

第 9 章 f : 多元函数微分学的几何应用

数学系 梁卓滨

2016-2017 学年 II

Outline

1. 曲线的切线、法平面

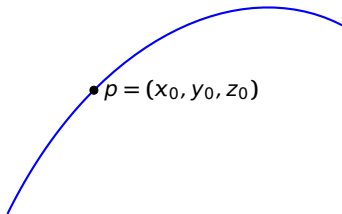
2. 曲面的切平面、法线

We are here now...

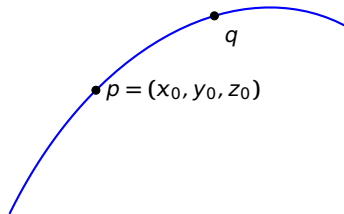
1. 曲线的切线、法平面

2. 曲面的切平面、法线

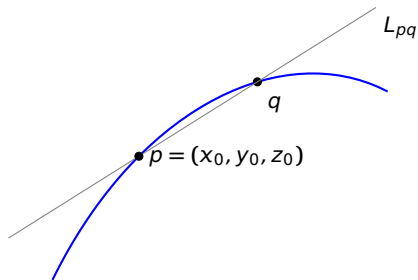
曲线的切线方程、法平面方程



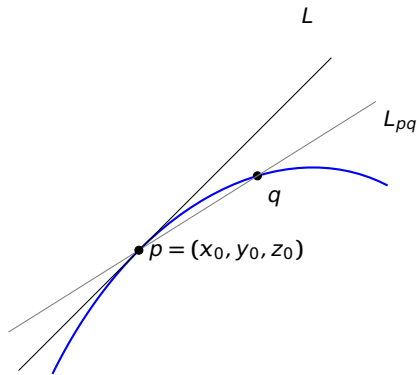
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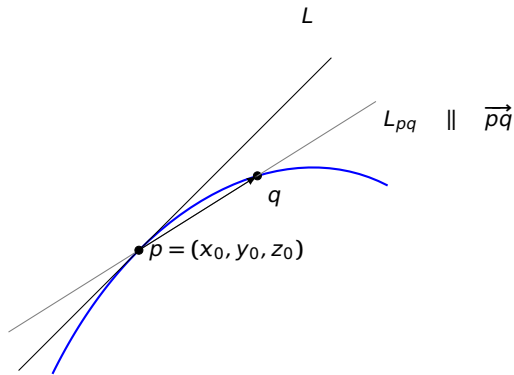
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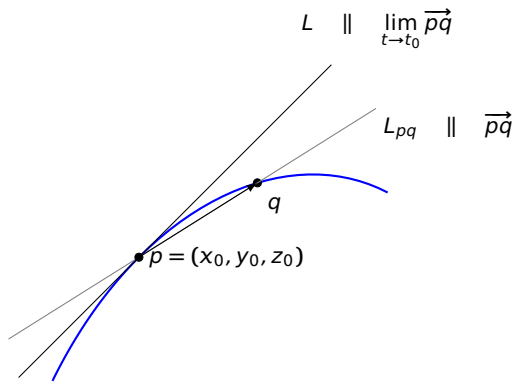
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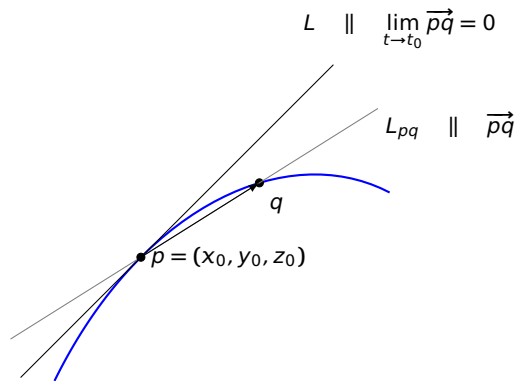
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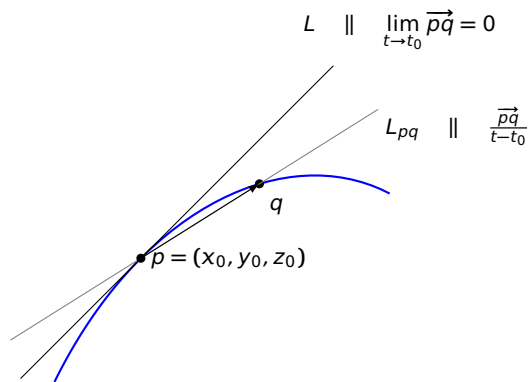
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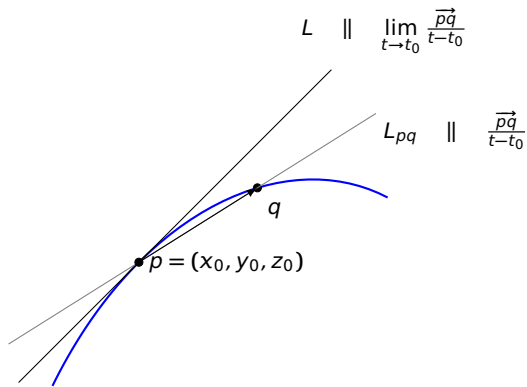
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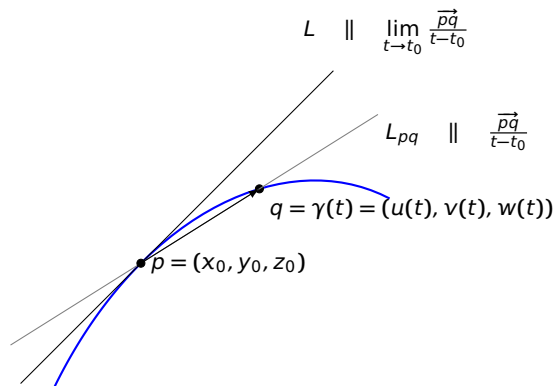
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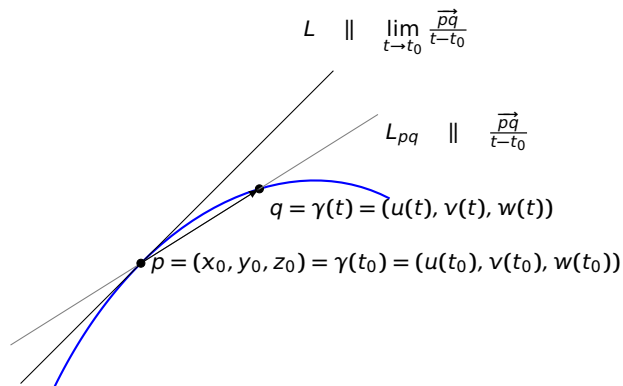
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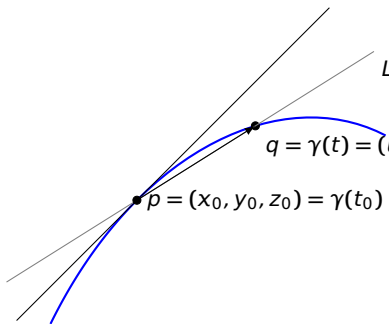
曲线的切线方程、法平面方程

$$L \parallel \lim_{t \rightarrow t_0} \frac{\vec{pq}}{t-t_0}$$

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$$q = \gamma(t) = (u(t), v(t), w(t))$$

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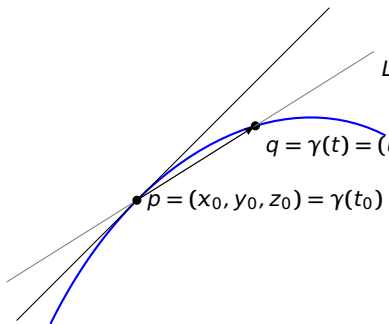
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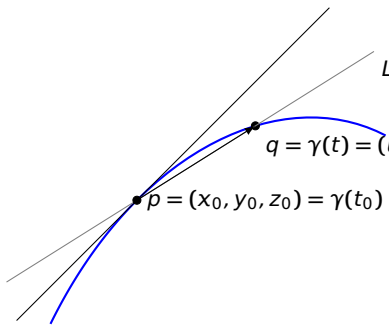
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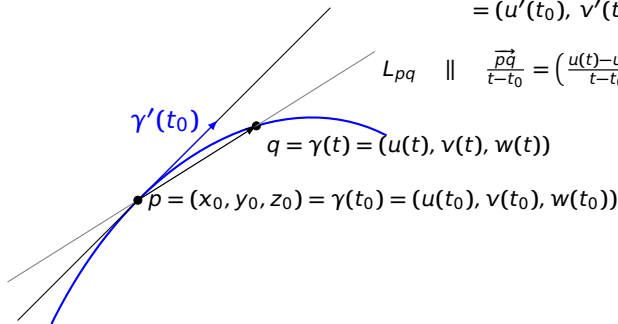
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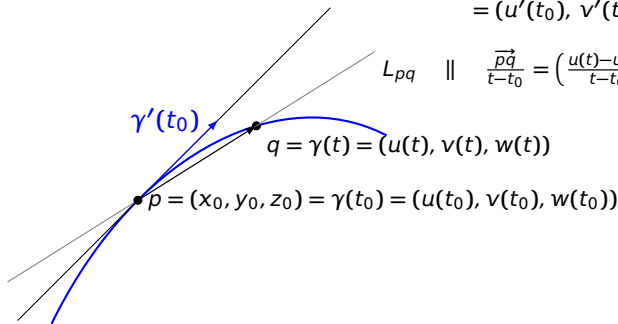
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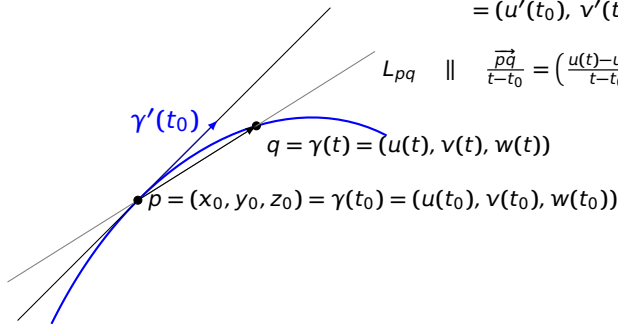


- 曲线的切线方程

曲线的切线方程、法平面方程

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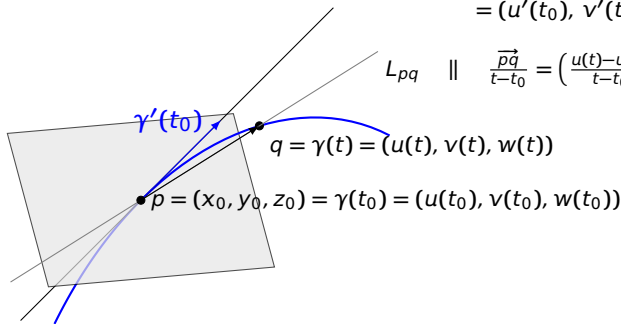
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$$\frac{x-x_0}{u'(t_0)} = \frac{y-y_0}{v'(t_0)} = \frac{z-z_0}{w'(t_0)}$$

曲线的切线方程、法平面方程

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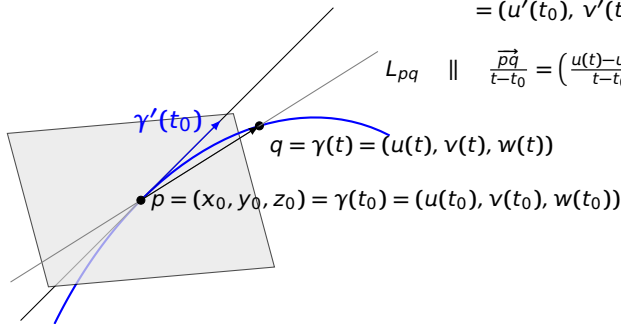
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- 曲线的切线方程

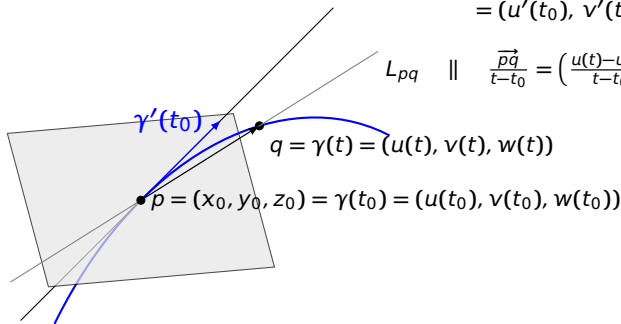
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- 曲线的法平面方程

$$u'(t_0)(x-x_0) + v'(t_0)(y-y_0) + w'(t_0)(z-z_0) = 0$$

例 求曲线 $\gamma(t) = (t, t^2, t^3)$ 在点 $(1, 1, 1)$ ($t = 1$) 处的切线及法平面的方程。

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- 曲线的法平面方程

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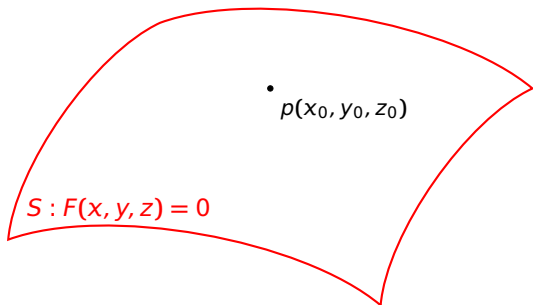
$$1 \cdot (x-1) + 2 \cdot (y-1) + 3 \cdot (z-1) = 0 \quad \Rightarrow \quad x + 2y + 3z - 6 = 0$$

We are here now...

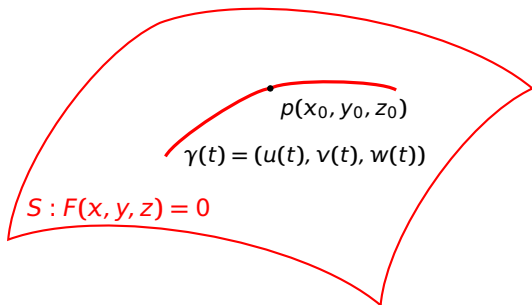
1. 曲线的切线、法平面

2. 曲面的切平面、法线

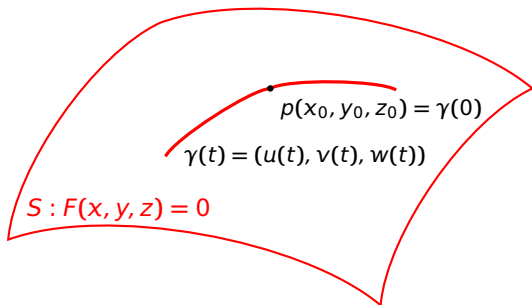
切平面



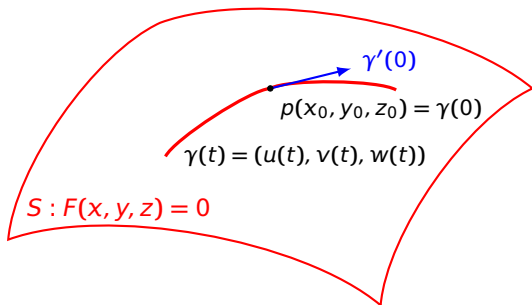
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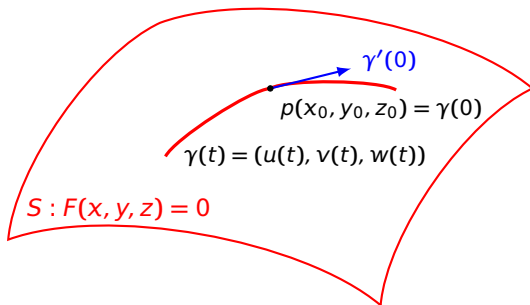
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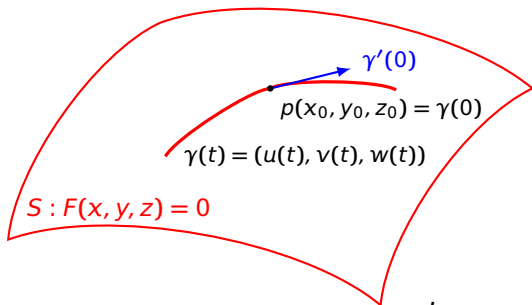


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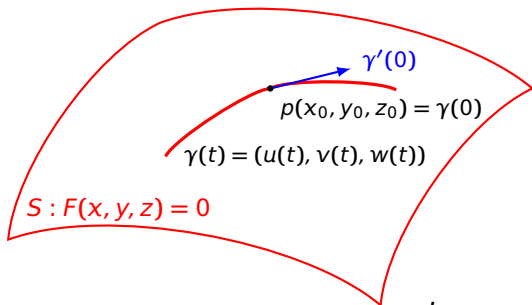
$$0 \equiv F(u(t), v(t), w(t))$$

切平面



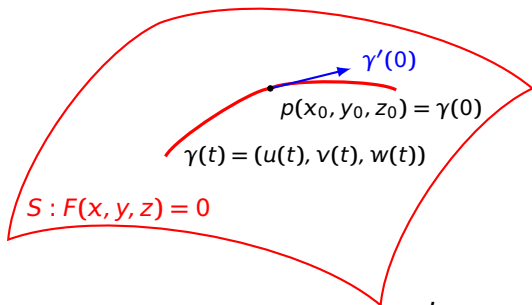
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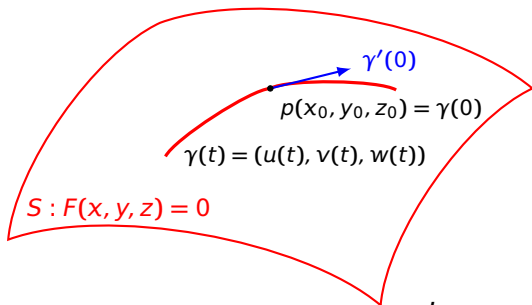
$$\begin{aligned} 0 \equiv F(u(t), v(t), w(t)) &\Rightarrow 0 = \frac{d}{dt} F(u(t), v(t), w(t)) \Big|_{t=0} \\ &= F_x \cdot u' + F_y \cdot v' + F_z \cdot w' \end{aligned}$$

切平面



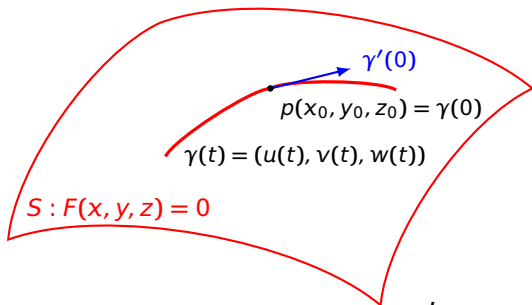
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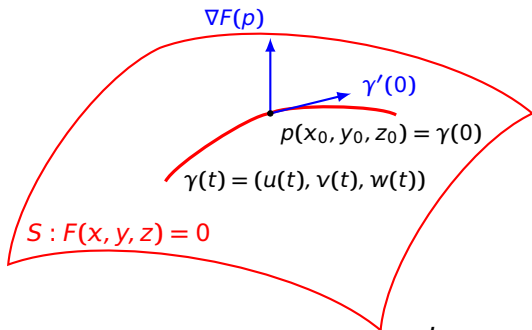
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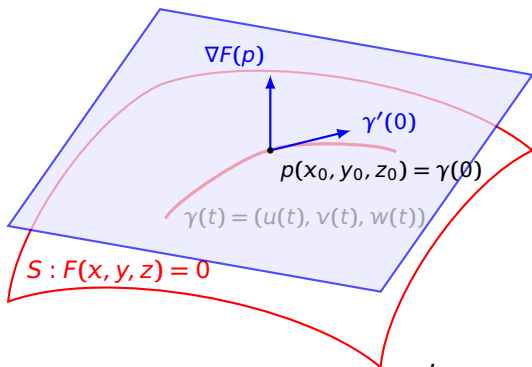
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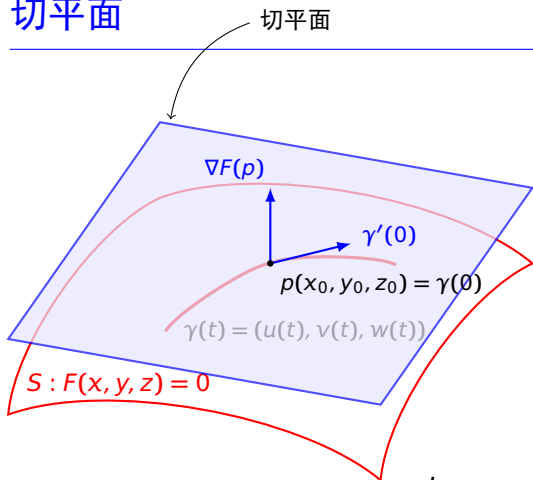
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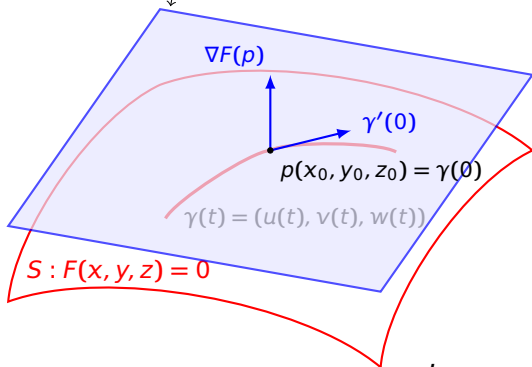
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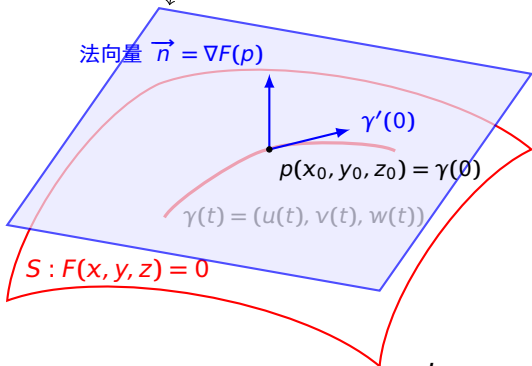
切平面 $F_x(p)(x - x_0) + F_y(p)(y - y_0) + F_z(p)(z - z_0) = 0$



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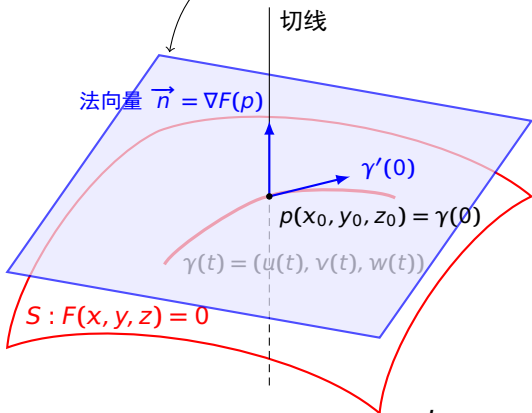
$$\text{切平面 } F_x(p)(x - x_0) + F_y(p)(y - y_0) + F_z(p)(z - z_0) = 0$$



$$\begin{aligned} 0 \equiv F(u(t), v(t), w(t)) &\Rightarrow 0 = \left. \frac{d}{dt} F(u(t), v(t), w(t)) \right|_{t=0} \\ &= F_x(p) \cdot u'(0) + F_y(p) \cdot v'(0) + F_z(p) \cdot w'(0) \\ &= \nabla F(p) \cdot \gamma'(0) \end{aligned}$$

切平面

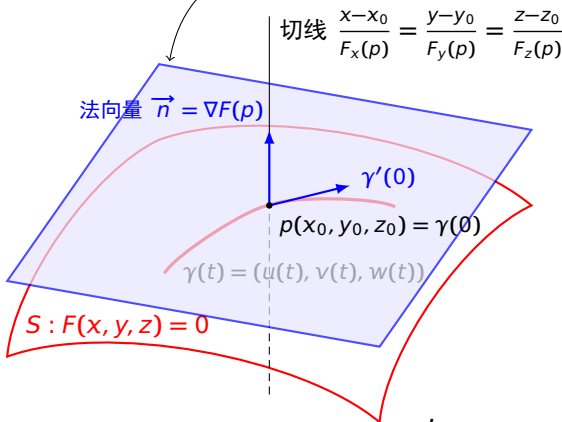
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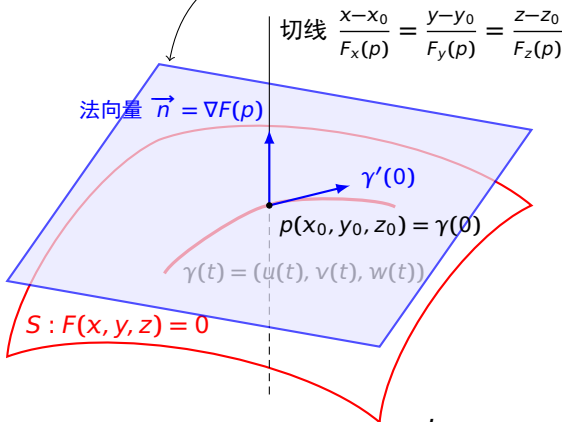
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例 求曲面 $3xy + z^2 = 4$ 在点 $(1, 1, 1)$ 处的切平面及法线的方程。

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$$3(x-1) + 3(y-1) + 2(z-1) = 0 \quad \Rightarrow \quad 3x + 3y + 2z - 8 = 0$$

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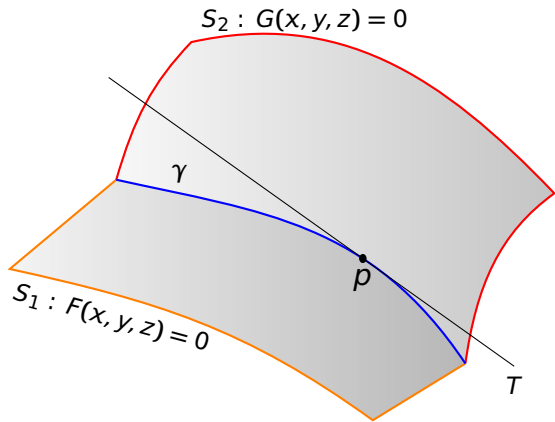
$$\vec{n}|_{(1, 1, 1)} = (3, 3, 2).$$

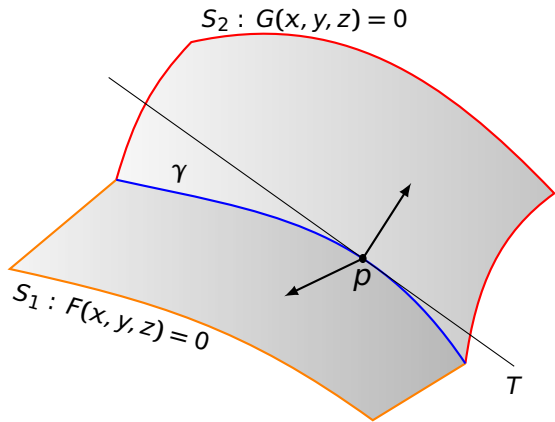
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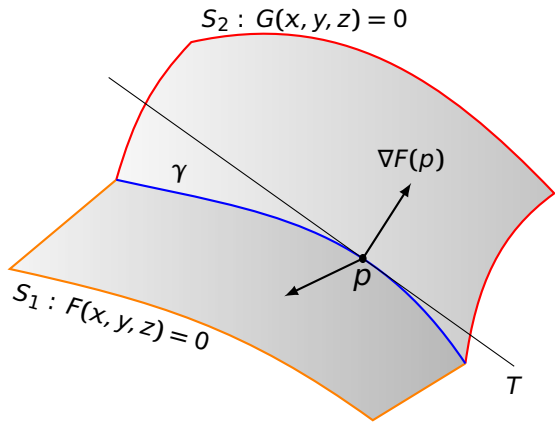
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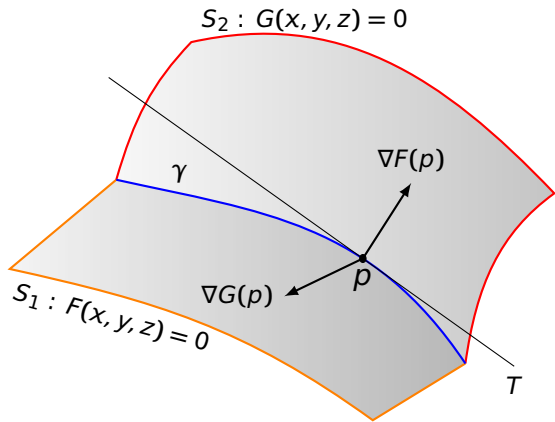
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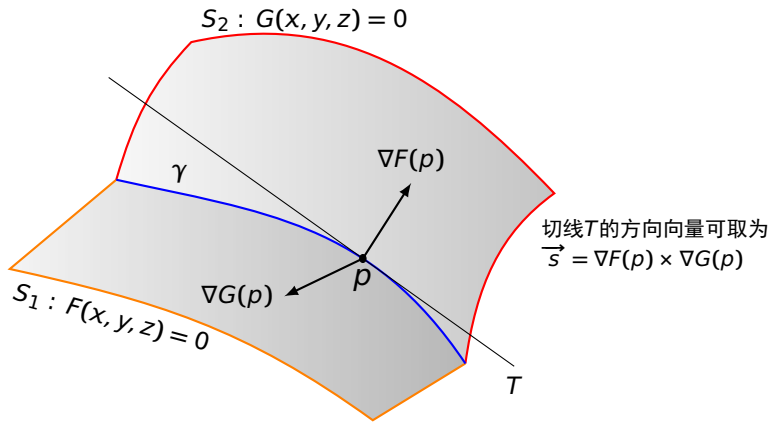
$$\frac{x-1}{3} = \frac{y-1}{3} = \frac{z-1}{2}$$

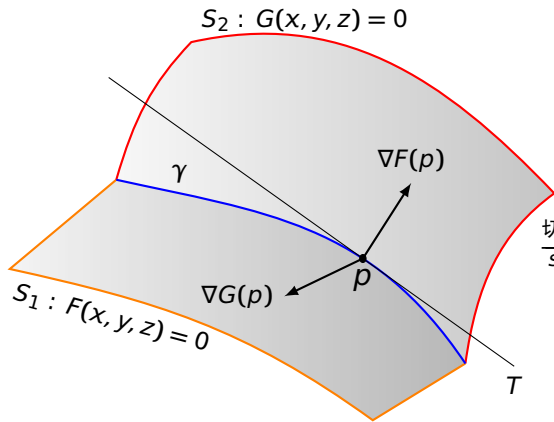






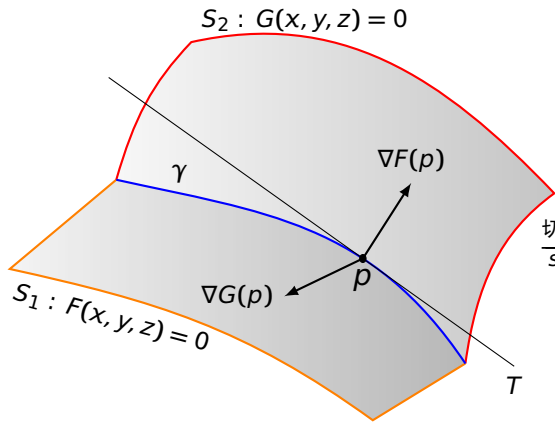






切线 T 的方向向量可取为
 $\vec{s} = \nabla F(p) \times \nabla G(p)$

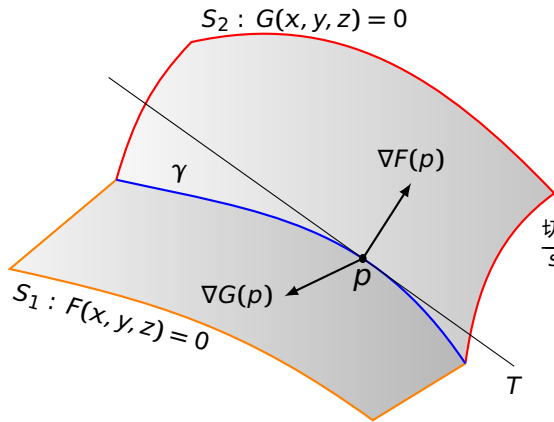
$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ F_x(p) & F_y(p) & F_z(p) \\ G_x(p) & G_y(p) & G_z(p) \end{vmatrix}$$



切线 T 的方向向量可取为

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 &= \left(\begin{vmatrix} F_y & F_z \\ G_y & G_z \end{vmatrix}_p, -\begin{vmatrix} F_x & F_z \\ G_x & G_z \end{vmatrix}_p, \begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix}_p \right)
 \end{aligned}$$

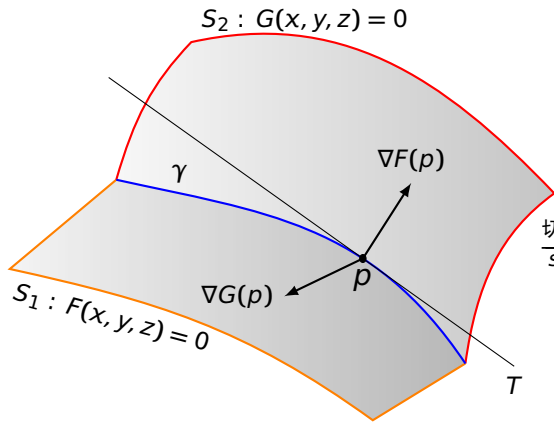


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- 切线方程:
- 法平面方程:



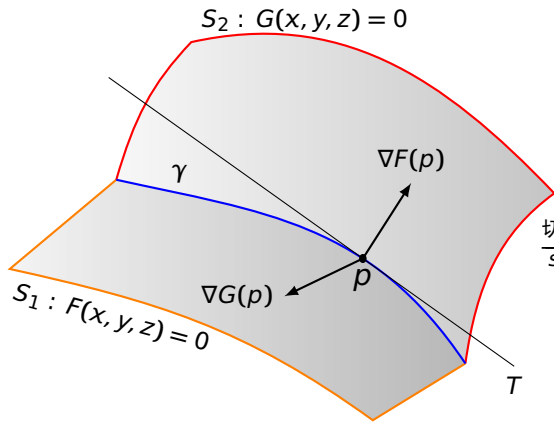
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● 切线方程:
$$\frac{x-x_0}{\begin{vmatrix} F_y & F_z \\ G_y & G_z \end{vmatrix}_p} = \frac{y-y_0}{-\begin{vmatrix} F_x & F_z \\ G_x & G_z \end{vmatrix}_p} = \frac{z-z_0}{\begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix}_p}$$

● 法平面方程:



切线 T 的方向向量可取为

$$\vec{s} = \nabla F(p) \times \nabla G(p)$$

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• 法平面方程:

$$\left| \begin{matrix} F_y & F_z \\ G_y & G_z \end{matrix} \right|_p (x-x_0) - \left| \begin{matrix} F_x & F_z \\ G_x & G_z \end{matrix} \right|_p (y-y_0) + \left| \begin{matrix} F_x & F_y \\ G_x & G_y \end{matrix} \right|_p (z-z_0) = 0$$

小结 曲线 $\begin{cases} F(x, y, z) = 0 \\ G(x, y, z) = 0 \end{cases}$ 上一点 $p(x_0, y_0, z_0)$ 处

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例 求曲线 $\begin{cases} x^2 + y^2 + z^2 = 6 \\ x + y + z = 0 \end{cases}$ 在点 $(1, -2, 1)$ 处的切线与法平面方程

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解 曲线在该点处的切线方向可取为

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ F_x & F_y & F_z \\ G_x & G_y & G_z \end{vmatrix}_p$$

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$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ F_x & F_y & F_z \\ G_x & G_y & G_z \end{vmatrix}_p = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}_{(1,-2,1)} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

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简单计，又不妨取为

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