第 11 章 α : 对弧长的曲线积分

数学系 梁卓滨

2018-2019 学年 II





We are here now...

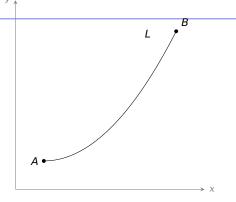
1. 对弧长的曲线积分: 概念与性质

2. 对弧长的曲线积分: 计算法

3. 对弧长的曲线积分: 空间曲线

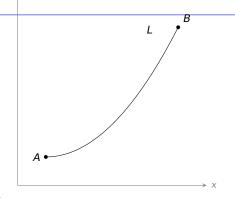
假设平面曲线 L

- 线密度为 μ(x, y)
- 质量为 m



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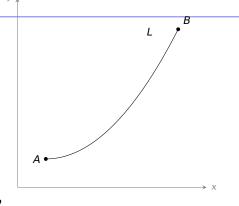


当曲线是均匀时(μ=常数),

• 当曲线非均匀时 ($\mu = \mu(x, y)$ 为 L 上函数)

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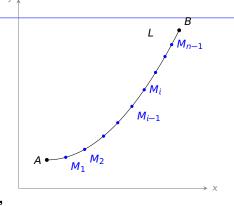
$$m = \mu \cdot \text{Length}(L)$$

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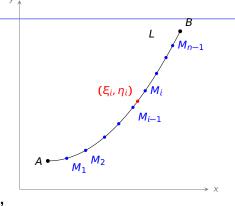
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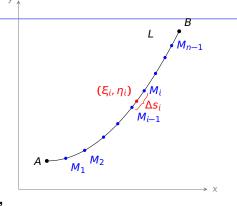
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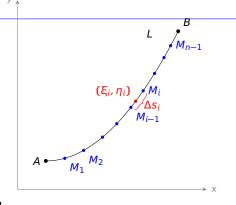
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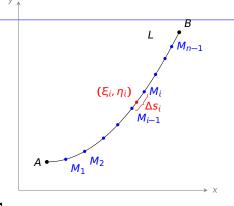
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$$\mu(\xi_i, \eta_i)\Delta s_i$$



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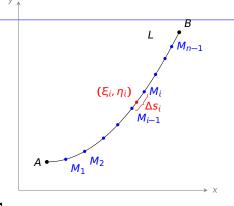
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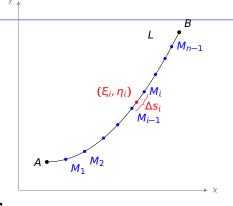
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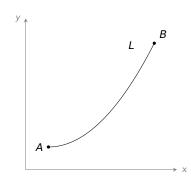
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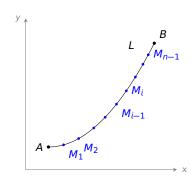
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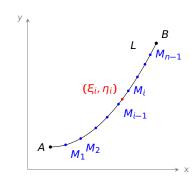
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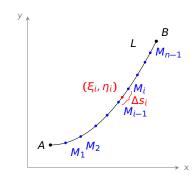
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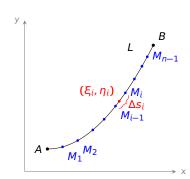
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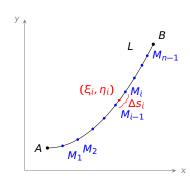
$$\sum_{i=1}^n f(\xi_i, \eta_i) \Delta s_i$$



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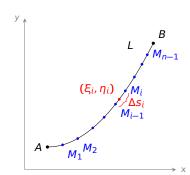


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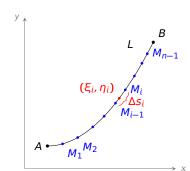
• 极限 $\lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_i, \eta_i) \Delta s_i$ 存在,



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- 极限 $\lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_i, \eta_i) \Delta s_i$ 存在,且极限
- 与上述 L 的划分、 $(ξ_i, η_i)$ 的选取无关,

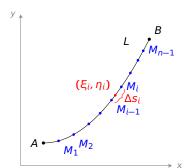


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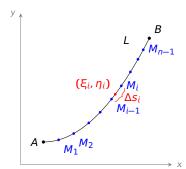
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称为 f(x, y) 在曲线 L 上的对弧长的曲线积分。



对弧长的曲线积分定义 设

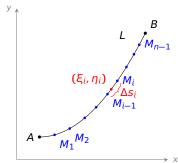
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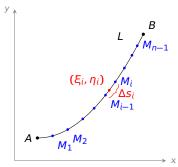
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注 对弧长的曲线积分的定义式与重积分的类似,故性质也类似



● 存在性 若 L 是分段光滑曲线, f(x, y) 在 L 上连续, 则

$$\int_{1} f(x, y) ds$$

存在。

● 存在性 若 L 是分段光滑曲线, f(x, y) 在 L 上连续, 则

$$\int_{L} f(x, y) ds$$

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• 线性性 $\int_{L} (\alpha f + \beta g) ds = \alpha \int_{L} f ds + \beta \int_{L} g ds$



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• 存在性 若 L 是分段光滑曲线,f(x, y) 在 L 上连续,则 $\int_{\Gamma} f(x, y) ds$

存在。

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- $\int_L 1ds = \text{Length}(L)$

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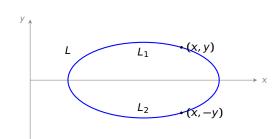
- 线性性 $\int_{L} (\alpha f + \beta g) ds = \alpha \int_{L} f ds + \beta \int_{L} g ds$
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- $\int_L 1ds = \text{Length}(L)$
- 若 f(x,y) ≤ g(x,y), 则

$$\int_L f(x,y)ds \le \int_L g(x,y)ds$$



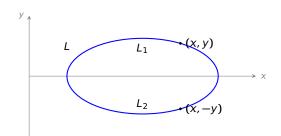


性质 设平面曲线 L 关于 x 轴对称,



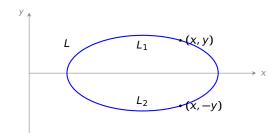
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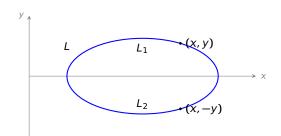


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• 若 f(x, y) 关于 y 是偶函数 (即: f(x, -y) = f(x, y)),则



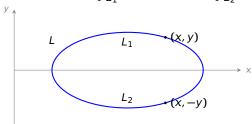
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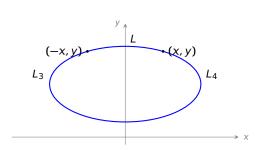
$$\int_{L} f(x, y) ds = 0$$

• 若 f(x, y) 关于 y 是偶函数 (即: f(x, -y) = f(x, y)),则

$$\int_{L} f(x, y) ds = 2 \int_{L_{1}} f(x, y) ds = 2 \int_{L_{2}} f(x, y) ds$$

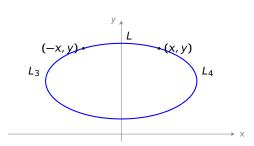


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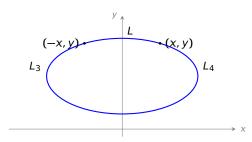
• 若 f(x,y) 关于 x 是奇函数 (即: f(-x,y) = -f(x,y)),则



积分的对称性

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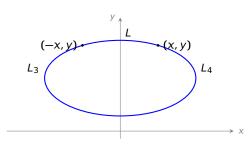
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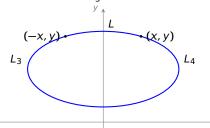
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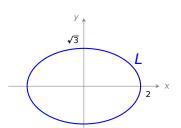
$$\int_{L} f(x, y) ds = 2 \int_{L_{3}} f(x, y) ds = 2 \int_{L_{4}} f(x, y) ds$$



计算

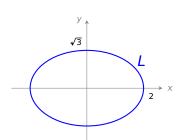
例 已知椭圆
$$L: \frac{x^2}{4} + \frac{y^2}{3} = 1$$
 的周长是 α ,
计算

$$\int_{1}^{\infty} 2xy + 3x^2 + 4y^2 ds$$



例 已知椭圆 $L: \frac{x^2}{4} + \frac{y^2}{3} = 1$ 的周长是 α , 计算

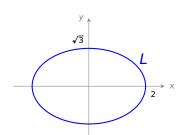
$$\int_{1} 2xy + 3x^2 + 4y^2 ds$$



原式 =
$$\int_{1}^{1} 2xyds + \int_{1}^{1} 12(\frac{x^{2}}{4} + \frac{y^{2}}{3})ds$$

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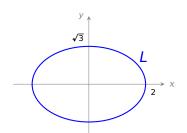


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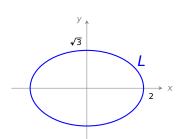
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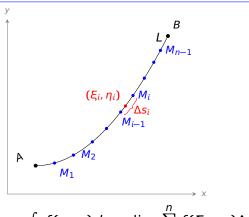


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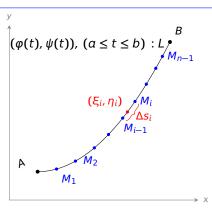
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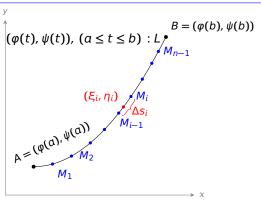
3. 对弧长的曲线积分:空间曲线



$$\int_L f(x, y) ds = \lim_{\lambda \to 0} \sum_{i=1}^n f(\xi_i, \eta_i) \Delta s_i$$

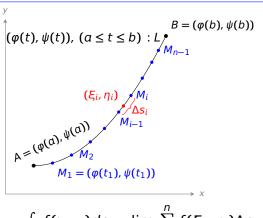


$$\int_{L} f(x, y) ds = \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_{i}, \eta_{i}) \Delta s_{i}$$

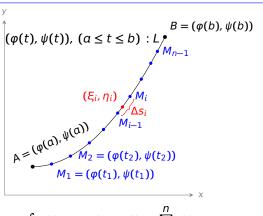


$$\int_{L} f(x, y) ds = \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_{i}, \eta_{i}) \Delta s_{i}$$





$$\int_L f(x, y) ds = \lim_{\lambda \to 0} \sum_{i=1}^n f(\xi_i, \eta_i) \Delta s_i$$



$$\int_L f(x, y) ds = \lim_{\lambda \to 0} \sum_{i=1}^n f(\xi_i, \eta_i) \Delta s_i$$

```
B = (\varphi(b), \psi(b))
(\varphi(t), \psi(t)), (a \le t \le b) : L^{\bullet}
A = (\varphi(\alpha), \psi(\alpha))
                                      M_{i-1} = (\varphi(t_{i-1}), \psi(t_{i-1}))
                              = (\varphi(t_2), \psi(t_2))
                 M_1 = (\varphi(t_1), \psi(t_1))
```

$$\int_{L} f(x, y) ds = \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_{i}, \eta_{i}) \Delta s_{i}$$



```
B = (\varphi(b), \psi(b))
(\varphi(t), \psi(t)), (a \le t \le b) : L^{\bullet}
                                           M_i = (\varphi(t_i), \psi(t_i))
A = (\phi(\alpha), \psi(\alpha))
                                      M_{i-1} = (\varphi(t_{i-1}), \psi(t_{i-1}))
                              = (\varphi(t_2), \psi(t_2))
                 M_1 = (\varphi(t_1), \psi(t_1))
```

$$\int_{L} f(x, y) ds = \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_{i}, \eta_{i}) \Delta s_{i}$$

$$B = (\varphi(b), \psi(b))$$

$$(\varphi(t), \psi(t)), (a \le t \le b) : L$$

$$M_{n-1} = (\varphi(t_{n-1}), \psi(t_{n-1}))$$

$$As_{i}$$

$$M_{i-1} = (\varphi(t_{i}), \psi(t_{i}))$$

$$As_{i}$$

$$M_{i-1} = (\varphi(t_{i-1}), \psi(t_{i-1}))$$

$$M_{1} = (\varphi(t_{1}), \psi(t_{1}))$$

$$M_{1} = (\varphi(t_{1}), \psi(t_{1}))$$

$$X$$

$$\int_{L} f(x, y) ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(\xi_{i}, \eta_{i}) \Delta s_{i}$$

$$\int_{L} f(x, y) ds = \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_{i}, \eta_{i}) \Delta s_{i}$$

```
B = (\varphi(b), \psi(b))
(\varphi(t), \psi(t)), (\alpha \leq t \leq b) : L^{\bullet}
                                                                   \Delta s_i \approx |M_{i-1}M_i|
A = (\varphi(\alpha), \psi(\alpha))
                                     M_{i-1} = (\varphi(t_{i-1}), \psi(t_{i-1}))
                               = (\varphi(t_2), \psi(t_2))
                  M_1 = (\varphi(t_1), \psi(t_1))
      \int_L f(x, y) ds = \lim_{\lambda \to 0} \sum_{i=1}^n f(\xi_i, \eta_i) \Delta s_i
```

```
(\varphi(t), \psi(t)), (\alpha \le t \le b) : L^{\bullet}
                                         i) M_i = (\varphi(t_i), \psi(t_i))

\Delta s_i \approx |M_{i-1}M_i|

M_{i-1} = (\varphi(t_{i-1}), \psi(t_{i-1})) = \sqrt{(\varphi(t_{i-1}) - \varphi(t_i))^2 + (\psi(t_{i-1}) - \psi(t_i))^2}
A = (\varphi(\alpha), \psi(\alpha))
                                    =(\varphi(t_2),\psi(t_2))
                    M_1 = (\varphi(t_1), \psi(t_1))
```

$$\int_L f(x, y) ds = \lim_{\lambda \to 0} \sum_{i=1}^n f(\xi_i, \eta_i) \Delta s_i$$

$$B = (\varphi(b), \psi(b))$$

$$(\varphi(t), \psi(t)), (a \le t \le b) : L^{\bullet}$$

$$M_{n-1} = (\varphi(t_{n-1}), \psi(t_{n-1}))$$

$$(\xi_{i}, \eta_{i}) \qquad M_{i} = (\varphi(t_{i}), \psi(t_{i}))$$

$$\Delta s_{i} \qquad \Delta s_{i} \approx |M_{i-1}M_{i}|$$

$$M_{i-1} = (\varphi(t_{i-1}), \psi(t_{i-1})) = \sqrt{(\varphi(t_{i-1}) - \varphi(t_{i}))^{2} + (\psi(t_{i-1}) - \psi(t_{i}))^{2}}$$

$$\approx \sqrt{(\varphi'(t_{i})(t_{i-1} - t_{i}))^{2} + (\psi'(t_{i})(t_{i-1} - t_{i}))^{2}}$$

$$M_{1} = (\varphi(t_{1}), \psi(t_{1}))$$

$$X$$

$$\int_{L} f(x, y) ds = \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_{i}, \eta_{i}) \Delta s_{i}$$

$$B = (\varphi(b), \psi(b))$$

$$(\varphi(t), \psi(t)), (a \le t \le b) : L^{\bullet}$$

$$M_{n-1} = (\varphi(t_{n-1}), \psi(t_{n-1}))$$

$$(\xi_{i}, \eta_{i}) \qquad M_{i} = (\varphi(t_{i}), \psi(t_{i}))$$

$$\Delta s_{i} \qquad \Delta s_{i} \approx |M_{i-1}M_{i}|$$

$$M_{i-1} = (\varphi(t_{i-1}), \psi(t_{i-1})) = \sqrt{(\varphi(t_{i-1}) - \varphi(t_{i}))^{2} + (\psi(t_{i-1}) - \psi(t_{i}))^{2}}$$

$$\approx \sqrt{(\varphi'(t_{i})(t_{i-1} - t_{i}))^{2} + (\psi'(t_{i})(t_{i-1} - t_{i}))^{2}}$$

$$M_{1} = (\varphi(t_{1}), \psi(t_{1})) \qquad = \sqrt{\varphi'(t_{i})^{2} + \psi'(t_{i})^{2}}(t_{i} - t_{i-1})$$

$$f(x, y)ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(\xi_{i}, \eta_{i}) \Delta s_{i}$$

$$\int_{L} f(x, y) ds = \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_{i}, \eta_{i}) \Delta s_{i}$$



$$B = (\varphi(b), \psi(b))$$

$$(\varphi(t), \psi(t)), (a \le t \le b) : L$$

$$M_{n-1} = (\varphi(t_{n-1}), \psi(t_{n-1}))$$

$$\Delta s_{i} \approx |M_{i-1}M_{i}|$$

$$M_{i-1} = (\varphi(t_{i-1}), \psi(t_{i-1})) = \sqrt{(\varphi(t_{i-1}) - \varphi(t_{i}))^{2} + (\psi(t_{i-1}) - \psi(t_{i}))^{2}}$$

$$\approx \sqrt{(\varphi'(t_{i})(t_{i-1} - t_{i}))^{2} + (\psi'(t_{i})(t_{i-1} - t_{i}))^{2}}$$

$$M_{1} = (\varphi(t_{1}), \psi(t_{1})) = \sqrt{\varphi'(t_{i})^{2} + \psi'(t_{i})^{2}}(t_{i} - t_{i-1})$$

$$= \sqrt{\varphi'(t_{i})^{2} + \psi'(t_{i})^{2}}\Delta t_{i}$$

$$\int_{L} f(x, y) ds = \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_{i}, \eta_{i}) \Delta s_{i}$$

$$B = (\varphi(b), \psi(b))$$

$$(\varphi(t), \psi(t)), (\alpha \le t \le b) : L^{\bullet}$$

$$M_{n-1} = (\varphi(t_{n-1}), \psi(t_{n-1}))$$

$$(\xi_{i}, \eta_{i}) \qquad M_{i} = (\varphi(t_{i}), \psi(t_{i}))$$

$$\Delta s_{i} \qquad \Delta s_{i} \approx |M_{i-1}M_{i}|$$

$$M_{i-1} = (\varphi(t_{i-1}), \psi(t_{i-1})) = \sqrt{(\varphi(t_{i-1}) - \varphi(t_{i}))^{2} + (\psi(t_{i-1}) - \psi(t_{i}))^{2}}$$

$$\approx \sqrt{(\varphi'(t_{i})(t_{i-1} - t_{i}))^{2} + (\psi'(t_{i})(t_{i-1} - t_{i}))^{2}}$$

$$= \sqrt{\varphi'(t_{i})^{2} + \psi'(t_{i})^{2}} (t_{i} - t_{i-1})$$

$$= \sqrt{\varphi'(t_{i})^{2} + \psi'(t_{i})^{2}} \Delta t_{i}$$

$$\int_{L} f(x, y) ds = \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_{i}, \eta_{i}) \Delta s_{i}$$

$$\sqrt{\varphi'(t_{i})^{2} + \psi'(t_{i})^{2}} \Delta t_{i}$$

$$B = (\varphi(b), \psi(b))$$

$$(\varphi(t), \psi(t)), (a \le t \le b) : L^{\bullet}$$

$$M_{n-1} = (\varphi(t_{n-1}), \psi(t_{n-1}))$$

$$(\xi_{i}, \eta_{i}) \qquad M_{i} = (\varphi(t_{i}), \psi(t_{i}))$$

$$\Delta s_{i} \qquad \Delta s_{i} \approx |M_{i-1}M_{i}|$$

$$M_{i-1} = (\varphi(t_{i-1}), \psi(t_{i-1})) = \sqrt{(\varphi(t_{i-1}) - \varphi(t_{i}))^{2} + (\psi(t_{i-1}) - \psi(t_{i}))^{2}}$$

$$\approx \sqrt{(\varphi'(t_{i})(t_{i-1} - t_{i}))^{2} + (\psi'(t_{i})(t_{i-1} - t_{i}))^{2}}$$

$$= \sqrt{\varphi'(t_{i})^{2} + \psi'(t_{i})^{2}} (t_{i} - t_{i-1})$$

$$= \sqrt{\varphi'(t_{i})^{2} + \psi'(t_{i})^{2}} \Delta t_{i}$$

$$f(\varphi(t_{i}), \psi(t_{i}), \psi(t_{i})) \sqrt{\varphi'(t_{i})^{2} + \psi'(t_{i})^{2}} \Delta t_{i}$$

$$B = (\varphi(b), \psi(b))$$

$$(\varphi(t), \psi(t)), (a \le t \le b) : L^{\bullet}$$

$$M_{n-1} = (\varphi(t_{n-1}), \psi(t_{n-1}))$$

$$(\xi_{i}, \eta_{i}) \qquad M_{i} = (\varphi(t_{i}), \psi(t_{i}))$$

$$\Delta s_{i} \qquad \Delta s_{i} \approx |M_{i-1}M_{i}|$$

$$M_{i-1} = (\varphi(t_{i-1}), \psi(t_{i-1})) = \sqrt{(\varphi(t_{i-1}) - \varphi(t_{i}))^{2} + (\psi(t_{i-1}) - \psi(t_{i}))^{2}}$$

$$\approx \sqrt{(\varphi'(t_{i})(t_{i-1} - t_{i}))^{2} + (\psi'(t_{i})(t_{i-1} - t_{i}))^{2}}$$

$$= \sqrt{\varphi'(t_{i})^{2} + \psi'(t_{i})^{2}} (t_{i} - t_{i-1})$$

$$= \sqrt{\varphi'(t_{i})^{2} + \psi'(t_{i})^{2}} \Delta t_{i}$$

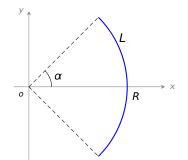
$$= \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_{i}, \eta_{i}) \Delta s_{i}$$

$$= \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\varphi(t_{i}), \psi(t_{i})) \sqrt{\varphi'(t_{i})^{2} + \psi'(t_{i})^{2}} \Delta t_{i}$$

从上述推导可知:

性质 设平面曲线 L 的参数方程为 $x = \varphi(t)$, $y = \psi(t)$, 则弧长元素 $ds = \sqrt{\varphi'(t)^2 + \psi'(t)^2} dt.$

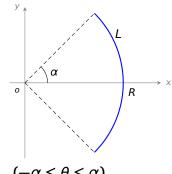
M1 计算 $\int_L y^2 ds$,其中曲线 L 如右图所示



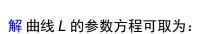
例 1 计算 $\int_{L} y^2 ds$, 其中曲线 L 如右图所 示

m 曲线 L 的参数方程可取为:

$$x = R\cos\theta$$
, $y = R\sin\theta$ $(-\alpha \le \theta \le \alpha)$

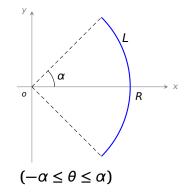


M_1 计算 $\int_L y^2 ds$,其中曲线 L 如右图所示

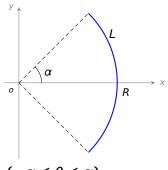


$$x = R \cos \theta$$
, $y = R \sin \theta$ $(-\alpha \le \theta \le \alpha)$

$$\int_{L} y^{2} ds = \int_{-\alpha}^{\alpha} R^{2} \sin^{2} \theta \cdot$$



 M_1 计算 $\int_L y^2 ds$,其中曲线 L 如右图所示



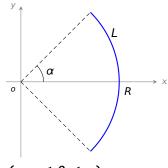
解曲线 L 的参数方程可取为:

$$x = R\cos\theta$$
, $y = R\sin\theta$ $(-\alpha \le \theta \le \alpha)$

所以

$$\int_{L} y^{2} ds = \int_{-\alpha}^{\alpha} R^{2} \sin^{2} \theta \cdot \sqrt{\left[(R \cos \theta)' \right]^{2} + \left[(R \sin \theta)' \right]^{2}} d\theta$$

 M_1 计算 $\int_L y^2 ds$,其中曲线 L 如右图所示

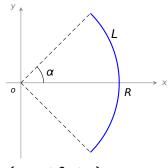


解曲线 L 的参数方程可取为:

$$x = R\cos\theta$$
, $y = R\sin\theta$ $(-\alpha \le \theta \le \alpha)$

$$\int_{L} y^{2} ds = \int_{-\alpha}^{\alpha} R^{2} \sin^{2} \theta \cdot \sqrt{\left[(R \cos \theta)' \right]^{2} + \left[(R \sin \theta)' \right]^{2}} d\theta$$
$$= \int_{-\alpha}^{\alpha} R^{2} \sin^{2} \theta \cdot R d\theta$$

 M_1 计算 $\int_L y^2 ds$,其中曲线 L 如右图所示



解曲线 L 的参数方程可取为:

$$x = R\cos\theta$$
, $y = R\sin\theta$ $(-\alpha \le \theta \le \alpha)$

所以

$$\int_{L} y^{2} ds = \int_{-\alpha}^{\alpha} R^{2} \sin^{2} \theta \cdot \sqrt{\left[(R \cos \theta)' \right]^{2} + \left[(R \sin \theta)' \right]^{2}} d\theta$$
$$= \int_{-\alpha}^{\alpha} R^{2} \sin^{2} \theta \cdot R d\theta = R^{3} \int_{-\alpha}^{\alpha} \frac{1}{2} (1 - \cos 2\theta) d\theta$$

$$M_1$$
计算 $\int_L y^2 ds$,其中曲线 L 如右图所

 $(-\alpha < \theta < \alpha)$

解 曲线 L 的参数方程可取为:

$$x = R\cos\theta$$
, $y = R\sin\theta$ $(-\alpha \le \theta \le \alpha)$

$$\int_{L} y^{2} ds = \int_{-\alpha}^{\alpha} R^{2} \sin^{2} \theta \cdot \sqrt{\left[(R \cos \theta)' \right]^{2} + \left[(R \sin \theta)' \right]^{2}} d\theta$$

$$= \int_{-\alpha}^{\alpha} R^{2} \sin^{2} \theta \cdot R d\theta = R^{3} \int_{-\alpha}^{\alpha} \frac{1}{2} (1 - \cos 2\theta) d\theta$$

$$= \frac{1}{2} R^{3} (\theta - \frac{1}{2} \sin 2\theta)$$

$$0$$
 1 计算 $\int_L y^2 ds$,其中曲线 L 如右图所

 \mathbf{H} 曲线 \mathbf{L} 的参数方程可取为:

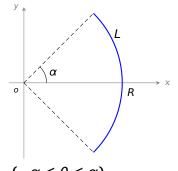
$$x = R\cos\theta, \quad y = R\sin\theta \quad (-\alpha \le \theta \le \alpha)$$

$$\int_{L} y^{2} ds = \int_{-\alpha}^{\alpha} R^{2} \sin^{2} \theta \cdot \sqrt{\left[(R \cos \theta)' \right]^{2} + \left[(R \sin \theta)' \right]^{2}} d\theta$$
$$= \int_{-\alpha}^{\alpha} R^{2} \sin^{2} \theta \cdot R d\theta = R^{3} \int_{-\alpha}^{\alpha} \frac{1}{2} (1 - \cos 2\theta) d\theta$$

 $= \frac{1}{2}R^3(\theta - \frac{1}{2}\sin 2\theta)\Big|^{\alpha}$



例 1 计算 $\int_{\mathcal{L}} y^2 ds$,其中曲线 L 如右图所



 \mathbf{H} 曲线 \mathbf{L} 的参数方程可取为:

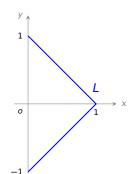
$$x = R\cos\theta, \quad y = R\sin\theta \quad (-\alpha \le \theta \le \alpha)$$

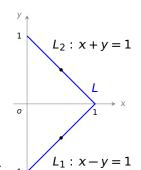
所以

$$\int_{L} y^{2} ds = \int_{-\alpha}^{\alpha} R^{2} \sin^{2} \theta \cdot \sqrt{\left[(R \cos \theta)' \right]^{2} + \left[(R \sin \theta)' \right]^{2}} d\theta$$

$$= \int_{-\alpha}^{\alpha} R^{2} \sin^{2} \theta \cdot R d\theta = R^{3} \int_{-\alpha}^{\alpha} \frac{1}{2} (1 - \cos 2\theta) d\theta$$

 $= \frac{1}{2}R^3(\theta - \frac{1}{2}\sin 2\theta)\Big|_{\alpha}^{\alpha} = R^3(\alpha - \frac{1}{2}\sin(2\alpha))$

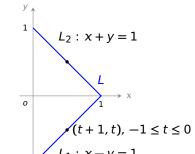




解

$$\int_{L} e^{x+y} ds = \int_{L_{1}} e^{x+y} ds + \int_{L_{2}} e^{x+y} ds _{-1}$$
 $L_{1}: x-y=1$

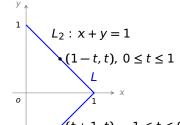




例 2 计算
$$\int_L e^{x+y} ds$$
,其中 L 如右图所示

$$\int_{L} e^{x+y} ds = \int_{L_{1}} e^{x+y} ds + \int_{L_{2}} e^{x+y} ds \Big|_{-1}$$
 $(t+1,t), -1$





$$\int_{L} e^{x+y} ds = \int_{L_{1}} e^{x+y} ds + \int_{L_{2}} e^{x+y} ds \Big|_{-1}$$
 $(t+1,t), -1$



$$\begin{array}{c|c}
 & L_2: x + y = 1 \\
 & (1 - t, t), \ 0 \le t \le 1
\end{array}$$

$$\begin{array}{c|c}
 & L \\
 & \downarrow \\
 & \downarrow \\
 & \downarrow \\
 & \downarrow \\
 & L_1: x - y = 1
\end{array}$$

$$\int_{L} e^{x+y} ds = \int_{L_{1}} e^{x+y} ds + \int_{L_{2}} e^{x+y} ds \Big|_{-1}$$

$$= \int_{-1}^{0} e^{2t+1} \cdot \sqrt{\left[(t+1)'\right]^{2} + \left[t'\right]^{2}} dt$$



例 2 计算
$$\int_{L} e^{x+y} ds$$
,其中 L 如右图所示

$$L_2: x + y = 1$$

$$(1 - t, t), 0 \le t \le 1$$

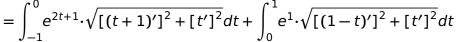
$$L_2: x + y = 1$$

$$L_3: x + y = 1$$

$$L_4: x + y = 1$$

$$\int_{L} e^{x+y} ds = \int_{L_{1}} e^{x+y} ds + \int_{L_{2}} e^{x+y} ds \Big|_{-1} \Big|_{L_{1}: x-y=1}$$

$$= \int_{L_{1}}^{0} e^{2t+1} \cdot \sqrt{[(t+1)']^{2} + [t']^{2}} dt + \int_{L_{2}}^{0} e^{1} \cdot \sqrt{[(1-t)']^{2} + [t']^{2}} dt$$





$$\begin{array}{c|c}
 & L_2: x + y = 1 \\
 & (1 - t, t), \ 0 \le t \le 1 \\
\hline
 & L_2: x + y = 1
\end{array}$$

$$\begin{array}{c|c}
 & L_2: x + y = 1 \\
 & L_2: x + y = 1
\end{array}$$

$$\int_{L} e^{x+y} ds = \int_{L_{1}} e^{x+y} ds + \int_{L_{2}} e^{x+y} ds \Big|_{-1} \Big|_{L_{1} : x-y=1}$$

$$= \int_{-1}^{0} e^{2t+1} \cdot \sqrt{\left[(t+1)'\right]^{2} + \left[t'\right]^{2}} dt + \int_{0}^{1} e^{1} \cdot \sqrt{\left[(1-t)'\right]^{2} + \left[t'\right]^{2}} dt$$

$$= \sqrt{2} \int_{-1}^{0} e^{2t+1} dt + \sqrt{2} \int_{0}^{1} e^{1} dt$$



第 11 章 a:对弧长的曲线积分

 $\int_{a}^{b} e^{x+y} ds = \int_{a}^{b} e^{x+y} ds + \int_{a}^{b} e^{x+y} ds = 1$

 $= \int_{-1}^{0} e^{2t+1} \cdot \sqrt{\left[(t+1)'\right]^2 + \left[t'\right]^2} dt + \int_{0}^{1} e^{1} \cdot \sqrt{\left[(1-t)'\right]^2 + \left[t'\right]^2} dt$ $=\sqrt{2}\int_{0}^{0}e^{2t+1}dt+\sqrt{2}\int_{0}^{1}e^{1}dt$

 $=\sqrt{2}\cdot\frac{1}{2}e^{2t+1}$



 $\int_{a}^{b} e^{x+y} ds = \int_{a}^{b} e^{x+y} ds + \int_{a}^{b} e^{x+y} ds = 1$

 $= \int_{-1}^{0} e^{2t+1} \cdot \sqrt{\left[(t+1)'\right]^2 + \left[t'\right]^2} dt + \int_{0}^{1} e^{1} \cdot \sqrt{\left[(1-t)'\right]^2 + \left[t'\right]^2} dt$ $=\sqrt{2}\int_{0}^{0}e^{2t+1}dt+\sqrt{2}\int_{0}^{1}e^{1}dt$

 $=\sqrt{2}\cdot\frac{1}{2}e^{2t+1}\Big|_{0}$

 $\int_{L} e^{x+y} ds = \int_{L_{1}} e^{x+y} ds + \int_{L_{2}} e^{x+y} ds \Big|_{-1} \Big|_{L_{1}: x-y=1}$

 $= \int_{-1}^{0} e^{2t+1} \cdot \sqrt{\left[(t+1)'\right]^2 + \left[t'\right]^2} dt + \int_{0}^{1} e^{1} \cdot \sqrt{\left[(1-t)'\right]^2 + \left[t'\right]^2} dt$ $=\sqrt{2}\int_{0}^{0}e^{2t+1}dt+\sqrt{2}\int_{0}^{1}e^{1}dt$

 $=\sqrt{2}\cdot\frac{1}{2}e^{2t+1}\Big|_{1}^{0}+\sqrt{2}e^{-t}$

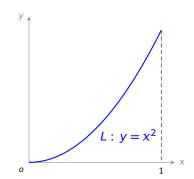


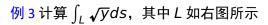
$$\int_{L} e^{x+y} ds = \int_{L_{1}} e^{x+y} ds + \int_{L_{2}} e^{x+y} ds \Big|_{-1} \Big|_{L_{1}: x-y=1}$$

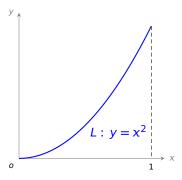
$$= \sqrt{2} \int_{-1}^{1} e^{2t+1} dt + \sqrt{2} \int_{0}^{1} e^{1} dt$$
$$= \sqrt{2} \cdot \frac{1}{2} e^{2t+1} \Big|_{0}^{1} + \sqrt{2} e = \frac{\sqrt{2}}{2} (3e - e^{-1})$$

$$= \int_{-1}^{0} e^{2t+1} \cdot \sqrt{\left[(t+1)'\right]^2 + \left[t'\right]^2} dt + \int_{0}^{1} e^{1} \cdot \sqrt{\left[(1-t)'\right]^2 + \left[t'\right]^2} dt$$
$$= \sqrt{2} \int_{-1}^{0} e^{2t+1} dt + \sqrt{2} \int_{0}^{1} e^{1} dt$$

 $=\sqrt{2}\int_{1}^{0}e^{2t+1}dt+\sqrt{2}\int_{0}^{1}e^{1}dt$

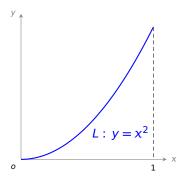






解曲线 L 的参数方程可取为:

$$x = t, \quad y = t^2 \qquad (0 \le t \le 1)$$

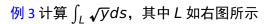


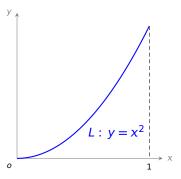
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$$\int_{t} \sqrt{y} ds = \int_{0}^{1} \sqrt{t^{2}} \cdot$$





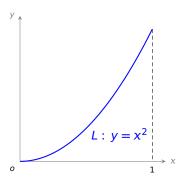


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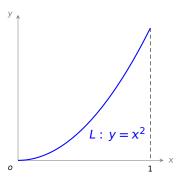
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第 11 草 a: 对弧长的曲线积分



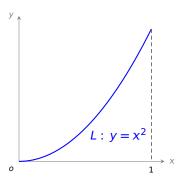
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第 11 章 α:对弧长的曲线积分



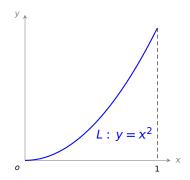
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$$\frac{u = 1 + 4t^{2}}{2} \int_{0}^{5} \sqrt{u} \cdot dt$$





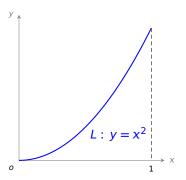
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$$\frac{u = 1 + 4t^{2}}{2} \int_{1}^{5} \sqrt{u} \cdot \frac{1}{8} du$$





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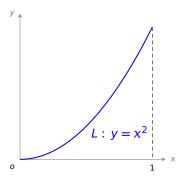
所以

$$\int_{L} \sqrt{y} ds = \int_{0}^{1} \sqrt{t^{2}} \cdot \sqrt{[t']^{2} + [(t^{2})']^{2}} dt = \int_{0}^{1} t \cdot \sqrt{1 + 4t^{2}} dt$$

$$\frac{u = 1 + 4t^{2}}{1 + 4t^{2}} \int_{1}^{5} \sqrt{u} \cdot \frac{1}{8} du = \frac{1}{8} \cdot \frac{2}{3} u^{\frac{3}{2}}$$



第 11 章 α:对弧长的曲线积分



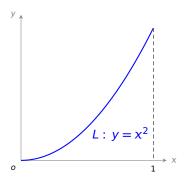
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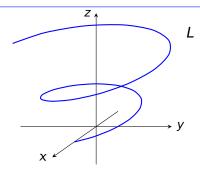


We are here now...

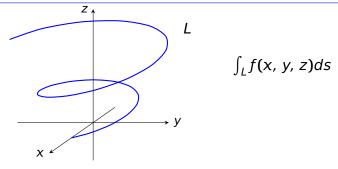
1. 对弧长的曲线积分: 概念与性质

2. 对弧长的曲线积分: 计算法

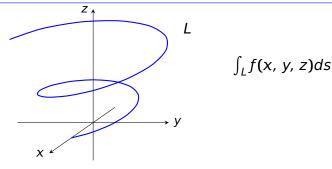
3. 对弧长的曲线积分: 空间曲线



 $\int_L f(x, y, z) ds$

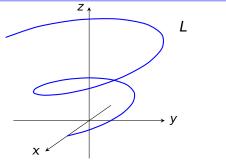


• 当 f(x, y, z) 是线密度时, $\int_L f(x, y, z) ds$ 表示曲线的质量



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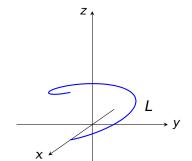


$$\int_L f(x, y, z) ds$$

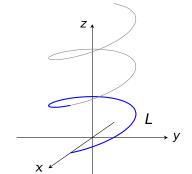
- 当 f(x, y, z) 是线密度时, $\int_L f(x, y, z) ds$ 表示曲线的质量
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$$\int_{a}^{b} f(x, y, z) ds = \int_{a}^{b} f(\varphi(t), \psi(t), \zeta(t)) \sqrt{\varphi'(t)^{2} + \psi'(t)^{2} + \zeta'(t)^{2}} dt$$

例 1 计算 $\int_L (x^2 + y^2 + z^2) ds$,其中 L 为 螺旋线 $x = a \cos t$, $y = a \sin t$, z = bt $(0 \le t \le 2\pi)$



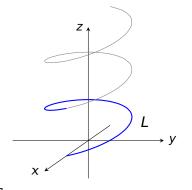
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$$\Re \int_{L} (x^{2} + y^{2} + z^{2}) ds$$

$$= \int_{0}^{2\pi} \left[(a\cos t)^{2} + (a\sin t)^{2} + (bt)^{2} \right] \cdot$$



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$$\sqrt{[(a\cos t)']^2 + [(a\sin t)']^2 + [(bt)']^2}dt$$



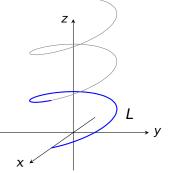
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$$= \int_0^{2\pi} [(a\cos t)^2 + (a\sin t)^2 + (bt)^2].$$

$$\sqrt{[(a\cos t)']^2 + [(a\sin t)']^2 + [(bt)']^2} dt$$

$$= \int_{0}^{2\pi} \left[a^{2} + b^{2} t^{2} \right] \cdot \sqrt{a^{2} + b^{2}} dt$$



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$$\sqrt{\left[(a\cos t)'\right]^2 + \left[(a\sin t)'\right]^2 + \left[(bt)'\right]^2} dt$$

$$= \int_0^{2\pi} \left[a^2 + b^2 t^2\right] \cdot \sqrt{a^2 + b^2} dt = \sqrt{a^2 + b^2} \cdot \left(a^2 t + \frac{1}{3}b^2 t^3\right) \Big|_0^{2\pi}$$



第 11 章 α:对弧长的曲线积分

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$$= \int_{0}^{2\pi} \left[(a\cos t)^{2} + (a\sin t)^{2} + (bt)^{2} \right].$$

$$\sqrt{[(a\cos t)']^2 + [(a\sin t)']^2 + [(bt)']^2}dt$$

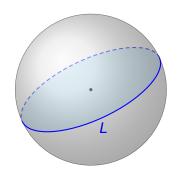
$$= \frac{2}{3}\pi\sqrt{a^2 + b^2} \cdot (3a^2 + 4b^2\pi^2)$$

 $= \int_{0}^{2\pi} \left[a^{2} + b^{2}t^{2} \right] \cdot \sqrt{a^{2} + b^{2}} dt = \sqrt{a^{2} + b^{2}} \cdot \left(a^{2}t + \frac{1}{3}b^{2}t^{3} \right) \Big|_{0}^{2\pi}$

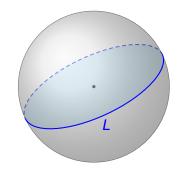


育11 章 α: 对弧长的曲线积分

例 2 计算 $\int_L x^2 ds$,其中 L 为球面 $x^2 + y^2 + z^2 = 1$ 与平面 x + y + z = 0 的交线。



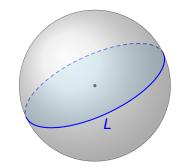
例 2 计算 $\int_L x^2 ds$,其中 L 为球面 $x^2 + y^2 + z^2 = 1$ 与平面 x + y + z = 0 的交线。



解由对称性可知:

$$\int_{L} x^2 ds = \int_{L} y^2 ds = \int_{L} z^2 ds$$

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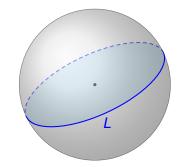


解由对称性可知:

$$\int_{L} x^{2} ds = \int_{L} y^{2} ds = \int_{L} z^{2} ds$$

$$\int_{1} x^{2} ds = \frac{1}{3} \int_{1} (x^{2} + y^{2} + z^{2}) ds$$

例 2 计算
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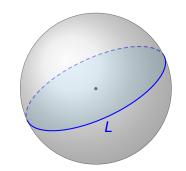


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$$\int_{L} x^{2} ds = \frac{1}{3} \int_{L} (x^{2} + y^{2} + z^{2}) ds = \frac{1}{3} \int_{L} 1 ds$$

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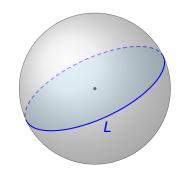
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解 由对称性可知:

$$\int_{I} x^{2} ds = \int_{I} y^{2} ds = \int_{I} z^{2} ds$$

$$\int_{L} x^{2} ds = \frac{1}{3} \int_{L} (x^{2} + y^{2} + z^{2}) ds = \frac{1}{3} \int_{L} 1 ds = \frac{1}{3} \text{Length}(L) = \frac{1}{3} \cdot 2\pi$$

