第 11 章 f: 高斯公式、斯托克斯公式

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Outline

1. 高斯公式

2. 斯托克斯公式



We are here now...

1. 高斯公式

2. 斯托克斯公式

定义 设
$$F = (P, Q, R)$$
 是空间中向量场,定义

$$divF := \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

称为向量场 F 的散度。

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例 1 计算向量场 $F = (x^2 + yz, y^2 + xz, z^2 + xy)$ 的散度。



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 的散度。

$$\operatorname{div} F = \frac{\partial}{\partial x}(x^2 + yz) + \frac{\partial}{\partial y}(y^2 + xz) + \frac{\partial}{\partial z}(z^2 + xy) = 2x + 2y + 2z.$$



例 2 计算梯度场 $\nabla \frac{1}{r}$ $(r = \sqrt{x^2 + y^2 + z^2})$ 的散度。

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$$\operatorname{div}\nabla \frac{1}{r}$$

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$$\nabla \frac{1}{r} = (\frac{\partial}{\partial x}r^{-1}, \frac{\partial}{\partial y}r^{-1}, \frac{\partial}{\partial z}r^{-1})$$

$$div \nabla \frac{1}{r}$$

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$$\nabla \frac{1}{r} = (\frac{\partial}{\partial x} r^{-1}, \frac{\partial}{\partial y} r^{-1}, \frac{\partial}{\partial z} r^{-1})$$
$$-r^{-2} \cdot r_{x}$$
$$\operatorname{div} \nabla \frac{1}{r}$$

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$$\nabla \frac{1}{r} = \left(\frac{\partial}{\partial x}r^{-1}, \frac{\partial}{\partial y}r^{-1}, \frac{\partial}{\partial z}r^{-1}\right)$$
$$= \left(-r^{-2} \cdot r_x, -r^{-2} \cdot r_y, -r^{-2} \cdot r_z\right)$$
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例 2 计算梯度场
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$$r_{x} = \frac{x}{r},$$

$$\nabla \frac{1}{r} = (\frac{\partial}{\partial x}r^{-1}, \frac{\partial}{\partial y}r^{-1}, \frac{\partial}{\partial z}r^{-1})$$

$$= (-r^{-2} \cdot r_{x}, -r^{-2} \cdot r_{y}, -r^{-2} \cdot r_{z})$$

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$$= (-r^{-2} \cdot r_{x}, -r^{-2} \cdot r_{y}, -r^{-2} \cdot r_{z}) = (-\frac{x}{r^{3}}, -\frac{y}{r^{3}}, -\frac{z}{r^{3}}),$$

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$$(-\frac{1}{r^{3}} + \frac{3x^{2}}{r^{5}})$$



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$$= (-\frac{1}{r^{3}} + \frac{3x^{2}}{r^{5}}) + (-\frac{1}{r^{3}} + \frac{3y^{2}}{r^{5}}) + (-\frac{1}{r^{3}} + \frac{3z^{2}}{r^{5}})$$



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$$= (-\frac{1}{r^{3}} + \frac{3x^{2}}{r^{5}}) + (-\frac{1}{r^{3}} + \frac{3y^{2}}{r^{5}}) + (-\frac{1}{r^{3}} + \frac{3z^{2}}{r^{5}})$$

$$= -\frac{3}{r^{3}} + \frac{3(x^{2} + y^{2} + z^{2})}{r^{5}}$$

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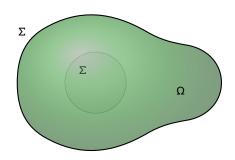
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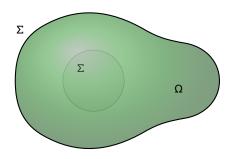
$$\operatorname{div} \nabla \frac{1}{r} = \frac{\partial}{\partial x}(-\frac{x}{r^{3}}) + \frac{\partial}{\partial y}(-\frac{y}{r^{3}}) + \frac{\partial}{\partial z}(-\frac{z}{r^{3}})$$

$$= \left(-\frac{1}{r^3} + \frac{3x^2}{r^5}\right) + \left(-\frac{1}{r^3} + \frac{3y^2}{r^5}\right) + \left(-\frac{1}{r^3} + \frac{3z^2}{r^5}\right)$$
$$= -\frac{3}{r^3} + \frac{3(x^2 + y^2 + z^2)}{r^5} = 0.$$

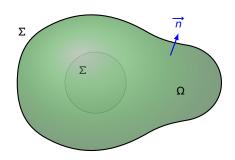
- 空间闭区域 Ω 的边界是分片光滑的闭曲面 Σ ,
- \overrightarrow{n} 是 Σ 的单位外法向量,



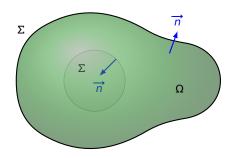
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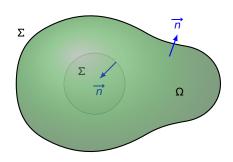
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- F = (P, Q, R) 是 Ω 中向量场,且 P, Q, R 具有一阶连续的偏导数,



定理(高斯公式) 假设

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则

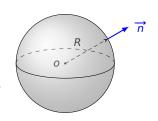
$$\iiint_{\Omega} \operatorname{div} F dv = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$

$$\sum_{n} \bigcap_{\Omega} \bigcap_{n} \bigcap_{n}$$



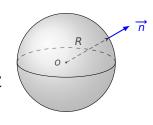
$$I = \iint_{\Sigma} 2x \, dy \, dz + y^2 \, dz \, dx + z^2 \, dx \, dy$$

其中定向曲面 Σ 是球面 $x^2 + y^2 + z^2 = R^2$,定 向取外侧



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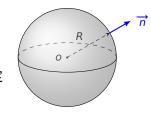


$$I = \iiint_{\Sigma} F \cdot \overrightarrow{n} \, dS$$



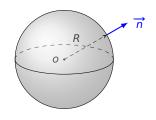
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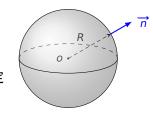
其中定向曲面 Σ 是球面 $x^2 + y^2 + z^2 = R^2$,定向取外侧



$$I = \underbrace{F = (2x, y^2, z^2)}_{\Gamma} \iint_{\Sigma} F \cdot \overrightarrow{n} dS = \underbrace{\overline{\text{高斯公式}}}_{\Omega} \iiint_{\Omega} \text{div} F dv$$

$$I = \iint_{\Sigma} 2x \, dy \, dz + y^2 \, dz \, dx + z^2 \, dx \, dy$$

其中定向曲面 Σ 是球面 $x^2 + y^2 + z^2 = R^2$,定向取外侧

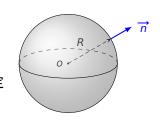


$$I = \frac{F = (2x, y^2, z^2)}{\int \int_{\Sigma} F \cdot \overrightarrow{n} dS} = \frac{\overline{\text{sh}公式}}{\int \int_{\Omega} \text{div} F dv}$$
$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} (2x) + \frac{\partial}{\partial y} (y^2) + \frac{\partial}{\partial z} (z^2) \right] dv$$



$$I = \iint_{\Sigma} 2x \, dy \, dz + y^2 \, dz \, dx + z^2 \, dx \, dy$$

其中定向曲面 Σ 是球面 $x^2 + y^2 + z^2 = R^2$,定向取外侧

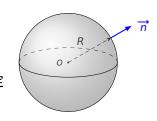


$$I = \frac{F = (2x, y^2, z^2)}{\int \int_{\Sigma} F \cdot \overrightarrow{n} \, dS} = \frac{\overrightarrow{\text{sh}} \triangle \overrightarrow{\text{sh}}}{\int \int_{\Omega} \text{div} F \, dv}$$
$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} (2x) + \frac{\partial}{\partial y} (y^2) + \frac{\partial}{\partial z} (z^2) \right] dv = \iiint_{\Omega} (2 + 2y + 2z) dv$$



$$I = \iint_{\Sigma} 2x \, dy \, dz + y^2 \, dz \, dx + z^2 \, dx \, dy$$

其中定向曲面 Σ 是球面 $x^2 + y^2 + z^2 = R^2$,定向取外侧



$$I \xrightarrow{F=(2x,y^2,z^2)} \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\overline{\text{sh}} \triangle \vec{x}} \iiint_{\Omega} \text{div} F \, dv$$

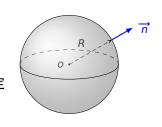
$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} (2x) + \frac{\partial}{\partial y} (y^2) + \frac{\partial}{\partial z} (z^2) \right] dv = \iiint_{\Omega} (2+2y+2z) \, dv$$

$$\xrightarrow{\text{对称性}} \iiint_{\Omega} 2 \, dv$$



$$I = \iint_{\Sigma} 2x \, dy \, dz + y^2 \, dz \, dx + z^2 \, dx \, dy$$

其中定向曲面 Σ 是球面 $x^2 + y^2 + z^2 = R^2$,定向取外侧



$$I \xrightarrow{F=(2x,y^2,z^2)} \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\underline{a}\underline{y} \cdot \underline{y}} \iiint_{\Omega} \operatorname{div} F \, dv$$

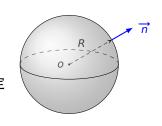
$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} (2x) + \frac{\partial}{\partial y} (y^2) + \frac{\partial}{\partial z} (z^2) \right] \, dv = \iiint_{\Omega} (2+2y+2z) \, dv$$

$$\xrightarrow{\underline{y}\underline{n}\underline{n}\underline{n}} \iiint_{\Omega} 2 \, dv = 2 \operatorname{Vol}(\Omega)$$



$$I = \iint_{\Sigma} 2x \, dy \, dz + y^2 \, dz \, dx + z^2 \, dx \, dy$$

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$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} (2x) + \frac{\partial}{\partial y} (y^2) + \frac{\partial}{\partial z} (z^2) \right] \, dv = \iiint_{\Omega} (2+2y+2z) \, dv$$

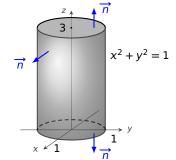
$$\xrightarrow{\text{对称性}} \iiint_{\Omega} 2 \, dv = 2 \text{Vol}(\Omega) = \frac{8}{3} \pi R^3$$



例2计算

$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$

其中定向曲面 Σ 是右图柱体的边界曲面



$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$

其中定向曲面 Σ 是右图柱体的边界曲面

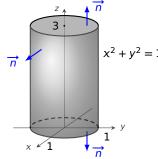
$$x^2 + y^2 = 1$$

$$I = \iiint_{\Sigma} F \cdot \overrightarrow{n} \, dS$$



$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$

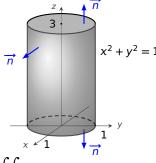
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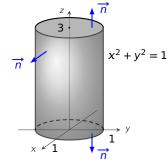
$$I = F = ((y-z)x,0,x-y)$$

$$\iint_{\Sigma} F \cdot \overrightarrow{n} dS = \overline{\text{S斯公式}} \iiint_{\Omega} \text{div} F dv$$



$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$

其中定向曲面 Σ 是右图柱体的边界曲面



$$I = \frac{F = ((y-z)x, 0, x-y)}{\int \int_{\Sigma} F \cdot \overrightarrow{n} dS} = \frac{\overline{\text{s斯公式}}}{\int \int_{\Omega} \text{div} F dv}$$
$$= \left[\int \int_{\Omega} \left[\frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv$$

$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$

其中定向曲面 Σ 是右图柱体的边界曲面

$$x^{2} + y^{2} = 1$$

$$I \xrightarrow{F = ((y-z)x, 0, x-y)} \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\overline{\text{sh}} \triangle x} \iiint_{\Omega} \text{div} F dv$$

$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dz$$



$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$

$$I \xrightarrow{F = ((y-z)x, 0, x-y)} \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{a}\underline{m} \triangle \underline{x}} \iiint_{\Omega} div F dv$$
$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dz$$



$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$

其中定向曲面 Σ 是右图柱体的边界曲面

$$x^{2} + y^{2} = 1$$

$$I = \frac{F = ((y-z)x, 0, x-y)}{\int \int_{\Sigma} F \cdot \overrightarrow{n} dS} = \frac{\overline{\sin} \Delta \overrightarrow{x}}{\int \int_{\Omega} \operatorname{div} F dv}$$

$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dz$$

$$= \frac{\overline{y} \pi t}{\int \int_{\Omega} -z dx dy dz} = \int \left[\int \int_{\Omega} -z dx dy \right] dz$$



$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$

其中定向曲面 Σ 是右图柱体的边界曲面

$$\overrightarrow{n}$$

$$x^2 + y^2 = 1$$

$$x + y^2 = 1$$

$$I \xrightarrow{F = ((y-z)x, 0, x-y)} \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{\operatorname{SH}} \subseteq X} \iiint_{\Omega} \operatorname{div} F dv$$

$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dz$$

$$\xrightarrow{\underline{\operatorname{MMM}}} \iiint_{\Omega} \left[-z dx dy dz \right] = \left[\iint_{\Omega} -z dx dy dz \right]$$



$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$

其中定向曲面 Σ 是右图柱体的边界曲面

$$\overrightarrow{n}$$

$$x^2 + y^2 = 1$$

$$x + y + y^2 = 1$$

$$I \xrightarrow{F = ((y-z)x, 0, x-y)} \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{a}\underline{m} \underline{C}\underline{C}} \iiint_{\Omega} div F dv$$

$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dz$$

$$\xrightarrow{\underline{M}\underline{M}\underline{C}} \iiint_{\Omega} -z dx dy dz = \int_{\Omega} \left[\iint_{\Omega} -z dx dy \right] dz$$



$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$

其中定向曲面 Σ 是右图柱体的边界曲面

$$x^{2} + y^{2} = 1$$

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$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$

其中定向曲面 Σ 是右图柱体的边界曲面

$$x^{2} + y^{2} = 1$$

$$x^{2} + y^{2} = 1$$

$$x^{2} + y^{2} = 1$$

$$I = \frac{F = ((y-z)x, 0, x-y)}{\int \int_{\Sigma} F \cdot \overrightarrow{n} dS} = \frac{\overline{\sin \Delta x}}{\int \int_{\Omega} \operatorname{div} F dv}$$

$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dz$$

$$= \frac{\overline{\sin \Delta x}}{\int \int \int -z dx dy dz} = \int_{0}^{3} \left[\int \int -z dx dy \right] dz$$



別
$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$

其中定向曲面 Σ 是右图柱体的边界曲面

$$x^{2} + y^{2} = 1$$

$$x^{2} + y^{2} = 1$$

$$x^{2} + y^{2} = 1$$

$$I \xrightarrow{F = ((y-z)x, 0, x-y)} \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\overline{\text{S}} \text{M} \triangle \vec{x}} \iiint_{\Omega} \text{div} F dv$$

$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dz$$

$$\xrightarrow{\underline{\text{M}} \text{M} \stackrel{\underline{\text{M}}}{\underline{\text{C}}}} \iiint_{\Omega} -z dx dy dz = \int_{0}^{3} \left[\iint_{D_{z}} -z dx dy \right] dz$$



$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$

其中定向曲面 Σ 是右图柱体的边界曲面

$$x^{2} + y^{2} = 1$$

$$x^{2} + y^{2} = 1$$

$$x^{2} + y^{2} = 1$$

$$I \xrightarrow{F = ((y-z)x, 0, x-y)} \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{a}\underline{m} \underline{\wedge}\underline{\wedge}\underline{\wedge}} \iiint_{\Omega} div F dv$$
$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dz$$

$$= \iiint_{\Omega} \left[\frac{1}{\partial x} ((y-z)x) + \frac{1}{\partial y} + \frac{1}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dz$$

$$= \iiint_{\Omega} \left[-z dx dy dz \right] = \int_{0}^{3} \left[\iint_{D_{z}} -z dx dy \right] dz - z |D_{z}|$$

 $-z|D_z|$

$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$

其中定向曲面 Σ 是右图柱体的边界曲面

$$x^{2} + y^{2} = 1$$

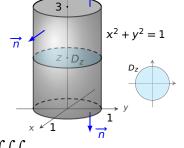
$$x^{2} + y^{2} = 1$$

$$x + y^{2} = 1$$



$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$

其中定向曲面 Σ 是右图柱体的边界曲面



$$I \xrightarrow{F = ((y-z)x, 0, x-y)} \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{a}\underline{n} \subseteq \underline{S}\underline{n}} \iiint_{\Omega} div F dv$$
$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dz$$

$$\frac{\forall m \neq 1}{m} \iiint_{\Omega} -z dx dy dz = \int_{0}^{3} \left[\iint_{D_{z}} -z dx dy \right] dz = \int_{0}^{3} \left[-z |D_{z}| \right] dz$$

$$= \int_{0}^{3} \left[-z \pi \right] dz$$



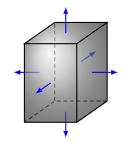
$$I = \iint_{\Sigma} (x - y) dx dy + (y - z) x dy dz$$

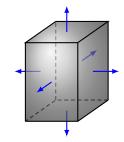
其中定向曲面 Σ 是右图柱体的边界曲面

I = F = ((y-z)x, 0, x-y) $\iiint_{\Sigma} F \cdot \overrightarrow{n} dS = \overline{\text{sh公式}} \iiint_{\Sigma} \text{div} F dv$ $= \iiint_{\Omega} \left[\frac{\partial}{\partial x} ((y-z)x) + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} (x-y) \right] dv = \iiint_{\Omega} y - z dx dy dz$

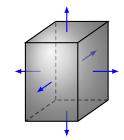
<u> 対称性</u> $\iiint_{\Omega} -z dx dy dz = \int_{0}^{3} \left[\iint_{\Omega} -z dx dy \right] dz = \int_{0}^{3} \left[-z |D_{z}| \right] dz$

 $= \int_0^3 \left[-z\pi \right] dz = -\frac{9}{2}\pi$



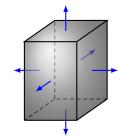


$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS$$

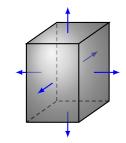


$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} dS \stackrel{\underline{=}\underline{\text{miss}}}{-} \iiint_{\Omega} \text{div} F dv$$





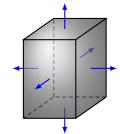
$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\underline{\underline{a}}\underline{\underline{m}}\underline{\underline{M}}} \iiint_{\Omega} \operatorname{div} F \, dv$$
$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} (x - y^2) + \frac{\partial}{\partial y} (y) + \frac{\partial}{\partial z} (z^3) \right] \, dv$$



$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\underline{a}\underline{m}\underline{\omega}\underline{\omega}} \iiint_{\Omega} \operatorname{div} F \, dv$$

$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} (x - y^2) + \frac{\partial}{\partial y} (y) + \frac{\partial}{\partial z} (z^3) \right] dv$$

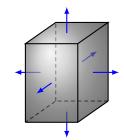
$$= \iiint_{\Omega} (2 + 3z^2) dx \, dy \, dz$$



$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\underline{\underline{a}}\underline{\underline{m}}\underline{\underline{M}}\underline{\underline{M}}} \iiint_{\Omega} \operatorname{div} F \, dv$$

$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} (x - y^2) + \frac{\partial}{\partial y} (y) + \frac{\partial}{\partial z} (z^3) \right] \, dv$$

$$= \iiint_{\Omega} (2 + 3z^2) \, dx \, dy \, dz = \int_{\Omega} \left[\int_{\Omega} \left[\int_{\Omega} (2 + 3z^2) \, dz \, dy \, dx \right] \, dx$$

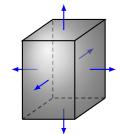


$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\underline{\underline{a}}\underline{\underline{m}}\underline{\underline{M}}\underline{\underline{M}}} \iiint_{\Omega} \operatorname{div} F \, dv$$

$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} (x - y^2) + \frac{\partial}{\partial y} (y) + \frac{\partial}{\partial z} (z^3) \right] \, dv$$

$$= \iiint_{\Omega} (2 + 3z^2) \, dx \, dy \, dz = \int_{0}^{1} \left[\int_{\Omega} \left[\int_{\Omega} (2 + 3z^2) \, dz \right] \, dy \right] \, dx$$

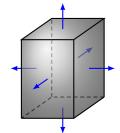




$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\underline{\underline{a}}\underline{\underline{m}}\underline{\underline{M}}\underline{\underline{M}}} \iiint_{\Omega} \operatorname{div} F \, dv$$

$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} (x - y^2) + \frac{\partial}{\partial y} (y) + \frac{\partial}{\partial z} (z^3) \right] \, dv$$

$$= \iiint_{\Omega} (2 + 3z^2) \, dx \, dy \, dz = \int_{\Omega}^{1} \left[\int_{1}^{2} \left[\int_{1}^{2} (2 + 3z^2) \, dz \right] \, dy \right] \, dx$$

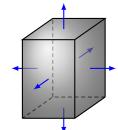


$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\underline{\underline{a}}\underline{\underline{m}}\underline{\underline{M}}\underline{\underline{M}}} \iiint_{\Omega} \operatorname{div} F \, dv$$

$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} (x - y^2) + \frac{\partial}{\partial y} (y) + \frac{\partial}{\partial z} (z^3) \right] \, dv$$

$$= \iiint_{\Omega} (2 + 3z^2) \, dx \, dy \, dz = \int_{\Omega}^{1} \left[\int_{1}^{2} \left[\int_{1}^{4} (2 + 3z^2) \, dz \right] \, dy \right] \, dx$$





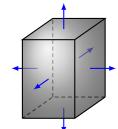
$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\underline{a}\underline{m}\underline{\omega}\underline{\omega}} \iiint_{\Omega} \operatorname{div} F \, dv$$

$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} (x - y^2) + \frac{\partial}{\partial y} (y) + \frac{\partial}{\partial z} (z^3) \right] \, dv$$

$$= \iiint_{\Omega} (2 + 3z^2) \, dx \, dy \, dz = \int_{0}^{1} \left[\int_{1}^{2} \left[\int_{1}^{4} (2 + 3z^2) \, dz \right] \, dy \right] \, dx$$

$$= \int_{0}^{1} 1 \, dx \cdot \int_{1}^{2} 1 \, dy \cdot \int_{1}^{4} (2 + 3z^2) \, dz$$





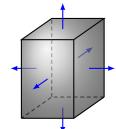
$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{a}\underline{m}\underline{c}\underline{m}} \iiint_{\Omega} \operatorname{div} F dv$$

$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} (x - y^2) + \frac{\partial}{\partial y} (y) + \frac{\partial}{\partial z} (z^3) \right] dv$$

$$= \iiint_{\Omega} (2 + 3z^2) dx dy dz = \int_{0}^{1} \left[\int_{1}^{2} \left[\int_{1}^{4} (2 + 3z^2) dz \right] dy \right] dx$$

$$= \int_{0}^{1} 1 dx \cdot \int_{1}^{2} 1 dy \cdot \int_{1}^{4} (2 + 3z^2) dz = 1 \cdot 1 \cdot (2z + z^3) \Big|_{1}^{4}$$





$$\Phi = \iint_{\Sigma} F \cdot \overrightarrow{n} \, dS \xrightarrow{\underline{a}\underline{y}\underline{y}} \iiint_{\Omega} \operatorname{div} F \, dv$$

$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} (x - y^2) + \frac{\partial}{\partial y} (y) + \frac{\partial}{\partial z} (z^3) \right] \, dv$$

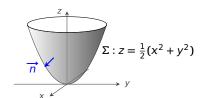
$$= \iiint_{\Omega} (2 + 3z^2) \, dx \, dy \, dz = \int_{0}^{1} \left[\int_{1}^{2} \left[\int_{1}^{4} (2 + 3z^2) \, dz \right] \, dy \right] \, dx$$

$$= \int_{0}^{1} 1 \, dx \cdot \int_{1}^{2} 1 \, dy \cdot \int_{1}^{4} (2 + 3z^2) \, dz = 1 \cdot 1 \cdot (2z + z^3) \Big|_{1}^{4} = 69$$



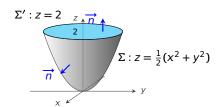
$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

其中定向曲面 Σ 是抛物面的一部分, 取单位外法向量,如图:



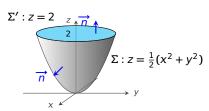
$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

其中定向曲面 Σ 是抛物面的一部分, 取单位外法向量,如图:



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其中定向曲面 Σ 是抛物面的一部分, 取单位外法向量,如图:

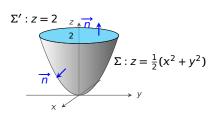


原式 =
$$\iint_{\Sigma \cup \Sigma'} F \cdot \overrightarrow{n} \, dS - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$



$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

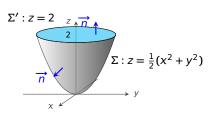
其中定向曲面 Σ 是抛物面的一部分, 取单位外法向量,如图:



原式 =
$$\iint_{\Sigma \cup \Sigma'} F \cdot \overrightarrow{n} \, dS - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$
$$= \iiint_{\Omega} \operatorname{div} F \, dV$$

$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

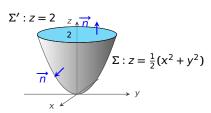
其中定向曲面 Σ 是抛物面的一部分, 取单位外法向量,如图:



原式 =
$$\iint_{\Sigma \cup \Sigma'} F \cdot \overrightarrow{n} \, dS - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$
$$= \iiint_{\Omega} \operatorname{div} F \, dv - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$

$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

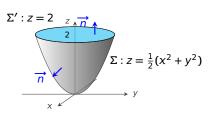
其中定向曲面 Σ 是抛物面的一部分, 取单位外法向量,如图:



原式 =
$$\iint_{\Sigma \cup \Sigma'} F \cdot \overrightarrow{n} \, dS - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$
$$= \iiint_{\Omega} \operatorname{div} F \, dv - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$
$$\underline{F} = (z^2 + x, 0, -z)$$

$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

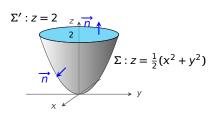
其中定向曲面 Σ 是抛物面的一部分, 取单位外法向量,如图:



原式 =
$$\iint_{\Sigma \cup \Sigma'} F \cdot \overrightarrow{n} \, dS - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$
$$= \iiint_{\Omega} \operatorname{div} F \, dv - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$
$$\underbrace{\frac{F = (z^2 + x, 0, -z)}{\operatorname{div} F = 0}}$$

$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

其中定向曲面 Σ 是抛物面的一部分, 取单位外法向量,如图:



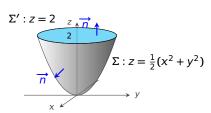
原式 =
$$\iint_{\Sigma \cup \Sigma'} F \cdot \overrightarrow{n} \, dS - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$
=
$$\iiint_{\Omega} \operatorname{div} F \, dv - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$

$$\frac{F = (z^2 + x, 0, -z)}{\operatorname{div} F = 0} - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$

例4计算

$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

其中定向曲面 Σ 是抛物面的一部分, 取单位外法向量,如图:



原式 =
$$\iint_{\Sigma \cup \Sigma'} F \cdot \overrightarrow{n} \, dS - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$
=
$$\iiint_{\Omega} \operatorname{div} F \, dv - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$

$$\frac{F = (z^2 + x, 0, -z)}{\operatorname{div} F = 0} - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$

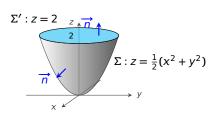
$$\overrightarrow{n} = (0, 0, 1)$$



例4计算

$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

其中定向曲面 Σ 是抛物面的一部分, 取单位外法向量,如图:



原式 =
$$\iint_{\Sigma \cup \Sigma'} F \cdot \overrightarrow{n} \, dS - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$
=
$$\iiint_{\Omega} \operatorname{div} F \, dv - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$

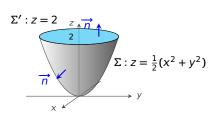
$$\frac{F = (z^2 + x, 0, -z)}{\operatorname{div} F = 0} - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$

$$\frac{\overrightarrow{n} = (0, 0, 1)}{F \cdot \overrightarrow{n} = -z}$$

例 4 计算

$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

其中定向曲面 Σ 是抛物面的一部分, 取单位外法向量,如图:



原式 =
$$\iint_{\Sigma \cup \Sigma'} F \cdot \overrightarrow{n} \, dS - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$
=
$$\iiint_{\Omega} \operatorname{div} F \, dv - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$

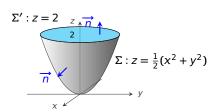
$$\frac{F = (z^2 + x, 0, -z)}{\operatorname{div} F = 0} - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$

$$\frac{\overrightarrow{n} = (0, 0, 1)}{F \cdot \overrightarrow{n} = -z} - \iint_{\Sigma'} -2 \, dS$$

例4计算

$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

其中定向曲面 Σ 是抛物面的一部分, 取单位外法向量,如图:



原式 =
$$\iint_{\Sigma \cup \Sigma'} F \cdot \overrightarrow{n} \, dS - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$
=
$$\iiint_{\Omega} \operatorname{div} F \, dv - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$

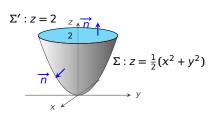
$$\frac{F = (z^2 + x, 0, -z)}{\operatorname{div} F = 0} - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$

$$\frac{\overrightarrow{n} = (0, 0, 1)}{F \cdot \overrightarrow{n} = -z} - \iint_{\Sigma'} -2 \, dS = 2 \operatorname{Area}(\Sigma')$$

例4计算

$$\iint_{\Sigma} (z^2 + x) dy dz - z dx dy$$

其中定向曲面 Σ 是抛物面的一部分, 取单位外法向量,如图:



原式 =
$$\iint_{\Sigma \cup \Sigma'} F \cdot \overrightarrow{n} \, dS - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$

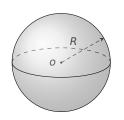
$$= \iiint_{\Omega} \operatorname{div} F \, dV - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$

$$\frac{F = (z^2 + x, 0, -z)}{\operatorname{div} F = 0} - \iint_{\Sigma'} F \cdot \overrightarrow{n} \, dS$$

$$\frac{\overrightarrow{n} = (0, 0, 1)}{F \cdot \overrightarrow{n} - z} - \iint_{\Sigma'} -2dS = 2\operatorname{Area}(\Sigma') = 8\pi$$

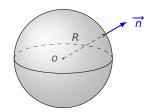
$$I = \iint_{\Sigma} (x^2 + y + z) dS$$

其中曲面 Σ 是球面 $x^2 + y^2 + z^2 = R^2$



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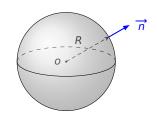


解

$$\iint_{\Sigma} (x^2 + y + z) dS$$

$$= \iint_{\Sigma} F \cdot \overrightarrow{n} dS \stackrel{\overline{=} \overline{\text{miss}}}{=} \iiint_{\Omega} \text{div} F dv$$

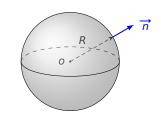
$$I = \iint_{\Sigma} (x^2 + y + z) dS$$



解 球面单位外法向量
$$\overrightarrow{n} = \frac{1}{R}(x, y, z)$$
,所以
$$\iint_{\Sigma} (x^2 + y + z) dS$$

$$= \iint_{\Gamma} F \cdot \overrightarrow{n} dS \xrightarrow{\overline{\text{Sh}} \subseteq \mathbb{Z}} \iiint_{\Gamma} \text{div} F dv$$

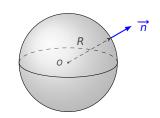
$$I = \iint_{\Sigma} (x^2 + y + z) dS$$



解 球面单位外法向量
$$\overrightarrow{n} = \frac{1}{R}(x, y, z),$$
 所以
$$\iint_{\Sigma} (x^2 + y + z) dS \qquad (, ,) \cdot \frac{1}{R}(x, y, z)$$

$$= \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\overline{\text{Sh}} \triangle X} \iiint_{\Omega} \text{div} F dv$$

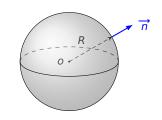
$$I = \iint_{\Sigma} (x^2 + y + z) dS$$



解 球面单位外法向量
$$\overrightarrow{n} = \frac{1}{R}(x, y, z)$$
,所以
$$\iint_{\Sigma} (x^2 + y + z) dS \qquad R(x, 1, 1) \cdot \frac{1}{R}(x, y, z)$$

$$= \iint_{\Gamma} F \cdot \overrightarrow{n} dS \xrightarrow{\overline{\text{SH}} \subseteq \mathbb{Z}} \iiint_{\Omega} \text{div} F dv$$

$$I = \iint_{\Sigma} (x^2 + y + z) dS$$

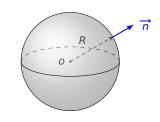


解 球面单位外法向量
$$\overrightarrow{n} = \frac{1}{R}(x, y, z)$$
,所以
$$\iint_{\Sigma} (x^2 + y + z) dS = \iint_{\Sigma} R(x, 1, 1) \cdot \frac{1}{R}(x, y, z) dS$$

$$= \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\overline{\text{B}} \text{M} \triangle \text{T}} \iiint_{\Omega} \text{div} F dV$$

$$I = \iint_{\Sigma} (x^2 + y + z) dS$$

其中曲面 Σ 是球面 $x^2 + y^2 + z^2 = R^2$



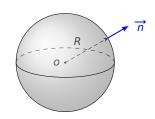
解 球面单位外法向量
$$\overrightarrow{n} = \frac{1}{R}(x, y, z)$$
,所以
$$\iint_{\Sigma} (x^2 + y + z) dS = \iint_{\Sigma} R(x, 1, 1) \cdot \frac{1}{R}(x, y, z) dS$$

$$= \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{n} + \underline{n}} \iiint_{\Omega} \operatorname{div} F dv$$

$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} (Rx) + \frac{\partial}{\partial y} (R) + \frac{\partial}{\partial z} (R) \right] dv$$

$$I = \iint_{\Sigma} (x^2 + y + z) dS$$

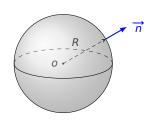
其中曲面 Σ 是球面 $x^2 + y^2 + z^2 = R^2$



解 球面单位外法向量
$$\overrightarrow{n} = \frac{1}{R}(x, y, z)$$
,所以
$$\iint_{\Sigma} (x^2 + y + z) dS = \iint_{\Sigma} R(x, 1, 1) \cdot \frac{1}{R}(x, y, z) dS$$
$$= \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{\text{S}} \underline{\text{M}} \triangle \underline{\text{T}}} \iiint_{\Omega} \text{div} F dv$$
$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} (Rx) + \frac{\partial}{\partial y} (R) + \frac{\partial}{\partial z} (R) \right] dv$$
$$= \iiint_{\Omega} R dx dy dz$$

$$I = \iint_{\Sigma} (x^2 + y + z) dS$$

其中曲面 Σ 是球面 $x^2 + y^2 + z^2 = R^2$



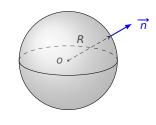
解 球面单位外法向量
$$\overrightarrow{n} = \frac{1}{R}(x, y, z)$$
,所以
$$\iint_{\Sigma} (x^2 + y + z) dS = \iint_{\Sigma} R(x, 1, 1) \cdot \frac{1}{R}(x, y, z) dS$$

$$= \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{a} \underline{n} \underline{n} \underline{n}} \iiint_{\Omega} \mathrm{div} F dv$$

$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} (Rx) + \frac{\partial}{\partial y} (R) + \frac{\partial}{\partial z} (R) \right] dv$$

$$= \iiint_{\Omega} R dx dy dz = R \mathrm{Vol}(\Omega)$$

$$I = \iint_{\Sigma} (x^2 + y + z) dS$$

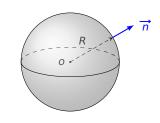


其中曲面 Σ 是球面 $x^2 + y^2 + z^2 = R^2$

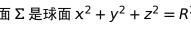
解 球面单位外法向量
$$\overrightarrow{n} = \frac{1}{R}(x, y, z)$$
,所以
$$\iint_{\Sigma} (x^2 + y + z) dS = \iint_{\Sigma} R(x, 1, 1) \cdot \frac{1}{R}(x, y, z) dS$$
$$= \iint_{\Sigma} F \cdot \overrightarrow{n} dS \xrightarrow{\underline{n} \underline{n} \underline{n}} \iiint_{\Omega} \operatorname{div} F dv$$
$$= \iiint_{\Omega} \left[\frac{\partial}{\partial x} (Rx) + \frac{\partial}{\partial y} (R) + \frac{\partial}{\partial z} (R) \right] dv$$
$$= \iiint_{\Omega} R dx dy dz = R \operatorname{Vol}(\Omega) = \frac{4}{3} \pi R^4$$

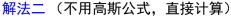
$$I = \iint_{\Sigma} (x^2 + y + z) dS$$

其中曲面 Σ 是球面 $x^2 + y^2 + z^2 = R^2$

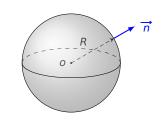


$$I = \iint_{\Sigma} (x^2 + y + z) dS$$



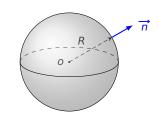


$$\iint_{\Sigma} (x^2 + y + z) dS \xrightarrow{\text{symbol}} \iint_{\Sigma} x^2 dS$$



$$I = \iint_{\Sigma} (x^2 + y + z) dS$$

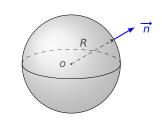
其中曲面 Σ 是球面 $x^2 + y^2 + z^2 = R^2$



$$\iint_{\Sigma} (x^2 + y + z) dS \xrightarrow{\text{span}} \iint_{\Sigma} x^2 dS = \frac{1}{3} \iint_{\Sigma} (x^2 + x^2 + x^2) dS$$

$$I = \iint_{\Sigma} (x^2 + y + z) dS$$

其中曲面 Σ 是球面 $x^2 + y^2 + z^2 = R^2$

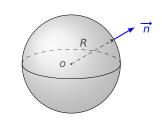


$$\iint_{\Sigma} (x^2 + y + z) dS \xrightarrow{\text{span}} \iint_{\Sigma} x^2 dS = \frac{1}{3} \iint_{\Sigma} (x^2 + x^2 + x^2) dS$$

$$\xrightarrow{\text{span}} \frac{1}{3} \iint_{\Sigma} (x^2 + y^2 + z^2) dS$$

$$I = \iint_{\Sigma} (x^2 + y + z) dS$$

其中曲面 Σ 是球面 $x^2 + y^2 + z^2 = R^2$



$$\iint_{\Sigma} (x^2 + y + z) dS \xrightarrow{\text{print}} \iint_{\Sigma} x^2 dS = \frac{1}{3} \iint_{\Sigma} (x^2 + x^2 + x^2) dS$$

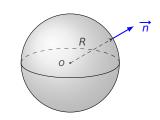
$$\xrightarrow{\text{print}} \frac{1}{3} \iint_{\Sigma} (x^2 + y^2 + z^2) dS$$

$$= \frac{1}{3} \iint_{\Sigma} R^2 dS$$



$$I = \iint_{\Sigma} (x^2 + y + z) dS$$

其中曲面 Σ 是球面 $x^2 + y^2 + z^2 = R^2$



$$\iint_{\Sigma} (x^2 + y + z) dS \xrightarrow{\text{対称性}} \iint_{\Sigma} x^2 dS = \frac{1}{3} \iint_{\Sigma} (x^2 + x^2 + x^2) dS$$

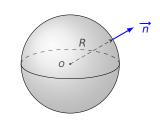
$$\xrightarrow{\text{対称性}} \frac{1}{3} \iint_{\Sigma} (x^2 + y^2 + z^2) dS$$

$$= \frac{1}{3} \iint_{\Sigma} R^2 dS = \frac{1}{3} R^2 \text{Area}(\Sigma)$$



$$I = \iint_{\Sigma} (x^2 + y + z) dS$$

其中曲面 Σ 是球面 $x^2 + y^2 + z^2 = R^2$



$$\iint_{\Sigma} (x^2 + y + z) dS \xrightarrow{\text{print}} \iint_{\Sigma} x^2 dS = \frac{1}{3} \iint_{\Sigma} (x^2 + x^2 + x^2) dS$$

$$\xrightarrow{\text{print}} \frac{1}{3} \iint_{\Sigma} (x^2 + y^2 + z^2) dS$$

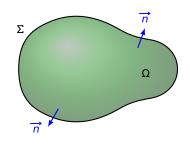
$$= \frac{1}{3} \iint_{\Sigma} R^2 dS = \frac{1}{3} R^2 \text{Area}(\Sigma) = \frac{4}{3} \pi R^4$$



高斯公式
$$\iiint_{\Omega} \operatorname{div} F dv = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$



假设 F = (P, Q, R) 是流体的速度向量场。



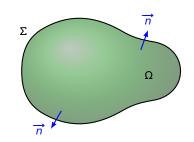
高斯公式 $\iiint_{\Omega} \operatorname{div} F dv = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$



假设 F = (P, Q, R) 是流体的速度向量场,则

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS$$

表示单位时间流向 Σ 外侧的通量。



高斯公式
$$\iiint_{\Omega} \operatorname{div} F dv = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$

 假设 F = (P, Q, R) 是流体的速度向 量场,则

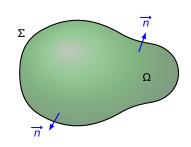
$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS$$

表示单位时间流向 Σ 外侧的通量。

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS > 0$$

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS < 0$$

高斯公式
$$\iiint_{\Omega} \operatorname{div} F dv = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$



假设 F = (P, Q, R) 是流体的速度向量场,则

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS$$

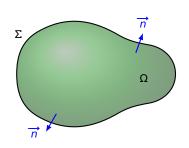
表示单位时间流向Σ外侧的通量。

• 进一步假设流体是不可压,则

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS > 0$$

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS < 0$$

高斯公式
$$\iiint_{\Omega} \operatorname{div} F dv = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$



 假设 F = (P, Q, R) 是流体的速度向 量场,则

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS$$

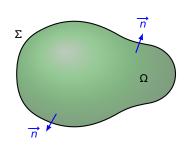
表示单位时间流向 Σ 外侧的通量。

• 进一步假设流体是不可压,则

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS > 0 \Rightarrow \Omega \, \text{内有 "source"}$$

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS < 0$$

高斯公式
$$\iiint_{\Omega} \operatorname{div} F dv = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$



假设 F = (P, Q, R) 是流体的速度向量场,则

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS$$

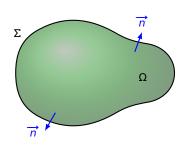
表示单位时间流向 Σ 外侧的通量。

• 进一步假设流体是不可压,则

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS > 0 \Rightarrow \Omega \, \text{内有 "source"}$$

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS < 0 \Rightarrow \Omega \, \text{内有 "sink"}$$

高斯公式
$$\iiint_{\Omega} \operatorname{div} F dv = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$$



 假设 F = (P, Q, R) 是流体的速度向 量场,则

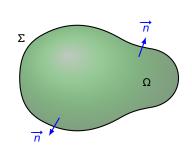
$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS$$

表示单位时间流向Σ外侧的通量。

• 进一步假设流体是不可压,则

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS > 0 \Rightarrow \Omega \, \text{内有 "source"}$$

$$\iint_{\Sigma} F \cdot \overrightarrow{n} \, dS < 0 \Rightarrow \Omega \, \text{内有 "sink"}$$



 $\mathbf{\hat{z}}$ 高斯公式 $\iiint_{\Omega} \operatorname{div} F dv = \iint_{\Sigma} F \cdot \overrightarrow{n} dS$ 表明: $\operatorname{div} F$ 反映这种"source"

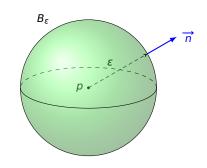
和"sink"的强度。



p •

div F(p)



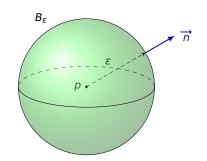


divF(p)



$$\iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \iiint_{B_{\varepsilon}} \operatorname{div} F \, dV$$



divF(p)



$$\iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$\iiint_{B_{\varepsilon}} \operatorname{div} F \, dV$$

$$= \operatorname{Vol}(B_{\varepsilon}) \operatorname{div} F(p_{\varepsilon})$$

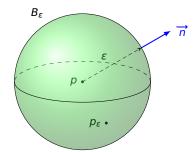




$$\frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} dS$$

$$= \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \iiint_{B_{\varepsilon}} \operatorname{div} F dV$$

$$= \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \operatorname{Vol}(B_{\varepsilon}) \operatorname{div} F(p_{\varepsilon})$$



divF(p)

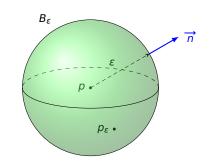
$$\frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \iiint_{B_{\varepsilon}} \operatorname{div} F \, dv$$

$$= \frac{1}{\operatorname{Vol}(B_{\varepsilon})} \cdot \operatorname{Vol}(B_{\varepsilon}) \operatorname{div} F(p_{\varepsilon})$$

$$= \operatorname{div} F(p_{\varepsilon})$$

$$\operatorname{div} F(p)$$



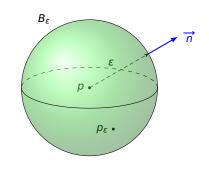
$$\lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \iiint_{B_{\varepsilon}} \text{div} F \, dv$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \text{Vol}(B_{\varepsilon}) \text{div} F(p_{\varepsilon})$$

$$= \lim_{\varepsilon \to 0^{+}} \text{div} F(p_{\varepsilon})$$

$$\text{div} F(p)$$



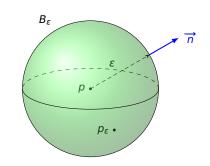
$$\lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \iiint_{B_{\varepsilon}} \text{div} F \, dV$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \text{Vol}(B_{\varepsilon}) \text{div} F(p_{\varepsilon})$$

$$= \lim_{\varepsilon \to 0^{+}} \text{div} F(p_{\varepsilon})$$

$$= \text{div} F(p)$$



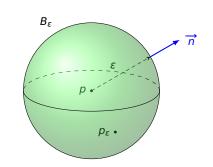
$$\lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \iiint_{B_{\varepsilon}} \text{div} F \, dv$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \text{Vol}(B_{\varepsilon}) \text{div} F(p_{\varepsilon})$$

$$= \lim_{\varepsilon \to 0^{+}} \text{div} F(p_{\varepsilon})$$

$$= \text{div} F(p)$$



- divF(p)>0时,
- divF(p)<0时,



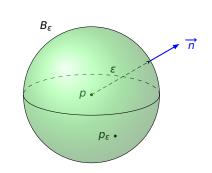
$$\lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \iiint_{B_{\varepsilon}} \text{div} F \, dv$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \text{Vol}(B_{\varepsilon}) \text{div} F(p_{\varepsilon})$$

$$= \lim_{\varepsilon \to 0^{+}} \text{div} F(p_{\varepsilon})$$

$$= \text{div} F(p)$$



- div*F*(*p*)>0 时,∫∫_{∂B}, *F* · *n* dS >0(ε 充分小),
- divF(p)<0时,



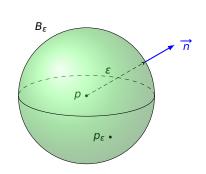
$$\lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \iiint_{B_{\varepsilon}} \text{div} F \, dV$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \text{Vol}(B_{\varepsilon}) \text{div} F(p_{\varepsilon})$$

$$= \lim_{\varepsilon \to 0^{+}} \text{div} F(p_{\varepsilon})$$

$$= \text{div} F(p)$$



- $\operatorname{div} F(p) > 0$ 时, $\iint_{\partial B_{\epsilon}} F \cdot \overrightarrow{n} \, dS > 0$ (ϵ 充分小),说明 p 点是 source
- divF(p)<0时,



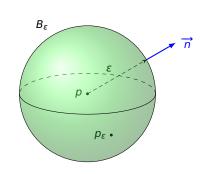
$$\lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \iiint_{B_{\varepsilon}} \text{div} F \, dv$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \text{Vol}(B_{\varepsilon}) \text{div} F(p_{\varepsilon})$$

$$= \lim_{\varepsilon \to 0^{+}} \text{div} F(p_{\varepsilon})$$

$$= \text{div} F(p)$$



- $\operatorname{div} F(p) > 0$ 时, $\iint_{\partial B_{\epsilon}} F \cdot \overrightarrow{n} \, dS > 0$ (ϵ 充分小),说明 p 点是 source
- $\operatorname{div} F(p) < 0$ 时, $\iint_{\partial B_{\epsilon}} F \cdot \overrightarrow{n} \, dS < 0 \ (\epsilon 充分小)$,



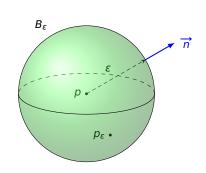
$$\lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \iint_{\partial B_{\varepsilon}} F \cdot \overrightarrow{n} \, dS$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \iiint_{B_{\varepsilon}} \text{div} F \, dv$$

$$= \lim_{\varepsilon \to 0^{+}} \frac{1}{\text{Vol}(B_{\varepsilon})} \cdot \text{Vol}(B_{\varepsilon}) \text{div} F(p_{\varepsilon})$$

$$= \lim_{\varepsilon \to 0^{+}} \text{div} F(p_{\varepsilon})$$

$$= \text{div} F(p)$$



- $\operatorname{div} F(p) > 0$ 时, $\iint_{\partial B_{\epsilon}} F \cdot \overrightarrow{n} \, dS > 0$ (ϵ 充分小),说明 p 点是 source
- $\operatorname{div} F(p) < 0$ 时, $\iint_{\partial B_{\epsilon}} F \cdot \overrightarrow{n} \, dS < 0$ (ϵ 充分小),说明 p 点是 sink



We are here now...

1. 高斯公式

2. 斯托克斯公式

定义 设
$$F = (P, Q, R)$$
 是空间中向量场,定义

$$\operatorname{rot} F := \left| \begin{array}{ccc} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{array} \right|$$

定义 设
$$F = (P, Q, R)$$
 是空间中向量场,定义

$$\operatorname{rot} F := \left| \begin{array}{ccc} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & O & R \end{array} \right| = \left(\left| \begin{array}{ccc} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{array} \right|, , ,$$

定义 设
$$F = (P, Q, R)$$
 是空间中向量场,定义

$$\operatorname{rot} F := \left| \begin{array}{ccc} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & O & R \end{array} \right| = \left(\left| \begin{array}{ccc} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{array} \right|, - \left| \begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{array} \right|,$$

定义 设 F = (P, Q, R) 是空间中向量场, 定义

$$\operatorname{rot} F := \left| \begin{array}{ccc} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{array} \right| = \left(\left| \begin{array}{ccc} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{array} \right|, - \left| \begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{array} \right|, \left| \begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{array} \right| \right)$$



定义 设
$$F = (P, Q, R)$$
 是空间中向量场,定义

$$\operatorname{rot} F := \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & Q \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} \right)$$
$$= (R_{y} - Q_{z}, \qquad , \qquad)$$



定义 设
$$F = (P, Q, R)$$
 是空间中向量场,定义

$$\operatorname{rot} F := \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} \right)$$
$$= (R_{y} - Q_{z}, P_{z} - R_{x},)$$



定义 设 F = (P, Q, R) 是空间中向量场,定义

$$\operatorname{rot} F := \left| \begin{array}{ccc} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{array} \right| = \left(\left| \begin{array}{ccc} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{array} \right|, - \left| \begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{array} \right|, \left| \begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{array} \right| \right)$$

$$= (R_y - Q_z, P_z - R_x, Q_x - P_y)$$

定义 设 F = (P, Q, R) 是空间中向量场,定义

$$\operatorname{rot} F := \left| \begin{array}{ccc} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{array} \right| = \left(\left| \begin{array}{ccc} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{array} \right|, - \left| \begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{array} \right|, \left| \begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{array} \right| \right)$$

 $= (R_{v} - Q_{z}, P_{z} - R_{x}, Q_{x} - P_{v})$

称为向量场 F 的旋度。



定义 设 F = (P, Q, R) 是空间中向量场,定义

$$\operatorname{rot} F := \left| \begin{array}{ccc} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{array} \right| = \left(\left| \begin{array}{ccc} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{array} \right|, - \left| \begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{array} \right|, \left| \begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{array} \right| \right)$$

 $= (R_{V} - Q_{Z}, P_{Z} - R_{X}, Q_{X} - P_{V})$

称为向量场 F 的旋度。

$$\operatorname{rot} F = \left| \begin{array}{ccc} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & e^{xz} \end{array} \right|$$

定义 设 F = (P, Q, R) 是空间中向量场,定义

$$\operatorname{rot} F := \left| \begin{array}{ccc} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{array} \right| = \left(\left| \begin{array}{ccc} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{array} \right|, - \left| \begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{array} \right|, \left| \begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{array} \right| \right)$$

 $= (R_{v} - Q_{z}, P_{z} - R_{x}, Q_{x} - P_{v})$

称为向量场 F 的旋度。

$$\operatorname{rot} F = \begin{vmatrix}
\overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
y & -x & e^{xz}
\end{vmatrix} = \left(\begin{vmatrix}
\frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
-x & e^{xz}
\end{vmatrix}, -\begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\
y & e^{xz}
\end{vmatrix}, \begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\
y & -x
\end{vmatrix}\right)$$



定义 设 F = (P, Q, R) 是空间中向量场, 定义

$$\operatorname{rot} F := \left| \begin{array}{ccc} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{array} \right| = \left(\left| \begin{array}{ccc} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{array} \right|, - \left| \begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{array} \right|, \left| \begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{array} \right| \right)$$

 $= (R_{v} - Q_{z}, P_{z} - R_{x}, Q_{x} - P_{v})$

称为向量场 F 的旋度。

$$\operatorname{rot} F = \begin{vmatrix}
\overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
y & -x & e^{XZ}
\end{vmatrix} = \left(\begin{vmatrix}
\frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
-x & e^{XZ}
\end{vmatrix}, -\begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\
y & e^{XZ}
\end{vmatrix}, \begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\
y & -x
\end{vmatrix}\right)$$



定义 设
$$F = (P, Q, R)$$
 是空间中向量场,定义

$$\operatorname{rot} F := \left| \begin{array}{ccc} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{array} \right| = \left(\left| \begin{array}{ccc} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{array} \right|, - \left| \begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{array} \right|, \left| \begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{array} \right| \right)$$

 $= (R_{v} - Q_{z}, P_{z} - R_{x}, Q_{x} - P_{v})$

称为向量场 F 的旋度。

$$\operatorname{rot} F = \begin{vmatrix}
\overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
y & -x & e^{XZ}
\end{vmatrix} = \left(\begin{vmatrix}
\frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
-x & e^{XZ}
\end{vmatrix}, -\begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\
y & e^{XZ}
\end{vmatrix}, \begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\
y & -x
\end{vmatrix}\right)$$

$$= (0, -ze^{xz},)$$



定义 设
$$F = (P, Q, R)$$
 是空间中向量场,定义

$$\operatorname{rot} F := \left| \begin{array}{ccc} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{array} \right| = \left(\left| \begin{array}{ccc} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{array} \right|, - \left| \begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{array} \right|, \left| \begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{array} \right| \right)$$

称为向量场 F 的旋度。

例 计算向量场 $F = (v_x - x_x, e^{xz})$ 的旋度。

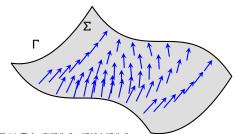
$$\operatorname{rot} F = \left| \begin{array}{ccc} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & e^{xz} \end{array} \right| = \left(\left| \begin{array}{ccc} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -x & e^{xz} \end{array} \right|, - \left| \begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ y & e^{xz} \end{array} \right|, \left| \begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ y & -x \end{array} \right| \right)$$

$$=(0, -ze^{xz}, -2)$$

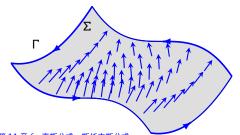
 $= (R_{v} - Q_{z}, P_{z} - R_{x}, Q_{x} - P_{v})$



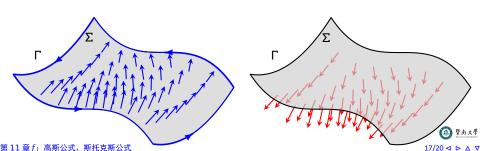
- Σ 是空间中分片光滑的定向曲面,选定单位外法向量 \overrightarrow{n} ,
- Γ是Σ的边界, 且赋予"边界定向",



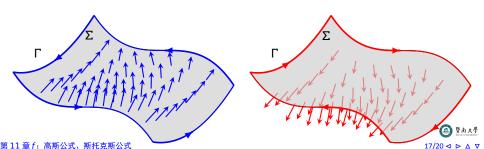
- Σ 是空间中分片光滑的定向曲面,选定单位外法向量 \overrightarrow{n} ,
- Γ是Σ的边界, 且赋予"边界定向",



- Σ 是空间中分片光滑的定向曲面,选定单位外法向量 \overrightarrow{n} ,
- Γ是Σ的边界, 且赋予"边界定向",



- Σ 是空间中分片光滑的定向曲面,选定单位外法向量 \overrightarrow{n} ,
- Γ 是 Σ 的边界, 且赋予 "边界定向",
- F = (P, Q, R) 是空间向量场,且 P, Q, R 具有一阶连续偏导数,

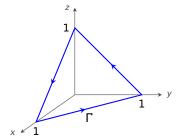


定理(斯托克斯公式) 假设

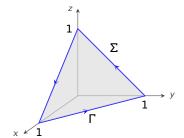
- Σ 是空间中分片光滑的定向曲面,选定单位外法向量 \overrightarrow{n} ,
- Γ是Σ的边界, 且赋予"边界定向",
- F = (P, Q, R) 是空间向量场,且 P, Q, R 具有一阶连续偏导数,

则成立: $\iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} \, dS = \int_{\Gamma} P dx + Q dy + R dz.$

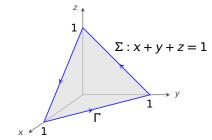
$$I = \int_{\Gamma} z dx + x dy + y dz$$



$$I = \int_{\Gamma} z dx + x dy + y dz$$

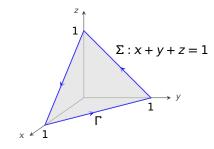


$$I = \int_{\Gamma} z dx + x dy + y dz$$



$$I = \int_{\Gamma} z dx + x dy + y dz$$

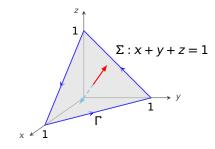
解设
$$F=(z,x,y)$$
,则



所以
$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} dS$$

$$I = \int_{\Gamma} z dx + x dy + y dz$$

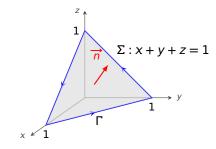
解设
$$F = (z, x, y)$$
,则



所以
$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} dS$$

$$I = \int_{\Gamma} z dx + x dy + y dz$$

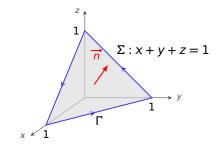
解设
$$F=(z,x,y)$$
,则



所以
$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} dS$$

$$I = \int_{\Gamma} z dx + x dy + y dz$$

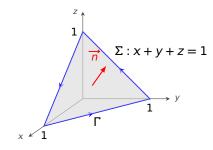
解设
$$F=(z,x,y)$$
,则



所以
$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1, 1, 1)}$$

$$I = \int_{\Gamma} z dx + x dy + y dz$$

解设
$$F = (z, x, y)$$
,则
$$rot F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix}$$



所以
$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)}$$

$$I = \int_{\Gamma} z dx + x dy + y dz$$

解设
$$F = (z, x, y)$$
,则

$$\operatorname{rot} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix}$$

以試利用斯托克斯公式计算
$$I = \int_{\Gamma} z dx + x dy + y dz$$
 中有向曲线 Γ 如图:
$$\mathbb{F} \oplus \{ \mathbf{z} \in \mathbf{z}, \mathbf{x}, \mathbf{y} \}, \mathbf{y} \}$$

$$\mathrm{rot} F = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathbf{z} & \mathbf{x} & \mathbf{y} \end{bmatrix}, - \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \mathbf{z} & \mathbf{x} \end{vmatrix}$$

所以
$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)}$$

$$I = \int_{\Gamma} z dx + x dy + y dz$$

解设
$$F=(z,x,y)$$
,则

$$\operatorname{rot} F = \left| \begin{array}{ccc} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{array} \right| = \left(\left| \begin{array}{ccc} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y \end{array} \right|, - \left| \begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ z & y \end{array} \right|, \left| \begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ z & x \end{array} \right| \right)$$

$$\begin{vmatrix} z & x & y \end{vmatrix}$$

所以
$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Gamma} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)}$$

$$I = \int_{\Gamma} z dx + x dy + y dz$$

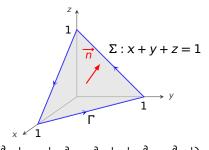
解设
$$F = (z, x, y)$$
,则

$$\operatorname{rot} F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial i} & \frac{\partial}{\partial i} & \frac{\partial}{\partial i} \end{vmatrix}$$

$$\operatorname{rot} F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \left(\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ z & y \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ z & x \end{vmatrix} \right)$$

$$= (1, 1, 1, 1)$$

所以
$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Gamma} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)}$$

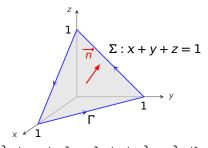


$$I = \int_{\Gamma} z dx + x dy + y dz$$

解设
$$F = (z, x, y)$$
,则

$$\operatorname{rot} F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \left(\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ z & y \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ z & x \end{vmatrix} \right)$$

$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Gamma} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)}$$

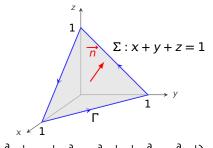


$$I = \int_{\Gamma} z dx + x dy + y dz$$

解设
$$F = (z, x, y)$$
,则

$$\operatorname{rot} F = \left| \begin{array}{ccc} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{array} \right| = \left(\left| \begin{array}{ccc} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y \end{array} \right|, - \left| \begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ z & y \end{array} \right|, \left| \begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ z & x \end{array} \right| \right)$$

$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)} \iint_{\Sigma} \sqrt{3} dS$$



$$I = \int_{\Gamma} z dx + x dy + y dz$$

解设
$$F=(z,x,y)$$
,则

Fig. F = (z, x, y), 则
$$rot F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \left(\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ z & y \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ z & x \end{vmatrix} \right)$$

$$=(1, 1, 1)$$

所以
$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)} \iint_{\Sigma} \sqrt{3} dS$$

$$=\sqrt{3}$$
Area(Σ)



$$I = \int_{\Gamma} z dx + x dy + y dz$$

解设
$$F = (z, x, y)$$
,则

$$\operatorname{rot} F = \begin{pmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{pmatrix} = \begin{pmatrix} \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ z & y \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ z & x \end{vmatrix} \end{pmatrix}$$

$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)} \iint_{\Sigma} \sqrt{3} dS$$

$$= \sqrt{3} \operatorname{Area}(\Sigma) = \sqrt{3} \cdot \frac{1}{2} \cdot \sqrt{2} \cdot \sqrt{2} \cdot \sin \frac{\pi}{3}$$



$$I = \int_{\Gamma} z dx + x dy + y dz$$

其中有向曲线 Γ如图:

解设
$$F = (z, x, y)$$
,则

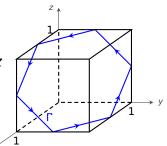
$$\operatorname{rot} F = \begin{pmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y \end{pmatrix}, - \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ z & y \end{pmatrix}, \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ z & x \end{pmatrix} \end{pmatrix}$$

$$\int_{\Gamma} z dx + x dy + y dz = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)} \iint_{\Sigma} \sqrt{3} dS$$

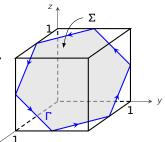
=(1, 1, 1)

$$= \sqrt{3} \text{Area}(\Sigma) = \sqrt{3} \cdot \frac{1}{2} \cdot \sqrt{2} \cdot \sqrt{2} \cdot \sin \frac{\pi}{3} = \frac{3}{2}$$

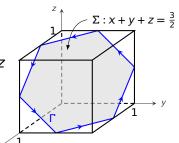
$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$



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其中有向曲线 Γ 如图:

解设
$$F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$
,则

所以
$$I = \iint_{\mathbb{R}^n} \operatorname{rot} F \cdot \overrightarrow{n} \, dS$$



 $\Sigma: x+y+z=\tfrac{3}{2}$

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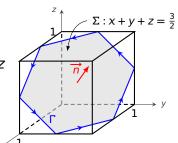


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$$I = \iint_{\Gamma} \operatorname{rot} F \cdot \overrightarrow{n} \, dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)}$$



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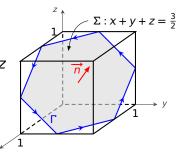
其中有向曲线 Γ如图:

解设
$$F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$
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$$\operatorname{rot} F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix}$$

所以

$$I = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} dS \stackrel{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)}{=}$$



$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$

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解设
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$$rot F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix} = (-2y - 2z, \qquad , \qquad)$$

所以

$$I = \iint_{\Gamma} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)}$$



 $\Sigma : x + y + z = \frac{3}{2}$

$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$

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所以

$$I = \iint_{\Gamma} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)}$$



 $\Sigma : x + y + z = \frac{3}{2}$

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所以

$$I = \iint_{\Gamma} \operatorname{rot} F \cdot \overrightarrow{n} \, dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)}$$



 $\Sigma: x+y+z=\tfrac{3}{2}$

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其中有向曲线 Γ 如图:

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$$F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$
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所以

$$I = \iint_{\Gamma} \operatorname{rot} F \cdot \overrightarrow{n} \, dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)} \frac{-4}{\sqrt{3}} \iint_{\Gamma} (x+y+z) dS$$



 $\Sigma: x+y+z=\tfrac{3}{2}$

$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$

解设
$$F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$
, 则

$$\operatorname{rot} F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix} = (-2y - 2z, -2z - 2x, -2x - 2y)$$

$$I = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} \, dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)} \frac{-4}{\sqrt{3}} \iint_{\Sigma} (x+y+z) dS = \frac{-4}{\sqrt{3}} \iint_{\Sigma} \frac{3}{2} dS$$



$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$

解设 $F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$,则

$$\overrightarrow{k}$$

$$\overrightarrow{k}$$

$$\operatorname{rot} F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix} = (-2y - 2z, -2z - 2x, -2x - 2y)$$

$$-x^2 x^2 - y^2$$

$$z^2 z^2 - x^2 x^2 - y$$

$$z^2 z^2 - x^2 x^2 - y$$

$$I = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} \, dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)} \frac{-4}{\sqrt{3}} \iint_{\Sigma} (x+y+z) dS = \frac{-4}{\sqrt{3}} \iint_{\Sigma} \frac{3}{2} dS$$

$$=-2\sqrt{3}$$
Area(Σ)



例 试利用斯托克斯公式计算
$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$

其中有向曲线 Γ 如图:

解设
$$F = (v^2 - z^2)$$

解设
$$F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$
,则

$$\vec{i}$$

$$\operatorname{rot} F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix} = (-2y - 2z, -2z - 2x, -2x - 2y)$$

$$y^2 - z$$

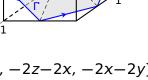
$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial z} =$$

$$\frac{\frac{\partial}{\partial y}}{-x^2} x^2 \frac{\frac{\partial}{\partial z}}{x^2 - y^2}$$

$$\left| \frac{\partial}{\partial z} \right| = (-2y -$$

 $I = \iiint_{-\infty} \cot F \cdot \overrightarrow{n} \, dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)} \frac{-4}{\sqrt{3}} \iiint_{\Sigma} (x+y+z) dS = \frac{-4}{\sqrt{3}} \iiint_{\Sigma} \frac{3}{2} dS$

 $\Sigma: x+y+z=\tfrac{3}{2}$



$$=-2\sqrt{3}$$
Area(Σ)



 $I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$

解设
$$F = (v^2 - z^2 + z^2)$$

$$\mathbf{W}$$
 设 $F = (v^2 - z^2)$

$$(z^2-x^2)$$

解设 $F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$,则

$$\overrightarrow{k}$$

$$\begin{vmatrix} \overrightarrow{k} \\ \frac{\partial}{\partial} \end{vmatrix} = (-2v^{-1})^{\frac{1}{2}}$$

$$rot F = \begin{vmatrix} \frac{\partial}{\partial x} \\ v^2 - z \end{vmatrix}$$

$$\operatorname{rot} F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix} = (-2y - 2z, -2z - 2x, -2x - 2y)$$

所以
$$|y^2 - z^2| z^2 - \lambda$$

$$-z^2 z^2 - x^2 x^2 - x^2$$

$$x^2 - x^2$$
 $x^2 - y^2$

$$\begin{vmatrix} \frac{\partial}{\partial z} \\ -y^2 \end{vmatrix} = (-2y -$$

$$-2x-2y$$

 $\Sigma : x + y + z = \frac{3}{2}$

$$I = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)} \frac{-4}{\sqrt{3}} \iint_{\Sigma} (x+y+z) dS = \frac{-4}{\sqrt{3}} \iint_{\Sigma} \frac{3}{2} dS$$
$$= -2\sqrt{3} \operatorname{Area}(\Sigma) = -2\sqrt{3} \cdot 6 \cdot \frac{1}{2} \cdot \sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2}} \cdot \sin \frac{\pi}{3}$$



 $I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$

解设
$$F = (y^2 - z^2)$$

解设
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解设
$$F = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$$
,则

 $= -2\sqrt{3}\operatorname{Area}(\Sigma) = -2\sqrt{3}\cdot 6\cdot \frac{1}{2}\cdot \sqrt{\frac{1}{2}\cdot \sin\frac{\pi}{3}} = -\frac{9}{2}$

$$\operatorname{rot} F = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix} = (-2y - 2z, -2z - 2x, -2x - 2y)$$

$$|y^2-z|$$
 所以

所以
$$I = \iint_{\Sigma} \operatorname{rot} F \cdot \overrightarrow{n} \, dS \xrightarrow{\overrightarrow{n} = \frac{1}{\sqrt{3}}(1,1,1)} \frac{-4}{\sqrt{3}} \iint_{\Sigma} (x+y+z) \, dS = \frac{-4}{\sqrt{3}} \iint_{\Sigma} \frac{3}{2} \, dS$$

$$m$$
 设 $F = (y^2 - z^2, z^2 - x^2)$

$$\overrightarrow{k}$$

$$\begin{vmatrix} \vec{k} \\ \frac{\partial}{\partial x} \end{vmatrix} = (-2y - 2y)$$

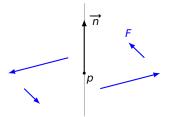
$$\begin{vmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{vmatrix} = (-2y-1)$$

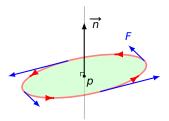
$$\begin{vmatrix} \frac{1}{2} \\ \frac{1}{2} \end{vmatrix} = (-2y-2)$$

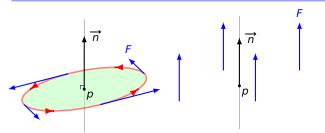
 $\Sigma : x + y + z = \frac{3}{2}$

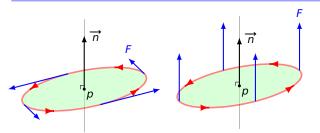


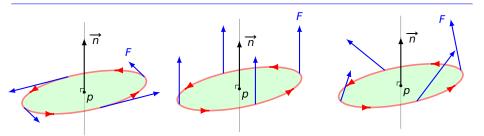


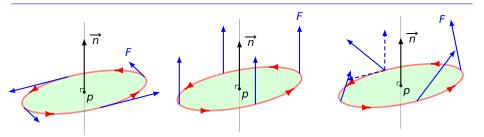


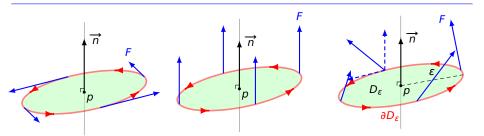


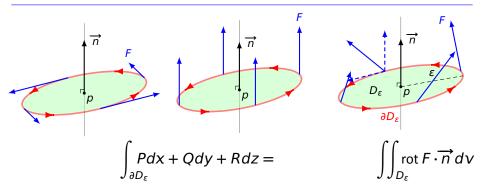


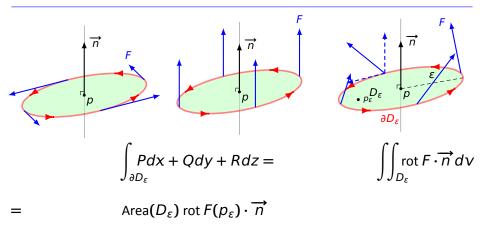


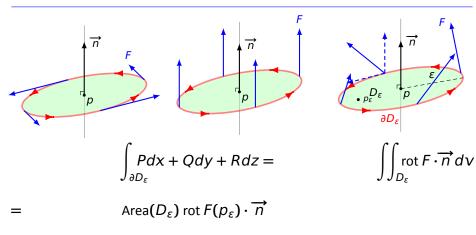




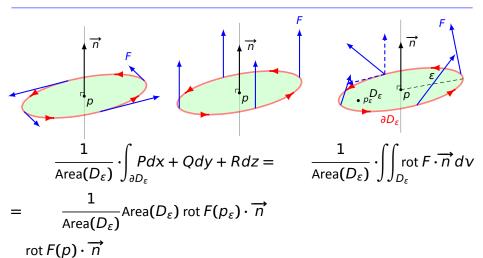




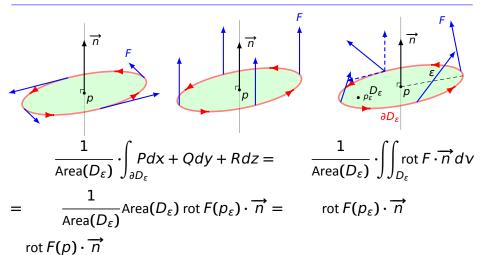




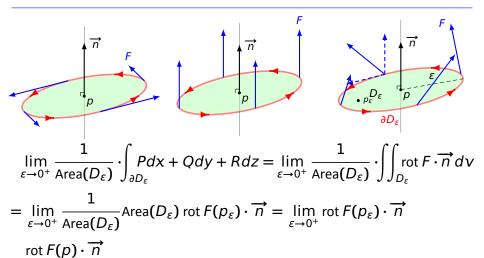
rot $F(p) \cdot \overrightarrow{n}$



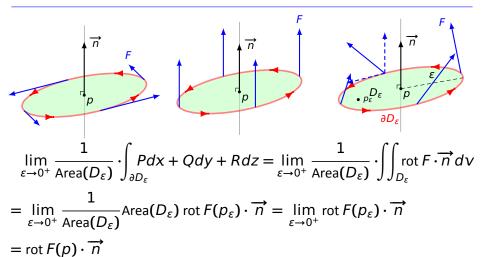




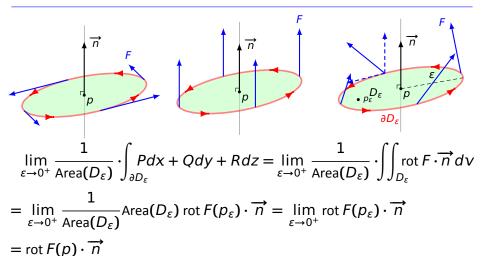






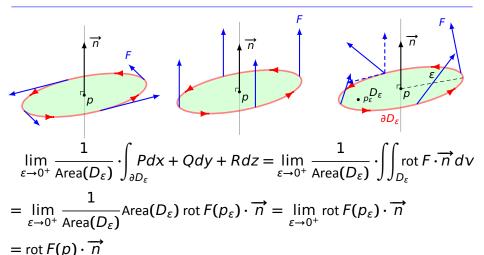






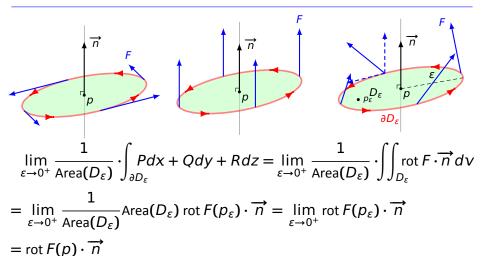
注 cot $F \neq 0$ 说明有旋,





注 $\cot F \neq 0$ 说明有旋,此时可认为 $F \neq p$ 点附近绕轴 $\overrightarrow{n} = \frac{\cot F}{|\cot F|}$ 旋转;





注
$$\cot F \neq 0$$
 说明有旋,此时可认为 $F \neq p$ 点附近绕轴 $\overrightarrow{n} = \frac{\cot F}{|\cot F|}$ 旋转;

 $\cot F = 0$ 说明无旋。