

§4.3 实对称矩阵的特征值和特征向量

数学系 梁卓滨

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本节内容

- ◇ 向量的内积
- ♣ 正交向量组，施密特正交化方法
- ♥ 正交矩阵
- ♠ 对称矩阵可对角化

向量内积

定义 \mathbb{R}^n 中两个向量 $\alpha = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ 和 $\beta = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ 的内积定义为:

$$\alpha^T \beta =$$

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$$= (-1) \times 2 + 1 \times 0 + 0 \times (-1) + 2 \times 3 = 4$$

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向量范数

定义

$$\|\alpha\| := \sqrt{\alpha^T \alpha} = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

称为向量的长度或范数。

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即

$$|a_1 b_1 + \cdots + a_n b_n| \leq \sqrt{a_1^2 + \cdots + a_n^2} \cdot \sqrt{b_1^2 + \cdots + b_n^2}$$

向量单位化

- 定义 长度为 1 的向量称为单位向量。

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- 设 $\alpha \neq 0$, 则 $\|\alpha\| \neq 0$, 向量 $\frac{1}{\|\alpha\|} \alpha$ 是单位向量:

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称 $\frac{1}{\|\alpha\|}\alpha$ 为 α 的**单位化**

例 将下列向量单位化

$$\alpha = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \beta = \begin{pmatrix} 2 \\ 2 \\ 4 \\ 5 \end{pmatrix}$$

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1. $\|\alpha\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14},$

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定义 若 $\alpha^T \beta = 0$, 则称 α, β 正交 (或垂直)

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正交向量组

定义 若 \mathbb{R}^n 中向量组 $\alpha_1, \alpha_2, \dots, \alpha_s$ 满足

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正交化

$\alpha_1, \alpha_2, \dots, \alpha_s$ (线性无关) $\longrightarrow \beta_1, \beta_2, \dots, \beta_s$ (等价, 两两正交)

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实现正交化步骤 (施密特正交化方法):

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例 1 将线性无关组 $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 3 \\ 3 \\ -1 \\ -1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix}$ 正交化

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解

$$\beta_1 =$$

$$\beta_2 =$$

$$\beta_3 =$$

例 1 将线性无关组 $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 3 \\ 3 \\ -1 \\ -1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix}$ 正交化

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解

$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \text{——} \beta_1$$

$$\beta_3 =$$

例 1 将线性无关组 $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 3 \\ 3 \\ -1 \\ -1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix}$ 正交化

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解

$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2^T \beta_1}{\|\beta_1\|^2} \beta_1$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3^T \beta_1}{\|\beta_1\|^2} \beta_1 - \frac{\alpha_3^T \beta_2}{\|\beta_2\|^2} \beta_2$$

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解

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} 3 \\ 3 \\ -1 \\ -1 \end{pmatrix} - \frac{10}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$

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$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} 3 \\ 3 \\ -1 \\ -1 \end{pmatrix} - \frac{4}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$

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$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$

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$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$

$$= \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix} - \frac{10}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{10}{8} \begin{pmatrix} 2 \\ 2 \\ -2 \\ -2 \end{pmatrix}$$

例 1 将线性无关组 $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 3 \\ 3 \\ -1 \\ -1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix}$ 正交化

解

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} 3 \\ 3 \\ -1 \\ -1 \end{pmatrix} - \frac{4}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -2 \\ -2 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$

$$= \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix} - \frac{1}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{4} \begin{pmatrix} 2 \\ 2 \\ -2 \\ -2 \end{pmatrix}$$

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$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$

$$= \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix} - \frac{12}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{12}{8} \begin{pmatrix} 2 \\ 2 \\ -2 \\ -2 \end{pmatrix}$$

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$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$

$$= \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix} - \frac{12}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{16} \begin{pmatrix} 2 \\ 2 \\ -2 \\ -2 \end{pmatrix}$$

例 1 将线性无关组 $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 3 \\ 3 \\ -1 \\ -1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix}$ 正交化

解

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

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$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$

$$= \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix} - \frac{12}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{-32}{16} \begin{pmatrix} 2 \\ 2 \\ -2 \\ -2 \end{pmatrix}$$

例 1 将线性无关组 $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 3 \\ 3 \\ -1 \\ -1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix}$ 正交化

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$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

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$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$

$$= \begin{pmatrix} -2 \\ 0 \\ 6 \\ 8 \end{pmatrix} - \frac{12}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{-32}{16} \begin{pmatrix} 2 \\ 2 \\ -2 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

例 2 将线性无关组 $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix}$ 正交化

例 2 将线性无关组 $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix}$ 正交化

解

$$\beta_1 =$$

$$\beta_2 =$$

$$\beta_3 =$$

例 2 将线性无关组 $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix}$ 正交化

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$$\beta_1 = \alpha_1$$

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例 2 将线性无关组 $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix}$ 正交化

解

$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \text{——} \beta_1$$

$$\beta_3 =$$

例 2 将线性无关组 $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix}$ 正交化

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$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

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解

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$

例 2 将线性无关组 $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix}$ 正交化

解

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$

例 2 将线性无关组 $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix}$ 正交化

解

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix} - \frac{6}{3} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$

例 2 将线性无关组 $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix}$ 正交化

解

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix} - \frac{6}{3} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$

例 2 将线性无关组 $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix}$ 正交化

解

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix} - \frac{6}{3} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$

$$= \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix} - \frac{6}{3} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \frac{0}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

例 2 将线性无关组 $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix}$ 正交化

解

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{(\alpha_2, \beta_1)}{(\beta_1, \beta_1)} \beta_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix} - \frac{6}{3} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{(\alpha_3, \beta_1)}{(\beta_1, \beta_1)} \beta_1 - \frac{(\alpha_3, \beta_2)}{(\beta_2, \beta_2)} \beta_2$$

$$= \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

例 2 将线性无关组 $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix}$ 正交化

解

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix} - \frac{6}{3} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$

$$= \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix} - \frac{6}{3} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \frac{0}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

例 2 将线性无关组 $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix}$ 正交化

解

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix} - \frac{6}{3} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$

$$= \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix} - \frac{6}{3} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

例 2 将线性无关组 $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix}$ 正交化

解

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix} - \frac{6}{3} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$

$$= \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix} - \frac{6}{3} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \frac{0}{3} \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

例 2 将线性无关组 $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix}$ 正交化

解

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix} - \frac{6}{3} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$

$$= \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix} - \frac{6}{3} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \frac{0}{3} \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

例 3 将线性无关组 $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ 正交化

例 3 将线性无关组 $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ 正交化

解

$$\beta_1 =$$

$$\beta_2 =$$

$$\beta_3 =$$

例 3 将线性无关组 $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ 正交化

解

$$\beta_1 = \alpha_1$$

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解

$$\beta_1 = \alpha_1$$

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$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \text{——} \beta_1$$

$$\beta_3 = \alpha_3 - \text{——} \beta_1 - \text{——} \beta_2$$

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解

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{(\alpha_2, \beta_1)}{(\beta_1, \beta_1)} \beta_1$$

$$\beta_3 = \alpha_3 - \frac{(\alpha_3, \beta_1)}{(\beta_1, \beta_1)} \beta_1 - \frac{(\alpha_3, \beta_2)}{(\beta_2, \beta_2)} \beta_2$$

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$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$

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$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix} - \frac{4}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 - \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2$$

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$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

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正交矩阵

定义 设 n 阶矩阵 Q 满足 $Q^T Q = I_n$, 则称 Q 是正交矩阵。

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1. 若 Q 为正交矩阵, 则 $|Q| = 1$ 或 $|Q| = -1$;

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1. 若 Q 为正交矩阵, 则 $|Q| = 1$ 或 $|Q| = -1$;
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3. 若 P, Q 为正交矩阵, 则 PQ 也是正交矩阵。

证明

1. $Q^T Q = I_n \Rightarrow 1 = |I_n| = |Q^T Q| = |Q^T| \cdot |Q| = |Q|^2 \Rightarrow |Q| = \pm 1$
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1. 若 Q 为正交矩阵, 则 $|Q| = 1$ 或 $|Q| = -1$;
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所以

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例 验证下列矩阵是否正交矩阵：

$$A_1 = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{pmatrix}, \quad A_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{pmatrix},$$

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答案 A_1 是正交矩阵

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答案 A_1 是正交矩阵， A_2 不是正交矩阵

实对称矩阵的特征值和特征向量

- 对任意 n 阶方阵:

1. 一定有 n 个特征值 (计算重数, 复数域内), 可能有非实数特征值
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- 对实对称矩阵, 总成立:

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注 由于正交矩阵满足 $Q^{-1} = Q^T$,

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也就是：设 A 为实对称矩阵，则一定存在可逆矩阵 P ，使得

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{pmatrix}$$

事实上，还可以进一步要求 P 是正交矩阵：

定理 设 A 为实对称矩阵，则一定存在正交矩阵 Q ，使得

$$Q^{-1}AQ = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{pmatrix}$$

注 由于正交矩阵满足 $Q^{-1} = Q^T$ ，上述等价于 $Q^T A Q = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{pmatrix}$

定理 实对称矩阵的对应于不同特征值的特征向量正交。

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证明 设 A 为实对称矩阵, $\lambda_1 \neq \lambda_2$ 为两特征值, α_1, α_2 为相应特征向量,

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注意 $\alpha_2^T A\alpha_1 = (\alpha_2^T A\alpha_1)^T =$

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$$0 = (\lambda_1 - \lambda_2) \alpha_2^T \alpha_1$$

由于 $\lambda_1 \neq \lambda_2$, 所以

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定理 设 A 为实对称矩阵，则存在正交矩阵 Q ，使得 $Q^{-1}AQ$ 为对角矩阵。

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解释示意图

不同 特征值	重 数	正交化	单位化
λ_1	n_1		
λ_2	n_2		
\vdots	\vdots		
λ_s	n_s		
共 n			
$ \lambda I - A = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_s)^{n_s}$			

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不同 特征值	重 数	$(\lambda_i I - A)x = 0$ 基础解系	正交化	单位化
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\vdots	\vdots	\vdots	\vdots	\vdots
λ_s	n_s	$\alpha_1^{(s)}, \dots, \alpha_{n_s}^{(s)}$	$\Rightarrow \beta_1^{(s)}, \dots, \beta_{n_s}^{(s)}$	$\Rightarrow \gamma_1^{(s)}, \dots, \gamma_{n_s}^{(s)}$
共 n		共 n 个无关特征向量		构成单位正交特 征向量
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共 n		共 n 个无关特征向量		构成单位正交特 征向量
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- 令 $P = (\alpha_1^{(1)}, \dots, \alpha_{n_s}^{(n_s)})$, 则 $P^{-1}AP = \Lambda$ 。但一般地, P 不是正交矩阵。
- 令 $Q = (\gamma_1^{(1)}, \dots, \gamma_{n_s}^{(n_s)})$,

定理 设 A 为实对称矩阵, 则存在正交矩阵 Q , 使得 $Q^{-1}AQ$ 为对角矩阵。

解释示意图

不同 特征值	重 数	$(\lambda_i I - A)x = 0$ 基础解系	正交化	单位化
λ_1	n_1	$\alpha_1^{(1)}, \dots, \alpha_{n_1}^{(1)}$	$\Rightarrow \beta_1^{(1)}, \dots, \beta_{n_1}^{(1)}$	$\Rightarrow \gamma_1^{(1)}, \dots, \gamma_{n_1}^{(1)}$
λ_2	n_2	$\alpha_1^{(2)}, \dots, \alpha_{n_2}^{(2)}$	$\Rightarrow \beta_1^{(2)}, \dots, \beta_{n_2}^{(2)}$	$\Rightarrow \gamma_1^{(2)}, \dots, \gamma_{n_2}^{(2)}$
\vdots	\vdots	\vdots	\vdots	\vdots
λ_s	n_s	$\alpha_1^{(s)}, \dots, \alpha_{n_s}^{(s)}$	$\Rightarrow \beta_1^{(s)}, \dots, \beta_{n_s}^{(s)}$	$\Rightarrow \gamma_1^{(s)}, \dots, \gamma_{n_s}^{(s)}$
共 n		共 n 个无关特征向量		构成单位正交特征向量

$$|\lambda I - A| = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_s)^{n_s}$$

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\vdots	\vdots	\vdots	\vdots	\vdots
λ_s	n_s	$\alpha_1^{(s)}, \dots, \alpha_{n_s}^{(s)}$	$\Rightarrow \beta_1^{(s)}, \dots, \beta_{n_s}^{(s)}$	$\Rightarrow \gamma_1^{(s)}, \dots, \gamma_{n_s}^{(s)}$
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例 1 求矩阵 $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ 的特征值与特征向量。

$$Q^{-1}AQ = \begin{pmatrix} * & \\ & * \end{pmatrix}$$

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 $\gamma_1 \qquad \gamma_2$

所以取 $Q = \underbrace{\begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}}_{Q: \text{正交阵}}$, 则 $Q^{-1}AQ = \begin{pmatrix} 1 & \\ & 3 \end{pmatrix}$

例 2 求矩阵 $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ 的特征值与特征向量。

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解 ● 特征方程: $0 = |\lambda I - A|$

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解 ● 特征方程: $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} =$

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解 ● 特征方程: $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2 - (-2)^2$

$$Q^{-1}AQ = \begin{pmatrix} * & \\ & * \end{pmatrix}$$

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解 ● 特征方程: $0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = (\lambda + 1)(\lambda - 3)$

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● $\lambda_1 = -1$

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所以取 $Q = \underbrace{\begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}}_{Q: \text{正交阵}}$, 则 $Q^{-1}AQ = \begin{pmatrix} 1 & \\ & 3 \end{pmatrix}$

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- $\lambda_3 = 5$, 特征向量 $\alpha_3 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

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所以取 $Q = \underbrace{\begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ 2/3 & 2/3 & 1/3 \\ 2/3 & -1/3 & -2/3 \\ 1/3 & -2/3 & 2/3 \end{pmatrix}}_{Q: \text{正交阵}}$, 则 $Q^{-1}AQ = \begin{pmatrix} -1 & & \\ & 2 & \\ & & 5 \end{pmatrix}$

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• $\lambda_1 = 1$ (二重), 特征向量

$$\begin{cases} \alpha_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \\ \alpha_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \end{cases}$$

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$$\begin{cases} \alpha_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \\ \alpha_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \end{cases}$$

• $\lambda_3 = 10$, 特征向量 $\alpha_3 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$

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$\gamma_1 \qquad \gamma_2 \qquad \gamma_3$

所以取 $Q = \underbrace{\begin{pmatrix} -2/\sqrt{5} & 2/3\sqrt{5} & 1/3 \\ 1/\sqrt{5} & 4/3\sqrt{5} & 2/3 \\ 0 & \sqrt{5}/3 & -2/3 \end{pmatrix}}_{Q: \text{正交阵}}$, 则 $Q^{-1}AQ = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 10 \end{pmatrix}$

例 3 $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix},$

$$Q^{-1}AQ = \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix}$$

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例 3 $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$, 特征方程: $0 = |\lambda I - A| = (\lambda + 1)^2(\lambda - 5)$ ► Det

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- $\lambda_1 = -1$ (二重)

- $\lambda_2 = 5$

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• $\lambda_1 = -1$ (二重), 特征向量: [▶ Detail](#)

$$\begin{cases} \alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \\ \alpha_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \end{cases} \xrightarrow[\text{正交化}]{\text{▶ Det}}$$

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• $\lambda_2 = 5$, 特征向量: ▶ Det $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \xrightarrow{\text{单位化}} \gamma_3 = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$

$$\text{取 } Q = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}, \text{ 则 } Q^{-1}AQ = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 5 \end{pmatrix}$$

Q : 正交阵

The End

- 求解特征方程

$$0 = |\lambda I - A| =$$

- 求解特征方程

$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ -2 & -2 & \lambda - 1 \end{vmatrix}$$

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$$\underline{\underline{r_3 - r_2}}$$

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$$\underline{\underline{r_3 - r_2}} \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -\lambda - 1 & \lambda + 1 \end{vmatrix}$$

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$$= (\lambda + 1) \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -1 & 1 \end{vmatrix}$$

- 求解特征方程

$$\begin{aligned}0 &= |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ -2 & -2 & \lambda - 1 \end{vmatrix} \\&\xrightarrow{\underline{\underline{r_3 - r_2}}} \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -\lambda - 1 & \lambda + 1 \end{vmatrix} \\&= (\lambda + 1) \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -1 & 1 \end{vmatrix} \xrightarrow{\underline{\underline{c_2 + c_3}}}\end{aligned}$$

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$$\begin{aligned} 0 = |\lambda I - A| &= \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ -2 & -2 & \lambda - 1 \end{vmatrix} \\ &\xrightarrow{r_3 - r_2} \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -\lambda - 1 & \lambda + 1 \end{vmatrix} \\ &= (\lambda + 1) \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -1 & 1 \end{vmatrix} \xrightarrow{c_2 + c_3} (\lambda + 1) \begin{vmatrix} \lambda - 1 & -4 & -2 \\ -2 & \lambda - 3 & -2 \\ 0 & 0 & 1 \end{vmatrix} \end{aligned}$$

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$$0 = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ -2 & -2 & \lambda - 1 \end{vmatrix}$$

$$\xrightarrow{r_3 - r_2} \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -\lambda - 1 & \lambda + 1 \end{vmatrix}$$

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$$= (\lambda + 1) \begin{vmatrix} \lambda - 1 & -4 \\ -2 & \lambda - 3 \end{vmatrix}$$

$$= (\lambda + 1)(\lambda^2 - 4\lambda - 5)$$

• 求解特征方程

$$\begin{aligned}0 &= |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ -2 & -2 & \lambda - 1 \end{vmatrix} \\&\xrightarrow{r_3 - r_2} \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -\lambda - 1 & \lambda + 1 \end{vmatrix} \\&= (\lambda + 1) \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ 0 & -1 & 1 \end{vmatrix} \xrightarrow{c_2 + c_3} (\lambda + 1) \begin{vmatrix} \lambda - 1 & -4 & -2 \\ -2 & \lambda - 3 & -2 \\ 0 & 0 & 1 \end{vmatrix} \\&= (\lambda + 1) \begin{vmatrix} \lambda - 1 & -4 \\ -2 & \lambda - 3 \end{vmatrix} \\&= (\lambda + 1)(\lambda^2 - 4\lambda - 5) \\&= (\lambda + 1)^2(\lambda - 5)\end{aligned}$$

- 当 $\lambda_1 = -1$, 求解 $(\lambda_1 I - A)x = 0$:

$$(-I - A : 0) =$$

► Back

- 当 $\lambda_1 = -1$, 求解 $(\lambda_1 I - A)x = 0$:

$$(-I - A : 0) = \left(\begin{array}{ccc|c} -2 & -2 & -2 & 0 \\ -2 & -2 & -2 & 0 \\ -2 & -2 & -2 & 0 \end{array} \right) \rightarrow$$

► Back

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所以

$$x_1 + x_2 + x_3 = 0$$

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所以

$$x_1 + x_2 + x_3 = 0 \quad \Rightarrow \quad x_1 = -x_2 - x_3$$

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所以

$$x_1 + x_2 + x_3 = 0 \Rightarrow x_1 = -x_2 - x_3$$

$$\text{基础解系: } \alpha_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

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▶ Back

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$$(-I - A : 0) = \left(\begin{array}{ccc|c} -2 & -2 & -2 & 0 \\ -2 & -2 & -2 & 0 \\ -2 & -2 & -2 & 0 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

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► Back

- 当 $\lambda_2 = 5$, 求解 $(\lambda_2 I - A)x = 0$:

$$(5I - A : 0) =$$

- 当 $\lambda_2 = 5$, 求解 $(\lambda_2 I - A)x = 0$:

$$(5I - A : 0) = \left(\begin{array}{ccc|c} 4 & -2 & -2 & 0 \\ -2 & 4 & -2 & 0 \\ -2 & -2 & 4 & 0 \end{array} \right)$$

- 当 $\lambda_2 = 5$, 求解 $(\lambda_2 I - A)x = 0$:

$$(5I - A : 0) = \left(\begin{array}{ccc|c} 4 & -2 & -2 & 0 \\ -2 & 4 & -2 & 0 \\ -2 & -2 & 4 & 0 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array} \right)$$

- 当 $\lambda_2 = 5$, 求解 $(\lambda_2 I - A)x = 0$:

$$(5I - A : 0) = \left(\begin{array}{ccc|c} 4 & -2 & -2 & 0 \\ -2 & 4 & -2 & 0 \\ -2 & -2 & 4 & 0 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array} \right)$$

$$\xrightarrow{r_1 \leftrightarrow r_3}$$

- 当 $\lambda_2 = 5$, 求解 $(\lambda_2 I - A)x = 0$:

$$(5I - A : 0) = \left(\begin{array}{ccc|c} 4 & -2 & -2 & 0 \\ -2 & 4 & -2 & 0 \\ -2 & -2 & 4 & 0 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array} \right)$$

$$\xrightarrow{r_1 \leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 1 & -2 & 1 & 0 \\ 2 & -1 & -1 & 0 \end{array} \right)$$

- 当 $\lambda_2 = 5$, 求解 $(\lambda_2 I - A)x = 0$:

$$(5I - A : 0) = \left(\begin{array}{ccc|c} 4 & -2 & -2 & 0 \\ -2 & 4 & -2 & 0 \\ -2 & -2 & 4 & 0 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array} \right)$$

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所以
$$\begin{cases} x_1 & -x_3 = 0 \end{cases}$$

- 当 $\lambda_2 = 5$, 求解 $(\lambda_2 I - A)x = 0$:

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所以
$$\begin{cases} x_1 - x_3 = 0 \\ x_2 - x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = x_3 \\ x_2 = x_3 \end{cases}$$

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基础解系: $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

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基础解系: $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

将线性无关组 $\alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ 正交化:

► Back

将线性无关组 $\alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ 正交化:

$$\beta_1 =$$

$$\beta_2 =$$

► Back

将线性无关组 $\alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ 正交化:

$$\beta_1 = \alpha_1$$

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► Back

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$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \text{——} \beta_1$$

► Back

将线性无关组 $\alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ 正交化:

$$\beta_1 = \alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1$$

► Back

将线性无关组 $\alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ 正交化:

$$\beta_1 = \alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}$$

► Back

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► Back

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► Back