# §1.4 克莱姆法则

数学系 梁卓滨

2018 - 2019 学年上学期



$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_2 + a_{22}y_2 = b_2 \end{cases}$$



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$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} , \quad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

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的解是

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$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$
 称为系数行列式

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•  $D_i$ : 将 D 的第 i 列换成常数项  $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ 



三元线性方程组 
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$$x_{3} = \frac{A_{11} + A_{12} + A_{13} + A$$



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$$\begin{vmatrix} a_{11} & \cdots & a_{1j-1} & a_{1,j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2,j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{n,j} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}$$

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### 定理(克莱姆法则) 线性方程组

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注 1 两个前提: (1) 未知元个数 = 方程个数; (2) 系数行列式  $D \neq 0$ 



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注 1 两个前提: (1) 未知元个数 = 方程个数; (2) 系数行列式  $D \neq 0$ 

注 2 若 D = 0,则方程或者无解、或者有无穷多解(以后详说)



克莱姆法则证明 (仅验证  $x_j = \frac{D_j}{D}$  是解,唯一性的证明要用到矩阵知识,略去。)

$$x_j = \frac{D_j}{D}$$

验证第 
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 条方程成立 ( $k = 1, 2, \dots, n$ ):

$$a_{k1}x_1 + \cdots + a_{kn}x_n =$$

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$$x_{j} = \frac{D_{j}}{D} = \frac{\begin{vmatrix} a_{21} & \cdots & a_{2j-1} & b_{2} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj-1} & b_{n} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}}$$

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 $b_k$ 

D

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**り 御** 暨南大学 克莱姆法则证明 (仅验证  $x_i = \frac{D_i}{D}$  是解,唯一性的证明要用到矩阵知识,略去。)  $\begin{vmatrix} a_{11} & \cdots & a_{1j-1} & b_1 & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & b_2 & a_{2j+1} & \cdots & a_{2n} \end{vmatrix}$ 

 $x_{j} = \frac{D_{j}}{D} = \frac{\begin{vmatrix} \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & b_{n} & a_{nj+1} & \cdots & a_{nn} \\ a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}} = \frac{b_{1}A_{1j} + b_{2} + \cdots + b_{n}}{D}$ 

$$a_{k1}x_1 + \cdots + a_{kn}x_n =$$

克莱姆法则证明 (仅验证  $x_i = \frac{D_i}{D}$  是解,唯一性的证明要用到矩阵知识,略去。)  $\begin{vmatrix} a_{11} & \cdots & a_{1j-1} & b_1 & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & b_2 & a_{2j+1} & \cdots & a_{2n} \end{vmatrix}$ 

 $x_{j} = \frac{D_{j}}{D} = \frac{\begin{vmatrix} \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & b_{n} & a_{nj+1} & \cdots & a_{nn} \\ a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}} = \frac{b_{1}A_{1j} + b_{2}A_{2j} + \cdots + b_{n}}{D}$ 

 $a_{k1}x_1 + \cdots + a_{kn}x_n =$ 



克莱姆法则证明 (仅验证  $x_i = \frac{D_i}{D}$  是解,唯一性的证明要用到矩阵知识,略去。)  $\begin{vmatrix} a_{11} & \cdots & a_{1j-1} & b_1 & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & b_2 & a_{2j+1} & \cdots & a_{2n} \end{vmatrix}$ 

 $x_{j} = \frac{D_{j}}{D} = \frac{\begin{vmatrix} \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & b_{n} & a_{nj+1} & \cdots & a_{nn} \\ a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}} = \frac{b_{1}A_{1j} + b_{2}A_{2j} + \cdots + b_{n}}{D}$ 

 $a_{k1}x_1 + \cdots + a_{kn}x_n =$ 



克莱姆法则证明 (仅验证  $x_j = \frac{D_j}{D}$  是解,唯一性的证明要用到矩阵知识,略去。)  $\begin{vmatrix} a_{11} & \cdots & a_{1j-1} & b_1 & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & b_2 & a_{2j+1} & \cdots & a_{2n} \end{vmatrix}$ 

 $x_{j} = \frac{D_{j}}{D} = \frac{\begin{vmatrix} \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & b_{n} & a_{nj+1} & \cdots & a_{nn} \\ a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}} = \frac{b_{1}A_{1j} + b_{2}A_{2j} + \cdots + b_{n}A_{nj}}{D}$ 

 $a_{k1}X_1 + \cdots + a_{kn}X_n =$ 





克莱姆法则证明 (仅验证  $x_i = \frac{D_i}{D}$  是解,唯一性的证明要用到矩阵知识,略去。)

$$x_{j} = \frac{D_{j}}{D} = \frac{\begin{vmatrix} a_{11} & \cdots & a_{1j-1} & b_{1} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & b_{2} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & b_{n} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}} = \frac{b_{1}A_{1j} + b_{2}A_{2j} + \cdots + b_{n}A_{nj}}{D}$$

$$\sum_{i=1}^{n} b_{i}A_{ij}$$

验证第 
$$k$$
 条方程成立( $k=1,2,\cdots,n$ ):

 $a_{k1}x_1 + \cdots + a_{kn}x_n =$ 



克莱姆法则证明 (仅验证  $x_i = \frac{D_i}{D}$  是解,唯一性的证明要用到矩阵知识,略去。)

$$x_{j} = \frac{D_{j}}{D} = \frac{\begin{vmatrix} a_{11} & \cdots & a_{1j-1} & b_{1} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & b_{2} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & b_{n} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}} = \frac{b_{1}A_{1j} + b_{2}A_{2j} + \cdots + b_{n}A_{nj}}{D}$$
$$= \frac{1}{D} \sum_{i=1}^{n} b_{i}A_{ij}$$

验证第 
$$k$$
 条方程成立( $k = 1, 2, \dots, n$ ):

 $a_{k1}x_1 + \cdots + a_{kn}x_n =$ 



克莱姆法则证明 (仅验证  $X_i = \frac{D_i}{D}$  是解,唯一性的证明要用到矩阵知识,略去。)

$$x_{j} = \frac{D_{j}}{D} = \frac{\begin{vmatrix} a_{11} & \cdots & a_{1j-1} & b_{1} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & b_{2} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj-1} & b_{n} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}} = \frac{b_{1}A_{1j} + b_{2}A_{2j} + \cdots + b_{n}A_{nj}}{D}$$

$$a_{k1}x_1 + \cdots + a_{kn}x_n = \sum_{i=1}^n a_{kj}x_j$$





 $=\frac{1}{D}\sum_{i}^{n}b_{i}A_{ij}$ 

克莱姆法则证明 (仅验证  $X_i = \frac{D_i}{D}$  是解,唯一性的证明要用到矩阵知识,略去。)  $a_{11} \cdots a_{1j-1} \xrightarrow{b_1} a_{1j+1} \cdots a_{1n}$  $a_{21} \cdots a_{2j-1} \xrightarrow{b_2} a_{2j+1} \cdots a_{2n}$ 

$$x_{j} = \frac{D_{j}}{D} = \frac{\begin{vmatrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & b_{n} & a_{nj+1} & \cdots & a_{nn} \\ a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}} = \frac{b_{1}A_{1j} + b_{2}A_{2j} + \cdots + b_{n}A_{nj}}{D}$$

$$= \frac{1}{D} \sum_{i=1}^{n} b_{i}A_{ij}$$
验证第  $k$  条方程成立  $(k = 1, 2, \cdots, n)$ :

 $a_{k1}x_1 + \dots + a_{kn}x_n = \sum_{i=1}^n a_{kj}x_j = \sum_{i=1}^n a_{kj} \left(\frac{1}{D}\sum_{i=1}^n b_i A_{ij}\right)$ 



克莱姆法则证明 (仅验证  $X_i = \frac{D_i}{D}$  是解,唯一性的证明要用到矩阵知识,略去。)  $a_{11} \cdots a_{1j-1} \xrightarrow{b_1} a_{1j+1} \cdots a_{1n}$  $a_{21} \cdots a_{2j-1} \xrightarrow{b_2} a_{2j+1} \cdots a_{2n}$ 

$$x_{j} = \frac{D_{j}}{D} = \frac{\begin{vmatrix} a_{21} & a_{2j-1} & a_{2} & a_{2j+1} & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & b_{n} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}} = \frac{b_{1}A_{1j} + b_{2}A_{2j} + \cdots + b_{n}A_{nj}}{D}$$

$$=\frac{1}{D}\sum_{i=1}^{n}b_{i}A_{ij}$$

验证第 k 条方程成立  $(k = 1, 2, \dots, n)$ :

$$a_{k1}x_1 + \dots + a_{kn}x_n = \sum_{i=1}^n a_{kj}x_j = \sum_{i=1}^n a_{kj} \left(\frac{1}{D}\sum_{i=1}^n b_i A_{ij}\right) = \frac{1}{D}\sum_{i=1}^n \sum_{i=1}^n a_{kj}b_i A_{ij}$$

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 $a_{11} \cdots a_{1j-1} \xrightarrow{b_1} a_{1j+1} \cdots a_{1n}$  $a_{21} \cdots a_{2j-1} \xrightarrow{b_2} a_{2j+1} \cdots a_{2n}$ 

克莱姆法则证明 (仅验证  $X_i = \frac{D_i}{D}$  是解,唯一性的证明要用到矩阵知识,略去。)

$$x_{j} = \frac{D_{j}}{D} = \frac{\begin{vmatrix} \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & b_{n} & a_{nj+1} & \cdots & a_{nn} \\ a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}} = \frac{b_{1}A_{1j} + b_{2}A_{2j} + \cdots + b_{n}A_{nj}}{D}$$

 $=\frac{1}{D}\sum_{i}^{n}b_{i}A_{ij}$ 

验证第 k 条方程成立  $(k = 1, 2, \dots, n)$ :

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$$a_{k1}x_1 + \dots + a_{kn}x_n = \sum_{i=1}^n a_{kj}x_j = \sum_{i=1}^n a_{kj} \left(\frac{1}{D}\sum_{i=1}^n b_i A_{ij}\right) = \frac{1}{D}\sum_{i=1}^n \sum_{i=1}^n a_{kj}b_i A_{ij}$$

 $=\frac{1}{D}\sum_{i=1}\sum_{i=1}a_{kj}b_iA_{ij}$ 

 $a_{11} \cdots a_{1j-1} \xrightarrow{b_1} a_{1j+1} \cdots a_{1n}$  $a_{21} \cdots a_{2j-1} \xrightarrow{b_2} a_{2j+1} \cdots a_{2n}$  $x_{j} = \frac{D_{j}}{D} = \frac{\begin{vmatrix} \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & b_{n} & a_{nj+1} & \cdots & a_{nn} \\ a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}} = \frac{b_{1}A_{1j} + b_{2}A_{2j} + \cdots + b_{n}A_{nj}}{D}$ 

克莱姆法则证明 (仅验证  $X_i = \frac{D_i}{D}$  是解,唯一性的证明要用到矩阵知识,略去。)

$$\begin{vmatrix} a_{21} & \cdots & a_{2j-1} & a_{2j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}$$

$$= \frac{1}{D} \sum_{i=1}^{n} b_i A_{ij}$$

验证第 k 条方程成立  $(k = 1, 2, \dots, n)$ :

 $a_{k1}x_1 + \dots + a_{kn}x_n = \sum_{i=1}^n a_{kj}x_j = \sum_{i=1}^n a_{kj} \left(\frac{1}{D}\sum_{i=1}^n b_i A_{ij}\right) = \frac{1}{D}\sum_{i=1}^n \sum_{i=1}^n a_{kj}b_i A_{ij}$ 

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 $a_{11} \cdots a_{1j-1} \xrightarrow{b_1} a_{1j+1} \cdots a_{1n}$  $a_{21} \cdots a_{2j-1} \xrightarrow{b_2} a_{2j+1} \cdots a_{2n}$  $x_{j} = \frac{D_{j}}{D} = \frac{\begin{vmatrix} \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & b_{n} & a_{nj+1} & \cdots & a_{nn} \\ a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}} = \frac{b_{1}A_{1j} + b_{2}A_{2j} + \cdots + b_{n}A_{nj}}{D}$ 

克莱姆法则证明 (仅验证  $X_i = \frac{D_i}{D}$  是解,唯一性的证明要用到矩阵知识,略去。)

$$=rac{1}{D}\sum_{i=1}^{n}b_{i}A_{ij}$$
  
验证第 $k$ 条方程成立( $k=1,2,\cdots,n$ ):

 $a_{k1}x_1 + \dots + a_{kn}x_n = \sum_{i=1}^n a_{kj}x_j = \sum_{i=1}^n a_{kj} \left(\frac{1}{D}\sum_{i=1}^n b_i A_{ij}\right) = \frac{1}{D}\sum_{i=1}^n \sum_{i=1}^n a_{kj}b_i A_{ij}$ 

 $\sum_{i=1}^{n} a_{kj} b_i A_{ij} = b_i \sum_{i=1}^{n} a_{kj} A_{ij}$ 

 $a_{11} \cdots a_{1j-1} \xrightarrow{b_1} a_{1j+1} \cdots a_{1n}$   $a_{21} \cdots a_{2j-1} \xrightarrow{b_2} a_{2j+1} \cdots a_{2n}$ 

克莱姆法则证明 (仅验证  $x_j = \frac{D_j}{D}$  是解,唯一性的证明要用到矩阵知识,略去。)

$$x_{j} = \frac{D_{j}}{D} = \frac{\begin{vmatrix} \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & b_{n} & a_{nj+1} & \cdots & a_{nn} \\ a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}} = \frac{b_{1}A_{1j} + b_{2}A_{2j} + \cdots + b_{n}A_{nj}}{D}$$

$$= \frac{1}{n} \sum_{j=1}^{n} b_{j}A_{jj}$$

 $=\frac{1}{D}\sum_{i}^{n}b_{i}A_{ij}$ 验证第 k 条方程成立 ( $k = 1, 2, \dots, n$ ):

 $a_{k1}x_1 + \dots + a_{kn}x_n = \sum_{i=1}^n a_{kj}x_j = \sum_{i=1}^n a_{kj} \left(\frac{1}{D}\sum_{i=1}^n b_i A_{ij}\right) = \frac{1}{D}\sum_{i=1}^n \sum_{i=1}^n a_{kj}b_i A_{ij}$ 

 $= \frac{1}{D} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{kj} b_i A_{ij} = b_i \sum_{j=1}^{n} a_{kj} A_{ij}$ 

 $a_{11} \cdots a_{1j-1} \xrightarrow{b_1} a_{1j+1} \cdots a_{1n}$   $a_{21} \cdots a_{2j-1} \xrightarrow{b_2} a_{2j+1} \cdots a_{2n}$  $x_{j} = \frac{D_{j}}{D} = \frac{\begin{vmatrix} \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & b_{n} & a_{nj+1} & \cdots & a_{nn} \\ a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}} = \frac{b_{1}A_{1j} + b_{2}A_{2j} + \cdots + b_{n}A_{nj}}{D}$ 

克莱姆法则证明 (仅验证  $X_i = \frac{D_i}{D}$  是解,唯一性的证明要用到矩阵知识,略去。)

$$=\frac{1}{D}\sum_{i=1}^{n}b_{i}A_{ij}$$
  
验证第  $k$  条方程成立( $k=1,2,\cdots,n$ ):

 $a_{k1}x_1 + \dots + a_{kn}x_n = \sum_{i=1}^n a_{kj}x_j = \sum_{i=1}^n a_{kj} \left(\frac{1}{D}\sum_{i=1}^n b_i A_{ij}\right) = \frac{1}{D}\sum_{i=1}^n \sum_{i=1}^n a_{kj}b_i A_{ij}$  $= \frac{1}{D} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{kj} b_i A_{ij} = \frac{1}{D} \sum_{i=1}^{n} b_i \sum_{j=1}^{n} a_{kj} A_{ij}$ 

 $a_{11} \cdots a_{1j-1} \xrightarrow{b_1} a_{1j+1} \cdots a_{1n}$   $a_{21} \cdots a_{2j-1} \xrightarrow{b_2} a_{2j+1} \cdots a_{2n}$  $x_{j} = \frac{D_{j}}{D} = \frac{\begin{vmatrix} \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & b_{n} & a_{nj+1} & \cdots & a_{nn} \\ a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}} = \frac{b_{1}A_{1j} + b_{2}A_{2j} + \cdots + b_{n}A_{nj}}{D}$ 

克莱姆法则证明 (仅验证  $x_j = \frac{D_j}{D}$  是解,唯一性的证明要用到矩阵知识,略去。)

$$=\frac{1}{D}\sum_{i=1}^{n}b_{i}A_{ij}$$
  
验证第 $k$ 条方程成立( $k=1,2,\cdots,n$ ):

 $a_{k1}x_1 + \dots + a_{kn}x_n = \sum_{i=1}^n a_{kj}x_j = \sum_{i=1}^n a_{kj} \left(\frac{1}{D}\sum_{i=1}^n b_i A_{ij}\right) = \frac{1}{D}\sum_{i=1}^n \sum_{i=1}^n a_{kj}b_i A_{ij}$ 

 $= \frac{1}{D} \sum_{i=1}^{n} \sum_{i=1}^{n} a_{kj} b_i A_{ij} = \frac{1}{D} \sum_{i=1}^{n} b_i \sum_{i=1}^{n} a_{kj} A_{ij} \qquad b_k \sum_{i=1}^{n} a_{kj} A_{kj}$ 

 $\begin{vmatrix} a_{11} & \cdots & a_{1j-1} & b_1 & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & b_2 & a_{2j+1} & \cdots & a_{2n} \end{vmatrix}$  $x_{j} = \frac{D_{j}}{D} = \frac{\begin{vmatrix} \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & b_{n} & a_{nj+1} & \cdots & a_{nn} \\ a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}} = \frac{b_{1}A_{1j} + b_{2}A_{2j} + \cdots + b_{n}A_{nj}}{D}$ 

克莱姆法则证明 (仅验证  $x_j = \frac{D_j}{D}$  是解,唯一性的证明要用到矩阵知识,略去。)

$$|a_{n1} \cdots a_{nj-1} a_{nj} a_{nj+1} \cdots a_{nn}|$$

$$= \frac{1}{D} \sum_{i=1}^{n} b_i A_{ij}$$
验证第  $k$  条方程成立( $k = 1, 2, \dots, n$ ):

$$a_{k1}x_1 + \dots + a_{kn}x_n = \sum_{j=1}^n a_{kj}x_j = \sum_{j=1}^n a_{kj} \left(\frac{1}{D}\sum_{i=1}^n b_i A_{ij}\right) = \frac{1}{D}\sum_{j=1}^n \sum_{i=1}^n a_{kj}b_i A_{ij}$$

 $= \frac{1}{D} \sum_{i=1}^{n} \sum_{k=1}^{n} a_{kj} b_i A_{ij} = \frac{1}{D} \sum_{i=1}^{n} b_i \sum_{k=1}^{n} a_{kj} A_{ij} = \frac{1}{D} \cdot b_k \sum_{k=1}^{n} a_{kj} A_{kj}$ 

克莱姆法则证明 (仅验证  $x_j = \frac{D_j}{D}$  是解,唯一性的证明要用到矩阵知识,略去。)  $a_{11} \cdots a_{1j-1} \xrightarrow{b_1} a_{1j+1} \cdots a_{1n}$   $a_{21} \cdots a_{2j-1} \xrightarrow{b_2} a_{2j+1} \cdots a_{2n}$  $x_{j} = \frac{D_{j}}{D} = \frac{\begin{vmatrix} \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & b_{n} & a_{nj+1} & \cdots & a_{nn} \\ a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}} = \frac{b_{1}A_{1j} + b_{2}A_{2j} + \cdots + b_{n}A_{nj}}{D}$  $=\frac{1}{D}\sum_{i}^{n}b_{i}A_{ij}$ 

验证第 k 条方程成立 ( $k = 1, 2, \dots, n$ ):

 $a_{k1}x_1 + \dots + a_{kn}x_n = \sum_{i=1}^n a_{kj}x_j = \sum_{i=1}^n a_{kj} \left(\frac{1}{D}\sum_{i=1}^n b_i A_{ij}\right) = \frac{1}{D}\sum_{i=1}^n \sum_{i=1}^n a_{kj}b_i A_{ij}$ 

 $=\frac{1}{D}\sum_{i=1}^{n}\sum_{i=1}^{n}a_{kj}b_{i}A_{ij}=\frac{1}{D}\sum_{i=1}^{n}b_{i}\sum_{i=1}^{n}a_{kj}A_{ij}=\frac{1}{D}\cdot b_{k}\sum_{i=1}^{n}a_{kj}A_{kj}=\frac{1}{D}\cdot b_{k}D_{0}b_{k}$ 

克莱姆法则证明 (仅验证  $x_j = \frac{D_j}{D}$  是解,唯一性的证明要用到矩阵知识,略去。)  $a_{11} \cdots a_{1j-1} \xrightarrow{b_1} a_{1j+1} \cdots a_{1n}$   $a_{21} \cdots a_{2j-1} \xrightarrow{b_2} a_{2j+1} \cdots a_{2n}$  $x_{j} = \frac{D_{j}}{D} = \frac{\begin{vmatrix} \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & b_{n} & a_{nj+1} & \cdots & a_{nn} \\ a_{11} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j-1} & a_{2j} & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}} = \frac{b_{1}A_{1j} + b_{2}A_{2j} + \cdots + b_{n}A_{nj}}{D}$  $=\frac{1}{D}\sum_{i}^{n}b_{i}A_{ij}$ 验证第 k 条方程成立 ( $k = 1, 2, \dots, n$ ):

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 $= \frac{1}{D} \sum_{i=1}^{n} \sum_{i=1}^{n} a_{kj} b_i A_{ij} = \frac{1}{D} \sum_{i=1}^{n} b_i \sum_{i=1}^{n} a_{kj} A_{ij} = \frac{1}{D} \cdot b_k \sum_{i=1}^{n} a_{kj} A_{kj} = \frac{1}{D} \cdot b_k D = b_k$ 

$$\bullet \begin{cases} x+y=1 \\ x+y=0 \end{cases}, D=\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}=0$$



• 
$$\begin{cases} x+y=1 \\ x+y=1 \end{cases}, D=\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}=0, 实质上只有一条方程 x+y=1,$$
 显然有无穷多解。

$$\begin{cases} x+y=1 \\ x+y=0 \end{cases}, D=\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}=0$$



• 
$$\begin{cases} x+y=1 \\ x+y=1 \end{cases}, D = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0, 实质上只有一条方程 x+y=1,$$
 显然有无穷多解。

• 
$$\begin{cases} x+y=1\\ x+y=0 \end{cases}, D=\begin{vmatrix} 1 & 1\\ 1 & 1 \end{vmatrix}=0, 方程组包含矛盾方程,显然无解。$$



#### 例 解线性方程组

$$\begin{cases} 2x_1 + x_2 - x_3 = 1\\ 3x_1 - x_2 - x_3 = -2\\ -x_1 + 2x_2 + x_3 = 6 \end{cases}$$

#### 练习 解线性方程组

$$\begin{cases} x_1 + x_2 = 90 \\ x_2 + x_3 = 86 \\ x_1 + x_3 = 80 \end{cases}$$

#### 例 解线性方程组

$$\begin{cases} 2x_1 + x_2 - x_3 = 1\\ 3x_1 - x_2 - x_3 = -2\\ -x_1 + 2x_2 + x_3 = 6 \end{cases}$$

提示 
$$D = -5$$
,  $D_1 = -5$ ,  $D_2 = -10$ ,  $D_3 = -15$ 

#### 练习 解线性方程组

$$\begin{cases} x_1 + x_2 = 90 \\ x_2 + x_3 = 86 \\ x_1 + x_3 = 80 \end{cases}$$

#### 例 解线性方程组

$$\begin{cases} 2x_1 + x_2 - x_3 = 1\\ 3x_1 - x_2 - x_3 = -2\\ -x_1 + 2x_2 + x_3 = 6 \end{cases}$$

提示 
$$D = -5$$
,  $D_1 = -5$ ,  $D_2 = -10$ ,  $D_3 = -15$ 

#### 练习 解线性方程组

$$\begin{cases} x_1 + x_2 = 90 \\ x_2 + x_3 = 86 \\ x_1 + x_3 = 80 \end{cases}$$

提示 
$$D = 2$$
,  $D_1 = 84$ ,  $D_2 = 96$ ,  $D_3 = 76$ 



#### 定理 齐次线性方程组

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = 0 \end{cases}$$

当系数行列式 
$$D \neq 0$$
 时,仅有零解( $x_1 = x_2 = \cdots = x_n = 0$ )

#### 定理 齐次线性方程组

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 当系数行列式  $D \neq 0$  时,仅有零解( $x_1 = x_2 = \cdots = x_n = 0$ )证明  $x_1 = x_2 = \cdots = x_n = 0$  显然是方程组的解

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另一方面,因为 $D \neq 0$ ,所以方程组有唯一解(克莱姆法则)

图 医南大学

#### 定理 齐次线性方程组

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = 0 \end{cases}$$

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另一方面,因为 $D \neq 0$ ,所以方程组有唯一解(克莱姆法则)

所以方程组除  $x_1 = x_2 = \cdots = x_n = 0$  外,没有其他解



§1.4 克莱姆法则

#### 定理 齐次线性方程组

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = 0 \end{cases}$$

当系数行列式 
$$D \neq 0$$
 时,仅有零解( $x_1 = x_2 = \cdots = x_n = 0$ )

证明 
$$x_1 = x_2 = \cdots = x_n = 0$$
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所以方程组除  $x_1 = x_2 = \cdots = x_n = 0$  外,没有其他解

#### 注

• 实际上,  $D \neq 0$   $\Rightarrow$  只有零解  $x_1 = x_2 = \cdots = x_n = 0$ 



#### 定理 齐次线性方程组

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = 0 \end{cases}$$

当系数行列式 
$$D \neq 0$$
 时,仅有零解( $x_1 = x_2 = \cdots = x_n = 0$ )

证明 
$$x_1 = x_2 = \cdots = x_n = 0$$
 显然是方程组的解

另一方面,因为 $D \neq 0$ ,所以方程组有唯一解(克莱姆法则)

所以方程组除  $x_1 = x_2 = \cdots = x_n = 0$  外,没有其他解

#### 注

• 实际上,  $D \neq 0 \iff$  只有零解  $x_1 = x_2 = \cdots = x_n = 0$ 



#### 定理 齐次线性方程组

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = 0 \end{cases}$$

当系数行列式 
$$D \neq 0$$
 时,仅有零解( $x_1 = x_2 = \cdots = x_n = 0$ )

证明 
$$X_1 = X_2 = \cdots = X_n = 0$$
 显然是方程组的解

另一方面,因为
$$D \neq 0$$
,所以方程组有唯一解(克莱姆法则)

所以方程组除  $x_1 = x_2 = \cdots = x_n = 0$  外,没有其他解

### 注

- 实际上,  $D \neq 0 \iff$  只有零解  $x_1 = x_2 = \cdots = x_n = 0$
- 若 D=0,方程有无穷多的解



例 齐次方程组 
$$\begin{cases} x_1 - 2x_2 = 0 \\ 2x_1 - 4x_2 = 0 \end{cases}$$
 的系数矩阵  $D = \begin{vmatrix} 1 & -2 \\ 2 & -4 \end{vmatrix}$ 



例 齐次方程组 
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例 齐次方程组 
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有无穷多的解。事实上,对任意的数 
$$k$$
,  $\begin{cases} x_1 = 2k \\ x_2 = k \end{cases}$  都是解。

例 判断线性方程组 
$$\begin{cases} 2x_1 + 3x_2 + 4x_3 + 5x_4 = 0 \\ 3x_1 + 4x_2 + 5x_3 + 5x_4 = 0 \\ 4x_1 + 5x_2 + 6x_3 + 6x_4 = 0 \\ 5x_1 + 6x_2 + 8x_3 + 9x_4 = 0 \end{cases}$$

是否只有零解



解	ļ				
	2	3	4 5 6 8	5	
	3	4	5	5	
	4	5	6	6	
	5	6	8	9	

$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 5 & 6 & 8 & 9 \end{vmatrix} \underline{r_4 - r_3}$$

$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 5 & 6 & 8 & 9 \end{vmatrix} \xrightarrow{\underline{r_4 - r_3}} \begin{vmatrix} \\ \\ \\ 1 & 1 & 2 & 3 \end{vmatrix}$$

2	3	4	5		2	3	4	5
3	4	5	5	$r_4-r_3$	3	4	5	5
4	5	6	6		4	5	6	6
5	6	8	9	<u>r<sub>4</sub>-r<sub>3</sub></u>	1	1	2	3

$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 5 & 6 & 8 & 9 \end{vmatrix} \xrightarrow{\underline{r_4 - r_3}} \begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 1 & 1 & 2 & 3 \end{vmatrix} \xrightarrow{\underline{c_2 - c_1}}_{\substack{c_3 - 2c_1 \\ c_4 - 3c_1}}$$

2	3	4	5		2	3	4	5		2
3	4	5	5	$r_4-r_3$	3	4	5	5	$c_2 - c_1$	3
4	5	6	6		4	5	6	6	$c_3 - 2c_1$	4
5	6	8	9		1	1	2	3		1

3	4	5		2	3	4	5		2	1
4	5	5	$r_4-r_3$	3	4	5	5	$c_2 - c_1$	3	1
5	6	6		4	5	6	6	$c_3 - 2c_1$	4	1
6	8	9		1	1	2	3	$c_4 - 3c_1$	1	0
	3 4 5 6	3 4 4 5 5 6 6 8	3 4 5 4 5 5 5 6 6 6 8 9	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$ \begin{array}{cccccccccccccccccccccccccccccccccccc$	$ \begin{vmatrix} 3 & 4 & 5 \\ 4 & 5 & 5 \\ 5 & 6 & 6 \\ 6 & 8 & 9 \end{vmatrix}                              $	$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$	$ \begin{vmatrix} 3 & 4 & 5 \\ 4 & 5 & 5 \\ 5 & 6 & 6 \\ 6 & 8 & 9 \end{vmatrix}                              $	$ \begin{vmatrix} 3 & 4 & 5 \\ 4 & 5 & 5 \\ 5 & 6 & 6 \\ 6 & 8 & 9 \end{vmatrix} \xrightarrow{\underline{r_4 - r_3}} \begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 1 & 1 & 2 & 3 \end{vmatrix} \xrightarrow{\underline{c_2 - c_1}} \begin{vmatrix} 2 \\ 3 \\ 4 \\ c_4 - 3c_1 \end{vmatrix} 1 $



Ì	2	3	4	5		2	3	4	5		12	1	0	
	_	_	•	٦,		-	_	•	٦,		-	_	U	
	3	4	5	5	$r_4-r_3$	3	4	5	5	$c_2-c_1$	3	1	-1	
	4	5	6	6		4	5	6	6	$c_3-2c_1$	4	1	<b>-</b> 2	
	5	6	8	9		1	1	2	3		1	0	0	



2	3	4	5		2	3	4	5		2	1	0	-1
3	4	5	5	$r_4-r_3$	3	4	5	5	$c_2 - c_1$	3	1	-1	<b>-4</b>
4	5	6	6		4	5	6	6	$c_3-2c_1$	4	1	<b>-</b> 2	<b>–</b> 6
5	6	8	9		1	1	2	3	$\frac{c_2 - c_1}{c_3 - 2c_1}$ $c_4 - 3c_1$	1	0	0	0



$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 5 & 6 & 8 & 9 \end{vmatrix} \xrightarrow{r_4 - r_3} \begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 1 & 1 & 2 & 3 \end{vmatrix} \xrightarrow{c_2 - c_1} \begin{vmatrix} 2 & 1 & 0 & -1 \\ 3 & 1 & -1 & -4 \\ 4 & 1 & -2 & -6 \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

$$= 1 \times (-1)^{4+1} \times \begin{vmatrix} 1 & 0 & -1 \\ 1 & -1 & -4 \\ 1 & -2 & -6 \end{vmatrix}$$



$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 5 & 6 & 8 & 9 \end{vmatrix} \xrightarrow{r_4 - r_3} \begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 1 & 1 & 2 & 3 \end{vmatrix} \xrightarrow{c_2 - c_1} \begin{vmatrix} 2 & 1 & 0 & -1 \\ 3 & 1 & -1 & -4 \\ 4 & 1 & -2 & -6 \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

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$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 5 & 6 & 8 & 9 \end{vmatrix} \xrightarrow{\underline{r_4 - r_3}} \begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 1 & 1 & 2 & 3 \end{vmatrix} \xrightarrow{\underline{c_2 - c_1}} \begin{vmatrix} 2 & 1 & 0 & -1 \\ 3 & 1 & -1 & -4 \\ 4 & 1 & -2 & -6 \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

$$= 1 \times (-1)^{4+1} \times \begin{vmatrix} 1 & 0 & -1 \\ 1 & -1 & -4 \\ 1 & -2 & -6 \end{vmatrix} \xrightarrow{c_3 + c_1} - \begin{vmatrix} 1 & 0 & 0 \\ 1 & -1 & -3 \\ 1 & -2 & -5 \end{vmatrix}$$



$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 5 & 6 & 8 & 9 \end{vmatrix} \xrightarrow{\underline{r_4 - r_3}} \begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 1 & 1 & 2 & 3 \end{vmatrix} \xrightarrow{\underline{c_2 - c_1}} \begin{vmatrix} 2 & 1 & 0 & -1 \\ 3 & 1 & -1 & -4 \\ 4 & 1 & -2 & -6 \\ 1 & 0 & 0 & 0 \end{vmatrix}$$
$$= 1 \times (-1)^{4+1} \times \begin{vmatrix} 1 & 0 & -1 \\ 1 & -1 & -4 \\ 1 & -2 & -6 \end{vmatrix} \xrightarrow{\underline{c_3 + c_1}} - \begin{vmatrix} 1 & 0 & 0 \\ 1 & -1 & -3 \\ 1 & -2 & -5 \end{vmatrix}$$

$$= - \begin{vmatrix} -1 & -3 \\ -2 & -5 \end{vmatrix}$$



$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 5 & 6 & 8 & 9 \end{vmatrix} \xrightarrow{\underline{r_4 - r_3}} \begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 1 & 1 & 2 & 3 \end{vmatrix} \xrightarrow{\underline{c_2 - c_1}} \begin{vmatrix} 2 & 1 & 0 & -1 \\ 3 & 1 & -1 & -4 \\ 4 & 1 & -2 & -6 \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

$$= 1 \times (-1)^{4+1} \times \begin{vmatrix} 1 & 0 & -1 \\ 1 & -1 & -4 \\ 1 & -2 & -6 \end{vmatrix} \xrightarrow{c_3 + c_1} - \begin{vmatrix} 1 & 0 & 0 \\ 1 & -1 & -3 \\ 1 & -2 & -5 \end{vmatrix}$$
$$= - \begin{vmatrix} -1 & -3 \\ -2 & -5 \end{vmatrix} = 1 \neq 0$$



$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 5 & 6 & 8 & 9 \end{vmatrix} \xrightarrow{\underline{r_4 - r_3}} \begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 \\ 4 & 5 & 6 & 6 \\ 1 & 1 & 2 & 3 \end{vmatrix} \xrightarrow{\underline{c_2 - c_1}} \begin{vmatrix} 2 & 1 & 0 & -1 \\ 3 & 1 & -1 & -4 \\ 4 & 1 & -2 & -6 \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

$$= 1 \times (-1)^{4+1} \times \begin{vmatrix} 1 & 0 & -1 \\ 1 & -1 & -4 \\ 1 & -2 & -6 \end{vmatrix} \xrightarrow{\underline{c_3 + c_1}} - \begin{vmatrix} 1 & 0 & 0 \\ 1 & -1 & -3 \\ 1 & -2 & -5 \end{vmatrix}$$

$$= - \begin{vmatrix} -1 & -3 \\ -2 & -5 \end{vmatrix} = 1 \neq 0$$

所以齐次线性方程组有唯一解



§1.4 克莱姆法则

的充分必要条件是 k 满足 \_\_\_\_



练习齐次线性方程组
$$\begin{cases} kx_1 & + x_4 = 0\\ x_1 + 2x_2 & - x_4 = 0\\ (k+2)x_1 - x_2 & + 4x_4 = 0\\ 2x_1 + x_2 + 3x_3 + kx_4 = 0 \end{cases}$$

的充分必要条件是 k 满足 \_\_\_\_

$$D = \begin{vmatrix} k & 0 & 0 & 1 \\ 1 & 2 & 0 & -1 \\ k+2 & -1 & 0 & 4 \\ 2 & 1 & 3 & k \end{vmatrix}$$



有非零解

练习 齐次线性方程组 
$$\begin{cases} kx_1 & + x_4 = 0 \\ x_1 + 2x_2 & - x_4 = 0 \\ (k+2)x_1 - x_2 & + 4x_4 = 0 \\ 2x_1 + x_2 + 3x_3 + kx_4 = 0 \end{cases}$$

有非零解

的充分必要条件是 k 满足  $_{\_\_\_}$ 

$$D = \begin{vmatrix} k & 0 & 0 & 1 \\ 1 & 2 & 0 & -1 \\ k+2 & -1 & 0 & 4 \\ 2 & 1 & 3 & k \end{vmatrix} = 3.$$



练习 齐次线性方程组 
$$\begin{cases} kx_1 + x_4 = 0 \\ x_1 + 2x_2 - x_4 = 0 \\ (k+2)x_1 - x_2 + 4x_4 = 0 \\ 2x_1 + x_2 + 3x_3 + kx_4 = 0 \end{cases}$$
 有非零解

的充分必要条件是 k 满足  $_{\_\_}$ 

$$D = \begin{vmatrix} k & 0 & 0 & 1 \\ 1 & 2 & 0 & -1 \\ k+2 & -1 & 0 & 4 \\ 2 & 1 & 3 & k \end{vmatrix} = 3 \cdot (-1)^{3+4} \begin{vmatrix} k & 0 & 1 \\ 1 & 2 & -1 \\ k+2 & -1 & 4 \end{vmatrix}$$



练习 齐次线性方程组 
$$\begin{cases} kx_1 & + x_4 = 0 \\ x_1 + 2x_2 & - x_4 = 0 \\ (k+2)x_1 - x_2 & + 4x_4 = 0 \\ 2x_1 + x_2 + 3x_3 + kx_4 = 0 \end{cases}$$
有非零解

的充分必要条件是 k 满足  $_{\_\_\_}$ 

$$D = \begin{vmatrix} k & 0 & 0 & 1 \\ 1 & 2 & 0 & -1 \\ k+2 & -1 & 0 & 4 \\ 2 & 1 & 3 & k \end{vmatrix} = 3 \cdot (-1)^{3+4} \begin{vmatrix} k & 0 & 1 \\ 1 & 2 & -1 \\ k+2 & -1 & 4 \end{vmatrix}$$

$$r_2 + r_1$$



$$D = \begin{vmatrix} k & 0 & 0 & 1 \\ 1 & 2 & 0 & -1 \\ k+2 & -1 & 0 & 4 \\ 2 & 1 & 3 & k \end{vmatrix} = 3 \cdot (-1)^{3+4} \begin{vmatrix} k & 0 & 1 \\ 1 & 2 & -1 \\ k+2 & -1 & 4 \end{vmatrix}$$

$$\frac{r_2 + r_1}{k} (-3) \cdot \begin{vmatrix} k & 0 & 1 \\ k & 0 & 1 \end{vmatrix}$$



$$D = \begin{vmatrix} k & 0 & 0 & 1 \\ 1 & 2 & 0 & -1 \\ k+2 & -1 & 0 & 4 \\ 2 & 1 & 3 & k \end{vmatrix} = 3 \cdot (-1)^{3+4} \begin{vmatrix} k & 0 & 1 \\ 1 & 2 & -1 \\ k+2 & -1 & 4 \end{vmatrix}$$

$$\frac{r_2 + r_1}{k} (-3) \cdot \begin{vmatrix} k & 0 & 1 \\ k+1 & 2 & 0 \end{vmatrix}$$



$$D = \begin{vmatrix} k & 0 & 0 & 1 \\ 1 & 2 & 0 & -1 \\ k+2 & -1 & 0 & 4 \\ 2 & 1 & 3 & k \end{vmatrix} = 3 \cdot (-1)^{3+4} \begin{vmatrix} k & 0 & 1 \\ 1 & 2 & -1 \\ k+2 & -1 & 4 \end{vmatrix}$$

$$\frac{r_2 + r_1}{r_2 + r_1} (-3) \begin{vmatrix} k & 0 & 1 \\ k+1 & 2 & 0 \end{vmatrix}$$

$$\frac{r_2 + r_1}{r_3 - 4r_1} (-3) \cdot \begin{vmatrix} k & 0 & 1 \\ k + 1 & 2 & 0 \end{vmatrix}$$

$$D = \begin{vmatrix} k & 0 & 0 & 1 \\ 1 & 2 & 0 & -1 \\ k+2 & -1 & 0 & 4 \\ 2 & 1 & 3 & k \end{vmatrix} = 3 \cdot (-1)^{3+4} \begin{vmatrix} k & 0 & 1 \\ 1 & 2 & -1 \\ k+2 & -1 & 4 \end{vmatrix}$$

$$\frac{r_2 + r_1}{k} (-3) \cdot \begin{vmatrix} k & 0 & 1 \\ k+1 & 2 & 0 \end{vmatrix}$$

$$\frac{r_2+r_1}{r_3-4r_1} (-3) \cdot \begin{vmatrix} k & 0 & 1 \\ k+1 & 2 & 0 \\ -3k+2 & -1 & 0 \end{vmatrix}$$

的充分必要条件是 k 满足 \_\_\_\_

$$D = \begin{vmatrix} k & 0 & 0 & 1 \\ 1 & 2 & 0 & -1 \\ k+2 & -1 & 0 & 4 \\ 2 & 1 & 3 & k \end{vmatrix} = 3 \cdot (-1)^{3+4} \begin{vmatrix} k & 0 & 1 \\ 1 & 2 & -1 \\ k+2 & -1 & 4 \end{vmatrix}$$

$$\frac{r_2 + r_1}{r_3 - 4r_1} (-3) \cdot \begin{vmatrix} k & 0 & 1 \\ k + 1 & 2 & 0 \\ -3k + 2 & -1 & 0 \end{vmatrix} = (-3) \cdot (-1)^{1+3} \cdot \begin{vmatrix} k + 1 & 2 \\ -3k + 2 & -1 \end{vmatrix}$$



$$D = \begin{vmatrix} k & 0 & 0 & 1 \\ 1 & 2 & 0 & -1 \\ k+2 & -1 & 0 & 4 \\ 2 & 1 & 3 & k \end{vmatrix} = 3 \cdot (-1)^{3+4} \begin{vmatrix} k & 0 & 1 \\ 1 & 2 & -1 \\ k+2 & -1 & 4 \end{vmatrix}$$
$$\frac{r_2 + r_1}{r_3 - 4r_1} (-3) \cdot \begin{vmatrix} k & 0 & 1 \\ k+1 & 2 & 0 \\ -3k+2 & -1 & 0 \end{vmatrix} = (-3) \cdot (-1)^{1+3} \cdot \begin{vmatrix} k+1 & 2 \\ -3k+2 & -1 \end{vmatrix}$$

=-3(5k-5)

$$D = \begin{vmatrix} k & 0 & 0 & 1 \\ 1 & 2 & 0 & -1 \\ k+2 & -1 & 0 & 4 \\ 2 & 1 & 3 & k \end{vmatrix} = 3 \cdot (-1)^{3+4} \begin{vmatrix} k & 0 & 1 \\ 1 & 2 & -1 \\ k+2 & -1 & 4 \end{vmatrix}$$
$$\frac{r_2 + r_1}{r_3 - 4r_1} (-3) \cdot \begin{vmatrix} k & 0 & 1 \\ k+1 & 2 & 0 \\ -3k+2 & -1 & 0 \end{vmatrix} = (-3) \cdot (-1)^{1+3} \cdot \begin{vmatrix} k+1 & 2 \\ -3k+2 & -1 \end{vmatrix}$$

=-3(5k-5)

有非零解当且仅当 D=0.

$$D = \begin{vmatrix} k & 0 & 0 & 1 \\ 1 & 2 & 0 & -1 \\ k+2 & -1 & 0 & 4 \\ 2 & 1 & 3 & k \end{vmatrix} = 3 \cdot (-1)^{3+4} \begin{vmatrix} k & 0 & 1 \\ 1 & 2 & -1 \\ k+2 & -1 & 4 \end{vmatrix}$$
$$\frac{r_2 + r_1}{r_3 - 4r_1} (-3) \cdot \begin{vmatrix} k & 0 & 1 \\ k+1 & 2 & 0 \\ -3k+2 & -1 & 0 \end{vmatrix} = (-3) \cdot (-1)^{1+3} \cdot \begin{vmatrix} k+1 & 2 \\ -3k+2 & -1 \end{vmatrix}$$

=-3(5k-5)有非零解当且仅当 D=0,当且仅当 k=1

