第11章 d:对面积的曲面积分

数学系 梁卓滨

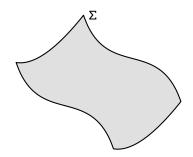
2019-2020 学年 II

Outline



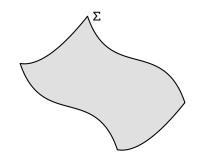
假设

- Σ 为空间中曲面
- 密度为 µ
- 质量为 m



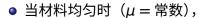
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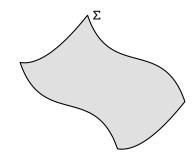
- Σ 为空间中曲面
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- 当材料均匀时(μ = 常数),



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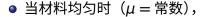
$$m = \mu \cdot Area(\Sigma)$$

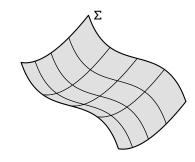
• 当材料非均匀时($\mu = \mu(x, y, z)$ 为 Σ 上函数),



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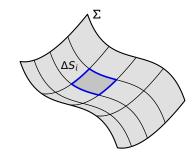




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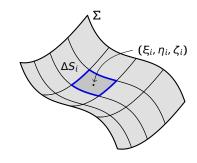
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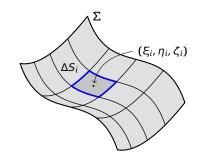
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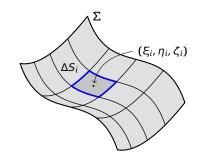


$$m = \mu \cdot Area(\Sigma)$$

$$\mu(\xi_i, \eta_i, \zeta_i)\Delta S_i$$

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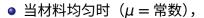


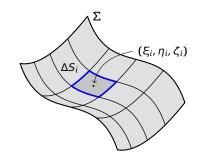
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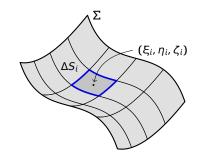


$$m = \mu \cdot \text{Area}(\Sigma)$$

$$\lim_{\lambda \to 0} \sum_{i=1}^{n} \mu(\xi_i, \, \eta_i, \, \zeta_i) \Delta S_i$$

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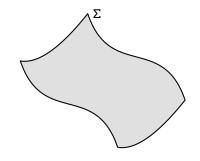


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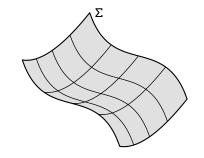
设

- Σ是空间中有界分片光滑曲面,
- f(x, y, z) 是 Σ 上的有界函数,



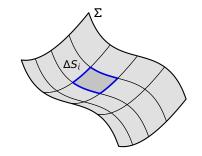
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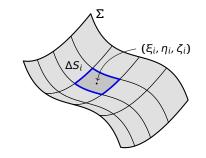
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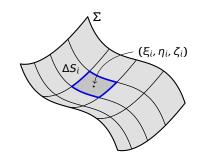
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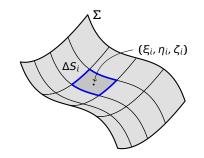
$$\sum_{i=1}^n f(\xi_i, \, \eta_i, \, \zeta_i) \Delta S_i$$



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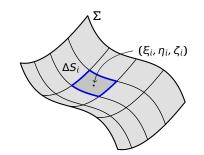


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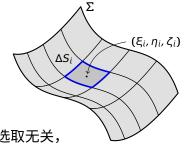
• 极限 $\lim_{\lambda \to 0} \sum_{i=1}^n f(\xi_i, \eta_i, \zeta_i) \Delta S_i$ 存在,



设

- Σ是空间中有界分片光滑曲面,
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- 且该极限与 Σ 的划分、(ξ_i , η_i , ζ_i) 的选取无关,



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则定义

$$\iint_{\Sigma} f(x, y, z) dS = \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_i, \eta_i, \zeta_i) \Delta S_i$$



 (ξ_i, η_i, ζ_i)

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即中有界分片光滑曲面,
$$(\xi_i, \eta_i, \zeta_i)$$
 之)是 Σ 上的有界函数,
$$\sum_{i=1}^n f(\xi_i, \eta_i, \zeta_i) \Delta S_i$$
 存在,
$$\mathbb{R} \to \Sigma$$
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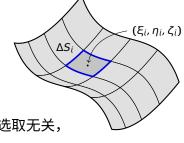
称为 f(x, y, z) 在 Σ 上对面积的曲面积分.

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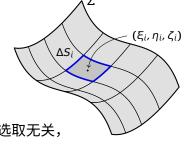
称为 f(x, y, z) 在 Σ 上 对面积的曲面积分 .dS 称为 面积元素 .

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称为 f(x, y, z) 在 Σ 上对面积的曲面积分 .dS 称为面积元素 .

注 对面积曲面积分的定义式与二重积分的类似,故性质也类似



存在性 若 f(x, y, z) 在有界曲面 Σ 上连续,则

$$\iint_{\Sigma} f(x, y, z) dS$$

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存在。

• 线性性 $\iint_{\Sigma} (\alpha f + \beta g) dS = \alpha \iint_{\Sigma} f dS + \beta \iint_{\Sigma} g dS$

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- 可加性 $\iint_{\Sigma} f(x, y, z) dS = \iint_{\Sigma_1} f(x, y, z) dS + \iint_{\Sigma_2} f(x, y, z) dS$

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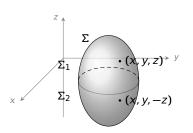
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- 若 $f(x, y, z) \leq g(x, y, z)$,则

$$\iint_{\Sigma} f(x, y, z) dS \le \iint_{\Sigma} g(x, y, z) dS$$



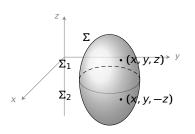
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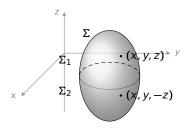




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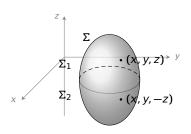


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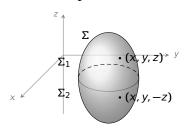
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$$\iint_{\Sigma} f(x, y, z) dS = 2 \iint_{\Sigma_1} f(x, y, z) dS = 2 \iint_{\Sigma_2} f(x, y, z) dS$$



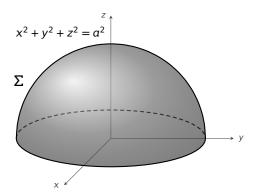
例 设曲面 Σ 为上半球面 $x^2 + y^2 + z^2 = \alpha^2$ ($z \ge 0$); Σ_1 为 Σ 在第一卦 限的部分. 则有 ()

(A)
$$\iint_{\Sigma} x dS = 4 \iint_{\Sigma_1} x dS$$

(B)
$$\iint_{\Sigma} y dS = 4 \iint_{\Sigma_1} y dS$$

(C)
$$\iint_{\Sigma} z dS = 4 \iint_{\Sigma_1} z dS$$

(D)
$$\iint_{\Sigma} xyzdS = 4 \iint_{\Sigma_1} xyzdS$$



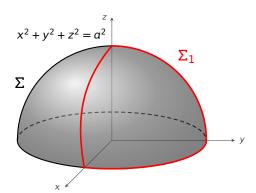
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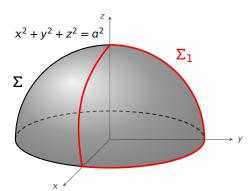
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解 由对称性:

$$\iint_{\Sigma} x^2 dS = \iint_{\Sigma} y^2 dS = \iint_{\Sigma} z^2 dS$$



解 由对称性:

$$\iint_{\Sigma} x^2 dS = \iint_{\Sigma} y^2 dS = \iint_{\Sigma} z^2 dS$$

$$\iint_{\Sigma} (x^2 + y^2) dS = 2 \iint_{\Sigma} x^2 dS$$



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$$\iint_{\Sigma} (x^2 + y^2) dS = 2 \iint_{\Sigma} x^2 dS$$
$$= \frac{2}{3} \left[\iint_{\Sigma} x^2 dS + \iint_{\Sigma} y^2 dS + \iint_{\Sigma} z^2 dS \right]$$



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$$= \frac{2}{3} \iint_{\Sigma} x^2 + y^2 + z^2 dS$$

$$= \frac{2}{3} \iint_{\Sigma} R^2 dS$$

解 由对称性:

$$\iint_{\Sigma} x^2 dS = \iint_{\Sigma} y^2 dS = \iint_{\Sigma} z^2 dS$$

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$$= \frac{2}{3} \iint_{\Sigma} x^2 + y^2 + z^2 dS$$

$$= \frac{2}{3} \iint_{\Sigma} R^2 dS = \frac{2}{3} R^2 \text{Area}(\Sigma)$$

解 由对称性:

$$\iint_{\Sigma} x^2 dS = \iint_{\Sigma} y^2 dS = \iint_{\Sigma} z^2 dS$$

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$$= \frac{2}{3} \iint_{\Sigma} x^2 + y^2 + z^2 dS$$

$$= \frac{2}{3} \iint_{\Sigma} R^2 dS = \frac{2}{3} R^2 \operatorname{Area}(\Sigma) = \frac{2}{3} R^2 \cdot 4\pi R^2$$



解 由对称性:

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$$= \frac{2}{3} \left[\iint_{\Sigma} x^{2} dS + \iint_{\Sigma} y^{2} dS + \iint_{\Sigma} z^{2} dS \right]$$

$$= \frac{2}{3} \iint_{\Sigma} x^{2} + y^{2} + z^{2} dS$$

$$= \frac{2}{3} \iint_{\Sigma} R^{2} dS = \frac{2}{3} R^{2} \text{Area}(\Sigma) = \frac{2}{3} R^{2} \cdot 4\pi R^{2} = \frac{8}{3} \pi R^{4}$$

$$\iint_{\Sigma} f(x, y, z) dS =$$

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$$\sum_{z} \sum_{z} z = z(x, y)$$

$$D_{xy}$$

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$$\sum_{z} \sum_{z} \sum_{z} z = z(x, y)$$

$$D_{xy}$$

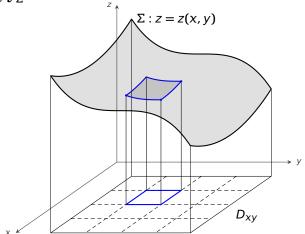
• 假设 Σ 是二元函数 z = z(x, y), $(x, y) \in D_{xy}$ 的图形,则

$$\iint_{\Sigma} f(x, y, z) dS =$$

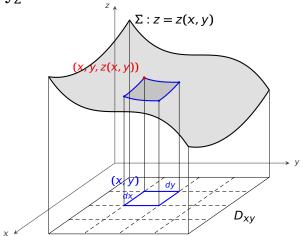
$$\sum_{z} \sum_{z} z = z(x, y)$$

 D_{xy}

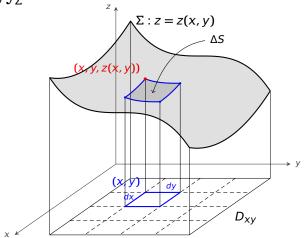
$$\iint_{\Sigma} f(x, y, z) dS =$$



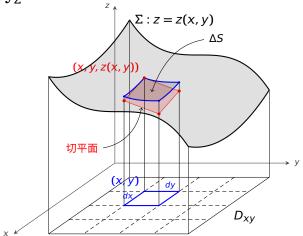
$$\iint_{\Sigma} f(x, y, z) dS =$$



$$\iint_{\Sigma} f(x, y, z) dS =$$



$$\iint_{\Sigma} f(x, y, z) dS =$$



• 假设 Σ 是二元函数 z = z(x, y), $(x, y) \in D_{xy}$ 的图形,则

$$\iint_{\Sigma} f(x, y, z) dS =$$

$$\sum_{z} \sum_{z} z = z(x, y)$$

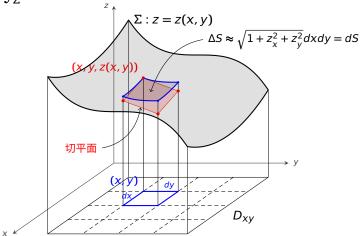
$$\Delta S \approx \sqrt{1 + z_{x}^{2} + z_{y}^{2}} dx dy$$
切平面

dy

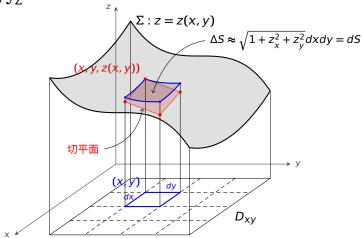
 D_{xy}



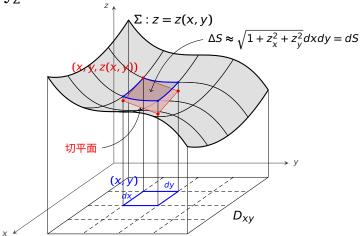
成及
$$Z$$
 定二元函数 $Z=Z(X,Y), (X,Y) \in D_{XY}$ 可图形,则
$$\iint_{\Sigma} f(x,y,z) dS =$$



$$\iint_{\Sigma} f(x, y, z) dS = f(x, y, z(x, y)) \cdot \sqrt{1 + z_{x}^{2} + z_{y}^{2}} dx dy$$



$$\iint_{\Sigma} f(x, y, z) dS = f(x, y, z(x, y)) \cdot \sqrt{1 + z_x^2 + z_y^2} dx dy$$



$$\iint_{\Sigma} f(x, y, z) dS = \sum f(x, y, z(x, y)) \cdot \sqrt{1 + z_{x}^{2} + z_{y}^{2}} dx dy$$

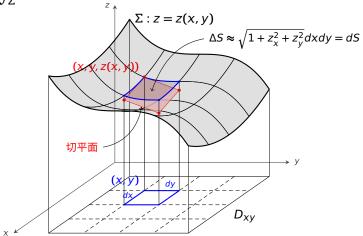
$$\Sigma : z = z(x, y)$$

$$\Delta S \approx \sqrt{1 + z_{x}^{2} + z_{y}^{2}} dx dy = dS$$

$$(x, y, z) dy$$

$$D_{xy}$$

$$\iint_{\Sigma} f(x, y, z) dS = \lim_{Z \to \infty} \sum_{x} f(x, y, z(x, y)) \cdot \sqrt{1 + z_x^2 + z_y^2} dx dy$$



$$\iint_{\Sigma} f(x, y, z) dS = \iint_{D_{xy}} f(x, y, z(x, y)) \cdot \sqrt{1 + z_x^2 + z_y^2} dx dy$$

$$\Sigma : z = z(x, y)$$

$$\Delta S \approx \sqrt{1 + z_x^2 + z_y^2} dx dy = dS$$

$$(x, y, z(x, y))$$

$$D_{xy}$$



$$\iint_{\Sigma} f(x, y, z) dS = \iint_{D_{xy}} f(x, y, z(x, y)) \cdot \sqrt{1 + z_{x}^{2} + z_{y}^{2}} dx dy$$

• 假设 Σ 是二元函数 z = z(x, y), $(x, y) \in D_{xy}$ 的图形,则

$$\iint_{\Sigma} f(x, y, z) dS = \iint_{D_{xy}} f(x, y, z(x, y)) \cdot \sqrt{1 + z_x^2 + z_y^2} dx dy$$

• 假设 Σ 是二元函数 y = y(x, z), $(x, z) \in D_{xz}$ 的图形,则 $\iint_{\Sigma} f(x, y, z) dS =$

● 假设 Σ 是二元函数 x = x(y, z), $(y, z) \in D_{yz}$ 的图形,则 $\iint_{\Sigma} f(x, y, z) dS =$

• 假设 Σ 是二元函数 z = z(x, y), $(x, y) \in D_{xy}$ 的图形,则

$$\iint_{\Sigma} f(x, y, z) dS = \iint_{D_{xy}} f(x, y, z(x, y)) \cdot \sqrt{1 + z_x^2 + z_y^2} dx dy$$

• 假设 Σ 是二元函数 y = y(x, z), $(x, z) \in D_{xz}$ 的图形,则

$$\iint_{\Sigma} f(x, y, z) dS = f(x, y(x, z), z)$$

● 假设 Σ 是二元函数 x = x(y, z), $(y, z) \in D_{yz}$ 的图形,则 $\iint_{\Sigma} f(x, y, z) dS =$

• 假设 Σ 是二元函数 z = z(x, y), $(x, y) \in D_{xy}$ 的图形,则

$$\iint_{\Sigma} f(x,y,z) dS = \iint_{D_{xy}} f(x,y,z(x,y)) \cdot \sqrt{1 + z_x^2 + z_y^2} dx dy$$

• 假设 Σ 是二元函数 y = y(x, z), $(x, z) \in D_{xz}$ 的图形,则

$$\iint_{\Sigma} f(x, y, z) dS = f(x, y(x, z), z) \cdot \sqrt{1 + y_x^2 + y_z^2} dx dz$$

● 假设 Σ 是二元函数 x = x(y, z), $(y, z) \in D_{yz}$ 的图形,则 $\iint_{\Sigma} f(x, y, z) dS =$

假设Σ是二元函数 z = z(x, y), (x, y) ∈ D_{xv} 的图形,则

$$\iint_{\Sigma} f(x, y, z) dS = \iint_{D_{xy}} f(x, y, z(x, y)) \cdot \sqrt{1 + z_x^2 + z_y^2} dx dy$$

• 假设 Σ 是二元函数 y = y(x, z), $(x, z) \in D_{xz}$ 的图形,则 $\iint_{\Sigma} f(x, y, z) dS = \iint_{\Sigma} f(x, y(x, z), z) \cdot \sqrt{1 + y_x^2 + y_z^2} dx dz$

$$\iint_{\Sigma} f(x, y, z) dS = \iint_{D_{xz}} f(x, y(x, z), z) \cdot \sqrt{1 + y_x^2 + y_z^2} dx dz$$

• 假设 Σ 是二元函数 x = x(y, z), $(y, z) \in D_{vz}$ 的图形,则 $\iint_{\Sigma} f(x, y, z) dS =$

• 假设 Σ 是二元函数 z = z(x, y), $(x, y) \in D_{xy}$ 的图形,则

$$\iint_{\Sigma} f(x, y, z) dS = \iint_{D_{xy}} f(x, y, z(x, y)) \cdot \sqrt{1 + z_x^2 + z_y^2} dx dy$$

$$\iint_{\Sigma} f(x, y, z) dS = \iint_{D_{XZ}} f(x, y(x, z), z) \cdot \sqrt{1 + y_{X}^{2} + y_{Z}^{2}} dx dz$$

• 假设 Σ 是二元函数 x = x(y, z), $(y, z) \in D_{yz}$ 的图形,则

$$\iint_{\Sigma} f(x, y, z) dS = f(x(y, z), y, z)$$

• 假设 Σ 是二元函数 z = z(x, y), $(x, y) \in D_{xy}$ 的图形,则

$$\iint_{\Sigma} f(x, y, z) dS = \iint_{D_{xy}} f(x, y, z(x, y)) \cdot \sqrt{1 + z_x^2 + z_y^2} dx dy$$

$$\iint_{\Sigma} f(x, y, z) dS = \iint_{D_{xz}} f(x, y(x, z), z) \cdot \sqrt{1 + y_x^2 + y_z^2} dx dz$$

• 假设 Σ 是二元函数 x = x(y, z), $(y, z) \in D_{yz}$ 的图形,则

$$\iint_{\Sigma} f(x, y, z) dS = f(x(y, z), y, z) \cdot \sqrt{1 + x_y^2 + x_z^2} dy dz$$



• 假设 Σ 是二元函数 z = z(x, y), $(x, y) \in D_{xy}$ 的图形,则

$$\iint_{\Sigma} f(x, y, z) dS = \iint_{D_{xy}} f(x, y, z(x, y)) \cdot \sqrt{1 + z_x^2 + z_y^2} dx dy$$

• 假设 Σ 是二元函数 y = y(x, z), $(x, z) \in D_{xz}$ 的图形,则 $\iint_{\Sigma} f(x, y, z) dS = \iint_{\Sigma} f(x, y(x, z), z) \cdot \sqrt{1 + y_x^2 + y_z^2} dx dz$

• 假设 Σ 是二元函数 x = x(y, z), $(y, z) \in D_{yz}$ 的图形,则 $\iint_{\Sigma} f(x, y, z) dS = \iint_{D} f(x(y, z), y, z) \cdot \sqrt{1 + x_{y}^{2} + x_{z}^{2}} dy dz$

• 假设 Σ 是二元函数 $z = z(x, y), (x, y) \in D_{xv}$ 的图形,则

$$\iint_{\Sigma} f(x, y, z) dS = \iint_{D_{xy}} f(x, y, z(x, y)) \cdot \sqrt{1 + z_x^2 + z_y^2} dx dy$$

• 假设 Σ 是二元函数 y = y(x, z), $(x, z) \in D_{xz}$ 的图形,则

$$\iint_{\Sigma} f(x, y, z) dS = \iint_{D_{xz}} f(x, y(x, z), z) \cdot \sqrt{1 + y_x^2 + y_z^2} dx dz$$

• 假设 Σ 是二元函数 x = x(y, z), $(y, z) \in D_{vz}$ 的图形,则

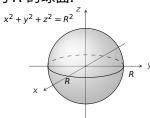
$$\iint_{\Sigma} f(x, y, z) dS = \iint_{D_{yz}} f(x(y, z), y, z) \cdot \sqrt{1 + x_y^2 + x_z^2} dy dz$$

注 对于复杂的曲面 Σ,尝试将其分解成若干部分 $Σ_1, \cdots, Σ_n$,每一部



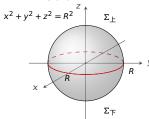
例 1 将对面积的曲面积分 $\iint_{\Sigma} f(x, y, z) dS$ 转换为重积分,其中 Σ 是球

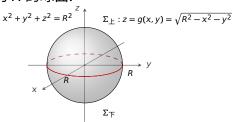
心在原点,半径为 R 的球面.

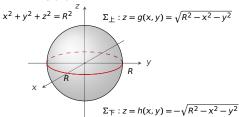


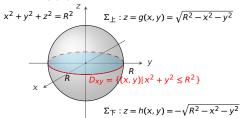
例 1 将对面积的曲面积分 $\iint_{\Sigma} f(x, y, z) dS$ 转换为重积分,其中 Σ 是球

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$$\Sigma_{\pm} : z = g(x, y) = \sqrt{R^2 - x^2 - y^2}$$

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$$\Sigma_{\pm} : z = h(x, y) = -\sqrt{R^2 - x^2 - y^2}$$

$$\iint_{\Sigma} f(x, y, z) dS = \iint_{\Sigma_{+}} f(x, y, z) dS + \iint_{\Sigma_{\mp}} f(x, y, z) dS$$

例 1 将对面积的曲面积分 $\iint_{\Sigma} f(x, y, z) dS$ 转换为重积分,其中 Σ 是球心在原点,半径为 R 的球面.

$$x^{2} + y^{2} + z^{2} = R^{2}$$

$$\Sigma_{\pm} : z = g(x, y) = \sqrt{R^{2} - x^{2} - y^{2}}$$

$$\sum_{K} y$$

$$x = \{x, y \mid x^{2} + y^{2} \le R^{2}\}$$

$$\Sigma_{\mp} : z = h(x, y) = -\sqrt{R^{2} - x^{2} - y^{2}}$$

$$\iint_{\Sigma} f(x, y, z) dS = \iint_{\Sigma_{\pm}} f(x, y, z) dS + \iint_{\Sigma_{\mp}} f(x, y, z) dS$$
$$= \iint_{D_{xy}} f(x, y, \sqrt{R^2 - x^2 - y^2}) \cdot \sqrt{1 + g_x^2 + g_y^2} dx dy$$

$$\Sigma_{\pm} : z = g(x, y) = \sqrt{R^2 - x^2 - y^2}$$

$$\Sigma_{\pm} : z = g(x, y) = \sqrt{R^2 - x^2 - y^2}$$

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$$\Sigma_{\mp} : z = h(x, y) = -\sqrt{R^2 - x^2 - y^2}$$

$$\iint_{\Sigma} f(x, y, z) dS = \iint_{\Sigma_{\pm}} f(x, y, z) dS + \iint_{\Sigma_{\mp}} f(x, y, z) dS
= \iint_{D_{xy}} f(x, y, \sqrt{R^2 - x^2 - y^2}) \cdot \sqrt{1 + g_x^2 + g_y^2} dx dy
+ \iint_{D_{xy}} f(x, y, -\sqrt{R^2 - x^2 - y^2}) \cdot \sqrt{1 + h_x^2 + h_y^2} dx dy$$



$$\Sigma_{\pm} : z = g(x, y) = \sqrt{R^2 - x^2 - y^2}$$

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$$\Sigma_{\pm} : z = g(x, y) = \sqrt{R^2 - x^2 - y^2}$$

$$\Sigma_{\pm} : z = h(x, y) = -\sqrt{R^2 - x^2 - y^2}$$

$$\iint_{\Sigma} f(x, y, z) dS = \iint_{\Sigma_{\pm}} f(x, y, z) dS + \iint_{\Sigma_{\mp}} f(x, y, z) dS
= \iint_{D_{xy}} f(x, y, \sqrt{R^2 - x^2 - y^2}) \cdot \frac{R}{\sqrt{R^2 - x^2 - y^2}} dx dy
+ \iint_{D} f(x, y, -\sqrt{R^2 - x^2 - y^2}) \cdot \sqrt{1 + h_x^2 + h_y^2} dx dy$$

$$x^{2} + y^{2} + z^{2} = R^{2}$$

$$\Sigma_{\pm} : z = g(x, y) = \sqrt{R^{2} - x^{2} - y^{2}}$$

$$X = D_{xy} = y(x, y) | x^{2} + y^{2} \le R^{2}$$

$$\Sigma_{\mp} : z = h(x, y) = -\sqrt{R^{2} - x^{2} - y^{2}}$$

$$\iint_{\Sigma} f(x, y, z) dS = \iint_{\Sigma_{\pm}} f(x, y, z) dS + \iint_{\Sigma_{\mp}} f(x, y, z) dS$$

$$= \iint_{D_{xy}} f(x, y, \sqrt{R^2 - x^2 - y^2}) \cdot \frac{R}{\sqrt{R^2 - x^2 - y^2}} dx dy$$

$$+ \iint_{D_{xy}} f(x, y, -\sqrt{R^2 - x^2 - y^2}) \cdot \frac{R}{\sqrt{R^2 - x^2 - y^2}} dx dy$$





例 1 将对面积的曲面积分 $\iint_{\Sigma} f(x, y, z) dS$ 转换为重积分,其中 Σ 是球 心在原点,半径为 R 的球面.

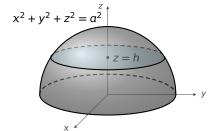
 $x^2 + v^2 + z^2 = R^2$ $\Sigma_+ : z = g(x, y) = \sqrt{R^2 - x^2 - y^2}$

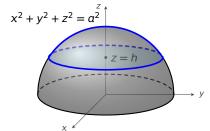
$$\int_{\Sigma} f(x, y, z) dS = \int_{\Sigma_{\pm}} f(x, y, z) dS + \int_{\Sigma_{\mp}} f(x, y, z) dS$$

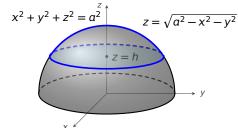
 $= \iint_{R} f(x, y, \sqrt{R^2 - x^2 - y^2}) \cdot \frac{R}{\sqrt{R^2 - x^2 - y^2}} dx dy$ $+\iint_{Dxy} f(x, y, -\sqrt{R^2 - x^2 - y^2}) \cdot \frac{R}{\sqrt{R^2 - x^2 - y^2}} dxdy$

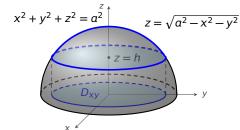
$$+ \int \int_{D_{xy}} \int (x, y, -\sqrt{R^2 - x^2 - y^2}) \cdot \frac{1}{\sqrt{R^2 - x^2 - y^2}} dx dy$$

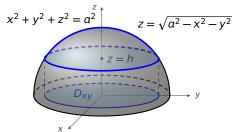
$$= \iint_{D_{xy}} \left[f(x, y, \sqrt{R^2 - x^2 - y^2}) + f(x, y, -\sqrt{R^2 - x^2 - y^2}) \right] \cdot \frac{R}{\sqrt{R^2 - x^2 - y^2}} dx dy \stackrel{\text{(a)}}{=} \frac{1}{2} \frac{1}{2}$$

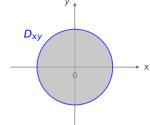


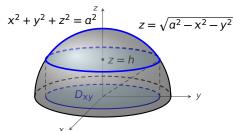


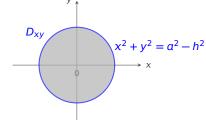


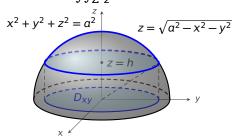


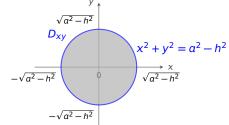


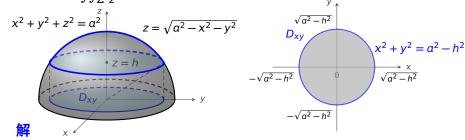








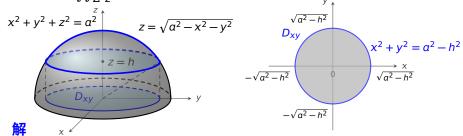






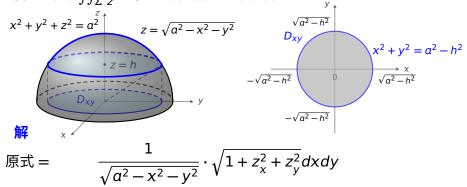


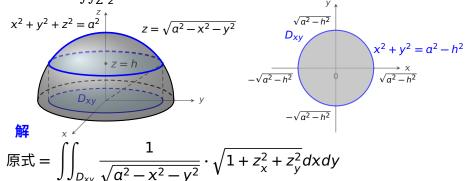
 $\sqrt{\alpha^2-x^2-y^2}$

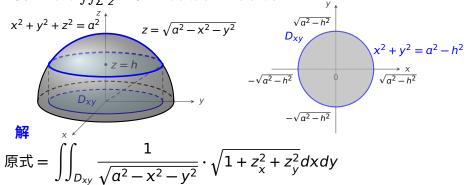


□ Man Section
 □

原式 =

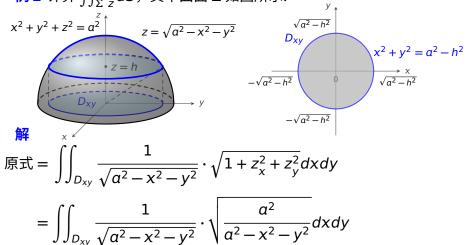




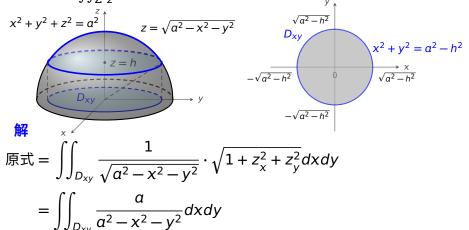


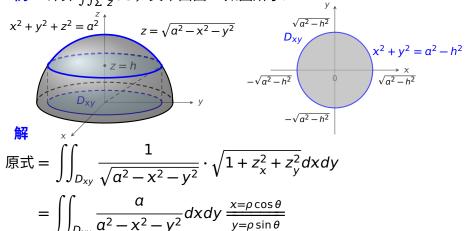
$$\cdot \sqrt{\frac{a^2}{a^2 - x^2 - y^2}}$$

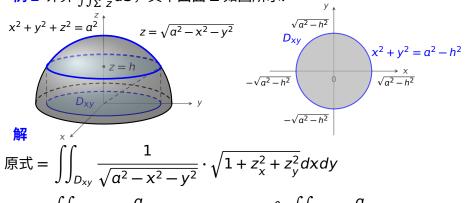












$$= \iint_{D_{xy}} \frac{\alpha}{\alpha^2 - x^2 - y^2} dx dy = \frac{x = \rho \cos \theta}{y = \rho \sin \theta} \iint_{D_{xy}} \frac{\alpha}{\alpha^2 - \rho^2} \cdot \rho d\rho d\theta$$



$$x^{2} + y^{2} + z^{2} = a^{2}$$
 $z = \sqrt{a^{2} - x^{2} - y^{2}}$
 $z = \sqrt{a^{2} - x^{2} - y^{2}}$
 $z = \sqrt{a^{2} - h^{2}}$
 $z = \sqrt{a^{2} - h^{2}}$

原式 =
$$\iint_{D_{xy}} \frac{1}{\sqrt{a^2 - x^2 - y^2}} \cdot \sqrt{1 + z_x^2 + z_y^2} dx dy$$

$$= \iint_{D_{xy}} \frac{a}{a^2 - x^2 - y^2} dx dy \frac{x = \rho \cos \theta}{y = \rho \sin \theta} \iint_{D_{xy}} \frac{a}{a^2 - \rho^2} \cdot \rho d\rho d\theta$$

$$= \iint_{D_{xy}} \frac{a}{a^2 - \rho^2} \cdot \rho d\rho d\theta$$



$$x^{2} + y^{2} + z^{2} = a^{2}$$

$$z = \sqrt{a^{2} - x^{2} - y^{2}}$$

$$z = \sqrt{a^{2} - x^{2} - y^{2}}$$

$$z = h$$

$$z = h$$

$$-\sqrt{a^{2} - h^{2}}$$

$$y$$

$$-\sqrt{a^{2} - h^{2}}$$

$$x^{2} + y^{2} = a^{2} - h^{2}$$

原式 =
$$\iint_{D_{xy}} \frac{1}{\sqrt{a^2 - x^2 - y^2}} \cdot \sqrt{1 + z_x^2 + z_y^2} dx dy$$

$$= \iint_{D_{xy}} \frac{a}{a^2 - x^2 - y^2} dx dy \frac{x = \rho \cos \theta}{y = \rho \sin \theta} \iint_{D_{xy}} \frac{a}{a^2 - \rho^2} \cdot \rho d\rho d\theta$$

$$= \int_0^{2\pi} \left[\int \frac{a}{a^2 - \rho^2} \cdot \rho d\rho \right] d\theta$$



$$x^{2} + y^{2} + z^{2} = a^{2}$$

$$z = \sqrt{a^{2} - x^{2} - y^{2}}$$

$$z = \sqrt{a^{2} - x^{2} - y^{2}}$$

$$-\sqrt{a^{2} - h^{2}}$$

$$x^{2} + y^{2} = a^{2} - h^{2}$$

$$-\sqrt{a^{2} - h^{2}}$$

$$x^{2} + y^{2} = a^{2} - h^{2}$$

$$-\sqrt{a^{2} - h^{2}}$$

$$x^{2} + y^{2} = a^{2} - h^{2}$$

$$x^{2} + y^{2} = a^{2} - h^{2}$$

$$x^{2} + y^{2} = a^{2} - h^{2}$$



$$x^{2} + y^{2} + z^{2} = a^{2}$$

$$z = \sqrt{a^{2} - x^{2} - y^{2}}$$

$$z = \sqrt{a^{2} - x^{2} - y^{2}}$$

$$-\sqrt{a^{2} - h^{2}}$$

$$x^{2} + y^{2} = a^{2} - h^{2}$$

$$-\sqrt{a^{2} - h^{2}}$$

$$x^{2} + y^{2} = a^{2} - h^{2}$$

$$-\sqrt{a^{2} - h^{2}}$$

$$1$$

$$\sqrt{1 + z^{2} + z^{2}} dy dy$$

原式 =
$$\iint_{D_{xy}} \frac{1}{\sqrt{a^2 - x^2 - y^2}} \cdot \sqrt{1 + z_x^2 + z_y^2} dx dy$$

$$= \iint_{D_{xy}} \frac{a}{a^2 - x^2 - y^2} dx dy \frac{x = \rho \cos \theta}{y = \rho \sin \theta} \iint_{D_{xy}} \frac{a}{a^2 - \rho^2} \cdot \rho d\rho d\theta$$

$$= \int_0^{2\pi} \left[\int_0^{\sqrt{a^2 - h^2}} \frac{a}{a^2 - \rho^2} \cdot \rho d\rho \right] d\theta = 2\pi \cdot$$



$$x^{2} + y^{2} + z^{2} = a^{\frac{z}{2}}$$

$$z = \sqrt{a^{2} - x^{2} - y^{2}}$$

$$D_{xy}$$

$$x^{2} + y^{2} = a^{2} - h^{2}$$

$$\sqrt{a^{2} - h^{2}}$$

$$\sqrt{a^{2} - h^{$$

$$= \int_0^{2\pi} \left[\int_0^{\sqrt{\alpha^2 - h^2}} \frac{\alpha}{\alpha^2 - \rho^2} \cdot \rho d\rho \right] d\theta \xrightarrow{u = \alpha^2 - \rho^2} 2\pi \cdot$$



$$x^{2} + y^{2} + z^{2} = a^{2}$$

$$z = \sqrt{a^{2} - x^{2} - y^{2}}$$

$$z = h$$

$$y$$

$$-\sqrt{a^{2} - h^{2}}$$

$$x^{2} + y^{2} = a^{2} - h^{2}$$

$$x^{2} + y^{2} = a^{2} - h^{2}$$

$$x^{2} + y^{2} = a^{2} - h^{2}$$

原式 =
$$\iint_{D_{xy}} \frac{1}{\sqrt{a^2 - x^2 - y^2}} \cdot \sqrt{1 + z_x^2 + z_y^2} dx dy$$

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$$= \int_0^{2\pi} \left[\int_0^{\sqrt{a^2 - h^2}} \frac{a}{a^2 - \rho^2} \cdot \rho d\rho \right] d\theta \xrightarrow{u = a^2 - \rho^2} 2\pi \cdot \frac{a}{u}$$



$$x^{2} + y^{2} + z^{2} = a^{2}$$

$$z = \sqrt{a^{2} - x^{2} - y^{2}}$$

$$z = \sqrt{a^{2} - x^{2} - y^{2}}$$

$$z = \sqrt{a^{2} - h^{2}}$$

$$-\sqrt{a^{2} - h^{2}}$$

$$x^{2} + y^{2} = a^{2} - h^{2}$$

$$-\sqrt{a^{2} - h^{2}}$$

原式 =
$$\iint_{D_{xy}} \frac{1}{\sqrt{a^2 - x^2 - y^2}} \cdot \sqrt{1 + z_x^2 + z_y^2} dx dy$$

$$= \iint_{D_{xy}} \frac{a}{a^2 - x^2 - y^2} dx dy \frac{x = \rho \cos \theta}{y = \rho \sin \theta} \iint_{D_{xy}} \frac{a}{a^2 - \rho^2} \cdot \rho d\rho d\theta$$

$$= \int_{0}^{2\pi} \left[\int_{0}^{\sqrt{a^2 - h^2}} \frac{a}{a^2 - \rho^2} \cdot \rho d\rho \right] d\theta \xrightarrow{u = a^2 - \rho^2} 2\pi \cdot \frac{a}{u} \cdot (-\frac{1}{2}) du$$



$$x^{2} + y^{2} + z^{2} = a^{2}$$

$$z = \sqrt{a^{2} - x^{2} - y^{2}}$$

$$z = h$$

$$y$$

$$-\sqrt{a^{2} - h^{2}}$$

$$x^{2} + y^{2} = a^{2} - h^{2}$$

$$x^{2} + y^{2} = a^{2} - h^{2}$$

$$x^{2} + y^{2} = a^{2} - h^{2}$$

原式 =
$$\iint_{D_{xy}} \frac{1}{\sqrt{a^2 - x^2 - y^2}} \cdot \sqrt{1 + z_x^2 + z_y^2} dx dy$$

$$= \iint_{D_{xy}} \frac{a}{a^2 - x^2 - y^2} dx dy \xrightarrow{\frac{x = \rho \cos \theta}{y = \rho \sin \theta}} \iint_{D_{xy}} \frac{a}{a^2 - \rho^2} \cdot \rho d\rho d\theta$$

$$= \int_{0}^{2\pi} \left[\int_{0}^{\sqrt{\alpha^{2}-h^{2}}} \frac{\alpha}{\alpha^{2}-\rho^{2}} \cdot \rho d\rho \right] d\theta \xrightarrow{u=\alpha^{2}-\rho^{2}} 2\pi \cdot \int_{a^{2}}^{h^{2}} \frac{\alpha}{u} \cdot (-\frac{1}{2}) du$$



例 2 计算 $\iint_{\Sigma} \frac{1}{z} dS$,其中曲面 Σ 如图所示. $x^2 + y^2 + z^2 = a^2$ $z = \sqrt{a^2 - x^2 - y^2}$ D_{xy} $x^2 + y^2 = a^2$

$$z = \sqrt{a^2 - x^2 - y^2}$$

$$z = \sqrt{a^2 - x^2 - y^2}$$

$$z = h$$

$$\sqrt{a^2 - h^2}$$

原式 =
$$\iint_{D_{xy}} \frac{1}{\sqrt{a^2 - x^2 - y^2}} \cdot \sqrt{1 + z_x^2 + z_y^2} dx dy$$

 $= \iint_{D_{xy}} \frac{a}{a^2 - x^2 - y^2} dx dy \xrightarrow{\frac{x = \rho \cos \theta}{y = \rho \sin \theta}} \iint_{D_{xy}} \frac{a}{a^2 - \rho^2} \cdot \rho d\rho d\theta$ $= \int_0^{2\pi} \left[\int_0^{\sqrt{a^2 - h^2}} \frac{a}{a^2 - \rho^2} \cdot \rho d\rho \right] d\theta \xrightarrow{u = a^2 - \rho^2} 2\pi \cdot \int_{a^2}^{h^2} \frac{a}{u} \cdot (-\frac{1}{2}) du$

$$= -\pi a \ln u \Big|_{a^2}^{h^2}$$

例 2 计算 $\iint_{\Sigma} \frac{1}{2} dS$,其中曲面 Σ 如图所示. $x^2 + y^2 + z^2 = a^2 \uparrow$ $\sqrt{a^2-h^2}$ $z = \sqrt{\alpha^2 - x^2 - v^2}$ $x^2 + y^2 = a^2 - h^2$ $\sqrt{a^2-h^2}$

原式 =
$$\iint_{D_{xy}} \frac{1}{\sqrt{a^2 - x^2 - y^2}} \cdot \sqrt{1 + z_x^2 + z_y^2} dx dy$$

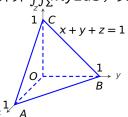
$$= \iint_{D_{xy}} \frac{a}{a^2 - x^2 - y^2} dx dy \frac{x = \rho \cos \theta}{y = \rho \sin \theta} \iint_{D_{xy}} \frac{a}{a^2 - \rho^2} \cdot \rho d\rho d\theta$$

$$= \int_{D_{xy}}^{2\pi} \left[\int_{-\infty}^{\sqrt{a^2 - h^2}} \frac{a}{a} \cdot \rho d\rho \right] d\theta \frac{u = a^2 - \rho^2}{a^2 - \rho^2} 2\pi \cdot \int_{-\infty}^{h^2} \frac{a}{a} \cdot (-\frac{1}{a}) d\mu$$

 $=\int_0^{2\pi} \left[\int_0^{\sqrt{a^2-h^2}} \frac{a}{a^2-\rho^2} \cdot \rho d\rho \right] d\theta \xrightarrow{u=a^2-\rho^2} 2\pi \cdot \int_{a^2}^{h^2} \frac{a}{u} \cdot (-\frac{1}{2}) du$

$$\int_{0}^{1} \left[\int_{0}^{1} d^{2} - \rho^{2} \right] \qquad \int_{a^{2}}^{1} d^{2} d$$

11d 曲面积分 11/13 ⊲ ⊳ ∆ ⊽ M 3 计算 $\iint_{\Sigma} xyzdS$,其中曲面 Σ 如图所示.



例 3 计算 $\iint_{\mathbb{Z}_{\lambda}} xyzdS$,其中曲面 Σ 如图所示.

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原式 =
$$\left(\iint_{\Delta OAB} + \iint_{\Delta OAC} + \iint_{\Delta OBC} + \iint_{\Delta ABC}\right) xyzdS = \iint_{\Delta ABC} xyzdS$$

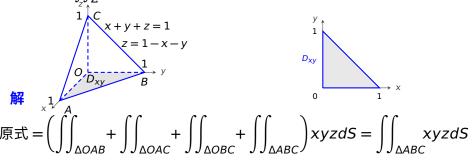


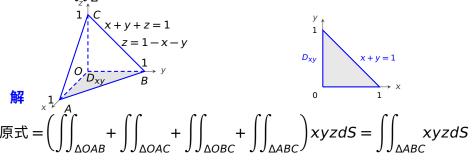
例 3 计算 $\iint_{\mathbb{Z}_{+}^{N}} xyzdS$,其中曲面 Σ 如图所示.

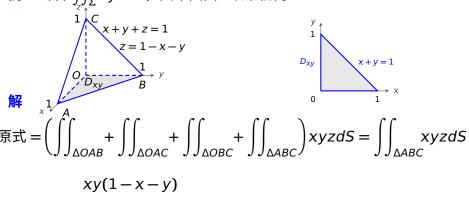
例 3 计算 $\iint_{\mathbb{Z}_{+}^{\infty}} xyzdS$,其中曲面 Σ 如图所示.

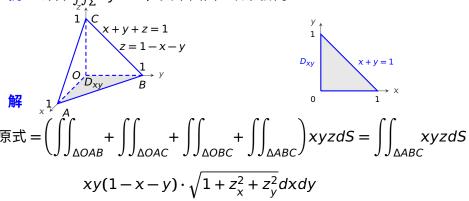
解
$$x + y + z = 1$$
 $z = 1 - x - y$

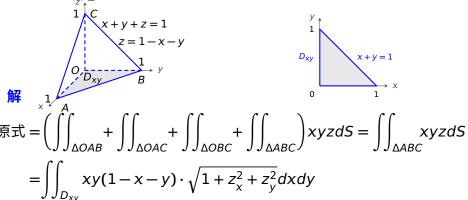
原式 = $\left(\iint_{\Delta OAB} + \iint_{\Delta OAC} + \iint_{\Delta OBC} + \iint_{\Delta ABC}\right) xyzdS = \iint_{\Delta ABC} xyzdS$



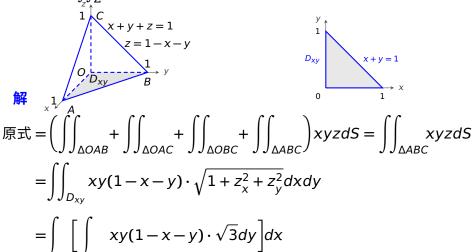




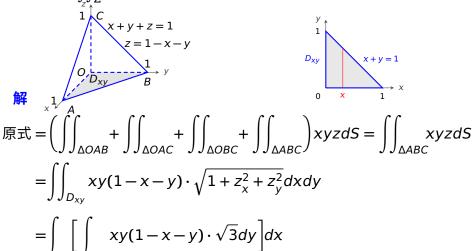




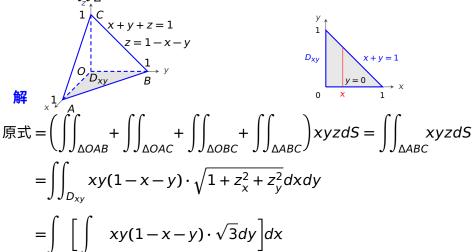




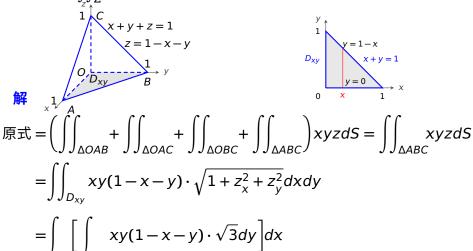




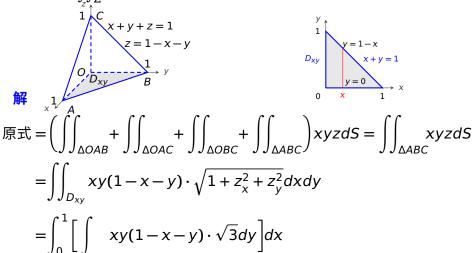




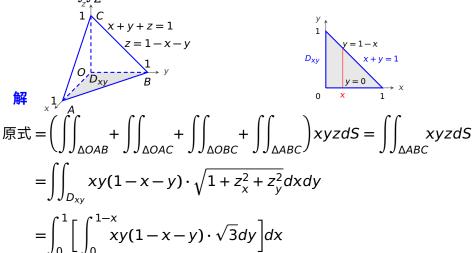














原式 =
$$\left(\iint_{\Delta OAB} + \iint_{\Delta OAC} + \iint_{\Delta OBC} + \iint_{\Delta ABC}\right) xyzdS = \iint_{\Delta ABC} xyzdS$$

= $\iint_{D_{xy}} xy(1-x-y) \cdot \sqrt{1+z_x^2+z_y^2} dxdy$
= $\int_0^1 \left[\int_0^{1-x} xy(1-x-y) \cdot \sqrt{3} dy\right] dx$
= $x\left[(1-x)\frac{y^2}{2} - \frac{1}{2}y^3\right]$



解
$$\int_{\Delta OAB}^{1} C x + y + z = 1$$
 $\int_{\Delta OAC}^{1} C x + y + z = 1$
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$$\begin{aligned}
& \left(\int \int_{\Delta OAB} \int \int_{\Delta OAC} \int \int_{\Delta OBC} \int \int_{\Delta ABC} \right)^{x} \\
&= \int \int_{D_{xy}} xy(1-x-y) \cdot \sqrt{1+z_{x}^{2}+z_{y}^{2}} dx dy \\
&= \int_{0}^{1} \left[\int_{0}^{1-x} xy(1-x-y) \cdot \sqrt{3} dy \right] dx \\
&= x \left[(1-x) \frac{y^{2}}{2} - \frac{1}{3} y^{3} \right]_{0}^{1-x}
\end{aligned}$$

例 3 计算 ∫∫_∇ xyzdS,其中曲面 Σ 如图所示.

原式 =
$$\left(\iint_{\Delta OAB} + \iint_{\Delta OAC} + \iint_{\Delta OBC} + \iint_{\Delta ABC} \right) xyzdS = \iint_{\Delta ABC} xyzdS$$

$$= \iint_{D_{xy}} xy(1-x-y) \cdot \sqrt{1+z_x^2+z_y^2} dxdy$$

$$= \int_0^1 \left[\int_0^{1-x} xy(1-x-y) \cdot \sqrt{3} dy \right] dx$$

$$= \sqrt{3} \int_0^1 x \left[(1-x) \frac{y^2}{2} - \frac{1}{3} y^3 \right]_0^{1-x} dx$$



例 3 计算 ∫∫_∇ xyzdS,其中曲面 Σ 如图所示.

解
$$\int_{\Delta OAB}^{1} C x + y + z = 1$$
 $z = 1 - x - y$

$$\int_{D_{xy}}^{y} \int_{D_{xy}}^{y} \int_{x + y = 1}^{y} \int$$

$$= \iint_{D_{xy}} xy(1-x-y) \cdot \sqrt{1+z_x^2+z_y^2} dx dy$$
$$= \int_0^1 \left[\int_0^{1-x} xy(1-x-y) \cdot \sqrt{3} dy \right] dx$$

 $= \sqrt{3} \int_{0}^{1} x \left[(1-x) \frac{y^{2}}{2} - \frac{1}{3} y^{3} \right]_{0}^{1-x} dx = \sqrt{3} \int_{0}^{1} \frac{1}{6} x (1-x)^{3} dx$



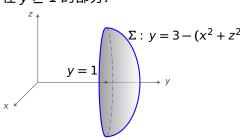
例 3 计算 $\iint_{\mathbb{Z}_{+}^{\Sigma}} xyzdS$,其中曲面 Σ 如图所示.

$$= \iint_{D_{xy}} xy(1-x-y) \cdot \sqrt{1+z_x^2+z_y^2} dxdy$$
$$= \int_0^1 \left[\int_0^{1-x} xy(1-x-y) \cdot \sqrt{3} dy \right] dx$$

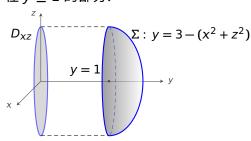
$$= \sqrt{3} \int_{0}^{1} x \left[(1-x) \frac{y^{2}}{2} - \frac{1}{3} y^{3} \right]_{0}^{1-x} dx = \sqrt{3} \int_{0}^{1} \frac{1}{6} x (1-x)^{3} dx = \frac{\sqrt{3}}{120}$$

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例 4 计算 $I = \iint_{\Sigma} (x^2 + z^2 + y) dS$,其中 Σ 是曲面 $y = 3 - (x^2 + z^2)$ 在 $y \ge 1$ 的部分.

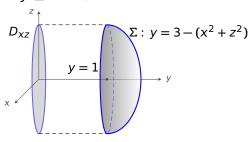


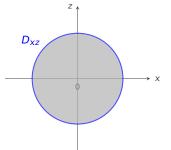
例 4 计算 $I = \iint_{\Sigma} (x^2 + z^2 + y) dS$,其中 Σ 是曲面 $y = 3 - (x^2 + z^2)$ 在 $y \ge 1$ 的部分.



例 4 计算 $I = \iint_{\Sigma} (x^2 + z^2 + y) dS$,其中 Σ 是曲面 $y = 3 - (x^2 + z^2)$

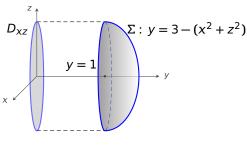
在 $y \ge 1$ 的部分.

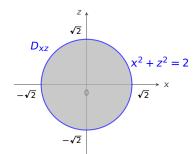




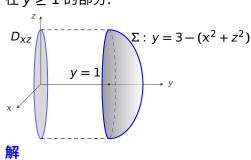
例 4 计算 $I = \iint_{\Sigma} (x^2 + z^2 + y) dS$,其中 Σ 是曲面 $y = 3 - (x^2 + z^2)$

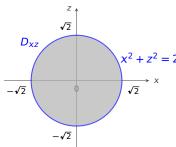
在 $y \ge 1$ 的部分.





例 4 计算 $I = \iint_{\Sigma} (x^2 + z^2 + y) dS$,其中 Σ 是曲面 $y = 3 - (x^2 + z^2)$ 在 $y \ge 1$ 的部分.

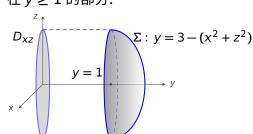


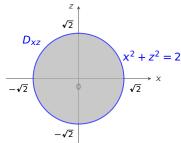


I =



例 4 计算 $I = \iint_{\Sigma} (x^2 + z^2 + y) dS$,其中 Σ 是曲面 $y = 3 - (x^2 + z^2)$ 在 $y \ge 1$ 的部分.



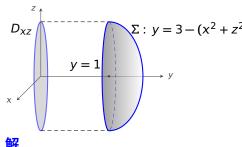


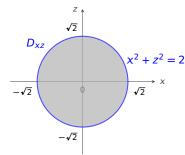
$$I = 3 \cdot \sqrt{1 + y_x^2 + y_z^2 dx dz}$$



例 4 计算 $I = \iint_{\Sigma} (x^2 + z^2 + y) dS$,其中 Σ 是曲面 $y = 3 - (x^2 + z^2)$

在 $y \ge 1$ 的部分.



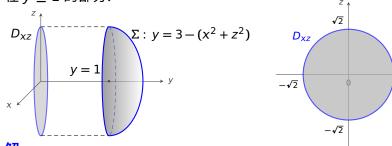


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$$I = \iint_{D_{xz}} 3 \cdot \sqrt{1 + y_x^2 + y_z^2} dx dz$$



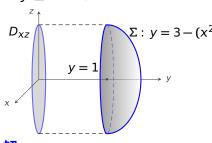
例 4 计算 $I = \iint_{\Sigma} (x^2 + z^2 + y) dS$,其中 Σ 是曲面 $y = 3 - (x^2 + z^2)$ 在 $y \ge 1$ 的部分.

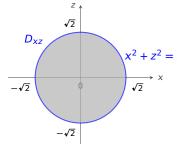


$$\mathbf{P} = \iint_{D \times z} 3 \cdot \sqrt{1 + y_x^2 + y_z^2} dx dz = \iint_{D \times z} 3 \cdot \sqrt{1 + 4x^2 + 4z^2} dx dz$$



例 4 计算 $I = \iint_{\Sigma} (x^2 + z^2 + y) dS$,其中 Σ 是曲面 $y = 3 - (x^2 + z^2)$ 在 $y \ge 1$ 的部分.





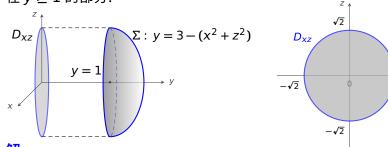
用

$$I = \iint_{D_{xz}} 3 \cdot \sqrt{1 + y_x^2 + y_z^2} dx dz = \iint_{D_{xz}} 3 \cdot \sqrt{1 + 4x^2 + 4z^2} dx dz$$

$$\frac{x = \rho \cos \theta}{z = \rho \sin \theta}$$



例 4 计算 $I = \iint_{\Sigma} (x^2 + z^2 + y) dS$,其中 Σ 是曲面 $y = 3 - (x^2 + z^2)$ 在 $y \ge 1$ 的部分.



$$I = \iint_{D_{xz}} 3 \cdot \sqrt{1 + y_x^2 + y_z^2} dx dz = \iint_{D_{xz}} 3 \cdot \sqrt{1 + 4x^2 + 4z^2} dx dz$$

$$\frac{x = \rho \cos \theta}{z = \rho \sin \theta} \iint_{D_{out}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho d\theta$$



例 4 计算 $I = \iint_{\Sigma} (x^2 + z^2 + y) dS$,其中 Σ 是曲面 $y = 3 - (x^2 + z^2)$

$$\sum_{x} : y = 3 - (x^2 + z^2)$$

$$y = 1$$

 $\frac{x = \rho \cos \theta}{z = \rho \sin \theta} \iint_{D_{x,y}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho d\theta = \int \left[\int 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho \right] d\theta$



例 4 计算 $I = \iint_{\Sigma} (x^2 + z^2 + y) dS$,其中 Σ 是曲面 $y = 3 - (x^2 + z^2)$

$$\sum_{x} y = 3 - (x^{2} + z^{2})$$

$$y = 1$$

$$y =$$

 $\frac{x = \rho \cos \theta}{z = \rho \sin \theta} \iint_{D_{xz}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho d\theta = \int_0^{2\pi} \left[\int 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho \right] d\theta$



例 4 计算 $I = \iint_{\Sigma} (x^2 + z^2 + y) dS$,其中 Σ 是曲面 $y = 3 - (x^2 + z^2)$

$$\sum : y = 3 - (x^2 + z^2)$$

$$y = 1$$

$$y$$

$$I = \iint_{D_{xz}} 3 \cdot \sqrt{1 + y_x^2 + y_z^2} dx dz = \iint_{D_{xz}} 3 \cdot \sqrt{1 + 4x^2 + 4z^2} dx dz$$

$$\frac{\int \int_{D_{xz}} \sqrt{1 + 4\rho^2} \cdot \rho d\rho d\theta}{\sum_{z=\rho \sin \theta} \int \int_{D_{xz}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho d\theta} = \int_0^{2\pi} \left[\int_0^{\sqrt{2}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho \right] d\theta$$



在 $y \ge 1$ 的部分. D_{XZ} y = 1 y = 1 y = 1 y = 1 $y = 3 - (x^2 + z^2)$ y = 1 y = 1 $y = 3 - (x^2 + z^2)$ y = 1

例 4 计算 $I = \iint_{\Sigma} (x^2 + z^2 + y) dS$,其中 Σ 是曲面 $y = 3 - (x^2 + z^2)$

$$\mathbf{H}$$

$$I = \iint_{D \times Z} 3 \cdot \sqrt{1 + y_x^2 + y_z^2} dx dz = \iint_{D \times Z} 3 \cdot \sqrt{1 + 4x^2 + 4z^2} dx dz$$

$$\frac{\sum_{z=\rho\cos\theta} \int_{D_{xz}} 3\sqrt{1+4\rho^2} \cdot \rho d\rho d\theta}{\sum_{z=\rho\sin\theta} \int_{D_{xz}} 3\sqrt{1+4\rho^2} \cdot \rho d\rho d\theta} = \int_0^{2\pi} \left[\int_0^{\sqrt{2}} 3\sqrt{1+4\rho^2} \cdot \rho d\rho \right] d\theta$$



在 $y \ge 1$ 的部分. D_{XZ} y = 1 y = 1 $\sum_{y=3}^{z} (x^2 + z^2)$ y = 1 $\sum_{y=3}^{z} (x^2 + z^2)$ y = 1

例 4 计算 $I = \iint_{\Sigma} (x^2 + z^2 + y) dS$,其中 Σ 是曲面 $y = 3 - (x^2 + z^2)$

$$\mathbf{H}$$

$$I = \iint_{D_{12}} 3 \cdot \sqrt{1 + y_x^2 + y_z^2} dx dz = \iint_{D_{12}} 3 \cdot \sqrt{1 + 4x^2 + 4z^2} dx dz$$

$$\frac{x = \rho \cos \theta}{z = \rho \sin \theta} \iint_{D_{xz}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho d\theta = \int_0^{2\pi} \left[\int_0^{\sqrt{2}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho \right] d\theta$$

$$\frac{u=1+4\rho^2}{2\pi}$$
 2π



在 $y \ge 1$ 的部分. D_{XZ} y = 1 y = 1 $\sum_{x=1}^{Z} (x^2 + z^2)$ y = 1 $\sum_{x=1}^{Z} (x^2 + z^2)$

例 4 计算 $I = \iint_{\Sigma} (x^2 + z^2 + y) dS$,其中 Σ 是曲面 $y = 3 - (x^2 + z^2)$

$$\mathbf{H}$$

$$I = \iint_{D \times Z} 3 \cdot \sqrt{1 + y_x^2 + y_z^2} dx dz = \iint_{D \times Z} 3 \cdot \sqrt{1 + 4x^2 + 4z^2} dx dz$$

$$\frac{x = \rho \cos \theta}{z = \rho \sin \theta} \iint_{D_{XZ}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho d\theta = \int_0^{2\pi} \left[\int_0^{\sqrt{2}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho \right] d\theta$$

 $\frac{u=1+4\rho^2}{2\pi}$ 2π . $3\sqrt{u}$



例 4 计算 $I = \iint_{\Sigma} (x^2 + z^2 + y) dS$,其中 Σ 是曲面 $y = 3 - (x^2 + z^2)$

$$\mathbf{P} I = \iint_{D_{xz}} 3 \cdot \sqrt{1 + y_x^2 + y_z^2} dx dz = \iint_{D_{xz}} 3 \cdot \sqrt{1 + 4x^2 + 4z^2} dx dz$$

$$\frac{x = \rho \cos \theta}{z = \rho \sin \theta} \iint_{D_{xz}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho d\theta = \int_0^{2\pi} \left[\int_0^{\sqrt{2}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho \right] d\theta$$

 $\frac{u=1+4\rho^2}{2\pi}$ $2\pi \cdot 3\sqrt{u} \cdot \frac{1}{8}du$

例 4 计算 $I = \int \int_{\Sigma} (x^2 + z^2 + y) dS$,其中 Σ 是曲面 $y = 3 - (x^2 + z^2)$

$$I = \iint_{D_{xz}} 3 \cdot \sqrt{1 + y_x^2 + y_z^2} dx dz = \iint_{D_{xz}} 3 \cdot \sqrt{1 + 4x^2 + 4z^2} dx dz$$

$$\xrightarrow{x = \rho \cos \theta} \left[\int_{D_{xz}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho d\theta = \int_{D_{xz}}^{2\pi} \left[\int_{0}^{\sqrt{2}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho d\theta \right] d\theta$$

$$I = \iint_{D_{xz}} 3 \cdot \sqrt{1 + y_x^2 + y_z^2} dx dz = \iint_{D_{xz}} 3 \cdot \sqrt{1 + 4x^2 + 4z^2} dx dz$$

$$\frac{x = \rho \cos \theta}{z = \rho \sin \theta} \iint_{D_{xz}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho d\theta = \int_0^{2\pi} \left[\int_0^{\sqrt{2}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho \right] d\theta$$

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例 4 计算 $I = \iint_{\Sigma} (x^2 + z^2 + y) dS$,其中 Σ 是曲面 $y = 3 - (x^2 + z^2)$

$$\mathbf{H} = \iint_{\mathbb{R}^{2}} 3 \cdot \sqrt{1 + y_{x}^{2} + y_{z}^{2}} dx dz = \iint_{\mathbb{R}^{2}} 3 \cdot \sqrt{1 + 4x^{2} + 4z^{2}} dx dz$$

$$\mathbf{E} = \iint_{D_{xz}} 3 \cdot \sqrt{1 + y_x^2 + y_z^2} dx dz = \iint_{D_{xz}} 3 \cdot \sqrt{1 + 4x^2 + 4z^2} dx dz$$

$$\underline{x = \rho \cos \theta} \left[\int_{D_{xz}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho d\theta = \int_{D_{xz}}^{2\pi} \left[\int_{0}^{\sqrt{2}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho \right] d\theta$$

$$\mathbf{P} = \int_{D_{xz}} 3 \cdot \sqrt{1 + y_x^2 + y_z^2} dx dz = \int_{D_{xz}} 3 \cdot \sqrt{1 + 4x^2 + 4z^2} dx dz$$

$$\frac{x = \rho \cos \theta}{z = \rho \sin \theta} \iint_{D_{xz}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho d\theta = \int_{0}^{2\pi} \left[\int_{0}^{\sqrt{2}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho \right] d\theta$$

 $\frac{u=1+4\rho^2}{2\pi} 2\pi \cdot \int_{1}^{9} 3\sqrt{u} \cdot \frac{1}{8} du = \frac{1}{2}\pi u^{\frac{3}{2}} \Big|_{1}^{9}$



例 4 计算 $I = \iint_{\Sigma} (x^2 + z^2 + y) dS$,其中 Σ 是曲面 $y = 3 - (x^2 + z^2)$

$$\mathbf{F} = \iint_{D} 3 \cdot \sqrt{1 + y_{x}^{2} + y_{z}^{2}} dx dz = \iint_{D} 3 \cdot \sqrt{1 + 4x^{2} + 4z^{2}} dx dz$$

$$\mathbf{H}$$

$$I = \iint_{D_{xz}} 3 \cdot \sqrt{1 + y_x^2 + y_z^2} dx dz = \iint_{D_{xz}} 3 \cdot \sqrt{1 + 4x^2 + 4z^2} dx dz$$

$$\frac{x = \rho \cos \theta}{2\pi \sin \theta} \iint_{D_{xz}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho d\theta = \int_{0}^{2\pi} \left[\int_{0}^{\sqrt{2}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho \right] d\theta$$

$$\frac{\sum_{z=\rho\cos\theta} \int \int_{D_{xz}} 3\sqrt{1+4\rho^2} \cdot \rho d\rho d\theta = \int_0^{2\pi} \left[\int_0^{\sqrt{2}} 3\sqrt{1+4\rho^2} \cdot \rho d\rho \right] d\theta$$

 $\frac{u=1+4\rho^2}{2\pi} 2\pi \cdot \int_{1}^{9} 3\sqrt{u} \cdot \frac{1}{8} du = \frac{1}{2}\pi u^{\frac{3}{2}} \Big|_{1}^{9} = 13\pi$



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