第 03 周作业解答

练习 1. 求不定积分
(1) $\int xe^{-x^2}dx$, (2) $\int \frac{x}{1+x^4}dx$, (3) $\int \frac{2x-1}{\sqrt{x^2-x+3}}dx$

解：1.

$$\int xe^{-x^2}dx = \frac{1}{2}\int e^{-x^2}dx^2 = \int e^{-x^2}\cdot(-\frac{1}{2})d(-x^2)$$
$$\xlongequal{u=-x^2} -\frac{1}{2}\int e^u du = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{-x^2} + C$$

2.

$$\int \frac{x}{1+x^4}dx = \int \frac{1}{1+x^4}\cdot\frac{1}{2}dx^2 \xlongequal{u=x^2} \frac{1}{2}\int \frac{1}{1+u^2}du = \frac{1}{2}\arctan u + C = \frac{1}{2}\arctan(x^2) + C$$

3.

$$\int \frac{2x-1}{\sqrt{x^2-x+3}}dx = \int \frac{1}{\sqrt{x^2-x+3}}d(x^2-x+3)$$
$$\xlongequal{u=x^2-x+3} \int \frac{1}{\sqrt{u}}du = \int u^{-1/2}du = 2u^{1/2} + C = 2(x^2-x+3)^{1/2} + C$$

练习 2. 求不定积分
(1) $\int \frac{1}{x\ln x}dx$, (2) $\int \frac{(\ln x)^{1/3}}{x}dx$.

解：1.

$$\int \frac{1}{x\ln x}dx = \int \frac{1}{\ln x}\cdot\frac{1}{x}dx = \int \frac{1}{\ln x}d(\ln x) \xlongequal{u=\ln x} \int \frac{1}{u}du = \ln|u| + C = \ln|\ln x| + C$$

2.

$$\int \frac{(\ln x)^{1/3}}{x}dx = \int (\ln x)^{1/3}d\ln x \xlongequal{u=\ln x} \int u^{1/3}du = \frac{3}{4}u^{4/3} + C = \frac{3}{4}(\ln x)^{4/3} + C$$

练习 3. 求不定积分
(1) $\int e^x\cos(e^x)dx$, (2) $\int \frac{e^x}{e^{2x}+1}dx$.

解：1.

$$\int e^x\cos(e^x)dx = \int \cos(e^x)de^x \xlongequal{u=e^x} \int \cos u du = \sin u + C = \sin e^x + C$$

2

$$\int \frac{e^x}{e^{2x}+1}dx = \int \frac{1}{e^{2x}+1}de^x \xlongequal{u=e^x} \int \frac{1}{u^2+1}du = \arctan u + C = \arctan e^x + C$$

练习 4. 求不定积分
(1) $\int e^{\cos x}\sin x dx$, (2) $\int \sin^4 x\cos x dx$.

解：1.

$$\int e^{\cos x}\sin x dx = -\int e^{\cos x} d\cos x \xlongequal{u=\cos x} -\int e^u du = -e^u + C = -e^{\cos x} + C$$

2.

$$\int \sin^4 x \cos x dx = \int \sin^4 x d\sin x \xlongequal{u=\sin x} \int u^4 du = \frac{1}{5}u^5 + C = \frac{1}{5}\sin^5 x + C$$

练习 5. 求不定积分 $\int \sin(\frac{1}{x})\frac{1}{x^2}dx$.

解：

$$\int \sin(\frac{1}{x})\frac{1}{x^2}dx = \int \sin(\frac{1}{x})\cdot(-1)d(\frac{1}{x}) \xlongequal{u=\frac{1}{x}} -\int \sin u du = \cos u + C = \cos\frac{1}{x} + C$$

练习 6. 求不定积分

(1) $\int x\sqrt{x+1}dx$, (2) $\int \frac{\sqrt{x-1}}{x}dx$, (3) $\int \frac{dx}{\sqrt{2x-3}-1}$, (4) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$.

解：1. 令 $t=\sqrt{x+1}$，则 $x=t^2-1$，$dx=d(t^2-1)=2tdt$，所以

$$\begin{aligned}\int x\sqrt{x+1}dx &= \int (t^2-1)\cdot t\cdot 2tdt = 2\int t^4 - t^2 dt = \frac{2}{5}t^5 - \frac{2}{3}t^3 + C\\ &= \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C.\end{aligned}$$

2. 令 $t=\sqrt{x-1}$，则 $x=t^2+1$，$dx=d(t^2+1)=2tdt$，所以

$$\begin{aligned}\int \frac{\sqrt{x-2}}{x}dx &= \int \frac{t}{t^2+1}\cdot 2tdt = 2\int \frac{t^2}{t^2+1}dt = 2\int \left(1 - \frac{1}{t^2+1}\right)dt\\ &= 2t - 2\arctan t + C\\ &= 2\sqrt{x-1} - 2\arctan\sqrt{x-1} + C.\end{aligned}$$

3. 令 $t=\sqrt{2x-3}-1$，则 $x=\frac{1}{2}\left[(t+1)^2+3\right]$，$dx=\frac{1}{2}d\left[(t+1)^2+3\right]=(t+1)dt$，所以

$$\begin{aligned}\int \frac{dx}{\sqrt{2x-3}-1} &= \int \frac{1}{t}\cdot(t+1)dt = \int 1 + \frac{1}{t}dt = t + \ln|t| + C\\ &= \sqrt{2x-3} + 1 + \ln\left(\sqrt{2x-3}-1\right) + C.\end{aligned}$$

4. 令 $t=\sqrt{x}$，则 $x=t^2$，$dx=dt^2=2tdt$，所以

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = \int \frac{e^t}{t}\cdot 2tdt = 2\int e^t dt = 2e^t + C = 2e^{\sqrt{x}} + C.$$