第 06 周作业解答

练习 1. 求下列定积分

1. $\int_{-2}^{2} f(x)dx$，其中 $f(x)=\begin{cases} x+1 & x>1 \\ 2 & x\le 1 \end{cases}$。
2. $\int_{0}^{2\pi} |\sin x|dx$。

解：(1)

$$\begin{aligned}\int_{-2}^{2} f(x)dx &= \int_{-2}^{1} f(x)dx+\int_{1}^{2} f(x)dx=\int_{-2}^{1} 2dx+\int_{1}^{2} x+1dx \\ &= 2x\big|_{-2}^{1}+(\frac{1}{2}x^2+x)\big|_{1}^{2}=2(1+2)+[(2+2)-(\frac{1}{2}+1)]=\frac{17}{2}\end{aligned}$$

(2)

$$\begin{aligned}\int_{0}^{2\pi} |\sin x|dx &= \int_{0}^{\pi} |\sin x|dx+\int_{\pi}^{2\pi} |\sin x|dx=\int_{0}^{\pi} \sin xdx+\int_{\pi}^{2\pi} -\sin xdx \\ &= -\cos x\big|_{0}^{\pi}+\cos x\big|_{\pi}^{2\pi}=-(-1-1)+[1-(-1)]=4\end{aligned}$$

练习 2. 计算不定积分：

$$(1)\ \int_{-1}^{2} 4x\sqrt{2x^2+1}dx;\quad (2)\ \int_{1}^{e} \frac{2+\ln x}{x}dx;\quad (3)\ \int_{1}^{2} \frac{e^{2t}}{\sqrt{e^{2t}-1}}dt;\quad (4)\ \int_{0}^{\frac{\pi}{2}} \cos^5 x\sin xdx$$

解：(1)

$$\begin{aligned}\int_{-1}^{2} 4x\sqrt{2x^2+1}dx &= 2\int_{-1}^{2} \sqrt{2x^2+1}d(x^2)=\int_{-1}^{2} \sqrt{2x^2+1}d(2x^2+1) \\ &\xlongequal{u=1+2x^2} \int_{3}^{9} u^{1/2}du=\frac{2}{3}u^{3/2}=\frac{2}{3}(9^{3/2}-3^{3/2})=18-2\sqrt{3}\end{aligned}$$

(2)

$$\int_{1}^{e} \frac{2+\ln x}{x}dx=\int_{1}^{e} (2+\ln x)d\ln x=\int_{1}^{e} (2+\ln x)d(2+\ln x) \xlongequal{u=2+\ln x} \int_{2}^{3} udu=\frac{1}{2}u^2\big|_{2}^{3}=\frac{1}{2}(3^2-2^2)=\frac{5}{2}$$

(3)

$$\int_{1}^{2} \frac{e^{2t}}{\sqrt{e^{2t}-1}}dt=\frac{1}{2}\int_{1}^{2} (e^{2t}-1)^{-\frac{1}{2}}d(e^{2t}-1) \xlongequal{u=e^{2t}-1} \frac{1}{2}\int_{e^2-1}^{e^4-1} u^{-1/2}du=\sqrt{u}\big|_{e^2-1}^{e^4-1}=\sqrt{e^4-1}-\sqrt{e^2-1}$$

(4)

$$\int_{0}^{\frac{\pi}{2}} \cos^5 x\sin xdx=-\int_{0}^{\frac{\pi}{2}} \cos^5 xd\cos x=-\frac{1}{6}\cos^6 x\big|_{0}^{\frac{\pi}{2}}=-\frac{1}{6}(0-1)=\frac{1}{6}$$

练习 3. 计算不定积分

$$(1)\ \int_2^5 \frac{1}{1+\sqrt{x-1}}dx; \quad (2)\ \int_4^9 \frac{\sqrt{x}}{\sqrt{x}-1}dx; \quad (3)\ \int_0^1 \frac{1}{1+e^x}dx$$

解：(1) 令 $t = 1+\sqrt{x-1}$，则 $x = (t-1)^2+1$，$dx = 2(t-1)dt$ 且 $t = 2...3$。所以

$$\int_2^5 \frac{1}{1+\sqrt{x-1}}dx = \int_2^3 \frac{1}{t}\cdot 2(t-1)dt = 2(t-\ln t)\big|_2^3 = 2-2\ln\frac{3}{2}$$

(2) 令 $t = \sqrt{x}-1$，则 $x = (t+1)^2$，$dx = 2(t+1)dt$ 且 $t = 1...2$。所以

$$\int_4^9 \frac{\sqrt{x}}{\sqrt{x}-1}dx = \int_1^2 \frac{t+1}{t}\cdot 2(t+1)dt = 2\int_1^2 (t+2+\frac{1}{t})dt = (t^2+4t+2\ln t)\big|_1^2 = 7+2\ln 2$$

(3) 令 $t = 1+e^x$，则 $x = \ln(t-1)$，$dx = \frac{1}{t-1}dt$ 且 $t = 2...1+e$。所以

$$\int_0^1 \frac{1}{1+e^x}dx = \int_2^{1+e} \frac{1}{t}\cdot\frac{1}{t-1}dt = \int_2^{1+e} \frac{1}{t-1}-\frac{1}{t}dt = \ln\frac{t-1}{t}\big|_2^{1+e} = \ln\frac{2e}{1+e}$$

另解：

$$\int_0^1 \frac{1}{1+e^x}dx = \int_0^1 \frac{1+e^x-e^x}{1+e^x}dx = \int_0^1 1dx - \int_0^1 \frac{e^x}{1+e^x}dx = x\big|_0^1 - \int_0^1 \frac{1}{1+e^x}de^x$$
$$= 1-\ln(1+e^x)\big|_0^1 = 1+\ln\frac{2}{1+e} = \ln\frac{2e}{1+e}$$

练习 4. 用分部积分法计算：

$$(1)\ \int_0^{\ln 2} xe^{-3x}dx; \quad (2)\ \int_{\frac{\pi}{2}}^{\pi} x\sin(2x)dx$$

解：(1)

$$\int_0^{\ln 2} xe^{-3x}dx = -\frac{1}{3}\int_0^{\ln 2} xde^{-3x} = -\frac{1}{3}\left(xe^{-3x}\big|_0^{\ln 2} - \int_0^{\ln 2} e^{-3x}dx\right)$$
$$= -\frac{1}{3}\left(\frac{1}{8}\ln 2 + \frac{1}{3}e^{-3x}\big|_0^{\ln 2}\right) = \frac{1}{24}\left(\frac{7}{3}-\ln 2\right)$$

(2)

$$\int_{\frac{\pi}{2}}^{\pi} x\sin(2x)dx = -\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} xd\cos(2x) = -\frac{1}{2}\left(x\cos(2x)\big|_{\frac{\pi}{2}}^{\pi} - \int_{\frac{\pi}{2}}^{\pi}\cos(2x)dx\right)$$
$$= -\frac{1}{2}\left(\frac{3}{2}\pi - \frac{1}{2}\sin(2x)\big|_{\frac{\pi}{2}}^{\pi}\right) = -\frac{3}{4}\pi$$

练习 5. 画出曲线 $y = x^3$ 与直线 $y = 1$，$x = 0$ 围成的区域，并求面积。

解：$A = \int_0^1 (1-x^3)dx = \left(x - \frac{1}{4}x^4\right)\big|_0^1 = (1-\frac{1}{4})-(0) = \frac{3}{4}$