

第 4 章 d : 有理函数的积分

数学系 梁卓滨

2019-2020 学年 I

Outline

有理函数

定义 设

$$P(x) = p_0x^n + p_1x^{n-1} + \cdots + p_{n-1}x + p_n$$

$$Q(x) = q_0x^m + q_1x^{m-1} + \cdots + q_{m-1}x + q_m$$

为多项式，并且没有公共零点（即 $P(x)$, $Q(x)$ 没有公因式），则

$$R(x) := \frac{P(x)}{Q(x)}$$

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真分式

$$\frac{4x + 1}{x^2 - 6x + 9}$$

假分式

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$$\frac{x^2 + 1}{x + 1} = \frac{x^2 - 1}{x + 1} + \frac{2}{x + 1} = x - 1 + \frac{2}{x + 1}$$

问题 本节讨论如下形式的有理函数的积分：

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- $\Delta > 0 \Rightarrow x^2 + px + q = (x - a)(x - b),$

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例 3 求不定积分 $\int \frac{4x+1}{x^2-6x+9} dx$.

解

$$\frac{A(x-3)+B}{(x-3)^2} = \frac{Ax-3A+B}{(x-3)^2} \Rightarrow \begin{cases} A = 4 \\ -3A + B = 1 \end{cases}$$

$$\frac{4x+1}{x^2-6x+9} = \frac{x+1}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2} \Rightarrow A = 4, B = 13$$

所以

$$\text{原式} = 4 \int \frac{1}{x-3} dx + 13 \int \frac{1}{(x-3)^2} dx$$

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$\Delta < 0$ 情形

例 4 求不定积分 $\int \frac{x-3}{x^2-x+1} dx$.

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解

$$\frac{x-3}{x^2-x+1} = \frac{A(x^2-x+1)'}{x^2-x+1} + \frac{B}{x^2-x+1}$$

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例 4 求不定积分 $\int \frac{x-3}{x^2-x+1} dx$.

解

$$\frac{x-3}{x^2-x+1} = \frac{A(x^2-x+1)'}{x^2-x+1} + \frac{\frac{2Ax-A+B}{x^2-x+1}}{x^2-x+1}$$

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例 4 求不定积分 $\int \frac{x-3}{x^2-x+1} dx$.

解

$$\frac{2Ax-A+B}{x^2-x+1} \Rightarrow \begin{cases} 2A = 1 \\ -A + B = -3 \end{cases}$$

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$$\text{原式} = \frac{1}{2} \int \frac{(x^2-x+1)'}{x^2-x+1} dx - \frac{5}{2} \int \frac{1}{x^2-x+1} dx$$

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$$\int \frac{(x^2-x+1)'}{x^2-x+1} dx = \int \frac{d(x^2-x+1)}{x^2-x+1} = \ln|x^2-x+1| + C$$

其中

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其中

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其中

$$\int \frac{1}{x^2 - x + 1} dx = \int \frac{1}{(x - \frac{1}{2})^2 + \frac{3}{4}} dx = \frac{4}{3} \int \frac{1}{\left(\frac{2}{\sqrt{3}}x - \frac{1}{\sqrt{3}}\right)^2 + 1} dx$$
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其中

$$\int \frac{1}{x^2 - x + 1} dx = \int \frac{1}{(x - \frac{1}{2})^2 + \frac{3}{4}} dx = \frac{4}{3} \int \frac{1}{(\frac{2}{\sqrt{3}}x - \frac{1}{\sqrt{3}})^2 + 1} dx$$

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所以

$$\begin{aligned} \text{原式} &= \frac{1}{2} \int \frac{(x^2 - x + 1)'}{x^2 - x + 1} dx - \frac{5}{2} \int \frac{1}{x^2 - x + 1} dx \\ &= \frac{1}{2} \ln|x^2 - x + 1| - \frac{5}{\sqrt{3}} \arctan\left(\frac{2}{\sqrt{3}}x - \frac{1}{\sqrt{3}}\right) + C \end{aligned}$$

例 5 求不定积分 $\int \frac{5x+6}{x^2+x+1} dx$.

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$$\frac{5x+6}{x^2+x+1} = \frac{A(x^2+x+1)'}{x^2+x+1} + \frac{\frac{2Ax+A+B}{x^2+x+1}}{x^2+x+1}$$

例 5 求不定积分 $\int \frac{5x+6}{x^2+x+1} dx$.

解

$$\frac{2Ax+A+B}{x^2+x+1} \Rightarrow \begin{cases} 2A = 5 \\ A + B = 6 \end{cases}$$

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其中

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所以

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