第12章 d: 函数展开成幂级数

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$$f(x) \stackrel{?}{=} a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

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$$n \cdot (n-1) \cdot \cdot \cdot (n-k+1) \cdot (x-x_0)^{n-k}$$

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$$= a_k \cdot k!$$

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$$= a_k \cdot k! + a_{k+1} \cdot (k+1) \cdots 2 \cdot (x - x_0)$$

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$$= a_k \cdot k! + a_{k+1} \cdot (k+1) \cdots 2 \cdot (x - x_0)$$

$$+ a_{k+2} \cdot (k+2) \cdots 3 \cdot (x - x_0)^2 + \cdots$$

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证明 逐项求 k 次导得:

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取 $x = x_0$ 得 $a_k = \frac{1}{k!} f^{(k)}(x_0)$



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$$f(x_0) f'(x_0)$$

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注 1 也就是,f(x) 至多能展成如下形式的幂级数:

$$f(x_0)+f'(x_0)(x-x_0)+\frac{1}{2!}f''(x_0)(x-x_0)^2+\cdots+\frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n+\cdots$$

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- 此级数称为 f(x) 在 x_0 处的 泰勒级数。

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- 记 p_n 为所有次数 ≤ n 的项之和(部分和),称为 n 次泰勒多项式

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注 2 $f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0) (x - x_0)^n$

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注 2 $f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0) (x - x_0)^n \quad \Leftrightarrow \quad f(x) = \lim_{n \to \infty} p_n(x)$

解 取
$$x_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

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当
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⇒
$$\bar{x}$$
 \$\text{\$\pi\$}\$ \$\pi\$\$ \$\pi\$ \$\p

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⇒
$$\$$$
 勒级数: $1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots + \frac{1}{n!}x^n + \dots$

注 n 次泰勒多项式是:

$$p_n(x) =$$



解取 $x_0 = 0$ 时,泰勒级数是

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⇒
$$\bar{x}$$
 \$\text{\$\pi\$}\$ \$\pi\$\$ \$\pi\$ \$\pi\$\$ \$\pi\$ \$\

 $\frac{3}{2!} + \frac{3!}{3!} + \cdots + \frac{n!}{n!} + \cdots$

注 n 次泰勒多项式是:

$$p_n(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots + \frac{1}{n!}x^n$$

例 2 求 $f(x) = \sin x$ 在 x = 0 处的泰勒级数。

$$\mathbf{H}$$
 取 $x_0 = 0$ 时,泰勒级数是

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当 $f(x) = \sin x$ 时,

	$f^{(n)}(x)$	$f^{(n)}(0)$
n = 0, 4, 8	sin x	0
<i>n</i> = 1, 5, 9	cosx	1
n = 2, 6, 10	— sin <i>x</i>	0
n = 3, 7, 11	— cos x	-1

解 取 $x_0 = 0$ 时,泰勒级数是

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当 $f(x) = \sin x$ 时,

	$f^{(n)}(x) = \sin(x + \frac{n}{2}\pi)$	$f^{(n)}(0)$
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所以泰勒级数是

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \cdots$$

例 2 求 $f(x) = \sin x$ 在 x = 0 处的泰勒级数。

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$$x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\frac{1}{7!}x^7+\frac{1}{9!}x^9-\frac{1}{11!}x^{11}+\cdots+(-1)^m\frac{1}{(2m+1)!}x^{2m+1}+\cdots$$

sin x 的泰勒级数是:

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sin x 的 n 次泰勒多项式是:

$$x - \frac{1}{3!}x^{3} + \frac{1}{5!}x^{5} - \frac{1}{7!}x^{7} + \frac{1}{9!}x^{9} - \frac{1}{11!}x^{11} + \dots + (-1)^{m} \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

$$p_1 = x$$
;

$$x - \frac{1}{3!}x^{3} + \frac{1}{5!}x^{5} - \frac{1}{7!}x^{7} + \frac{1}{9!}x^{9} - \frac{1}{11!}x^{11} + \dots + (-1)^{m} \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

$$p_1 = p_2 = x$$
;

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

$$p_1 = p_2 = x;$$

 $p_3 = x - \frac{1}{3!}x^3;$

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$$p_3 = p_4 = x - \frac{1}{3!}x^3;$$

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

$$p_1 = p_2 = x;$$

$$p_3 = p_4 = x - \frac{1}{3!}x^3;$$

$$p_5 = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5;$$

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

$$p_1 = p_2 = x;$$

$$p_3 = p_4 = x - \frac{1}{3!}x^3;$$

$$p_5 = p_6 = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5;$$

$$x - \frac{1}{3!}x^{3} + \frac{1}{5!}x^{5} - \frac{1}{7!}x^{7} + \frac{1}{9!}x^{9} - \frac{1}{11!}x^{11} + \dots + (-1)^{m} \frac{1}{(2m+1)!}x^{2m+1} + \dots$$

sin x 的 n 次泰勒多项式是:

 $p_1 = p_2 = x$;

$$p_3 = p_4 = x - \frac{1}{3!}x^3;$$

$$p_5 = p_6 = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5;$$

$$\vdots$$

 p_{2m+1}



$$\frac{1}{2}$$
 $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{7}{2}$

 $p_1 = p_2 = x$;

 p_{2m+1}

 $x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$

$$1_{x_{3}}$$
 $1_{x_{5}}$ $1_{x_{7}}$

sin x 的 n 次泰勒多项式是:

 $p_3 = p_4 = x - \frac{1}{3!}x^3;$

 $p_5 = p_6 = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5;$

 $= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1}$

$$\frac{1}{2}$$
 $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$

 $p_1 = p_2 = x$;

 $x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + \dots$

sin x 的 n 次泰勒多项式是:

 $p_3 = p_4 = x - \frac{1}{3!}x^3;$

 $p_5 = p_6 = x - \frac{1}{2!}x^3 + \frac{1}{5!}x^5;$

 $p_{2m+1} = p_{2m+2} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1}$

- sin x 的泰勒级数是:

解 取
$$x_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = \cos x$$
时,

解 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = \cos x$ 时,

	$f^{(n)}(x)$	$f^{(n)}(0)$
n = 0, 4, 8	cosx	1
<i>n</i> = 1, 5, 9	— sin <i>x</i>	0
n = 2, 6, 10	— cos x	-1
n = 3, 7, 11	sin x	0

解 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = \cos x$ 时,

	$f^{(n)}(x) = \cos(x + \frac{n}{2}\pi)$	f ⁽ⁿ⁾ (0)
n = 0, 4, 8	cosx	1
n = 1, 5, 9	— sin <i>x</i>	0
n = 2, 6, 10	— cos <i>x</i>	-1
<i>n</i> = 3, 7, 11	sin x	0

解 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = \cos x$ 时,

	$f^{(n)}(x) = \cos(x + \frac{n}{2}\pi)$	$f^{(n)}(0) = \cos(\frac{n}{2}\pi)$
n = 0, 4, 8	cosx	1
$n = 1, 5, 9 \dots$	— sin <i>x</i>	0
n = 2, 6, 10	— cos <i>x</i>	-1
n = 3, 7, 11	sin x	0

$$\mathbf{H}$$
 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = \cos x$ 时,

	$f^{(n)}(x) = \cos(x + \frac{n}{2}\pi)$	$f^{(n)}(0) = \cos(\frac{n}{2}\pi)$
n = 0, 4, 8	cosx	1
n = 1, 5, 9	— sin <i>x</i>	0
n = 2, 6, 10	— cos x	-1
<i>n</i> = 3, 7, 11	sin x	0

所以泰勒级数是

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \cdots$$



解 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = \cos x$ 时,

	$f^{(n)}(x) = \cos(x + \frac{n}{2}\pi)$	$f^{(n)}(0) = \cos(\frac{n}{2}\pi)$
n = 0, 4, 8	cosx	1
n = 1, 5, 9	— sin <i>x</i>	0
<i>n</i> = 2, 6, 10	— cos x	-1
<i>n</i> = 3, 7, 11	sin x	0

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$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$



$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

$$p_0 = 1;$$

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

$$p_0 = p_1 = 1$$
;

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

$$p_0 = p_1 = 1;$$

 $p_2 = 1 - \frac{1}{2!}x^2;$

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

● cos x 的 n 次泰勒多项式是:

$$p_2 = p_3 = 1 - \frac{1}{2!}x^2;$$

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

• cos x 的 n 次泰勒多项式是:

$$p_2 = p_3 = 1 - \frac{1}{2!}x^2;$$

$$p_4 = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4;$$



$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

● cos x 的 n 次泰勒多项式是:

$$p_2 = p_3 = 1 - \frac{1}{2!}x^2;$$

 $p_4 = p_5 = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4;$

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

cos x 的 n 次泰勒多项式是:

 $p_0 = p_1 = 1$;

$$p_{2} = p_{3} = 1 - \frac{1}{2!}x^{2};$$

$$p_{4} = p_{5} = 1 - \frac{1}{2!}x^{2} + \frac{1}{4!}x^{4};$$
:

 $p_{2m}(x)$



$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

cos x 的 n 次泰勒多项式是:

 $p_0 = p_1 = 1$;

$$p_2 = p_3 = 1 - \frac{1}{2!}x^2;$$

$$p_4 = p_5 = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4;$$

 $=1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\cdots+(-1)^m\frac{1}{(2m)!}x^{2m}$







 $p_{2m}(x)$

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + \dots$$

 $p_{2m}(x) = p_{2m+1}(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots + (-1)^m \frac{1}{(2m)!}x^{2m}$

cos x 的 n 次泰勒多项式是:

$$p_2 = p_3 = 1 - \frac{1}{2!}x^2;$$

$$p_4 = p_5 = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4;$$





 $\mathbf{H} \mathbf{H} \mathbf{X}_0 = \mathbf{0} \mathbf{H}$,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = \ln(1+x)$$
时,

解 取
$$x_0 = 0$$
 时,泰勒级数是
$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = \ln(1+x)$$
时,
 $f = \ln(1+x)$, $f' = \frac{1}{1+x}$,

解 取
$$x_0 = 0$$
 时,泰勒级数是
$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = \ln(1+x)$$
时,
 $f = \ln(1+x)$, $f' = \frac{1}{1+x}$, $f'' = \frac{(-1)}{(1+x)^2}$,

解 取
$$x_0 = 0$$
 时,泰勒级数是
$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = \ln(1+x)$$
时,
 $f = \ln(1+x)$, $f' = \frac{1}{1+x}$, $f'' = \frac{(-1)}{(1+x)^2}$, $f''' = \frac{2}{(1+x)^3}$,

解 取
$$x_0 = 0$$
 时,泰勒级数是
$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

$$f^{(4)} = \frac{-2 \cdot 3}{(1+x)^4},$$

解 取
$$x_0 = 0$$
 时,泰勒级数是
$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

$$f^{(4)} = \frac{-2 \cdot 3}{(1+x)^4}, \quad f^{(5)} = \frac{2 \cdot 3 \cdot 4}{(1+x)^5}, \dots$$

 \mathbf{H} 取 $\mathbf{x}_0 = \mathbf{0}$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = \ln(1+x)$ 时, $f = \ln(1+x)$, $f' = \frac{1}{1+x}$, $f'' = \frac{(-1)}{(1+x)^2}$, $f''' = \frac{2}{(1+x)^3}$,

$$f = \ln(1+x), \quad f' = \frac{1}{1+x}, \quad f'' = \frac{(-1)^2}{(1+x)^2}, \quad f''' = \frac{1}{(1+x)^3},$$
$$f^{(4)} = \frac{-2 \cdot 3}{(1+x)^4}, \quad f^{(5)} = \frac{2 \cdot 3 \cdot 4}{(1+x)^5}, \dots, f^{(n)} = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}, \dots$$

解 取
$$x_0 = 0$$
 时,泰勒级数是
$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

$$f(0) + f'(0)x + \frac{f'(0)}{2!}x^2 + \frac{f'(0)}{3!}x^3 + \dots + \frac{f'(0)}{n!}x^n + \dots$$

当
$$f(x) = \ln(1+x)$$
 时,
 $f = \ln(1+x)$, $f' = \frac{1}{1+x}$, $f'' = \frac{(-1)}{(1+x)^2}$, $f''' = \frac{2}{(1+x)^3}$,
 $f^{(4)} = \frac{-2 \cdot 3}{(1+x)^4}$, $f^{(5)} = \frac{2 \cdot 3 \cdot 4}{(1+x)^5}$, ..., $f^{(n)} = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$, ...

所以
$$\frac{1}{n!}f^{(n)}(0) = \frac{(-1)^{n-1}}{n}$$
,

例 $4 \, \bar{x} f(x) = \ln(1+x) \, \bar{x} = 0 \,$ 处泰勒级数。

m 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = \ln(1+x)$$
时,

$$f = \ln(1+x), \quad f' = \frac{1}{1+x}, \quad f'' = \frac{(-1)}{(1+x)^2}, \quad f''' = \frac{2}{(1+x)^3},$$

$$f^{(4)} = \frac{-2 \cdot 3}{(1+x)^4}, \quad f^{(5)} = \frac{2 \cdot 3 \cdot 4}{(1+x)^5}, \dots, f^{(n)} = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}, \dots$$

$$\text{MU} \frac{1}{n!} f^{(n)}(0) = \frac{(-1)^{n-1}}{n}, \quad \text{\$$ $\overline{4}$ $\overline{$$

所以
$$\frac{1}{n!} f^{(n)}(0) = \frac{1}{n}$$
 , 泰朝级数是
$$x - \frac{1}{2} x^2 + \frac{1}{3} x^3 - \frac{1}{4} x^4 + \cdots$$

例 $4 \, \bar{x} f(x) = \ln(1+x) \, \bar{x} = 0 \,$ 处泰勒级数。

解 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = \ln(1+x)$ 时,

$$f = \ln(1+x), \quad f' = \frac{1}{1+x}, \quad f'' = \frac{(-1)}{(1+x)^2}, \quad f''' = \frac{2}{(1+x)^3},$$

$$f^{(4)} = \frac{-2 \cdot 3}{(1+x)^4}, \quad f^{(5)} = \frac{2 \cdot 3 \cdot 4}{(1+x)^5}, \dots, f^{(n)} = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}, \dots$$

$$x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots$$

 \mathbf{H} 取 $\mathbf{x}_0 = \mathbf{0}$ 时,泰勒级数是

解 取
$$X_0 = 0$$
 的, 泰剌级数 $f''(0)$

 $f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$

当 $f(x) = \ln(1+x)$ 时,

所以 $\frac{1}{n!}f^{(n)}(0) = \frac{(-1)^{n-1}}{n}$,泰勒级数是

例 4 求 $f(x) = \ln(1 + x)$ 在 x = 0 处泰勒级数。

 $f^{(4)} = \frac{-2 \cdot 3}{(1+x)^4}, \quad f^{(5)} = \frac{2 \cdot 3 \cdot 4}{(1+x)^5}, \dots, f^{(n)} = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}, \dots$

 $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots$

 $f = \ln(1+x), \quad f' = \frac{1}{1+x}, \quad f'' = \frac{(-1)}{(1+x)^2}, \quad f''' = \frac{2}{(1+x)^3},$

注 n 次泰勒多项式是: $p_n(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots + (-1)^{n-1}\frac{1}{n}x^n$

解 取
$$x_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = (1+x)^{\alpha}$$
时,

解 取
$$x_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = (1+x)^{\alpha}$$
 时,
 $f = (1+x)^{\alpha}$, $f' = \alpha(1+x)^{\alpha-1}$,

$$\mathbf{H}$$
 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = (1+x)^{\alpha}$$
时,

$$f = (1+x)^{\alpha}$$
, $f' = \alpha(1+x)^{\alpha-1}$, $f'' = \alpha(\alpha-1)(1+x)^{\alpha-2}$,

$$\mathbf{H}$$
 取 $\mathbf{x}_0 = \mathbf{0}$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = (1+x)^{\alpha}$$
 时,

$$f = (1+x)^{\alpha}, \quad f' = \alpha(1+x)^{\alpha-1}, \quad f'' = \alpha(\alpha-1)(1+x)^{\alpha-2},$$

$$\ldots, f^{(n)} = \alpha(\alpha - 1) \cdots (\alpha - n + 1)(1 + x)^{\alpha - n}, \cdots$$

$$\mathbf{H}$$
 取 $x_0 = 0$ 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = (1+x)^{\alpha}$$
 时,

$$f = (1+x)^{\alpha}, \quad f' = \alpha(1+x)^{\alpha-1}, \quad f'' = \alpha(\alpha-1)(1+x)^{\alpha-2},$$

$$\ldots, f^{(n)} = \alpha(\alpha - 1) \cdots (\alpha - n + 1)(1 + x)^{\alpha - n}, \cdots$$

所以
$$\frac{1}{n!}f^{(n)}(0) = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$$
,

解
$$\mathbf{x}_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = (1+x)^{\alpha}$$
 时,

$$f = (1+x)^{\alpha}$$
, $f' = \alpha(1+x)^{\alpha-1}$, $f'' = \alpha(\alpha-1)(1+x)^{\alpha-2}$,

$$\ldots, f^{(n)} = \alpha(\alpha - 1) \cdots (\alpha - n + 1)(1 + x)^{\alpha - n}, \cdots$$

所以
$$\frac{1}{n!}f^{(n)}(0) = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$$
,泰勒级数是

$$1 + \alpha x + \frac{\alpha(\alpha - 1)}{2!}x^2 + \cdots$$

解 取
$$x_0 = 0$$
 时,泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当
$$f(x) = (1+x)^{\alpha}$$
 时,

$$f = (1+x)^{\alpha}, \quad f' = \alpha(1+x)^{\alpha-1}, \quad f'' = \alpha(\alpha-1)(1+x)^{\alpha-2},$$

$$\ldots, f^{(n)} = \alpha(\alpha - 1) \cdots (\alpha - n + 1)(1 + x)^{\alpha - n}, \cdots$$

所以
$$\frac{1}{n!}f^{(n)}(0) = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$$
,泰勒级数是

$$1 + \alpha x + \frac{\alpha(\alpha - 1)}{2!}x^2 + \dots + \frac{\alpha(\alpha - 1)\cdots(\alpha - n + 1)}{n!}x^n + \dots$$

 $\mathbf{m} \mathbf{n} \mathbf{x}_0 = \mathbf{0} \mathbf{n}$, 泰勒级数是

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{2!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

当 $f(x) = (1+x)^{\alpha}$ 时,

$$f = (1+x)^{\alpha}, \quad f' = \alpha(1+x)^{\alpha-1}, \quad f'' = \alpha(\alpha-1)(1+x)^{\alpha-2},$$

$$\ldots, f^{(n)} = \alpha(\alpha - 1) \cdots (\alpha - n + 1)(1 + x)^{\alpha - n}, \cdots$$

所以
$$\frac{1}{n!}f^{(n)}(0) = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$$
,泰勒级数是
$$\alpha(\alpha-1) = \alpha(\alpha-1)\cdots(\alpha-n+1)$$

$$1 + \alpha x + \frac{\alpha(\alpha - 1)}{2!}x^2 + \dots + \frac{\alpha(\alpha - 1)\cdots(\alpha - n + 1)}{n!}x^n + \dots$$

注 n 次泰勒多项式是:

$$p_n(x) =$$



 $\mathbf{H} \mathbf{H} \mathbf{X}_0 = \mathbf{0} \mathbf{H}$,泰勒级数是

当 $f(x) = (1+x)^{\alpha}$ 时,

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

例 5 求 $f(x) = (1 + x)^{\alpha}$ 在 x = 0 处的 n 次泰勒多项式 $p_n(x)$

 $f = (1+x)^{\alpha}$, $f' = \alpha(1+x)^{\alpha-1}$, $f'' = \alpha(\alpha-1)(1+x)^{\alpha-2}$ $f^{(n)} = \alpha(\alpha - 1) \cdots (\alpha - n + 1)(1 + x)^{\alpha - n}, \cdots$

$$\dots, f^{(n)} = \alpha(\alpha - 1) \cdots (\alpha - n + 1)(1 + x)^{\alpha - n}, \dots$$
所以 $\frac{1}{n!} f^{(n)}(0) = \frac{\alpha(\alpha - 1) \cdots (\alpha - n + 1)}{n!}$,泰勒级数是

所以
$$\frac{1}{n!}f^{(n)}(0) = \frac{\alpha(\alpha-1)}{n!}$$
, 泰朝级数是
$$1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \dots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n + \dots$$

注 n 次泰勒多项式是: $p_n(x) = 1 + \alpha x + \frac{\alpha(\alpha - 1)}{2!}x^2 + \dots + \frac{\alpha(\alpha - 1)\cdots(\alpha - n + 1)}{n!}x^n$

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0) (x - x_0)^n$$

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0) (x - x_0)^n \iff f(x) = \lim_{n \to \infty} p_n(x)$$

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$$(\Leftrightarrow R_n(x) = f(x) - p_n(x))$$

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$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0) (x - x_0)^n \iff f(x) = \lim_{n \to \infty} p_n(x)$$

$$\Leftrightarrow \lim_{n \to \infty} [f(x) - p_n(x)] = 0$$

$$(R_n(x) = f(x) - p_n(x))$$

$$\Leftrightarrow \lim_{n \to \infty} R_n(x) = 0$$

注 $R_n(x) = f(x) - p_n(x)$,或者 $f(x) = p_n(x) + R_n(x)$,刻画了原函数 f(x) 与其泰勒多项式 $p_n(x)$ 的差异。



第 12 章 d:函数展开成幂级数

回忆 泰勒中值定理 1 若 f 具有 n 阶导数,则

$$R_n(x) = o((x-x_0)^n).$$

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特别地,

$$f(x) = p_n(x) + R_n(x)$$

$$= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + o((x - x_0)^n)$$

 $e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + o(x^{n})$

 $(1+x)^{\alpha} = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \dots + \frac{\alpha(\alpha-1) \cdots (\alpha-n+1)}{n!} x^n + o(x^n)$

 $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots + (-1)^{n-1}\frac{1}{n}x^n + o(x^n)$

 $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots + (-1)^m \frac{1}{(2m)!}x^{2m} + o(x^{2m+1})$

 $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots + (-1)^m \frac{1}{(2m+1)!}x^{2m+1} + o(x^{2m+2})$

例求 $\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$, $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1-x)]}$

例求
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
, $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1-x)]}$

$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$

例求
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
, $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1-x)]}$

$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$$



例求
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
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$$= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3}$$

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$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
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$$= \lim_{x \to 0} \frac{\frac{1}{3!}x^3 + o(x^4)}{x^3} - 1$$

例求
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
, $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1-x)]}$

$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x}, \quad \lim_{x \to 0} \frac{x^2 [x + \ln(1 - x)]}{x^2 [x + \ln(1 - x)]}$$

$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$$

$$\sin^{3} x \qquad x \to 0$$

$$= \lim_{x \to 0} \frac{\frac{1}{3}x^{3} + o(x^{4})}{x^{3}} = \frac{1}{3}$$

$$= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3} = \frac{1}{3}$$

$$\cos x - e^{-\frac{x^2}{2}}$$

$$\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2 [x + \ln(1 - x)]}$$

$$= \lim_{x \to 0} \frac{1}{x^{2} [x + \ln(1 - x)]}$$

$$= \lim_{x \to 0} \frac{\left[\frac{1}{x^{2} [x + (1 - x)]} \right] - \left[\frac{1}{x^{2} [x + (1 - x)]} \right]}{x^{2} [x + (1 - x)]}$$



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例求
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
, $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1-x)]}$

$$\lim_{x \to 0} \frac{1}{\sin^3 x}, \lim_{x \to 0} \frac{1}{x^2 [x + \ln(1 - x)]}$$

$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!} x^3 + o(x^4)\right] - x \left[1 - \frac{1}{2!} x^2 + o(x^3)\right]}{x^3}$$

$$= \lim_{x \to 0} \frac{\frac{1}{3} x^3 + o(x^4)}{x^3} = \frac{1}{3}$$

$$\lim_{x \to 0} x^3$$

$$\lim_{x \to 0} \cos x - e^{-\frac{x^2}{2}}$$

$$\lim_{x \to 0} \frac{1}{x^{2} [x + \ln(1 - x)]}$$

$$= \lim_{x \to 0} \frac{\left[\frac{1}{x^{2} [x + (-x - \frac{1}{2}x^{2} + o(x^{2}))]} \right]}{x^{2} [x + (-x - \frac{1}{2}x^{2} + o(x^{2}))]}$$



例求
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
, $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1-x)]}$

$$\lim_{x \to 0} \frac{\sin^3 x}{\sin^3 x}, \frac{\sin^3 x}{x^2 [x + \ln(1 - x)]}$$

$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$$

$$\frac{1}{2}x^3 + o(x^4) = 1$$

$$= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3} = \frac{1}{3}$$

$$\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2 [x + \ln(1 - x)]}$$

$$= \lim_{x \to 0} \frac{\left[1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + o(x^5)\right] - \left[\frac{1}{2!}x^2 + \frac{1}{4!}x^4 + o(x^5)\right]}{x^2 [x + \left(-x - \frac{1}{2}x^2 + o(x^2)\right)]}$$

例求
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
, $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1-x)]}$

例 求
$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
, $\lim_{x \to 0} \frac{\cos x - e^{-2x}}{x^2 [x + \ln(1 - x)]}$

$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$$

$$= \lim_{x \to 0} \frac{3^{x}}{x^{2}}$$

$$\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^{2}}{2}}}{x^{2}[x + \ln(1 - x)]}$$

$$e^{-\frac{x^2}{2}}$$

$$\frac{1}{1}$$

$$\frac{1}{1-x}$$

$$(1-x)$$

$$= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3} = \frac{1}{3}$$

$$\lim_{x \to 0} \frac{\cos x - e^{-\frac{1}{2}}}{x^2 [x + \ln(1 - x)]}$$

$$= \lim_{x \to 0} \frac{\left[1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + o(x^5)\right] - \left[1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + o(x^4)\right]}{x^2 [x + \left(-x - \frac{1}{2}x^2 + o(x^2)\right)]}$$

$$\sin x - x \cos x$$

$$\sin x - x \cos x$$

例求 $\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$, $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2 [x + \ln(1-x)]}$

$$\lim_{x \to 0} \sin^2 x \qquad \lim_{x \to 0} x^{-1} x^{+1} x^{-1} x^{-1}$$

$$\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^2 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + \frac{$$

 $\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$

 $= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3} = \frac{1}{3}$

 $\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2 [x + \ln(1 - x)]}$

 $= \lim_{x \to 0} \frac{\left[1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + o(x^5)\right] - \left[1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + o(x^4)\right]}{x^2 \left[x + \left(-x - \frac{1}{2}x^2 + o(x^2)\right)\right]}$

 $= \lim_{x \to 0} \frac{-\frac{1}{12}x^4 + o(x^4)}{-\frac{1}{2}x^4 + o(x^4)}$



例求
$$\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$$
, $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x + \ln(1 - x)]}$

 $\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$ $= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3} = \frac{1}{3}$

$$\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2 [x + \ln(1 - x)]}$$

$$\left[1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 + o(x^5) \right] - \left[1 - \frac{1}{2} x^2 + \frac{1}{8} x^4 + o(x^4) \right]$$

 $= \lim_{x \to 0} \frac{\left[1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + o(x^5)\right] - \left[1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + o(x^4)\right]}{x^2 \left[x + \left(-x - \frac{1}{2}x^2 + o(x^2)\right)\right]}$ $- \frac{1}{2}x^4 + o(x^4)$ $- \frac{1}{2}x^4 + o(x^4)$

$$= \lim_{x \to 0} \frac{-\frac{1}{12}x^4 + o(x^4)}{-\frac{1}{2}x^4 + o(x^4)} = \lim_{x \to 0} \frac{-\frac{1}{12} + o(x^4)/x^4}{-\frac{1}{2} + o(x^4)/x^4}$$

$$\frac{\sin x - x \cos x}{\sin^3 x}$$

例求 $\lim_{x\to 0} \frac{\sin x - x \cos x}{\sin^3 x}$, $\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2 [x + \ln(1-x)]}$

$$\lim_{x \to 0} \frac{\sin^3 x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + \frac{1}{3!}x^$$

 $\lim_{x \to 0} \frac{\sin x - x \cos x}{\sin^3 x} = \lim_{x \to 0} \frac{\left[x - \frac{1}{3!}x^3 + o(x^4)\right] - x\left[1 - \frac{1}{2!}x^2 + o(x^3)\right]}{x^3}$

 $= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^4)}{x^3} = \frac{1}{3}$

$$\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^2 [x + \ln(1 - x)]}$$

$$= \lim_{x \to 0} \frac{\left[1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + o(x^5)\right] - \left[1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + o(x^4)\right]}{x^2 \left[x + \left(-x - \frac{1}{2}x^2 + o(x^2)\right)\right]}$$

$$= \lim_{x \to 0} \frac{L^{\perp}}{}$$

$$= \lim_{X \to 0} \frac{-\frac{1}{12}x^4 + o(x^4)}{-\frac{1}{2}x^4 + o(x^4)} = \lim_{X \to 0} \frac{-\frac{1}{12} + o(x^4)/x^4}{-\frac{1}{2} + o(x^4)/x^4} = \frac{1}{6}$$
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泰勒中值定理 2 若 f 具有 n+1 阶导数,则

$$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) (x - x_0)^{n+1}$$

其中 ξ 是 x_0 与 x 之间的某个值

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$$(1-\theta)x_0+\theta x$$

其中 ξ 是 x_0 与 x 之间的某个值, $0 < \theta < 1$ 。

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注

1. ξ (以及 θ) 不是固定不变的,而是随 x 和 n 的改变而变化。

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注

- 1. ξ (以及 θ) 不是固定不变的,而是随 x 和 n 的改变而变化。
- 2. 当 $x_0 = 0$ 时,则余项可写成

$$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1}, \qquad (0 < \theta < 1)$$



$$e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty).$$

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证明

1. 只需证明对任意 x,成立 $\lim_{n\to\infty} R_n(x) = 0$ 。

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$$e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty).$$

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(日知级数 $\sum \frac{|x|^{n+1}}{n} |x|^{n+1} = 0$)

(已知级数 $\sum_{(n+1)}^{|x|^{n+1}}$ 收敛,所以一般项 $\frac{|x|^{n+1}}{(n+1)!} \to 0$)

$$e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty).$$

证明

- 1. 只需证明对任意 x, 成立 $\lim_{n \to \infty} R_n(x) = 0$ 。
- 2. 由泰勒中值定理 2.

$$|R_n(x)| = \left| \frac{1}{(n+1)!} f^{(n+1)}(\theta x) x^{n+1} \right| = \left| \frac{e^{\theta x} x^{n+1}}{(n+1)!} \right| \le \frac{|x|^{n+1}}{(n+1)!} e^{|x|} \to 0$$

$$(已知级数 \sum_{n=1}^{|x|^{n+1}} |\psi_n|, \quad \text{MU} - \text{MV} \frac{|x|^{n+1}}{(n+1)!} \to 0)$$

注 $e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$



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• 至此,我们知道 e^x , $\sin x$, $\cos x$ 以及 $\frac{1}{1+x}$ 是等于其泰勒级数,即

$$e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty)$$

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$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots + (-1)^n x^n + \dots, x \in (-1, 1)$$

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• 利用上述结果,及逐项积分公式,可进一步求出

ln(1+x), arctan x

的幂级数展开。



性质 成立
$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots$$
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证明 1. 幂级数的收敛域是 (-1, 1],故上式至多对 $x \in (-1, 1]$ 成立。

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$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n.$$



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 $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^{n}.$ $\overrightarrow{\Lambda}$

$$n=0$$
 $n=1$ $n=1$

(这是*f*(1)=

= S(1)

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 $=\sum_{n=1}^{\infty}(-1)^n\frac{1}{n+1}x^{n+1}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n$

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$$\lim_{x \to 1^{-}} S(x) = S(1)$$



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In(1+x) = x -
$$\frac{1}{2}$$
x² + $\frac{1}{3}$ x³ + ···+ (-1)ⁿ⁻¹ $\frac{1}{n}$ xⁿ + ··· , x ∈ (-1, 1].

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$$= \sum_{n=0}^\infty (-1)^n \frac{1}{n+1} x^{n+1} = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n} x^n$$

3. 注意到 $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$ 收敛域是 (-1, 1], 由连续性, 当 x = 1 时也成

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性质 成立 $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots + (-1)^n \frac{1}{2n+1}x^{2n+1} + \dots, x \in [-1,1].$

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 2.77 年 1 3. 注意到 $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$ 收敛域是 $[-1, 1]$, 由连续性, 当 $x=\pm 1$ 时也

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第 12 章 d:函数展开成幂级数

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第 12 章 d:函数展开成幂级数

 $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots + (-1)^n \frac{1}{2n+1}x^{2n+1} + \dots, x \in [-1,1].$

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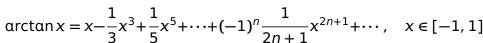
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 $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots + (-1)^n \frac{1}{2n+1}x^{2n+1} + \dots, \quad x \in [-1, 1]$

注 $\mathbf{x} = 1$,则得到

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注 取
$$x = 1$$
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$$\pi$$
 ,则得到 π , π ,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$$

 $\cos x = 1 - \tfrac{1}{2!} x^2 + \tfrac{1}{4!} x^4 - \tfrac{1}{6!} x^6 + \dots + (-1)^n \tfrac{1}{(2n)!} x^{2n} + \dots , \ x \in (-\infty, \infty)$

• 至此, 得出如下常用函数的幂级数展开式:

 $e^{x} = 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \dots + \frac{1}{n!}x^{n} + \dots, \quad x \in (-\infty, \infty)$

 $\sin x = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 + \dots + (-1)^n \frac{1}{(2n+1)!} x^{2n+1} + \dots, \ x \in (-\infty, \infty)$

 $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots + (-1)^n x^n + \dots, x \in (-1, 1)$

 $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + (-1)^{n-1}\frac{1}{n}x^n + \dots, \quad x \in (-1, 1]$ $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots + (-1)^n \frac{1}{2n+1}x^{2n+1} + \dots, x \in [-1,1]$

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• 用上述结果, 及逐项求导、积分公式, 可求更多函数的泰勒级数展开

 $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots + (-1)^n x^n + \dots, x \in (-1, 1)$

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例 1 把函数 $f(x) = (1-x)\ln(1+x)$ 展开成 x 的幂级数。

 $=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}x^n-\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}x^{n+1}$

 $=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}x^n-\sum_{n=2}^{\infty}(-1)^{n-2}\frac{1}{n-1}x^n$

$$= x + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{1}{n} x^n - \sum_{n=2}^{\infty} (-1)^{n-2} \frac{1}{n-1} x^n$$

$$= x + \sum_{n=2}^{\infty} \left(\frac{(-1)^{n-1}}{n} - \frac{(-1)^n}{n-1} \right) x^n$$



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$$\cos t = 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6 + \dots + (-1)^n \frac{1}{(2n)!}t^{2n} + \dots, \ t \in (-\infty, \infty)$$

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$$

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所以当
$$x \in (-\infty, \infty)$$
时,

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$$
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例 2 把函数 $f(x) = \cos^2 x$ 展开成 x 的幂级数。

解利用

$$\cos t = 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6 + \dots + (-1)^n \frac{1}{(2n)!}t^{2n} + \dots, \ t \in (-\infty, \infty)$$

所以当
$$x \in (-\infty, \infty)$$
时,

$$\cos^{2} x = \frac{1}{2} + \frac{1}{2} \cos 2x$$

$$= \frac{1}{2} + \frac{1}{2} \sum_{n=0}^{\infty} (-1)^{n} \frac{1}{(2n)!} (2x)^{2n}$$

$$= \frac{1}{2} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^{n} 2^{2n}}{(2n)!} x^{2n}$$

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所以当 $x \in (-\infty, \infty)$ 时,



第 12 章 d: 函数展开成幂级数

解 1. 注意到
$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$
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2. 利用 $\frac{1}{1-t} = 1 + t + t^2 + t^3 + \dots + t^n + \dots$, $t \in (-1, 1)$ 将 $\frac{1}{v+1}$, $\frac{1}{v+2}$ 分别展开成 (x+4) 的幂级数:

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其中 $\left|\frac{x+4}{3}\right| = \left|\frac{t}{3}\right| < 1$,即 -7 < x < -1。

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其中 $\left|\frac{x+4}{2}\right| = \left|\frac{t}{2}\right| < 1$,即-6 < x < -2。

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3. 所以 -6 < x < -2 时

解 1. 注意到 $\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$.

例 3 把函数 $f(x) = \frac{1}{x^2 + 3x + 2}$ 展开成 (x + 4) 的幂级数。

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$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+1}$$

2. 利用
$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \dots + t^n + \dots$$
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$$x+1$$
 $t-3$ -3 $1-\frac{t}{3}$ $3 \underset{n=0}{\longleftarrow} 3^n$ $\underset{n=0}{\longrightarrow} 3^{n+1}$ 其中 $|\frac{x+4}{3}| = |\frac{t}{3}| < 1$,即 $-7 < x < -1$ 。

1 1 1 1 $\sum_{n=0}^{\infty} t^n$ $\sum_{n=0}^{\infty} (x+4)^n$

其中
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其中 $\left|\frac{x+4}{2}\right| = \left|\frac{t}{2}\right| < 1$,即 $-6 < x < -2$ 。

 $\frac{1}{x^2 + 3x + 2} = \frac{1}{x + 1} - \frac{1}{x + 2} = \sum_{n=0}^{\infty} \left(\frac{1}{2^{n+1}} - \frac{1}{3^{n+1}} \right) (x + 4)_{0}^{n}$