v - \cos v) + 1}, \quad v_y = \frac{\begin{vmatrix} e^u + \sin v & 0 \\ e^u - \cos v & 1 \end{vmatrix}}{J} = \frac{e^u + \sin v}{ue^u(\sin v - \cos v) + u}$$