第 11 章 d:对面积的曲面积分

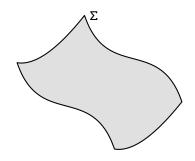
数学系 梁卓滨

2017-2018 学年 II



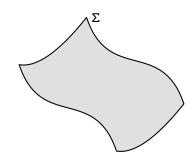
假设

- Σ 为空间中曲面
- 密度为 μ
- 质量为 m



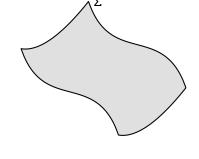
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- 当材料均匀时(μ=常数),



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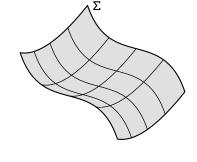
当材料均匀时(μ=常数),

$$m = \mu \cdot Area(\Sigma)$$

• 当材料非均匀时 (μ = μ(x, y, z) 为 Σ 上函数),

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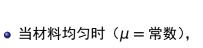


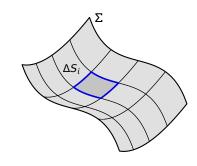
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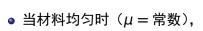


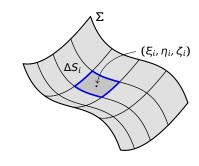


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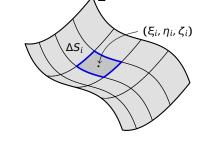




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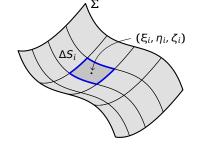
当材料均匀时(μ=常数),

$$m = \mu \cdot Area(\Sigma)$$

$$\mu(\xi_i, \eta_i, \zeta_i)\Delta S_i$$

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● 当材料均匀时(µ=常数),

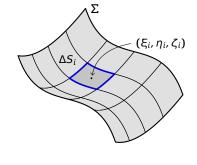
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$$\sum_{i=1}^{n} \mu(\xi_i, \, \eta_i, \, \zeta_i) \Delta S_i$$



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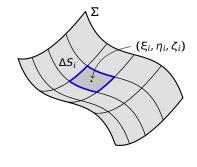
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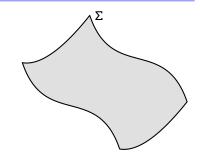
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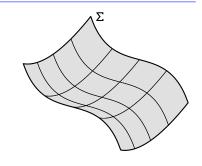
设

- Σ是空间中有界分片光滑曲面,
- f(x, y, z) 是 Σ 上的有界函数,



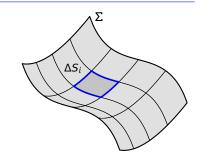
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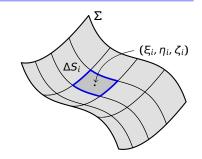
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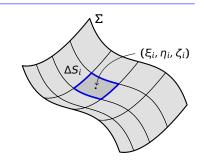
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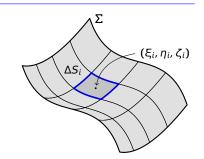
$$\sum_{i=1}^n f(\xi_i, \, \eta_i, \, \zeta_i) \Delta S_i$$



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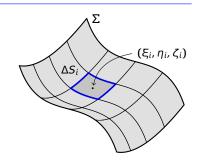


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若

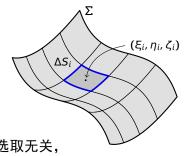
• 极限 $\lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_i, \eta_i, \zeta_i) \Delta S_i$ 存在,



设

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- 极限 $\lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_i, \eta_i, \zeta_i) \Delta S_i$ 存在,
- 且该极限与 Σ 的划分、 (ξ_i, η_i, ζ_i) 的选取无关,

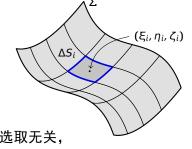


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则定义

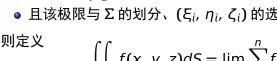
$$\iint_{\Sigma} f(x, y, z) dS = \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_i, \eta_i, \zeta_i) \Delta S_i$$

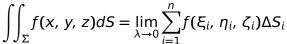
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称为 f(x, y, z) 在 Σ 上对面积的曲面积分。

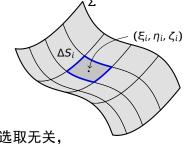
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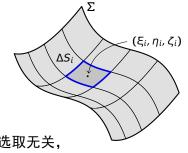
称为 f(x, y, z) 在 Σ 上对面积的曲面积分。 dS 称为面积元素。

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注 对面积曲面积分的定义式与二重积分的类似,故性质也类似

存在性 若 f(x, y, z) 在有界曲面 Σ 上连续,则

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• 线性性 $\iint_{\Sigma} (\alpha f + \beta g) dS = \alpha \iint_{\Sigma} f dS + \beta \iint_{\Sigma} g dS$

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- 可加性 $\iint_{\Sigma} f(x, y, z) dS = \iint_{\Sigma_1} f(x, y, z) dS + \iint_{\Sigma_2} f(x, y, z) dS$

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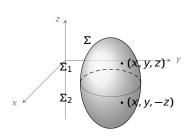
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- $\iint_{\Sigma} 1dS = \text{Area}(\Sigma)$
- 若 $f(x, y, z) \leq g(x, y, z)$,则

$$\iint_{\Sigma} f(x, y, z) dS \le \iint_{\Sigma} g(x, y, z) dS$$

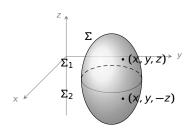


性质 设曲面 Σ 关于 xoy 坐标面对称,



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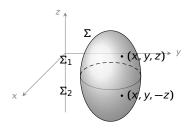
• 若f(x, y, z) 关于z 是奇函数(即: f(x, y, -z) = -f(x, y, z)),则





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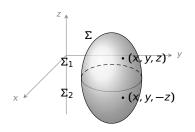


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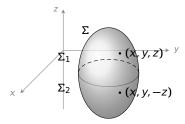
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$$\iint_{\Sigma} f(x, y, z) dS = 2 \iint_{\Sigma_1} f(x, y, z) dS = 2 \iint_{\Sigma_2} f(x, y, z) dS$$





例 设曲面 Σ 为上半球面 $x^2 + y^2 + z^2 = \alpha^2$ ($z \ge 0$); Σ_1 为 Σ 在第一卦

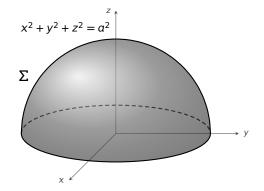
限的部分。则有(

(A)
$$\iint_{\Sigma} x dS = 4 \iint_{\Sigma_1} x dS$$

(B)
$$\iint_{\Sigma} y dS = 4 \iint_{\Sigma_1} y dS$$

(C)
$$\iint_{\Sigma} z dS = 4 \iint_{\Sigma_1} z dS$$

(D)
$$\iint_{\Sigma} xyzdS = 4 \iint_{\Sigma_1} xyzdS$$



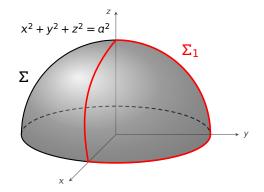
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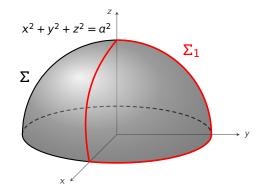
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$$\iint_{\Sigma} x^2 dS = \iint_{\Sigma} y^2 dS = \iint_{\Sigma} z^2 dS$$

解由对称性:

所以

$$\iint_{\Sigma} x^2 dS = \iint_{\Sigma} y^2 dS = \iint_{\Sigma} z^2 dS$$

$$\iint_{\Sigma} (x^2 + y^2) dS = 2 \iint_{\Sigma} x^2 dS$$

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所以
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$$= \frac{2}{3} \left[\iint_{\Sigma} x^2 dS + \iint_{\Sigma} y^2 dS + \iint_{\Sigma} z^2 dS \right]$$

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$$= \frac{2}{3} \iint_{\Sigma} x^2 + y^2 + z^2 dS$$

$$= \frac{2}{3} \iint_{\Sigma} R^2 dS$$

$$\iint_{\Sigma} x^2 dS = \iint_{\Sigma} y^2 dS = \iint_{\Sigma} z^2 dS$$

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$$= \frac{2}{3} \iint_{\Sigma} x^2 + y^2 + z^2 dS$$

$$= \frac{2}{3} \iint_{\Sigma} R^2 dS = \frac{2}{3} R^2 \text{Area}(\Sigma)$$

$$\iint_{\Sigma} x^2 dS = \iint_{\Sigma} y^2 dS = \iint_{\Sigma} z^2 dS$$

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$$= \frac{2}{3} \iint_{\Sigma} x^2 + y^2 + z^2 dS$$

$$= \frac{2}{3} \iiint_{\pi} R^2 dS = \frac{2}{3} R^2 \text{Area}(\Sigma) = \frac{2}{3} R^2 \cdot 4\pi R^2$$



$$\iint_{\Sigma} x^2 dS = \iint_{\Sigma} y^2 dS = \iint_{\Sigma} z^2 dS$$

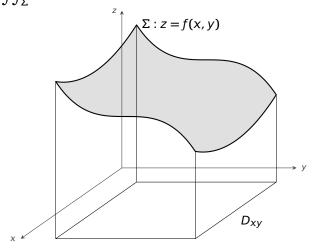
所以
$$\iint_{\mathbb{R}} (x^2 + y^2) dS = 2 \iint_{\mathbb{R}} x^2 dS$$

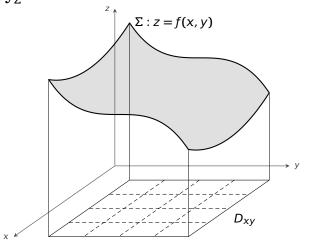
$$= \frac{2}{3} \left[\iint_{\Sigma} x^2 dS + \iint_{\Sigma} y^2 dS + \iint_{\Sigma} z^2 dS \right]$$

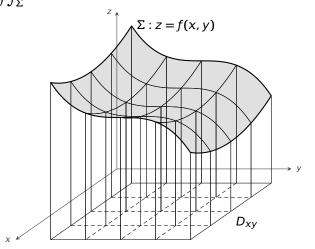
$$= \frac{2}{3} \iint_{\Sigma} x^2 + y^2 + z^2 dS$$

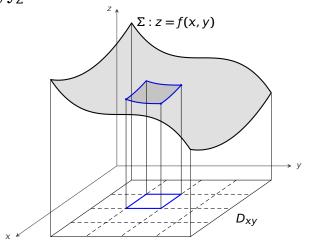
$$+z^2dS$$

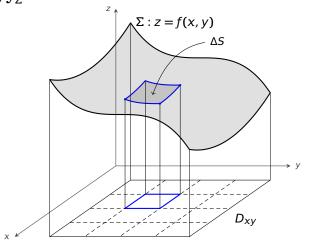
$$= \frac{2}{3} \iint_{\Sigma} R^2 dS = \frac{2}{3} R^2 \text{Area}(\Sigma) = \frac{2}{3} R^2 \cdot 4\pi R^2 = \frac{8}{3} \pi R^4$$

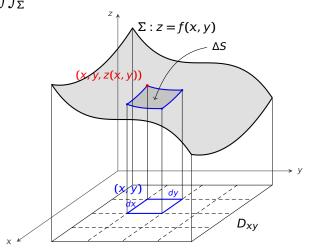


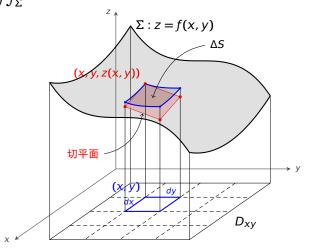


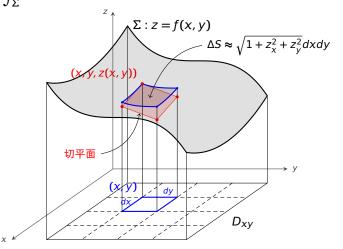


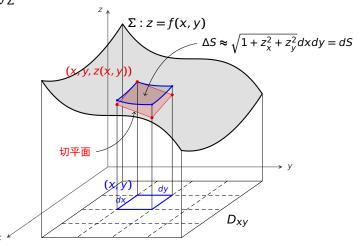






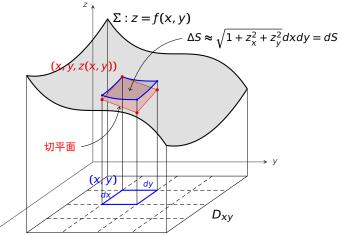




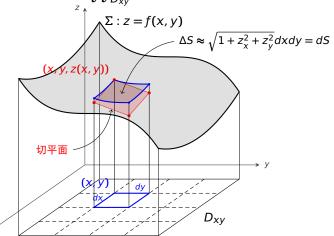


• 假设 Σ 是二元函数 z=z(x,y), $(x,y)\in D_{xy}$ 的图形,则

$$\iint_{\Sigma} f(x, y, z) dS = f(x, y, z(x, y)) \cdot \sqrt{1 + z_x^2 + z_y^2} dx dy$$

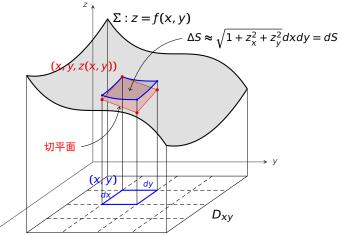


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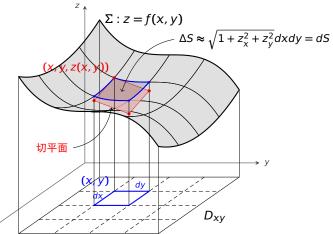
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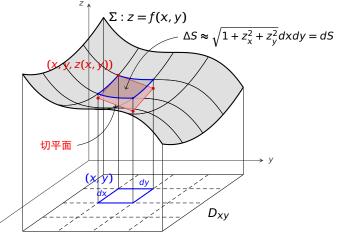
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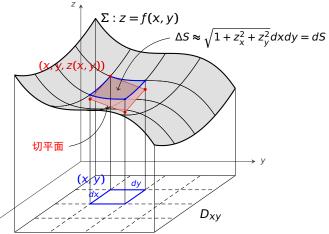
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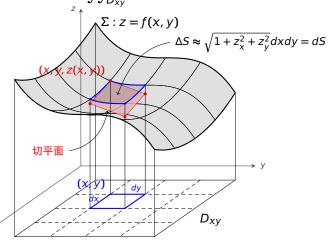


• 假设 Σ 是二元函数 z = z(x, y), $(x, y) \in D_{xy}$ 的图形,则

$$\iint_{\Sigma} f(x, y, z) dS = \lim_{X \to \infty} \sum_{x} f(x, y, z(x, y)) \cdot \sqrt{1 + z_x^2 + z_y^2} dx dy$$



• 假设 Σ 是二元函数 $z=z(x,y), (x,y) \in D_{xy}$ 的图形,则 $\iint_{\Sigma} f(x,y,z) dS = \iint_{D_{xy}} f(x,y,z(x,y)) \cdot \sqrt{1+z_x^2+z_y^2} dx dy$



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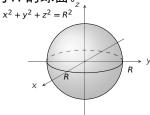
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 Σ_k 都分别是某个二元函数的图形



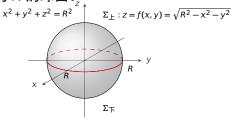
例 1 将对面积的曲面积分 $\iint_{\Sigma} f(x, y, z) dS$ 转换为重积分,其中 Σ 是球

心在原点,半径为 R 的球面。 $_{_{z}}$

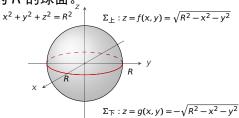


心在原点,半径为 R 的球面。 $x^2 + y^2 + z^2 = R^2$ Σ_{\perp} Σ_{\perp}

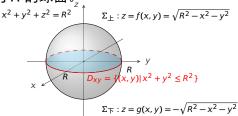
心在原点,半径为R的球面。 $_{_{7}}$

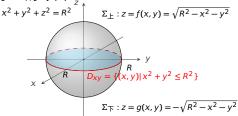


心在原点,半径为R的球面。



心在原点,半径为R的球面。





$$\iint_{\Sigma} f(x, y, z) dS = \iint_{\Sigma_{\pm}} f(x, y, z) dS + \iint_{\Sigma_{\mp}} f(x, y, z) dS$$

心在原点,半径为R的球面。 $_{z}$

$$\sum_{x^{2} + y^{2} + z^{2} = R^{2}} \sum_{\pm} \sum_{z = f(x, y) = \sqrt{R^{2} - x^{2} - y^{2}}} \sum_{x = f(x, y) = f(x, y) = f(x, y) = f(x, y)} \sum_{x = f(x, y) = f(x, y) = f(x, y) = f(x, y)} \sum_{x = f(x, y) = f(x, y) = f(x, y) = f(x, y)} \sum_{x = f(x, y) = f(x, y) = f(x, y) = f(x, y)} \sum_{x = f(x, y) = f(x, y) = f(x, y)} \sum_{x = f(x, y) = f(x, y) = f(x, y)} \sum_{x = f(x, y) = f(x, y) = f(x, y)} \sum_{x = f(x, y)} \sum_{x = f(x, y) = f(x, y)} \sum_{x = f(x, y)}$$

$$\iint_{\Sigma} f(x, y, z) dS = \iint_{\Sigma_{\pm}} f(x, y, z) dS + \iint_{\Sigma_{\mp}} f(x, y, z) dS$$
$$= \iint_{D_{xy}} f(x, y, \sqrt{\alpha^2 - x^2 - y^2}) \cdot \sqrt{1 + f_x^2 + f_y^2} dx dy$$

$$\sum_{x^{2}+y^{2}+z^{2}=R^{2}} \sum_{x^{2}+y^{2}+z^{2}=R^{2}} \sum_{x^{2}+y^{2}+z^{2}=R^{2}} \sum_{x^{2}+y^{2}=R^{2}} \sum_$$

$$\iint_{\Sigma} f(x, y, z) dS = \iint_{\Sigma_{\pm}} f(x, y, z) dS + \iint_{\Sigma_{\mp}} f(x, y, z) dS$$

$$= \iint_{D_{xy}} f(x, y, \sqrt{a^2 - x^2 - y^2}) \cdot \sqrt{1 + f_x^2 + f_y^2} dx dy$$

$$+ \iint_{D_{xy}} f(x, y, -\sqrt{a^2 - x^2 - y^2}) \cdot \sqrt{1 + g_x^2 + g_y^2} dx dy$$

心在原点,半径为
$$R$$
 的球面。 $_{z}$ $x^{2}+y^{2}+z^{2}=R^{2}$ $\Sigma_{\pm}:z=f(x,y)=\sqrt{R^{2}-x^{2}-y^{2}}$ $\Sigma_{\pm}:z=g(x,y)=\sqrt{R^{2}-x^{2}-y^{2}}$ $\Sigma_{\mp}:z=g(x,y)=-\sqrt{R^{2}-x^{2}-y^{2}}$

$$\iint_{\Sigma} f(x, y, z) dS = \iint_{\Sigma_{\pm}} f(x, y, z) dS + \iint_{\Sigma_{\mp}} f(x, y, z) dS
= \iint_{D_{xy}} f(x, y, \sqrt{a^2 - x^2 - y^2}) \cdot \frac{R}{\sqrt{R^2 - x^2 - y^2}} dx dy
+ \iint_{\Sigma} f(x, y, -\sqrt{a^2 - x^2 - y^2}) \cdot \sqrt{1 + g_x^2 + g_y^2} dx dy$$

心在原点,半径为
$$R$$
 的球面。 z
 $x^2 + y^2 + z^2 = R^2$
 $\sum_{\pm} : z = f(x, y) = \sqrt{R^2 - x^2 - y^2}$
 $\sum_{\pm} : z = g(x, y) = -\sqrt{R^2 - x^2 - y^2}$

$$\iint_{\Sigma} f(x, y, z) dS = \iint_{\Sigma_{\pm}} f(x, y, z) dS + \iint_{\Sigma_{\mp}} f(x, y, z) dS
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$$\sum_{x^{2} + y^{2} + z^{2} = R^{2}} \sum_{\underline{L}} : z = f(x, y) = \sqrt{R^{2} - x^{2} - y^{2}}$$

$$\sum_{\underline{L}} : z = f(x, y) = \sqrt{R^{2} - x^{2} - y^{2}}$$

$$\sum_{\underline{L}} : z = f(x, y) = \sqrt{R^{2} - x^{2} - y^{2}}$$

$$\sum_{\underline{L}} : z = g(x, y) = \sqrt{R^{2} - x^{2} - y^{2}}$$

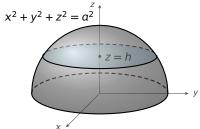
$$\iint_{\Sigma} f(x, y, z) dS = \iint_{\Sigma_{\pm}} f(x, y, z) dS + \iint_{\Sigma_{\mp}} f(x, y, z) dS$$

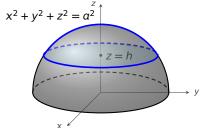
$$= \iint_{D_{xy}} f(x, y, \sqrt{\alpha^2 - x^2 - y^2}) \cdot \frac{R}{\sqrt{R^2 - x^2 - y^2}} dxdy$$

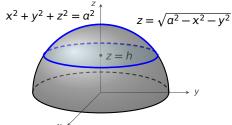
$$+ \iint_{D_{xy}} f(x, y, -\sqrt{\alpha^2 - x^2 - y^2}) \cdot \frac{R}{\sqrt{R^2 - x^2 - y^2}} dxdy$$

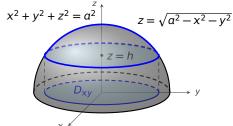
$$= \iint_{D_{xy}} f(x, y, \sqrt{\alpha^2 - x^2 - y^2}) \cdot \frac{R}{\sqrt{R^2 - x^2 - y^2}} dx dy$$
$$+ \iint_{D_{xy}} f(x, y, -\sqrt{\alpha^2 - x^2 - y^2}) \cdot \frac{R}{\sqrt{R^2 - x^2 - y^2}} dx dy$$

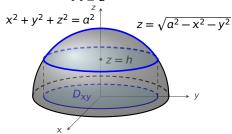
 $= \iint_{D_{XY}} \left[f(x,y,\sqrt{a^2 - x^2 - y^2}) + f(x,y,-\sqrt{a^2 - x^2 - y^2}) \right] \cdot \frac{R}{\sqrt{R^2 - x^2 - y^2}} dxdy$

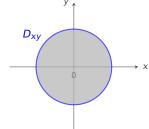


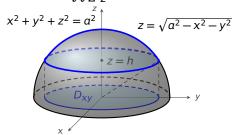


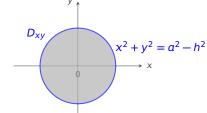


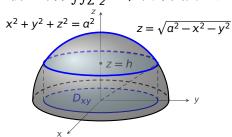


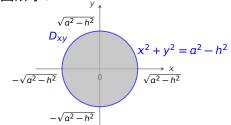


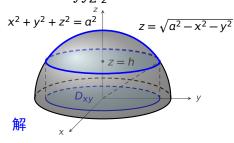


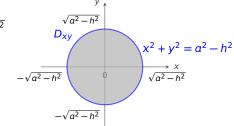




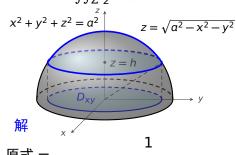




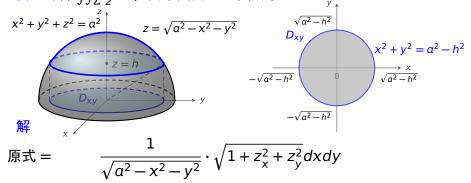


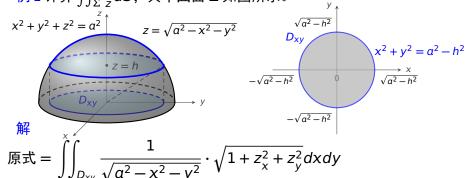


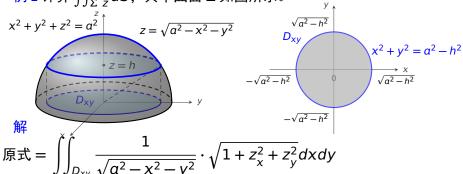
原式 =



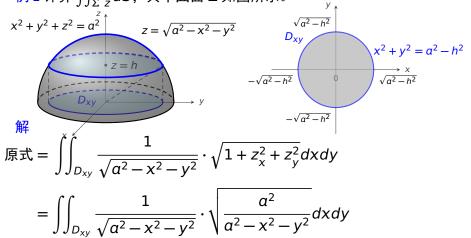
原式 = $\frac{1}{\sqrt{a^2 - x^2 - y^2}}$



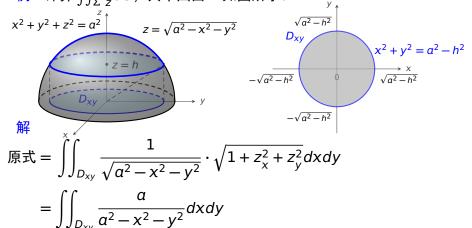


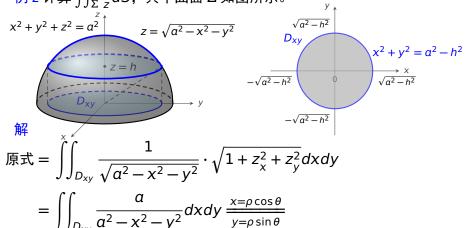


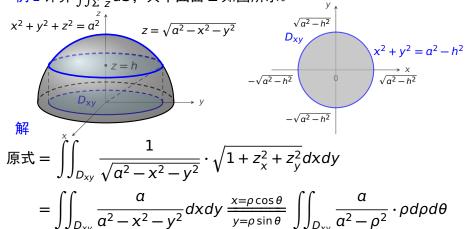
$$\cdot \sqrt{\frac{a^2}{a^2 - x^2 - y^2}}$$

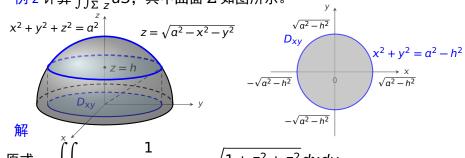








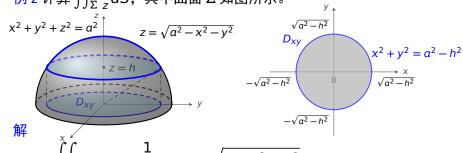




原式 =
$$\iint_{D_{xy}} \frac{1}{\sqrt{a^2 - x^2 - y^2}} \cdot \sqrt{1 + z_x^2 + z_y^2} dx dy$$
$$= \iint_{D_{xy}} \frac{a}{a^2 - x^2 - y^2} dx dy \frac{x = \rho \cos \theta}{y = \rho \sin \theta} \iint_{D_{xy}} \frac{a}{a^2 - \rho^2} \cdot \rho d\rho d\theta$$

$$= \int \left[\int \frac{a}{a^2 - \rho^2} \cdot \rho d\rho \right] d\theta$$





原式 =
$$\iint_{D_{xy}} \frac{1}{\sqrt{a^2 - x^2 - y^2}} \cdot \sqrt{1 + z_x^2 + z_y^2} dx dy$$
$$= \iint_{D_{xy}} \frac{a}{a^2 - x^2 - y^2} dx dy \frac{x = \rho \cos \theta}{y = \rho \sin \theta} \iint_{D_{xy}} \frac{a}{a^2 - \rho^2} \cdot \rho d\rho d\theta$$

$$= \int_0^{2\pi} \left[\int \frac{\alpha}{\alpha^2 - \rho^2} \cdot \rho d\rho \right] d\theta$$



$$x^{2} + y^{2} + z^{2} = a^{2}$$

$$z = \sqrt{a^{2} - x^{2} - y^{2}}$$

$$z = \sqrt{a^{2} - x^{2} - y^{2}}$$

$$-\sqrt{a^{2} - h^{2}}$$

$$x^{2} + y^{2} = a^{2} - h^{2}$$

$$-\sqrt{a^{2} - h^{2}}$$

$$1$$

原式 =
$$\iint_{D_{xy}} \frac{1}{\sqrt{a^2 - x^2 - y^2}} \cdot \sqrt{1 + z_x^2 + z_y^2} dx dy$$

$$= \iint_{D_{xy}} \frac{a}{a^2 - x^2 - y^2} dx dy \frac{x = \rho \cos \theta}{y = \rho \sin \theta} \iint_{D_{xy}} \frac{a}{a^2 - \rho^2} \cdot \rho d\rho d\theta$$

$$= \int_0^{2\pi} \left[\int_0^{\sqrt{a^2 - h^2}} \frac{a}{a^2 - \rho^2} \cdot \rho d\rho \right] d\theta$$



$$x^{2} + y^{2} + z^{2} = a^{2}$$

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$$\sqrt{a^{2} - h^{2}}$$

$$\sqrt{a^{2} - h^{2}}$$

$$\sqrt{a^{2} - h^{2}}$$

原式 =
$$\iint_{D_{xy}} \frac{1}{\sqrt{a^2 - x^2 - y^2}} \cdot \sqrt{1 + z_x^2 + z_y^2} dx dy$$

$$= \iint_{D_{xy}} \frac{a}{a^2 - x^2 - y^2} dx dy \frac{x = \rho \cos \theta}{y = \rho \sin \theta} \iint_{D_{xy}} \frac{a}{a^2 - \rho^2} \cdot \rho d\rho d\theta$$

$$= \int_0^{2\pi} \left[\int_0^{\sqrt{a^2 - h^2}} \frac{a}{a^2 - \rho^2} \cdot \rho d\rho \right] d\theta = 2\pi.$$



アンドナリテンスはあり、 共下面間 2 対配所が。
$$x^{2} + y^{2} + z^{2} = a^{2}$$

$$z = \sqrt{a^{2} - x^{2} - y^{2}}$$

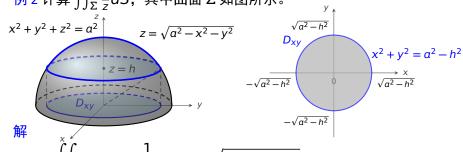
$$z = h$$

$$-\sqrt{a^{2} - h^{2}}$$

原式 =
$$\frac{1}{\sqrt{x^{2} - x^{2} - x^{2}}} \cdot \sqrt{1 + z_{x}^{2} + z_{y}^{2}} dx dy$$

原式 =
$$\iint_{D_{xy}} \frac{1}{\sqrt{a^2 - x^2 - y^2}} \cdot \sqrt{1 + z_x^2 + z_y^2} dx dy$$
=
$$\iint_{D_{xy}} \frac{a}{a^2 - x^2 - y^2} dx dy \frac{x = \rho \cos \theta}{y = \rho \sin \theta} \iint_{D_{xy}} \frac{a}{a^2 - \rho^2} \cdot \rho d\rho d\theta$$
=
$$\int_0^{2\pi} \left[\int_0^{\sqrt{a^2 - h^2}} \frac{a}{a^2 - \rho^2} \cdot \rho d\rho \right] d\theta \frac{u = a^2 - \rho^2}{2\pi} 2\pi \cdot \theta d\theta d\theta$$



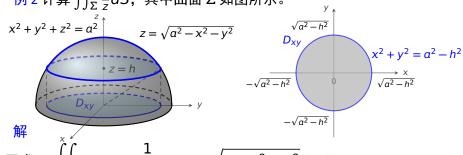


原式 =
$$\iint_{D_{xy}} \frac{1}{\sqrt{a^2 - x^2 - y^2}} \cdot \sqrt{1 + z_x^2 + z_y^2} dx dy$$

$$= \iint_{D_{xy}} \frac{a}{a^2 - x^2 - y^2} dx dy \frac{x = \rho \cos \theta}{y = \rho \sin \theta} \iint_{D_{xy}} \frac{a}{a^2 - \rho^2} \cdot \rho d\rho d\theta$$

$$= \int_0^{2\pi} \left[\int_0^{\sqrt{a^2 - h^2}} \frac{a}{a^2 - \rho^2} \cdot \rho d\rho \right] d\theta \xrightarrow{u = a^2 - \rho^2} 2\pi \cdot \frac{a}{u}$$



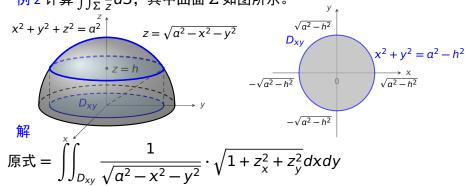


原式 =
$$\iint_{D_{xy}} \frac{1}{\sqrt{a^2 - x^2 - y^2}} \cdot \sqrt{1 + z_x^2 + z_y^2} dx dy$$

$$= \iint_{D_{xy}} \frac{a}{a^2 - x^2 - y^2} dx dy \xrightarrow{\frac{x = \rho \cos \theta}{y = \rho \sin \theta}} \iint_{D_{xy}} \frac{a}{a^2 - \rho^2} \cdot \rho d\rho d\theta$$

$$= \int_{0}^{2\pi} \left[\int_{0}^{\sqrt{a^2 - h^2}} \frac{a}{a^2 - \rho^2} \cdot \rho d\rho \right] d\theta \xrightarrow{u = a^2 - \rho^2} 2\pi \cdot \frac{a}{u} \cdot (-\frac{1}{2}) du$$





$$\iint_{D_{xy}} \sqrt{a^2 - x^2 - y^2} \sqrt{x} dx dy = \iint_{D_{xy}} \frac{a}{a^2 - x^2 - y^2} dx dy = \iint_{D_{xy}} \frac{a}{a^2 - \rho^2} \cdot \rho d\rho d\theta$$

$$= \iint_{D_{xy}} \frac{1}{\alpha^2 - x^2 - y^2} dx dy = \underbrace{\prod_{y=\rho \sin \theta}}_{y=\rho \sin \theta} \iint_{D_{xy}} \frac{1}{\alpha^2 - \rho^2} \cdot \rho d\rho d\theta$$

$$= \int_0^{2\pi} \left[\int_0^{\sqrt{\alpha^2 - h^2}} \frac{\alpha}{\alpha^2 - \rho^2} \cdot \rho d\rho \right] d\theta = \underbrace{\prod_{y=\rho \sin \theta}}_{y=\rho \sin \theta} 2\pi \cdot \int_{\alpha^2}^{h^2} \frac{\alpha}{u} \cdot (-\frac{1}{2}) du$$



例 2 计算 $\iint_{\Sigma} \frac{1}{z} dS$, 其中曲面 Σ 如图所示。 $x^2 + y^2 + z^2 = a^2 \uparrow$ $\sqrt{a^2 - h^2}$ $z = \sqrt{a^2 - x^2 - y^2}$ $x^2 + v^2 = a^2 - h^2$

$$z = \sqrt{a^2 - x^2 - y^2}$$

$$z = \sqrt{a^2 - x^2 - y^2}$$

$$x^2 + y^2 = a^2 - h$$

$$\sqrt{a^2 - h^2}$$

$$x^2 + y^2 = a^2 - h$$

$$\sqrt{a^2 - h^2}$$

$$\sqrt{a^2 - h^2}$$

$$\sqrt{a^2 - h^2}$$

$$\sqrt{a^2 - h^2}$$

$$= \iint_{D_{xy}} \frac{a}{a^2 - x^2 - y^2} dx dy \xrightarrow{\frac{x = \rho \cos \theta}{y = \rho \sin \theta}} \iint_{D_{xy}} \frac{a}{a^2 - \rho^2} \cdot \rho d\rho d\theta$$

$$= \int_{D_{xy}}^{2\pi} \left[\int_{-\infty}^{\sqrt{a^2 - h^2}} \frac{a}{a} \cdot \rho d\rho \right] d\theta \xrightarrow{u = a^2 - \rho^2} 2\pi \cdot \int_{-\infty}^{h^2} \frac{a}{a} \cdot (-\frac{1}{a}) d\mu$$

$$= \int_{0}^{2\pi} \left[\int_{0}^{\sqrt{a^{2}-h^{2}}} \frac{a}{a^{2}-\rho^{2}} \cdot \rho d\rho \right] d\theta \xrightarrow{u=a^{2}-\rho^{2}} 2\pi \cdot \int_{a^{2}}^{h^{2}} \frac{a}{u} \cdot (-\frac{1}{2}) du$$

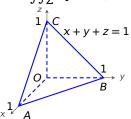
$$= -\pi a \ln u \Big|_{a^{2}}^{h^{2}}$$



例 2 计算 $\iint_{\Sigma} \frac{1}{z} dS$, 其中曲面 Σ 如图所示。 $x^2 + y^2 + z^2 = a^{\frac{z}{2}}$ $z = \sqrt{a^2 - x^2 - y^2}$ D_{xy} $x^2 + y^2 = a^2 - h^2$ $-\sqrt{a^2 - h^2}$

 $= \iint_{D_{xy}} \frac{a}{a^2 - x^2 - y^2} dx dy \xrightarrow{\frac{x = \rho \cos \theta}{y = \rho \sin \theta}} \iint_{D_{xy}} \frac{a}{a^2 - \rho^2} \cdot \rho d\rho d\theta$ $= \int_0^{2\pi} \left[\int_0^{\sqrt{a^2 - h^2}} \frac{a}{a^2 - \rho^2} \cdot \rho d\rho \right] d\theta \xrightarrow{u = a^2 - \rho^2} 2\pi \cdot \int_{a^2}^{h^2} \frac{a}{u} \cdot (-\frac{1}{2}) du$

 $= -\pi \alpha \ln u \Big|_{\alpha^2}^{h^2} = 2\pi \alpha \ln \frac{\alpha}{h}$



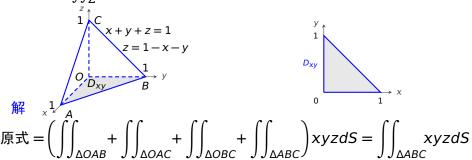
$$\mathbf{F}_{A} = \left(\iint_{\Delta OAB} + \iint_{\Delta OAC} + \iint_{\Delta OBC} + \iint_{\Delta ABC} \right) xyzdS$$

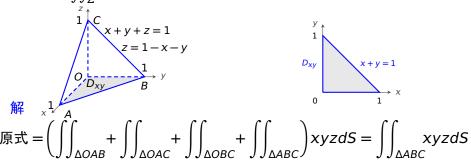
$$\mathbf{p}$$
 展式 = $\left(\iint_{\Delta OAB} + \iint_{\Delta OAC} + \iint_{\Delta OBC} + \iint_{\Delta ABC} xyzdS = \iint_{\Delta ABC} xyzdS \right)$

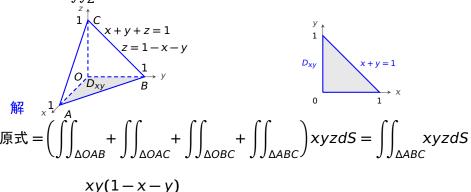
原式 =
$$\left(\iint_{\Delta OAB} + \iint_{\Delta OAC} + \iint_{\Delta OBC} + \iint_{\Delta ABC} xyzdS = \iint_{\Delta ABC} xyzdS$$

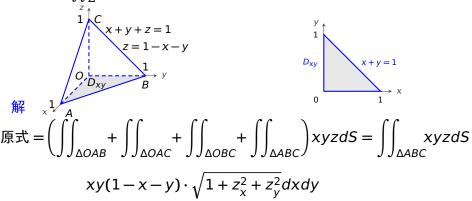
$$x + y + z = 1$$
 $z = 1 - x - y$

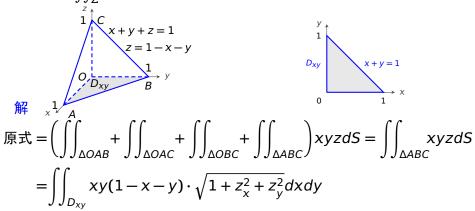
原式 = $\left(\iint_{\Delta OAB} + \iint_{\Delta OAC} + \iint_{\Delta OBC} + \iint_{\Delta ABC} \right) xyzdS = \iint_{\Delta ABC} xyzdS$

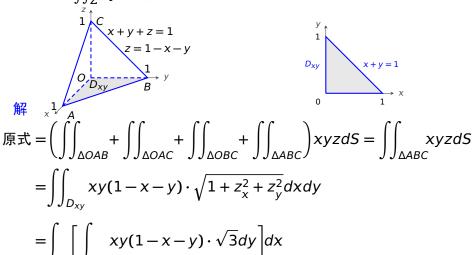




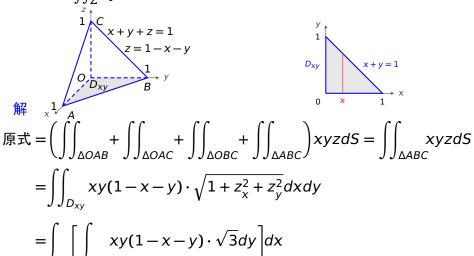




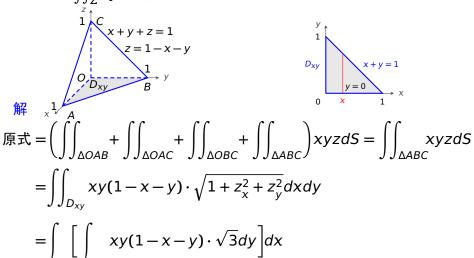




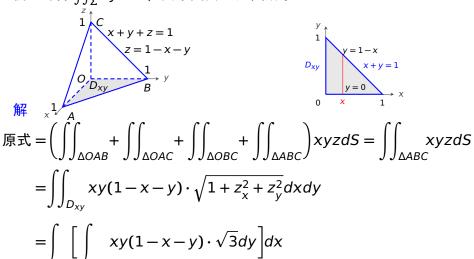




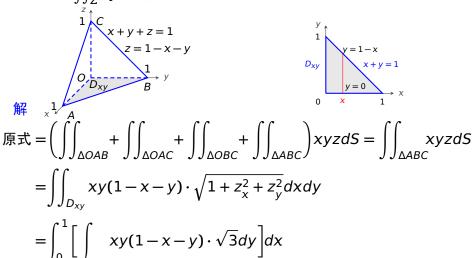




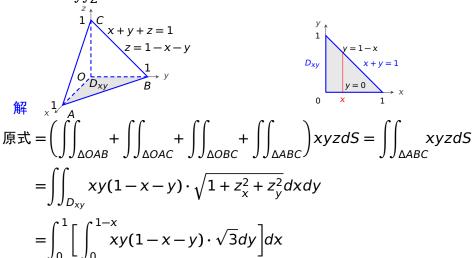












解
$$x^{1}A$$

原式 = $\left(\iint_{\Delta OAB} + \iint_{\Delta OAC} + \iint_{\Delta OBC} + \iint_{\Delta ABC}\right) xyzdS = \iint_{\Delta ABC} xyzdS$

$$= \iint_{\Delta OAB} \int_{\Delta OAC} \int_{\Delta OBC} \int_{\Delta OBC} \int_{\Delta ABC} \int_{\Delta ABC} \int_{\Delta ABC} \int_{\Delta OBC} \int_$$



解
$$x^{1}A$$

原式 = $\left(\iint_{\Delta OAB} + \iint_{\Delta OAC} + \iint_{\Delta OBC} + \iint_{\Delta ABC}\right) xyzdS = \iint_{\Delta ABC} xyzdS$

$$= \iint_{D_{xy}} xy(1-x-y) \cdot \sqrt{1+z_x^2+z_y^2} dx dy$$

$$= \int_0^1 \left[\int_0^{1-x} xy(1-x-y) \cdot \sqrt{3} dy \right] dx$$

$$= x \left[(1-x) \frac{y^2}{2} - \frac{1}{3} y^3 \right]_0^{1-x}$$



解
$$x^{1}$$
 A

原式 = $\left(\iint_{\Delta OAB} + \iint_{\Delta OAC} + \iint_{\Delta OBC} + \iint_{\Delta ABC}\right) xyzdS = \iint_{\Delta ABC} xyzdS$

$$= \iint_{D_{xy}} xy(1-x-y) \cdot \sqrt{1+z_{x}^{2}+z_{y}^{2}} dxdy$$

$$= \int_{0}^{1} \left[\int_{0}^{1-x} xy(1-x-y) \cdot \sqrt{3} dy\right] dx$$

$$= \sqrt{3} \int_{0}^{1} x \left[(1-x)\frac{y^{2}}{2} - \frac{1}{3}y^{3}\right]_{0}^{1-x} dx$$



$$= \iint_{D_{xy}} xy(1-x-y) \cdot \sqrt{1+z_x^2+z_y^2} dxdy$$
$$= \int_0^1 \left[\int_0^{1-x} xy(1-x-y) \cdot \sqrt{3} dy \right] dx$$

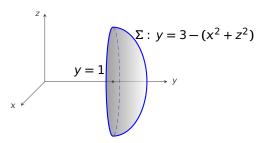
 $= \sqrt{3} \int_{0}^{1} x \left[(1-x) \frac{y^{2}}{2} - \frac{1}{3} y^{3} \right]_{0}^{1-x} dx = \sqrt{3} \int_{0}^{1} \frac{1}{6} x (1-x)^{3} dx$

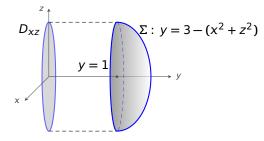
 $=\sqrt{3}\int_{0}^{1}x\left[(1-x)\frac{y^{2}}{2}-\frac{1}{3}y^{3}\right]_{0}^{1-x}dx=\sqrt{3}\int_{0}^{1}\frac{1}{6}x(1-x)^{3}dx=\frac{\sqrt{3}}{120}$

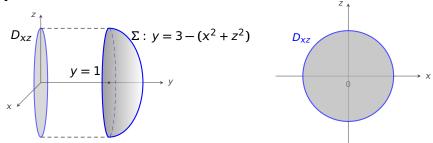
$$= \iint_{D_{xy}} xy(1-x-y) \cdot \sqrt{1+z_{x}^{2}+z_{y}^{2}} dx dy$$
$$= \int_{0}^{1} \left[\int_{0}^{1-x} xy(1-x-y) \cdot \sqrt{3} dy \right] dx$$

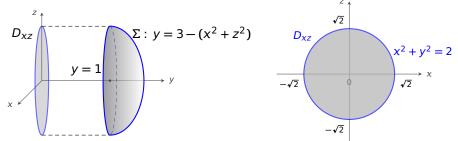
11章 d:对面积的曲面积分

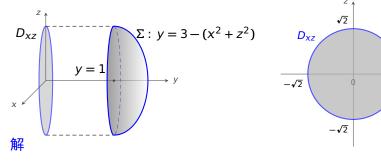
13 ⊲ ⊳ Δ

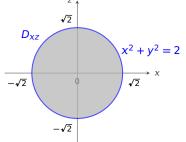




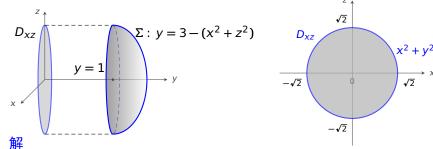






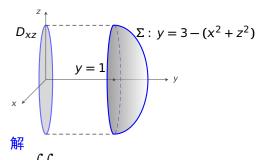


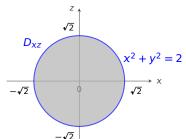
I =

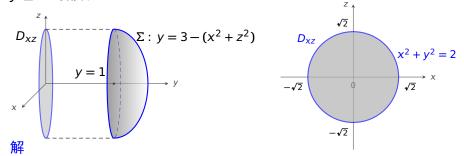


 $I = 3 \cdot \sqrt{1 + y_x^2 + y_z^2 dx dz}$

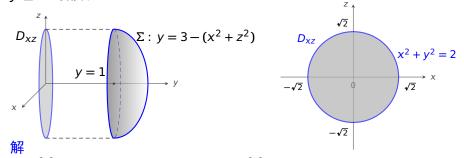






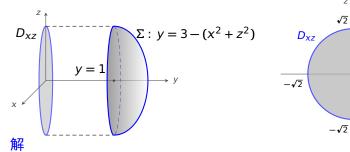


 $I = \iint_{D_{xz}} 3 \cdot \sqrt{1 + y_x^2 + y_z^2} dx dz = \iint_{D_{xz}} 3 \cdot \sqrt{1 + 4x^2 + 4z^2} dx dz$



 $I = \iint_{C} 3 \cdot \sqrt{1 + y_{x}^{2} + y_{z}^{2}} dx dz = \iint_{C} 3 \cdot \sqrt{1 + 4x^{2} + 4z^{2}} dx dz$ $x=\rho\cos\theta$

$$\frac{x=\rho\cos\theta}{z=\rho\sin\theta}$$



$$D_{XZ}$$

$$-\sqrt{2}$$

$$0$$

$$\sqrt{2}$$

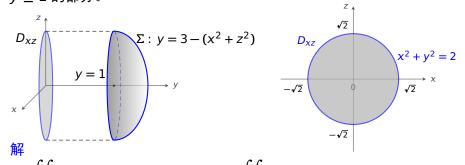
$$x^{2} + y^{2} = 2$$

$$\sqrt{2}$$

$$I = \iint_{D_{xz}} 3 \cdot \sqrt{1 + y_x^2 + y_z^2} dx dz = \iint_{D_{xz}} 3 \cdot \sqrt{1 + 4x^2 + 4z^2} dx dz$$

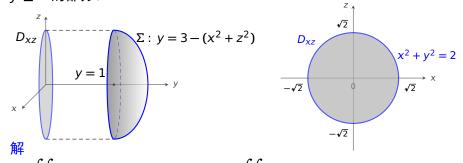
$$\frac{x = \rho \cos \theta}{z = \rho \sin \theta} \iint_{D_{xz}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho d\theta$$





$$\prod_{z=\rho\cos\theta} 3\cdot\sqrt{1+y_x^2+y_z^2}dxdz = \iint_{D_{xz}} 3\cdot\sqrt{1+4x^2+4z^2}dxdz$$

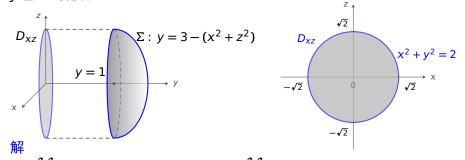
$$\frac{x=\rho\cos\theta}{z=\rho\sin\theta} \iint_{D_{xz}} 3\sqrt{1+4\rho^2}\cdot\rho d\rho d\theta = \left[\int_{D_{xz}} 3\sqrt{1+4\rho^2}\cdot\rho d\rho\right]d\theta$$



$$I = \iint_{D_{XZ}} 3 \cdot \sqrt{1 + y_X^2 + y_Z^2} dx dz = \iint_{D_{XZ}} 3 \cdot \sqrt{1 + 4x^2 + 4z^2} dx dz$$

$$\frac{x = \rho \cos \theta}{z = \rho \sin \theta} \iint_{D_{xz}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho d\theta = \int_{0}^{2\pi} \left[\int_{D_{xz}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho \right] d\theta$$

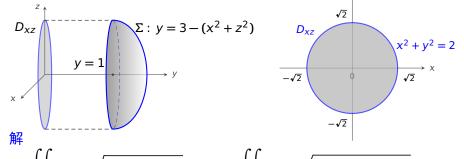




$$\frac{R}{I} = \iint_{D_{XZ}} 3 \cdot \sqrt{1 + y_X^2 + y_Z^2} dx dz = \iint_{D_{XZ}} 3 \cdot \sqrt{1 + 4x^2 + 4z^2} dx dz$$

$$\frac{x = \rho \cos \theta}{z = \rho \sin \theta} \iint_{D_{XZ}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho d\theta = \int_{0}^{2\pi} \left[\int_{0}^{\sqrt{2}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho \right] d\theta$$

 $y \ge 1$ 的部分。



$$\widetilde{H}$$

$$I = \iint_{D_{xz}} 3 \cdot \sqrt{1 + y_x^2 + y_z^2} dx dz = \iint_{D_{xz}} 3 \cdot \sqrt{1 + 4x^2 + 4z^2} dx dz$$

$$\frac{x = \rho \cos \theta}{z = \rho \sin \theta} \iint_{D_{xz}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho d\theta = \int_{0}^{2\pi} \left[\int_{0}^{\sqrt{2}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho \right] d\theta$$



 $y \ge 1$ 的部分。 D_{XZ}

例 4 计算 $I = \int_{\Sigma} (x^2 + z^2 + y) dS$,其中 Σ 是曲面 $y = 3 - (x^2 + z^2)$ 在

$$\sum_{x} y = 3 - (x^{2} + z^{2})$$

$$y = 1$$

$$y =$$

 $I = \iint_{\Omega} 3 \cdot \sqrt{1 + y_x^2 + y_z^2} dx dz = \iint_{\Omega} 3 \cdot \sqrt{1 + 4x^2 + 4z^2} dx dz$

$$I = \iint_{D_{xz}} 3 \cdot \sqrt{1 + y_x^2 + y_z^2} dx dz = \iint_{D_{xz}} 3 \cdot \sqrt{1 + 4x^2 + 4z^2} dx dz$$

$$\frac{x = \rho \cos \theta}{z = \rho \sin \theta} \iint_{D_{xz}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho d\theta = \int_{0}^{2\pi} \left[\int_{0}^{\sqrt{2}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho \right] d\theta$$

 $y \ge 1$ 的部分。

例 4 计算 $I = \iint_{\Sigma} (x^2 + z^2 + y) dS$, 其中 Σ 是曲面 $y = 3 - (x^2 + z^2)$ 在

$$\sum_{x} y = 3 - (x^{2} + z^{2})$$

$$y = 1$$

$$y =$$

$$\mathbf{H}$$

$$I = \iint_{D_{xz}} 3 \cdot \sqrt{1 + y_x^2 + y_z^2} dx dz = \iint_{D_{xz}} 3 \cdot \sqrt{1 + 4x^2 + 4z^2} dx dz$$

$$\underline{x = \rho \cos \theta} \left[\int_{0}^{\infty} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho d\theta = \int_{0}^{2\pi} \left[\int_{0}^{\sqrt{2}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho \right] d\theta$$

 $\frac{x = \rho \cos \theta}{z = \rho \sin \theta} \iint_{D} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho d\theta = \int_{0}^{2\pi} \left[\int_{0}^{\sqrt{2}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho \right] d\theta$

 $y \ge 1$ 的部分。 D_{XZ}

$$\frac{1}{\sqrt{2}} = \iint_{D_{xz}} 3 \cdot \sqrt{1 + y_x^2 + y_z^2} dx dz = \iint_{D_{xz}} 3 \cdot \sqrt{1 + 4x^2 + 4z^2} dx dz$$

$$\mathbf{H}$$

$$I = \iint_{D_{xz}} 3 \cdot \sqrt{1 + y_x^2 + y_z^2} dx dz = \iint_{D_{xz}} 3 \cdot \sqrt{1 + 4x^2 + 4z^2} dx dz$$

$$\frac{x = \rho \cos \theta}{z = \rho \sin \theta} \iint_{D_{xz}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho d\theta = \int_0^{2\pi} \left[\int_0^{\sqrt{2}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho \right] d\theta$$

$$\frac{u=1+4\rho^2}{2\pi} 2\pi \cdot 3\sqrt{u} \cdot \frac{1}{8} du$$

 $y \ge 1$ 的部分。 D_{xz}

$$y = 1$$

$$y =$$

$$\mathbf{H}$$

$$I = \iint_{D_{xz}} 3 \cdot \sqrt{1 + y_x^2 + y_z^2} dx dz = \iint_{D_{xz}} 3 \cdot \sqrt{1 + 4x^2 + 4z^2} dx dz$$

$$\frac{x = \rho \cos \theta}{z = \rho \sin \theta} \iint_{D_{xz}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho d\theta = \int_0^{2\pi} \left[\int_0^{\sqrt{2}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho \right] d\theta$$

$$\frac{u=1+4\rho^2}{2\pi} 2\pi \cdot \int_1^9 3\sqrt{u} \cdot \frac{1}{8} du$$

 $y \ge 1$ 的部分。 D_{XZ}

$$y = 1$$

$$\int_{D_{xz}} 3 \cdot \sqrt{1 + y_x^2 + y_z^2} dx dz = \iint_{D_{xz}} 3 \cdot \sqrt{1 + 4x^2 + 4z^2} dx dz$$

$$\mathbf{H}$$

$$I = \iint_{D_{xz}} 3 \cdot \sqrt{1 + y_x^2 + y_z^2} dx dz = \iint_{D_{xz}} 3 \cdot \sqrt{1 + 4x^2 + 4z^2} dx dz$$

$$\frac{x = \rho \cos \theta}{z = \rho \sin \theta} \iint_{D_{xz}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho d\theta = \int_0^{2\pi} \left[\int_0^{\sqrt{2}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho \right] d\theta$$

$$\frac{x = \rho \cos \theta}{z = \rho \sin \theta} \iint_{D_{xz}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho d\theta = \int_0^{\infty} \left[\int_0^{\infty} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho \right] d\theta$$

$$\frac{u = 1 + 4\rho^2}{2\pi i} 2\pi \cdot \left[\int_0^{\infty} 3\sqrt{u} \cdot \frac{1}{8} du = \frac{1}{2}\pi u^{\frac{3}{2}} \right]_1^9$$

 $y \ge 1$ 的部分。

例 4 计算 $I = \int_{\Sigma} (x^2 + z^2 + y) dS$,其中 Σ 是曲面 $y = 3 - (x^2 + z^2)$ 在

$$y = 1$$

$$-\sqrt{2}$$

$$= \iint_{D_{XZ}} 3 \cdot \sqrt{1 + y_x^2 + y_z^2} dx dz = \iint_{D_{XZ}} 3 \cdot \sqrt{1 + 4x^2 + 4z^2} dx dz$$

$$\frac{x}{z} = \iint_{D_{xz}} 3 \cdot \sqrt{1 + y_x^2 + y_z^2} dx dz = \iint_{D_{xz}} 3 \cdot \sqrt{1 + 4x^2 + 4z^2} dx dz$$

$$\frac{x = \rho \cos \theta}{z = \rho \sin \theta} \iint_{D_{xz}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho d\theta = \int_0^{2\pi} \left[\int_0^{\sqrt{2}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho \right] d\theta$$

$$\frac{1}{z} = \iint_{D_{xz}} 3 \cdot \sqrt{1 + y_x^2 + y_z^2} dx dz = \iint_{D_{xz}} 3 \cdot \sqrt{1 + 4x^2 + 4z^2} dx dz$$

$$\frac{x = \rho \cos \theta}{z = \rho \sin \theta} \iint_{D_{xz}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho d\theta = \int_0^{2\pi} \left[\int_0^{\sqrt{2}} 3\sqrt{1 + 4\rho^2} \cdot \rho d\rho \right] d\theta$$

 $\frac{u=1+4\rho^2}{2\pi \cdot \left[\int_{-1}^{9} 3\sqrt{u} \cdot \frac{1}{8} du = \frac{1}{2} \pi u^{\frac{3}{2}} \right]_{1}^{9} = 13\pi$