

## 第 07 周作业解答

练习 1. 将 4 阶方阵  $M$  作如下分块

$$M = \left( \begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ \hline 1 & 0 & -1 & 0 \\ 0 & 1 & -3 & 1 \end{array} \right) = \begin{pmatrix} A & O \\ I & -A \end{pmatrix}$$

请按此分块方式计算  $M^2$ 。

解

$$M^2 = \begin{pmatrix} A & O \\ I & -A \end{pmatrix} \begin{pmatrix} A & O \\ I & -A \end{pmatrix} = \begin{pmatrix} AA + OI & AO + O(-A) \\ IA + (-A)I & IO + (-A)(-A) \end{pmatrix} = \begin{pmatrix} A^2 & O \\ O & A^2 \end{pmatrix}$$

而

$$A^2 = \begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

所以

$$M = \begin{pmatrix} I & O \\ O & I \end{pmatrix} = \left( \begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right)$$

练习 2. 将矩阵  $A, B$  作如下分块

$$A = \left( \begin{array}{cc|cc} 5 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ \hline 3 & 1 & 0 & 0 \end{array} \right) = \begin{pmatrix} A_1 & 2I \\ 3 & A_2 \end{pmatrix}, \quad B = \left( \begin{array}{cc|cc} 2 & 7 & 0 & 0 \\ -1 & 0 & 1 & 4 \\ \hline 0 & -1 & 2 & 5 \end{array} \right) = \begin{pmatrix} B_1 & O \\ -I & B_2 \end{pmatrix},$$

请按此分块方式计算乘积  $AB$ 。

解

$$AB = \begin{pmatrix} A_1 & 2I \\ 3 & A_2 \end{pmatrix} \begin{pmatrix} B_1 & O \\ -I & B_2 \end{pmatrix} = \begin{pmatrix} A_1 B_1 + 2I(-I) & A_1 O + 2I B_2 \\ 3O + A_2(-I) & 3O + A_2 B_2 \end{pmatrix} = \begin{pmatrix} A_1 B_1 - 2I & 2B_2 \\ 3B_1 - A_2 & A_2 B_2 \end{pmatrix}$$

其中

$$A_1 B_1 - 2I = \begin{pmatrix} 5 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 7 \\ -1 & 0 \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 35 \\ 0 & -2 \end{pmatrix}, \quad A_2 B_2 = \begin{pmatrix} 1 & 0 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 10 & 29 \end{pmatrix}$$

所以

$$M = \left( \begin{array}{cc|cc} 8 & 35 & 2 & 8 \\ 0 & -2 & 4 & 10 \\ \hline 1 & 4 & 1 & 4 \\ 10 & 29 & 2 & 5 \end{array} \right)$$

练习 3. 设

$$M = \begin{pmatrix} O_{r \times s} & A_{r \times r} \\ B_{s \times s} & O_{s \times r} \end{pmatrix}$$

其中  $A, B$  分别为  $r, s$  阶可逆方阵, 求  $M$  的逆矩阵  $M^{-1}$ 。

解设

$$M^{-1} = \begin{pmatrix} U_{s \times r} & V_{s \times s} \\ W_{r \times r} & X_{r \times s} \end{pmatrix}$$

应有

$$MM^{-1} = \begin{pmatrix} O_{r \times s} & A_{r \times r} \\ B_{s \times s} & O_{s \times r} \end{pmatrix} \begin{pmatrix} U_{s \times r} & V_{s \times s} \\ W_{r \times r} & X_{r \times s} \end{pmatrix} = \begin{pmatrix} OU + AW & OV + AX \\ BU + OW & BV + OX \end{pmatrix} = \begin{pmatrix} AW & AX \\ BU & BV \end{pmatrix} = \begin{pmatrix} I_r & O_{r \times s} \\ O_{s \times r} & I_s \end{pmatrix}$$

故须

$$AW = I, \quad AX = O, \quad BU = O, \quad BV = I$$

利用  $A, B$  可逆条件, 可解出

$$W = A^{-1}, \quad X = O, \quad U = O, \quad V = B^{-1}$$

所以

$$M^{-1} = \begin{pmatrix} O & B^{-1} \\ A^{-1} & O \end{pmatrix}$$

**练习 4.** 用初等变换将下列矩阵化为等价标准形:

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 2 \\ 3 & -3 & 1 \\ -2 & 2 & -4 \end{pmatrix}$$

解

$$\begin{aligned} A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 2 \\ 3 & -3 & 1 \\ -2 & 2 & -4 \end{pmatrix} &\xrightarrow{c_1 \leftrightarrow c_3} \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 1 \\ 1 & -3 & 3 \\ -4 & 2 & -2 \end{pmatrix} \xrightarrow{\substack{r_2 - 2r_1 \\ r_3 - r_1 \\ r_4 + 4r_1}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -3 & 3 \\ 0 & 2 & -2 \end{pmatrix} \\ &\xrightarrow{\substack{r_3 - 3r_2 \\ r_4 + 2r_2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{c_3 + c_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{(-1) \times c_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

**练习 5.** 用初等行变换求下列矩阵  $A, B, C, D$  的逆矩阵:

$$A = \begin{pmatrix} 2 & 2 & 3 \\ 3 & 1 & 3 \\ -1 & 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & a_3 \\ a_4 & 0 & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(其中  $a_i \neq 0, i = 1, 2, 3, 4$ )

解

$$\begin{aligned}
 (A:I) &= \left( \begin{array}{ccc|ccc} 2 & 2 & 3 & 1 & 0 & 0 \\ 3 & 1 & 3 & 0 & 1 & 0 \\ -1 & 2 & 1 & 0 & 0 & 1 \end{array} \right) \xrightarrow{r_1 \leftrightarrow r_3} \left( \begin{array}{ccc|ccc} -1 & 2 & 1 & 0 & 0 & 1 \\ 3 & 1 & 3 & 0 & 1 & 0 \\ 2 & 2 & 3 & 1 & 0 & 0 \end{array} \right) \xrightarrow{(-1) \times r_1} \left( \begin{array}{ccc|ccc} 1 & -2 & -1 & 0 & 0 & -1 \\ 3 & 1 & 3 & 0 & 1 & 0 \\ 2 & 2 & 3 & 1 & 0 & 0 \end{array} \right) \\
 &\xrightarrow[r_3-2r_1]{r_2-3r_1} \left( \begin{array}{ccc|ccc} 1 & -2 & -1 & 0 & 0 & -1 \\ 0 & 7 & 6 & 0 & 1 & 3 \\ 0 & 6 & 5 & 1 & 0 & 2 \end{array} \right) \xrightarrow{r_2-r_3} \left( \begin{array}{ccc|ccc} 1 & -2 & -1 & 0 & 0 & -1 \\ 0 & 1 & 1 & -1 & 1 & 1 \\ 0 & 6 & 5 & 1 & 0 & 2 \end{array} \right) \\
 &\xrightarrow{r_3-6r_2} \left( \begin{array}{ccc|ccc} 1 & -2 & -1 & 0 & 0 & -1 \\ 0 & 1 & 1 & -1 & 1 & 1 \\ 0 & 0 & -1 & 7 & -6 & -4 \end{array} \right) \xrightarrow{(-1) \times r_3} \left( \begin{array}{ccc|ccc} 1 & -2 & -1 & 0 & 0 & -1 \\ 0 & 1 & 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & -7 & 6 & 4 \end{array} \right) \\
 &\xrightarrow[r_1+r_3]{r_2-r_3} \left( \begin{array}{ccc|ccc} 1 & -2 & 0 & -7 & 6 & 3 \\ 0 & 1 & 0 & 6 & -5 & -3 \\ 0 & 0 & 1 & -7 & 6 & 4 \end{array} \right) \xrightarrow{r_1+2r_2} \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & -4 & -3 \\ 0 & 1 & 0 & 6 & -5 & -3 \\ 0 & 0 & 1 & -7 & 6 & 4 \end{array} \right)
 \end{aligned}$$

所以  $A^{-1} = \begin{pmatrix} 5 & -4 & -3 \\ 6 & -5 & -3 \\ -7 & 6 & 4 \end{pmatrix}$ 。

$$\begin{aligned}
 (B:I) &= \left( \begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & -1 & 1 & 1 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & 1 & 0 & 0 & 0 & 1 \end{array} \right) \xrightarrow[r_4+r_3]{r_2+r_1} \left( \begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 & 1 \end{array} \right) \\
 &\xrightarrow{\frac{1}{2} \times r_4} \left( \begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{array} \right) \xrightarrow[r_1-r_4]{r_3-2r_4} \left( \begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & 2 & 2 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{array} \right) \\
 &\xrightarrow{\frac{1}{2} \times r_3} \left( \begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & 2 & 2 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{array} \right) \xrightarrow[r_1-r_3]{r_2-2r_3} \left( \begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{array} \right) \\
 &\xrightarrow{\frac{1}{2} \times r_2} \left( \begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{array} \right) \xrightarrow{r_1-r_2} \left( \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{array} \right)
 \end{aligned}$$

所以  $B^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$ 。

$$\begin{aligned}
 (C:I) &= \left( \begin{array}{cccc|cccc} 0 & a_1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & a_2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & a_3 & 0 & 0 & 1 & 0 \\ a_4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right) \xrightarrow{r_4 \leftrightarrow r_3} * \xrightarrow{r_3 \leftrightarrow r_2} * \xrightarrow{r_2 \leftrightarrow r_1} \left( \begin{array}{cccc|cccc} a_4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & a_1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & a_2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & a_3 & 0 & 0 & 1 & 0 \end{array} \right) \\
 &\xrightarrow[\frac{1}{a_3} \times r_4]{\frac{1}{a_4} \times r_1} \left( \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{a_4} \\ 0 & 1 & 0 & 0 & \frac{1}{a_1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{a_2} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \frac{1}{a_3} & 0 \end{array} \right)
 \end{aligned}$$

$$\text{所以 } C^{-1} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{a_4} \\ \frac{1}{a_1} & 0 & 0 & 0 \\ 0 & \frac{1}{a_2} & 0 & 0 \\ 0 & 0 & \frac{1}{a_3} & 0 \end{pmatrix}$$

$$(D:I) = \left( \begin{array}{cccc|cccc} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right) \xrightarrow[r_3-2 \times r_4]{\begin{array}{l} r_1-4 \times r_4 \\ r_2-3 \times r_4 \end{array}} \left( \begin{array}{cccc|cccc} 1 & 2 & 3 & 0 & 1 & 0 & 0 & -4 \\ 0 & 1 & 2 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right)$$

$$\xrightarrow[r_2-2 \times r_3]{r_1-3 \times r_3} \left( \begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 1 & 0 & -3 & 2 \\ 0 & 1 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right) \xrightarrow{r_1-2 \times r_2} \left( \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right)$$

$$\text{所以 } D^{-1} = \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

练习 6. 求  $A = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 2 & 0 & \cdots & 0 \\ 1 & 2 & 3 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & n \end{pmatrix}$  的逆矩阵。

解

$$(A:I) = \left( \begin{array}{ccccc|ccccc} 1 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 2 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\ 1 & 2 & 3 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots & & & & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & n & 0 & 0 & 0 & \cdots & 0 \end{array} \right) \xrightarrow[r_3-r_1, \dots, r_n-r_1]{r_2-r_1} \left( \begin{array}{ccccc|ccccc} 1 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 & -1 & 1 & 0 & \cdots & 0 \\ 0 & 2 & 3 & \cdots & 0 & -1 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots & \vdots & & & \ddots & \vdots \\ 0 & 2 & 3 & \cdots & n & -1 & 0 & 0 & \cdots & 1 \end{array} \right)$$

$$\xrightarrow[r_4-r_2, \dots, r_n-r_2]{r_3-r_2} \left( \begin{array}{ccccc|ccccc} 1 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 & -1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 3 & \cdots & 0 & 0 & -1 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots & \vdots & & & \ddots & \vdots \\ 0 & 0 & 3 & \cdots & n & 0 & -1 & 0 & \cdots & 1 \end{array} \right) \cdots \rightarrow \left( \begin{array}{ccccc|ccccc} 1 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 & -1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 3 & \cdots & 0 & 0 & -1 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots & \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & n & 0 & 0 & 0 & \cdots & 1 \end{array} \right)$$

$$\xrightarrow[\frac{1}{3} \times r_3, \dots, \frac{1}{n} \times r_n]{\frac{1}{2} \times r_2} \left( \begin{array}{ccccc|ccccc} 1 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & \cdots & 0 \\ \vdots & & & \ddots & \vdots & \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & \frac{1}{n} \end{array} \right)$$

所以

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} & \cdots & 0 & 0 \\ \vdots & & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{n-1} & 0 \\ 0 & 0 & 0 & \cdots & -\frac{1}{n} & \frac{1}{n} \end{pmatrix}.$$

以下两题是附加题，做出来的同学下次课交，可以加分。注意解答过程要详细。

**练习 7.** 求出一个 2 阶方阵  $A$ ，满足  $A^{17} = I_2$ ，且  $A \neq I_2$ 。

**解**  $A = \begin{pmatrix} \cos \frac{2\pi}{17} & -\sin \frac{2\pi}{17} \\ \sin \frac{2\pi}{17} & \cos \frac{2\pi}{17} \end{pmatrix}$ 。(回忆:  $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}^n = \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix}$ )

**练习 8.** 设分块矩阵  $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$  中子块  $A_{11}$  和  $A_{22}$  为方阵，并且  $A_{11}$  可逆。求出矩阵  $X$  和  $Y$  满足

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} I & O \\ X & I \end{pmatrix} \begin{pmatrix} A_{11} & \\ & S \end{pmatrix} \begin{pmatrix} I & Y \\ O & I \end{pmatrix}$$

其中  $S = A_{22} - A_{21}A_{11}^{-1}A_{12}$ 。

假设  $S$  也是可逆，导出用  $S^{-1}$ ， $A_{11}^{-1}$  及  $A$  的子块计算  $A^{-1}$  的一个公式。

**解设**

$$\begin{aligned} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} &= \begin{pmatrix} I & O \\ A_{21}A_{11}^{-1} & I \end{pmatrix} \begin{pmatrix} A_{11} & \\ & S \end{pmatrix} \begin{pmatrix} I & A_{11}^{-1}A_{12} \\ O & I \end{pmatrix} \\ \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{-1} &= \begin{pmatrix} I & A_{11}^{-1}A_{12} \\ O & I \end{pmatrix}^{-1} \begin{pmatrix} A_{11} & \\ & S \end{pmatrix} \begin{pmatrix} I & O \\ A_{21}A_{11}^{-1} & I \end{pmatrix}^{-1} \\ &= \begin{pmatrix} I & -A_{11}^{-1}A_{12} \\ O & I \end{pmatrix} \begin{pmatrix} A_{11} & \\ & S \end{pmatrix} \begin{pmatrix} I & O \\ -A_{21}A_{11}^{-1} & I \end{pmatrix} \end{aligned}$$