§4.3 实对称矩阵的特征值和特征向量

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本节内容

- ◇ 向量的内积
- ♣ 正交向量组,施密特正交化方法
- ♡ 正交矩阵
- ♠ 对称矩阵可对角化

定义
$$\mathbb{R}^n$$
 中两个向量 $\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$ 和 $\beta = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b \end{pmatrix}$ 的内积定义为:

$$\alpha^T \beta =$$

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, $(k$ 是实数)

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$$(\alpha + \beta)^T \gamma = \alpha^T \gamma + \beta^T \gamma$$

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另证 $\alpha^T \beta = (\alpha^T \beta)^T = \beta^T (\alpha^T)^T =$



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定义

$$||\alpha|| := \sqrt{\alpha^T \alpha} = \sqrt{\alpha_1^2 + \alpha_2^2 + \dots + \alpha_n^2}$$

称为向量的长度或范数。

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, $\beta = \begin{pmatrix} -1 \\ 3 \\ 1 \\ 5 \end{pmatrix}$

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$$|\alpha^T \beta| \le ||\alpha|| \cdot ||\beta||$$

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$$|\alpha^T \beta| \le ||\alpha|| \cdot ||\beta||$$

即

$$|a_1b_1 + \dots + a_nb_n| \le \sqrt{a_1^2 + \dots + a_n^2} \cdot \sqrt{b_1^2 + \dots + b_n^2}$$

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• 设 $\alpha \neq 0$,则 $||\alpha|| \neq 0$,向量 $\frac{1}{||\alpha||} \alpha$ 是单位向量:



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 $\pi \frac{1}{||\alpha||} \alpha$ 为 α 的单位化



$$\alpha = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \beta = \begin{pmatrix} 2 \\ 2 \\ 4 \\ 5 \end{pmatrix}$$

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解

1.
$$||\alpha|| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$
,

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解

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$$||\alpha|| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$
, 所以的 α 单位化为:
$$\frac{1}{||\alpha||} \alpha = \frac{1}{\sqrt{14}} \begin{pmatrix} 1\\2\\3 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{14}\\2/\sqrt{14}\\3/\sqrt{14} \end{pmatrix}$$

$$\alpha = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \beta = \begin{pmatrix} 2 \\ 2 \\ 4 \\ 5 \end{pmatrix}$$

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$$||\beta|| = \sqrt{2^2 + 2^2 + 4^2 + 5^2} = \sqrt{49} = 7$$
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2. $||\beta|| = \sqrt{2^2 + 2^2 + 4^2 + 5^2} = \sqrt{49} = 7$, 所以的 β 单位化为:

$$\frac{1}{||\beta||}\beta = \frac{1}{7} \begin{pmatrix} 2\\2\\4\\5 \end{pmatrix} = \begin{pmatrix} 2/7\\2/7\\4/7\\5/7 \end{pmatrix}$$



定义 若 $\alpha^T \beta = 0$, 则称 α , β 正交(或垂直)

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例 零向量与任意向量正交:

 $0^T \alpha$

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$$\varepsilon_1^T \varepsilon_2 = 0$$
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定义 若 \mathbb{R}^n 中向量组 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 满足

- 1. 每个向量非零: $\alpha_i \neq 0$, i = 1, 2, ..., s
- 2. 两两正交: $\alpha_i^T \alpha_j = 0$, $i \neq j$

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证明设

$$k_1\alpha_1 + k_2\alpha_2 + \cdots + k_s\alpha_s = 0$$

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$$k_1 = k_2 = \cdots = k_s = 0$$

正交化

 $\alpha_1, \alpha_2, \ldots, \alpha_s$ (线性无关) $\longrightarrow \beta_1, \beta_2, \ldots, \beta_s$ (等价, 两两正交)



 $\alpha_1, \alpha_2, \ldots, \alpha_s$ (线性无关) $\xrightarrow{\mathbb{E}^{\Sigma(k)}} \beta_1, \beta_2, \ldots, \beta_s$ (等价, 两两正交)



$$\alpha_1, \alpha_2, \ldots, \alpha_s$$
(线性无关) $\xrightarrow{\mathbb{E}^{\chi \ell}} \beta_1, \beta_2, \ldots, \beta_s$ (等价,两两正交) 实现正交化步骤(施密特正交化方法):

$$\beta_1 =$$

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例 1 将线性无关组 $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 3 \\ 3 \\ -1 \\ -1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix}$ 正交化

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$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2^T \beta_1}{||\beta_1||^2} \beta_1$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3^T \beta_1}{||\beta_1||^2} \beta_1 - \frac{\alpha_3^T \beta_2}{||\beta_2||^2} \beta_2$$

例 1 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1\\1\\1\\1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 3\\3\\-1\\-1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} -2\\0\\6\\8 \end{pmatrix}$ 正交化

$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \cdots - \beta_1$$

$$\beta_3 = \alpha_3 - \cdots - \beta_1 - \cdots - \beta_2$$

例 1 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1\\1\\1\\1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 3\\3\\-1\\-1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} -2\\0\\6\\8 \end{pmatrix}$ 正交化

$$\beta_1 = \alpha_1 = \begin{pmatrix} \frac{1}{1} \\ \frac{1}{1} \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \cdots - \beta_1$$

$$\beta_3 = \alpha_3 - \cdots - \beta_1 - \cdots - \beta_2$$



例 1 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 3 \\ 3 \\ -1 \\ -1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix}$ 正交化

$$\beta_1 = \alpha_1 = \begin{pmatrix} \frac{1}{1} \\ \frac{1}{1} \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\beta_1}{\beta_1} = \begin{pmatrix} \frac{3}{3} \\ -\frac{1}{1} \end{pmatrix} - -\begin{pmatrix} \frac{1}{1} \\ \frac{1}{1} \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \dots - \beta_1 - \dots - \beta_2$$



例 1 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 3 \\ 3 \\ -1 \\ -1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix}$ 正交化

$$\beta_1 = \alpha_1 = \begin{pmatrix} \frac{1}{1} \\ \frac{1}{1} \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\beta_1}{1} = \begin{pmatrix} \frac{3}{3} \\ -\frac{1}{1} \end{pmatrix} - \frac{1}{4} \begin{pmatrix} \frac{1}{1} \\ \frac{1}{1} \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \dots - \beta_1 - \dots - \beta_2$$



例 1 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 3 \\ 3 \\ -1 \\ -1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix}$ 正交化

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1\\1\\1\\1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{3}{-1} = \begin{pmatrix} 3\\3\\-1\\-1 \end{pmatrix} - \frac{4}{4} \begin{pmatrix} 1\\1\\1\\1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\beta_1 - \beta_2}{-1} = \frac{\beta_2}{-1}$$

例 1 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 3 \\ 3 \\ -1 \\ -1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix}$ 正交化

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1\\1\\1\\1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \dots - \beta_1 = \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} - \frac{4}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -2 \\ -2 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \cdots - \beta_1 - \cdots - \beta_2$$



例 1 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1\\1\\1\\1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 3\\3\\-1\\-1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} -2\\0\\6\\8 \end{pmatrix}$ 正交化

$$\beta_1 = \alpha_1 = \begin{pmatrix} \frac{1}{1} \\ \frac{1}{1} \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{3}{-1} - \frac{4}{4} \begin{pmatrix} 1\\1\\1\\1 \end{pmatrix} = \begin{pmatrix} 2\\2\\-2\\-2 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix} - - - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - - - \begin{pmatrix} 2 \\ 2 \\ -2 \\ -2 \end{pmatrix}$$

 $\beta_3 = \alpha_3 - \cdots - \beta_1 - \cdots - \beta_2$



例 1 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1\\1\\1\\1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 3\\3\\-1\\-1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} -2\\0\\6\\8 \end{pmatrix}$ 正交化

$$\beta_1 = \alpha_1 = \begin{pmatrix} \frac{1}{1} \\ \frac{1}{1} \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\beta_1}{-1} - \frac{4}{4} \begin{pmatrix} \frac{1}{1} \\ \frac{1}{1} \end{pmatrix} = \begin{pmatrix} \frac{2}{2} \\ -\frac{2}{2} \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\beta_1 - \beta_2}{\beta_1 - \beta_2}$$

$$= \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix} - \frac{1}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{2}{2} \begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix}$$



例 1 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1\\1\\1\\1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 3\\3\\-1\\-1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} -2\\0\\6\\8 \end{pmatrix}$ 正交化

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1\\1\\1\\1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \dots - \beta_1 = \begin{pmatrix} 3 \\ 3 \\ -1 \\ -1 \end{pmatrix} - \frac{4}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -2 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix} - \frac{12}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - - - \begin{pmatrix} 2 \\ 2 \\ -2 \\ -2 \end{pmatrix}$$

 $\beta_3 = \alpha_3 - \cdots - \beta_1 - \cdots - \beta_2$



例 1 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 3 \\ 3 \\ -1 \\ -1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix}$ 正交化

$$\beta_1 = \alpha_1 = \begin{pmatrix} \frac{1}{1} \\ \frac{1}{1} \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\beta_1}{1} = \begin{pmatrix} \frac{3}{3} \\ -\frac{1}{1} \end{pmatrix} - \frac{4}{4} \begin{pmatrix} \frac{1}{1} \\ \frac{1}{1} \end{pmatrix} = \begin{pmatrix} \frac{2}{2} \\ -\frac{2}{3} \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\beta_1 - \beta_2}{\beta_1 - \beta_2}$$

$$= \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix} - \frac{12}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{2}{16} \begin{pmatrix} 2 \\ 2 \\ -2 \\ -2 \end{pmatrix}$$



例 1 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1\\1\\1\\1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 3\\3\\-1\\-1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} -2\\0\\6\\8 \end{pmatrix}$ 正交化

$$\beta_1 = \alpha_1 = \begin{pmatrix} \frac{1}{1} \\ \frac{1}{1} \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\beta_1}{-1} = \begin{pmatrix} \frac{3}{3} \\ -\frac{1}{-1} \end{pmatrix} - \frac{4}{4} \begin{pmatrix} \frac{1}{1} \\ \frac{1}{1} \end{pmatrix} = \begin{pmatrix} \frac{2}{2} \\ -\frac{2}{-2} \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix} - \frac{12}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{-32}{16} \begin{pmatrix} 2 \\ 2 \\ -2 \\ -2 \end{pmatrix}$$



例 1 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 3 \\ 3 \\ -1 \\ -1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix}$ 正交化

$$\beta_1 = \alpha_1 = \begin{pmatrix} \frac{1}{1} \\ \frac{1}{1} \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\beta_1}{-1} = \begin{pmatrix} \frac{3}{3} \\ -\frac{1}{-1} \end{pmatrix} - \frac{4}{4} \begin{pmatrix} \frac{1}{1} \\ \frac{1}{1} \end{pmatrix} = \begin{pmatrix} \frac{2}{2} \\ -\frac{2}{-2} \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\beta_1 - \beta_2}{\beta_1 - \beta_2}$$

$$= \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix} - \frac{12}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{-32}{16} \begin{pmatrix} 2 \\ 2 \\ -2 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$



例 2 将线性无关组 $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ 正交化

例 2 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1\\1\\0\\1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 3\\2\\1\\1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2\\1\\1\\3 \end{pmatrix}$ 正交化

$$\beta_1 =$$

$$\beta_2 =$$

$$\beta_3 =$$

例 2 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ 正交化

$$\beta_1 = \alpha_1$$

$$\beta_2 =$$

$$\beta_3 =$$

例 2 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ 正交化

$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \cdots - \beta_1$$

$$\beta_3 =$$

例 2 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ 正交化

$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \cdots - \beta_1$$

$$\beta_3 = \alpha_3 - \cdots - \beta_1 - \cdots - \beta_2$$

例 2 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix}$ 正交化

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1\\1\\0\\1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \cdots - \beta_1$$

$$\beta_3 = \alpha_3 - \cdots - \beta_1 - \cdots - \beta_2$$

例 2 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1\\1\\0\\1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 3\\2\\1\\1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2\\1\\1\\3 \end{pmatrix}$ 正交化

$$\beta_1 = \alpha_1 = \begin{pmatrix} \frac{1}{1} \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \dots - \beta_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} - - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \cdots - \beta_1 - \cdots - \beta_2$$

例 2 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1\\1\\0\\1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 3\\2\\1\\1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2\\1\\1\\3 \end{pmatrix}$ 正交化

$$\beta_1 = \alpha_1 = \begin{pmatrix} \frac{1}{1} \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{3}{2} - \frac{3}{2} - \frac{1}{3} - \frac{1}{3} \begin{pmatrix} \frac{1}{1} \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \cdots - \beta_1 - \cdots - \beta_2$$

例 2 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1\\1\\0\\1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 3\\2\\1\\1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2\\1\\1\\3 \end{pmatrix}$ 正交化

$$\beta_1 = \alpha_1 = \begin{pmatrix} \frac{1}{1} \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \dots - \beta_1 = \begin{pmatrix} \frac{3}{2} \\ \frac{1}{1} \end{pmatrix} - \frac{6}{3} \begin{pmatrix} \frac{1}{1} \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \cdots - \beta_1 - \cdots - \beta_2$$

例 2 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix}$ 正交化

$$\beta_1 = \alpha_1 = \begin{pmatrix} \frac{1}{1} \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \dots - \beta_1 = \begin{pmatrix} \frac{3}{2} \\ \frac{1}{1} \end{pmatrix} - \frac{6}{3} \begin{pmatrix} \frac{1}{1} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{0} \\ \frac{1}{-1} \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \cdots - \beta_1 - \cdots - \beta_2$$

例 2 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ 正交化

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \dots - \beta_1 = \begin{pmatrix} \frac{3}{2} \\ \frac{1}{1} \end{pmatrix} - \frac{6}{3} \begin{pmatrix} \frac{1}{1} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{0} \\ 1 \\ -1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\beta_1 - \beta_2}{\beta_1 - \beta_2}$$

$$= \begin{pmatrix} \frac{2}{1} \\ \frac{1}{3} \end{pmatrix} - - \begin{pmatrix} \frac{1}{1} \\ \frac{1}{0} \end{pmatrix} - - \begin{pmatrix} \frac{1}{0} \\ \frac{1}{-1} \end{pmatrix}$$



例 2 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ 正交化

$$\beta_1 = \alpha_1 = \begin{pmatrix} \frac{1}{0} \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\beta_1}{1} = \begin{pmatrix} \frac{3}{2} \\ \frac{1}{1} \end{pmatrix} - \frac{6}{3} \begin{pmatrix} \frac{1}{1} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{0} \\ \frac{1}{-1} \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\beta_1}{1} - \frac{\beta_2}{1}$$

$$= \begin{pmatrix} 2\\1\\1\\3 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 1\\1\\0\\1 \end{pmatrix} - - \begin{pmatrix} 1\\0\\1\\-1 \end{pmatrix}$$



例 2 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ 正交化

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1\\1\\0\\1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \dots - \beta_1 = \begin{pmatrix} \frac{3}{2} \\ \frac{1}{1} \end{pmatrix} - \frac{6}{3} \begin{pmatrix} \frac{1}{1} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{0} \\ 1 \\ -1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\beta_1 - \beta_2}{\beta_1 - \beta_2}$$

$$= \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix} - \frac{6}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - - \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$



例 2 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ 正交化

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1\\1\\0\\1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{3}{2} - \frac{6}{3} \begin{pmatrix} 1\\1\\0\\1 \end{pmatrix} - \frac{6}{3} \begin{pmatrix} 1\\0\\1\\-1 \end{pmatrix} = \begin{pmatrix} 1\\0\\1\\-1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\beta_1 - \beta_2}{\beta_1 - \beta_2}$$

$$= \begin{pmatrix} \frac{2}{1} \\ \frac{1}{3} \end{pmatrix} - \frac{6}{3} \begin{pmatrix} \frac{1}{1} \\ \frac{1}{0} \\ \frac{1}{1} \end{pmatrix} - \frac{1}{3} \begin{pmatrix} \frac{1}{0} \\ \frac{1}{-1} \end{pmatrix}$$



例 2 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ 正交化

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1\\1\\0\\1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{3}{2} - \frac{6}{3} \begin{pmatrix} 1\\1\\1 \end{pmatrix} - \frac{6}{3} \begin{pmatrix} 1\\1\\0\\1 \end{pmatrix} = \begin{pmatrix} 1\\0\\1\\-1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\beta_1 - \beta_2}{\beta_1 - \beta_2}$$

$$= \begin{pmatrix} \frac{2}{1} \\ \frac{1}{3} \end{pmatrix} - \frac{6}{3} \begin{pmatrix} \frac{1}{1} \\ \frac{1}{0} \\ \frac{1}{1} \end{pmatrix} - \frac{0}{3} \begin{pmatrix} \frac{1}{0} \\ \frac{1}{-1} \end{pmatrix}$$



例 2 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ 正交化

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1\\1\\0\\1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{3}{2} - \frac{6}{3} \begin{pmatrix} \frac{1}{1} \\ \frac{1}{1} \end{pmatrix} - \frac{6}{3} \begin{pmatrix} \frac{1}{1} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{0} \\ 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2\\1\\1 \end{pmatrix} - \frac{6}{3} \begin{pmatrix} 1\\1\\0 \end{pmatrix} - \frac{0}{3} \begin{pmatrix} 1\\0\\1 \end{pmatrix} = \begin{pmatrix} 0\\-1\\1 \end{pmatrix}$$



例 3 将线性无关组 $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ 正交化

例 3 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ 正交化

$$\beta_1 =$$

$$\beta_2 =$$

$$\beta_3 =$$

例 3 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ 正交化

$$\beta_1 = \alpha_1$$

$$\beta_2 =$$

$$\beta_3 =$$

例 3 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1\\1\\1 \end{pmatrix}$$
 , $\alpha_2 = \begin{pmatrix} 0\\1\\2\\1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} -1\\0\\1\\1 \end{pmatrix}$ 正交化

$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \cdots - \beta_1$$

$$\beta_3 =$$

例 3 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1\\1\\1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 0\\1\\2\\1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} -1\\0\\1\\1 \end{pmatrix}$ 正交化

$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \cdots - \beta_1$$

$$\beta_3 = \alpha_3 - \cdots - \beta_1 - \cdots - \beta_2$$



例 3 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ 正交化

$$\beta_1 = \alpha_1 = \begin{pmatrix} \frac{1}{1} \\ \frac{1}{1} \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \cdots - \beta_1$$

$$\beta_3 = \alpha_3 - \cdots - \beta_1 - \cdots - \beta_2$$

例 3 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ 正交化

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1\\1\\1\\1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \dots - \beta_1 = \begin{pmatrix} 0 \\ \frac{1}{2} \\ 1 \end{pmatrix} - - \begin{pmatrix} \frac{1}{1} \\ \frac{1}{1} \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \cdots - \beta_1 - \cdots - \beta_2$$

例 3 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ 正交化

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1\\1\\1\\1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \dots - \beta_1 = \begin{pmatrix} 0 \\ \frac{1}{2} \\ 1 \end{pmatrix} - \frac{1}{4} \begin{pmatrix} \frac{1}{1} \\ \frac{1}{1} \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \cdots - \beta_1 - \cdots - \beta_2$$

例 3 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ 正交化

$$\beta_1 = \alpha_1 = \begin{pmatrix} \frac{1}{1} \\ \frac{1}{1} \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \dots - \beta_1 = \begin{pmatrix} 0 \\ \frac{1}{2} \\ 1 \end{pmatrix} - \frac{4}{4} \begin{pmatrix} \frac{1}{1} \\ \frac{1}{1} \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \cdots - \beta_1 - \cdots - \beta_2$$

例 3 将线性无关组
$$\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ 正交化

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正交矩阵

定义 设 n 阶矩阵 Q 满足 $Q^TQ = I_n$,则称 Q 是正交矩阵。

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$$Q^{T}Q = \begin{pmatrix} \alpha_{1}^{T} \\ \alpha_{2}^{T} \\ \vdots \\ \alpha_{n}^{T} \end{pmatrix} (\alpha_{1} \alpha_{2} \dots \alpha_{n}) = \begin{pmatrix} \alpha_{1}^{T} \alpha_{1} & \alpha_{1}^{T} \alpha_{2} & \cdots & \alpha_{1}^{T} \alpha_{n} \\ \alpha_{2}^{T} \alpha_{1} & \alpha_{2}^{T} \alpha_{2} & \cdots & \alpha_{2}^{T} \alpha_{n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n}^{T} \alpha_{1} & \alpha_{n}^{T} \alpha_{2} & & \end{pmatrix}$$

定理 n 阶矩阵 Q 是正交矩阵的充分必要条件是: Q 的列(行)向量组是单位正交向量组。

证明 设 $Q = (\alpha_1 \alpha_2 \dots \alpha_n)$,则

$$Q^{T}Q = \begin{pmatrix} \alpha_{1}^{T} \\ \alpha_{2}^{T} \\ \vdots \\ \alpha_{n}^{T} \end{pmatrix} (\alpha_{1} \alpha_{2} \dots \alpha_{n}) = \begin{pmatrix} \alpha_{1}^{T} \alpha_{1} & \alpha_{1}^{T} \alpha_{2} & \cdots & \alpha_{1}^{T} \alpha_{n} \\ \alpha_{2}^{T} \alpha_{1} & \alpha_{2}^{T} \alpha_{2} & \cdots & \alpha_{2}^{T} \alpha_{n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n}^{T} \alpha_{1} & \alpha_{n}^{T} \alpha_{2} & \cdots & \alpha_{n}^{T} \alpha_{n} \end{pmatrix}$$

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所以

$$O^TO = I$$



定理 n 阶矩阵 Q 是正交矩阵的充分必要条件是: Q 的列(行)向量组是单位正交向量组。

证明 设
$$Q = (\alpha_1 \alpha_2 \dots \alpha_n)$$
,则

$$Q^{T}Q = \begin{pmatrix} \alpha_{1}^{T} \\ \alpha_{2}^{T} \\ \vdots \\ \alpha_{n}^{T} \end{pmatrix} (\alpha_{1} \alpha_{2} \dots \alpha_{n}) = \begin{pmatrix} \alpha_{1}^{T} \alpha_{1} & \alpha_{1}^{T} \alpha_{2} & \cdots & \alpha_{1}^{T} \alpha_{n} \\ \alpha_{2}^{T} \alpha_{1} & \alpha_{2}^{T} \alpha_{2} & \cdots & \alpha_{2}^{T} \alpha_{n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n}^{T} \alpha_{1} & \alpha_{n}^{T} \alpha_{2} & \cdots & \alpha_{n}^{T} \alpha_{n} \end{pmatrix}$$

所以

$$Q^{T}Q = I \quad \Leftrightarrow \quad \begin{cases} \alpha_{i}^{T}\alpha_{i} = 1, \\ \alpha_{i}^{T}\alpha_{j} = 0, \end{cases}$$



定理 n 阶矩阵 O 是正交矩阵的充分必要条件是:O 的列(行)向量组是 单位正交向量组。

证明 设
$$Q = (\alpha_1 \alpha_2 \dots \alpha_n)$$
,则

$$Q^{T}Q = \begin{pmatrix} \alpha_{1}^{T} \\ \alpha_{2}^{T} \\ \vdots \\ \alpha_{n}^{T} \end{pmatrix} (\alpha_{1} \alpha_{2} \dots \alpha_{n}) = \begin{pmatrix} \alpha_{1}^{T} \alpha_{1} & \alpha_{1}^{T} \alpha_{2} & \cdots & \alpha_{1}^{T} \alpha_{n} \\ \alpha_{2}^{T} \alpha_{1} & \alpha_{2}^{T} \alpha_{2} & \cdots & \alpha_{2}^{T} \alpha_{n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n}^{T} \alpha_{1} & \alpha_{n}^{T} \alpha_{2} & \cdots & \alpha_{n}^{T} \alpha_{n} \end{pmatrix}$$

所以

$$Q^{T}Q = I \iff \begin{cases} \alpha_{i}^{T}\alpha_{i} = 1, & (i = 1, 2, ..., n) \\ \alpha_{i}^{T}\alpha_{j} = 0, & (i \neq j; i, j = 1, 2, ..., n) \end{cases}$$





$$A_1 = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{pmatrix}, \quad A_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{pmatrix},$$

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提示 验证: 列向量组是单位正交向量组



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提示 验证: 列向量组是单位正交向量组

答案 A1 是正交矩阵

$$A_1 = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{pmatrix}, \quad A_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{pmatrix},$$

提示验证:列向量组是单位正交向量组

答案 A_1 是正交矩阵, A_2 不是正交矩阵

- 对任意 n 阶方阵:
 - 1. 一定有 n 个特征值 (计算重数,复数域内),可能有非实数特征值
 - 2. 不一定能对角化

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例
$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
 的特征值方程是

$$0 = |\lambda I - A| =$$

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例
$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
 的特征值方程是
$$0 = |\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ 1 & \lambda \end{vmatrix} = \lambda^2 + 1$$

- 对任意 n 阶方阵:
 - 1. 一定有 n 个特征值 (计算重数,复数域内),可能有非实数特征值
 - 2. 不一定能对角化

$$MA = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
 的特征值方程是

$$0 = |\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ 1 & \lambda \end{vmatrix} = \lambda^2 + 1$$

所以特征值是 $\lambda_1 = -\sqrt{-1}$, $\lambda_2 = \sqrt{-1}$.



- 对任意 n 阶方阵:
 - 1. 一定有 n 个特征值 (计算重数, 复数域内), 可能有非实数特征值
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所以特征值是 $\lambda_1 = -\sqrt{-1}$, $\lambda_2 = \sqrt{-1}$.

- 对实对称矩阵,总成立:
 - 1. 定理 实对称矩阵的特征值都是实数。
 - 2. 定理 实对称矩阵一定可以对角化。



也就是:设A为实对称矩阵,则一定存在可逆矩阵P,使得

$$P^{-1}AP = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ & \ddots \\ & & \lambda_n \end{pmatrix}$$

也就是:设A为实对称矩阵,则一定存在可逆矩阵P,使得

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事实上,还可以进一步要求 P 是正交矩阵:

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也就是:设A为实对称矩阵,则一定存在可逆矩阵P,使得

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注 由于正交矩阵满足 $Q^{-1} = Q^T$,

也就是:设A为实对称矩阵,则一定存在可逆矩阵P,使得

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注 由于正交矩阵满足 $Q^{-1} = Q^T$,上述等价于 $Q^T A Q = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix}$

$$\alpha_2^T \alpha_1 = 0$$



$$A\alpha_1 = \lambda_1\alpha_1$$

$$A\alpha_2 = \lambda_2\alpha_2$$

$$\alpha_2^T \alpha_1 = 0$$

$$A\alpha_1 = \lambda_1 \alpha_1 \quad \Rightarrow \quad \alpha_2^T A \alpha_1 = \lambda_1 \alpha_2^T \alpha_1$$

 $A\alpha_2 = \lambda_2 \alpha_2$

$$\alpha_2^T \alpha_1 = 0$$

$$A\alpha_1 = \lambda_1 \alpha_1 \quad \Rightarrow \quad \alpha_2^T A \alpha_1 = \lambda_1 \ \alpha_2^T \alpha_1$$

$$A\alpha_2 = \lambda_2 \alpha_2 \quad \Rightarrow \quad \alpha_1^T A \alpha_2 = \lambda_2 \ \alpha_1^T \alpha_2$$

$$\alpha_2^T \alpha_1 = 0$$



$$A\alpha_1 = \lambda_1 \alpha_1 \quad \Rightarrow \quad \alpha_2^T A \alpha_1 = \lambda_1 \alpha_2^T \alpha_1$$

$$A\alpha_2 = \lambda_2 \alpha_2 \quad \Rightarrow \quad \alpha_1^T A \alpha_2 = \lambda_2 \alpha_1^T \alpha_2$$

$$\alpha_2^T \alpha_1 = 0$$



$$A\alpha_{1} = \lambda_{1}\alpha_{1} \Rightarrow \alpha_{2}^{T}A\alpha_{1} = \lambda_{1}\alpha_{2}^{T}\alpha_{1}$$

$$A\alpha_{2} = \lambda_{2}\alpha_{2} \Rightarrow \alpha_{1}^{T}A\alpha_{2} = \lambda_{2}\alpha_{1}^{T}\alpha_{2}$$

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$$A\alpha_{2} = \lambda_{2}\alpha_{2} \quad \Rightarrow \quad \alpha_{1}^{T}A\alpha_{2} = \lambda_{2} \quad \alpha_{1}^{T}\alpha_{2}$$

注意
$$\alpha_2^T A \alpha_1 = (\alpha_2^T A \alpha_1)^T =$$

$$\alpha_2^T \alpha_1 = 0$$

$$A\alpha_{1} = \lambda_{1}\alpha_{1} \implies \alpha_{2}^{T}A\alpha_{1} = \lambda_{1}\alpha_{2}^{T}\alpha_{1}$$

$$A\alpha_{2} = \lambda_{2}\alpha_{2} \implies \alpha_{1}^{T}A\alpha_{2} = \lambda_{2}\alpha_{1}^{T}\alpha_{2}$$

注意
$$\alpha_2^T A \alpha_1 = \left(\alpha_2^T A \alpha_1\right)^T = \alpha_1^T A^T \left(\alpha_2^T\right)^T =$$

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$$\alpha_2^T A \alpha_1 = (\alpha_2^T A \alpha_1)^T = \alpha_1^T A^T (\alpha_2^T)^T = \alpha_1^T A \alpha_2$$

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注意
$$\alpha_2^T A \alpha_1 = \left(\alpha_2^T A \alpha_1\right)^T = \alpha_1^T A^T \left(\alpha_2^T\right)^T = \alpha_1^T A \alpha_2$$
,两式相减得
$$0 = (\lambda_1 - \lambda_2) \alpha_2^T \alpha_1$$

$$\alpha_2^T \alpha_1 = 0$$



证明 设 A 为实对称矩阵, $\lambda_1 \neq \lambda_2$ 为两特征值, α_1 , α_2 为相应特征向量,则

$$A\alpha_{1} = \lambda_{1}\alpha_{1} \quad \Rightarrow \quad \boxed{\alpha_{2}^{T}A\alpha_{1}} = \lambda_{1} \boxed{\alpha_{2}^{T}\alpha_{1}}$$

$$A\alpha_{2} = \lambda_{2}\alpha_{2} \quad \Rightarrow \quad \boxed{\alpha_{1}^{T}A\alpha_{2}} = \lambda_{2} \boxed{\alpha_{1}^{T}\alpha_{2}}$$

注意
$$\alpha_2^T A \alpha_1 = \left(\alpha_2^T A \alpha_1\right)^T = \alpha_1^T A^T \left(\alpha_2^T\right)^T = \alpha_1^T A \alpha_2$$
,两式相减得
$$0 = (\lambda_1 - \lambda_2) \alpha_2^T \alpha_1$$

由于 $\lambda_1 \neq \lambda_2$, 所以

$$\alpha_2^T \alpha_1 = 0$$



	不同 特征值	重 数	正交化	单位化		
	λ_1	n_1				
	λ_2	n ₂				
	:	÷				
	λ_{s}	ns				
		共n				
$ \lambda I - A = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_s)^{n_s}$						

不同 特征值	重 数	$(\lambda_i I - A)x = 0$ 基础解系	正交化	单位化		
λ_1	n_1					
λ_2	n_2					
÷	÷					
λ_{s}	ns					
	共 n					
$ \lambda I - A = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_s)^{n_s}$						

不同 特征值	重 数	$(\lambda_i I - A)x = 0$ 基础解系	正交化	单位化			
λ_1	n ₁	$\alpha_1^{(1)}, \cdots, \alpha_{n_1}^{(1)}$					
λ_2	n_2						
÷	÷						
λ_{s}	ns						
	共 n						
$ \lambda I - A = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_s)^{n_s}$							

不同 特征值	重 数	$(\lambda_i I - A)x = 0$ 基础解系	正交化	单位化			
λ_1	n_1	$\alpha_1^{(1)}, \cdots, \alpha_{n_1}^{(1)}$					
λ_2	n_2	$\alpha_1^{(2)},\cdots,\alpha_{n_2}^{(2)}$					
:	÷						
λ_{s}	ns						
	共 n						
$ \lambda I - A = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_s)^{n_s}$							

不同 特征值	重 数	$(\lambda_i I - A)x = 0$ 基础解系	正交化	单位化
λ_1	n_1	$\alpha_1^{(1)}, \cdots, \alpha_{n_1}^{(1)}$		
λ_2	n_2	$\alpha_1^{(2)}, \cdots, \alpha_{n_2}^{(2)}$		
:	į	÷		
$\lambda_{\scriptscriptstyle \mathcal{S}}$	ns	$\alpha_1^{(s)}, \cdots, \alpha_{n_s}^{(s)}$		
	共 n			
$ \lambda I - A $	$=(\lambda -\lambda$	$(\lambda_1)^{n_1}(\lambda-\lambda_2)^{n_2}\cdots(\lambda_n)^{n_n}$	$-\lambda_s)^{n_s}$	



	不同 特征值	重 数	$(\lambda_i I - A)x = 0$ 基础解系	正交化	单位化	
	λ_1	n ₁	$\alpha_1^{(1)}, \cdots, \alpha_{n_1}^{(1)}$			
	λ_2	n_2	$\alpha_1^{(2)}, \cdots, \alpha_{n_2}^{(2)}$			
	:	:	:			
	λ_s	ns	$\alpha_1^{(s)}, \cdots, \alpha_{n_s}^{(s)}$			
		共 n	共 <i>n</i> 个无关特征向量			
$ \lambda I - A = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_s)^{n_s}$						

解释示意图

不同 特征值	重 数	(λ _i I – A)x = 0 基础解系	正交化	单位化
λ_1	n_1	$\alpha_1^{(1)}, \cdots, \alpha_{n_1}^{(1)}$		
λ_2	n_2	$\alpha_1^{(2)}, \cdots, \alpha_{n_2}^{(2)}$		
:	:	i:		
λ_s	ns	$\alpha_1^{(s)}, \cdots, \alpha_{n_s}^{(s)}$		
	共n	共 n 个无关特征向量		

•
$$\Leftrightarrow P = (\alpha_1^{(1)}, \dots, \alpha_{n_c}^{(n_s)}), \ \text{MI} \ P^{-1}AP = \Lambda_o$$



解释示意图

不同 特征值	重 数	$(\lambda_i I - A)x = 0$ 基础解系	正交化	单位化
λ_1	n_1	$\alpha_1^{(1)}, \cdots, \alpha_{n_1}^{(1)}$		
λ_2	n_2	$\alpha_1^{(2)}, \cdots, \alpha_{n_2}^{(2)}$		
÷	÷	:		
λ_s	ns	$\alpha_1^{(s)}, \cdots, \alpha_{n_s}^{(s)}$		
	共 n	共 <i>n</i> 个无关特征向量		

• 令 $P = (\alpha_1^{(1)}, \dots, \alpha_{n_s}^{(n_s)})$,则 $P^{-1}AP = \Lambda$ 。但一般地,P 不是正交矩阵。



解释示意图

不同 特征值	重 数	$(\lambda_i I - A)x = 0$ 基础解系	正交化	单位化
λ_1	n_1	$\alpha_1^{(1)}, \cdots, \alpha_{n_1}^{(1)}$	$\Rightarrow \beta_1^{(1)}, \cdots, \beta_{n_1}^{(1)}$	
λ_2	n ₂	$\alpha_1^{(2)}, \cdots, \alpha_{n_2}^{(2)}$		
:	•	i i		
λ_s	ns	$\alpha_1^{(s)}, \cdots, \alpha_{n_s}^{(s)}$		
	共 n	共n个无关特征向量		

• 令 $P = (\alpha_1^{(1)}, \dots, \alpha_{n_s}^{(n_s)})$,则 $P^{-1}AP = \Lambda$ 。但一般地,P 不是正交矩阵。



解释示意图

不同 特征值	重 数	$(\lambda_i I - A)x = 0$ 基础解系		正交化		单位化	
λ_1	n_1	$\alpha_1^{(1)}, \cdots, \alpha_{n_1}^{(1)}$	⇒	$\beta_1^{(1)}, \cdots, \beta_{n_1}^{(1)}$	⇒	$\gamma_1^{(1)}, \cdots, \gamma_{n_1}^{(1)}$	
λ_2	n ₂	$\alpha_1^{(2)}, \cdots, \alpha_{n_2}^{(2)}$					
÷	÷	i:					
λ_s	ns	$\alpha_1^{(s)}, \cdots, \alpha_{n_s}^{(s)}$					
	共 n	共 n 个无关特征向量					

$$|\lambda I - A| = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_s)^{n_s}$$

• 令 $P = (\alpha_1^{(1)}, \dots, \alpha_{n_s}^{(n_s)})$,则 $P^{-1}AP = \Lambda$ 。但一般地,P 不是正交矩阵。



解释示意图

不同 特征值	重 数	$(\lambda_i I - A)x = 0$ 基础解系		正交化		单位化
λ_1	n_1	$\alpha_1^{(1)}, \cdots, \alpha_{n_1}^{(1)}$	⇒	$\beta_1^{(1)}, \cdots, \beta_{n_1}^{(1)}$	⇒	$\gamma_1^{(1)}, \cdots, \gamma_{n_1}^{(1)}$
λ_2	n ₂	$\alpha_1^{(2)}, \cdots, \alpha_{n_2}^{(2)}$	⇒	$\beta_1^{(2)}, \cdots, \beta_{n_2}^{(2)}$		
÷	:	÷				
λ_{s}	ns	$\alpha_1^{(s)}, \cdots, \alpha_{n_s}^{(s)}$				
	共 n	共 n 个无关特征向量				

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解释示意图

不同 特征值	重 数	$(\lambda_i I - A)x = 0$ 基础解系		正交化		单位化
λ_1	n_1	$\alpha_1^{(1)}, \cdots, \alpha_{n_1}^{(1)}$	⇒	$\beta_1^{(1)}, \cdots, \beta_{n_1}^{(1)}$	⇒	$\gamma_1^{(1)}, \cdots, \gamma_{n_1}^{(1)}$
λ_2	n ₂	$\alpha_1^{(2)}, \cdots, \alpha_{n_2}^{(2)}$	\Rightarrow	$\beta_1^{(2)},\cdots,\beta_{n_2}^{(2)}$	\Rightarrow	$\gamma_1^{(2)}, \cdots, \gamma_{n_2}^{(2)}$
:	÷	:				:
λ_s	ns	$\alpha_1^{(s)}, \cdots, \alpha_{n_s}^{(s)}$				
	共 n	共 n 个无关特征向量	1			

• 令 $P = (\alpha_1^{(1)}, \dots, \alpha_{n_s}^{(n_s)})$,则 $P^{-1}AP = \Lambda$ 。但一般地,P 不是正交矩阵。



 $|\lambda I - A| = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_s)^{n_s}$

解释示意图

不同 特征值	重 数	$(\lambda_i I - A)x = 0$ 基础解系		正交化		单位化
λ_1	n_1	$\alpha_1^{(1)}, \cdots, \alpha_{n_1}^{(1)}$	⇒	$\beta_1^{(1)}, \cdots, \beta_{n_1}^{(1)}$	⇒	$\gamma_1^{(1)}, \cdots, \gamma_{n_1}^{(1)}$
λ_2	n ₂	$\alpha_1^{(2)}, \cdots, \alpha_{n_2}^{(2)}$	\Rightarrow	$\beta_1^{(2)},\cdots,\beta_{n_2}^{(2)}$	\Rightarrow	$\gamma_1^{(2)}, \cdots, \gamma_{n_2}^{(2)}$
:	÷	:		÷		:
λ_s	ns	$\alpha_1^{(s)}, \cdots, \alpha_{n_s}^{(s)}$	\Rightarrow	$\beta_1^{(s)}, \cdots, \beta_{n_s}^{(s)}$		
	共 n	共 n 个无关特征向				

• 令 $P = (\alpha_1^{(1)}, \dots, \alpha_{n_s}^{(n_s)})$,则 $P^{-1}AP = \Lambda$ 。但一般地,P 不是正交矩阵。



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解释示意图

不同 特征值	重 数	$(\lambda_i I - A)x = 0$ 基础解系		正交化		单位化	
λ_1	n_1	$\alpha_1^{(1)}, \cdots, \alpha_{n_1}^{(1)}$	⇒	$\beta_1^{(1)}, \cdots, \beta_{n_1}^{(1)}$	⇒	$\gamma_1^{(1)}, \cdots, \gamma_{n_1}^{(1)}$	
λ_2	n_2	$\alpha_1^{(2)}, \cdots, \alpha_{n_2}^{(2)}$	⇒	$\beta_1^{(2)},\cdots,\beta_{n_2}^{(2)}$	⇒	$\gamma_1^{(2)}, \cdots, \gamma_{n_2}^{(2)}$	
÷	:	:		:		:	
λ_s	ns	$\alpha_1^{(s)}, \cdots, \alpha_{n_s}^{(s)}$	\Rightarrow	$\beta_1^{(s)}, \cdots, \beta_{n_s}^{(s)}$	\Rightarrow	$\gamma_1^{(s)}, \cdots, \gamma_{n_s}^{(s)}$	
	共n	共 n 个无关特征向	量				

$$|\lambda I - A| = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_s)^{n_s}$$

• 令 $P = (\alpha_1^{(1)}, \dots, \alpha_{p_c}^{(n_s)})$,则 $P^{-1}AP = \Lambda$ 。但一般地,P 不是正交矩阵。



解释示意图

不同 特征值	重 数	$(\lambda_i I - A)x = 0$ 基础解系		正交化		单位化
λ_1	n_1	$\alpha_1^{(1)}, \cdots, \alpha_{n_1}^{(1)}$	⇒	$\beta_1^{(1)}, \cdots, \beta_{n_1}^{(1)}$	⇒	$\gamma_1^{(1)}, \cdots, \gamma_{n_1}^{(1)}$
λ_2	n ₂	$\alpha_1^{(2)}, \cdots, \alpha_{n_2}^{(2)}$	\Rightarrow	$\beta_1^{(2)},\cdots,\beta_{n_2}^{(2)}$	\Rightarrow	$\gamma_1^{(2)}, \cdots, \gamma_{n_2}^{(2)}$
:	÷	:		:		:
λ_s	ns	$\alpha_1^{(s)}, \cdots, \alpha_{n_s}^{(s)}$	⇒	$\beta_1^{(s)}, \cdots, \beta_{n_s}^{(s)}$	\Rightarrow	$\gamma_1^{(s)}, \cdots, \gamma_{n_s}^{(s)}$
	共n	共 n 个无关特征向	里			构成单位正交特 征向量

$$a \diamondsuit B = (a^{(1)} \dots a^{(n_s)}) \text{ III } B^{-1}AB = A \text{ } (B - \text{With } B \text{ } A \text{ } B \text{ }$$

 $|\lambda I - A| = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_s)^{n_s}$

• 令 $P = (\alpha_1^{(1)}, \dots, \alpha_{n_s}^{(n_s)})$,则 $P^{-1}AP = \Lambda$ 。但一般地,P 不是正交矩阵。



 $(\lambda_i I - A)x = 0$

基础解系

 $|\lambda I - A| = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_s)^{n_s}$

解释示意图

不同

特征值

 λ_1

λ ~

数

 n_1

na

 $\bullet \Leftrightarrow Q = (\gamma_1^{(1)}, \cdots, \gamma_n^{(n_s)}),$

实对称矩阵的特征值和特征向量

7.2	112	\mathbf{u}_1 , \mathbf{u}_{n_2}	~	ρ_1 ,, ρ_{n_2}	~	r_1 , r_{n_2}
÷	÷	÷		:		÷
λ_{s}	ns	$\alpha_1^{(s)}, \cdots, \alpha_{n_s}^{(s)}$	\Rightarrow	$\beta_1^{(s)}, \cdots, \beta_{n_s}^{(s)}$	\Rightarrow	$\gamma_1^{(s)}, \cdots, \gamma_{n_s}^{(s)}$
	共 n	共n个无关特征向抗	量			构成单位正交特 征向量

• 令 $P = (\alpha_1^{(1)}, \dots, \alpha_n^{(n_s)})$, 则 $P^{-1}AP = \Lambda$ 。但一般地,P 不是正交矩阵。

正交化.

 $\alpha_1^{(1)}, \cdots, \alpha_{n_1}^{(1)} \Rightarrow \beta_1^{(1)}, \cdots, \beta_{n_1}^{(1)} \Rightarrow \gamma_1^{(1)}, \cdots, \gamma_{n_1}^{(1)}$

 $\alpha^{(2)} \cdots \alpha^{(2)} \Rightarrow \beta^{(2)} \cdots \beta^{(2)} \Rightarrow \gamma^{(2)} \cdots \gamma^{(2)}$

单位化

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 $(\lambda_i I - A)x = 0$

基础解系

 $|\lambda I - A| = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_s)^{n_s}$

正交化.

 $\alpha_1^{(1)}, \cdots, \alpha_{n_1}^{(1)} \Rightarrow \beta_1^{(1)}, \cdots, \beta_{n_1}^{(1)} \Rightarrow \gamma_1^{(1)}, \cdots, \gamma_{n_1}^{(1)}$

 $\alpha_1^{(2)}, \cdots, \alpha_{n_2}^{(2)} \Rightarrow \beta_1^{(2)}, \cdots, \beta_{n_2}^{(2)} \Rightarrow \gamma_1^{(2)}, \cdots, \gamma_{n_2}^{(2)}$

解释示意图

不同

特征值

 λ_1

 λ_2

数

 n_1

 n_2

			2		2		2
	:	÷	:		:		:
	λ_s	ns	$\alpha_1^{(s)}, \cdots, \alpha_{n_s}^{(s)}$	\Rightarrow	$\beta_1^{(s)}, \cdots, \beta_{n_s}^{(s)}$	\Rightarrow	$\gamma_1^{(s)}, \cdots, \gamma_{n_s}^{(s)}$
-		共 n	共 n 个无关特征向	量			构成单位正交特 征向量

• 令 $P = (\alpha_1^{(1)}, \dots, \alpha_n^{(n_s)})$,则 $P^{-1}AP = \Lambda$ 。但一般地,P 不是正交矩阵。

• $\Leftrightarrow Q = (\gamma_1^{(1)}, \dots, \gamma_{n_s}^{(n_s)}), \ \mathbb{M} \ Q^{-1}AQ = \Lambda,$

单位化

解释示意图

特征值	数	基础解系				
λ_1	n_1	$\alpha_1^{(1)}, \cdots, \alpha_{n_1}^{(1)}$	⇒	$\beta_1^{(1)}, \cdots, \beta_{n_1}^{(1)}$	⇒	$\gamma_1^{(1)}, \cdots, \gamma_{n_1}^{(1)}$
λ_2	n ₂	$\alpha_1^{(2)}, \cdots, \alpha_{n_2}^{(2)}$	⇒	$\beta_1^{(2)}, \cdots, \beta_{n_2}^{(2)}$	\Rightarrow	$\gamma_1^{(2)}, \cdots, \gamma_{n_2}^{(2)}$

正交化

$$\lambda_s$$
 n_s $\alpha_1^{(s)}, \cdots, \alpha_{n_s}^{(s)}$ \Rightarrow $\beta_1^{(s)}, \cdots, \beta_{n_s}^{(s)}$ \Rightarrow $\gamma_1^{(s)}, \cdots, \gamma_{n_s}^{(s)}$
 共 n 共 n 个无关特征向量

 $|\lambda I - A| = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_s)^{n_s}$

不同 重 $(\lambda_i I - A)x = 0$

构成单位正交特 征向量

单位化

• 令 $P = (\alpha_1^{(1)}, \dots, \alpha_{n_s}^{(n_s)})$,则 $P^{-1}AP = \Lambda$ 。但一般地,P 不是正交矩阵。

• 令 $Q = (\gamma_1^{(1)}, \dots, \gamma_n^{(n_s)})$,则 $Q^{-1}AQ = \Lambda$,并且 Q 是正交矩阵。

 M_1 求矩阵 $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ 的特征值与特征向量。

$$Q^{-1}AQ = \begin{pmatrix} * \\ * \end{pmatrix}$$

 M_1 求矩阵 $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ 的特征值与特征向量。

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例 1 求矩阵 $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ 的特征值与特征向量。 $\mathbf{H} \bullet \mathbf{H}$ 特征方程: $\mathbf{0} = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = \mathbf{H}$

$$Q^{-1}AQ = \begin{pmatrix} * \\ * \end{pmatrix}$$

例 1 求矩阵 $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ 的特征值与特征向量。

解 ● 特征方程:
$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^2 - 1$$

$$Q^{-1}AQ = \left(\begin{array}{c} * \\ * \end{array} \right)$$

例 1 求矩阵 $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ 的特征值与特征向量。

解 ● 特征方程:
$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$$

$$Q^{-1}AQ = \left(\begin{array}{c} * \\ * \end{array} \right)$$

 \bullet $\lambda_1 = 1$

解•特征方程:
$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$$

例 1 求矩阵 $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ 的特征值与特征向量。

 \bullet $\lambda_2 = 3$

$$Q^{-1}AQ = \left(\begin{array}{c} * \\ * \end{array} \right)$$



解・特征方程:
$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$$
• $\lambda_1 = 1$

例 1 求矩阵 $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ 的特征值与特征向量。

•
$$\lambda_2 = 3$$



 $Q^{-1}AQ = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

解 • 特征方程:
$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$$

• $\lambda_1 = 1$, $\pi K (\lambda_1 I - A) x = 0$:

例 1 求矩阵 $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ 的特征值与特征向量。

•
$$\lambda_2 = 3$$
, \overline{x} $\mathbb{R}(\lambda_2 I - A)x = 0$:

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$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$$

• $\lambda_1 = 1$. 求解 $(\lambda_1 I - A)x = 0$:

$$(1I - A \vdots 0) = \begin{pmatrix} -1 - 1 & | 0 \\ -1 - 1 & | 0 \end{pmatrix}$$

$$Q^{-1}AQ = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

例 1 求矩阵 $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ 的特征值与特征向量。 解 ● 特征方程: $\hat{0} = |\hat{\lambda I} - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$

解 • 特征方程:
$$0 = |\lambda I - A| = |A - 1|$$

• $\lambda_1 = 1$. 求解 $(\lambda_1 I - A)x = 0$:

$$(1I - A \vdots 0) = \begin{pmatrix} -1 - 1 & | & 0 \\ -1 - 1 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix}$$

$$Q^{-1}AQ = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

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$$(1I - A \vdots 0) = \begin{pmatrix} -1 - 1 & 0 \\ -1 - 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{array}{c} x_1 + x_2 = 0 \\ \end{array}$$

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例 1 求矩阵 $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ 的特征值与特征向量。 解 ● 特征方程: $\hat{0} = |\hat{\lambda I} - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$

$$\mathbf{M}$$
 ● 特征方程: $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$

$$(1I - A : 0) = \begin{pmatrix} -1 - 1 & | & 0 \\ -1 - 1 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix} \qquad \begin{matrix} x_1 + x_2 & = & 0 \\ x_1 & = & -x_2 \end{matrix}$$

$$Q^{-1}AQ = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

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$$\mathbf{M} \bullet$$
 特征方程: $0 = |\lambda I - A| = \begin{vmatrix} 1 & 1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$

• $\lambda_1 = 1$, $\pi M (\lambda_1 I - A) x = 0$:

$$(1I - A : 0) = \begin{pmatrix} -1 - 1 & 0 \\ -1 - 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{array}{c} x_1 + x_2 = 0 \\ \vdots \\ x_1 = -x_2 \end{array}$$

基础解系: $\alpha_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

• $\lambda_2 = 3$,求解 $(\lambda_2 I - A)x = 0$:

$$Q^{-1}AQ = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

例 1 求矩阵 $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ 的特征值与特征向量。 $\mathbf{M} \bullet \text{ 特征方程: } 0 = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$

* 特征方程:
$$0 = |AI - A| = |-1|\lambda - 2| = (\lambda - 1)(\lambda - 3)$$

• $\lambda_1 = 1$, \overline{x} $\mathbf{R}(\lambda_1 I - A)x = 0$:

$$(1I - A : 0) = \begin{pmatrix} -1 - 1 & | & 0 \\ -1 - 1 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix} \qquad \begin{array}{c} x_1 + x_2 = 0 \\ \downarrow \\ x_1 = -x_2 \end{array}$$

基础解系:
$$\alpha_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$Q^{-1}AQ = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

例 1 求矩阵 $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ 的特征值与特征向量。 解 • 特征方程: $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$

特征方性: $0 - |\Lambda I - \Lambda| - |-1 \lambda - 2| - |\Lambda - I| \Lambda - 3|$

• $\lambda_1 = 1$, \overline{x} $\mathbf{R}(\lambda_1 I - A)x = 0$:

$$(1I - A : 0) = \begin{pmatrix} -1 - 1 & 0 \\ -1 - 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{cases} x_1 + x_2 = 0 \\ x_1 = -x_2 \end{cases}$$

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$$(3I - A \stackrel{\cdot}{\cdot} 0) = \begin{pmatrix} 1 & -1 & | & 0 \\ -1 & 1 & | & 0 \end{pmatrix}$$

$$Q^{-1}AQ = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$



例 1 求矩阵 $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ 的特征值与特征向量。

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$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$$

• $\lambda_1 = 1$. 求解 $(\lambda_1 I - A)x = 0$:

$$(1I - A \vdots 0) = \begin{pmatrix} -1 - 1 & 0 \\ -1 - 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{cases} x_1 + x_2 = 0 \\ x_1 = -x_2 \end{cases}$$

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$$Q^{-1}AQ = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$



例 1 求矩阵 $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ 的特征值与特征向量。 解 • 特征方程: $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$

 $\frac{1}{2} + \frac{2\pi}{3} \frac{1}{2} + \frac{2\pi}{3} \frac{1}{3} \frac{1}{3$

•
$$\lambda_1 = 1$$
, \overline{x} $\mathbf{R}(\lambda_1 I - A)\mathbf{X} = 0$:

$$(1I - A : 0) = \begin{pmatrix} -1 - 1 & 0 \\ -1 - 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{matrix} x_1 + x_2 = 0 \\ \psi \\ x_1 = -x_2 \end{matrix}$$

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$$(3I - A : 0) = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{array}{c} x_1 - x_2 = 0 \\ \end{array}$$

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例 1 求矩阵 $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ 的特征值与特征向量。

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•
$$\lambda_2 = 3$$
, \overline{x} $\mathbf{R}(\lambda_2 I - A)\mathbf{x} = 0$:

$$(3I - A \vdots 0) = \begin{pmatrix} 1 & -1 & | & 0 \\ -1 & 1 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix} \qquad \begin{array}{c} x_1 - x_2 & = 0 \\ x_1 & = x_2 \end{array}$$

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例 1 求矩阵 $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ 的特征值与特征向量。 解 ● 特征方程: $\hat{0} = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$

$$\mathbf{H}$$
 ● 特征方程: $0 = |\lambda I - A| = \begin{vmatrix} 1 & 1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$

• $\lambda_1 = 1$. 求解 $(\lambda_1 I - A)x = 0$:

$$(1I - A : 0) = \begin{pmatrix} -1 - 1 & 0 \\ -1 - 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{cases} x_1 + x_2 = 0 \\ x_1 = -x_2 \end{cases}$$

基础解系: $\alpha_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ • $\lambda_2 = 3$, 求解 $(\lambda_2 I - A)x = 0$:

5,
$$\chi_{HF} (\Lambda_2 I - A) \chi = 0$$
:
. $(1 - 1|0) (1 - 1|0) \chi_1$

$$(3I - A : 0) = \begin{pmatrix} 1 & -1 & | & 0 \\ -1 & 1 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix} \qquad \begin{array}{c} x_1 - x_2 & = 0 \\ x_1 & = x_2 \end{array}$$

基础解系: $\alpha_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$Q^{-1}AQ = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$



例 1 求矩阵 $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ 的特征值与特征向量。 解 ● 特征方程: $\hat{0} = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3)$

特 特征方性:
$$0 = |\Lambda I - A| = |-1| \lambda - 2| = (\Lambda - I)(\Lambda - 3)$$

• $\lambda_1 = 1$. 求解 $(\lambda_1 I - A)x = 0$:

$$(1I - A \vdots 0) = \begin{pmatrix} -1 - 1 & 0 \\ -1 - 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{matrix} x_1 + x_2 & 0 \\ \downarrow \\ x_1 & -x_2 \end{matrix}$$

基础解系: $\alpha_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$(3I - A \vdots 0) = \begin{pmatrix} 1 & -1 & | & 0 \\ -1 & 1 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix} \qquad \begin{matrix} x_1 - x_2 & = & 0 \\ x_1 & = & x_2 \end{matrix}$$
 基础解系: $\alpha_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \xrightarrow{\hat{\mu} \oplus \mathcal{U}} \qquad \gamma_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

基础解系:
$$\alpha_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
 $\xrightarrow{\text{FIGITO}}$ $\gamma_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

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解 ● 特征方程: $0 = |\lambda I - A| = |\lambda - 2| = (\lambda - 1)(\lambda - 3)$ • $\lambda_1 = 1$, $\pi M(\lambda_1 I - A) X = 0$:

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例 1 求矩阵 $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ 的特征值与特征向量。

基础解系:
$$\alpha_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

• $\lambda_2 = 3$,求解 $(\lambda_2 I - A)x = 0$:

 $(3I - A \vdots 0) = \begin{pmatrix} 1 & -1 & | & 0 \\ -1 & 1 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix} \qquad \begin{array}{c} x_1 - x_2 & = & 0 \\ x_1 & = & x_2 \end{array}$ 基础解系: $\alpha_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

 γ_1 所以取 $Q = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$,则 $Q^{-1}AQ = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

例 2 求矩阵 $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ 的特征值与特征向量。

$$Q^{-1}AQ = \begin{pmatrix} * \\ * \end{pmatrix}$$



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例 2 求矩阵 $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ 的特征值与特征向量。

解 ● 特征方程:
$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2 - (-2)^2$$

$$Q^{-1}AQ = \left(\begin{array}{c} * \\ * \end{array} \right)$$

例 2 求矩阵 $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ 的特征值与特征向量。

解 ● 特征方程:
$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = (\lambda + 1)(\lambda - 3)$$

$$Q^{-1}AQ = \left(\begin{array}{c} * \\ * \end{array} \right)$$



例 2 求矩阵 $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ 的特征值与特征向量。 $\mathbf{E} \cdot \mathbf{E} = \begin{bmatrix} \lambda & -1 & -2 \\ 2 & 1 \end{bmatrix}$

解● 特征方程:
$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = (\lambda + 1)(\lambda - 3)$$

$$\bullet \ \lambda_1 = -1$$

$$\lambda_2 = 3$$

$$Q^{-1}AQ = \begin{pmatrix} * \\ * \end{pmatrix}$$



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$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = (\lambda + 1)(\lambda - 3)$$

•
$$\lambda_2 = 3$$
, $x \in (\lambda_2 I - A)x = 0$:

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解 ● 特征方程:
$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = (\lambda + 1)(\lambda - 3)$$

• $\lambda_1 = -1$, $\pi k (\lambda_1 I - A) x = 0$:

$$(1I - A \vdots 0) = \begin{pmatrix} -2 - 2 & | & 0 \\ -2 - 2 & | & 0 \end{pmatrix}$$

• $\lambda_2 = 3$, \overline{x} M ($\lambda_2 I - A$)X = 0:

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解 ● 特征方程:
$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = (\lambda + 1)(\lambda - 3)$$

• 特征方程: $0 = |\lambda I - A| = |-2|\lambda - 1| = |\lambda + 1|(\lambda - 3)$ • $\lambda_1 = -1$,求解 $(\lambda_1 I - A)x = 0$:

$$(1I - A \vdots 0) = \begin{pmatrix} -2 - 2 & | & 0 \\ -2 - 2 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix}$$

• $\lambda_2 = 3$, $\Re (\lambda_2 I - A)x = 0$:

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$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = (\lambda + 1)(\lambda - 3)$$

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$$\alpha_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

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$$(3I - A \stackrel{?}{\cdot} 0) = \begin{pmatrix} 2 & -2 & | 0 \\ -2 & 2 & | 0 \end{pmatrix}$$

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解 ● 特征方程:
$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = (\lambda + 1)(\lambda - 3)$$

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, χ μ $(\lambda_1 I - A)\chi = 0$

$$(1I - A \vdots 0) = \begin{pmatrix} -2 & -2 & | & 0 \\ -2 & -2 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix} \qquad \begin{matrix} x_1 + x_2 & = & 0 \\ x_1 & = & -x_2 \end{matrix}$$

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解 • 特征方程:
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• $\lambda_1 = -1$, 求解 $(\lambda_1 I - A)x = 0$:

$$(1I - A : 0) = \begin{pmatrix} -2 - 2 & 0 \\ -2 - 2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{matrix} x_1 + x_2 & 0 \\ x_1 & 0 & 0 \\ x_1 & 0 & 0 \end{matrix}$$

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例
$$1A = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 3 \end{pmatrix}$$

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$$1A = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 3 \end{pmatrix}$$
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- $\bullet \lambda_1 = -1,$
- $\lambda_2 = 2$,
- $\lambda_3 = 5$,

$$Q^{-1}AQ = \begin{pmatrix} * & \\ & * \\ & & * \end{pmatrix}$$



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$$1A = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 3 \end{pmatrix}$$
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$$1A = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 3 \end{pmatrix}$$
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•
$$\lambda_3 = 5$$
,

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•
$$\lambda_2 = 2$$
, 特征向量 $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$

•
$$\lambda_3 = 5$$
, 特征向量 $\alpha_3 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

$$Q^{-1}AQ = \begin{pmatrix} -1 & 2 \\ & 5 \end{pmatrix}$$



例
$$1A = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 3 \end{pmatrix}$$
, 特征方程: $0 = |\lambda I - A| = (\lambda + 1)(\lambda - 2)(\lambda - 5)$

•
$$\lambda_1 = -1$$
,特征向量 $\alpha_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ 单位化 $\gamma_1 = \begin{pmatrix} 2/3 \\ 2/3 \\ 1/3 \end{pmatrix}$

•
$$\lambda_2 = 2$$
, 特征向量 $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$

•
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例
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•
$$\lambda_3 = 5$$
, 特征向量 $\alpha_3 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

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例
$$1A = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 3 \end{pmatrix}$$
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, 特征向量 $\alpha_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ $\xrightarrow{\text{$\neq \text{d}}\text{$\neq \text{$d$}}}$ $\gamma_1 = \begin{pmatrix} 2/3 \\ 2/3 \\ 1/3 \end{pmatrix}$

•
$$\lambda_2 = 2$$
, 特征向量 $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$ 单位化 $\gamma_2 = \begin{pmatrix} 2/3 \\ -1/3 \\ -2/3 \end{pmatrix}$

•
$$\lambda_3 = 5$$
,特征向量 $\alpha_3 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ 单位化 $\gamma_3 = \begin{pmatrix} 1/3 \\ -2/3 \\ 2/3 \end{pmatrix}$

$$Q^{-1}AQ = \begin{pmatrix} -1 & 2 \\ 5 & 5 \end{pmatrix}$$



例
$$1A = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 3 \end{pmatrix}$$
, 特征方程: $0 = |\lambda I - A| = (\lambda + 1)(\lambda - 2)(\lambda - 5)$

•
$$\lambda_1 = -1$$
,特征向量 $\alpha_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_1 = \begin{pmatrix} 2/3 \\ 2/3 \\ 1/3 \end{pmatrix}$

•
$$\lambda_2 = 2$$
, 特征向量 $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$ 单位化 $\gamma_2 = \begin{pmatrix} 2/3 \\ -1/3 \\ -2/3 \end{pmatrix}$

•
$$\lambda_3 = 5$$
,特征向量 $\alpha_3 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ 单位化 $\gamma_3 = \begin{pmatrix} 1/3 \\ -2/3 \\ 2/3 \end{pmatrix}$

所以取
$$Q = \underbrace{\begin{pmatrix} 2/3 & 2/3 & 1/3 \\ 2/3 & -1/3 & -2/3 \\ 1/3 & -2/3 & 2/3 \end{pmatrix}}_{Q: F 交際}, \quad 则 Q^{-1}AQ = \begin{pmatrix} -1 & 2 & 2 & 2 \\ & 2 & 5 & 2 \end{pmatrix}$$



例
$$2A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{pmatrix}$$

$$Q^{-1}AQ = \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix}$$





例
$$2A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{pmatrix}$$
,特征方程: $0 = |\lambda I - A| = (\lambda - 1)^2(\lambda - 10)$

$$Q^{-1}AQ = \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix}$$

$$Q^{-1}AQ = \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix}$$

例 $2A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{pmatrix}$, 特征方程: $0 = |\lambda I - A| = (\lambda - 1)^2(\lambda - 10)$

•
$$\lambda_3 = 10$$

• $\lambda_1 = 1$ (二重)

• $\lambda_3 = 10$

• $\lambda_1 = 1$ (二重)

例 $2A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{pmatrix}$, 特征方程: $0 = |\lambda I - A| = (\lambda - 1)^2(\lambda - 10)$

 $Q^{-1}AQ = \begin{pmatrix} 1 \\ 1 \\ 10 \end{pmatrix}$

• $\lambda_3 = 10$

λ₁ = 1 (二重). 特征向量

 $\begin{cases} \alpha_1 = \begin{pmatrix} -2\\1\\0 \end{pmatrix} \\ \alpha_2 = \begin{pmatrix} 2\\0\\1 \end{pmatrix} \end{cases}$

 $Q^{-1}AQ = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

例 $2A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{pmatrix}$, 特征方程: $0 = |\lambda I - A| = (\lambda - 1)^2(\lambda - 10)$

$$\alpha_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

λ₁ = 1 (二重). 特征向量

$$\begin{cases}
\alpha_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \\
\alpha_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}
\end{cases}$$

•
$$\lambda_3 = 10$$
,特征向量 $\alpha_3 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$



实对称矩阵的特征值和特征向

例 $2A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ 2 & -4 & 5 \end{pmatrix}$, 特征方程: $0 = |\lambda I - A| = (\lambda - 1)^2(\lambda - 10)$

 $Q^{-1}AQ = \begin{pmatrix} 1 & 1 & 1 \\ & 1 & 1 \end{pmatrix}$

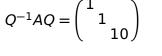
例 $2A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ 2 & 5 & 5 \end{pmatrix}$, 特征方程: $0 = |\lambda I - A| = (\lambda - 1)^2(\lambda - 10)$

$$\left(\begin{array}{c}
\alpha_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}\right)_{\mathbb{E}^{\frac{1}{2}}} \left(\beta_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}\right)$$

λ₁ = 1 (二重). 特征向量

$$\begin{cases}
\alpha_1 = \begin{pmatrix} -2\\1\\0 \end{pmatrix} \xrightarrow{\mathbb{E}^{\frac{1}{2}}(\mathbb{R}^2)} \\
\alpha_2 = \begin{pmatrix} 2\\0\\1 \end{pmatrix} & \beta_1 = \begin{pmatrix} -2\\1\\0 \end{pmatrix} \\
\beta_2 = \begin{pmatrix} 2/5\\4/5\\1 \end{pmatrix}
\end{cases}$$

•
$$\lambda_3 = 10$$
,特征向量 $\alpha_3 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$





例
$$2A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{pmatrix}$$
, 特征方程: $0 = |\lambda I - A| = (\lambda - 1)^2 (\lambda - 10)$

• $\lambda_1 = 1$ (二重),特征向量

$$\begin{cases} \alpha_1 = \begin{pmatrix} -2\\1\\0 \end{pmatrix} \xrightarrow{\text{EXC}} \begin{cases} \beta_1 = \begin{pmatrix} -2\\1\\0 \end{pmatrix} \xrightarrow{\text{With}} \begin{cases} \gamma_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} -2\\1\\0 \end{pmatrix} \\ \alpha_2 = \begin{pmatrix} 2\\0\\1 \end{pmatrix} \end{cases} \end{cases} \begin{cases} \beta_2 = \begin{pmatrix} 2/5\\4/5\\1 \end{pmatrix} \end{cases} \begin{cases} \beta_2 = \begin{pmatrix} 2/5\\4/5\\1 \end{pmatrix} \end{cases}$$

$$\bullet \ \lambda_3 = 10, \ \text{特征向量} \\ \alpha_3 = \begin{pmatrix} 1\\2\\-2 \end{pmatrix} \end{cases}$$

3 实对称矩阵的特征值和特征向量

例
$$2A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{pmatrix}$$
, 特征方程: $0 = |\lambda I - A| = (\lambda - 1)^2 (\lambda - 10)$

λ₁ = 1 (二重). 特征向量

$$\begin{cases} \alpha_1 = \begin{pmatrix} -2\\1\\0 \end{pmatrix} \xrightarrow{\text{EXK}} \begin{cases} \beta_1 = \begin{pmatrix} -2\\1\\0 \end{pmatrix} \xrightarrow{\text{$\frac{1}{4}$}} \begin{cases} \gamma_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} -2\\1\\0 \end{pmatrix} \end{cases} \\ \alpha_2 = \begin{pmatrix} 2\\0\\1 \end{pmatrix} & \beta_2 = \begin{pmatrix} 2/5\\4/5\\1 \end{pmatrix} & \gamma_2 = \frac{5}{3\sqrt{5}} \begin{pmatrix} 2/5\\4/5\\1 \end{pmatrix} \end{cases}$$

$$\bullet \ \lambda_3 = 10, \ \text{$\%$} \text{$$\text{$$$$$$$in ϕ}$} \text{$$$$$$$$$$} \alpha_3 = \begin{pmatrix} 1\\2\\-2 \end{pmatrix} & \xrightarrow{\text{$\frac{1}{4}$}} \text{$$$$$$$$$$$$$$$$$$$$$$$$$$$} \gamma_3 = \begin{pmatrix} 1/3\\2/3\\-2/3 \end{pmatrix}$$

 $Q^{-1}AQ = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$



例 $2A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ 2 & 5 & 5 \end{pmatrix}$, 特征方程: $0 = |\lambda I - A| = (\lambda - 1)^2(\lambda - 10)$ λ₁ = 1 (二重), 特征向量

$$\alpha_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{cases} \alpha_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \xrightarrow{\text{正交化}} \begin{cases} \beta_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \xrightarrow{\text{单位化}} \begin{cases} \gamma_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \end{cases} \\ \alpha_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} & \begin{cases} \beta_2 = \begin{pmatrix} 2/5 \\ 4/5 \\ 1 \end{cases} \end{cases} \end{cases} \begin{cases} \gamma_2 = \frac{5}{3\sqrt{5}} \begin{pmatrix} 2/5 \\ 4/5 \\ 1 \end{cases} \end{cases}$$

$$\bullet \ \lambda_3 = 10, \ \text{特征向量} \\ \alpha_3 = \begin{pmatrix} 1 \\ 2 \\ -2 \\ \gamma_3 \end{cases} & \xrightarrow{\text{单位化}} \qquad \gamma_3 = \begin{pmatrix} 1/3 \\ 2/3 \\ -2/3 \end{pmatrix} \end{cases}$$

$$\text{所以取 } Q = \begin{pmatrix} -2/\sqrt{5} & 2/3 & \sqrt{5} & 1/3 \\ 1/\sqrt{5} & 4/3 & \sqrt{5} & 2/3 \\ 0 & \sqrt{5}/3 & -2/3 \end{pmatrix}, \ \text{刚 } Q^{-1}AQ = \begin{pmatrix} 1 \\ 1 \\ 10 \end{pmatrix}$$

例
$$3A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$
,

 $Q^{-1}AQ = \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix}$

例
$$3A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$
,特征方程: $0 = |\lambda I - A| =$



 $Q^{-1}AQ = \left(\begin{array}{cc} * & \\ & * \\ & \end{array}\right)$

例 3
$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$
,特征方程: $0 = |\lambda I - A| = (\lambda + 1)^2 (\lambda - 5)$ Det

 $Q^{-1}AQ = \left(\begin{array}{cc} * & \\ & * \\ & \end{array}\right)$

 $Q^{-1}AQ = \begin{pmatrix} * \\ * \\ * \end{pmatrix}$

例 3 $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$,特征方程: $0 = |\lambda I - A| = (\lambda + 1)^2 (\lambda - 5)$ Del

 \bullet $\lambda_2 = 5$

• $\lambda_1 = -1$ (二重)

$$Q^{-1}AQ = \begin{pmatrix} -1 \\ -1 \\ 5 \end{pmatrix}$$

例 3 $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$,特征方程: $0 = |\lambda I - A| = (\lambda + 1)^2 (\lambda - 5)$ Del

• $\lambda_1 = -1$ (二重)

 $\begin{cases} \alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \\ \alpha_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

例 $3A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$,特征方程: $0 = |\lambda I - A| = (\lambda + 1)^2(\lambda - 5)$ **Dell**

λ₁ = −1 (二重), 特征向量: ▶ Detail

•
$$\lambda_2 = 5$$

 $Q^{-1}AQ = \begin{pmatrix} -1 & \\ & -1 & \\ & 5 \end{pmatrix}$

 $\begin{cases} \alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \\ \alpha_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \end{cases}$



λ₁ = −1 (二重), 特征向量: ▶ Detail

• $\lambda_2 = 5$,特征向量: $\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

例 3 $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$,特征方程: $0 = |\lambda I - A| = (\lambda + 1)^2 (\lambda - 5)$ Del

 $Q^{-1}AQ = \begin{pmatrix} -1 \\ -1 \\ 5 \end{pmatrix}$

λ₁ = −1 (二重), 特征向量: ▶ Detail

$$\begin{cases}
\alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \xrightarrow{\mathbb{E}^{\frac{1}{2}} \mathbb{E}^{\frac{1}{2}}} \\
\alpha_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} \alpha_2 & 1 \end{pmatrix}$$
• $\lambda_2 = 5$,特征向量: • Det $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$



 $Q^{-1}AQ = \begin{pmatrix} -1 & \\ & -1 & \\ & & 5 \end{pmatrix}$

例 $3A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$,特征方程: $0 = |\lambda I - A| = (\lambda + 1)^2(\lambda - 5)$ **Dell**

例 3 $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$,特征方程: $0 = |\lambda I - A| = (\lambda + 1)^2 (\lambda - 5)$ **Det**

$$\begin{cases}
\alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \xrightarrow{\mathbb{E}^{\frac{1}{2}} \mathbb{E}^{\frac{1}{2}}}
\begin{cases}
\beta_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \\
\alpha_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}
\end{cases}$$

$$\beta_2 = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}$$

•
$$\lambda_2 = 5$$
,特征向量: $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

② ½ 1. 29/34 < ⊳

 $Q^{-1}AQ = \begin{pmatrix} -1 & \\ & -1 & \\ & & \end{pmatrix}$

例
$$3A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$
, 特征方程: $0 = |\lambda I - A| = (\lambda + 1)^2 (\lambda - 5)$ **Det**

• $\lambda_1 = -1$ (二重) ,特征向量: **Detail** ($-1/\sqrt{2}$)





 $Q^{-1}AQ = \begin{pmatrix} -1 & \\ & -1 & \\ & 5 \end{pmatrix}$

例 3
$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$
, 特征方程: $0 = |\lambda I - A| = (\lambda + 1)^2(\lambda - 5)$ Det $\lambda_1 = -1$ (二重), 特征向量: Detail

•
$$\lambda_1 = -1$$
 (二重) ,特征向量: • Detail
$$\begin{cases} \alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} & \beta_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \\ \alpha_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} & \text{Det} \end{cases} \begin{cases} \beta_1 = \begin{pmatrix} -1/2 \\ 1 \\ 0 \end{pmatrix} & \frac{\text{单位化}}{\text{4c}} \end{cases} \begin{cases} \gamma_1 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \\ \gamma_2 = \begin{pmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix} \end{cases}$$

$$\begin{bmatrix} \alpha_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \begin{bmatrix} \beta_2 = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \end{bmatrix}$$
• $\lambda_2 = 5$, 特征向量: • $\alpha_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\frac{\hat{\mu}\hat{\alpha}}{\alpha_3}$

•
$$\lambda_2 = 5$$
,特征向量: $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_3 = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$



 $Q^{-1}AQ = \begin{pmatrix} -1 & \\ & -1 & \\ & 5 \end{pmatrix}$

例
$$3A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$
, 特征方程: $0 = |\lambda I - A| = (\lambda + 1)^2 (\lambda - 5)$ Det $\lambda_1 = -1$ (二重) , 特征向量: Detail
$$\begin{pmatrix} \alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} & \begin{pmatrix} \beta_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} & \begin{pmatrix} \gamma_1 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \end{pmatrix}$$

$$\lambda_1 = -1$$
 (二重) ,特征向量: • Detail
$$\begin{cases} \alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} & \text{ } \end{cases}$$
 $\beta_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ $\beta_2 = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}$ $\beta_2 = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}$ $\beta_3 = 5$ 特征向量: • Detail $\beta_4 = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}$ $\beta_4 = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}$ $\beta_5 = \begin{pmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix}$

• $\lambda_2 = 5$,特征向量: $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_3 = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$

取
$$Q = \underbrace{\begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}}_{\text{N}}$$
, 则 $Q^{-1}AQ = \begin{pmatrix} -1 & -1 & 0 \\ -1 & 0 & 1/\sqrt{3} & 0 \end{pmatrix}$



———The End———

$$0 = |\lambda I - A| =$$



$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ -2 & -2 & \lambda - 1 \end{vmatrix}$$

$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ -2 & -2 & \lambda - 1 \end{vmatrix}$$

$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ -2 & -2 & \lambda - 1 \end{vmatrix}$$

$$\frac{r_3 - r_2}{2} \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -\lambda - 1 & \lambda + 1 \end{vmatrix}$$



$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ -2 & -2 & \lambda - 1 \end{vmatrix}$$
$$\frac{r_3 - r_2}{2} \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -\lambda - 1 & \lambda + 1 \end{vmatrix}$$
$$= (\lambda + 1) \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -1 & 1 \end{vmatrix}$$

$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ -2 & -2 & \lambda - 1 \end{vmatrix}$$

$$\frac{r_3 - r_2}{2} \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -\lambda - 1 & \lambda + 1 \end{vmatrix}$$

$$= (\lambda + 1) \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -1 & 1 \end{vmatrix} \frac{c_2 + c_3}{2}$$

$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ -2 & -2 & \lambda - 1 \end{vmatrix}$$

$$\frac{r_3 - r_2}{2} \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -\lambda - 1 & \lambda + 1 \end{vmatrix}$$

$$= (\lambda + 1) \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -1 & 1 \end{vmatrix} \xrightarrow{c_2 + c_3} (\lambda + 1) \begin{vmatrix} \lambda - 1 & -4 & -2 \\ -2 & \lambda - 3 & -2 \\ 0 & 0 & 1 \end{vmatrix}$$



$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ -2 & -2 & \lambda - 1 \end{vmatrix}$$

$$\frac{r_3 - r_2}{2} \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -\lambda - 1 & \lambda + 1 \end{vmatrix}$$

$$= (\lambda + 1) \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -1 & 1 \end{vmatrix} \stackrel{c_2 + c_3}{=} (\lambda + 1) \begin{vmatrix} \lambda - 1 & -4 & -2 \\ -2 & \lambda - 3 & -2 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (\lambda + 1) \begin{vmatrix} \lambda - 1 & -4 \\ -2 & \lambda - 3 \end{vmatrix}$$



$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ -2 & -2 & \lambda - 1 \end{vmatrix}$$

$$\frac{r_3 - r_2}{2} \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -\lambda - 1 & \lambda + 1 \end{vmatrix}$$

$$= (\lambda + 1) \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -1 & 1 \end{vmatrix} \frac{c_2 + c_3}{2} (\lambda + 1) \begin{vmatrix} \lambda - 1 & -4 & -2 \\ -2 & \lambda - 3 & -2 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (\lambda + 1) \begin{vmatrix} \lambda - 1 & -4 \\ -2 & \lambda - 3 \end{vmatrix}$$





 $=(\lambda+1)(\lambda^2-4\lambda-5)$

$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix}$$

$$\frac{r_3 - r_2}{2} \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -\lambda - 1 & \lambda + 1 \end{vmatrix}$$

$$= (\lambda + 1) \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -1 & 1 \end{vmatrix} \frac{c_2 + c_3}{2} (\lambda + 1) \begin{vmatrix} \lambda - 1 & -4 & -2 \\ -2 & \lambda - 3 & -2 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (\lambda + 1) \begin{vmatrix} \lambda - 1 & -4 \\ -2 & \lambda - 3 \end{vmatrix}$$

$$= (\lambda + 1)(\lambda^2 - 4\lambda - 5)$$



 $=(\lambda+1)^2(\lambda-5)$

•
$$\exists \lambda_1 = -1$$
, $\forall M (\lambda_1 I - A) x = 0$:

$$(-I - A : 0) =$$





$$(-I-A:0) = \begin{pmatrix} -2 & -2 & -2 & 0 \\ -2 & -2 & -2 & 0 \\ -2 & -2 & -2 & 0 \end{pmatrix} \rightarrow$$





$$(-I-A:0) = \begin{pmatrix} -2 & -2 & -2 & | & 0 \\ -2 & -2 & -2 & | & 0 \\ -2 & -2 & -2 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$





$$(-I - A \vdots 0) = \begin{pmatrix} -2 & -2 & -2 & | & 0 \\ -2 & -2 & -2 & | & 0 \\ -2 & -2 & -2 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$x_1 + x_2 + x_3 = 0$$





$$(-I-A:0) = \begin{pmatrix} -2 & -2 & -2 & | & 0 \\ -2 & -2 & -2 & | & 0 \\ -2 & -2 & -2 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$x_1 + x_2 + x_3 = 0 \Rightarrow x_1 = -x_2 - x_3$$





$$(-I-A:0) = \begin{pmatrix} -2 & -2 & -2 & | & 0 \\ -2 & -2 & -2 & | & 0 \\ -2 & -2 & -2 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$x_1 + x_2 + x_3 = 0$$
 \Rightarrow $x_1 = -x_2 - x_3$ 基础解系: $\alpha_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



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• $\exists \lambda_2 = 5$, $\forall x \in (\lambda_2 I - A)x = 0$:

$$(5I - A : 0) =$$

• 当 $\lambda_2 = 5$,求解 $(\lambda_2 I - A)x = 0$:

$$(5I - A \vdots 0) = \begin{pmatrix} 4 & -2 & -2 & | & 0 \\ -2 & 4 & -2 & | & 0 \\ -2 & -2 & 4 & | & 0 \end{pmatrix}$$

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$$(5I - A \vdots 0) = \begin{pmatrix} 4 & -2 & -2 & 0 \\ -2 & 4 & -2 & 0 \\ -2 & -2 & 4 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & -1 & -1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{pmatrix}$$



• 当 $\lambda_2 = 5$, 求解 $(\lambda_2 I - A)x = 0$:

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$$r_1 \leftrightarrow r_3$$



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所以
$$\begin{cases} x_1 & -x_3 = 0 \end{cases}$$





$$(5I - A \vdots 0) = \begin{pmatrix} 4 & -2 & -2 & | & 0 \\ -2 & 4 & -2 & | & 0 \\ -2 & -2 & 4 & | & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & -1 & -1 & | & 0 \\ 1 & -2 & 1 & | & 0 \\ 1 & 1 & -2 & | & 0 \end{pmatrix}$$

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所以
$$\begin{cases} x_1 & -x_3 = 0 \\ x_2 - x_3 = 0 \end{cases}$$





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所以
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$$\begin{pmatrix} 1 & 1 & -2 & 0 \\ 1 & 1 & -2 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{r_1 - r_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

所以 $\begin{cases} x_1 & -x_3 = 0 \\ x_2 - x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = x_3 \\ x_2 = x_3 \end{cases}$

基础解系:
$$\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



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所以 $\begin{cases} x_1 & -x_3 = 0 \\ x_2 - x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = x_3 \\ x_2 = x_3 \end{cases}$

基础解系:
$$\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$



将线性无关组
$$\alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$
, $\alpha_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ 正交化:





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$$\beta_1 =$$

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$$\beta_2 = \alpha_2 - \cdots - \beta_1$$

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$$\beta_1 = \alpha_1 = \left(\begin{array}{c} -1\\1\\0 \end{array}\right)$$

$$\beta_2 = \alpha_2 - \dots - \beta_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - - \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$





将线性无关组
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▶ Back

