第 12 章 d: 函数展开成幂级数

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Outline



$$f(x) \stackrel{?}{=} a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

$$f(x) \neq a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

$$f(x) \stackrel{?}{=} a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

$$f^{(k)}(x) = \left[\sum_{n=0}^{\infty} a_n (x - x_0)^n\right]^{(k)}$$

$$f(x) \neq a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

$$f^{(k)}(x) = \left[\sum_{n=0}^{\infty} a_n (x - x_0)^n\right]^{(k)} = \sum_{n=0}^{\infty} \left[a_n (x - x_0)^n\right]^{(k)}$$

$$f(x) \neq a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

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$$n \cdot (n-1) \cdots (n-k+1) \cdot (x-x_0)^{n-k}$$



$$f(x) \neq a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

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$$f^{(k)}(x) = \left[\sum_{n=0}^{\infty} a_n (x - x_0)^n\right]^{(k)} = \sum_{n=0}^{\infty} \left[a_n (x - x_0)^n\right]^{(k)}$$
$$= \sum_{n=0}^{\infty} a_n \cdot n \cdot (n-1) \cdots (n-k+1) \cdot (x - x_0)^{n-k}$$

$$f(x) \stackrel{?}{=} a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

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$$f^{(k)}(x) = \left[\sum_{n=0}^{\infty} a_n (x - x_0)^n\right]^{(k)} = \sum_{n=0}^{\infty} \left[a_n (x - x_0)^n\right]^{(k)}$$
$$= \sum_{n=k}^{\infty} a_n \cdot n \cdot (n-1) \cdots (n-k+1) \cdot (x - x_0)^{n-k}$$
$$= a_k \cdot k!$$

$$f(x) \stackrel{?}{=} a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

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$$= \sum_{n=k}^{\infty} a_n \cdot n \cdot (n-1) \cdots (n-k+1) \cdot (x - x_0)^{n-k}$$
$$= a_k \cdot k! + a_{k+1} \cdot (k+1) \cdots 2 \cdot (x - x_0)$$

$$f(x) \stackrel{?}{=} a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

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$$= \sum_{n=k}^{\infty} a_n \cdot n \cdot (n-1) \cdots (n-k+1) \cdot (x - x_0)^{n-k}$$

$$= a_k \cdot k! + a_{k+1} \cdot (k+1) \cdots 2 \cdot (x - x_0)$$

$$+ a_{k+2} \cdot (k+2) \cdots 3 \cdot (x - x_0)^2 + \cdots$$

$$f(x) \neq a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

证明 逐项求 k 次导得:

$$f^{(k)}(x) = \left[\sum_{n=0}^{\infty} a_n (x - x_0)^n\right]^{(k)} = \sum_{n=0}^{\infty} \left[a_n (x - x_0)^n\right]^{(k)}$$
$$= \sum_{n=k}^{\infty} a_n \cdot n \cdot (n-1) \cdots (n-k+1) \cdot (x - x_0)^{n-k}$$
$$= a_k \cdot k! + a_{k+1} \cdot (k+1) \cdots 2 \cdot (x - x_0)$$

 $+ a_{k+2} \cdot (k+2) \cdots 3 \cdot (x-x_0)^2 + \cdots$

取 $x = x_0$ 得 $a_k = \frac{1}{k!} f^{(k)}(x_0)$



$$f(x) \stackrel{?}{=} a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

$$f(x) \neq a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

注 1

 $f(x_0)$

$$f(x) \neq a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

$$f(x_0)$$
 $f'(x_0)$

$$f(x) \neq a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

$$f(x_0)$$
 $f'(x_0)$ $\frac{1}{2!}f''(x_0)$

$$f(x) \neq a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

$$f(x_0)$$
 $f'(x_0)$ $\frac{1}{2!}f''(x_0)$ \cdots $\frac{1}{n!}f^{(n)}(x_0)$

$$f(x) \neq a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = \frac{1}{n!} f^{(n)}(x_0).$$

注 1 也就是, f(x) 至多能展成如下形式的幂级数:

$$f(x_0)+f'(x_0)(x-x_0)+\frac{1}{2!}f''(x_0)(x-x_0)^2+\cdots+\frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n+\cdots$$

$$f(x) \neq a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

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 $f(x_0)+f'(x_0)(x-x_0)+\frac{1}{2!}f''(x_0)(x-x_0)^2+\cdots+\frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n+\cdots$

- 此级数称为 f(x) 在 x_0 处的 泰勒级数。

$$f(x) \neq a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots + a_n(x-x_0)^n + \cdots$$

性质 若 f(x) 能展成上述幂级数,则

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$$f(x_0)+f'(x_0)(x-x_0)+\frac{1}{2!}f''(x_0)(x-x_0)^2+\cdots+\frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n+\cdots$$

- 此级数称为 *f*(x) 在 x₀ 处的 泰勒级数。
- 级数前 n+1 项的部分和记为 p_n ,称为 n 次泰勒多项式

$$f(x) \neq a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots + a_n(x-x_0)^n + \dots$$

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- 此级数称为 $f(x)$ 在 x_0 处的 泰勒级数。

- 级数前 n+1 项的部分和记为 p_n ,称为 n 次泰勒多项式
- 注 2

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0) (x - x_0)^n$$



$$f(x) \stackrel{?}{=} a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

 $a_n = \frac{1}{n!} f^{(n)}(x_0).$

性质 若 f(x) 能展成上述幂级数,则

$$a_n = -\frac{1}{n!} f^{(n)}(x)$$

注 1 也就是,f(x) 至多能展成如下形式的幂级数:

$$f(x_0)+f'(x_0)(x-x_0)+\frac{1}{2!}f''(x_0)(x-x_0)^2+\cdots+\frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n+\cdots$$
 - 此级数称为 $f(x)$ 在 x_0 处的 泰勒级数。

- 级数前 n+1 项的部分和记为 p_n ,称为 n 次泰勒多项式

 $f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0) (x - x_0)^n \iff f(x) = \lim_{n \to \infty} p_n(x)$

解 取
$$x_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

解 取
$$x_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = e^x$$
 时,

$$f(x) = f'(x) = f''(x) = f'''(x) = \dots = f^{(n)}(x) = e^x$$

解 取
$$x_0 = 0$$
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$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

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$$\Rightarrow f(0) = f'(0) = f''(0) = f'''(0) = \dots = f^{(n)}(0) = 1$$

解 取
$$x_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = e^x$$
 时,

$$f(x) = f'(x) = f''(x) = f'''(x) = \dots = f^{(n)}(x) = e^x$$

$$\Rightarrow f(0) = f'(0) = f''(0) = f'''(0) = \dots = f^{(n)}(0) = 1$$

⇒
$$\bar{x}$$
 \$\text{\$\pi\$}\$ \$\pi\$\$ \$\pi\$ \$\pi\$\$ \$\pi\$

解 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = e^x$ 时,

$$f(x) = f'(x) = f''(x) = f'''(x) = \dots = f^{(n)}(x) = e^x$$

$$f(0) = f'(0) = f''(0) = f'''(0) = \dots = f^{(n)}(0) = 1$$

$$\Rightarrow f(0) = f'(0) = f''(0) = f'''(0) = \dots = f^{(n)}(0) = 1$$
1 1 1 1

⇒
$$\bar{x}$$
 \$\text{\$\pi\$}\$ \$\tag{\pi}\$ \$\ta

注 n 次泰勒多项式是:

$$p_n(x) =$$

解 取 $x_0 = 0$ 时, 泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = e^x$ 时,

$$f(x) = f'(x) = f''(x) = f'''(x) = \dots = f^{(n)}(x) = e^{x}$$

$$\Rightarrow f(0) = f'(0) = f''(0) = f'''(0) = \dots = f^{(n)}(0) = 1$$

⇒
$$\bar{x}$$
 \$\text{\$\pi\$}\$ \$\pi\$\$ \$\pi\$ \$\pi\$\$ \$\pi\$ \$\

$$2!$$
 $3!$ $n!$

注 n 次泰勒多项式是:

$$p_n(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots + \frac{1}{n!}x^n$$



解 取
$$x_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = \sin x$$
 时,

解 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = \sin x$ 时,

	$f^{(n)}(x)$	$f^{(n)}(0)$
n = 0, 4, 8	sin x	0
<i>n</i> = 1, 5, 9	cosx	1
n = 2, 6, 10	— sin <i>x</i>	0
n = 3, 7, 11	— cos x	-1

解 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = \sin x$ 时,

	$f^{(n)}(x) = \sin(x + \frac{n}{2}\pi)$	$f^{(n)}(0)$
n = 0, 4, 8	sin x	0
n = 1, 5, 9	cosx	1
n = 2, 6, 10	— sin <i>x</i>	0
n = 3, 7, 11	— cos x	-1

解 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = \sin x$ 时,

	$f^{(n)}(x) = \sin(x + \frac{n}{2}\pi)$	$f^{(n)}(0) = \sin(\frac{n}{2}\pi)$
n = 0, 4, 8	sin x	0
<i>n</i> = 1, 5, 9	cosx	1
n = 2, 6, 10	— sin <i>x</i>	0
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	$f^{(n)}(x) = \sin(x + \frac{n}{2}\pi)$	$f^{(n)}(0) = \sin(\frac{n}{2}\pi)$
n = 0, 4, 8	sin x	0
<i>n</i> = 1, 5, 9	cosx	1
n = 2, 6, 10	— sin <i>x</i>	0
<i>n</i> = 3, 7, 11	— cos <i>x</i>	-1

所以泰勒级数是

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \cdots$$

解 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = \sin x$ 时,

	$f^{(n)}(x) = \sin(x + \frac{n}{2}\pi)$	$f^{(n)}(0) = \sin(n\pi)$
	$\int \nabla f(x) = \sin(x + \frac{1}{2}\pi)$	$\int \sqrt{(0)} = \sin(\frac{\pi}{2}n)$
n = 0, 4, 8	sin x	0
$n = 1, 5, 9 \dots$	cosx	1
<i>n</i> = 2, 6, 10	— sin <i>x</i>	0
<i>n</i> = 3, 7, 11	- cos <i>x</i>	-1

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$$x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\frac{1}{7!}x^7+\frac{1}{9!}x^9-\frac{1}{11!}x^{11}+\cdots+(-1)^m\frac{1}{(2m+1)!}x^{2m+1}+\cdots$$

$$x - \frac{1}{3!}x^{3} + \frac{1}{5!}x^{5} - \frac{1}{7!}x^{7} + \frac{1}{9!}x^{9} - \frac{1}{11!}x^{11} + \dots + (-1)^{m} \frac{1}{(2m+1)!}x^{2m+1} + \dots$$



$$x - \frac{1}{3!}x^{3} + \frac{1}{5!}x^{5} - \frac{1}{7!}x^{7} + \frac{1}{9!}x^{9} - \frac{1}{11!}x^{11} + \dots + (-1)^{m} \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

$$p_1 = x$$
;

$$x - \frac{1}{3!}x^{3} + \frac{1}{5!}x^{5} - \frac{1}{7!}x^{7} + \frac{1}{9!}x^{9} - \frac{1}{11!}x^{11} + \dots + (-1)^{m} \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

$$p_1 = p_2 = x$$
;

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

$$p_1 = p_2 = x;$$

 $p_3 = x - \frac{1}{2!}x^3;$

$$p_3 = x - \frac{1}{3!}x^3;$$

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

sin x 的 n 次泰勒多项式是:

$$p_3 = p_4 = x - \frac{1}{3!}x^3;$$

 $p_1 = p_2 = x$;

$$p_3 = p_4 = x - \frac{1}{3!}x^3$$

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

$$p_1 = p_2 = x;$$

$$p_3 = p_4 = x - \frac{1}{3!}x^3;$$

$$p_5 = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5;$$



$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

sinx的n次泰勒多项式是:

$$p_1 = p_2 = x;$$

$$p_3 = p_4 = x - \frac{1}{3!}x^3;$$

$$p_5 = p_6 = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5;$$

$$-\frac{1}{2!}x^3 + \frac{1}{5!}x^5$$
;



$$x - \frac{1}{3!}x^{3} + \frac{1}{5!}x^{5} - \frac{1}{7!}x^{7} + \frac{1}{9!}x^{9} - \frac{1}{11!}x^{11} + \dots + (-1)^{m} \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

sin x 的 n 次泰勒多项式是:

 $p_1 = p_2 = x$;

$$p_3 = p_4 = x - \frac{1}{3!}x^3;$$

$$p_5 = p_6 = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5;$$
:

 p_{2m+1}

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

 $p_1 = p_2 = x$;

 p_{2m+1}

$$\frac{1}{3}$$
 $\frac{1}{5}$ $\frac{1}{5}$ $\frac{1}{5}$

sinx的n次泰勒多项式是:

 $p_3 = p_4 = x - \frac{1}{3!}x^3;$

 $p_5 = p_6 = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5;$

 $= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1}$

- sinx 的泰勒级数是:

$$\frac{1}{3}$$
 $\frac{1}{5}$ $\frac{1}{5}$

 $p_1 = p_2 = x$;

 $x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$

- sinx 的泰勒级数是:

sinx的n次泰勒多项式是:

 $p_3 = p_4 = x - \frac{1}{3!}x^3;$

 $p_5 = p_6 = x - \frac{1}{31}x^3 + \frac{1}{51}x^5;$

 $p_{2m+1} = p_{2m+2} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1}$

解 取
$$x_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = \cos x$$
 时,

解 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = \cos x$ 时,

	$f^{(n)}(x)$	$f^{(n)}(0)$
n = 0, 4, 8	cosx	1
n = 1, 5, 9	— sin <i>x</i>	0
n = 2, 6, 10	— cos x	-1
n = 3, 7, 11	sin x	0

解 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = \cos x$ 时,

	$f^{(n)}(x) = \cos(x + \frac{n}{2}\pi)$	$f^{(n)}(0)$
n = 0, 4, 8	cosx	1
n = 1, 5, 9	— sin <i>x</i>	0
n = 2, 6, 10	— cos x	-1
n = 3, 7, 11	sin x	0

解 取 $x_0 = 0$ 时,泰勒级数是

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当 $f(x) = \cos x$ 时,

	$f^{(n)}(x) = \cos(x + \frac{n}{2}\pi)$	$f^{(n)}(0) = \cos(\frac{n}{2}\pi)$
n = 0, 4, 8	cosx	1
n = 1, 5, 9	— sin <i>x</i>	0
<i>n</i> = 2, 6, 10	— cos x	-1
n = 3, 7, 11	sin x	0

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n = 0, 4, 8	cosx	1
$n = 1, 5, 9 \dots$	— sin <i>x</i>	0
n = 2, 6, 10	— cos x	-1
<i>n</i> = 3, 7, 11	sin x	0

所以泰勒级数是

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \cdots$$

 \mathbf{m} 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = \cos x$ 时,

	$f^{(n)}(x) = \cos(x + \frac{n}{2}\pi)$	$f^{(n)}(0) = \cos(\frac{n}{2}\pi)$
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cos x 的 n 次泰勒多项式是:



$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

cos x 的 n 次泰勒多项式是:

$$p_0 = 1;$$

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

cos x 的 n 次泰勒多项式是:

$$p_0 = p_1 = 1$$
;

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

• cos x 的 n 次泰勒多项式是:

$$p_0 = p_1 = 1;$$

 $p_2 = 1 - \frac{1}{2!}x^2;$

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

• cos x 的 n 次泰勒多项式是:

$$p_0 = p_1 = 1;$$

 $p_2 = p_3 = 1 - \frac{1}{2!}x^2;$

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

• cos x 的 n 次泰勒多项式是:

$$p_0 = p_1 = 1;$$

$$p_2 = p_3 = 1 - \frac{1}{2!}x^2;$$

$$p_4 = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4;$$



$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

• cos x 的 n 次泰勒多项式是:

 $p_0 = p_1 = 1$;

$$p_2 = p_3 = 1 - \frac{1}{2!}x^2;$$

 $p_4 = p_5 = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4;$



$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

• cos x 的 n 次泰勒多项式是:

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 $p_{2m}(x)$

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

cos x 的 n 次泰勒多项式是:

$$p_0 = p_1 = 1;$$

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$$p_4 = p_5 = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4;$$

 $=1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\cdots+(-1)^m\frac{1}{(2m)!}x^{2m}$

 $p_{2m}(x)$

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

 $p_{2m}(x) = p_{2m+1}(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots + (-1)^m \frac{1}{(2m)!}x^{2m}$

cos x 的 n 次泰勒多项式是:

 $p_0 = p_1 = 1$;

$$p_2 = p_3 = 1 - \frac{1}{2!}x^2;$$

$$p_4 = p_5 = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4;$$

.

 \mathbf{H} 取 $\mathbf{x}_0 = \mathbf{0}$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = \ln(1+x)$ 时,

解 取
$$x_0 = 0$$
 时,泰勒级数是
$$f'''(0) = f'''(0) = f^{(n)}(0)$$

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

$$\stackrel{\text{def}}{=} f(x) = \ln(1+x) \text{ Bd.}$$

当
$$f(x) = \ln(1+x)$$
 时,
 $f = \ln(1+x)$, $f' = \frac{1}{1+x}$,

$$\mathbf{m} \mathbf{n} \mathbf{x}_0 = \mathbf{0} \mathbf{n}$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = \ln(1+x)$$
 时,
 $f = \ln(1+x)$, $f' = \frac{1}{1+x}$, $f'' = \frac{(-1)}{(1+x)^2}$,



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 时,
 $f = \ln(1+x)$, $f' = \frac{1}{1+x}$, $f'' = \frac{(-1)}{(1+x)^2}$, $f''' = \frac{2}{(1+x)^3}$,

$$f^{(4)} = \frac{-2 \cdot 3}{(1+x)^4},$$



解 取
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 时,泰勒级数是
$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

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$$f^{(4)} = \frac{-2 \cdot 3}{(1+x)^4}, \quad f^{(5)} = \frac{2 \cdot 3 \cdot 4}{(1+x)^5}, \dots,$$



解 取
$$x_0 = 0$$
 时,泰勒级数是 $f''(0)$ $f'''(0)$

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

$$f = \ln(1+x), \quad f' = \frac{1}{1+x}, \quad f'' = \frac{(-1)}{(1+x)^2}, \quad f''' = \frac{2}{(1+x)^3},$$

$$f^{(4)} = \frac{-2 \cdot 3}{(1+x)^4}, \quad f^{(5)} = \frac{2 \cdot 3 \cdot 4}{(1+x)^5}, \dots, f^{(n)} = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}, \dots$$

解 取
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当
$$f(x) = \ln(1+x)$$
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 $f = \ln(1+x)$, $f' = \frac{1}{1+x}$, $f'' = \frac{(-1)}{(1+x)^2}$, $f''' = \frac{2}{(1+x)^3}$,

$$f^{(4)} = \frac{-2 \cdot 3}{(1+x)^4}, \quad f^{(5)} = \frac{2 \cdot 3 \cdot 4}{(1+x)^5}, \dots, f^{(n)} = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}, \dots$$

所以
$$\frac{1}{n!}f^{(n)}(0) = \frac{(-1)^{n-1}}{n!}$$
,

例 求 $f(x) = \ln(1 + x)$ 在 x = 0 处泰勒级数。

解 取
$$x_0 = 0$$
 时,泰勒级数是
$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
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所以
$$\frac{1}{n!}f^{(n)}(0) = \frac{(-1)^{n-1}}{n}$$
,泰勒级数是
$$x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{1}{4}x^4 + \cdots$$



例 求 $f(x) = \ln(1+x)$ 在 x = 0 处泰勒级数。

解 取
$$x_0 = 0$$
 时,泰勒级数是

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 时,

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所以
$$\frac{1}{n!}f^{(n)}(0) = \frac{(-1)^{n-1}}{n}$$
, 泰勒级数是

$$x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots$$

解 取 $x_0 = 0$ 时,泰勒级数是

例 求 $f(x) = \ln(1+x)$ 在 x = 0 处泰勒级数。

$$\mathbf{H}$$
 取 $X_0 = 0$ 时,泰朝级数 $f''(0)$

 $f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{2!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$ 当 $f(x) = \ln(1+x)$ 时,

当
$$f(x) = \ln(1+x)$$
 时,
 $f = \ln(1+x)$, $f' = \frac{1}{1+x}$, $f'' = \frac{(-1)}{(1+x)^2}$, $f''' = \frac{2}{(1+x)^3}$,

$$f^{(4)} = \frac{-2 \cdot 3}{(1+x)^4}, \quad f^{(5)} = \frac{2 \cdot 3 \cdot 4}{(1+x)^5}, \dots, f^{(n)} = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}, \dots$$

$$f^{(4)} = \frac{-2 \cdot 3}{(1+x)^4}, \quad f^{(5)} = \frac{2 \cdot 3 \cdot 4}{(1+x)^5}, \dots, f^{(n)} = \frac{(-1)^n (n-1)!}{(1+x)^n}, \dots$$

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所以
$$\frac{1}{n!}f^{(n)}(0) = \frac{(-1)^{n-1}}{n}$$
,泰勒级数是
$$x - -x^2 + -x^3 - -x^4 + \dots + (-1)^{n-1} - x^n + \dots$$

 $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots$

注 n 次泰勒多项式是: $p_n(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots + (-1)^{n-1}\frac{1}{n}x^n$

解 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = (1+x)^{\alpha}$$
 时,

解 取
$$x_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = (1+x)^{\alpha}$$
 时,
 $f = (1+x)^{\alpha}$, $f' = \alpha(1+x)^{\alpha-1}$,

解 取 $x_0 = 0$ 时,泰勒级数是

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当
$$f(x) = (1+x)^{\alpha}$$
 时,

$$f = (1+x)^{\alpha}$$
, $f' = \alpha(1+x)^{\alpha-1}$, $f'' = \alpha(\alpha-1)(1+x)^{\alpha-2}$,

解 取
$$x_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

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$$\ldots, f^{(n)} = \alpha(\alpha - 1) \cdots (\alpha - n + 1)(1 + x)^{\alpha - n}, \cdots$$

所以
$$\frac{1}{n!}f^{(n)}(0) = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$$
,

解 取
$$x_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = (1+x)^{\alpha}$$
 时,

$$f = (1+x)^{\alpha}, \quad f' = \alpha(1+x)^{\alpha-1}, \quad f'' = \alpha(\alpha-1)(1+x)^{\alpha-2},$$

$$\ldots, f^{(n)} = \alpha(\alpha - 1) \cdots (\alpha - n + 1)(1 + x)^{\alpha - n}, \cdots$$

所以
$$\frac{1}{n!}f^{(n)}(0) = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$$
,泰勒级数是 $\alpha(\alpha-1)$

$$1 + \alpha x + \frac{\alpha(\alpha - 1)}{2!}x^2 + \cdots$$

解 取
$$x_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = (1+x)^{\alpha}$$
 时.

$$f = (1+x)^{\alpha}, \quad f' = \alpha(1+x)^{\alpha-1}, \quad f'' = \alpha(\alpha-1)(1+x)^{\alpha-2},$$

$$\dots, f^{(n)} = \alpha(\alpha - 1) \cdots (\alpha - n + 1)(1 + x)^{\alpha - n}, \dots$$
所以 $\frac{1}{n!} f^{(n)}(0) = \frac{\alpha(\alpha - 1) \cdots (\alpha - n + 1)}{n!}$,泰勒级数是

$$1 + \alpha x + \frac{\alpha(\alpha - 1)}{\alpha(\alpha - 1)} x^2 + \dots + \frac{\alpha(\alpha - 1) \cdots (\alpha - n + 1)}{\alpha(\alpha - 1)} x^n + \dots$$

$$1 + \alpha x + \frac{\alpha(\alpha - 1)}{2!}x^2 + \dots + \frac{\alpha(\alpha - 1)\cdots(\alpha - n + 1)}{n!}x^n + \dots$$

 \mathbf{H} 取 $\mathbf{X}_0 = \mathbf{0}$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{2!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = (1+x)^{\alpha}$ 时,

$$f = (1+x)^{\alpha}, \quad f' = \alpha(1+x)^{\alpha-1}, \quad f'' = \alpha(\alpha-1)(1+x)^{\alpha-2},$$

...,
$$f^{(n)} = \alpha(\alpha - 1) \cdots (\alpha - n + 1)(1 + x)^{\alpha - n}, \cdots$$

所以
$$\frac{1}{n!}f^{(n)}(0) = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$$
,泰勒级数是
$$1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n + \cdots$$

注 n 次泰勒多项式是:

$$p_n(x) =$$



 $f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{2!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$

 \mathbf{H} 取 $x_0 = 0$ 时,泰勒级数是

例 求 $f(x) = (1+x)^{\alpha}$ 在 x = 0 处的 n 次泰勒多项式 $p_n(x)$

 $\dots, f^{(n)} = \alpha(\alpha - 1) \cdots (\alpha - n + 1)(1 + x)^{\alpha - n}, \dots$

 $1 + \alpha x + \frac{\alpha(\alpha - 1)}{2!} x^2 + \dots + \frac{\alpha(\alpha - 1) \cdots (\alpha - n + 1)}{2!} x^n + \dots$

所以 $\frac{1}{n!}f^{(n)}(0) = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$, 泰勒级数是

 $f = (1+x)^{\alpha}$, $f' = \alpha(1+x)^{\alpha-1}$, $f'' = \alpha(\alpha-1)(1+x)^{\alpha-2}$

注 *n* 次泰勒多项式是:

当 $f(x) = (1+x)^{\alpha}$ 时,

 $p_n(x) = 1 + \alpha x + \frac{\alpha(\alpha - 1)}{2!}x^2 + \dots + \frac{\alpha(\alpha - 1)\cdots(\alpha - n + 1)}{n!}x^n$

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0) (x - x_0)^n$$

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0) (x - x_0)^n \iff f(x) = \lim_{n \to \infty} p_n(x)$$

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$$(R_n(x) = f(x) - p_n(x))$$

$$\Leftrightarrow \lim_{n \to \infty} R_n(x) = 0$$

注 $R_n(x) = f(x) - p_n(x)$,或者 $f(x) = p_n(x) + R_n(x)$,刻画了原函数 f(x) 与其泰勒多项式 $p_n(x)$ 的差异。



回忆 泰勒中值定理 1 若 f 具有 n 阶导数,则

$$R_n(x) = o((x-x_0)^n).$$

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$$= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + o((x - x_0)^n)$$

 $e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + o(x^{n})$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots + (-1)^{n-1}\frac{1}{n}x^n + o(x^n)$$

$$\alpha(\alpha - 1) \qquad \alpha(\alpha - 1) \dots (\alpha - n + 1)$$

 $(1+x)^{\alpha} = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \dots + \frac{\alpha(\alpha-1) \cdots (\alpha-n+1)}{n!} x^n + o(x^n)$

 $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + o(x^{2m+2})$

 $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + o(x^{2m+1})$

例求 $\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$, $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1-x)]}$

例求
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
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例求
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
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$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!} x^3 + o(x^4) \right] - x \left[1 - \frac{1}{2!} x^2 + o(x^3) \right]}{x^3}$$



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例 求
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
, $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1-x)]}$

例 求
$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
, $\lim_{x \to 0} \frac{\cos x - e^{-2x}}{x^2 [x + \ln(1 - x)]}$
 $\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + \frac{1}{3!}x^3$

$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$$

$$= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3} = \frac{1}{3}$$

$$\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1 - x)]}$$

$$= \lim_{x \to 0} \frac{1}{x^{2} [x + \ln(1 - x)]}$$

$$= \lim_{x \to 0} \frac{\left[\frac{1}{x^{2} [x + (1 - x)]} \right]}{x^{2} [x + (1 - x)]}$$



例求
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
, $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1-x)]}$

$$\lim_{x \to 0} \frac{\sin^3 x}{\sin^3 x}, \frac{\sin^3 x}{x^2 [x + \ln(1 - x)]}$$

$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$$

$$= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3} = \frac{1}{3}$$

$$\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2 [x + \ln(1 - x)]}$$

$$n(1-x)$$

$$(1-x)$$

$$-\ln(1-x)$$

$$= \lim_{x \to 0} \frac{\left[1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + o(x^5)\right] - \left[}{x^2 \left[x + \left(\frac{1}{2!}x^2 + \frac{1}{4!}x^4 + o(x^5)\right]\right]} \right]$$

例 求
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
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$$= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3} = \frac{1}{3}$$

$$\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2 [x + \ln(1 - x)]}$$

$$= \lim_{x \to 0} \frac{\left[1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + o(x^5)\right] - \left[1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + o(x^4)\right]}{x^2 [x + (y^2)]}$$



例 求
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
, $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1-x)]}$

$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$$
$$= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3} = \frac{1}{3}$$

$$-e^{-\frac{x^2}{2}}$$

$$\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2 [x + \ln(1 - x)]}$$

$$= \lim_{x \to 0} \frac{\left[1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + o(x^5)\right] - \left[1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + o(x^4)\right]}{x^2 [x + \left(-x - \frac{1}{2}x^2 + o(x^2)\right)]}$$



解

 $\frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$

 $= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3} = \frac{1}{2}$

 $\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2 [x + \ln(1 - x)]}$

 $= \lim_{x \to 0} \frac{\left[1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + o(x^5)\right] - \left[1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + o(x^4)\right]}{x^2 \left[x + \left(-x - \frac{1}{2}x^2 + o(x^2)\right)\right]}$ $= \lim_{x \to 0} \frac{-\frac{1}{12}x^4 + o(x^4)}{-\frac{1}{2}x^4 + o(x^4)}$

解

例 求 $\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$, $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1-x)]}$

 $\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$

 $= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3} = \frac{1}{2}$

 $\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2 [x + \ln(1 - x)]}$

 $= \lim_{x \to 0} \frac{\left[1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + o(x^5)\right] - \left[1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + o(x^4)\right]}{x^2 \left[x + \left(-x - \frac{1}{2}x^2 + o(x^2)\right)\right]}$

 $= \lim_{x \to 0} \frac{-\frac{1}{12}x^4 + o(x^4)}{-\frac{1}{2}x^4 + o(x^4)} = \lim_{x \to 0} \frac{-\frac{1}{12} + o(x^4)/x^4}{-\frac{1}{2} + o(x^4)/x^4}$

函数展开成幂级数

例 求
$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
, $\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2 [x + \ln(1 - x)]}$

$$\lim_{x \to \infty} \frac{\sin x - x \cos x}{\sin x} = \lim_{x \to \infty} \frac{\left[x - \frac{1}{3!}x^3 + c^2\right]}{x^2 [x + \ln(1 - x)]}$$

$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$

 $\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$ $= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3} = \frac{1}{2}$

$$\lim_{x\to 0} \frac{\cos x - e^{-x}}{x^2 + \ln(1)}$$

 $\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2 [x + \ln(1 - x)]}$ $= \lim_{x \to 0} \frac{\left[1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + o(x^5)\right] - \left[1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + o(x^4)\right]}{x^2 \left[x + \left(-x - \frac{1}{2}x^2 + o(x^2)\right)\right]}$

 $= \lim_{x \to 0} \frac{-\frac{1}{12}x^4 + o(x^4)}{-\frac{1}{2}x^4 + o(x^4)} = \lim_{x \to 0} \frac{-\frac{1}{12} + o(x^4)/x^4}{-\frac{1}{2} + o(x^4)/x^4} = \frac{1}{6}$ 函数展开成幂级数



泰勒中值定理 2 若 f 具有 n+1 阶导数,则

$$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) (x - x_0)^{n+1}$$

其中 ξ 是 x_0 与 x 之间的某个值

$$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) (x - x_0)^{n+1}$$

$$(1-\theta)x_0+\theta x$$

其中 ξ 是 x_0 与 x 之间的某个值, $0 < \theta < 1$ 。

$$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) (x - x_0)^{n+1}$$

$$\stackrel{or}{=} \frac{1}{(n+1)!} f^{(n+1)}((1-\theta)x_0 + \theta x) (x - x_0)^{n+1},$$

其中 ξ 是 x_0 与 x 之间的某个值, $0 < \theta < 1$ 。

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注

1. ξ(以及 θ) 不是固定不变的,而是随 x 和 n 的改变而变化。

$$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) (x - x_0)^{n+1}$$

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其中 ξ 是 x_0 与 x 之间的某个值, $0 < \theta < 1$ 。

注

- 1 ξ (以及 θ) 不是固定不变的,而是随 x 和 n 的改变而变化。
- 2. 当 $x_0 = 0$ 时,则余项可写成

$$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1}, \qquad (0 < \theta < 1)$$



$$e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty).$$

$$e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty).$$

证明

1. 只需证明对任意 x,成立 $\lim_{n\to\infty} R_n(x) = 0$ 。

$$e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty).$$

- 1. 只需证明对任意 x,成立 $\lim_{n\to\infty} R_n(x) = 0$ 。
- 2. 由泰勒中值定理 2.

$$|R_n(x)| = \left| \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1} \right|$$

$$e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty).$$

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$$e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty).$$

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其中
$$x \in (-\infty, \infty)$$
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$$|R_n(x)| = \left| \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1} \right| = \left| \frac{\sin\left(\theta x + \frac{n+1}{2}\pi\right) x^{n+1}}{(n+1)!} \right|$$



$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

其中 $x \in (-\infty, \infty)$ 。

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$$|R_n(x)| = \left| \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1} \right| = \left| \frac{\sin\left(\theta x + \frac{n+1}{2}\pi\right) x^{n+1}}{(n+1)!} \right|$$

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• 至此,我们知道 e^x , $\sin x$, $\cos x$ 以及 $\frac{1}{1+x}$ 是等于其泰勒级数,即

$$e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty)$$

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ln(1+x), arctan x



的幂级数展开。

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$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$$

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(这是
$$f(1) = \lim_{x \to 1^{-}} \ln(1+x)$$

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$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1} \frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$$

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 $\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n} x^{n}.$ (这是 $f(1) = \lim_{x \to 1^{-}} \ln(1+x) = \lim_{x \to 1^{-}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n = \lim_{x \to 1^{-}} S(x) = S(1)$)

证明 1. 幂级数的收敛域是 [-1, 1], 故上式至多对 $x \in [-1, 1]$ 成立。

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$$\arctan x = \int_0^x \frac{1}{1+t^2} dt = \int_0^x \sum_{n=0}^\infty (-1)^n t^{2n} dt = \sum_{n=0}^\infty (-1)^n \int_0^x t^{2n} dt$$

$$=\sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}x^{2n+1}$$

3. 注意到 $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$ 收敛域是 [-1, 1], 由连续性, 当 $x = \pm 1$ 时

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 $(如f(1) = \lim arctan x$

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 $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots + (-1)^n \frac{1}{2n+1}x^{2n+1} + \dots, x \in [-1,1].$

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$$注$$
 取 $x=1$,则得到

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$$



 $e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty)$ $\sin x = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 + \dots + (-1)^n \frac{1}{(2n+1)!} x^{2n+1} + \dots, \ x \in (-\infty, \infty)$

• 至此, 得出如下常用函数的幂级数展开式:

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots + (-1)^n \frac{1}{(2n)!}x^{2n} + \dots, \ x \in (-\infty, \infty)$$

 $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots + (-1)^n x^n + \dots, x \in (-1, 1)$

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 $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1]$

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• 用上述结果, 及逐项求导、积分公式, 可求更多函数的泰勒级数展开

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1].$$

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$$x \in (-1, 1]$$
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$$\sum_{n=1}^{\infty} (-1)^{n-2} \frac{1}{n-1} x^n$$

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解 利用

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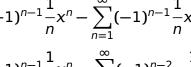
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 $(1-x)\ln(1+x) = (1-x)\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n$

例 把函数 $f(x) = (1-x) \ln(1+x)$ 展开成 x 的幂级数。



$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n - \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^{n+1}$$
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 $= x + \sum_{n=0}^{\infty} \left(\frac{(-1)^{n-1}}{n} - \frac{(-1)^n}{n-1} \right) x^n$

 $= x + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{1}{n} x^n - \sum_{n=2}^{\infty} (-1)^{n-2} \frac{1}{n-1} x^n$



解 利用

$$\cos t = 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6 + \dots + (-1)^n \frac{1}{(2n)!}t^{2n} + \dots, \ t \in (-\infty, \infty)$$

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$$

解 利用

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 $=1+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n}2^{2n}}{(2n)!}x^{2n}$

解 利用 $\cos t = 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6 + \dots + (-1)^n \frac{1}{(2n)!}t^{2n} + \dots, \ t \in (-\infty, \infty)$

所以当 $x \in (-\infty, \infty)$ 时,

第 12 章 d: 函数展开成幂级数

26/27 ▷ ▷ ▷ ♡

例 把函数 $f(x) = \frac{1}{x^2 + 3x + 2}$ 展开成 (x + 4) 的幂级数。

解 1. 注意到
$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$
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$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \dots + t^n + \dots, t \in (-1, 1)$$

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, $\frac{1}{x+2}$ 分别展开成 $(x+4)$ 的幂级数:

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$$* \quad \frac{1}{x+1} = \frac{1}{t-3}$$

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其中
$$|\frac{t}{3}| < 1$$



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其中 $\left|\frac{x+4}{3}\right| = \left|\frac{t}{3}\right| < 1$,即 -7 < x < -1。



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$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$
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$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots + t^n + \cdots, t \in (-1, 1)$$

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其中 $\left|\frac{x+4}{2}\right| = \left|\frac{t}{2}\right| < 1$,即 -6 < x < -2。

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例 把函数 $f(x) = \frac{1}{x^2 + 3x + 2}$ 展开成 (x + 4) 的幂级数。

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$$\frac{1}{x^2 + 3x + 2} = \frac{1}{x + 1} - \frac{1}{x + 2} = \sum_{n=0}^{\infty} \left(\frac{1}{2^{n+1}} - \frac{1}{3^{n+1}} \right) (x + 4)^n$$