# 第 11 章 f: 高斯公式、斯托克斯公式

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### **Outline**

1. 高斯公式

2. 斯托克斯公式

### We are here now...

1. 高斯公式

2. 斯托克斯公式

定义设
$$F = (P, Q, R)$$
是空间中向量场,定义

$$divF := \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

称为向量场 F 的散度。

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$$divF = \frac{\partial}{\partial x}(x^2 + yz) + \frac{\partial}{\partial y}(y^2 + xz) + \frac{\partial}{\partial z}(z^2 + xy) = 2x + 2y + 2z.$$



例 2 计算梯度场  $\nabla \frac{1}{r}$   $(r = \sqrt{x^2 + y^2 + z^2})$  的散度。

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$$\nabla \frac{1}{r} = (\frac{\partial}{\partial x} r^{-1}, \frac{\partial}{\partial y} r^{-1}, \frac{\partial}{\partial z} r^{-1})$$

$$div \nabla \frac{1}{r}$$

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$$-r^{-2} \cdot r_{x}$$
$$\operatorname{div} \nabla \frac{1}{r}$$



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$$= \left(-r^{-2} \cdot r_x, -r^{-2} \cdot r_y, -r^{-2} \cdot r_z\right)$$

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$$r_{x} = \frac{x}{r},$$

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$$= \left(-r^{-2} \cdot r_{x}, -r^{-2} \cdot r_{y}, -r^{-2} \cdot r_{z}\right) = \left(-\frac{x}{r^{3}}, -\frac{y}{r^{3}}, -\frac{z}{r^{3}}\right),$$

$$= 1$$

 $\operatorname{div}\nabla \frac{1}{r}$ 

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$$= (-r^{-2} \cdot r_{x}, -r^{-2} \cdot r_{y}, -r^{-2} \cdot r_{z}) = (-\frac{x}{r^{3}}, -\frac{y}{r^{3}}, -\frac{z}{r^{3}}),$$

$$\operatorname{div} \nabla \frac{1}{r} = \frac{\partial}{\partial x}(-\frac{x}{r^{3}}) + \frac{\partial}{\partial y}(-\frac{y}{r^{3}}) + \frac{\partial}{\partial z}(-\frac{z}{r^{3}})$$

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$$(-\frac{1}{r^{3}} + \frac{3x^{2}}{r^{5}})$$

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$$= (-\frac{1}{r^{3}} + \frac{3x^{2}}{r^{5}}) + (-\frac{1}{r^{3}} + \frac{3y^{2}}{r^{5}}) + (-\frac{1}{r^{3}} + \frac{3z^{2}}{r^{5}})$$

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 $= \left(-\frac{1}{r^3} + \frac{3x^2}{r^5}\right) + \left(-\frac{1}{r^3} + \frac{3y^2}{r^5}\right) + \left(-\frac{1}{r^3} + \frac{3z^2}{r^5}\right)$  $=-\frac{3}{x^3}+\frac{3(x^2+y^2+z^2)}{x^5}$ 

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  $(r = \sqrt{x^2 + y^2 + z^2})$  的散度。

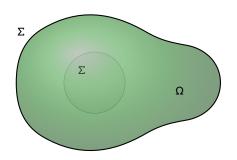
$$r_{x} = \frac{x}{r}, \qquad r_{y} = \frac{y}{r}, \qquad r_{z} = \frac{z}{r},$$

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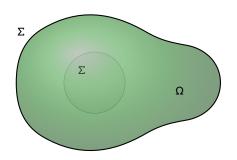
$$= (-r^{-2} \cdot r_{x}, -r^{-2} \cdot r_{y}, -r^{-2} \cdot r_{z}) = (-\frac{x}{r^{3}}, -\frac{y}{r^{3}}, -\frac{z}{r^{3}}),$$

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$$= \left( -\frac{1}{r^3} + \frac{3x^2}{r^5} \right) + \left( -\frac{1}{r^3} + \frac{3y^2}{r^5} \right) + \left( -\frac{1}{r^3} + \frac{3z^2}{r^5} \right)$$
$$= -\frac{3}{r^3} + \frac{3(x^2 + y^2 + z^2)}{r^5} = 0.$$

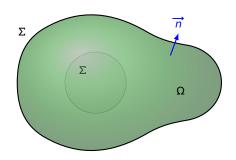
- 空间闭区域  $\Omega$  的边界是分片光滑的闭曲面  $\Sigma$ ,
- $\overrightarrow{n}$  是  $\Sigma$  的单位外法向量,



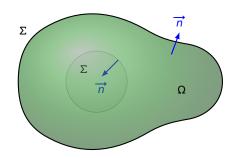
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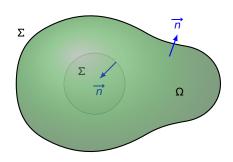
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- F = (P, Q, R) 是  $\Omega$  中向量场,且 P, Q, R 具有一阶连续的偏导数,





#### 定理(高斯公式) 假设

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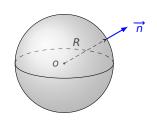
则

$$\iiint_{\Omega} \operatorname{div} F dv = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$

$$\sum_{\overrightarrow{n}} Q$$

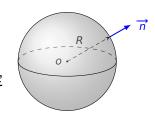
$$I = \iint_{\Sigma} 2x \, dy \, dz + y^2 \, dz \, dx + z^2 \, dx \, dy$$

其中定向曲面  $\Sigma$  是球面  $x^2 + y^2 + z^2 = R^2$ ,定 向取外侧



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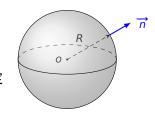


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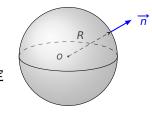
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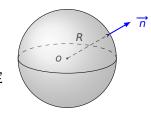


$$I = \underbrace{F = (2x, y^2, z^2)}_{\Gamma} \iint_{\Sigma} F \cdot \overrightarrow{n} dS = \underbrace{\overline{\text{高斯公式}}}_{\Omega} \iint_{\Omega} \text{div} F dv$$



$$I = \iint_{\Sigma} 2x \, dy \, dz + y^2 \, dz \, dx + z^2 \, dx \, dy$$

其中定向曲面  $\Sigma$  是球面  $x^2 + y^2 + z^2 = R^2$ ,定向取外侧

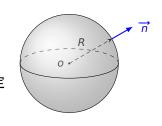


$$I = \frac{F = (2x, y^2, z^2)}{\int \int_{\Sigma} F \cdot \overrightarrow{n} \, dS} = \frac{\overline{\text{sh公式}}}{\int \int_{\Omega} \text{div} F \, dv}$$
$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} (2x) + \frac{\partial}{\partial y} (y^2) + \frac{\partial}{\partial z} (z^2) \right] dv$$



$$I = \iint_{\Sigma} 2x \, dy \, dz + y^2 \, dz \, dx + z^2 \, dx \, dy$$

其中定向曲面  $\Sigma$  是球面  $x^2 + y^2 + z^2 = R^2$ ,定向取外侧

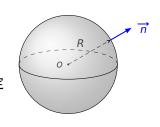


$$I = \frac{F = (2x, y^2, z^2)}{\iint_{\Sigma} F \cdot \overrightarrow{n} dS} = \frac{\overrightarrow{\text{sh}} \triangle \overrightarrow{x}}{\iint_{\Omega} \text{div} F dv}$$
$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} (2x) + \frac{\partial}{\partial y} (y^2) + \frac{\partial}{\partial z} (z^2) \right] dv = \iiint_{\Omega} (2 + 2y + 2z) dv$$



$$I = \iint_{\Sigma} 2x \, dy \, dz + y^2 \, dz \, dx + z^2 \, dx \, dy$$

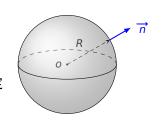
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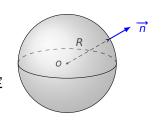
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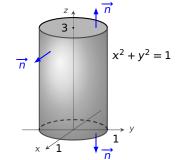
$$I = \frac{F = (2x, y^2, z^2)}{\int \int_{\Sigma} F \cdot \overrightarrow{n} \, dS} = \frac{\overrightarrow{\text{sh} \triangle x}}{\int \int_{\Omega} \text{div} F \, dv}$$
$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} (2x) + \frac{\partial}{\partial y} (y^2) + \frac{\partial}{\partial z} (z^2) \right] dv = \iiint_{\Omega} (2 + 2y + 2z) dv$$
$$= \frac{\cancel{\text{MRM}}}{\int \int_{\Omega} 2 \, dv} = 2 \text{Vol}(\Omega) = \frac{8}{3} \pi R^3$$



### 例2计算

$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$

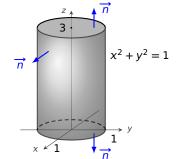
其中定向曲面  $\Sigma$  是右图柱体的边界曲面



$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$

其中定向曲面  $\Sigma$  是右图柱体的边界曲面

$$I = \int \int_{\Sigma} F \cdot \overrightarrow{n} \, dS$$



$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$

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$$\overrightarrow{n}$$

$$x^2 + y^2 = 1$$

$$x + y^2 = 1$$



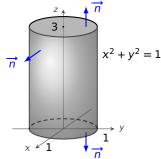
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# $x^{2} + y^{2} = 3$



$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$
  
其中定向曲面  $\Sigma$  是右图柱体的边界曲面



$$I \xrightarrow{F=((y-z)x,0,x-y)} \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\overline{\text{synch}}} \iiint_{\Omega} \text{div} F \, dv$$
$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] \, dv$$



$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$

其中定向曲面 Σ 是右图柱体的边界曲面

$$x^{2} + y^{2} = 1$$

$$I = F = ((y-z)x, 0, x-y)$$
 
$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS = \overline{\text{SMAT}} \iiint_{\Omega} \text{div} F \, dv$$

$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dz$$



$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$

解 
$$I = F = ((y-z)x,0,x-y)$$
  $\int \int_{\mathbb{R}} F \cdot \overrightarrow{n} dS = \overline{S} = \int_{\mathbb{R}} \int_{$ 

$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dz$$



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$$= \frac{\overline{\text{sh} \Delta x}}{\int \int_{\Omega} -z dx dy dz} = \int \left[ \int \int_{\Omega} -z dx dy \right] dz$$



 $\{11 \, \text{章} \, f \colon$  高斯公式、斯托克斯公式

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$$x^{2} + y^{2} = 1$$

$$I = \frac{F = ((y-z)x, 0, x-y)}{\int \int_{\Sigma} F \cdot \overrightarrow{n} dS} = \frac{\overline{\sin} \Delta \overrightarrow{x}}{\int \int_{\Omega} \operatorname{div} F dv}$$

$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dz$$

$$= \frac{\overline{y} \pi t}{\int \int_{\Omega} -z dx dy dz} = \int \left[ \int \int_{\Omega} -z dx dy \right] dz$$



$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$
  
其中定向曲面  $\Sigma$  是右图柱体的边界曲面

$$x^{2} + y^{2} = 1$$

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$$I \xrightarrow{F = ((y-z)x, 0, x-y)} \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\underline{a}\underline{m} \underline{\Delta}\underline{d}} \iiint_{\Omega} \operatorname{div} F \, dv$$

$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z \, dx \, dy \, dz$$

$$\xrightarrow{\underline{M}\underline{M}\underline{d}} \iiint_{\Omega} -z \, dx \, dy \, dz = \int_{\Omega} \left[ \iint_{\Omega} -z \, dx \, dy \, dz \right] dz$$



$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$
  
其中定向曲面  $\Sigma$  是右图柱体的边界曲面

$$\overrightarrow{R}$$

$$x^2 + y^2 = 1$$

$$x + y^2 = 1$$

$$x + y^2 = 1$$



$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$
  
其中定向曲面  $\Sigma$  是右图柱体的边界曲面

$$x^{2} + y^{2} = 1$$

$$x^{2} + y^{2} = 1$$

$$x^{2} + y^{2} = 1$$

$$I \xrightarrow{F = ((y-z)x, 0, x-y)} \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\overline{a} \underline{y} \cdot \Delta z} \iiint_{\Omega} \operatorname{div} F \, dv$$

$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] \, dv = \iiint_{\Omega} y - z \, dx \, dy \, dz$$

$$\xrightarrow{\overline{y} \underline{y} \underline{y} \cdot \underline{y}} \left[ \iint_{\Omega} -z \, dx \, dy \, dz \right] \, dz$$



$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$
  
其中定向曲面  $\Sigma$  是右图柱体的边界曲面

$$x^{2} + y^{2} = 1$$

$$x^{2} + y^{2} = 1$$

$$x^{2} + y^{2} = 1$$

$$I \xrightarrow{F = ((y-z)x, 0, x-y)} \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\overline{a} \underline{m} \triangle \underline{x}} \iiint_{\Omega} \operatorname{div} F \, dv$$

$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z \, dx \, dy \, dz$$

$$\xrightarrow{\underline{M} \underline{n} \underline{n}} \iiint_{\Omega} -z \, dx \, dy \, dz = \int_{0}^{3} \left[ \iint_{D_{z}} -z \, dx \, dy \right] dz$$



$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$
  
其中定向曲面  $\Sigma$  是右图柱体的边界曲面

$$x^{2} + y^{2} = 1$$

$$x^{2} + y^{2} = 1$$

$$x + y^{2} = 1$$

$$I = \frac{F = ((y - z)x, 0, x - y)}{\int \int_{\Sigma} F \cdot \overrightarrow{n} dS} = \frac{\overrightarrow{\text{sh}} \triangle \overrightarrow{\text{sh}}}{\int \int_{\Omega} \text{div} F dV}$$
$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} ((y - z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x - y) \right] dV = \iiint_{\Omega} y - z dx dy dz$$

$$= \iiint_{\Omega} \left[ \frac{1}{\partial x} ((y-z)x) + \frac{1}{\partial y} + \frac{1}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dz$$

$$= \iiint_{\Omega} \left[ \frac{1}{\partial x} ((y-z)x) + \frac{1}{\partial y} + \frac{1}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dz$$

$$= \iiint_{\Omega} \left[ -z dx dy dz \right] = \int_{0}^{3} \left[ \iint_{\Omega} -z dx dy \right] dz = -z |D_{z}|$$



$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$
  
其中定向曲面  $\Sigma$  是右图柱体的边界曲面

$$x^{2} + y^{2} = 1$$

$$x^{2} + y^{2} = 1$$

$$x^{2} + y^{2} = 1$$

$$I \xrightarrow{F = ((y-z)x, 0, x-y)} \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{a}\underline{n} \underline{n}} \iiint_{\Omega} div F dv$$
$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dz$$

<u> 対称性</u>  $\iiint_{\Omega} -z dx dy dz = \int_{\Omega}^{3} \left[ \iint_{\Omega} -z dx dy \right] dz =$ 

$$=$$
  $-z\pi$ 



$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$
  
其中定向曲面  $\Sigma$  是右图柱体的边界曲面

$$x^{2} + y^{2} = 1$$

$$x^{2} + y^{2} = 1$$

$$x^{2} + y^{2} = 1$$

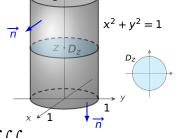
$$I \xrightarrow{F = ((y-z)x, 0, x-y)} \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{a}\underline{n} \underline{n}} \iiint_{\Omega} \operatorname{div} F dv$$
$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dz$$

 $\frac{\exists x}{\exists x} = \left[ \int_{0}^{1} \left[ \int_{0}^{1} -z dx dy dz \right] dz \right] = \left[ \int_{0}^{3} \left[ -z |D_{z}| \right] dz \right]$ 

$$=$$
  $-z\pi$ 



$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$
  
其中定向曲面  $\Sigma$  是右图柱体的边界曲面



$$I \xrightarrow{F = ((y-z)x, 0, x-y)} \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{a}\underline{m} \underline{\wedge}\underline{\wedge}\underline{\wedge}} \iiint_{\Omega} div F dv$$
$$= \iiint_{\Sigma} \left[ \frac{\partial}{\partial u} ((y-z)x) + \frac{\partial}{\partial u} (u-z) + \frac{\partial}{\partial u} (u-z) \right] dv = \iiint_{\Omega} dv = \iiint_{\Omega$$

 $= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dz$  $\frac{\exists x}{\exists x} = \left[ \int_{0}^{1} \left[ \int_{0}^{1} -z dx dy dz \right] dz \right] = \left[ \int_{0}^{3} \left[ -z |D_{z}| \right] dz \right]$ 

 $= \int_{a}^{3} \left[ -z\pi \right] dz$ 



$$I = \iint_{\mathbb{R}} (x - y) dx dy + (y - z) x dy dz$$

其中定向曲面 Σ 是右图柱体的边界曲面

$$x^{2} + y^{2} = 1$$

$$x + y^{2} = 1$$

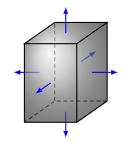
$$x + y^{2} = 1$$

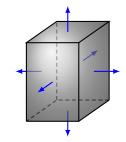
I = F = ((y-z)x, 0, x-y)  $\iiint_{\Sigma} F \cdot \overrightarrow{n} dS = \overline{\text{sh公式}} \iiint_{\Sigma} \text{div} F dv$ 

 $= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dz$  $\frac{\exists x}{\exists x} = \left[ \int_{0}^{1} \left[ \int_{0}^{1} -z dx dy dz \right] dz \right] = \int_{0}^{3} \left[ -z |D_{z}| dz \right] dz$ 

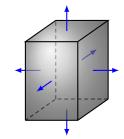
 $= \int_{0}^{3} \left[ -z\pi \right] dz = -\frac{9}{2}\pi$ 



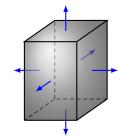




$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS$$

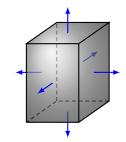


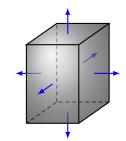
$$Φ = \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{a} \underline{m} \triangle \underline{\exists}} \iiint_{\Omega} div F dv$$



$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\underline{\underline{a}}\underline{\underline{m}}\underline{\underline{M}}} \iiint_{\Omega} \operatorname{div} F \, dv$$
$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} (x - y^2) + \frac{\partial}{\partial y} (y) + \frac{\partial}{\partial z} (z^3) \right] \, dv$$





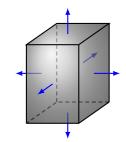


$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\underline{a}\underline{m}\underline{\omega}\underline{\omega}} \iiint_{\Omega} \operatorname{div} F \, dv$$

$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} (x - y^2) + \frac{\partial}{\partial y} (y) + \frac{\partial}{\partial z} (z^3) \right] dv$$

$$= \iiint_{\Omega} (2 + 3z^2) dx \, dy \, dz = \int_{\Omega} \left[ \int_{\Omega} \left[ \int_{\Omega} (2 + 3z^2) dz \, dy \, dx \right] dx$$



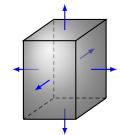


$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\underline{=}\underline{\text{M}} \triangle \underline{\mathcal{A}}} \iiint_{\Omega} \text{div} F \, dv$$

$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} (x - y^2) + \frac{\partial}{\partial y} (y) + \frac{\partial}{\partial z} (z^3) \right] \, dv$$

$$= \iiint_{\Omega} (2 + 3z^2) \, dx \, dy \, dz = \int_{\Omega}^{1} \left[ \int_{\Omega} \left[ \int_{\Omega} (2 + 3z^2) \, dz \right] \, dy \right] \, dx$$



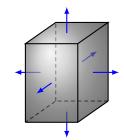


$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\underline{=}\underline{\text{M}} \triangle \underline{\mathcal{A}}} \iiint_{\Omega} \text{div} F \, dv$$

$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} (x - y^2) + \frac{\partial}{\partial y} (y) + \frac{\partial}{\partial z} (z^3) \right] \, dv$$

$$= \iiint_{\Omega} (2 + 3z^2) \, dx \, dy \, dz = \int_{\Omega}^{1} \left[ \int_{1}^{2} \left[ \int_{1}^{2} (2 + 3z^2) \, dz \right] \, dy \right] \, dx$$



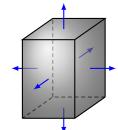


$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\underline{=}\underline{\text{M}}\underline{\text{M}}\underline{\text{M}}\underline{\text{M}}} \iiint_{\Omega} \text{div} F \, dv$$

$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} (x - y^2) + \frac{\partial}{\partial y} (y) + \frac{\partial}{\partial z} (z^3) \right] \, dv$$

$$= \iiint_{\Omega} (2 + 3z^2) \, dx \, dy \, dz = \int_{\Omega}^{1} \left[ \int_{1}^{2} \left[ \int_{1}^{4} (2 + 3z^2) \, dz \right] \, dy \right] \, dx$$





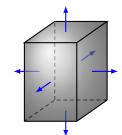
$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{a}\underline{m}\underline{\omega}\underline{\omega}} \iiint_{\Omega} \operatorname{div} F dv$$

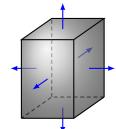
$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} (x - y^2) + \frac{\partial}{\partial y} (y) + \frac{\partial}{\partial z} (z^3) \right] dv$$

$$= \iiint_{\Omega} (2 + 3z^2) dx dy dz = \int_{0}^{1} \left[ \int_{1}^{2} \left[ \int_{1}^{4} (2 + 3z^2) dz \right] dy \right] dx$$

$$= \int_{0}^{1} 1 dx \cdot \int_{1}^{2} 1 dy \cdot \int_{1}^{4} (2 + 3z^2) dz$$



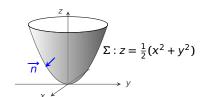






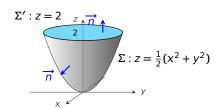
$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

其中定向曲面  $\Sigma$  是抛物面的一部分, 取单位外法向量,如图:



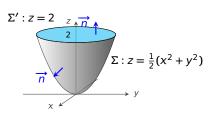
$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

其中定向曲面  $\Sigma$  是抛物面的一部分, 取单位外法向量,如图:



$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

其中定向曲面 Σ 是抛物面的一部分, 取单位外法向量,如图:

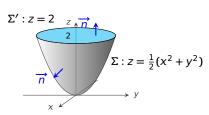


原式 = 
$$\iint_{\Sigma \cup \Sigma'} F \cdot \overrightarrow{n} \, dS - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$



$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

其中定向曲面 Σ 是抛物面的一部分, 取单位外法向量,如图:



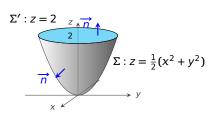
 $\mathbf{m}$  如图补充平面  $\mathbf{\Sigma}'$ ,则  $\mathbf{\Sigma} \cup \mathbf{\Sigma}'$  构成  $\mathbf{3}$  维区域  $\mathbf{\Omega}$  边界,应用高斯公式:

原式 = 
$$\iint_{\Sigma \cup \Sigma'} F \cdot \overrightarrow{n} \, dS - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$
$$= \iiint_{\Omega} \operatorname{div} F \, dV$$



$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

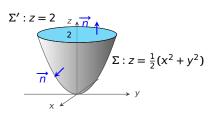
其中定向曲面 Σ 是抛物面的一部分, 取单位外法向量,如图:



原式 = 
$$\iint_{\Sigma \cup \Sigma'} F \cdot \overrightarrow{n} \, dS - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$
$$= \iiint_{\Omega} \operatorname{div} F \, dv - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$

$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

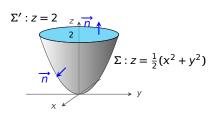
其中定向曲面 Σ 是抛物面的一部分, 取单位外法向量,如图:



原式 = 
$$\iint_{\Sigma \cup \Sigma'} F \cdot \overrightarrow{n} \, dS - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$
$$= \iiint_{\Omega} \operatorname{div} F \, dv - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$
$$\underbrace{F = (z^2 + x, 0, -z)}_{}$$

$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

其中定向曲面 Σ 是抛物面的一部分, 取单位外法向量,如图:

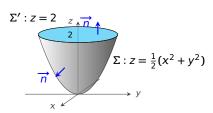


原式 = 
$$\iint_{\Sigma \cup \Sigma'} F \cdot \overrightarrow{n} \, dS - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$
$$= \iiint_{\Omega} \operatorname{div} F \, dv - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$
$$\underbrace{\underbrace{F = (z^2 + x, 0, -z)}_{\text{div} F = 0}}$$

#### 例4计算

$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

其中定向曲面 Σ 是抛物面的一部分, 取单位外法向量,如图:

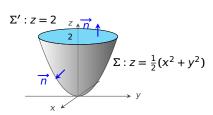


原式 = 
$$\iint_{\Sigma \cup \Sigma'} F \cdot \overrightarrow{n} \, dS - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$
$$= \iiint_{\Omega} \operatorname{div} F \, dv - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$
$$\frac{F = (z^2 + x, 0, -z)}{\operatorname{div} F = 0} - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$



$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

其中定向曲面 Σ 是抛物面的一部分, 取单位外法向量,如图:



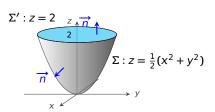
 $\mathbf{m}$  如图补充平面  $\mathbf{\Sigma}'$ ,则  $\mathbf{\Sigma} \cup \mathbf{\Sigma}'$  构成  $\mathbf{3}$  维区域  $\mathbf{\Omega}$  边界,应用高斯公式:

原式 = 
$$\iint_{\Sigma \cup \Sigma'} F \cdot \overrightarrow{n} \, dS - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$
$$= \iiint_{\Omega} \operatorname{div} F \, dv - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$
$$\frac{F = (z^2 + x, 0, -z)}{\operatorname{div} F = 0} - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$
$$\overrightarrow{n} = (0, 0, 1)$$



$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

其中定向曲面 Σ 是抛物面的一部分, 取单位外法向量,如图:



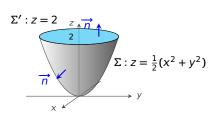
原式 = 
$$\iint_{\Sigma \cup \Sigma'} F \cdot \overrightarrow{n} \, dS - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$
= 
$$\iint_{\Omega} \operatorname{div} F \, dv - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$

$$\frac{F = (z^2 + x, 0, -z)}{\operatorname{div} F = 0} - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$

$$\frac{\overrightarrow{n} = (0, 0, 1)}{F \cdot \overrightarrow{n} - z}$$

$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

其中定向曲面 Σ 是抛物面的一部分, 取单位外法向量,如图:



原式 = 
$$\iint_{\Sigma \cup \Sigma'} F \cdot \overrightarrow{n} \, dS - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$
= 
$$\iiint_{\Omega} \operatorname{div} F \, dv - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$

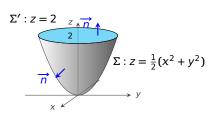
$$\frac{F = (z^2 + x, 0, -z)}{\operatorname{div} F = 0} - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$

$$\frac{\overrightarrow{n} = (0, 0, 1)}{F \cdot \overrightarrow{n} = -z} - \iint_{\Sigma'} -2 \, dS$$

例4计算

$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

其中定向曲面 Σ 是抛物面的一部分, 取单位外法向量,如图:



原式 = 
$$\iint_{\Sigma \cup \Sigma'} F \cdot \overrightarrow{n} \, dS - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$
= 
$$\iiint_{\Omega} \operatorname{div} F \, dv - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$

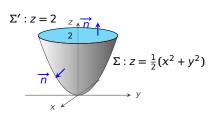
$$\frac{F = (z^2 + x, 0, -z)}{\operatorname{div} F = 0} - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$

$$\frac{\overrightarrow{n} = (0, 0, 1)}{F \cdot \overrightarrow{n} = -z} - \iint_{\Sigma'} -2 \, dS = 2 \operatorname{Area}(\Sigma')$$

例4计算

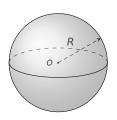
$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

其中定向曲面 Σ 是抛物面的一部分, 取单位外法向量,如图:

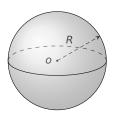


原式 = 
$$\iint_{\Sigma \cup \Sigma'} F \cdot \overrightarrow{n} dS - \iint_{\Sigma'} F \cdot \overrightarrow{n} dS$$
  
=  $\iiint_{\Omega} \operatorname{div} F dv - \iint_{\Sigma'} F \cdot \overrightarrow{n} dS$   
 $\frac{F = (z^2 + x, 0, -z)}{\operatorname{div} F = 0} - \iint_{\Sigma'} F \cdot \overrightarrow{n} dS$   
 $\frac{\overrightarrow{n} = (0, 0, 1)}{F \cdot \overrightarrow{n} = -z} - \iint_{\Sigma'} -2dS = 2\operatorname{Area}(\Sigma') = 8\pi$ 

$$V = \frac{R}{3}S.$$

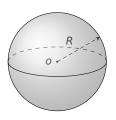


$$V = \frac{R}{3}S.$$



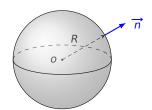
$$V = \iiint_{\Omega} 1 dv$$

$$V = \frac{R}{3}S.$$

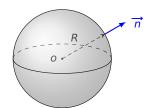


$$V = \iiint_{\Omega} 1 dv = \iiint_{\Omega} \operatorname{div} F dv$$

$$V = \frac{R}{3}S$$
.

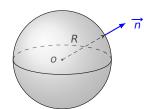


$$V = \frac{R}{3}S$$
.



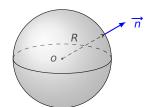
$$V = \iiint_{\Omega} 1 dv \frac{F = (x, y, z)}{\iiint_{\Omega} \text{div} F dv} = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$

$$V = \frac{R}{3}S$$
.



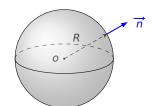
$$V = \iiint_{\Omega} 1 dv \frac{F = (x, y, z)}{\text{div} F = 3}$$
  $\iiint_{\Omega} \text{div} F dv = \overline{\text{S斯公式}}$   $\iint_{\Sigma} F \cdot \overrightarrow{n} dS$ 

$$V = \frac{R}{3}S$$
.



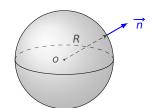
$$V = \iiint_{\Omega} 1 dv \frac{F = (x, y, z)}{\text{div} F = 3} \frac{1}{3} \iiint_{\Omega} \text{div} F dv \frac{\overline{\text{高斯公式}}}{\int_{\Sigma} F \cdot \overrightarrow{n} dS}$$

$$V = \frac{R}{3}S$$
.



$$V = \iiint_{\Omega} 1 dv \frac{F = (x, y, z)}{\text{div}F = 3} \frac{1}{3} \iiint_{\Omega} \text{div}F dv = \frac{\overline{\text{s}} \underline{\text{m}} \underline{\text{s}} \underline{\text{m}}}{3} \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$

$$V = \frac{R}{3}S$$
.



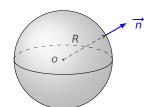
解

$$V = \iiint_{\Omega} 1 dv \frac{F = (x, y, z)}{\text{div}F = 3} \frac{1}{3} \iiint_{\Omega} \text{div}F dv \stackrel{\overline{\text{sh}公式}}{=} \frac{1}{3} \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$

$$\overrightarrow{n} = \frac{1}{R}(x, y, z)$$

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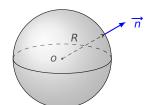
$$V = \frac{R}{3}S$$
.



$$V = \iiint_{\Omega} 1 dv \frac{F = (x, y, z)}{\text{div}F = 3} \frac{1}{3} \iiint_{\Omega} \text{div}F dv \frac{\overline{\text{sin}} \cdot \overline{\text{div}}}{3} \frac{1}{3} \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$

$$\overrightarrow{F \cdot n} = \frac{1}{P}(x^2 + y^2 + z^2)$$

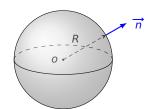
$$V = \frac{R}{3}S.$$



$$V = \iiint_{\Omega} 1 dv \frac{F = (x, y, z)}{\text{div}F = 3} \frac{1}{3} \iiint_{\Omega} \text{div}F dv \frac{\overline{\text{sin}} \cdot \overline{\text{div}}}{3} \frac{1}{3} \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$

$$F \cdot \overrightarrow{n} = \frac{1}{p} (x^2 + y^2 + z^2) = R$$

$$V = \frac{R}{3}S.$$



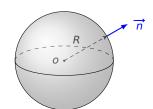
解

$$V = \iiint_{\Omega} 1 dv \frac{F = (x, y, z)}{\text{div}F = 3} \frac{1}{3} \iiint_{\Omega} \text{div}F dv \frac{\overline{\text{sin}} \Delta \Delta}{\overline{\text{div}}F} \frac{1}{3} \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$

$$\frac{\overrightarrow{n} = \frac{1}{R}(x, y, z)}{F \cdot \overrightarrow{n} = \frac{1}{R}(x^2 + y^2 + z^2) = R} \frac{1}{3} \iint_{\Sigma} R dS$$

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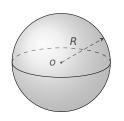
$$V = \frac{R}{3}S.$$



$$V = \iiint_{\Omega} 1 dv \frac{F = (x, y, z)}{\text{div}F = 3} \frac{1}{3} \iiint_{\Omega} \text{div}F dv \xrightarrow{\underline{n} = \frac{1}{R}(x, y, z)} \frac{1}{3} \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$

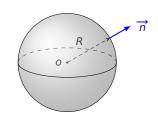
$$\frac{\overrightarrow{n} = \frac{1}{R}(x, y, z)}{\overrightarrow{F \cdot \overrightarrow{n}} = \frac{1}{R}(x^2 + y^2 + z^2) = R} \frac{1}{3} \iint_{\Sigma} R dS = \frac{1}{3} RS$$

$$I = \iint_{\Sigma} (x^2 + y + z) dS$$



$$I = \iint_{\Sigma} (x^2 + y + z) dS$$

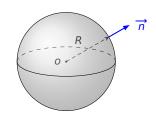
JJΣ 其中曲面 Σ 是球面  $x^2 + y^2 + z^2 = R^2$ 



$$\iint_{\Sigma} (x^2 + y + z) dS$$

$$= \iint_{\Sigma} F \cdot \overrightarrow{n} dS \stackrel{\overline{\text{高斯公式}}}{=====} \iiint_{\Omega} \text{div} F dv$$

$$I = \iint_{\Sigma} (x^2 + y + z) dS$$



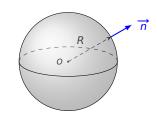
解 球面单位外法向量 
$$\overrightarrow{n} = \frac{1}{R}(x, y, z)$$
,所以 
$$\iint_{\Sigma} (x^2 + y + z) dS$$
 
$$= \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{\text{Sh}} \subseteq \mathbb{Z}} \iiint_{\Omega} \text{div} F dv$$

$$I = \iint_{\Sigma} (x^2 + y + z) dS$$

R

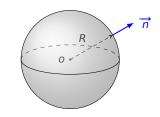
解 球面单位外法向量 
$$\overrightarrow{n}=\frac{1}{R}(x,y,z)$$
,所以 
$$\iint_{\Sigma}(x^2+y+z)dS \qquad ( , , )\cdot\frac{1}{R}(x,y,z)$$
 
$$=\iint_{\Sigma}F\cdot\overrightarrow{n}\,dS \xrightarrow{\overline{\mathrm{sh}}\Delta\Xi}\iiint_{\Omega}\mathrm{div}FdV$$

$$I = \iint_{\Sigma} (x^2 + y + z) dS$$



解 球面单位外法向量 
$$\overrightarrow{n} = \frac{1}{R}(x, y, z),$$
 所以 
$$\iint_{\Sigma} (x^2 + y + z) dS \qquad \qquad R(x, 1, 1) \cdot \frac{1}{R}(x, y, z)$$
 
$$= \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\overline{\text{Sh}} \triangle X} \iiint_{\Omega} \text{div} F dv$$

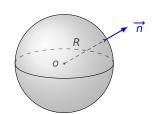
$$I = \iint_{\Sigma} (x^2 + y + z) dS$$



解 球面单位外法向量 
$$\overrightarrow{n} = \frac{1}{R}(x, y, z)$$
,所以 
$$\iint_{\Sigma} (x^2 + y + z) dS = \iint_{\Sigma} R(x, 1, 1) \cdot \frac{1}{R}(x, y, z) dS$$
 
$$= \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{a}\underline{m} \subseteq \mathbb{Z}} \iiint_{\Omega} \mathrm{div} F dv$$

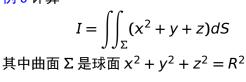
$$I = \iint_{\Sigma} (x^2 + y + z) dS$$

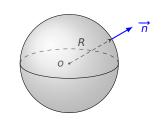
其中曲面  $\Sigma$  是球面  $x^2 + y^2 + z^2 = R^2$ 



解 球面单位外法向量 
$$\overrightarrow{n} = \frac{1}{R}(x, y, z)$$
,所以 
$$\iint_{\Sigma} (x^2 + y + z) dS = \iint_{\Sigma} R(x, 1, 1) \cdot \frac{1}{R}(x, y, z) dS$$
 
$$= \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{n} + \Delta + \Delta} \iiint_{\Omega} \text{div} F dV$$
 
$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} (Rx) + \frac{\partial}{\partial y} (R) + \frac{\partial}{\partial z} (R) \right] dV$$

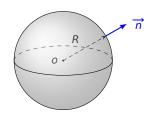
$$I = \iint_{\Sigma} (x^2 + y + z) dS$$





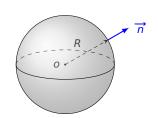
解 球面单位外法向量 
$$\overrightarrow{n} = \frac{1}{R}(x, y, z)$$
,所以 
$$\iint_{\Sigma} (x^2 + y + z) dS = \iint_{\Sigma} R(x, 1, 1) \cdot \frac{1}{R}(x, y, z) dS$$
 
$$= \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{\text{sh}\Delta\Delta\Delta}} \iiint_{\Omega} \text{div} F dV$$
 
$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} (Rx) + \frac{\partial}{\partial y} (R) + \frac{\partial}{\partial z} (R) \right] dV$$
 
$$= \iiint_{\Omega} R dx dy dz$$

$$I = \iint_{\Sigma} (x^2 + y + z) dS$$
  
其中曲面  $\Sigma$  是球面  $x^2 + y^2 + z^2 = R^2$ 



解 球面单位外法向量 
$$\overrightarrow{n} = \frac{1}{R}(x, y, z)$$
,所以 
$$\iint_{\Sigma} (x^2 + y + z) dS = \iint_{\Sigma} R(x, 1, 1) \cdot \frac{1}{R}(x, y, z) dS$$
 
$$= \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{a} \underline{n} \underline{n} \underline{n}} \iiint_{\Omega} \mathrm{div} F dv$$
 
$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} (Rx) + \frac{\partial}{\partial y} (R) + \frac{\partial}{\partial z} (R) \right] dv$$
 
$$= \iiint_{\Omega} R dx dy dz = R \mathrm{Vol}(\Omega)$$

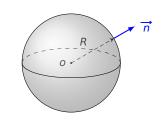
$$I = \iint_{\Sigma} (x^2 + y + z) dS$$
  
其中曲面  $\Sigma$  是球面  $x^2 + y^2 + z^2 = R^2$ 



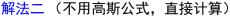
解 球面单位外法向量 
$$\overrightarrow{n} = \frac{1}{R}(x, y, z)$$
,所以 
$$\iint_{\Sigma} (x^2 + y + z) dS = \iint_{\Sigma} R(x, 1, 1) \cdot \frac{1}{R}(x, y, z) dS$$
$$= \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{n} + \underline{n} + \underline{n}} \iiint_{\Omega} \operatorname{div} F dV$$
$$= \iiint_{\Omega} \left[ \frac{\partial}{\partial x} (Rx) + \frac{\partial}{\partial y} (R) + \frac{\partial}{\partial z} (R) \right] dV$$
$$= \iiint_{\Omega} R dx dy dz = R \operatorname{Vol}(\Omega) = \frac{4}{3} \pi R^4$$

$$I = \iint_{\Sigma} (x^2 + y + z) dS$$

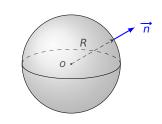
其中曲面 Σ 是球面  $x^2 + y^2 + z^2 = R^2$ 



$$I = \iint_{\Sigma} (x^2 + y + z) dS$$

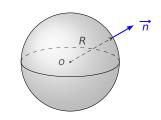


$$\iint_{\Sigma} (x^2 + y + z) dS \xrightarrow{\text{symptot}} \iint_{\Sigma} x^2 dS$$



$$I = \iint_{\Sigma} (x^2 + y + z) dS$$

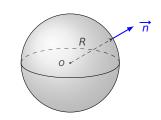
其中曲面 Σ 是球面  $x^2 + y^2 + z^2 = R^2$ 



$$\iint_{\Sigma} (x^2 + y + z) dS \xrightarrow{\text{span}} \iint_{\Sigma} x^2 dS = \frac{1}{3} \iint_{\Sigma} (x^2 + x^2 + x^2) dS$$

$$I = \iint_{\Sigma} (x^2 + y + z) dS$$

其中曲面 Σ 是球面  $x^2 + y^2 + z^2 = R^2$ 

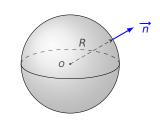


$$\iint_{\Sigma} (x^2 + y + z) dS \xrightarrow{\text{span}} \iint_{\Sigma} x^2 dS = \frac{1}{3} \iint_{\Sigma} (x^2 + x^2 + x^2) dS$$

$$\xrightarrow{\text{span}} \frac{1}{3} \iint_{\Sigma} (x^2 + y^2 + z^2) dS$$

$$I = \iint_{\Sigma} (x^2 + y + z) dS$$

其中曲面 Σ 是球面  $x^2 + y^2 + z^2 = R^2$ 



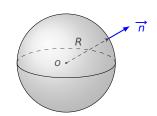
$$\iint_{\Sigma} (x^2 + y + z) dS \xrightarrow{\text{spinite}} \iint_{\Sigma} x^2 dS = \frac{1}{3} \iint_{\Sigma} (x^2 + x^2 + x^2) dS$$

$$\xrightarrow{\text{spinite}} \frac{1}{3} \iint_{\Sigma} (x^2 + y^2 + z^2) dS$$

$$= \frac{1}{3} \iint_{\Sigma} R^2 dS$$

$$I = \iint_{\Sigma} (x^2 + y + z) dS$$

其中曲面 Σ 是球面  $x^2 + y^2 + z^2 = R^2$ 



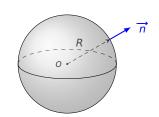
$$\iint_{\Sigma} (x^2 + y + z) dS \xrightarrow{\text{print}} \iint_{\Sigma} x^2 dS = \frac{1}{3} \iint_{\Sigma} (x^2 + x^2 + x^2) dS$$

$$\xrightarrow{\text{print}} \frac{1}{3} \iint_{\Sigma} (x^2 + y^2 + z^2) dS$$

$$= \frac{1}{3} \iint_{\Sigma} R^2 dS = \frac{1}{3} R^2 \text{Area}(\Sigma)$$

$$I = \iint_{\Sigma} (x^2 + y + z) dS$$

其中曲面 Σ 是球面  $x^2 + y^2 + z^2 = R^2$ 



$$\iint_{\Sigma} (x^2 + y + z) dS \xrightarrow{\text{print}} \iint_{\Sigma} x^2 dS = \frac{1}{3} \iint_{\Sigma} (x^2 + x^2 + x^2) dS$$

$$\xrightarrow{\text{print}} \frac{1}{3} \iint_{\Sigma} (x^2 + y^2 + z^2) dS$$

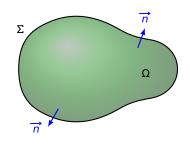
$$= \frac{1}{3} \iint_{\Sigma} R^2 dS = \frac{1}{3} R^2 \text{Area}(\Sigma) = \frac{4}{3} \pi R^4$$



高斯公式 
$$\iiint_{\Omega} \operatorname{div} F dv = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$



• 假设 F = (P, Q, R) 是流体的速度向量场,



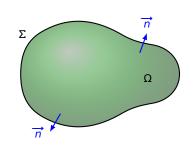
高斯公式  $\iiint_{\Omega} \operatorname{div} F dv = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$ 



假设 F = (P, Q, R) 是流体的速度向量场,则

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS$$

表示单位时间流向  $\Sigma$  外侧的通量。



高斯公式 
$$\iiint_{\Omega} \operatorname{div} F dv = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$



 假设 F = (P, Q, R) 是流体的速度向 量场,则

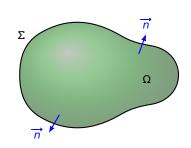
$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS$$

表示单位时间流向Σ外侧的通量。

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS > 0$$

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS < 0$$

高斯公式 
$$\iiint_{\Omega} \operatorname{div} F dv = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$



 假设 F = (P, Q, R) 是流体的速度向 量场,则

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS$$

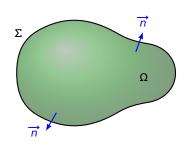
表示单位时间流向Σ外侧的通量。

• 进一步假设流体是不可压,则

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS > 0$$

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS < 0$$

高斯公式 
$$\iiint_{\Omega} \operatorname{div} F dv = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$



 假设 F = (P, Q, R) 是流体的速度向 量场,则

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS$$

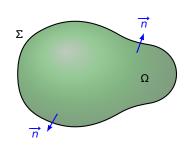
表示单位时间流向  $\Sigma$  外侧的通量。

• 进一步假设流体是不可压,则

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS > 0 \Rightarrow \Omega \, \text{内有 "source"}$$

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS < 0$$

高斯公式 
$$\iiint_{\Omega} \operatorname{div} F dv = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$



 假设 F = (P, Q, R) 是流体的速度向 量场,则

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS$$

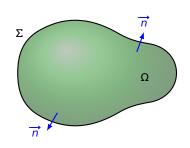
表示单位时间流向  $\Sigma$  外侧的通量。

• 进一步假设流体是不可压,则

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS > 0 \Rightarrow \Omega \, \text{内有 "source"}$$

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS < 0 \Rightarrow \Omega \, \text{内有 "sink"}$$

高斯公式 
$$\iiint_{\Omega} \operatorname{div} F dv = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$



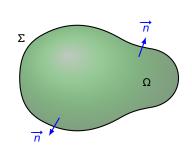
 假设 F = (P, Q, R) 是流体的速度向 量场,则

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS$$

表示单位时间流向Σ外侧的通量。

• 进一步假设流体是不可压,则

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS > 0 \Rightarrow \Omega \, \text{内有 "source"}$$
 
$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS < 0 \Rightarrow \Omega \, \text{内有 "sink"}$$



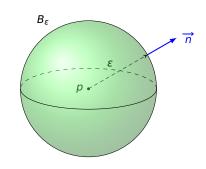
 $\mathbf{\hat{z}}$  高斯公式  $\iiint_{\Omega} \operatorname{div} F dv = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$  表明: $\operatorname{div} F$  反映这种"source"

和"sink"的强度。



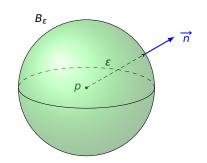
p •





$$\iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \iiint_{B_{\varepsilon}} \operatorname{div} F \, dV$$





$$\iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$\iiint_{B_{\varepsilon}} \operatorname{div} F \, dV$$

$$= \operatorname{Vol}(B_{\varepsilon}) \operatorname{div} F(p_{\varepsilon})$$

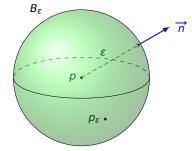




$$\frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \iiint_{B_{\varepsilon}} \text{div} F \, dV$$

$$= \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \text{Vol}(B_{\varepsilon}) \text{div} F(p_{\varepsilon})$$



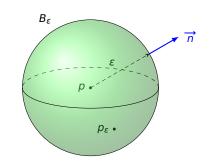
$$\frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \iiint_{B_{\varepsilon}} \operatorname{div} F \, dv$$

$$= \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \operatorname{Vol}(B_{\varepsilon}) \operatorname{div} F(p_{\varepsilon})$$

$$= \operatorname{div} F(p_{\varepsilon})$$

$$\operatorname{div} F(p)$$



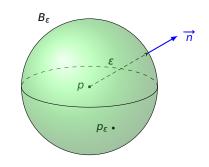
$$\lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \iiint_{B_{\varepsilon}} \text{div} F \, dv$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \text{Vol}(B_{\varepsilon}) \text{div} F(p_{\varepsilon})$$

$$= \lim_{\varepsilon \to 0^{+}} \text{div} F(p_{\varepsilon})$$

$$\text{div} F(p)$$



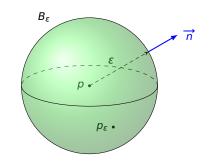
$$\lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \iiint_{B_{\varepsilon}} \text{div} F \, dV$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \text{Vol}(B_{\varepsilon}) \text{div} F(p_{\varepsilon})$$

$$= \lim_{\varepsilon \to 0^{+}} \text{div} F(p_{\varepsilon})$$

$$= \text{div} F(p)$$



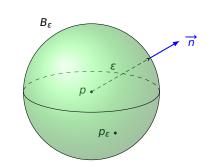
$$\lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \iiint_{B_{\varepsilon}} \text{div} F \, dv$$

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$$= \lim_{\varepsilon \to 0^{+}} \text{div} F(p_{\varepsilon})$$

$$= \text{div} F(p)$$



- divF(p)>0时,
- divF(p)<0时,

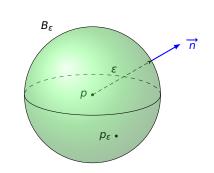
$$\lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

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$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \text{Vol}(B_{\varepsilon}) \text{div} F(p_{\varepsilon})$$

$$= \lim_{\varepsilon \to 0^{+}} \text{div} F(p_{\varepsilon})$$

$$= \text{div} F(p)$$



- div*F*(*p*)>0 时,∫∫<sub>∂B</sub>, *F* · *n* dS >0(ε 充分小),
- divF(p)<0时,



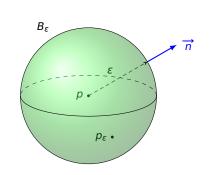
$$\lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \iiint_{B_{\varepsilon}} \text{div} F \, dv$$

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$$= \lim_{\varepsilon \to 0^{+}} \text{div} F(p_{\varepsilon})$$

$$= \text{div} F(p)$$



- $\operatorname{div} F(p) > 0$  时,  $\iint_{\partial B_{\epsilon}} F \cdot \overrightarrow{n} \, dS > 0$  ( $\epsilon$  充分小),说明 p 点是 source
- divF(p)<0时,



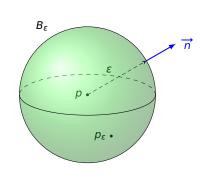
$$\lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \iiint_{B_{\varepsilon}} \text{div} F \, dv$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \text{Vol}(B_{\varepsilon}) \text{div} F(p_{\varepsilon})$$

$$= \lim_{\varepsilon \to 0^{+}} \text{div} F(p_{\varepsilon})$$

$$= \text{div} F(p)$$



- $\operatorname{div} F(p) > 0$  时, $\iint_{\partial B_{\epsilon}} F \cdot \overrightarrow{n} \, dS > 0$ ( $\epsilon$  充分小),说明 p 点是 source
- $\operatorname{div} F(p) < 0$  时,  $\iint_{\partial B_{\epsilon}} F \cdot \overrightarrow{n} \, dS < 0 \ (\epsilon 充分小)$ ,



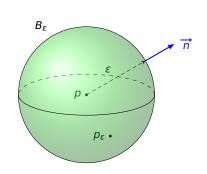
$$\lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \iiint_{B_{\varepsilon}} \text{div} F \, dv$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \text{Vol}(B_{\varepsilon}) \text{div} F(p_{\varepsilon})$$

$$= \lim_{\varepsilon \to 0^{+}} \text{div} F(p_{\varepsilon})$$

$$= \text{div} F(p)$$



- $\operatorname{div} F(p) > 0$  时, $\iint_{\partial B_{\epsilon}} F \cdot \overrightarrow{n} \, dS > 0$ ( $\epsilon$  充分小),说明 p 点是 source
- $\operatorname{div} F(p) < 0$  时,  $\iint_{\partial B_{\epsilon}} F \cdot \overrightarrow{n} \, dS < 0$  ( $\epsilon$  充分小),说明 p 点是  $\operatorname{sink}$



#### We are here now...

1. 高斯公式

2. 斯托克斯公式

定义 设 
$$F = (P, Q, R)$$
 是空间中向量场,定义

$$\operatorname{rot} F := \left| \begin{array}{ccc} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{array} \right|$$

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$$\operatorname{rot} F := \left| \begin{array}{ccc} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & O & R \end{array} \right| = \left( \left| \begin{array}{ccc} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{array} \right|, - \left| \begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{array} \right|,$$



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定义 设 
$$F = (P, Q, R)$$
 是空间中向量场,定义

$$\operatorname{rot} F := \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left( \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & Q \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} \right)$$

$$= (R_{y} - Q_{z}, , )$$



定义 设 
$$F = (P, Q, R)$$
 是空间中向量场,定义

$$\operatorname{rot} F := \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left( \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} \right)$$
$$= (R_{y} - Q_{z}, P_{z} - R_{x}, )$$



定义 设 F = (P, Q, R) 是空间中向量场,定义

$$\operatorname{rot} F := \left| \begin{array}{ccc} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{array} \right| = \left( \left| \begin{array}{ccc} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{array} \right|, - \left| \begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{array} \right|, \left| \begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{array} \right| \right)$$

$$= (R_y - Q_z, P_z - R_x, Q_x - P_y)$$



定义 设 F = (P, Q, R) 是空间中向量场,定义

$$\operatorname{rot} F := \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left( \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & Q \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} \right)$$
$$= (R_{V} - Q_{Z}, P_{Z} - R_{X}, Q_{X} - P_{Y})$$

称为向量场 F 的旋度。

例 计算向量场  $F = (y, -x, e^{xz})$  的旋度。



定义 设 F = (P, Q, R) 是空间中向量场,定义

$$\operatorname{rot} F := \left| \begin{array}{ccc} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{array} \right| = \left( \left| \begin{array}{ccc} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{array} \right|, - \left| \begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{array} \right|, \left| \begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{array} \right| \right)$$

 $= (R_{V} - Q_{Z}, P_{Z} - R_{X}, Q_{X} - P_{V})$ 

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例 计算向量场  $F = (v, -x, e^{xz})$  的旋度。

$$\operatorname{rot} F = \left| \begin{array}{ccc} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & e^{xz} \end{array} \right|$$

定义 设 F = (P, Q, R) 是空间中向量场, 定义

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\overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\
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y & -x & e^{xz}
\end{vmatrix} = \left(\begin{vmatrix}
\frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
-x & e^{xz}
\end{vmatrix}, -\begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\
y & e^{xz}
\end{vmatrix}, \begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\
y & -x
\end{vmatrix}\right)$$



定义 设 F = (P, Q, R) 是空间中向量场, 定义

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\frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\
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$$= (0, -ze^{xz}, )$$



定义 设 
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称为向量场 F 的旋度。

例 计算向量场 
$$F = (v, -x, e^{xz})$$
 的旋度。

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y & -x & e^{XZ}
\end{vmatrix} = \left(\begin{vmatrix}
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-x & e^{XZ}
\end{vmatrix}, -\begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\
y & e^{XZ}
\end{vmatrix}, \begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\
y & -x
\end{vmatrix}\right)$$

$$= (0, -ze^{xz}, -2)$$

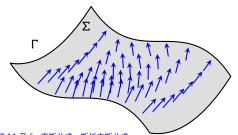
 $= (R_{v} - Q_{z}, P_{z} - R_{x}, Q_{x} - P_{v})$ 



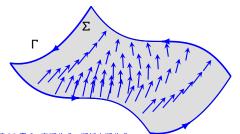
#### 斯托克斯公式

#### 定理(斯托克斯公式) 假设

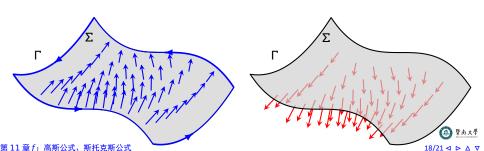
- $\Sigma$  是空间中分片光滑的定向曲面,选定单位法向量场  $\overrightarrow{n}$ ,
- Γ是Σ的边界, 且赋予"边界定向",



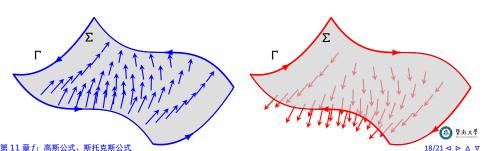
- $\Sigma$  是空间中分片光滑的定向曲面,选定单位法向量场  $\overrightarrow{n}$ ,
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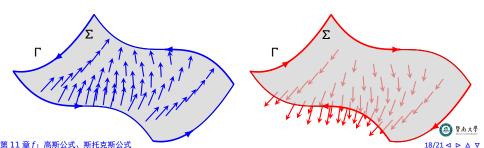
- $\Sigma$  是空间中分片光滑的定向曲面,选定单位法向量场  $\overrightarrow{n}$  ,
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- $\Sigma$  是空间中分片光滑的定向曲面,选定单位法向量场  $\overrightarrow{n}$ ,
- $\Gamma$ 是  $\Sigma$  的边界, 且赋予 "边界定向",
- F = (P, Q, R) 是空间向量场,且 P, Q, R 具有一阶连续偏导数,

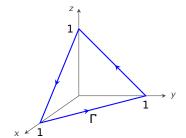


### 定理(斯托克斯公式) 假设

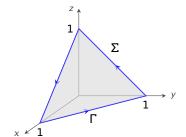
- $\Sigma$  是空间中分片光滑的定向曲面,选定单位法向量场  $\overrightarrow{n}$  ,
- $\Gamma$ 是  $\Sigma$  的边界, 且赋予"边界定向",
- F = (P, Q, R) 是空间向量场,且 P, Q, R 具有一阶连续偏导数,

则成立:  $\iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} \, dS = \int_{\Gamma} P dx + Q dy + R dz.$ 

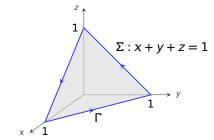
$$I = \int_{\Gamma} z dx + x dy + y dz$$



$$I = \int_{\Gamma} z dx + x dy + y dz$$

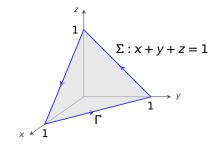


$$I = \int_{\Gamma} z dx + x dy + y dz$$



$$I = \int_{\Gamma} z dx + x dy + y dz$$

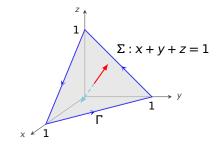
解设
$$F = (z, x, y)$$
,则



所以
$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} dS$$

$$I = \int_{\Gamma} z dx + x dy + y dz$$

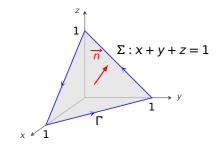
解设
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所以
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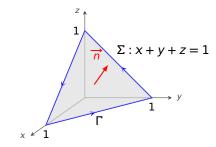
解设
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所以
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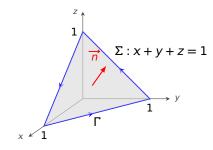
解设
$$F=(z,x,y)$$
,则



所以
$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)}$$

$$I = \int_{\Gamma} z dx + x dy + y dz$$

解设
$$F = (z, x, y)$$
,则 
$$rot F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix}$$



所以
$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)}$$

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$$\operatorname{rot} F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \left( \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ z & y \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ z & x \end{vmatrix} \right)$$

$$\Sigma: x + y + z = 1$$

$$\begin{bmatrix} \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} \end{bmatrix}, \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} \end{bmatrix}, \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} \end{bmatrix}$$

所以
$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)}$$

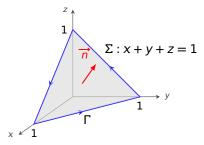
$$I = \int_{\Gamma} z dx + x dy + y dz$$

解设
$$F = (z, x, y)$$
,则

$$\operatorname{rot} F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \left( \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ z & y \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ z & x \end{vmatrix} \right)$$

$$=($$

$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Gamma} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1, 1, 1)}$$



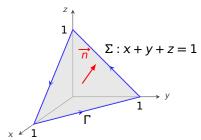
$$I = \int_{\Gamma} z dx + x dy + y dz$$

解设
$$F=(z,x,y)$$
,则

$$\operatorname{rot} F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \left( \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ z & y \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ z & x \end{vmatrix} \right)$$

$$= \left( \left| \begin{array}{cc} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y \end{array} \right| \right.$$

$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Gamma} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)}$$



$$I = \int_{\Gamma} z dx + x dy + y dz$$

解设
$$F=(z,x,y)$$
,则

$$\operatorname{rot} F = \begin{bmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \end{array}$$

$$\operatorname{rot} F = \begin{pmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{pmatrix} = \begin{pmatrix} \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ z & y \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ z & x \end{vmatrix} \end{pmatrix}$$

$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Gamma} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)}$$

$$I = \int_{\Gamma} z dx + x dy + y dz$$

解设
$$F = (z, x, y)$$
,则

$$\operatorname{rot} F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \left( \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ z & y \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ z & x \end{vmatrix} \right)$$

$$=\left(\left|\begin{array}{c} \overline{d} \\ \overline{d} \end{array}\right|\right)$$

$$\Sigma: x + y + z = 1$$

$$\begin{bmatrix} x & y & y \\ y & y \\ y & y \end{bmatrix}$$

$$=(1, 1, 1)$$

所以
$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)} \iint_{\Sigma} \sqrt{3} dS$$

$$I = \int_{\Gamma} z dx + x dy + y dz$$

$$\mathbf{H}$$
 设  $F = (z, x, y)$ , 则

$$\operatorname{rot} F = \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \vec{z} & \vec{x} & \vec{y} \end{pmatrix} = \begin{pmatrix} \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \vec{x} & \vec{y} \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ \vec{z} & \vec{y} \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \vec{z} & \vec{x} \end{vmatrix} \end{pmatrix}$$

$$=(1, 1, 1)$$

所以
$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)} \iint_{\Sigma} \sqrt{3} dS$$

$$=\sqrt{3}$$
Area( $\Sigma$ )



$$I = \int_{\Gamma} z dx + x dy + y dz$$

其中有向曲线 Γ如图:

解设
$$F = (z, x, y)$$
,则

Figure 1. The proof of the following form of 
$$F = \begin{pmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{pmatrix} = \begin{pmatrix} \begin{vmatrix} \overrightarrow{\partial} & \overrightarrow{\partial} & \overrightarrow{\partial} \\ \overrightarrow{\partial y} & \frac{\partial}{\partial z} \\ x & y \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ z & y \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ z & x \end{vmatrix} \end{pmatrix}$$

$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)} \iint_{\Sigma} \sqrt{3} dS$$

=(1, 1, 1)

$$= \sqrt{3} \operatorname{Area}(\Sigma) = \sqrt{3} \cdot \frac{1}{2} \cdot \sqrt{2} \cdot \sqrt{2} \cdot \sin \frac{\pi}{3}$$



$$I = \int_{\Gamma} z dx + x dy + y dz$$

其中有向曲线 Γ如图:

解设
$$F = (z, x, y)$$
,则

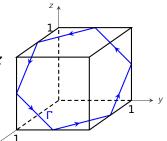
Figure 1. The proof of the following form of 
$$F = \begin{pmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{pmatrix} = \begin{pmatrix} \begin{vmatrix} \overrightarrow{\partial} & \overrightarrow{\partial} & \overrightarrow{\partial} \\ \overrightarrow{\partial y} & \frac{\partial}{\partial z} \\ x & y \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ z & y \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ z & x \end{vmatrix} \end{pmatrix}$$

$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)} \iint_{\Sigma} \sqrt{3} dS$$

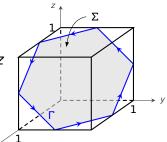
=(1, 1, 1)

$$= \sqrt{3} \text{Area}(\Sigma) = \sqrt{3} \cdot \frac{1}{2} \cdot \sqrt{2} \cdot \sqrt{2} \cdot \sin \frac{\pi}{3} = \frac{3}{2}$$

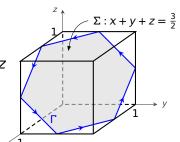
$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$



$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$



$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$



$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$

其中有向曲线 Γ 如图:

解设
$$F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$
,则

所以
$$I = \iint_{\mathbb{R}^n} \operatorname{rot} F \cdot \overrightarrow{n} \, dS$$



$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$

其中有向曲线 Γ 如图:

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$$I = \iint_{\Gamma} \operatorname{rot} F \cdot \overrightarrow{n} \, dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)}$$



$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$

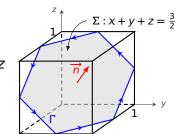
其中有向曲线 Γ如图:

解设
$$F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$
,则

$$\operatorname{rot} F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix}$$

所以

$$I = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)}$$



$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$

其中有向曲线 Γ如图:

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$$F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$
,则
$$rot F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix} = (-2y - 2z, \qquad , \qquad )$$

所以

$$I = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} \, dS \stackrel{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)}{=}$$



 $\Sigma : x + y + z = \frac{3}{2}$ 

$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$

其中有向曲线 Γ如图:

解设
$$F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$
,则
$$rot F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix} = (-2y - 2z, -2z - 2x,$$

所以

$$I = \iint_{\Gamma} \operatorname{rot} F \cdot \overrightarrow{n} \, dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)}$$



$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$

其中有向曲线 Γ如图:

解设
$$F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$
, 则
$$rot F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix} = (-2y - 2z, -2z - 2x, -2x - 2y)$$

所以

$$I = \iint_{\Gamma} \operatorname{rot} F \cdot \overrightarrow{n} \, dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)}$$



$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$

其中有向曲线 Γ 如图:

解设 
$$F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$
, 则
$$rot F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix} = (-2y - 2z, -2z - 2x, -2x - 2y)$$

所以

$$I = \iint_{\Gamma} \operatorname{rot} F \cdot \overrightarrow{n} \, dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)} \frac{-4}{\sqrt{3}} \iint_{\Gamma} (x+y+z) dS$$



$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$

其中有向曲线 Γ如图:

解设 
$$F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$
, 则
$$rot F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix} = (-2y - 2z, -2z - 2x, -2x - 2y)$$

所以

$$I = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} \, dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1, 1, 1)} \frac{-4}{\sqrt{3}} \iint_{\Sigma} (x + y + z) dS = \frac{-4}{\sqrt{3}} \iint_{\Sigma} \frac{3}{2} dS$$



 $I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$ 

解设
$$F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$
,则

解设
$$F = (y^2 - z^2, z^2)$$

解设
$$F = (y^2 - z^2, z^2)$$

$$\mathbf{H} \oplus F = (y^2 - z^2, z^2)$$

$$\mathfrak{C} F = (y^2 - z^2, z^2 - x)$$

$$\overrightarrow{i}$$
  $\overrightarrow{i}$   $\overrightarrow{k}$ 

$$\operatorname{rot} F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix} = (-2y - 2z, -2z - 2x, -2x - 2y)$$

$$\frac{\partial}{\partial y}$$
  $\frac{\partial}{\partial z}$ 

$$\frac{\partial}{\partial y}$$
  $\frac{\partial}{\partial z}$   $-x^2$   $x^2$   $-y$ 

$$-x^{2} x^{2} - x^{2} - y$$

所以
$$I = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} \, dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)} \frac{-4}{\sqrt{3}} \iint_{\Sigma} (x+y+z) \, dS = \frac{-4}{\sqrt{3}} \iint_{\Sigma} \frac{3}{2} \, dS$$

$$-x^2 x^2 - y$$

$$x^2$$
  $x^2 - y^2$ 

$$| = (-2y - y^2) |$$

$$=-2\sqrt{3}$$
Area $(\Sigma)$ 

例 2 试利用斯托克斯公式计算
$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$

解设
$$F = (v^2 - z^2)$$

解设
$$F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$
,则

$$\begin{array}{cccc}
& & & \downarrow & \downarrow & \downarrow \\
& & & & \downarrow & \downarrow & \downarrow \\
& & & & & \downarrow & \downarrow & \downarrow
\end{array}$$

$$\overrightarrow{j}$$
  $\overrightarrow{k}$ 

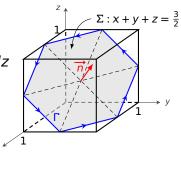
$$\operatorname{rot} F = \left| \begin{array}{c} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} \end{array} \right|$$

$$\operatorname{rot} F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix} = (-2y - 2z, -2z - 2x, -2x - 2y)$$

|y²-z² z²-x² x²-y²|
所以
$$I = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} \, dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)} \frac{-4}{\sqrt{3}} \iint_{\Sigma} (x+y+z) dS = \frac{-4}{\sqrt{3}} \iint_{\Sigma} \frac{3}{2} dS$$

$$JJ\Sigma$$

$$=-2\sqrt{3}$$
Area( $\Sigma$ )





例 2 试利用斯托克斯公式计算
$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$

其中有向曲线 Γ如图:

$$\mathbf{w}$$
 设  $F = (v^2 - z^2)$ 

解设
$$F = (y^2 - z^2, z^2)$$

解设
$$F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$
,则

$$\overrightarrow{y} - \overrightarrow{z}, \overrightarrow{z} - \overrightarrow{x}, \overrightarrow{x} - \overrightarrow{z}$$

$$\overrightarrow{j}$$
  $\overrightarrow{k}$ 

解设
$$F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$
,则
$$rot F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix} = (-2y - 2z, -2z - 2x, -2x - 2y)$$
所以

$$y^2 - z^2 z^2 - x$$
  
所以

$$\operatorname{rot} F = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ y^2 - z^2 & z^2 - x^2 \end{vmatrix}$$

$$x^2 - y^2$$

$$2 - y^2$$

$$-2z-2x, -2x-$$

$$\frac{1}{3} \int_{\Sigma} \frac{1}{2}$$

 $\Sigma : x + y + z = \frac{3}{2}$ 

$$I = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} \, dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)} \frac{-4}{\sqrt{3}} \iint_{\Sigma} (x+y+z) dS = \frac{-4}{\sqrt{3}} \iint_{\Sigma} \frac{3}{2} dS$$
$$= -2\sqrt{3} \operatorname{Area}(\Sigma) = -2\sqrt{3} \cdot 6 \cdot \frac{1}{2} \cdot \sqrt{\frac{1}{2} \cdot \sqrt{\frac{1}{2}} \cdot \sin \frac{\pi}{3}}$$

例 2 试利用斯托克斯公式计算
$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$

其中有向曲线 Γ如图:

解设
$$F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$
,则

解设
$$F = (y^2 - z^2, z^2 - x^2, x^2 - y^2), 则$$

$$\begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \end{vmatrix}$$

$$F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} = (-2y - 1)$$

$$y^2 -$$

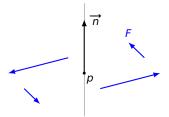
$$\operatorname{rot} F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix} = (-2y - 2z, -2z - 2x, -2x - 2y)$$

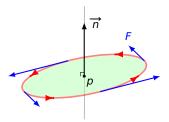
$$I = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} \, dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)} \frac{-4}{\sqrt{3}} \iint_{\Sigma} (x+y+z) dS = \frac{-4}{\sqrt{3}} \iint_{\Sigma} \frac{3}{2} dS$$
$$= -2\sqrt{3}\operatorname{Area}(\Sigma) = -2\sqrt{3} \cdot 6 \cdot \frac{1}{2} \cdot \sqrt{\frac{1}{2} \cdot \sin \frac{\pi}{3}} = -\frac{9}{2}$$

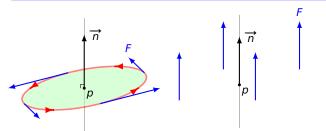


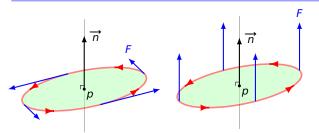
 $\Sigma : x + y + z = \frac{3}{2}$ 

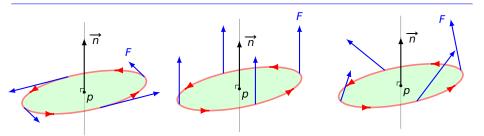


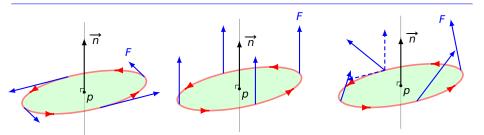


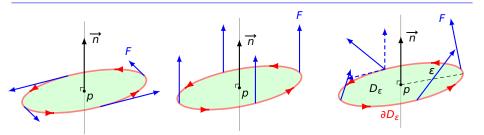


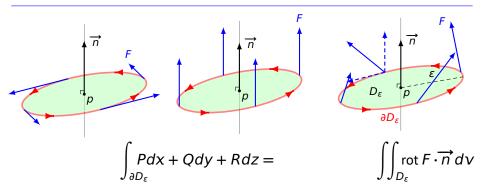


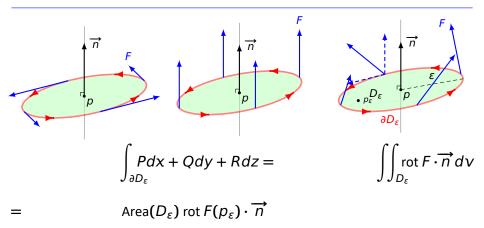


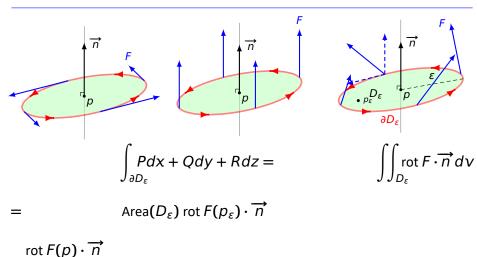




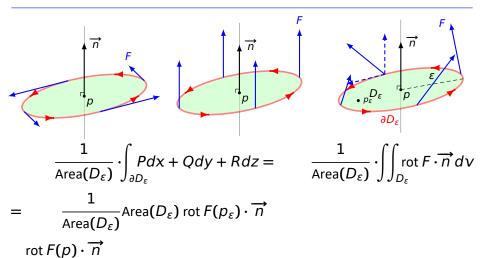




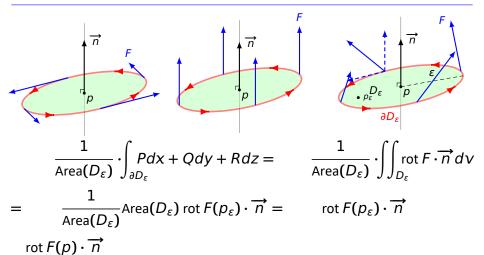




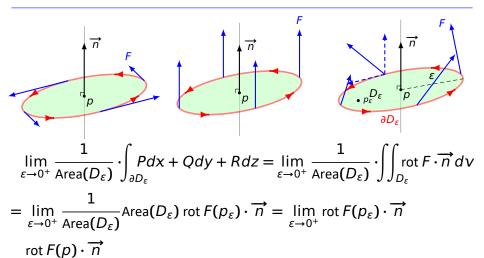




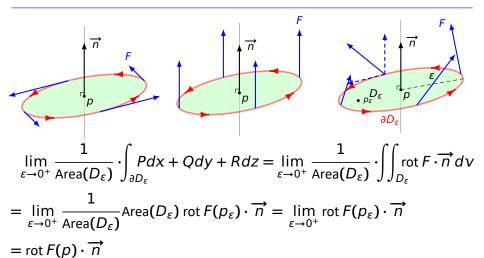




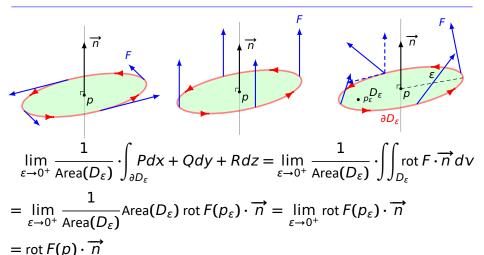






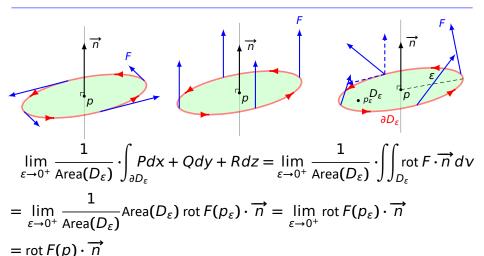






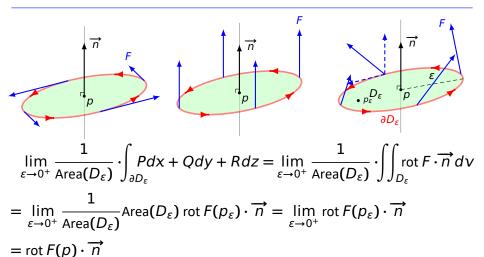
注 cot  $F \neq 0$  说明有旋,





注  $\cot F \neq 0$  说明有旋,此时可认为 F 在 p 点附近绕轴  $\overrightarrow{n} = \frac{\cot F}{|\cot F|}$  旋转;





注  $\cot F \neq 0$  说明有旋,此时可认为  $F \propto p$  点附近绕轴  $\overrightarrow{n} = \frac{\cot F}{|\cot F|}$  旋转;

 $\cot F = 0$  说明无旋。

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