第 07 周作业解答

练习 1. 用分部积分法计算：

$$(1)\ \int_{\frac{\pi}{2}}^{\pi} x\sin(2x)dx;\quad (2)\ \int_1^2 x^2\ln x dx;\quad (3)\ \int_0^1 x^3e^{-x^2}dx$$

解：(1)

$$\begin{aligned}\int_{\frac{\pi}{2}}^{\pi} x\sin(2x)dx &= -\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} xd\cos(2x) = -\frac{1}{2}\left(x\cos(2x)\Big|_{\frac{\pi}{2}}^{\pi} - \int_{\frac{\pi}{2}}^{\pi}\cos(2x)dx\right)\\ &= -\frac{1}{2}\left(\frac{3}{2}\pi - \frac{1}{2}\sin(2x)\Big|_{\frac{\pi}{2}}^{\pi}\right) = -\frac{3}{4}\pi\end{aligned}$$

(2)

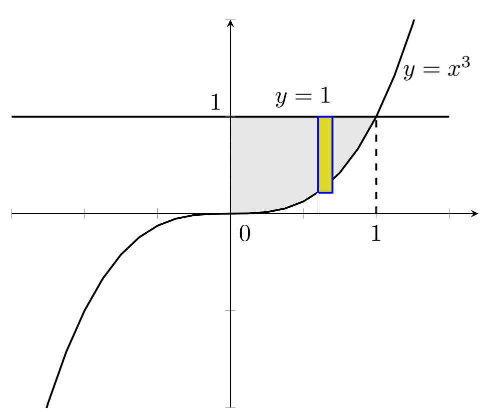
$$\begin{aligned}\int_1^2 x^2\ln x dx &= \frac{1}{3}\int_1^2 \ln x dx^3 = \frac{1}{3}\left(x^3\ln x\Big|_1^2 - \int_1^2 x^3 d\ln x\right)\\ &= \frac{1}{3}\left(8\ln 2 - \int_1^2 x^3\cdot\frac{1}{x}dx\right) = \frac{1}{3}\left(8\ln 2 - \frac{1}{3}x^3\Big|_1^2\right) = \frac{8}{3}\ln 2 - \frac{7}{9}\end{aligned}$$

(3)

$$\begin{aligned}\int_0^1 x^3e^{-x^2}dx &= \frac{1}{2}\int_0^1 x^2e^{-x^2}dx^2 = \frac{1}{2}\int_0^1 te^{-t}dt\\ &= -\frac{1}{2}\int_0^1 tde^{-t} = -\frac{1}{2}\left(te^{-t}\Big|_0^1 - \int_0^1 e^{-t}dt\right)\\ &= -\frac{1}{2}\left(e^{-1} + e^{-t}\Big|_0^1\right) = \frac{1}{2} - \frac{1}{e}\end{aligned}$$

练习 2. 画出曲线 $y = x^3$ 与直线 $y = 1$，$x = 0$ 围成的区域，并求面积。

解：$A = \int_0^1(1-x^3)dx = \left(x - \frac{1}{4}x^4\right)\Big|_0^1 = (1-\frac{1}{4}) - (0) = \frac{3}{4}$



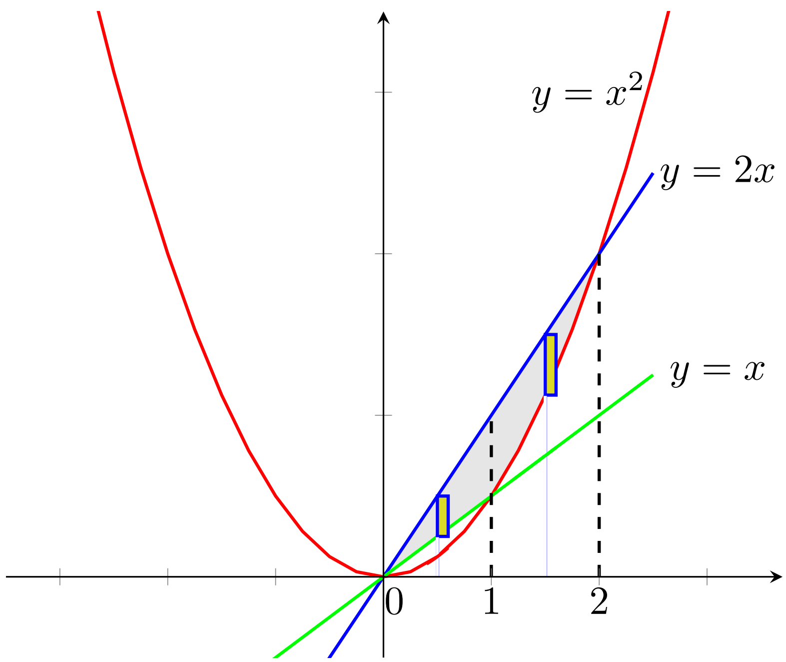
练习 3. 画出曲线 $y = x^2$ 与 $y = 2 - x^2$ 所围成的区域，并求面积。

解：$A = \int_{-1}^{1}(2 - x^2 - x^2)dx = \left(2x - \frac{2}{3}x^3\right)\big|_{-1}^{1} = (2 - \frac{2}{3}) - (-2 + \frac{2}{3}) = \frac{8}{3}$

练习 4. 画出曲线 $y = x^2$ 与直线 $y = x$，$y = 2x$ 围成的区域，并求面积。
提示：可能需要将区域划分成两部分，分别求面积。

解：

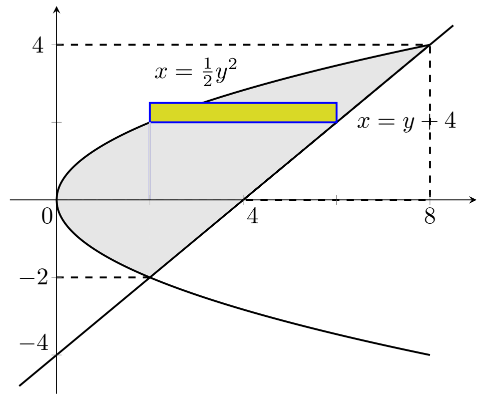
$$\begin{aligned} A &= \int_0^1 (2x - x)dx + \int_1^2 (2x - x^2)dx \\ &= \frac{1}{2}x^2\big|_0^1 + \left(-\frac{1}{3}x^3 + x^2\right)\big|_1^2 \\ &= \frac{7}{6} \end{aligned}$$



练习 5. 画出曲线 $y^2 = 2x$ 与直线 $y = x - 4$ 围成的区域，并求面积。

解：

$$A = \int_{-2}^{4} [y + 4 - \frac{1}{2}y^2]dy = \left(-\frac{1}{6}y^3 + \frac{1}{2}y^2 + 4y\right)\big|_{-2}^{4} = (-\frac{32}{3} + 8 + 16) - (\frac{4}{3} + 2 - 8) = 18$$



练习 6. 设 $f(x)=\int_1^x e^{-t^2}dt$，试利用分部积分公式计算 $\int_0^1 f(x)dx$。

解:

$$\int_0^1 f(x)dx = f(x)x\Big|_0^1 - \int_0^1 xdf(x) = [f(1)-0] - \int_0^1 xf'(x)dx$$

注意到

$$f(1)=\int_1^1 e^{-t^2}dt = 0, \qquad f'(x)=e^{-x^2}$$

所以

$$\int_0^1 f(x)dx = -\int_0^1 xe^{-x^2}dx = -\frac{1}{2}\int_0^1 e^{-x^2}dx^2 \overset{u=x^2}{=\!=\!=} -\frac{1}{2}\int_0^1 e^{-u}du = \frac{1}{2}e^{-u}\Big|_0^1 = \frac{1}{2}(\frac{1}{e}-1).$$