§3.2　向量与向量组的线性组合

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向量

- n 维行向量

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- 零向量 $O = (0, 0, \cdots, 0)$

向量的线性运算

- 设 $\alpha = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$, $\beta = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$, $k \in \mathbb{R}$, 则

$$\alpha + \beta = \qquad , \quad \alpha - \beta = \qquad , \quad k\alpha =$$

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- 行向量类似

线性组合问题

- 给定向量组

$$\alpha_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \ \alpha_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \ \ldots, \ \alpha_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

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$$k_1\alpha_1 + k_2\alpha_2 + \cdots + k_n\alpha_n$$

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线性组合问题

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设 $k_1, k_2, \cdots, k_n$ 为任意数，则称

$$k_1 \alpha_1 + k_2 \alpha_2 + \cdots + k_n \alpha_n$$

为向量组 $\alpha_1, \alpha_2, \cdots, \alpha_n$ 的线性组合 。

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- 问题 给定向量 $\beta = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$，问 β 能否由向量组 $\alpha_1, \alpha_2, \cdots, \alpha_n$

的线性表示？

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  的线性表示？即：是否存在数 $k_1, k_2, \ldots, k_n$ 使得：

$$\beta = k_1\alpha_1 + k_2\alpha_2 + \cdots + k_n\alpha_n ?$$

线性组合问题

例 判断 β 能否由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式 $\beta = k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3$。

- **(1)** 问

$$\beta \begin{pmatrix} 2 \\ -7 \\ 5 \end{pmatrix} \qquad \alpha_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \alpha_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \alpha_3 \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

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$$\underset{\beta}{\begin{pmatrix} 2 \\ -7 \\ 5 \end{pmatrix}} = \underline{\quad} \underset{\alpha_1}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}} + \underline{\quad} \underset{\alpha_2}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}} + \underline{\quad} \underset{\alpha_3}{\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}}$$

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- (1) 问

$$\begin{pmatrix} 2 \\ -7 \\ 5 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \underline{\quad} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \underline{\quad} \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

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$$\begin{pmatrix} 2 \\ -7 \\ 5 \end{pmatrix} = \underline{2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \underline{-7} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \underline{\quad} \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

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$$\underset{\beta}{\begin{pmatrix} 2 \\ -7 \\ 5 \end{pmatrix}} = 2\underset{\alpha_1}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}} + -7\underset{\alpha_2}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}} + \frac{5}{2}\underset{\alpha_3}{\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}}$$

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例 判断 β 能否由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式 $\beta = k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3$。

- (1) 问

$$\underset{\beta}{\begin{pmatrix} 2 \\ -7 \\ 5 \end{pmatrix}} = \underset{\alpha_1}{2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}} + \underset{\alpha_2}{-7\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}} + \underset{\alpha_3}{\frac{5}{2}\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}}$$

所以 $\beta = 2\alpha_1 - 7\alpha_2 + \frac{5}{2}\alpha_3$；

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$$\begin{matrix} \beta & & \alpha_1 & & \alpha_2 & & \alpha_3 \end{matrix}$$

$$\begin{pmatrix} 2 \\ -7 \\ 5 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (-7)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \frac{5}{2}\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

所以 $\beta = 2\alpha_1 - 7\alpha_2 + \frac{5}{2}\alpha_3$；$\beta$ 能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表出

- **(2)** 问

$$\begin{matrix} \beta & & \alpha_1 & & \alpha_2 & & \alpha_3 \end{matrix}$$

$$\begin{pmatrix} 2 \\ -7 \\ 5 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

线性组合问题

例 判断 β 能否由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式 $\beta = k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3$。

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例 判断 β 能否由 $\alpha_1,\ \alpha_2,\ \alpha_3$ 线性表示，若能，写出线性表示等式 $\beta = k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3$。

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$$\begin{pmatrix} 2 \\ -7 \\ 5 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (-7)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \frac{5}{2}\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

所以 $\beta = 2\alpha_1 - 7\alpha_2 + \frac{5}{2}\alpha_3$；$\beta$ 能由 $\alpha_1,\ \alpha_2,\ \alpha_3$ 线性表出

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$$\begin{pmatrix} 2 \\ -7 \\ 5 \end{pmatrix} \neq \underline{}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \underline{}\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + \underline{}\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

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所以 $\beta = 2\alpha_1 - 7\alpha_2 + \frac{5}{2}\alpha_3$；$\beta$ 能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表出

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所以 β 不能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表出！

线性组合问题

例 设 $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$; 及 $\beta = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix}$, 问

$$\begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix} = \underline{} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix} + \underline{} \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix} + \underline{} \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

即：β 能否由 $\alpha_1, \alpha_2, \alpha_3$ 线性表出？如果能，线性表达式是什么？

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问题

- **一般地，如何判断 β 能否由 $\alpha_1, \alpha_2, \ldots, \alpha_n$ 线性表出？**

线性组合问题

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即：β 能否由 α_1, α_2, α_3 线性表出？如果能，线性表达式是什么？

问题

- 一般地，如何判断 β 能否由 α_1, α_2, $\ldots$, α_n 线性表出？
- 如果能线性表出，如何求出 k_1, k_2, $\ldots$, k_n 使

$$k_1\alpha_1 + k_2\alpha_2 + \cdots + k_n\alpha_n = \beta \, ?$$

线性表示的判别

定理 β 可由 $\alpha_1, \alpha_2, \cdots, \alpha_n$ 线性表示 $\Longleftrightarrow$

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$$
\overset{\alpha_1}{\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}}
\quad
\overset{\alpha_2}{\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}}
\quad \cdots \quad
\overset{\alpha_n}{\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}}
\quad
\overset{\beta}{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}
$$

线性表示的判别

定理 β 可由 $\alpha_1, \alpha_2, \cdots, \alpha_n$ 线性表示 $\Longleftrightarrow (\alpha_1 \alpha_2 \cdots \alpha_n) \quad (\alpha_1 \alpha_2 \cdots \alpha_n \beta)$

$$
\overset{\alpha_1}{\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}} \quad
\overset{\alpha_2}{\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}} \quad \cdots \quad
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\overset{\beta}{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}
$$

线性表示的判别

定理 β 可由 $\alpha_1, \alpha_2, \cdots, \alpha_n$ 线性表示 $\Leftrightarrow r(\alpha_1 \alpha_2 \cdots \alpha_n) = r(\alpha_1 \alpha_2 \cdots \alpha_n \beta)$

$$
\overset{\alpha_1}{\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}}
\quad
\overset{\alpha_2}{\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}}
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\quad
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定理 β 可由 $\alpha_1, \alpha_2, \cdots, \alpha_n$ 线性表示 $\Leftrightarrow r(\alpha_1\alpha_2\cdots\alpha_n) = r(\alpha_1\alpha_2\cdots\alpha_n\beta)$

证明 β 可由 $\alpha_1, \alpha_2, \cdots, \alpha_n$ 线性表示

$$\Leftrightarrow \quad \overset{\alpha_1}{\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}} \quad \overset{\alpha_2}{\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}} \quad \cdots \quad \overset{\alpha_n}{\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}} \quad \overset{\beta}{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}$$

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定理 β 可由 $\alpha_1, \alpha_2, \cdots, \alpha_n$ 线性表示 $\Leftrightarrow r(\alpha_1 \alpha_2 \cdots \alpha_n) = r(\alpha_1 \alpha_2 \cdots \alpha_n \beta)$

证明 β 可由 $\alpha_1, \alpha_2, \cdots, \alpha_n$ 线性表示

$$\Leftrightarrow \quad k_1 \overset{\alpha_1}{\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}} + k_2 \overset{\alpha_2}{\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}} + \cdots + k_n \overset{\alpha_n}{\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}} = \overset{\beta}{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}$$

线性表示的判别

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$$\Leftrightarrow \quad \begin{pmatrix} \overset{\alpha_1}{a_{11}} & \overset{\alpha_2}{a_{12}} & \cdots & \overset{\alpha_n}{a_{1n}} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} = \overset{\beta}{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}$$

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$$\Leftrightarrow \quad \underbrace{\overset{\alpha_1 \quad \alpha_2 \quad \quad \alpha_n}{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}}_{A} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} = \overset{\beta}{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}$$

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$$\Leftrightarrow \quad Ax = \beta \text{ 有解}$$

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$$\Leftrightarrow r(A) = r(A \vdots \beta)$$

线性表示的判别

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$$\Leftrightarrow \quad r(A) = r(A \vdots \beta) \qquad (\alpha_1\alpha_2\cdots\alpha_n) \qquad (\alpha_1\alpha_2\cdots\alpha_n\beta)$$

线性表示的判别

定理 β 可由 $\alpha_1, \alpha_2, \cdots, \alpha_n$ 线性表示 $\Leftrightarrow r(\alpha_1\alpha_2\cdots\alpha_n) = r(\alpha_1\alpha_2\cdots\alpha_n\beta)$

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$$\Leftrightarrow \quad Ax = \beta \text{有解}$$

$$\Leftrightarrow \quad r(A) = r(A\mathbin{\vdots}\beta) \quad \Leftrightarrow \quad r(\alpha_1\alpha_2\cdots\alpha_n) = r(\alpha_1\alpha_2\cdots\alpha_n\beta)$$

初等行变换求线性表示问题——例 1

例 判断 β 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

(1)

$$(\alpha_1 \ \ \alpha_2 \ \ \alpha_3 \,|\, \beta) = \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{pmatrix}$$

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- $\qquad r(\alpha_1\alpha_2\alpha_3) = \quad , \ \ r(\alpha_1\alpha_2\alpha_3\beta) = \quad ,$

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- $r(\alpha_1\alpha_2\alpha_3) = \quad , \ r(\alpha_1\alpha_2\alpha_3\beta) = 3,$

例 判断 β 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

(1)

$$(\begin{array}{ccc|c} \alpha_1 & \alpha_2 & \alpha_3 & \end{array}) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & \\ 0 & -1 & 2 & \\ 1 & 1 & 0 & \\ 2 & -2 & 1 & \end{array} \right) \xrightarrow{\text{初等行变换}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & \end{array} \right)$$

- $r(\alpha_1\alpha_2\alpha_3) = \quad , \quad r(\alpha_1\alpha_2\alpha_3\beta) = 3,$

例 判断 β 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

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$$(\alpha_1 \ \alpha_2 \ \alpha_3 \mid \ ) = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 1 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix} \xrightarrow{\text{初等行变换}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

- 所以 $r(\alpha_1 \alpha_2 \alpha_3) = 3$, $r(\alpha_1 \alpha_2 \alpha_3 \beta) = 3$,

初等行变换求线性表示问题——例 1

例 判断 β 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

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$$(\alpha_1 \ \ \alpha_2 \ \ \alpha_3 \mid \beta) = \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{pmatrix} \xrightarrow{\text{初等行变换}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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初等行变换求线性表示问题——例 1

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- 所以 $r(\alpha_1\alpha_2\alpha_3) = 3$，$r(\alpha_1\alpha_2\alpha_3\beta) = 3$，成立

$$r(\alpha_1\alpha_2\alpha_3) = r(\alpha_1\alpha_2\alpha_3\beta)$$

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初等行变换求线性表示问题——例 1

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初等行变换求线性表示问题——例 1

例 判断 β 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

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(\ \alpha_1 \quad \alpha_2 \quad \alpha_3 \ | \ \beta \ ) = \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{pmatrix} \xrightarrow{\text{初等行变换}} \begin{array}{cccc} \alpha_1' & \alpha_2' & \alpha_3' & \beta' \end{array} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

- 所以 $r(\alpha_1\alpha_2\alpha_3) = 3$, $r(\alpha_1\alpha_2\alpha_3\beta) = 3$, 成立

$$
r(\alpha_1\alpha_2\alpha_3) = r(\alpha_1\alpha_2\alpha_3\beta)
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β 可由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示。

初等行变换求线性表示问题——例 1

例 判断 β 是否能由 α_1, α_2, α_3 线性表示，若能，写出线性表示等式。

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(\,\alpha_1 \ \ \alpha_2 \ \ \alpha_3 \,|\, \beta\,) = \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{pmatrix} \xrightarrow{\text{初等行变换}} \begin{array}{c} \begin{matrix} \alpha_1' & \alpha_2' & \alpha_3' & \beta' \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}
$$

- 所以 $r(\alpha_1 \alpha_2 \alpha_3) = 3$, $r(\alpha_1 \alpha_2 \alpha_3 \beta) = 3$, 成立

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β 可由 α_1, α_2, α_3 线性表示。

- 显然 $\beta' = \alpha_1' - \alpha_2' + \alpha_3'$，

初等行变换求线性表示问题——例 1

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$$

- 所以 $r(\alpha_1\alpha_2\alpha_3) = 3$，$r(\alpha_1\alpha_2\alpha_3\beta) = 3$，成立

$$
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$$

β 可由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示。

- 显然 $\beta' = \alpha_1' - \alpha_2' + \alpha_3'$，是否也有 $\beta = \alpha_1 - \alpha_2 + \alpha_3$？

初等行变换求线性表示问题——例 1

例 判断 β 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

(1)

$$(\alpha_1 \ \alpha_2 \ \alpha_3 \,|\, \beta) = \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{pmatrix} \xrightarrow{\text{初等行变换}} \begin{matrix} \alpha_1' & \alpha_2' & \alpha_3' & \beta' \\ \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{matrix}$$

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初等行变换求线性表示问题——例 1

例 判断 β 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

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注 可证明：作初等行变换不改变列与列之间的"线性关系"。

例 判断 β 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

(2)

$$(\alpha_1 \quad \alpha_2 \quad \alpha_3 \mid \beta) = \begin{pmatrix} 1 & 2 & -1 & 4 \\ 2 & -1 & 3 & 3 \\ -1 & 1 & -2 & 0 \\ 5 & 1 & 4 & 11 \end{pmatrix}$$

例 判断 β 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

(2)

$$(\alpha_1 \ \alpha_2 \ \alpha_3 \,|\, \beta) = \begin{pmatrix} 1 & 2 & -1 & 4 \\ 2 & -1 & 3 & 3 \\ -1 & 1 & -2 & 0 \\ 5 & 1 & 4 & 11 \end{pmatrix}$$

- $r(\alpha_1 \alpha_2 \alpha_3) = \quad , \ r(\alpha_1 \alpha_2 \alpha_3 \beta) = \quad ,$

初等行变换求线性表示问题——例 2

例 判断 β 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

(2)

$$(\alpha_1 \ \alpha_2 \ \alpha_3 \,|\, \beta) = \left(\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 2 & -1 & 3 & 3 \\ -1 & 1 & -2 & 0 \\ 5 & 1 & 4 & 11 \end{array} \right) \xrightarrow{\text{初等行变换}} \left(\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

- $r(\alpha_1 \alpha_2 \alpha_3) = \quad , \ r(\alpha_1 \alpha_2 \alpha_3 \beta) = \quad ,$

例 判断 β 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

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$$(\alpha_1 \ \alpha_2 \ \alpha_3 \,|\, \beta) = \begin{pmatrix} 1 & 2 & -1 & 4 \\ 2 & -1 & 3 & 3 \\ -1 & 1 & -2 & 0 \\ 5 & 1 & 4 & 11 \end{pmatrix} \xrightarrow{\text{初等行变换}} \begin{pmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- $r(\alpha_1 \alpha_2 \alpha_3) = \quad$, $r(\alpha_1 \alpha_2 \alpha_3 \beta) = \quad$,

初等行变换求线性表示问题——例 2

例 判断 β 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

(2)

$$(\ \alpha_1\ \ \alpha_2\ \ \alpha_3\ |\ \beta\ ) = \begin{pmatrix} 1 & 2 & -1 & 4 \\ 2 & -1 & 3 & 3 \\ -1 & 1 & -2 & 0 \\ 5 & 1 & 4 & 11 \end{pmatrix} \xrightarrow{\text{初等行变换}} \begin{pmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- $r(\alpha_1\alpha_2\alpha_3) = \quad , \ r(\alpha_1\alpha_2\alpha_3\beta) = 3,$

例 判断 β 是否能由 α_1, α_2, α_3 线性表示，若能，写出线性表示等式。

(2)

$$(\alpha_1 \ \alpha_2 \ \alpha_3 \,|\, \beta) = \begin{pmatrix} 1 & 2 & -1 & \Big| & 4 \\ 2 & -1 & 3 & \Big| & 3 \\ -1 & 1 & -2 & \Big| & 0 \\ 5 & 1 & 4 & \Big| & 11 \end{pmatrix} \xrightarrow{\text{初等行变换}} \begin{pmatrix} 1 & 2 & -1 & \Big| & 4 \\ 0 & 1 & -1 & \Big| & 1 \\ 0 & 0 & 0 & \Big| & 1 \\ 0 & 0 & 0 & \Big| & 0 \end{pmatrix}$$

- $r(\alpha_1 \alpha_2 \alpha_3) = 2$, $r(\alpha_1 \alpha_2 \alpha_3 \beta) = 3$,

初等行变换求线性表示问题——例 2

例 判断 β 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

(2)

$$(\alpha_1 \ \alpha_2 \ \alpha_3 \,|\, \beta) = \left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 2 & -1 & 3 & 3 \\ -1 & 1 & -2 & 0 \\ 5 & 1 & 4 & 11 \end{array} \right) \xrightarrow{\text{初等行变换}} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

- 所以 $r(\alpha_1\alpha_2\alpha_3) = 2$，$r(\alpha_1\alpha_2\alpha_3\beta) = 3$，成立

$$r(\alpha_1\alpha_2\alpha_3) \neq r(\alpha_1\alpha_2\alpha_3\beta)$$

初等行变换求线性表示问题——例 2

例 判断 β 是否能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，若能，写出线性表示等式。

(2)

$$(\alpha_1 \ \alpha_2 \ \alpha_3 \,|\, \beta) = \begin{pmatrix} 1 & 2 & -1 & 4 \\ 2 & -1 & 3 & 3 \\ -1 & 1 & -2 & 0 \\ 5 & 1 & 4 & 11 \end{pmatrix} \xrightarrow{\text{初等行变换}} \begin{pmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- 所以 $r(\alpha_1\alpha_2\alpha_3) = 2$，$r(\alpha_1\alpha_2\alpha_3\beta) = 3$，成立

$$r(\alpha_1\alpha_2\alpha_3) \neq r(\alpha_1\alpha_2\alpha_3\beta)$$

β 不能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示。

初等行变换求线性表示问题——总结

问题 β 能否由 $\alpha_1, \alpha_2, \ldots, \alpha_n$ 线性表示？若能，写出线性表示等式。

步骤

初等行变换求线性表示问题——总结

问题 β 能否由 $\alpha_1, \alpha_2, \ldots, \alpha_n$ 线性表示？若能，写出线性表示等式。

步骤 作初等 行 变换：

$$(\, \alpha_1 \ \ \alpha_2 \ \ \cdots \ \ \alpha_n \, | \, \beta \,)$$

初等行变换求线性表示问题——总结

问题 β 能否由 $\alpha_1, \alpha_2, \ldots, \alpha_n$ 线性表示？若能，写出线性表示等式。

步骤 作初等 **行** 变换：

$$(\ \alpha_1 \ \ \alpha_2 \ \ \cdots \ \ \alpha_n \,|\, \beta \ ) \ \xrightarrow{\text{初等行变换}}$$

初等行变换求线性表示问题——总结

问题 β 能否由 $\alpha_1, \alpha_2, \ldots, \alpha_n$ 线性表示？若能，写出线性表示等式。

步骤 作初等 行 变换：

$$(\alpha_1 \ \alpha_2 \ \cdots \ \alpha_n \,|\, \beta) \xrightarrow{\text{初等行变换}} (\alpha_1' \ \alpha_2' \ \cdots \ \alpha_n' \,|\, \beta')_{\substack{\text{（简化）}\\\text{阶梯型矩阵}}}$$

初等行变换求线性表示问题——总结

问题 β 能否由 $\alpha_1, \alpha_2, \ldots, \alpha_n$ 线性表示？若能，写出线性表示等式。

步骤 作初等 行 变换：

$$(\ \alpha_1 \ \ \alpha_2 \ \ \cdots \ \ \alpha_n \,|\, \beta \ ) \ \xrightarrow{\text{初等行变换}} \ (\ \alpha_1' \ \ \alpha_2' \ \ \cdots \ \ \alpha_n' \,|\, \beta' \ )_{\substack{\text{（简化）}\\\text{阶梯型矩阵}}}$$

1.

   β 由 $\alpha_1, \alpha_2, \ldots, \alpha_n$ 线性表示 $\iff$ $r(\alpha_1 \cdots \alpha_n) = r(\alpha_1 \cdots \alpha_n \beta)$

初等行变换求线性表示问题——总结

问题 β 能否由 $\alpha_1, \alpha_2, \ldots, \alpha_n$ 线性表示？若能，写出线性表示等式。

步骤 作初等 **行** 变换：

$$(\; \alpha_1 \;\; \alpha_2 \; \cdots \; \alpha_n \,|\, \beta \;) \xrightarrow{\text{初等行变换}} (\; \alpha_1' \;\; \alpha_2' \; \cdots \; \alpha_n' \,|\, \beta' \;)^{\text{(简化)}}_{\text{阶梯型矩阵}}$$

1.

$$\beta \text{由} \alpha_1, \alpha_2, \ldots, \alpha_n \text{线性表示} \iff r(\alpha_1 \cdots \alpha_n) = r(\alpha_1 \cdots \alpha_n \beta)$$

$$\Updownarrow$$

$$r(\alpha_1' \cdots \alpha_n') = r(\alpha_1' \cdots \alpha_n' \beta')$$

初等行变换求线性表示问题——总结

问题 β 能否由 $\alpha_1, \alpha_2, \ldots, \alpha_n$ 线性表示？若能，写出线性表示等式。

步骤 作初等 行 变换：

$$(\begin{array}{cccc|c} \alpha_1 & \alpha_2 & \cdots & \alpha_n & \beta \end{array}) \xrightarrow{\text{初等} \color{red}\text{行}\color{black}\text{变换}} (\begin{array}{cccc|c} \alpha_1' & \alpha_2' & \cdots & \alpha_n' & \beta' \end{array})_{\substack{(简化)\\ \text{阶梯型矩阵}}}$$

1.

   β 由 $\alpha_1, \alpha_2, \ldots, \alpha_n$ 线性表示 $\iff$ $r(\alpha_1 \cdots \alpha_n) = r(\alpha_1 \cdots \alpha_n \beta)$

   $$\Updownarrow$$

   $$r(\alpha_1' \cdots \alpha_n') = r(\alpha_1' \cdots \alpha_n' \beta')$$

2. 行变换前后列与列的线性关系不变，即：

初等行变换求线性表示问题——总结

问题 β 能否由 $\alpha_1, \alpha_2, \ldots, \alpha_n$ 线性表示？若能，写出线性表示等式。

步骤 作初等 行 变换：

$$(\alpha_1 \ \alpha_2 \ \cdots \ \alpha_n \,|\, \beta) \xrightarrow{\text{初等\color{red}行\color{black}变换}} (\alpha'_1 \ \alpha'_2 \ \cdots \ \alpha'_n \,|\, \beta')^{(简化)}_{阶梯型矩阵}$$

1.
$$\beta 由 \alpha_1, \alpha_2, \ldots, \alpha_n 线性表示 \iff r(\alpha_1 \cdots \alpha_n) = r(\alpha_1 \cdots \alpha_n \beta)$$

$$\updownarrow$$

$$r(\alpha'_1 \cdots \alpha'_n) = r(\alpha'_1 \cdots \alpha'_n \beta')$$

2. 行变换前后列与列的线性关系不变，即：

$$\beta' = k_1 \alpha'_1 + \cdots + k_n \alpha'_n \quad \Rightarrow$$

初等行变换求线性表示问题——总结

问题 β 能否由 $\alpha_1, \alpha_2, \ldots, \alpha_n$ 线性表示？若能，写出线性表示等式。

步骤 作初等 **行** 变换：

$$(\alpha_1 \ \alpha_2 \ \cdots \ \alpha_n \,|\, \beta) \xrightarrow{\text{初等行变换}} (\alpha_1' \ \alpha_2' \ \cdots \ \alpha_n' \,|\, \beta')^{\text{（简化）}}_{\text{阶梯型矩阵}}$$

1.

$$\beta \text{由} \alpha_1, \alpha_2, \ldots, \alpha_n \text{线性表示} \iff r(\alpha_1 \cdots \alpha_n) = r(\alpha_1 \cdots \alpha_n \beta)$$

$$\Updownarrow$$

$$r(\alpha_1' \cdots \alpha_n') = r(\alpha_1' \cdots \alpha_n' \beta')$$

2. 行变换前后列与列的线性关系不变，即：

$$\beta' = k_1 \alpha_1' + \cdots + k_n \alpha_n' \implies \beta = k_1 \alpha_1 + \cdots + k_n \alpha_n$$

例 1 $\beta = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix}$ 能否由 $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ 线性表示?

解

例 1 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$ 能否由 $\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$ 线性表示？

解 $\quad \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \beta$

$$\left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\0 & -1 & 2 & 3\\1 & 1 & 0 & 0\\2 & -2 & 1 & 5\end{array}\right)$$

例 1 $\beta = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix}$ 能否由 $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ 线性表示？

解 $\quad \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \beta$

$$\begin{pmatrix} 1 & 2 & 3 & \bigg| & 2 \\ 0 & -1 & 2 & \bigg| & 3 \\ 1 & 1 & 0 & \bigg| & 0 \\ 2 & -2 & 1 & \bigg| & 5 \end{pmatrix} \xrightarrow[r_4 - 2r_1]{r_3 - r_1}$$

例 1 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$ 能否由 $\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$ 线性表示？

解 $\quad\alpha_1\quad\alpha_2\quad\alpha_3\quad\beta$

$$\begin{pmatrix}1 & 2 & 3 & 2\\0 & -1 & 2 & 3\\1 & 1 & 0 & 0\\2 & -2 & 1 & 5\end{pmatrix}\xrightarrow[r_4-2r_1]{r_3-r_1}\left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\0 & -1 & 2 & 3\\ & & & \\ & & & \end{array}\right)$$

例 1 $\beta=\begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix}$ 能否由 $\alpha_1=\begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_2=\begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix}$, $\alpha_3=\begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ 线性表示？

解 $\quad \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \beta$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{array}\right) \xrightarrow[r_4-2r_1]{r_3-r_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 0 & -1 & -3 & -2 \end{array}\right)$$

例 1 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$ 能否由 $\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$ 线性表示？

解 $\quad\alpha_1\quad\alpha_2\quad\alpha_3\quad\beta$

$$\begin{pmatrix}1 & 2 & 3 & \Big| & 2\\0 & -1 & 2 & \Big| & 3\\1 & 1 & 0 & \Big| & 0\\2 & -2 & 1 & \Big| & 5\end{pmatrix}\xrightarrow[r_4-2r_1]{r_3-r_1}\begin{pmatrix}1 & 2 & 3 & \Big| & 2\\0 & -1 & 2 & \Big| & 3\\0 & -1 & -3 & \Big| & -2\\0 & -6 & -5 & \Big| & 1\end{pmatrix}$$

例 1 $\beta = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix}$ 能否由 $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ 线性表示？

解 $\quad \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \beta$

$$\begin{pmatrix} 1 & 2 & 3 & \bigm| & 2 \\ 0 & -1 & 2 & \bigm| & 3 \\ 1 & 1 & 0 & \bigm| & 0 \\ 2 & -2 & 1 & \bigm| & 5 \end{pmatrix} \xrightarrow[r_4 - 2r_1]{r_3 - r_1} \begin{pmatrix} 1 & 2 & 3 & \bigm| & 2 \\ 0 & -1 & 2 & \bigm| & 3 \\ 0 & -1 & -3 & \bigm| & -2 \\ 0 & -6 & -5 & \bigm| & 1 \end{pmatrix} \xrightarrow{(-1) \times r_2}$$

例 1 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$ 能否由 $\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$ 线性表示？

解 $\quad\alpha_1\quad\alpha_2\quad\alpha_3\quad\beta$

$$\begin{pmatrix}1 & 2 & 3 & 2\\0 & -1 & 2 & 3\\1 & 1 & 0 & 0\\2 & -2 & 1 & 5\end{pmatrix} \xrightarrow[r_4-2r_1]{r_3-r_1} \left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\0 & -1 & 2 & 3\\0 & -1 & -3 & -2\\0 & -6 & -5 & 1\end{array}\right) \xrightarrow{(-1)\times r_2} \left(\begin{array}{ccc|c}1 & 2 & 3 & 2\\0 & 1 & -2 & -3\\0 & -1 & -3 & -2\\0 & -6 & -5 & 1\end{array}\right)$$

例 1 $\beta = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix}$ 能否由 $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ 线性表示？

解 $\quad \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \beta$

$$
\begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{pmatrix} \xrightarrow[r_4 - 2r_1]{r_3 - r_1} \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 0 & -1 & -3 & -2 \\ 0 & -6 & -5 & 1 \end{pmatrix} \xrightarrow{(-1) \times r_2} \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & -1 & -3 & -2 \\ 0 & -6 & -5 & 1 \end{pmatrix}
$$

$$
\xrightarrow[r_4 + 6r_2]{r_3 + r_2}
$$

例 1 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$ 能否由 $\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$ 线性表示？

解 $\quad\alpha_1\quad\alpha_2\quad\alpha_3\quad\beta$

$$\left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\1&1&0&0\\2&-2&1&5\end{array}\right)\xrightarrow[r_4-2r_1]{r_3-r_1}\left(\begin{array}{ccc|c}1&2&3&2\\0&-1&2&3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)\xrightarrow{(-1)\times r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\0&-1&-3&-2\\0&-6&-5&1\end{array}\right)$$

$$\xrightarrow[r_4+6r_2]{r_3+r_2}\left(\begin{array}{ccc|c}1&2&3&2\\0&1&-2&-3\\&&&\\&&&\end{array}\right)$$

例 1 $\beta = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix}$ 能否由 $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ 线性表示？

解

$$\begin{matrix} \alpha_1 & \alpha_2 & \alpha_3 & \beta \end{matrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{pmatrix} \xrightarrow[r_4-2r_1]{r_3-r_1} \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 0 & -1 & -3 & -2 \\ 0 & -6 & -5 & 1 \end{pmatrix} \xrightarrow{(-1) \times r_2} \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & -1 & -3 & -2 \\ 0 & -6 & -5 & 1 \end{pmatrix}$$

$$\xrightarrow[r_4+6r_2]{r_3+r_2} \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & -5 & -5 \end{pmatrix}$$

例 1 $\beta = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix}$ 能否由 $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ 线性表示？

解

$$
\begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \beta \end{array}
$$

$$
\left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{array}\right)
\xrightarrow[r_4-2r_1]{r_3-r_1}
\left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 0 & -1 & -3 & -2 \\ 0 & -6 & -5 & 1 \end{array}\right)
\xrightarrow{(-1)\times r_2}
\left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & -1 & -3 & -2 \\ 0 & -6 & -5 & 1 \end{array}\right)
$$

$$
\xrightarrow[r_4+6r_2]{r_3+r_2}
\left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & -5 & -5 \\ 0 & 0 & -17 & -17 \end{array}\right)
$$

例 1 $\beta = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix}$ 能否由 $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ 线性表示？

解
$$\begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \beta \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{array} \right) \xrightarrow[r_4-2r_1]{r_3-r_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 0 & -1 & -3 & -2 \\ 0 & -6 & -5 & 1 \end{array} \right) \xrightarrow{(-1)\times r_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & -1 & -3 & -2 \\ 0 & -6 & -5 & 1 \end{array} \right)$$

$$\xrightarrow[r_4+6r_2]{r_3+r_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & -5 & -5 \\ 0 & 0 & -17 & -17 \end{array} \right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array} \right)$$

例 1 $\beta = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix}$ 能否由 $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ 线性表示？

解 $\quad \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \beta$

$$\begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{pmatrix} \xrightarrow[r_4 - 2r_1]{r_3 - r_1} \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 0 & -1 & -3 & -2 \\ 0 & -6 & -5 & 1 \end{pmatrix} \xrightarrow{(-1) \times r_2} \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & -1 & -3 & -2 \\ 0 & -6 & -5 & 1 \end{pmatrix}$$

$$\xrightarrow[r_4 + 6r_2]{r_3 + r_2} \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & -5 & -5 \\ 0 & 0 & -17 & -17 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \xrightarrow{r_4 - r_3}$$

例 1 $\beta = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix}$ 能否由 $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ 线性表示？

解 $\quad \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \beta$

$$\begin{pmatrix} 1 & 2 & 3 & \bigm| & 2 \\ 0 & -1 & 2 & \bigm| & 3 \\ 1 & 1 & 0 & \bigm| & 0 \\ 2 & -2 & 1 & \bigm| & 5 \end{pmatrix} \xrightarrow[r_4-2r_1]{r_3-r_1} \begin{pmatrix} 1 & 2 & 3 & \bigm| & 2 \\ 0 & -1 & 2 & \bigm| & 3 \\ 0 & -1 & -3 & \bigm| & -2 \\ 0 & -6 & -5 & \bigm| & 1 \end{pmatrix} \xrightarrow{(-1)\times r_2} \begin{pmatrix} 1 & 2 & 3 & \bigm| & 2 \\ 0 & 1 & -2 & \bigm| & -3 \\ 0 & -1 & -3 & \bigm| & -2 \\ 0 & -6 & -5 & \bigm| & 1 \end{pmatrix}$$

$$\xrightarrow[r_4+6r_2]{r_3+r_2} \begin{pmatrix} 1 & 2 & 3 & \bigm| & 2 \\ 0 & 1 & -2 & \bigm| & -3 \\ 0 & 0 & -5 & \bigm| & -5 \\ 0 & 0 & -17 & \bigm| & -17 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 & \bigm| & 2 \\ 0 & 1 & -2 & \bigm| & -3 \\ 0 & 0 & 1 & \bigm| & 1 \\ 0 & 0 & 1 & \bigm| & 1 \end{pmatrix} \xrightarrow{r_4-r_3} \begin{pmatrix} 1 & 2 & 3 & \bigm| & 2 \\ 0 & 1 & -2 & \bigm| & -3 \\ 0 & 0 & 1 & \bigm| & 1 \\ 0 & 0 & 0 & \bigm| & 0 \end{pmatrix}$$

例 1 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$ 能否由 $\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$ 线性表示？

解 $\quad \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \beta$

$$\begin{pmatrix}1&2&3&\bigm|&2\\0&-1&2&\bigm|&3\\1&1&0&\bigm|&0\\2&-2&1&\bigm|&5\end{pmatrix}\xrightarrow[r_4-2r_1]{r_3-r_1}\begin{pmatrix}1&2&3&\bigm|&2\\0&-1&2&\bigm|&3\\0&-1&-3&\bigm|&-2\\0&-6&-5&\bigm|&1\end{pmatrix}\xrightarrow{(-1)\times r_2}\begin{pmatrix}1&2&3&\bigm|&2\\0&1&-2&\bigm|&-3\\0&-1&-3&\bigm|&-2\\0&-6&-5&\bigm|&1\end{pmatrix}$$

$$\xrightarrow[r_4+6r_2]{r_3+r_2}\begin{pmatrix}1&2&3&\bigm|&2\\0&1&-2&\bigm|&-3\\0&0&-5&\bigm|&-5\\0&0&-17&\bigm|&-17\end{pmatrix}\longrightarrow\begin{pmatrix}1&2&3&\bigm|&2\\0&1&-2&\bigm|&-3\\0&0&1&\bigm|&1\\0&0&1&\bigm|&1\end{pmatrix}\xrightarrow{r_4-r_3}\begin{pmatrix}1&2&3&\bigm|&2\\0&1&-2&\bigm|&-3\\0&0&1&\bigm|&1\\0&0&0&\bigm|&0\end{pmatrix}$$

$$\xrightarrow[r_1-3r_3]{r_2-2r_3}$$

例 1 $\beta = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix}$ 能否由 $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ 线性表示？

解　$\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \beta$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{array}\right) \xrightarrow[r_4-2r_1]{r_3-r_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 0 & -1 & -3 & -2 \\ 0 & -6 & -5 & 1 \end{array}\right) \xrightarrow{(-1)\times r_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & -1 & -3 & -2 \\ 0 & -6 & -5 & 1 \end{array}\right)$$

$$\xrightarrow[r_4+6r_2]{r_3+r_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & -5 & -5 \\ 0 & 0 & -17 & -17 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow{r_4-r_3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[r_1-3r_3]{r_2-2r_3} \left(\begin{array}{ccc|c} & & & \\ & & & \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

例1 $\beta = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix}$ 能否由 $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ 线性表示？

解 $\quad \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \beta$

$$\begin{pmatrix} 1 & 2 & 3 & | & 2 \\ 0 & -1 & 2 & | & 3 \\ 1 & 1 & 0 & | & 0 \\ 2 & -2 & 1 & | & 5 \end{pmatrix} \xrightarrow[r_4-2r_1]{r_3-r_1} \begin{pmatrix} 1 & 2 & 3 & | & 2 \\ 0 & -1 & 2 & | & 3 \\ 0 & -1 & -3 & | & -2 \\ 0 & -6 & -5 & | & 1 \end{pmatrix} \xrightarrow{(-1)\times r_2} \begin{pmatrix} 1 & 2 & 3 & | & 2 \\ 0 & 1 & -2 & | & -3 \\ 0 & -1 & -3 & | & -2 \\ 0 & -6 & -5 & | & 1 \end{pmatrix}$$

$$\xrightarrow[r_4+6r_2]{r_3+r_2} \begin{pmatrix} 1 & 2 & 3 & | & 2 \\ 0 & 1 & -2 & | & -3 \\ 0 & 0 & -5 & | & -5 \\ 0 & 0 & -17 & | & -17 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 & | & 2 \\ 0 & 1 & -2 & | & -3 \\ 0 & 0 & 1 & | & 1 \\ 0 & 0 & 1 & | & 1 \end{pmatrix} \xrightarrow{r_4-r_3} \begin{pmatrix} 1 & 2 & 3 & | & 2 \\ 0 & 1 & -2 & | & -3 \\ 0 & 0 & 1 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$\xrightarrow[r_1-3r_3]{r_2-2r_3} \begin{pmatrix} 0 & 1 & 0 & | & -1 \\ 0 & 0 & 1 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

例 1 $\beta = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix}$ 能否由 $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ 线性表示？

解 $\quad \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \beta$

$$\begin{pmatrix} 1 & 2 & 3 & | & 2 \\ 0 & -1 & 2 & | & 3 \\ 1 & 1 & 0 & | & 0 \\ 2 & -2 & 1 & | & 5 \end{pmatrix} \xrightarrow[r_4 - 2r_1]{r_3 - r_1} \begin{pmatrix} 1 & 2 & 3 & | & 2 \\ 0 & -1 & 2 & | & 3 \\ 0 & -1 & -3 & | & -2 \\ 0 & -6 & -5 & | & 1 \end{pmatrix} \xrightarrow{(-1) \times r_2} \begin{pmatrix} 1 & 2 & 3 & | & 2 \\ 0 & 1 & -2 & | & -3 \\ 0 & -1 & -3 & | & -2 \\ 0 & -6 & -5 & | & 1 \end{pmatrix}$$

$$\xrightarrow[r_4 + 6r_2]{r_3 + r_2} \begin{pmatrix} 1 & 2 & 3 & | & 2 \\ 0 & 1 & -2 & | & -3 \\ 0 & 0 & -5 & | & -5 \\ 0 & 0 & -17 & | & -17 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 & | & 2 \\ 0 & 1 & -2 & | & -3 \\ 0 & 0 & 1 & | & 1 \\ 0 & 0 & 1 & | & 1 \end{pmatrix} \xrightarrow{r_4 - r_3} \begin{pmatrix} 1 & 2 & 3 & | & 2 \\ 0 & 1 & -2 & | & -3 \\ 0 & 0 & 1 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$\xrightarrow[r_1 - 3r_3]{r_2 - 2r_3} \begin{pmatrix} 1 & 2 & 0 & | & -1 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 1 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

例 1 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$ 能否由 $\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$ 线性表示？

解 $\quad\alpha_1\quad\alpha_2\quad\alpha_3\quad\beta$

$$\begin{pmatrix}1 & 2 & 3 & 2\\ 0 & -1 & 2 & 3\\ 1 & 1 & 0 & 0\\ 2 & -2 & 1 & 5\end{pmatrix}\xrightarrow[r_4-2r_1]{r_3-r_1}\begin{pmatrix}1 & 2 & 3 & 2\\ 0 & -1 & 2 & 3\\ 0 & -1 & -3 & -2\\ 0 & -6 & -5 & 1\end{pmatrix}\xrightarrow{(-1)\times r_2}\begin{pmatrix}1 & 2 & 3 & 2\\ 0 & 1 & -2 & -3\\ 0 & -1 & -3 & -2\\ 0 & -6 & -5 & 1\end{pmatrix}$$

$$\xrightarrow[r_4+6r_2]{r_3+r_2}\begin{pmatrix}1 & 2 & 3 & 2\\ 0 & 1 & -2 & -3\\ 0 & 0 & -5 & -5\\ 0 & 0 & -17 & -17\end{pmatrix}\longrightarrow\begin{pmatrix}1 & 2 & 3 & 2\\ 0 & 1 & -2 & -3\\ 0 & 0 & 1 & 1\\ 0 & 0 & 1 & 1\end{pmatrix}\xrightarrow{r_4-r_3}\begin{pmatrix}1 & 2 & 3 & 2\\ 0 & 1 & -2 & -3\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0\end{pmatrix}$$

$$\xrightarrow[r_1-3r_3]{r_2-2r_3}\begin{pmatrix}1 & 2 & 0 & -1\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0\end{pmatrix}\xrightarrow{r_1-2r_2}$$

例 1 $\beta = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix}$ 能否由 $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ 线性表示？

解

$$
\begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \beta \end{array}
$$

$$
\left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 5 \end{array}\right) \xrightarrow[r_4-2r_1]{r_3-r_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 3 \\ 0 & -1 & -3 & -2 \\ 0 & -6 & -5 & 1 \end{array}\right) \xrightarrow{(-1)\times r_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & -1 & -3 & -2 \\ 0 & -6 & -5 & 1 \end{array}\right)
$$

$$
\xrightarrow[r_4+6r_2]{r_3+r_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & -5 & -5 \\ 0 & 0 & -17 & -17 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow{r_4-r_3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)
$$

$$
\xrightarrow[r_1-3r_3]{r_2-2r_3} \left(\begin{array}{ccc|c} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{r_1-2r_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)
$$

例 1 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$ 能否由 $\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$ 线性表示？

解 $\quad \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \beta$

$$\begin{pmatrix}1&2&3&\Big|&2\\0&-1&2&\Big|&3\\1&1&0&\Big|&0\\2&-2&1&\Big|&5\end{pmatrix}\xrightarrow[r_4-2r_1]{r_3-r_1}\begin{pmatrix}1&2&3&\Big|&2\\0&-1&2&\Big|&3\\0&-1&-3&\Big|&-2\\0&-6&-5&\Big|&1\end{pmatrix}\xrightarrow{(-1)\times r_2}\begin{pmatrix}1&2&3&\Big|&2\\0&1&-2&\Big|&-3\\0&-1&-3&\Big|&-2\\0&-6&-5&\Big|&1\end{pmatrix}$$

$$\xrightarrow[r_4+6r_2]{r_3+r_2}\begin{pmatrix}1&2&3&\Big|&2\\0&1&-2&\Big|&-3\\0&0&-5&\Big|&-5\\0&0&-17&\Big|&-17\end{pmatrix}\rightarrow\begin{pmatrix}1&2&3&\Big|&2\\0&1&-2&\Big|&-3\\0&0&1&\Big|&1\\0&0&1&\Big|&1\end{pmatrix}\xrightarrow{r_4-r_3}\begin{pmatrix}1&2&3&\Big|&2\\0&1&-2&\Big|&-3\\0&0&1&\Big|&1\\0&0&0&\Big|&0\end{pmatrix}$$

$$\xrightarrow[r_1-3r_3]{r_2-2r_3}\begin{pmatrix}1&2&0&\Big|&-1\\0&1&0&\Big|&-1\\0&0&1&\Big|&1\\0&0&0&\Big|&0\end{pmatrix}\xrightarrow{r_1-2r_2}\begin{pmatrix}1&0&0&\Big|&1\\0&1&0&\Big|&-1\\0&0&1&\Big|&1\\0&0&0&\Big|&0\end{pmatrix}$$

例 1 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$ 能否由 $\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$ 线性表示？

解 $\quad\alpha_1\quad\alpha_2\quad\alpha_3\quad\beta$

$$\begin{pmatrix}1 & 2 & 3 & 2\\0 & -1 & 2 & 3\\1 & 1 & 0 & 0\\2 & -2 & 1 & 5\end{pmatrix}\xrightarrow[r_4-2r_1]{r_3-r_1}\begin{pmatrix}1 & 2 & 3 & 2\\0 & -1 & 2 & 3\\0 & -1 & -3 & -2\\0 & -6 & -5 & 1\end{pmatrix}\xrightarrow{(-1)\times r_2}\begin{pmatrix}1 & 2 & 3 & 2\\0 & 1 & -2 & -3\\0 & -1 & -3 & -2\\0 & -6 & -5 & 1\end{pmatrix}$$

$$\xrightarrow[r_4+6r_2]{r_3+r_2}\begin{pmatrix}1 & 2 & 3 & 2\\0 & 1 & -2 & -3\\0 & 0 & -5 & -5\\0 & 0 & -17 & -17\end{pmatrix}\longrightarrow\begin{pmatrix}1 & 2 & 3 & 2\\0 & 1 & -2 & -3\\0 & 0 & 1 & 1\\0 & 0 & 1 & 1\end{pmatrix}\xrightarrow{r_4-r_3}\begin{pmatrix}1 & 2 & 3 & 2\\0 & 1 & -2 & -3\\0 & 0 & 1 & 1\\0 & 0 & 0 & 0\end{pmatrix}$$

$$\xrightarrow[r_1-3r_3]{r_2-2r_3}\begin{pmatrix}1 & 2 & 0 & -1\\0 & 1 & 0 & -1\\0 & 0 & 1 & 1\\0 & 0 & 0 & 0\end{pmatrix}\xrightarrow{r_1-2r_2}\begin{pmatrix}1 & 0 & 0 & 1\\0 & 1 & 0 & -1\\0 & 0 & 1 & 1\\0 & 0 & 0 & 0\end{pmatrix}$$

所以 $r(\alpha_1\alpha_2\alpha_3)=r(\alpha_1\alpha_2\alpha_3\beta)$,

例 1 $\beta = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 5 \end{pmatrix}$ 能否由 $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ 线性表示？

解 $\quad \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \beta$

$$\begin{pmatrix} 1 & 2 & 3 & | & 2 \\ 0 & -1 & 2 & | & 3 \\ 1 & 1 & 0 & | & 0 \\ 2 & -2 & 1 & | & 5 \end{pmatrix} \xrightarrow[r_4 - 2r_1]{r_3 - r_1} \begin{pmatrix} 1 & 2 & 3 & | & 2 \\ 0 & -1 & 2 & | & 3 \\ 0 & -1 & -3 & | & -2 \\ 0 & -6 & -5 & | & 1 \end{pmatrix} \xrightarrow{(-1) \times r_2} \begin{pmatrix} 1 & 2 & 3 & | & 2 \\ 0 & 1 & -2 & | & -3 \\ 0 & -1 & -3 & | & -2 \\ 0 & -6 & -5 & | & 1 \end{pmatrix}$$

$$\xrightarrow[r_4 + 6r_2]{r_3 + r_2} \begin{pmatrix} 1 & 2 & 3 & | & 2 \\ 0 & 1 & -2 & | & -3 \\ 0 & 0 & -5 & | & -5 \\ 0 & 0 & -17 & | & -17 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 & | & 2 \\ 0 & 1 & -2 & | & -3 \\ 0 & 0 & 1 & | & 1 \\ 0 & 0 & 1 & | & 1 \end{pmatrix} \xrightarrow{r_4 - r_3} \begin{pmatrix} 1 & 2 & 3 & | & 2 \\ 0 & 1 & -2 & | & -3 \\ 0 & 0 & 1 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$\xrightarrow[r_1 - 3r_3]{r_2 - 2r_3} \begin{pmatrix} 1 & 2 & 0 & | & -1 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 1 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \xrightarrow{r_1 - 2r_2} \begin{pmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 1 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

所以 $r(\alpha_1 \alpha_2 \alpha_3) = r(\alpha_1 \alpha_2 \alpha_3 \beta)$，能线性表示

例 1 $\beta=\begin{pmatrix}2\\3\\0\\5\end{pmatrix}$ 能否由 $\alpha_1=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$, $\alpha_2=\begin{pmatrix}2\\-1\\1\\-2\end{pmatrix}$, $\alpha_3=\begin{pmatrix}3\\2\\0\\1\end{pmatrix}$ 线性表示？

解 $\quad \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \beta$

$$\begin{pmatrix}1 & 2 & 3 & \bigm| & 2\\0 & -1 & 2 & \bigm| & 3\\1 & 1 & 0 & \bigm| & 0\\2 & -2 & 1 & \bigm| & 5\end{pmatrix}\xrightarrow[r_4-2r_1]{r_3-r_1}\begin{pmatrix}1 & 2 & 3 & \bigm| & 2\\0 & -1 & 2 & \bigm| & 3\\0 & -1 & -3 & \bigm| & -2\\0 & -6 & -5 & \bigm| & 1\end{pmatrix}\xrightarrow{(-1)\times r_2}\begin{pmatrix}1 & 2 & 3 & \bigm| & 2\\0 & 1 & -2 & \bigm| & -3\\0 & -1 & -3 & \bigm| & -2\\0 & -6 & -5 & \bigm| & 1\end{pmatrix}$$

$$\xrightarrow[r_4+6r_2]{r_3+r_2}\begin{pmatrix}1 & 2 & 3 & \bigm| & 2\\0 & 1 & -2 & \bigm| & -3\\0 & 0 & -5 & \bigm| & -5\\0 & 0 & -17 & \bigm| & -17\end{pmatrix}\rightarrow\begin{pmatrix}1 & 2 & 3 & \bigm| & 2\\0 & 1 & -2 & \bigm| & -3\\0 & 0 & 1 & \bigm| & 1\\0 & 0 & 1 & \bigm| & 1\end{pmatrix}\xrightarrow{r_4-r_3}\begin{pmatrix}1 & 2 & 3 & \bigm| & 2\\0 & 1 & -2 & \bigm| & -3\\0 & 0 & 1 & \bigm| & 1\\0 & 0 & 0 & \bigm| & 0\end{pmatrix}$$

$$\xrightarrow[r_1-3r_3]{r_2-2r_3}\begin{pmatrix}1 & 2 & 0 & \bigm| & -1\\0 & 1 & 0 & \bigm| & -1\\0 & 0 & 1 & \bigm| & 1\\0 & 0 & 0 & \bigm| & 0\end{pmatrix}\xrightarrow{r_1-2r_2}\begin{pmatrix}1 & 0 & 0 & \bigm| & 1\\0 & 1 & 0 & \bigm| & -1\\0 & 0 & 1 & \bigm| & 1\\0 & 0 & 0 & \bigm| & 0\end{pmatrix}$$

所以 $r(\alpha_1\alpha_2\alpha_3) = r(\alpha_1\alpha_2\alpha_3\beta)$，能线性表示，且 $\beta = \alpha_1 - \alpha_2 + \alpha_3$

例 2 $\beta = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 6 \end{pmatrix}$ 能否由 $\alpha_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} -1 \\ 3 \\ -2 \\ 4 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 4 \\ 3 \\ 0 \\ 11 \end{pmatrix}$ 线性表示？

解

例 2 $\beta = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 6 \end{pmatrix}$ 能否由 $\alpha_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} -1 \\ 3 \\ -2 \\ 4 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 4 \\ 3 \\ 0 \\ 11 \end{pmatrix}$ 线性表示？

解

$$\begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \beta \end{array}$$

$$\left(\begin{array}{ccc|c} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{array} \right)$$

例 2 $\beta = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 6 \end{pmatrix}$ 能否由 $\alpha_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} -1 \\ 3 \\ -2 \\ 4 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 4 \\ 3 \\ 0 \\ 11 \end{pmatrix}$ 线性表示？

解
$$\begin{matrix} \alpha_1 & \alpha_2 & \alpha_3 & \beta \end{matrix}$$
$$\left(\begin{array}{ccc|c} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{array} \right) \xrightarrow{r_1 \leftrightarrow r_3}$$

例 2 $\beta = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 6 \end{pmatrix}$ 能否由 $\alpha_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} -1 \\ 3 \\ -2 \\ 4 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 4 \\ 3 \\ 0 \\ 11 \end{pmatrix}$ 线性表示？

解

$$\begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \beta \end{array}$$

$$\left(\begin{array}{ccc|c} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{array} \right) \xrightarrow{r_1 \leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ -1 & 3 & 3 & 2 \\ 2 & -1 & 4 & 1 \\ 1 & 4 & 11 & 6 \end{array} \right)$$

例 2 $\beta = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 6 \end{pmatrix}$ 能否由 $\alpha_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} -1 \\ 3 \\ -2 \\ 4 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 4 \\ 3 \\ 0 \\ 11 \end{pmatrix}$ 线性表示？

解

$$\begin{matrix} \alpha_1 & \alpha_2 & \alpha_3 & \beta \end{matrix}$$

$$\left(\begin{array}{ccc|c} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{array} \right) \xrightarrow{r_1 \leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ -1 & 3 & 3 & 2 \\ 2 & -1 & 4 & 1 \\ 1 & 4 & 11 & 6 \end{array} \right)$$

$$\xrightarrow[\substack{r_3 - 2r_1 \\ r_4 - r_1}]{r_2 + r_1}$$

例 2 $\beta=\begin{pmatrix} 1 \\ 2 \\ -1 \\ 6 \end{pmatrix}$ 能否由 $\alpha_1=\begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2=\begin{pmatrix} -1 \\ 3 \\ -2 \\ 4 \end{pmatrix}$, $\alpha_3=\begin{pmatrix} 4 \\ 3 \\ 0 \\ 11 \end{pmatrix}$ 线性表示？

解

$$\begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \beta \end{array}$$

$$\left(\begin{array}{ccc|c} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{array} \right) \xrightarrow{r_1 \leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ -1 & 3 & 3 & 2 \\ 2 & -1 & 4 & 1 \\ 1 & 4 & 11 & 6 \end{array} \right)$$

$$\xrightarrow[\substack{r_3 - 2r_1 \\ r_4 - r_1}]{r_2 + r_1} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ & & & \\ & & & \\ & & & \end{array} \right)$$

例 2 $\beta = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 6 \end{pmatrix}$ 能否由 $\alpha_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} -1 \\ 3 \\ -2 \\ 4 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 4 \\ 3 \\ 0 \\ 11 \end{pmatrix}$ 线性表示?

解

$$
\begin{array}{cccc}
\alpha_1 & \alpha_2 & \alpha_3 & \beta
\end{array}
$$

$$
\left(\begin{array}{ccc|c}
2 & -1 & 4 & 1 \\
-1 & 3 & 3 & 2 \\
1 & -2 & 0 & -1 \\
1 & 4 & 11 & 6
\end{array}\right)
\xrightarrow{r_1 \leftrightarrow r_3}
\left(\begin{array}{ccc|c}
1 & -2 & 0 & -1 \\
-1 & 3 & 3 & 2 \\
2 & -1 & 4 & 1 \\
1 & 4 & 11 & 6
\end{array}\right)
$$

$$
\xrightarrow[\substack{r_3 - 2r_1 \\ r_4 - r_1}]{r_2 + r_1}
\left(\begin{array}{ccc|c}
1 & -2 & 0 & -1 \\
0 & 1 & 3 & 1 \\
 & & & \\
 & & &
\end{array}\right)
$$

例 2 $\beta = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 6 \end{pmatrix}$ 能否由 $\alpha_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} -1 \\ 3 \\ -2 \\ 4 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 4 \\ 3 \\ 0 \\ 11 \end{pmatrix}$ 线性表示？

解

$$\begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \beta \end{array}$$

$$\left(\begin{array}{ccc|c} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{array}\right) \xrightarrow{r_1 \leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ -1 & 3 & 3 & 2 \\ 2 & -1 & 4 & 1 \\ 1 & 4 & 11 & 6 \end{array}\right)$$

$$\xrightarrow[\substack{r_3 - 2r_1 \\ r_4 - r_1}]{r_2 + r_1} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 3 & 4 & 3 \end{array}\right)$$

例 2 $\beta=\begin{pmatrix}1\\2\\-1\\6\end{pmatrix}$ 能否由 $\alpha_1=\begin{pmatrix}2\\-1\\1\\1\end{pmatrix}$, $\alpha_2=\begin{pmatrix}-1\\3\\-2\\4\end{pmatrix}$, $\alpha_3=\begin{pmatrix}4\\3\\0\\11\end{pmatrix}$ 线性表示？

解

$$
\begin{array}{cccc}
\alpha_1 & \alpha_2 & \alpha_3 & \beta
\end{array}
$$

$$
\left(\begin{array}{ccc|c}
2 & -1 & 4 & 1\\
-1 & 3 & 3 & 2\\
1 & -2 & 0 & -1\\
1 & 4 & 11 & 6
\end{array}\right)
\xrightarrow{r_1 \leftrightarrow r_3}
\left(\begin{array}{ccc|c}
1 & -2 & 0 & -1\\
-1 & 3 & 3 & 2\\
2 & -1 & 4 & 1\\
1 & 4 & 11 & 6
\end{array}\right)
$$

$$
\xrightarrow[\substack{r_3-2r_1\\r_4-r_1}]{r_2+r_1}
\left(\begin{array}{ccc|c}
1 & -2 & 0 & -1\\
0 & 1 & 3 & 1\\
0 & 3 & 4 & 3\\
0 & 6 & 11 & 7
\end{array}\right)
$$

例 2 $\beta = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 6 \end{pmatrix}$ 能否由 $\alpha_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} -1 \\ 3 \\ -2 \\ 4 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 4 \\ 3 \\ 0 \\ 11 \end{pmatrix}$ 线性表示？

解

$$
\begin{array}{cccc}
\alpha_1 & \alpha_2 & \alpha_3 & \beta
\end{array}
$$

$$
\left(\begin{array}{ccc|c} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{array} \right) \xrightarrow{r_1 \leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ -1 & 3 & 3 & 2 \\ 2 & -1 & 4 & 1 \\ 1 & 4 & 11 & 6 \end{array} \right)
$$

$$
\xrightarrow[\substack{r_3 - 2r_1 \\ r_4 - r_1}]{r_2 + r_1} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 3 & 4 & 3 \\ 0 & 6 & 11 & 7 \end{array} \right) \xrightarrow[\substack{r_4 - 6r_2}]{r_3 - 3r_2}
$$

例2 $\beta=\begin{pmatrix}1\\2\\-1\\6\end{pmatrix}$ 能否由 $\alpha_1=\begin{pmatrix}2\\-1\\1\\1\end{pmatrix}$, $\alpha_2=\begin{pmatrix}-1\\3\\-2\\4\end{pmatrix}$, $\alpha_3=\begin{pmatrix}4\\3\\0\\11\end{pmatrix}$ 线性表示？

解

$$\begin{array}{cccc}\alpha_1 & \alpha_2 & \alpha_3 & \beta\end{array}$$

$$\left(\begin{array}{ccc|c}2 & -1 & 4 & 1\\-1 & 3 & 3 & 2\\1 & -2 & 0 & -1\\1 & 4 & 11 & 6\end{array}\right) \xrightarrow{r_1 \leftrightarrow r_3} \left(\begin{array}{ccc|c}1 & -2 & 0 & -1\\-1 & 3 & 3 & 2\\2 & -1 & 4 & 1\\1 & 4 & 11 & 6\end{array}\right)$$

$$\xrightarrow[\substack{r_3-2r_1\\r_4-r_1}]{r_2+r_1} \left(\begin{array}{ccc|c}1 & -2 & 0 & -1\\0 & 1 & 3 & 1\\0 & 3 & 4 & 3\\0 & 6 & 11 & 7\end{array}\right) \xrightarrow[\substack{r_4-6r_2}]{r_3-3r_2} \left(\begin{array}{ccc|c}1 & -2 & 0 & -1\\0 & 1 & 3 & 1\\ & & & \\ & & & \end{array}\right)$$

例 2 $\beta=\begin{pmatrix} 1 \\ 2 \\ -1 \\ 6 \end{pmatrix}$ 能否由 $\alpha_1=\begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2=\begin{pmatrix} -1 \\ 3 \\ -2 \\ 4 \end{pmatrix}$, $\alpha_3=\begin{pmatrix} 4 \\ 3 \\ 0 \\ 11 \end{pmatrix}$ 线性表示？

解

$$\begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \beta \end{array}$$

$$\left(\begin{array}{ccc|c} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{array}\right) \xrightarrow{r_1 \leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ -1 & 3 & 3 & 2 \\ 2 & -1 & 4 & 1 \\ 1 & 4 & 11 & 6 \end{array}\right)$$

$$\xrightarrow[\substack{r_3-2r_1 \\ r_4-r_1}]{r_2+r_1} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 3 & 4 & 3 \\ 0 & 6 & 11 & 7 \end{array}\right) \xrightarrow[\substack{r_4-6r_2}]{r_3-3r_2} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -5 & 0 \end{array}\right)$$

例 2 $\beta = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 6 \end{pmatrix}$ 能否由 $\alpha_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} -1 \\ 3 \\ -2 \\ 4 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 4 \\ 3 \\ 0 \\ 11 \end{pmatrix}$ 线性表示？

解

$$\begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \beta \end{array}$$

$$\left(\begin{array}{ccc|c} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{array} \right) \xrightarrow{r_1 \leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ -1 & 3 & 3 & 2 \\ 2 & -1 & 4 & 1 \\ 1 & 4 & 11 & 6 \end{array} \right)$$

$$\xrightarrow[\substack{r_3-2r_1 \\ r_4-r_1}]{r_2+r_1} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 3 & 4 & 3 \\ 0 & 6 & 11 & 7 \end{array} \right) \xrightarrow[\substack{r_4-6r_2}]{r_3-3r_2} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & -7 & 1 \end{array} \right)$$

例 2 $\beta = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 6 \end{pmatrix}$ 能否由 $\alpha_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} -1 \\ 3 \\ -2 \\ 4 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 4 \\ 3 \\ 0 \\ 11 \end{pmatrix}$ 线性表示？

解

$$
\begin{array}{cccc}
\alpha_1 & \alpha_2 & \alpha_3 & \beta
\end{array}
$$

$$
\left(\begin{array}{ccc|c}
2 & -1 & 4 & 1 \\
-1 & 3 & 3 & 2 \\
1 & -2 & 0 & -1 \\
1 & 4 & 11 & 6
\end{array}\right)
\xrightarrow{r_1 \leftrightarrow r_3}
\left(\begin{array}{ccc|c}
1 & -2 & 0 & -1 \\
-1 & 3 & 3 & 2 \\
2 & -1 & 4 & 1 \\
1 & 4 & 11 & 6
\end{array}\right)
$$

$$
\xrightarrow[\substack{r_3-2r_1 \\ r_4-r_1}]{r_2+r_1}
\left(\begin{array}{ccc|c}
1 & -2 & 0 & -1 \\
0 & 1 & 3 & 1 \\
0 & 3 & 4 & 3 \\
0 & 6 & 11 & 7
\end{array}\right)
\xrightarrow[\substack{r_4-6r_2}]{r_3-3r_2}
\left(\begin{array}{ccc|c}
1 & -2 & 0 & -1 \\
0 & 1 & 3 & 1 \\
0 & 0 & -5 & 0 \\
0 & 0 & -7 & 1
\end{array}\right)
$$

$$
\xrightarrow{-\frac{1}{5} \times r_3}
$$

例 2 $\beta = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 6 \end{pmatrix}$ 能否由 $\alpha_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} -1 \\ 3 \\ -2 \\ 4 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 4 \\ 3 \\ 0 \\ 11 \end{pmatrix}$ 线性表示？

解

$$\begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \beta \end{array}$$

$$\left(\begin{array}{ccc|c} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{array} \right) \xrightarrow{r_1 \leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ -1 & 3 & 3 & 2 \\ 2 & -1 & 4 & 1 \\ 1 & 4 & 11 & 6 \end{array} \right)$$

$$\xrightarrow[\substack{r_3-2r_1 \\ r_4-r_1}]{r_2+r_1} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 3 & 4 & 3 \\ 0 & 6 & 11 & 7 \end{array} \right) \xrightarrow[\substack{r_4-6r_2}]{r_3-3r_2} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & -7 & 1 \end{array} \right)$$

$$\xrightarrow{-\frac{1}{5} \times r_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -7 & 1 \end{array} \right)$$

例2 $\beta=\begin{pmatrix}1\\2\\-1\\6\end{pmatrix}$ 能否由 $\alpha_1=\begin{pmatrix}2\\-1\\1\\1\end{pmatrix}$, $\alpha_2=\begin{pmatrix}-1\\3\\-2\\4\end{pmatrix}$, $\alpha_3=\begin{pmatrix}4\\3\\0\\11\end{pmatrix}$ 线性表示？

解

$$
\begin{array}{cccc}
\alpha_1 & \alpha_2 & \alpha_3 & \beta
\end{array}
$$

$$
\left(\begin{array}{ccc|c}
2 & -1 & 4 & 1 \\
-1 & 3 & 3 & 2 \\
1 & -2 & 0 & -1 \\
1 & 4 & 11 & 6
\end{array}\right)
\xrightarrow{r_1\leftrightarrow r_3}
\left(\begin{array}{ccc|c}
1 & -2 & 0 & -1 \\
-1 & 3 & 3 & 2 \\
2 & -1 & 4 & 1 \\
1 & 4 & 11 & 6
\end{array}\right)
$$

$$
\xrightarrow[\substack{r_3-2r_1 \\ r_4-r_1}]{r_2+r_1}
\left(\begin{array}{ccc|c}
1 & -2 & 0 & -1 \\
0 & 1 & 3 & 1 \\
0 & 3 & 4 & 3 \\
0 & 6 & 11 & 7
\end{array}\right)
\xrightarrow[r_4-6r_2]{r_3-3r_2}
\left(\begin{array}{ccc|c}
1 & -2 & 0 & -1 \\
0 & 1 & 3 & 1 \\
0 & 0 & -5 & 0 \\
0 & 0 & -7 & 1
\end{array}\right)
$$

$$
\xrightarrow{-\frac{1}{5}\times r_3}
\left(\begin{array}{ccc|c}
1 & -2 & 0 & -1 \\
0 & 1 & 3 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & -7 & 1
\end{array}\right)
\xrightarrow{r_4+7r_3}
$$

例 2 $\beta = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 6 \end{pmatrix}$ 能否由 $\alpha_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} -1 \\ 3 \\ -2 \\ 4 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 4 \\ 3 \\ 0 \\ 11 \end{pmatrix}$ 线性表示？

解

$$\begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \beta \end{array}$$

$$\left(\begin{array}{ccc|c} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{array}\right) \xrightarrow{r_1 \leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ -1 & 3 & 3 & 2 \\ 2 & -1 & 4 & 1 \\ 1 & 4 & 11 & 6 \end{array}\right)$$

$$\xrightarrow[\substack{r_3-2r_1 \\ r_4-r_1}]{r_2+r_1} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 3 & 4 & 3 \\ 0 & 6 & 11 & 7 \end{array}\right) \xrightarrow[r_4-6r_2]{r_3-3r_2} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & -7 & 1 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{5} \times r_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -7 & 1 \end{array}\right) \xrightarrow{r_4+7r_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

例 2 $\beta=\begin{pmatrix} 1 \\ 2 \\ -1 \\ 6 \end{pmatrix}$ 能否由 $\alpha_1=\begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2=\begin{pmatrix} -1 \\ 3 \\ -2 \\ 4 \end{pmatrix}$, $\alpha_3=\begin{pmatrix} 4 \\ 3 \\ 0 \\ 11 \end{pmatrix}$ 线性表示？

解

$$\begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \beta \end{array}$$

$$\left(\begin{array}{ccc|c} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{array}\right) \xrightarrow{r_1 \leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ -1 & 3 & 3 & 2 \\ 2 & -1 & 4 & 1 \\ 1 & 4 & 11 & 6 \end{array}\right)$$

$$\xrightarrow[\substack{r_3-2r_1 \\ r_4-r_1}]{r_2+r_1} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 3 & 4 & 3 \\ 0 & 6 & 11 & 7 \end{array}\right) \xrightarrow[\substack{r_4-6r_2}]{r_3-3r_2} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & -7 & 1 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{5} \times r_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -7 & 1 \end{array}\right) \xrightarrow{r_4+7r_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

例 2 $\beta = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 6 \end{pmatrix}$ 能否由 $\alpha_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} -1 \\ 3 \\ -2 \\ 4 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 4 \\ 3 \\ 0 \\ 11 \end{pmatrix}$ 线性表示？

解

$$\begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \beta \end{array}$$

$$\left(\begin{array}{ccc|c} 2 & -1 & 4 & 1 \\ -1 & 3 & 3 & 2 \\ 1 & -2 & 0 & -1 \\ 1 & 4 & 11 & 6 \end{array}\right) \xrightarrow{r_1 \leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ -1 & 3 & 3 & 2 \\ 2 & -1 & 4 & 1 \\ 1 & 4 & 11 & 6 \end{array}\right)$$

$$\xrightarrow[\substack{r_3 - 2r_1 \\ r_4 - r_1}]{r_2 + r_1} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 3 & 4 & 3 \\ 0 & 6 & 11 & 7 \end{array}\right) \xrightarrow[\substack{r_4 - 6r_2}]{r_3 - 3r_2} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & -7 & 1 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{5} \times r_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -7 & 1 \end{array}\right) \xrightarrow{r_4 + 7r_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

可见 $r(\alpha_1 \alpha_2 \alpha_3 \beta) = 4 > 3 = r(\alpha_1 \alpha_2 \alpha_3)$,

例 2 $\beta=\begin{pmatrix}1\\2\\-1\\6\end{pmatrix}$ 能否由 $\alpha_1=\begin{pmatrix}2\\-1\\1\\1\end{pmatrix}$, $\alpha_2=\begin{pmatrix}-1\\3\\-2\\4\end{pmatrix}$, $\alpha_3=\begin{pmatrix}4\\3\\0\\11\end{pmatrix}$ 线性表示？

解

$$
\begin{array}{cccc}
\alpha_1 & \alpha_2 & \alpha_3 & \beta
\end{array}
$$

$$
\left(\begin{array}{ccc|c}
2 & -1 & 4 & 1\\
-1 & 3 & 3 & 2\\
1 & -2 & 0 & -1\\
1 & 4 & 11 & 6
\end{array}\right)
\xrightarrow{r_1\leftrightarrow r_3}
\left(\begin{array}{ccc|c}
1 & -2 & 0 & -1\\
-1 & 3 & 3 & 2\\
2 & -1 & 4 & 1\\
1 & 4 & 11 & 6
\end{array}\right)
$$

$$
\xrightarrow[\substack{r_3-2r_1\\r_4-r_1}]{r_2+r_1}
\left(\begin{array}{ccc|c}
1 & -2 & 0 & -1\\
0 & 1 & 3 & 1\\
0 & 3 & 4 & 3\\
0 & 6 & 11 & 7
\end{array}\right)
\xrightarrow[r_4-6r_2]{r_3-3r_2}
\left(\begin{array}{ccc|c}
1 & -2 & 0 & -1\\
0 & 1 & 3 & 1\\
0 & 0 & -5 & 0\\
0 & 0 & -7 & 1
\end{array}\right)
$$

$$
\xrightarrow{-\frac{1}{5}\times r_3}
\left(\begin{array}{ccc|c}
1 & -2 & 0 & -1\\
0 & 1 & 3 & 1\\
0 & 0 & 1 & 0\\
0 & 0 & -7 & 1
\end{array}\right)
\xrightarrow{r_4+7r_3}
\left(\begin{array}{ccc|c}
1 & -2 & 0 & -1\\
0 & 1 & 3 & 1\\
0 & 0 & 1 & 0\\
0 & 0 & 0 & 1
\end{array}\right)
$$

可见 $r(\alpha_1\alpha_2\alpha_3\beta) = 4 > 3 = r(\alpha_1\alpha_2\alpha_3)$，所以不能线性表示。

向量组的线性组合

定义 设有两个向量组

$$(A): \quad \alpha_1, \alpha_2, \ldots, \alpha_s$$

$$(B): \quad \beta_1, \beta_2, \ldots, \beta_t$$

向量组的线性组合

定义 设有两个向量组

$$(A): \quad \alpha_1, \alpha_2, \ldots, \alpha_s$$

$$(B): \quad \beta_1, \beta_2, \ldots, \beta_t$$

如果中 (A) 中每一向量均可由 (B) 线性表示，则称向量组 (A) 可由向量组 (B) 线性表示。

向量组的线性组合

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例 向量组 α_1, α_2 可由向量组 $\beta_1, \beta_2, \beta_3$ 线性表示

向量组的线性组合

定义 设有两个向量组

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例　向量组 α_1, α_2 可由向量组 $\beta_1, \beta_2, \beta_3$ 线性表示

$$\overset{\text{不妨设}}{\Longrightarrow} \begin{cases} \alpha_1 = & \beta_1 + & \beta_2 + & \beta_3 \\ \alpha_2 = & \beta_1 + & \beta_2 + & \beta_3 \end{cases}$$

向量组的线性组合

定义 设有两个向量组

$$(A): \quad \alpha_1, \alpha_2, \ldots, \alpha_s$$

$$(B): \quad \beta_1, \beta_2, \ldots, \beta_t$$

如果中 (A) 中每一向量均可由 (B) 线性表示，则称向量组 (A) 可由向量组 (B) 线性表示。

例   向量组 α_1, α_2 可由向量组 $\beta_1, \beta_2, \beta_3$ 线性表示

$$\xrightarrow{\text{不妨设}} \begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 + a_{31}\beta_3 \\ \alpha_2 = \quad\ \beta_1 + \quad\ \beta_2 + \quad\ \beta_3 \end{cases}$$

向量组的线性组合

定义 设有两个向量组

$$(A): \quad \alpha_1, \alpha_2, \ldots, \alpha_s$$

$$(B): \quad \beta_1, \beta_2, \ldots, \beta_t$$

如果中 (A) 中每一向量均可由 (B) 线性表示，则称向量组 (A) 可由向量组 (B)线性表示。

例　　　向量组 α_1, α_2 可由向量组 $\beta_1, \beta_2, \beta_3$ 线性表示

$\xRightarrow{\text{不妨设}}$ $\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 + a_{31}\beta_3 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 + a_{32}\beta_3 \end{cases}$

向量组的线性组合

定义 设有两个向量组

$$(A): \quad \alpha_1, \alpha_2, \ldots, \alpha_s$$

$$(B): \quad \beta_1, \beta_2, \ldots, \beta_t$$

如果中 (A) 中每一向量均可由 (B) 线性表示，则称向量组 (A) 可由向量组 (B) 线性表示。

例 　　 向量组 α_1, α_2 可由向量组 $\beta_1, \beta_2, \beta_3$ 线性表示

$$\overset{\text{不妨设}}{\Longrightarrow} \begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 + a_{31}\beta_3 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 + a_{32}\beta_3 \end{cases}$$

$$\overset{\text{改写为}}{\Longrightarrow} (\alpha_1, \alpha_2) = (\beta_1, \beta_2, \beta_3) \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

向量组的线性组合

定义 设有两个向量组

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注 1 一般地，向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 可由向量组 $\beta_1, \beta_2, \ldots, \beta_t$ 线性表示，当且仅当存在矩阵 $A_{t \times s}$ 满足：

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$$\alpha_j = \quad \beta_1 + \quad \beta_2 + \cdots + \quad \beta_t$$

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$$\alpha_j = a_{1j}\beta_1 + a_{2j}\beta_2 + \cdots + a_{tj}\beta_t$$

其中的系数就是 A 的第 j 列 $\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{sj} \end{pmatrix}$。

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$$= (\beta_1, \beta_2, \ldots, \beta_t) \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ a_{t1} & a_{t2} & \cdots & a_{ts} \end{pmatrix}$$

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注 3 反之，若 $P = QA$，则说明 P 的列向量组可由的 Q 列向量组线性表示

定理（向量组线性表示的传递性） 假设向量组 (A), (B), (C) 满足：(A) 可由 (B) 线性表示，(B) 可由 (C) 线性表示，则 (A) 可由 (C) 线性表示。

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证明 设向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 可由向量组 $\beta_1, \beta_2, \ldots, \beta_t$ 线性表示：

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将第 2 式代入第 1 式，可得

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$$(\alpha_1, \alpha_2, \ldots, \alpha_s) = (\gamma_1, \gamma_2, \ldots, \gamma_k)\underbrace{B_{k \times t}A_{t \times s}}_{C_{k \times s}}$$

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所以向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 可由向量组 $\gamma_1, \gamma_2, \ldots, \gamma_k$ 线性表示。

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所以向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 可由向量组 $\gamma_1, \gamma_2, \ldots, \gamma_k$ 线性表示。
（并且，线性组合的系数就是矩阵 C 的列。）

例 $\left.\begin{array}{l} \alpha_1,\ \alpha_2 由\beta_1,\ \beta_2 线性表示 \\ \beta_1,\ \beta_2 由\gamma_1,\ \gamma_2,\ \gamma_3 线性表示 \end{array}\right\} \Rightarrow \alpha_1,\ \alpha_2 由\gamma_1,\ \gamma_2,\ \gamma_3 线性表示$

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具体地，设

$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases}$$

例
$$\left.\begin{array}{l} \alpha_1, \ \alpha_2 \text{由} \beta_1, \ \beta_2 \text{线性表示} \\ \beta_1, \ \beta_2 \text{由} \gamma_1, \ \gamma_2, \ \gamma_3 \text{线性表示} \end{array}\right\} \Rightarrow \alpha_1, \ \alpha_2 \text{由} \gamma_1, \ \gamma_2, \ \gamma_3 \text{线性表示}$$

具体地，设

$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases}$$

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases}$$

例 $\left.\begin{array}{l}\alpha_1,\ \alpha_2\ \text{由}\beta_1,\ \beta_2\text{线性表示}\\ \beta_1,\ \beta_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示}\end{array}\right\} \Rightarrow \alpha_1,\ \alpha_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示}$

具体地，设

$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases}$$

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则

$\alpha_1 =$

$\alpha_2 =$

例
$$\left.\begin{array}{l} \alpha_1,\ \alpha_2 由 \beta_1,\ \beta_2 线性表示 \\ \beta_1,\ \beta_2 由 \gamma_1,\ \gamma_2,\ \gamma_3 线性表示 \end{array}\right\} \Rightarrow \alpha_1,\ \alpha_2 由 \gamma_1,\ \gamma_2,\ \gamma_3 线性表示$$

具体地，设

$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases}$$

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则

$$\alpha_1 = a_{11}(\qquad\qquad) + a_{21}(\qquad\qquad)$$

$$\alpha_2 =$$

例 $\left.\begin{array}{l} \alpha_1,\ \alpha_2\text{由}\beta_1,\ \beta_2\text{线性表示} \\ \beta_1,\ \beta_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示} \end{array}\right\} \Rightarrow \alpha_1,\ \alpha_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示}$

具体地，设

$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases}$$

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases}$$

则

$$\alpha_1 = a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(\qquad\qquad)$$

$$\alpha_2 =$$

例 $\left.\begin{array}{l}\alpha_1,\ \alpha_2 由 \beta_1,\ \beta_2 线性表示\\ \beta_1,\ \beta_2 由 \gamma_1,\ \gamma_2,\ \gamma_3 线性表示\end{array}\right\} \Rightarrow \alpha_1,\ \alpha_2 由 \gamma_1,\ \gamma_2,\ \gamma_3 线性表示$

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则

$$\alpha_1 = a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)$$

$$\alpha_2 =$$

例
$$\left.\begin{array}{l}\alpha_1,\ \alpha_2 由 \beta_1,\ \beta_2 线性表示\\ \beta_1,\ \beta_2 由 \gamma_1,\ \gamma_2,\ \gamma_3 线性表示\end{array}\right\} \Rightarrow \alpha_1,\ \alpha_2 由 \gamma_1,\ \gamma_2,\ \gamma_3 线性表示$$

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则

$$\alpha_1 = a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)$$

$$= (\qquad\qquad)\gamma_1 + (\qquad\qquad)\gamma_2 + (\qquad\qquad)\gamma_3$$

$$\alpha_2 =$$

例 $\left.\begin{array}{l} \alpha_1,\ \alpha_2 \text{由} \beta_1,\ \beta_2 \text{线性表示} \\ \beta_1,\ \beta_2 \text{由} \gamma_1,\ \gamma_2,\ \gamma_3 \text{线性表示} \end{array}\right\} \Rightarrow \alpha_1,\ \alpha_2 \text{由} \gamma_1,\ \gamma_2,\ \gamma_3 \text{线性表示}$

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则

$$\alpha_1 = a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)$$

$$= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (\qquad\qquad)\gamma_2 + (\qquad\qquad)\gamma_3$$

$$\alpha_2 =$$

例 $\left.\begin{array}{l}\alpha_1,\ \alpha_2\text{由}\beta_1,\ \beta_2\text{线性表示}\\ \beta_1,\ \beta_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示}\end{array}\right\} \Rightarrow \alpha_1,\ \alpha_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示}$

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则

$$\alpha_1 = a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)$$

$$= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (\qquad\qquad)\gamma_3$$

$$\alpha_2 =$$

例 $\left.\begin{array}{l}\alpha_1,\ \alpha_2\text{由}\beta_1,\ \beta_2\text{线性表示}\\\beta_1,\ \beta_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示}\end{array}\right\}\Rightarrow\alpha_1,\ \alpha_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示}$

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则

$$\alpha_1=a_{11}(b_{11}\gamma_1+b_{21}\gamma_2+b_{31}\gamma_3)+a_{21}(b_{12}\gamma_1+b_{22}\gamma_2+b_{32}\gamma_3)$$

$$=(a_{11}b_{11}+a_{21}b_{12})\gamma_1+(a_{11}b_{21}+a_{21}b_{22})\gamma_2+(a_{11}b_{31}+a_{21}b_{32})\gamma_3$$

$$\alpha_2=$$

例 $\left.\begin{array}{l} \alpha_1,\ \alpha_2 由 \beta_1,\ \beta_2 线性表示 \\ \beta_1,\ \beta_2 由 \gamma_1,\ \gamma_2,\ \gamma_3 线性表示 \end{array}\right\} \Rightarrow \alpha_1,\ \alpha_2 由 \gamma_1,\ \gamma_2,\ \gamma_3 线性表示$

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则

$$\alpha_1 = a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)$$

$$= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3$$

$$\alpha_2 = a_{12}(\qquad\qquad\qquad) + a_{22}(\qquad\qquad\qquad)$$

例 $\left. \begin{array}{l} \alpha_1,\ \alpha_2 \text{由} \beta_1,\ \beta_2 \text{线性表示} \\ \beta_1,\ \beta_2 \text{由} \gamma_1,\ \gamma_2,\ \gamma_3 \text{线性表示} \end{array} \right\} \Rightarrow \alpha_1,\ \alpha_2 \text{由} \gamma_1,\ \gamma_2,\ \gamma_3 \text{线性表示}$

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则

$$\alpha_1 = a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)$$

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$$\alpha_2 = a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(\qquad\qquad\qquad\qquad)$$

例 $\left.\begin{array}{l}\alpha_1,\ \alpha_2\text{由}\beta_1,\ \beta_2\text{线性表示}\\ \beta_1,\ \beta_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示}\end{array}\right\} \Rightarrow \alpha_1,\ \alpha_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示}$

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$$\alpha_2 = a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)$$

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$$\alpha_1 = a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)$$

$$= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3$$

$$\alpha_2 = a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)$$

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则

$$\alpha_1 = a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)$$

$$= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3$$

$$\alpha_2 = a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)$$

$$= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (\qquad\qquad)\gamma_3$$

例 $\left.\begin{array}{l}\alpha_1,\ \alpha_2 \text{由} \beta_1,\ \beta_2 \text{线性表示}\\ \beta_1,\ \beta_2 \text{由} \gamma_1,\ \gamma_2,\ \gamma_3 \text{线性表示}\end{array}\right\} \Rightarrow \alpha_1,\ \alpha_2 \text{由} \gamma_1,\ \gamma_2,\ \gamma_3 \text{线性表示}$

具体地，设

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则

$$\alpha_1 = a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)$$

$$= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3$$

$$\alpha_2 = a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)$$

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例 $\left.\begin{array}{l}\alpha_1,\ \alpha_2 由 \beta_1,\ \beta_2 线性表示 \\ \beta_1,\ \beta_2 由 \gamma_1,\ \gamma_2,\ \gamma_3 线性表示\end{array}\right\} \Rightarrow \alpha_1,\ \alpha_2 由 \gamma_1,\ \gamma_2,\ \gamma_3 线性表示$

具体地，设

$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases}$$

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases}$$

则

$$\begin{aligned} \alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3 \\ &= c_{11}\gamma_1 + \\ \alpha_2 &= a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (a_{12}b_{31} + a_{22}b_{32})\gamma_3 \end{aligned}$$

例 $\left.\begin{array}{l}\alpha_1,\ \alpha_2 由 \beta_1,\ \beta_2 线性表示 \\ \beta_1,\ \beta_2 由 \gamma_1,\ \gamma_2,\ \gamma_3 线性表示\end{array}\right\} \Rightarrow \alpha_1,\ \alpha_2 由 \gamma_1,\ \gamma_2,\ \gamma_3 线性表示$

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则

$$\alpha_1 = a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)$$

$$= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3$$

$$= c_{11}\gamma_1 + c_{21}\gamma_2 +$$

$$\alpha_2 = a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)$$

$$= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (a_{12}b_{31} + a_{22}b_{32})\gamma_3$$

例 $\left. \begin{array}{l} \alpha_1,\ \alpha_2 \text{由} \beta_1,\ \beta_2 \text{线性表示} \\ \beta_1,\ \beta_2 \text{由} \gamma_1,\ \gamma_2,\ \gamma_3 \text{线性表示} \end{array} \right\} \Rightarrow \alpha_1,\ \alpha_2 \text{由} \gamma_1,\ \gamma_2,\ \gamma_3 \text{线性表示}$

具体地，设

$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases}$$

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases}$$

则

$$\alpha_1 = a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)$$

$$= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3$$

$$= c_{11}\gamma_1 + c_{21}\gamma_2 + c_{31}\gamma_3$$

$$\alpha_2 = a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)$$

$$= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (a_{12}b_{31} + a_{22}b_{32})\gamma_3$$

例 $\left. \begin{array}{l} \alpha_1, \, \alpha_2 由 \beta_1, \, \beta_2 线性表示 \\ \beta_1, \, \beta_2 由 \gamma_1, \, \gamma_2, \, \gamma_3 线性表示 \end{array} \right\} \Rightarrow \alpha_1, \, \alpha_2 由 \gamma_1, \, \gamma_2, \, \gamma_3 线性表示$

具体地，设
$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases}$$

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases}$$

则
$$\begin{aligned} \alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3 \\ &= c_{11}\gamma_1 + c_{21}\gamma_2 + c_{31}\gamma_3 \\ \alpha_2 &= a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (a_{12}b_{31} + a_{22}b_{32})\gamma_3 \\ &= c_{12}\gamma_1 + \end{aligned}$$

例 $\left.\begin{array}{l}\alpha_1,\ \alpha_2 \text{由} \beta_1,\ \beta_2 \text{线性表示}\\ \beta_1,\ \beta_2 \text{由} \gamma_1,\ \gamma_2,\ \gamma_3 \text{线性表示}\end{array}\right\} \Rightarrow \alpha_1,\ \alpha_2 \text{由} \gamma_1,\ \gamma_2,\ \gamma_3 \text{线性表示}$

具体地，设
$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases}$$

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases}$$

则
$$\alpha_1 = a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)$$
$$= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3$$
$$= c_{11}\gamma_1 + c_{21}\gamma_2 + c_{31}\gamma_3$$
$$\alpha_2 = a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)$$
$$= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (a_{12}b_{31} + a_{22}b_{32})\gamma_3$$
$$= c_{12}\gamma_1 + c_{22}\gamma_2 +$$

例 $\left.\begin{array}{l} \alpha_1,\ \alpha_2 \text{由} \beta_1,\ \beta_2 \text{线性表示} \\ \beta_1,\ \beta_2 \text{由} \gamma_1,\ \gamma_2,\ \gamma_3 \text{线性表示} \end{array}\right\} \Rightarrow \alpha_1,\ \alpha_2 \text{由} \gamma_1,\ \gamma_2,\ \gamma_3 \text{线性表示}$

具体地，设
$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases}$$

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases}$$

则
$$\begin{aligned} \alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3 \\ &= c_{11}\gamma_1 + c_{21}\gamma_2 + c_{31}\gamma_3 \\ \alpha_2 &= a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (a_{12}b_{31} + a_{22}b_{32})\gamma_3 \\ &= c_{12}\gamma_1 + c_{22}\gamma_2 + c_{32}\gamma_3 \end{aligned}$$

例 $\left.\begin{array}{l}\alpha_1,\ \alpha_2\text{由}\beta_1,\ \beta_2\text{线性表示}\\ \beta_1,\ \beta_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示}\end{array}\right\} \Rightarrow \alpha_1,\ \alpha_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示}$

具体地，设

$$\begin{cases} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{cases}$$

$$\begin{cases} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{cases}$$

则

$$\begin{aligned} \alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3 \\ &= c_{11}\gamma_1 + c_{21}\gamma_2 + c_{31}\gamma_3 \\ \alpha_2 &= a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (a_{12}b_{31} + a_{22}b_{32})\gamma_3 \\ &= c_{12}\gamma_1 + c_{22}\gamma_2 + c_{32}\gamma_3 \end{aligned}$$

其中

$$(c_{ij}) = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{12} & a_{12}b_{11} + a_{22}b_{12} \\ a_{11}b_{21} + a_{21}b_{22} & a_{12}b_{21} + a_{22}b_{22} \\ a_{11}b_{31} + a_{21}b_{32} & a_{12}b_{31} + a_{22}b_{32} \end{pmatrix}$$

例 $\left.\begin{array}{l}\alpha_1,\ \alpha_2\text{由}\beta_1,\ \beta_2\text{线性表示}\\ \beta_1,\ \beta_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示}\end{array}\right\}\Rightarrow\alpha_1,\ \alpha_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示}$

具体地，设
$$\begin{cases}\alpha_1 = a_{11}\beta_1 + a_{21}\beta_2\\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2\end{cases}$$

$$\begin{cases}\beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3\\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3\end{cases}$$

则

$$\begin{aligned}\alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)\\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3\\ &= c_{11}\gamma_1 + c_{21}\gamma_2 + c_{31}\gamma_3\\ \alpha_2 &= a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)\\ &= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (a_{12}b_{31} + a_{22}b_{32})\gamma_3\\ &= c_{12}\gamma_1 + c_{22}\gamma_2 + c_{32}\gamma_3\end{aligned}$$

其中

$$(c_{ij}) = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{12} & a_{12}b_{11} + a_{22}b_{12}\\ a_{11}b_{21} + a_{21}b_{22} & a_{12}b_{21} + a_{22}b_{22}\\ a_{11}b_{31} + a_{21}b_{32} & a_{12}b_{31} + a_{22}b_{32}\end{pmatrix} = \begin{pmatrix} b_{11} & b_{12}\\ b_{21} & b_{22}\\ b_{31} & b_{32}\end{pmatrix}\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$$

例 $\left.\begin{array}{l} \alpha_1, \alpha_2 由 \beta_1, \beta_2 线性表示 \\ \beta_1, \beta_2 由 \gamma_1, \gamma_2, \gamma_3 线性表示 \end{array}\right\} \Rightarrow \alpha_1, \alpha_2 由 \gamma_1, \gamma_2, \gamma_3 线性表示$

具体地，设

$$\left\{\begin{array}{l} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{array}\right. \Rightarrow (\alpha_1, \alpha_2) = (\beta_1, \beta_2)\left(\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array}\right)$$

$$\left\{\begin{array}{l} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{array}\right.$$

则

$$\begin{aligned} \alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3 \\ &= c_{11}\gamma_1 + c_{21}\gamma_2 + c_{31}\gamma_3 \\ \alpha_2 &= a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (a_{12}b_{31} + a_{22}b_{32})\gamma_3 \\ &= c_{12}\gamma_1 + c_{22}\gamma_2 + c_{32}\gamma_3 \end{aligned}$$

其中

$$(c_{ij}) = \left(\begin{array}{cc} a_{11}b_{11} + a_{12}b_{12} & a_{12}b_{11} + a_{22}b_{12} \\ a_{11}b_{21} + a_{21}b_{22} & a_{12}b_{21} + a_{22}b_{22} \\ a_{11}b_{31} + a_{21}b_{32} & a_{12}b_{31} + a_{22}b_{32} \end{array}\right) = \left(\begin{array}{cc} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{array}\right)\left(\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array}\right)$$

例 $\left.\begin{array}{l} \alpha_1,\ \alpha_2 \text{由} \beta_1,\ \beta_2 \text{线性表示} \\ \beta_1,\ \beta_2 \text{由} \gamma_1,\ \gamma_2,\ \gamma_3 \text{线性表示} \end{array}\right\} \Rightarrow \alpha_1,\ \alpha_2 \text{由} \gamma_1,\ \gamma_2,\ \gamma_3 \text{线性表示}$

具体地，设

$$\left\{\begin{array}{l} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{array}\right. \Rightarrow (\alpha_1,\ \alpha_2) = (\beta_1,\ \beta_2)\left(\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array}\right)$$

$$\left\{\begin{array}{l} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{array}\right. \Rightarrow (\beta_1,\ \beta_2) = (\gamma_1,\ \gamma_2,\ \gamma_3)\left(\begin{array}{cc} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{array}\right)$$

则

$$\alpha_1 = a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)$$

$$= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3$$

$$= c_{11}\gamma_1 + c_{21}\gamma_2 + c_{31}\gamma_3$$

$$\alpha_2 = a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)$$

$$= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (a_{12}b_{31} + a_{22}b_{32})\gamma_3$$

$$= c_{12}\gamma_1 + c_{22}\gamma_2 + c_{32}\gamma_3$$

其中

$$(c_{ij}) = \left(\begin{array}{cc} a_{11}b_{11} + a_{12}b_{12} & a_{12}b_{11} + a_{22}b_{12} \\ a_{11}b_{21} + a_{21}b_{22} & a_{12}b_{21} + a_{22}b_{22} \\ a_{11}b_{31} + a_{21}b_{32} & a_{12}b_{31} + a_{22}b_{32} \end{array}\right) = \left(\begin{array}{cc} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{array}\right)\left(\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array}\right)$$

例
$$\left.\begin{array}{l} \alpha_1,\ \alpha_2\text{由}\beta_1,\ \beta_2\text{线性表示}\\ \beta_1,\ \beta_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示} \end{array}\right\} \Rightarrow \alpha_1,\ \alpha_2\text{由}\gamma_1,\ \gamma_2,\ \gamma_3\text{线性表示}$$

具体地，设
$$\left\{\begin{array}{l} \alpha_1 = a_{11}\beta_1 + a_{21}\beta_2\\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2 \end{array}\right. \Rightarrow (\alpha_1,\ \alpha_2) = (\beta_1,\ \beta_2)\underbrace{\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22} \end{pmatrix}}_{A}$$

$$\left\{\begin{array}{l} \beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3\\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3 \end{array}\right. \Rightarrow (\beta_1,\ \beta_2) = (\gamma_1,\ \gamma_2,\ \gamma_3)\begin{pmatrix} b_{11} & b_{12}\\ b_{21} & b_{22}\\ b_{31} & b_{32} \end{pmatrix}$$

则
$$\begin{aligned} \alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)\\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3\\ &= c_{11}\gamma_1 + c_{21}\gamma_2 + c_{31}\gamma_3 \end{aligned}$$

$$\begin{aligned} \alpha_2 &= a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3)\\ &= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (a_{12}b_{31} + a_{22}b_{32})\gamma_3\\ &= c_{12}\gamma_1 + c_{22}\gamma_2 + c_{32}\gamma_3 \end{aligned}$$

其中
$$(c_{ij}) = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{12} & a_{12}b_{11} + a_{22}b_{12}\\ a_{11}b_{21} + a_{21}b_{22} & a_{12}b_{21} + a_{22}b_{22}\\ a_{11}b_{31} + a_{21}b_{32} & a_{12}b_{31} + a_{22}b_{32} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12}\\ b_{21} & b_{22}\\ b_{31} & b_{32} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22} \end{pmatrix}$$

例 $\left.\begin{array}{l}\alpha_1,\,\alpha_2 由 \beta_1,\,\beta_2 线性表示 \\ \beta_1,\,\beta_2 由 \gamma_1,\,\gamma_2,\,\gamma_3 线性表示\end{array}\right\} \Rightarrow \alpha_1,\,\alpha_2 由 \gamma_1,\,\gamma_2,\,\gamma_3 线性表示$

具体地，设

$$\left\{\begin{array}{l}\alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2\end{array}\right. \Rightarrow (\alpha_1,\,\alpha_2) = (\beta_1,\,\beta_2)\underbrace{\left(\begin{array}{cc}a_{11} & a_{12} \\ a_{21} & a_{22}\end{array}\right)}_{A}$$

$$\left\{\begin{array}{l}\beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3\end{array}\right. \Rightarrow (\beta_1,\,\beta_2) = (\gamma_1,\,\gamma_2,\,\gamma_3)\underbrace{\left(\begin{array}{cc}b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32}\end{array}\right)}_{B}$$

则

$$\begin{aligned}\alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3 \\ &= c_{11}\gamma_1 + c_{21}\gamma_2 + c_{31}\gamma_3\end{aligned}$$

$$\begin{aligned}\alpha_2 &= a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (a_{12}b_{31} + a_{22}b_{32})\gamma_3 \\ &= c_{12}\gamma_1 + c_{22}\gamma_2 + c_{32}\gamma_3\end{aligned}$$

其中

$$(c_{ij}) = \left(\begin{array}{cc}a_{11}b_{11} + a_{12}b_{12} & a_{12}b_{11} + a_{22}b_{12} \\ a_{11}b_{21} + a_{21}b_{22} & a_{12}b_{21} + a_{22}b_{22} \\ a_{11}b_{31} + a_{21}b_{32} & a_{12}b_{31} + a_{22}b_{32}\end{array}\right) = \left(\begin{array}{cc}b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32}\end{array}\right)\left(\begin{array}{cc}a_{11} & a_{12} \\ a_{21} & a_{22}\end{array}\right)$$

例 $\left.\begin{array}{l}\alpha_1,\ \alpha_2\ 由\beta_1,\ \beta_2\ 线性表示 \\ \beta_1,\ \beta_2\ 由\gamma_1,\ \gamma_2,\ \gamma_3\ 线性表示\end{array}\right\} \Rightarrow \alpha_1,\ \alpha_2\ 由\gamma_1,\ \gamma_2,\ \gamma_3\ 线性表示$

具体地，设

$$\begin{cases}\alpha_1 = a_{11}\beta_1 + a_{21}\beta_2 \\ \alpha_2 = a_{12}\beta_1 + a_{22}\beta_2\end{cases} \Rightarrow (\alpha_1,\ \alpha_2) = (\beta_1,\ \beta_2)\underbrace{\begin{pmatrix}a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}}_{A}$$

$$\begin{cases}\beta_1 = b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3 \\ \beta_2 = b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3\end{cases} \Rightarrow (\beta_1,\ \beta_2) = (\gamma_1,\ \gamma_2,\ \gamma_3)\underbrace{\begin{pmatrix}b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32}\end{pmatrix}}_{B}$$

则

$$\begin{aligned}\alpha_1 &= a_{11}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{21}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{11}b_{11} + a_{21}b_{12})\gamma_1 + (a_{11}b_{21} + a_{21}b_{22})\gamma_2 + (a_{11}b_{31} + a_{21}b_{32})\gamma_3 \\ &= c_{11}\gamma_1 + c_{21}\gamma_2 + c_{31}\gamma_3\end{aligned}$$

$$\begin{aligned}\alpha_2 &= a_{12}(b_{11}\gamma_1 + b_{21}\gamma_2 + b_{31}\gamma_3) + a_{22}(b_{12}\gamma_1 + b_{22}\gamma_2 + b_{32}\gamma_3) \\ &= (a_{12}b_{11} + a_{22}b_{12})\gamma_1 + (a_{12}b_{21} + a_{22}b_{22})\gamma_2 + (a_{12}b_{31} + a_{22}b_{32})\gamma_3 \\ &= c_{12}\gamma_1 + c_{22}\gamma_2 + c_{32}\gamma_3\end{aligned}$$

其中

$$(c_{ij}) = \begin{pmatrix}a_{11}b_{11} + a_{12}b_{12} & a_{12}b_{11} + a_{22}b_{12} \\ a_{11}b_{21} + a_{21}b_{22} & a_{12}b_{21} + a_{22}b_{22} \\ a_{11}b_{31} + a_{21}b_{32} & a_{12}b_{31} + a_{22}b_{32}\end{pmatrix} = \begin{pmatrix}b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32}\end{pmatrix}\begin{pmatrix}a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix} = BA$$

定义 设有两个向量组

$$(A): \quad \alpha_1, \alpha_2, \ldots, \alpha_s$$

$$(B): \quad \beta_1, \beta_2, \ldots, \beta_t$$

如果 (A) 与 (B) 可相互线性表示，则称向量组 (A) 与 (B) 等价。