第 09 周作业解答

练习 1. 求解线性方程组 $\begin{cases} x_1+ & 2x_2+ & x_3+ & x_4+ & x_5= & 1 \\ 2x_1+ & 4x_2+ & 3x_3+ & x_4+ & x_5= & 3 \\ -x_1- & 2x_2+ & x_3+ & 3x_4- & 3x_5= & 7 \\ & & 2x_3+ & 5x_4- & 2x_5= & 9 \end{cases}$

解对增广矩阵作初等行变换：

$$(A\vdots b)=\left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{array}\right) \xrightarrow[r_3+r_1]{r_2-2r_1} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{array}\right) \xrightarrow[r_4-2r_2]{r_3-2r_2} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 6 & 0 & 6 \\ 0 & 0 & 0 & 7 & 0 & 7 \end{array}\right)$$

$$\xrightarrow[\frac{1}{7}\times r_4]{\frac{1}{6}\times r_3} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right) \xrightarrow{r_4-r_3} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[r_1-r_3]{r_2+r_3} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{r_1-r_2} \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

可见 $r(A)=r(A\vdots b)=3<5$，所以原方程组有无穷多的解，包含 $5-3=2$ 个自由变量。事实上，通过上述简化的阶梯型矩阵，可知原方程等价于

$$\begin{cases} x_1+ \quad 2x_2 \qquad\qquad + \ 2x_5 = -2 \\ \qquad\qquad x_3 \qquad - \ x_5 = 2 \\ \qquad\qquad\qquad x_4 \qquad = 1 \end{cases} \Rightarrow \begin{cases} x_1=-2-2x_2-2x_5 \\ x_3=2+x_5 \\ x_4=1 \end{cases}$$

所以通解是

$$\begin{cases} x_1=-2-2c_1-2c_2 \\ x_2=c_1 \\ x_3=2+c_2 \\ x_4=1 \\ x_5=c_2 \end{cases} \qquad (c_1,\ c_2\text{为任意常数})$$

用向量形式表示则是

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 2 \\ 1 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

练习 2. 问 k 取何值时，方程组 $\begin{cases} x_1 + x_2 + kx_3 = 4 \\ -x_1 + kx_2 + x_3 = k^2 \\ x_1 - x_2 + 2x_3 = -4 \end{cases}$ 有唯一解、无穷多解、无解。并且有解时，求出全部解。

解对增广矩阵作初等行变换：

$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 1 & k & 4 \\ -1 & k & 1 & k^2 \\ 1 & -1 & 2 & -4 \end{array}\right) \xrightarrow[r_3 - r_1]{r_2 + r_1} \left(\begin{array}{ccc|c} 1 & 1 & k & 4 \\ 0 & k+1 & k+1 & k^2+4 \\ 0 & -2 & 2-k & -8 \end{array}\right) \xrightarrow{r_3 \leftrightarrow r_2} \left(\begin{array}{ccc|c} 1 & 1 & k & 4 \\ 0 & -2 & 2-k & -8 \\ 0 & k+1 & k+1 & k^2+4 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{2} \times r_2} \left(\begin{array}{ccc|c} 1 & 1 & k & 4 \\ 0 & 1 & \frac{1}{2}k-1 & 4 \\ 0 & k+1 & k+1 & k^2+4 \end{array}\right) \xrightarrow[r_1 - r_2]{r_3 - (k+1) \times r_2} \left(\begin{array}{ccc|c} 1 & 0 & \frac{1}{2}k+1 & 0 \\ 0 & 1 & \frac{1}{2}k-1 & 4 \\ 0 & 0 & \frac{1}{2}(k+1)(4-k) & k(k-4) \end{array}\right)$$

- 当 $k \neq -1$ 且 $k \neq 4$ 时，$r(A) = r(A \vdots b) = 3 =$ 未知量个数，方程组有唯一解。此时

$$(A \vdots b) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & \frac{1}{2}k+1 & 0 \\ 0 & 1 & \frac{1}{2}k-1 & 4 \\ 0 & 0 & \frac{1}{2}(k+1)(4-k) & k(k-4) \end{array}\right) \xrightarrow{\frac{2}{(k+1)(4-k)} \times r_3} \left(\begin{array}{ccc|c} 1 & 0 & \frac{1}{2}k+1 & 0 \\ 0 & 1 & \frac{1}{2}k-1 & 4 \\ 0 & 0 & 1 & -\frac{2k}{k+1} \end{array}\right)$$

$$\xrightarrow[r_2 - (\frac{1}{2}k-1) \times r_3]{r_1 - (\frac{1}{2}k+1) \times r_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{k(k+2)}{k+1} \\ 0 & 1 & 0 & \frac{k^2+2k+4}{k+1} \\ 0 & 0 & 1 & -\frac{2k}{k+1} \end{array}\right)$$

所以

$$\begin{cases} x_1 = \frac{k^2+2k}{k+1} \\ x_2 = \frac{k^2+2k+4}{k+1} \\ x_3 = -\frac{2k}{k+1} \end{cases}$$

- 当 $k = -1$ 时

$$(A \vdots b) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & \frac{1}{2}k+1 & 0 \\ 0 & 1 & \frac{1}{2}k-1 & 4 \\ 0 & 0 & \frac{1}{2}(k+1)(4-k) & k(k-4) \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{3}{2} & 4 \\ 0 & 0 & 0 & 5 \end{array}\right)$$

可见 $r(A) = 2 < 3 = r(A \vdots b)$，此时方程无解。

- 当 $k = 4$ 时

$$(A \vdots b) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & \frac{1}{2}k+1 & 0 \\ 0 & 1 & \frac{1}{2}k-1 & 4 \\ 0 & 0 & \frac{1}{2}(k+1)(4-k) & k(k-4) \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

可见 $r(A) = r(A \vdots b) = 2 <$ 未知量个数3，方程组有无穷多的解，包含 $3 - 2 = 1$ 个自由变量。事实上，通过上述简化的阶梯型矩阵，可知原方程等价于

$$\begin{cases} x_1 \qquad + 3x_3 = 0 \\ \qquad x_2 + x_3 = 4 \end{cases} \Rightarrow \begin{cases} x_1 = -3x_3 \\ x_3 = 4 - x_3 \end{cases}$$

所以通解是

$$\begin{cases} x_1 = -3c \\ x_2 = 4 - c \quad (c\text{为任意常数}) \\ x_3 = c \end{cases}$$

用向量形式表示则是

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} + c \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}$$