§3.4 向量组的秩

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2018 - 2019 学年上学期



 $\alpha_1, \alpha_2, \ldots, \alpha_s$

逐个剔除 $lpha_1,lpha_2,\ldots,lpha_s$ 能被其余向量线性表示的向量

逐个剔除 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 能被其余向量线性表示的向量 $\alpha_1, \alpha_2, \ldots, \alpha_s$

逐个剔除

 $lpha_1,lpha_2,\ldots,lpha_s$ 能被其余向量线性表示的向量 $lpha_{j_1},lpha_{j_2},\ldots,lpha_{j_r}$

逐个剔除

 $\alpha_1, \alpha_2, \ldots, \alpha_s \xrightarrow{\text{fixt} \text{ fixed by a local points}} \alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}$ (极大无关组)

逐个剔除

 $lpha_1,lpha_2,\ldots,lpha_s \xrightarrow{ ext{fk被其余向量线性表示的向量}} lpha_{j_1},lpha_{j_2},\ldots,lpha_{j_r}$ (极大无关组)

例 求
$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ 的一个极大无关组。

逐个剔除

 $\alpha_1, \alpha_2, \ldots, \alpha_s \xrightarrow{\text{fiwtfshold fixed fixed$

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$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
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 $\alpha_1, \alpha_2, \alpha_3, \alpha_4$

逐个剔除

$$\alpha_1, \alpha_2, \ldots, \alpha_s \xrightarrow{\text{fiwtfsno} = style fixed style$$

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$$\stackrel{\mathbf{p}}{\alpha_1}, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_4 = 2\alpha_1 + 0\alpha_2 + 0\alpha_3}$$



逐个剔除

$$oldsymbol{lpha}_{S} \stackrel{ ext{twistand}}{=\!=\!=\!=\!=\!=\!=\!=\!=}$$

 $lpha_1,lpha_2,\ldots,lpha_s \xrightarrow{ ilde{ text{tkat}} ilde{ text{tkat}} ilde{ text{tkat}} ilde{ text{tat}} ilde{ text{d}}} lpha_{j_1},lpha_{j_2},\ldots,lpha_{j_r}$ (极大无关组)

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$$\alpha_1, \alpha_2, \alpha_3, \alpha_4$$
 $\alpha_4 = 2\alpha_1 + 0\alpha_2 + 0\alpha_3$
剔除 α_4



逐个剔除

 $lpha_1,lpha_2,\ldots,lpha_s \xrightarrow{ ilde{ text{tkat}} ilde{ text{tkat}} ilde{ text{tkat}} ilde{ text{tat}} ilde{ text{d}}} lpha_{j_1},lpha_{j_2},\ldots,lpha_{j_r}$ (极大无关组)

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$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_4 = 2\alpha_1 + 0\alpha_2 + 0\alpha_3} \alpha_1, \alpha_2, \alpha_3$$



逐个剔除

 $\alpha_1, \alpha_2, \ldots, \alpha_s \xrightarrow{\text{fiwtfsnold} \atop \text{直到不能再剔除为止}} \alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r} \quad (极大无关组)$

例 求
$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
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$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_4 = 2\alpha_1 + 0\alpha_2 + 0\alpha_3}$$
 $\alpha_1, \alpha_2, \alpha_3 \xrightarrow{\alpha_3 = \alpha_1 + \alpha_2}$



逐个剔除

 $lpha_1,lpha_2,\ldots,lpha_s \xrightarrow{ ilde{ text{tkat}} ilde{ text{tkat}} ilde{ text{tkat}} ilde{ text{tat}} ilde{ text{d}}} lpha_{j_1},lpha_{j_2},\ldots,lpha_{j_r}$ (极大无关组)

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$$\stackrel{\textstyle R}{\alpha_1, \alpha_2, \alpha_3, \alpha_4} \xrightarrow{\alpha_4 = 2\alpha_1 + 0\alpha_2 + 0\alpha_3}
\stackrel{\textstyle \alpha_1, \alpha_2, \alpha_3} \xrightarrow{\alpha_3 = \alpha_1 + \alpha_2}
\xrightarrow{\textstyle \operatorname{slip}(\alpha_4)}$$



逐个剔除

 $\alpha_1, \alpha_2, \ldots, \alpha_s \xrightarrow{\text{fiwtfshold fixed fixed$

例 求
$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ 的一个极大无关组。



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 $lpha_1,lpha_2,\ldots,lpha_s \xrightarrow{ ilde{ text{tkat}} ilde{ text{tkat}} ilde{ text{tkat}} ilde{ text{tat}} ilde{ text{dist}} i$

例 求
$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
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 $egin{array}{c} \mathbf{p} \\ \alpha_1, \alpha_2, \alpha_3, \alpha_4 \end{array} \xrightarrow[]{\alpha_4 = 2\alpha_1 + 0\alpha_2 + 0\alpha_3} \quad \alpha_1, \alpha_2, \alpha_3 \xrightarrow[]{\alpha_3 = \alpha_1 + \alpha_2} \quad \alpha_1, \alpha_2 \xrightarrow[]{\mathrm{RK}} \quad \alpha_1, \alpha_2 \xrightarrow[]{\mathrm{RK}} \quad \alpha_2, \alpha_3 \xrightarrow[]{\mathrm{RK}} \quad \alpha_3, \alpha_4 \xrightarrow[]{\mathrm{RK}} \quad \alpha_4, \alpha_4 \xrightarrow[]{\mathrm{R$



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 $\alpha_1, \alpha_2, \ldots, \alpha_s \xrightarrow{\text{fiwt}, \text{sholl}} \text{fiwt}, \alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}$ (极大无关组)

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$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
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$$egin{array}{c} \mathbf{p} \\ \mathbf{p} \\ \mathbf{p} \\ \mathbf{p} \\ \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4 \end{array} \xrightarrow{\alpha_4 = 2\alpha_1 + 0\alpha_2 + 0\alpha_3} \quad \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3 \xrightarrow{\alpha_3 = \alpha_1 + \alpha_2} \quad \mathbf{q}_1, \mathbf{q}_2 \xrightarrow{\mathrm{RK}} \\ \mathbf{q} \\ \mathbf{q}$$

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4$$



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 $lpha_1,lpha_2,\ldots,lpha_s \xrightarrow{ ilde{ t k被其余向量线性表示的向量}} lpha_{j_1},lpha_{j_2},\ldots,lpha_{j_r}$ (极大无关组)

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 $egin{array}{c} \mathbf{p} \\ \alpha_1, \alpha_2, \alpha_3, \alpha_4 \end{array} \xrightarrow[]{\alpha_4 = 2\alpha_1 + 0\alpha_2 + 0\alpha_3} \quad \alpha_1, \alpha_2, \alpha_3 \xrightarrow[]{\alpha_3 = \alpha_1 + \alpha_2} \quad \alpha_1, \alpha_2 \xrightarrow[]{\mathrm{RK}} \quad \alpha_1, \alpha_2 \xrightarrow[]{\mathrm{RK}} \quad \alpha_2, \alpha_3 \xrightarrow[]{\mathrm{RK}} \quad \alpha_3, \alpha_4 \xrightarrow[]{\mathrm{RK}} \quad \alpha_4, \alpha_4 \xrightarrow[]{\mathrm{R$

 $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_1 = -\alpha_2 + \alpha_3 + 0\alpha_4}$



诼个剔除

$$lpha_1,lpha_2,\ldots,lpha_s \xrightarrow{ ext{fk被其条向量线性表示的向量}} lpha_{j_1},lpha_{j_2},\ldots,lpha_{j_r}$$
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$$egin{array}{c} \mu \\ \alpha_1, \alpha_2, \alpha_3, \alpha_4 & \xrightarrow{\alpha_4 = 2\alpha_1 + 0\alpha_2 + 0\alpha_3} & \alpha_1, \alpha_2, \alpha_3 & \xrightarrow{\alpha_3 = \alpha_1 + \alpha_2} & \alpha_1, \alpha_2 & \xrightarrow{\mathrm{RK}} \\ \hline \end{array}$$
 》
 $\lambda_1, \alpha_2, \alpha_3, \alpha_4 & \xrightarrow{\alpha_4 = 2\alpha_1 + 0\alpha_2 + 0\alpha_3} & \alpha_1, \alpha_2, \alpha_3 & \xrightarrow{\beta_1 \in \mathcal{A}_1} & \alpha_1, \alpha_2 & \xrightarrow{\beta_1 \in \mathcal{A}_2} & \alpha_2, \alpha_3 & \xrightarrow{\beta_1 \in \mathcal{A}_3} & \alpha_1, \alpha_2 & \xrightarrow{\beta_1 \in \mathcal{A}_3} & \alpha_2, \alpha_3 & \xrightarrow{\beta_1 \in \mathcal{A}_3} & \alpha_2, \alpha_3 & \xrightarrow{\beta_1 \in \mathcal{A}_3} & \alpha_3, \alpha_4 & \xrightarrow{\beta_1 \in \mathcal{A}_3} & \alpha_4 & \xrightarrow{\beta_1 \in$

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_1 = -\alpha_2 + \alpha_3 + 0\alpha_4}$$
 $\xrightarrow{\text{slip}(\alpha_1)}$

逐个剔除

 $lpha_1,lpha_2,\ldots,lpha_s \xrightarrow{ ilde{ text{tkat}} ilde{ text{tkat}} ilde{ text{tkat}} ilde{ text{tat}} ilde{ text{d}}} lpha_{j_1},lpha_{j_2},\ldots,lpha_{j_r}$ (极大无关组)

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 $egin{array}{c} \mathbf{p} \\ \alpha_1, \alpha_2, \alpha_3, \alpha_4 \end{array} \xrightarrow[]{\alpha_4 = 2\alpha_1 + 0\alpha_2 + 0\alpha_3} \quad \alpha_1, \alpha_2, \alpha_3 \xrightarrow[]{\alpha_3 = \alpha_1 + \alpha_2} \quad \alpha_1, \alpha_2 \xrightarrow[]{\mathrm{Re}} \quad \alpha_1, \alpha_2 \xrightarrow[]{\mathrm{Re}} \quad \alpha_1, \alpha_2 \xrightarrow[]{\mathrm{Re}} \quad \alpha_2, \alpha_3 \xrightarrow[]{\mathrm{Re}} \quad \alpha_3, \alpha_4 \xrightarrow[]{\mathrm{Re}} \quad \alpha_4 = 2\alpha_1 + 0\alpha_2 + 0\alpha_3 \xrightarrow[]{\mathrm{Re}} \quad \alpha_4, \alpha_4 \xrightarrow[]{\mathrm{Re}} \quad$

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_1 = -\alpha_2 + \alpha_3 + 0\alpha_4} \alpha_2, \alpha_3, \alpha_4$$



诼个剔除

 $lpha_1,lpha_2,\ldots,lpha_s \xrightarrow{ ilde{ t k被其余向量线性表示的向量}} lpha_{j_1},lpha_{j_2},\ldots,lpha_{j_r}$ (极大无关组)

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$$egin{array}{c} \mathbf{p} \\ \alpha_1, \alpha_2, \alpha_3, \alpha_4 \end{array} \xrightarrow{\alpha_4 = 2\alpha_1 + 0\alpha_2 + 0\alpha_3} \quad \alpha_1, \alpha_2, \alpha_3 \xrightarrow{\alpha_3 = \alpha_1 + \alpha_2} \quad \alpha_1, \alpha_2 \xrightarrow{\mathrm{KE}} \quad \mathbf{p} \\ \mathrm{Milk}(\alpha_4) \quad \mathrm{$$

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_1 = -\alpha_2 + \alpha_3 + 0\alpha_4} \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2 = \alpha_3 - \frac{1}{2}\alpha_4}$$

诼个剔除

 $lpha_1,lpha_2,\ldots,lpha_s \xrightarrow{ ilde{ text{tkat}} ilde{ text{tkat}} ilde{ text{tkat}} ilde{ text{tat}} ilde{ text{dist}} i$

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$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
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$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_1 = -\alpha_2 + \alpha_3 + 0\alpha_4} \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2 = \alpha_3 - \frac{1}{2}\alpha_4}$$
 $\xrightarrow{\text{slip}(\alpha_1)}$



逐个剔除

 $lpha_1,lpha_2,\ldots,lpha_s \xrightarrow{ ilde{ text{tikty}} ilde{ text{tikty}} ilde{ text{tikty}} ilde{ text{dist}}} lpha_{j_1},lpha_{j_2},\ldots,lpha_{j_r}$ (极大无关组)

例 求
$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
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 $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_1 = -\alpha_2 + \alpha_3 + 0\alpha_4} \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2 = \alpha_3 - \frac{1}{2}\alpha_4} \alpha_3, \alpha_4$

逐个剔除

 $lpha_1,lpha_2,\ldots,lpha_s \xrightarrow{ ilde{ t k m j s nol 2} ilde{ t k m j s nol 2}} ilde{ alpha_{j_1},lpha_{j_2},\ldots,lpha_{j_r}}$ (极大无关组)

例 求
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 $egin{array}{c} \mathbf{p} \\ \boldsymbol{\alpha}_1, \, \boldsymbol{\alpha}_2, \, \boldsymbol{\alpha}_3, \, \boldsymbol{\alpha}_4 & \xrightarrow{\alpha_4 = 2\alpha_1 + 0\alpha_2 + 0\alpha_3} & \boldsymbol{\alpha}_1, \, \boldsymbol{\alpha}_2, \, \boldsymbol{\alpha}_3 & \xrightarrow{\alpha_3 = \alpha_1 + \alpha_2} & \boldsymbol{\alpha}_1, \, \boldsymbol{\alpha}_2 & \overset{\mathrm{d}}{\mathrm{H}} \\ & & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\$

 $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_1 = -\alpha_2 + \alpha_3 + 0\alpha_4} \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2 = \alpha_3 - \frac{1}{2}\alpha_4} \alpha_3, \alpha_4 \xrightarrow{\text{KE}}$

 $\alpha_1, \alpha_2, \ldots, \alpha_s \xrightarrow{\text{fiwtfsnold} \atop \text{直到不能再剔除为止}} \alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r} \quad (极大无关组)$

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$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ 的一个极大无关组。

 $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_1 = -\alpha_2 + \alpha_3 + 0\alpha_4} \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2 = \alpha_3 - \frac{1}{2}\alpha_4} \alpha_3, \alpha_4 \xrightarrow{\text{MX}} \alpha_5$

 $\alpha_1, \alpha_2, \alpha_3, \alpha_4$



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 $\alpha_1, \alpha_2, \ldots, \alpha_s \xrightarrow{\text{fiwtJsholightersholighter}} \alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r} \quad (极大无关组)$

例 求
$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ 的一个极大无关组。

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 $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_1 = -\alpha_2 + \alpha_3 + 0\alpha_4} \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2 = \alpha_3 - \frac{1}{2}\alpha_4} \alpha_3, \alpha_4 \xrightarrow{\text{MX}} \alpha_3, \alpha_4 \xrightarrow{\text{MX}} \alpha_3, \alpha_4 \xrightarrow{\text{MX}} \alpha_4$

 $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2=0\alpha_1+\alpha_3-\frac{1}{2}\alpha_4}$



诼个剔除

 $lpha_1,lpha_2,\ldots,lpha_s \xrightarrow{ ilde{ t k被其余向量线性表示的向量}} lpha_{j_1},lpha_{j_2},\ldots,lpha_{j_r}$ (极大无关组)

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 $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_1 = -\alpha_2 + \alpha_3 + 0\alpha_4} \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2 = \alpha_3 - \frac{1}{2}\alpha_4} \alpha_3, \alpha_4 \xrightarrow{\text{Mb}} \alpha_5$

 $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2 = 0\alpha_1 + \alpha_3 - \frac{1}{2}\alpha_4}$ $\xrightarrow{\text{spik}\alpha_2}$



 $\alpha_1, \alpha_2, \ldots, \alpha_s \xrightarrow{\text{fiwtJsholightersholighter}} \alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r} \quad (极大无关组)$

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$$egin{array}{c} egin{array}{c} egin{array}{c} eta \\ eta \\$$

 $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_1 = -\alpha_2 + \alpha_3 + 0\alpha_4} \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2 = \alpha_3 - \frac{1}{2}\alpha_4} \alpha_3, \alpha_4 \xrightarrow{\text{MX}} \alpha_5$

 $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2 = 0 \alpha_1 + \alpha_3 - \frac{1}{2} \alpha_4} \alpha_1, \alpha_3, \alpha_4$



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 $\alpha_1, \alpha_2, \ldots, \alpha_s \xrightarrow{\text{fiwtJsho} = \text{states}} \text{fiwtJsho} \alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r} \quad (极大无关组)$

例 求
$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ 的一个极大无关组。

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_1 = -\alpha_2 + \alpha_3 + 0\alpha_4} \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2 = \alpha_3 - \frac{1}{2}\alpha_4} \alpha_3, \alpha_4 \xrightarrow{\alpha_5 = 0\alpha_5 + \alpha_5} \alpha_5$$

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2 = 0\alpha_1 + \alpha_3 - \frac{1}{2}\alpha_4} \alpha_1, \alpha_3, \alpha_4 \xrightarrow{\alpha_4 = 2\alpha_1 + 0\alpha_4}$$



 $\alpha_1, \alpha_2, \ldots, \alpha_s \xrightarrow{\text{fiwtJsholightersholighter}} \alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r} \quad (极大无关组)$

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$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
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$$\begin{array}{c} \alpha_{1},\alpha_{2},\alpha_{3},\alpha_{4} \xrightarrow{\alpha_{1}=-\alpha_{2}+\alpha_{3}+0\alpha_{4}} & \alpha_{2},\alpha_{3},\alpha_{4} \xrightarrow{\alpha_{2}=\alpha_{3}-\frac{1}{2}\alpha_{4}} & \alpha_{3},\alpha_{4} \xrightarrow{\text{$\frac{1}{2}$}} & \alpha_{3},\alpha_{4} \xrightarrow{\text{$\frac{1}{2}$}} & \alpha_{4},\alpha_{2},\alpha_{3},\alpha_{4} & \alpha_{4}=2\alpha_{1}+0\alpha_{4}} \\ \alpha_{1},\alpha_{2},\alpha_{3},\alpha_{4} \xrightarrow{\alpha_{2}=0\alpha_{1}+\alpha_{3}-\frac{1}{2}\alpha_{4}} & \alpha_{1},\alpha_{3},\alpha_{4} \xrightarrow{\alpha_{4}=2\alpha_{1}+0\alpha_{4}} & \alpha_{1},\alpha_{2},\alpha_{3},\alpha_{4} \xrightarrow{\text{$\frac{1}{2}$}} & \alpha_{1},\alpha_{2},\alpha_{3},\alpha_{4} & \alpha_{2}=\alpha_{1}+\alpha_{2}+\alpha_{2}+\alpha_{3}+\alpha_{4}+\alpha_{2}+\alpha_{4}$$

 $\alpha_1, \alpha_2, \ldots, \alpha_s \xrightarrow{\text{fiwtJsholightersholighter}} \alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r} \quad (极大无关组)$

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$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ 的一个极大无关组。

 $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_1 = -\alpha_2 + \alpha_3 + 0\alpha_4} \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2 = \alpha_3 - \frac{1}{2}\alpha_4} \alpha_3, \alpha_4 \xrightarrow{\text{MX}} \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\text{MX}} \alpha_4, \alpha_5 \xrightarrow{\text{MX}} \alpha_5, \alpha_5 \xrightarrow{\text{MX}} \alpha_5 \xrightarrow{\text{$

 $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2 = 0\alpha_1 + \alpha_3 - \frac{1}{2}\alpha_4} \alpha_1, \alpha_3, \alpha_4 \xrightarrow{\alpha_4 = 2\alpha_1 + 0\alpha_4} \alpha_1, \alpha_3$



 $\alpha_1, \alpha_2, \ldots, \alpha_s \xrightarrow{\text{fixting fixed fix$

例 求
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 $egin{array}{c} \mathbf{p} \\ \alpha_1, \alpha_2, \alpha_3, \alpha_4 & \xrightarrow{\alpha_4 = 2\alpha_1 + 0\alpha_2 + 0\alpha_3} \\ \hline \betalike \alpha_4 & \alpha_1, \alpha_2, \alpha_3 & \xrightarrow{\alpha_3 = \alpha_1 + \alpha_2} \\ \hline \betalike \alpha_3 & \alpha_1, \alpha_2 & \cot \alpha_2 \\ \hline \betalike \alpha_3 & \alpha_2, \alpha_3 & \cot \alpha_3 \\ \hline \betalike \alpha_3 & \alpha_3, \alpha_4 & \cot \alpha_4 \\ \hline \betalike \alpha_4 & \cot \alpha_2, \alpha_3 & \cot \alpha_3 \\ \hline \betalike \alpha_4 & \cot \alpha_4, \alpha_4 & \cot \alpha_4 \\ \hline \betalike \alpha_4 & \cot \alpha_4, \alpha_5 & \cot \alpha_4 \\ \hline \betalike \alpha_4 & \cot \alpha_5, \alpha_5 & \cot \alpha_5, \alpha_5 \\ \hline \betalike \alpha_5 & \cot \alpha_5, \alpha_5 & \cot \alpha_5, \alpha_5 & \cot \alpha_5, \alpha_5 \\ \hline \betalike \alpha_5 & \cot \alpha_5, \alpha_5 & \cot \alpha_5, \alpha_5 & \cot \alpha_5, \alpha_5 \\ \hline \betalike \alpha_5 & \cot \alpha_5, \alpha_5 & \cot \alpha_5, \alpha_5 & \cot \alpha_5, \alpha_5 \\ \hline \betalike \alpha_5 & \cot \alpha_5, \alpha_5 & \cot \alpha_5, \alpha_5 & \cot \alpha_5, \alpha_5 & \cot \alpha_5, \alpha_5 \\ \hline \betalike \alpha_5 & \cot \alpha_5, \alpha_5 & \cot \alpha_5, \alpha_5 & \cot \alpha_5, \alpha_5 & \cot \alpha_5, \alpha_5 \\ \hline \betalike \alpha_5 & \cot \alpha_5, \alpha_5 \\ \hline \betalike \alpha_5 & \cot \alpha_5, \alpha_5 & \cot \alpha$

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还有其他极大无关组吗?



 $\alpha_1, \alpha_2, \ldots, \alpha_s \xrightarrow{\text{fixitation}} \text{fixing fixed f$

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还有其他极大无关组吗?

定理 α_{j_1} , α_{j_2} , . . . , α_{j_r} 是 α_1 , α_2 , . . . , α_s 的极大无关组,当且仅当

- α_{j1}, α_{j2}, ..., α_{jr} 线性无关
- α_1 , α_2 , \cdots , α_s 中每个向量都可由 α_{j_1} , α_{j_2} , \ldots , α_{j_r} 线性表示

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$$\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}; \qquad \beta_{k_1}, \beta_{k_2}, \ldots, \beta_{k_t}$$

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极大无关组的性质

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例设
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定义 向量组 α_1 , α_2 , ..., α_s 的极大无关组所包含向量的个数,称向量组的秩,记为:

$$r(\alpha_1, \alpha_2, \ldots, \alpha_s)$$

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设

$$\begin{pmatrix} a_1 & \alpha_2 & \alpha_n \\ a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (\alpha_1 \alpha_2 \cdots \alpha_n)$$

$$r(\alpha_1, \alpha_2, \ldots, \alpha_n)$$

设
$$A_{m\times n} = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_n \\ \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix} = (\alpha_1 \alpha_2 \cdots \alpha_n)$$

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定理
$$r(A) = r(\alpha_1, \alpha_2, \ldots, \alpha_n)$$



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定义

• $r(\alpha_1, \alpha_2, \ldots, \alpha_n)$ 称为 A 的列秩;

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$$r(A) = r(\alpha_1, \alpha_2, \ldots, \alpha_n)$$



设
$$a_1 \quad a_2 \quad a_n$$

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定义

r(α₁, α₂, ..., α_n) 称为 A 的列秩;

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设
$$A_{m\times n} = \begin{array}{cccc} \beta_1 & \alpha_1 & \alpha_2 & \alpha_n \\ \beta_2 & \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \beta_2 & \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_m & \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{array} \right) = (\alpha_1 \alpha_2 \cdots \alpha_n)$$

定义

• $r(\alpha_1, \alpha_2, \ldots, \alpha_n)$ 称为 A 的列秩;

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定义

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定义

- r(α₁, α₂,..., α_n) 称为 A 的列秩;
- r(β₁, β₂,...,β_m) 称为 A 的行秩;

定理
$$r(A) = r(\alpha_1, \alpha_2, \ldots, \alpha_n)$$



设
$$A_{m \times n} = \begin{array}{cccc} \beta_1 & \alpha_1 & \alpha_2 & \alpha_n \\ \beta_2 & \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_m & \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{array} \right) = (\alpha_1 \alpha_2 \cdots \alpha_n) = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{pmatrix}$$

定义

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定理
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问题 给出 m 维的向量组 α_1 , α_2 , \cdots , α_n , 如何求出其一组极大无关组?

步骤

问题 给出 m 维的向量组 $lpha_1$, $lpha_2$, \cdots , $lpha_n$,如何求出其一组极大无关组 ?

步骤
$$1. A_{m \times n} = \begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_n \\
\alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\
\alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn}
\end{pmatrix}$$

问题 给出 m 维的向量组 α_1 , α_2 , \cdots , α_n , 如何求出其一组极大无关组?

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$$1. \ \, A_{m \times n} = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_n \\ \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix} \xrightarrow{\eta \$ \uparrow - \psi }$$
简化的阶梯型矩阵

问题 给出 m 维的向量组 $lpha_1$, $lpha_2$, \cdots , $lpha_n$, 如何求出其一组极大无关组 ?

步骤
$$1. \ \, A_{m \times n} = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_n \\ \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix} \xrightarrow{\eta \not \approx f - g \not p}$$
简化的阶梯型矩阵

2. 通过简化的阶梯型矩阵, 求出 r(A)。

问题 给出 m 维的向量组 $lpha_1$, $lpha_2$, \cdots , $lpha_n$, 如何求出其一组极大无关组 ?

步骤
$$1. \ \, A_{m \times n} = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_n \\ \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix} \xrightarrow{\eta \not \in \widehat{T}_{\mathfrak{D}} \not \to}$$
简化的阶梯型矩阵

2. 通过简化的阶梯型矩阵,求出 *r(A)*。

利用 $r(\alpha_1, \alpha_2, ..., \alpha_n) = r(A)$,得出极大无关组所包含向量的个数



问题 给出 m 维的向量组 $lpha_1$, $lpha_2$, \cdots , $lpha_n$, 如何求出其一组极大无关组 ?

步骤
$$1. \ A_{m \times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \xrightarrow{\eta \circledast \uparrow - \psi}$$
简化的阶梯型矩阵

- 2. 通过简化的阶梯型矩阵,求出 r(A)。 利用 $r(\alpha_1,\alpha_2,\ldots,\alpha_n)=r(A)$,得出极大无关组所包含向量的个数
- 3. 通过简化的阶梯型矩阵,容易看出线性无关的 r(A) 列,这就找到一组极大无关组



问题 给出 m 维的向量组 $lpha_1$, $lpha_2$, \cdots , $lpha_n$, 如何求出其一组极大无关组 ?

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简化的阶梯型矩阵

- 2. 通过简化的阶梯型矩阵,求出 r(A)。 利用 $r(\alpha_1, \alpha_2, \ldots, \alpha_n) = r(A)$,得出极大无关组所包含向量的个数
- 3. 通过简化的阶梯型矩阵,容易看出线性无关的 *r(A)* 列,这就找到一组极大无关组
- 4. 通过简化的阶梯型矩阵,容易看出其余列如何用该选定极大无关组 线性表示

例 1 求向量组 $\alpha_1 = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$ 的一个极

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$$\begin{pmatrix} 2 & 1 & 2 & 3 \\ 4 & 1 & 3 & 5 \\ 2 & 0 & 1 & 2 \end{pmatrix} \frac{r_2 - 2r_1}{r_3 - r_1}$$

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$$\begin{pmatrix} 2 & 1 & 2 & 3 \\ 4 & 1 & 3 & 5 \\ 2 & 0 & 1 & 2 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 2 & 1 & 2 & 3 \\ 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 \end{pmatrix} \longrightarrow$$

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$$\begin{pmatrix}
2 & 1 & 2 & 3 \\
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\end{pmatrix}
\xrightarrow[r_3-r_1]{r_2-2r_1}
\begin{pmatrix}
2 & 1 & 2 & 3 \\
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\end{pmatrix}
\longrightarrow
\begin{pmatrix}
2 & 1 & 2 & 3 \\
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\end{pmatrix}$$

$$r_1-r_2$$



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\end{pmatrix}
\longrightarrow
\begin{pmatrix}
2 & 1 & 2 & 3 \\
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

$$\xrightarrow{r_1-r_2}
\begin{pmatrix}
2 & 0 & 1 & 2 \\
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\end{pmatrix}
\xrightarrow{\frac{1}{2} \times r_1}$$



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\end{pmatrix}
\longrightarrow
\begin{pmatrix}
2 & 1 & 2 & 3 \\
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

$$\xrightarrow{r_1-r_2} \left(\begin{array}{cccc} 2 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\frac{1}{2} \times r_1} \left(\begin{array}{cccc} 1 & 0 & \frac{1}{2} & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

所以

•
$$r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2$$
;

● 整角大寿

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整局大学

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$$\begin{pmatrix}
2 & 1 & 2 & 3 \\
4 & 1 & 3 & 5 \\
2 & 0 & 1 & 2
\end{pmatrix}
\xrightarrow[r_3-r_1]{r_2-2r_1}
\begin{pmatrix}
2 & 1 & 2 & 3 \\
0 & -1 & -1 & -1 \\
0 & -1 & -1 & -1
\end{pmatrix}
\longrightarrow
\begin{pmatrix}
2 & 1 & 2 & 3 \\
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

$$\xrightarrow{r_1-r_2} \left(\begin{array}{cccc} 2 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\frac{1}{2} \times r_1} \left(\begin{array}{cccc} 1 & 0 & \frac{1}{2} & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

所以

- $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2$;
- α₁, α₂ 是极大无关组;

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$$\begin{pmatrix} 2 & 1 & 2 & 3 \\ 4 & 1 & 3 & 5 \\ 2 & 0 & 1 & 2 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 2 & 1 & 2 & 3 \\ 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{r_1-r_2} \left(\begin{array}{cccc} 2 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\frac{1}{2} \times r_1} \left(\begin{array}{cccc} 1 & 0 & \frac{1}{2} & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

所以

- $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2;$
- α₁, α₂ 是极大无关组;
- $\alpha_3 = \frac{1}{2}\alpha_1 + \alpha_2$, $\alpha_4 = \alpha_1 + \alpha_2$

例 2 求向量组
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$
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$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 - r_1}$$

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$$\begin{pmatrix}
1 & 0 & 1 & 2 \\
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1 & 1 & 0 & 3 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow[r_3-r_1]{r_2-2r_1}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
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\end{pmatrix}$$

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\end{pmatrix}
\xrightarrow[r_3-r_1]{r_2-2r_1}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 1 & -1 & 1 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow[r_4-2r_2]{r_3-r_2}$$

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1 & 1 & 0 & 3 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow[r_3-r_1]{r_2-2r_1}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 1 & -1 & 1 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow[r_4-2r_2]{r_3-r_2}
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$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow{r_3 - r_2} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$\xrightarrow{-3r_3}$$



例 2 求向量组
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$$\begin{pmatrix}
1 & 0 & 1 & 2 \\
2 & 1 & 1 & 4 \\
1 & 1 & 0 & 3 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow[r_3-r_1]{r_2-2r_1}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 1 & -1 & 1 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow[r_4-2r_2]{r_3-r_2}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 3
\end{pmatrix}$$

$$\frac{r_4 - 3r_3}{r_1 - 2r_3} \left(\begin{array}{cccc}
1 & 0 & 1 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{array} \right)$$

例 2 求向量组
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$$\begin{pmatrix}
1 & 0 & 1 & 2 \\
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1 & 1 & 0 & 3 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow[r_3-r_1]{r_2-2r_1}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 1 & -1 & 1 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow[r_4-2r_2]{r_3-r_2}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 3
\end{pmatrix}$$

$$\frac{r_{4}-3r_{3}}{r_{1}-2r_{3}}
\begin{pmatrix}
1 & 0 & 1 & 0 \\
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$$\begin{pmatrix}
1 & 0 & 1 & 2 \\
2 & 1 & 1 & 4 \\
1 & 1 & 0 & 3 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow[r_{3}-r_{1}]{r_{3}-r_{1}}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 1 & -1 & 1 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow[r_{4}-2r_{2}]{r_{3}-r_{2}}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}$$

$$\xrightarrow[r_{4}-3r_{3}]{r_{1}-2r_{3}}
\begin{pmatrix}
1 & 0 & 1 & 0 \\
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0 & 0 & 0 & 1 \\
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\end{pmatrix}
\xrightarrow[r_3-r_1]{r_2-2r_1}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 1 & -1 & 1 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow[r_4-2r_2]{r_3-r_2}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 3
\end{pmatrix}$$

$$\frac{r_4 - 3r_3}{r_1 - 2r_3} \begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

所以
•
$$r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 3$$
;

例 2 求向量组
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$
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\xrightarrow[r_3-r_1]{r_2-2r_1}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 1 & -1 & 1 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow[r_4-2r_2]{r_3-r_2}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 3
\end{pmatrix}$$

$$\xrightarrow[r_1-2r_3]{r_1-2r_3} \left(\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

所以
•
$$r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 3$$
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例 2 求向量组
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$
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\end{pmatrix}
\xrightarrow[r_4-2r_3]{r_3-r_2}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
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\end{pmatrix}
\xrightarrow[r_4-2r_3]{r_3-r_2}
\begin{pmatrix}
1 & 0 & 1 & 0 \\
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0 & 0 & 0 & 0
\end{pmatrix}$$

所以
•
$$r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 3$$
:

α₁, α₂, α₄ 是极大无关组;



例 2 求向量组
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一个极大无关组;并把其余向量用该极大无关组线性表示。

所以
$$r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 3$$
:

$$\alpha_3 = \alpha_1 - \alpha_2$$



例 3 求向量组
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个

例 3 求向量组
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个

例 3 求向量组
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \xrightarrow[r_4-4r_1]{r_2-2r_1} \xrightarrow[r_4-4r_1]{r_4-4r_1}$$

例 3 求向量组
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个

$$\begin{pmatrix}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 5 \\
3 & 4 & 5 & 6 \\
4 & 5 & 6 & 7
\end{pmatrix}
\xrightarrow[r_4-4r_1]{r_2-2r_1}
\begin{pmatrix}
1 & 2 & 3 & 4 \\
0 & -1 & -2 & -3 \\
0 & -2 & -4 & -6 \\
0 & -3 & -6 & -9
\end{pmatrix}$$

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$$r_3-2r_2$$

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\end{pmatrix}$$

$$\xrightarrow[r_{4}-3r_{2}]{r_{4}-3r_{2}}
\begin{pmatrix}
1 & 2 & 3 & 4 \\
0 & -1 & -2 & -3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

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0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\longrightarrow
\begin{pmatrix}
1 & 0 & -1 & -2 \\
0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 \\
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\end{pmatrix}$$



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\begin{pmatrix}
1 & 2 & 3 & 4 \\
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0 & 0 & 0 & 0 \\
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\end{pmatrix}
\longrightarrow
\begin{pmatrix}
1 & 0 & -1 & -2 \\
0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

所以
•
$$r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2$$
;

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\end{pmatrix}
\xrightarrow[r_{4}-4r_{1}]{r_{2}-2r_{1}}
\begin{pmatrix}
1 & 2 & 3 & 4 \\
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\end{pmatrix}$$

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\begin{pmatrix}
1 & 2 & 3 & 4 \\
0 & -1 & -2 & -3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\longrightarrow
\begin{pmatrix}
1 & 0 & -1 & -2 \\
0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

所以
• $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2$;



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$$\alpha_1 = \begin{pmatrix} 1\\2\\3\\4 \end{pmatrix}$$
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$$\xrightarrow{r_3-2r_2} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

•
$$r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2$$
;

α₁, α₂ 是极大无关组;

例 3 求向量组
$$\alpha_1 = \begin{pmatrix} 1\\2\\3\\4 \end{pmatrix}$$
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$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \xrightarrow[r_{4}-4r_{1}]{r_{2}-2r_{1}} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{pmatrix}$$

$$\xrightarrow[r_{3}-2r_{2}]{r_{4}-3r_{2}} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

•
$$r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2;$$

$$\alpha_3 = -\alpha_1 + 2\alpha_2$$



所以

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$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \xrightarrow[r_{2}-2r_{1}]{r_{3}-3r_{1}} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{pmatrix}$$

$$\xrightarrow[r_{3}-2r_{2}]{r_{4}-3r_{2}} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

•
$$r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2;$$

•
$$\alpha_3 = -\alpha_1 + 2\alpha_2$$
, $\alpha_4 = -2\alpha_1 + 3\alpha_2$



所以

证明设

$$r_1 = r(\alpha_1, \alpha_2, \dots, \alpha_s),$$

$$r_2 = r(\beta_1, \beta_2, \dots, \beta_t),$$

$$r_1 = r(\alpha_1, \alpha_2, \ldots, \alpha_s), \quad \alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_{r_1}}$$
 是极大无关组 $r_2 = r(\beta_1, \beta_2, \ldots, \beta_t),$

$$r_1 = r(\alpha_1, \alpha_2, ..., \alpha_s), \quad \alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_{r_1}}$$
 是极大无关组 $r_2 = r(\beta_1, \beta_2, ..., \beta_t), \quad \beta_{j_1}, \beta_{j_2}, ..., \beta_{j_{r_2}}$ 是极大无关组

$$r_1 = r(\alpha_1, \alpha_2, \ldots, \alpha_s), \quad \alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_{r_1}}$$
 是极大无关组 $r_2 = r(\beta_1, \beta_2, \ldots, \beta_t), \quad \beta_{j_1}, \beta_{j_2}, \ldots, \beta_{j_{r_2}}$ 是极大无关组 注意到 $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_{r_1}}$ 能由 $\beta_{j_1}, \beta_{j_2}, \ldots, \beta_{j_{r_2}}$ 线性表示,

$$r_1 = r(\alpha_1, \alpha_2, ..., \alpha_s)$$
, $\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_{r_1}}$ 是极大无关组 $r_2 = r(\beta_1, \beta_2, ..., \beta_t)$, $\beta_{j_1}, \beta_{j_2}, ..., \beta_{j_{r_2}}$ 是极大无关组 注意到 $\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_{r_1}}$ 能由 $\beta_{j_1}, \beta_{j_2}, ..., \beta_{j_{r_2}}$ 线性表示,所以 $r_1 \leq r_2$ 。

证明设

$$r_1 = r(\alpha_1, \alpha_2, ..., \alpha_s), \quad \alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_{r_1}}$$
 是极大无关组 $r_2 = r(\beta_1, \beta_2, ..., \beta_t), \quad \beta_{j_1}, \beta_{j_2}, ..., \beta_{j_{r_2}}$ 是极大无关组 注意到 $\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_{r_1}}$ 能由 $\beta_{j_1}, \beta_{j_2}, ..., \beta_{j_{r_2}}$ 线性表示,所以 $r_1 \leq r_2$ 。

定理 设有向量组 (A): $\alpha_1, \alpha_2, \ldots, \alpha_s$

(B): $\beta_1, \beta_2, \ldots, \beta_t$

若它们等价,

证明设

$$r_1 = r(\alpha_1, \alpha_2, ..., \alpha_s), \quad \alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_{r_1}}$$
 是极大无关组 $r_2 = r(\beta_1, \beta_2, ..., \beta_t), \quad \beta_{j_1}, \beta_{j_2}, ..., \beta_{j_{r_2}}$ 是极大无关组 注意到 $\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_{r_1}}$ 能由 $\beta_{j_1}, \beta_{j_2}, ..., \beta_{j_{r_2}}$ 线性表示,所以 $r_1 \leq r_2$ 。

定理 设有向量组
$$(A)$$
: $\alpha_1, \alpha_2, \ldots, \alpha_s$ (B) : $\beta_1, \beta_2, \ldots, \beta_t$

若它们等价,则 $r(\alpha_1, \alpha_2, \ldots, \alpha_s) = r(\beta_1, \beta_2, \ldots, \beta_t)$ 。



$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}$$

证明 设
$$AB = C_{m \times s}$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}$$

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$$(\gamma_1 \ \gamma_2 \cdots \ \gamma_s) = (\alpha_1 \ \alpha_2 \cdots \ \alpha_n) \begin{pmatrix} b_{11} \ b_{12} \cdots b_{1s} \\ b_{21} \ b_{22} \cdots b_{2s} \\ \vdots \ \vdots \ \ddots \ \vdots \\ b_{n1} \ b_{n2} \cdots b_{ns} \end{pmatrix}$$

证明 设
$$AB = C_{m \times s}$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}$$

即

$$(\gamma_1 \ \gamma_2 \cdots \ \gamma_s) = (\alpha_1 \ \alpha_2 \cdots \ \alpha_n) \begin{pmatrix} b_{11} \ b_{12} \cdots b_{1s} \\ b_{21} \ b_{22} \cdots b_{2s} \\ \vdots \ \vdots \ \ddots \ \vdots \\ b_{n1} \ b_{n2} \cdots b_{ns} \end{pmatrix}$$

$$\Rightarrow \quad \gamma_1 = b_{11}\alpha_1 + b_{21}\alpha_2 + \dots + b_{n1}\alpha_n$$

证明 设
$$AB = C_{m \times s}$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}$$

即

$$(\gamma_1 \ \gamma_2 \cdots \ \gamma_s) = (\alpha_1 \ \alpha_2 \cdots \ \alpha_n) \begin{pmatrix} b_{11} \ b_{12} \cdots b_{1s} \\ b_{21} \ b_{22} \cdots b_{2s} \\ \vdots \ \vdots \ \ddots \ \vdots \\ b_{n1} \ b_{n2} \cdots b_{ns} \end{pmatrix}$$

$$\Rightarrow$$
 $\gamma_1 = b_{11}\alpha_1 + b_{21}\alpha_2 + \cdots + b_{n1}\alpha_n$ 等等

证明 设
$$AB = C_{m \times s}$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}$$

即

$$(\gamma_1 \ \gamma_2 \cdots \ \gamma_s) = (\alpha_1 \ \alpha_2 \cdots \ \alpha_n) \begin{pmatrix} b_{11} \ b_{12} \cdots b_{1s} \\ b_{21} \ b_{22} \cdots b_{2s} \\ \vdots \ \vdots \ \ddots \ \vdots \\ b_{n1} \ b_{n2} \cdots b_{ns} \end{pmatrix}$$

$$\Rightarrow \quad \gamma_1 = b_{11}\alpha_1 + b_{21}\alpha_2 + \dots + b_{n1}\alpha_n \quad 等等$$

可见 $\gamma_1, \ldots, \gamma_s$ 由 $\alpha_1, \ldots, \alpha_n$ 线性表示,

证明 设
$$AB = C_{m \times s}$$

$$\underbrace{\begin{pmatrix}
c_{11} & c_{12} & \cdots & c_{1s} \\
c_{21} & c_{22} & \cdots & c_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
c_{m1} & c_{m2} & \cdots & c_{ms}
\end{pmatrix}}_{C} = \underbrace{\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}}_{A} \begin{pmatrix}
b_{11} & b_{12} & \cdots & b_{1s} \\
b_{21} & b_{22} & \cdots & b_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
b_{n1} & b_{n2} & \cdots & b_{ns}
\end{pmatrix}}_{B}$$

即

$$(\gamma_1 \ \gamma_2 \cdots \ \gamma_s) = (\alpha_1 \ \alpha_2 \cdots \ \alpha_n) \begin{pmatrix} b_{11} \ b_{12} \cdots b_{1s} \\ b_{21} \ b_{22} \cdots b_{2s} \\ \vdots \ \vdots \ \ddots \ \vdots \\ b_{n1} \ b_{n2} \cdots b_{ns} \end{pmatrix}$$

 $\gamma_1 = b_{11}\alpha_1 + b_{21}\alpha_2 + \cdots + b_{n1}\alpha_n$ 等等

可见
$$\gamma_1, \ldots, \gamma_s$$
 由 $\alpha_1, \ldots, \alpha_n$ 线性表示,所以

$$r(\gamma_1, \ldots, \gamma_s) \leq r(\alpha_1, \ldots, \alpha_n)$$



证明 设
$$AB = C_{mxs}$$

$$\underbrace{\begin{pmatrix}
c_{11} & c_{12} & \cdots & c_{1s} \\
c_{21} & c_{22} & \cdots & c_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
c_{m1} & c_{m2} & \cdots & c_{ms}
\end{pmatrix}}_{C} = \underbrace{\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}}_{A} \begin{pmatrix}
b_{11} & b_{12} & \cdots & b_{1s} \\
b_{21} & b_{22} & \cdots & b_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
b_{n1} & b_{n2} & \cdots & b_{ns}
\end{pmatrix}}_{B}$$

即

$$(\gamma_1 \ \gamma_2 \cdots \gamma_s) = (\alpha_1 \ \alpha_2 \cdots \alpha_n) \begin{pmatrix} b_{11} \ b_{12} \cdots b_{1s} \\ b_{21} \ b_{22} \cdots b_{2s} \\ \vdots \ \vdots \ \ddots \ \vdots \\ b_{n1} \ b_{n2} \cdots b_{ns} \end{pmatrix}$$

$$\Rightarrow \gamma_1 = b_{11}\alpha_1 + b_{21}\alpha_2 + \cdots + b_{n1}\alpha_n \quad \text{\reff}$$

可见
$$\gamma_1, \ldots, \gamma_s$$
 由 $\alpha_1, \ldots, \alpha_n$ 线性表示,所以

$$r(\gamma_1, \ldots, \gamma_s) \le r(\alpha_1, \ldots, \alpha_n) = r(A)$$

证明 设
$$AB = C_{m \times s}$$

$$\underbrace{\begin{pmatrix}
c_{11} & c_{12} & \cdots & c_{1s} \\
c_{21} & c_{22} & \cdots & c_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
c_{m1} & c_{m2} & \cdots & c_{ms}
\end{pmatrix}}_{C} = \underbrace{\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}}_{A} \begin{pmatrix}
b_{11} & b_{12} & \cdots & b_{1s} \\
b_{21} & b_{22} & \cdots & b_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
b_{n1} & b_{n2} & \cdots & b_{ns}
\end{pmatrix}}_{B}$$

即

$$(\gamma_1 \ \gamma_2 \cdots \gamma_s) = (\alpha_1 \ \alpha_2 \cdots \ \alpha_n) \begin{pmatrix} b_{11} \ b_{12} \cdots b_{1s} \\ b_{21} \ b_{22} \cdots b_{2s} \\ \vdots \ \vdots \ \ddots \ \vdots \\ b_{n1} \ b_{n2} \cdots b_{ns} \end{pmatrix}$$

$$\Rightarrow \gamma_1 = b_{11}\alpha_1 + b_{21}\alpha_2 + \cdots + b_{n1}\alpha_n \quad \text{\reff}$$

可见
$$\gamma_1, \ldots, \gamma_s$$
 由 $\alpha_1, \ldots, \alpha_n$ 线性表示,所以

$$r(AB) = r(\gamma_1, \ldots, \gamma_s) \le r(\alpha_1, \ldots, \alpha_n) = r(A)$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}^{\beta_{1}}$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}^{\beta_{1}}$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}^{\beta_{1}}$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{\beta_{n}}^{\beta_{1}}$$

$$\underbrace{\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{bmatrix}}_{C} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{bmatrix}}_{B}^{\beta_{1}}$$

$$\underbrace{\begin{bmatrix}
c_{11} & c_{12} & \cdots & c_{1s} \\
c_{21} & c_{22} & \cdots & c_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
c_{m1} & c_{m2} & \cdots & c_{ms}
\end{bmatrix}}_{C} = \underbrace{\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}}_{A} \underbrace{\begin{bmatrix}
b_{11} & b_{12} & \cdots & b_{1s} \\
b_{21} & b_{22} & \cdots & b_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
b_{n1} & b_{n2} & \cdots & b_{ns}
\end{bmatrix}}_{\beta_{n}}^{\beta_{1}}$$

$$\frac{\delta_{1}}{\delta_{2}} \begin{pmatrix} c_{11} & c_{12} \cdots c_{1s} \\ c_{21} & c_{22} \cdots c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} \cdots c_{ms} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} \cdots a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \cdots b_{1s} \\ b_{21} & b_{22} \cdots b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} \cdots b_{ns} \end{pmatrix} \beta_{n}$$

$$\frac{\delta_{1}}{\delta_{2}} \left(\begin{array}{ccc} C_{11} & C_{12} & \cdots & C_{1s} \\ C_{21} & C_{22} & \cdots & C_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ C_{m1} & C_{m2} & \cdots & C_{ms} \end{array} \right) = \left(\begin{array}{ccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{array} \right) \left(\begin{array}{ccc} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{array} \right) \beta_{1}$$

$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$$

$$\Rightarrow \quad \delta_1 = a_{11}\beta_1 + a_{12}\beta_2 + \dots + a_{1n}\beta_n$$

证明 设 $AB = C_{mxs}$

$$\frac{\delta_{1}}{\delta_{2}} \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix} \begin{pmatrix} \beta_{1} \\ \beta_{2} \end{pmatrix}$$

$$\begin{pmatrix} \delta_{1} \\ \delta_{2} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \end{pmatrix} \begin{pmatrix} \beta_{1} \\ \beta_{2} \end{pmatrix}$$

即

$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$$

$$\Rightarrow \quad \delta_1 = a_{11}\beta_1 + a_{12}\beta_2 + \dots + a_{1n}\beta_n \quad$$
 等等

证明 设 $AB = C_{m \times s}$

$$\frac{\delta_{1}}{\delta_{2}} \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix} \begin{pmatrix} \beta_{1} \\ \beta_{n} \\ \beta_$$

即

$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$$

$$\Rightarrow \quad \delta_1 = a_{11}\beta_1 + a_{12}\beta_2 + \dots + a_{1n}\beta_n \quad \text{\refs}$$

可见 $\delta_1, \ldots, \delta_m$ 由 β_1, \ldots, β_n 线性表示,

证明 设 $AB = C_{m \times s}$

$$\underbrace{ \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{bmatrix}}_{C} = \underbrace{ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{A} \underbrace{ \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{bmatrix}}_{\beta_{n}} _{\beta_{n}}$$

即

$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$$

$$\Rightarrow \quad \delta_1 = a_{11}\beta_1 + a_{12}\beta_2 + \cdots + a_{1n}\beta_n \quad \text{\$}$$

可见 $\delta_1, \ldots, \delta_m$ 由 β_1, \ldots, β_n 线性表示,所以

$$r(\delta_1, \ldots, \delta_m) \leq r(\beta_1, \ldots, \beta_n)$$

证明 设 $AB = C_{m \times s}$

$$\frac{\delta_{1}}{\delta_{2}} \left(\begin{array}{ccc} C_{11} & C_{12} & \cdots & C_{1s} \\ C_{21} & C_{22} & \cdots & C_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ C_{m1} & C_{m2} & \cdots & C_{ms} \end{array} \right) = \underbrace{ \left(\begin{array}{ccc} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{array} \right)}_{A} \underbrace{ \left(\begin{array}{ccc} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{array} \right)}_{B}^{\beta_{1}}$$

即

$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$$

$$\Rightarrow \quad \delta_1 = a_{11}\beta_1 + a_{12}\beta_2 + \dots + a_{1n}\beta_n \quad$$
\$\$

可见 $\delta_1, \ldots, \delta_m$ 由 β_1, \ldots, β_n 线性表示,所以

$$r(\delta_1, \ldots, \delta_m) \le r(\beta_1, \ldots, \beta_n) = r(B)$$

证明 设 $AB = C_{m \times s}$

$$\frac{\delta_{1}}{\delta_{2}} \underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{\beta_{n}}_{\beta_{n}}$$

即

$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$$

$$\Rightarrow \quad \delta_1 = a_{11}\beta_1 + a_{12}\beta_2 + \dots + a_{1n}\beta_n \quad$$
\$

可见 $\delta_1, \ldots, \delta_m$ 由 β_1, \ldots, β_n 线性表示,所以

$$r(AB) = r(\delta_1, \ldots, \delta_m) \le r(\beta_1, \ldots, \beta_n) = r(B)$$