# §3.4 向量组的秩

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 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 

逐个剔除  $lpha_1,lpha_2,\ldots,lpha_s$  能被其余向量线性表示的向量

逐个剔除  $\alpha_1, \alpha_2, \ldots, \alpha_s$  能被其余向量线性表示的向量  $\alpha_1, \alpha_2, \ldots, \alpha_s$ 

逐个剔除

 $lpha_1,lpha_2,\ldots,lpha_s$  能被其余向量线性表示的向量  $lpha_{j_1},lpha_{j_2},\ldots,lpha_{j_r}$ 

直到不能再剔除为止

逐个剔除

 $\alpha_1, \alpha_2, \ldots, \alpha_s \xrightarrow{\text{fixt} \text{ fixed by a local points}} \alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}$  (极大无关组)

逐个剔除

 $lpha_1,lpha_2,\ldots,lpha_s \xrightarrow{ ext{fk被其余向量线性表示的向量}} lpha_{j_1},lpha_{j_2},\ldots,lpha_{j_r}$  (极大无关组)

例 求 
$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
,  $\alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ 的一个极大无关组。

逐个剔除

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$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
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 $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ 



逐个剔除

 $lpha_1,lpha_2,\ldots,lpha_s \xrightarrow{ ilde{ text{tkd}}{ ilde{ text{tkd}}}{ ilde{ text{text}}{ ilde{ text{text}}}} lpha_{j_1},lpha_{j_2},\ldots,lpha_{j_r}$  (极大无关组)

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$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_4 = 2\alpha_1 + 0\alpha_2 + 0\alpha_3}$$



逐个剔除

$$\alpha_1, \alpha_2, \ldots, \alpha_s$$
能被其余向量线性表示的向量  $\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}$  (极大无关组) 直到不能再剔除为止

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$$\alpha_1, \alpha_2, \alpha_3, \alpha_4$$
 $\alpha_4 = 2\alpha_1 + 0\alpha_2 + 0\alpha_3$ 
剔除 $\alpha_4$ 



逐个剔除

 $lpha_1,lpha_2,\ldots,lpha_s \xrightarrow{ ilde{ text{tkat}} ilde{ text{tkat}} ilde{ text{tkat}} ilde{ text{tat}} ilde{ text{d}}} lpha_{j_1},lpha_{j_2},\ldots,lpha_{j_r}$  (极大无关组)

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$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_4 = 2\alpha_1 + 0\alpha_2 + 0\alpha_3} \alpha_1, \alpha_2, \alpha_3$$



逐个剔除

 $\alpha_1, \alpha_2, \ldots, \alpha_s \xrightarrow{\text{fiwtfsnold} \atop \text{直到不能再剔除为止}} \alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r} \quad (极大无关组)$ 

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$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_4 = 2\alpha_1 + 0\alpha_2 + 0\alpha_3}$$
 $\alpha_1, \alpha_2, \alpha_3 \xrightarrow{\alpha_3 = \alpha_1 + \alpha_2}$ 



逐个剔除

 $lpha_1,lpha_2,\ldots,lpha_s \xrightarrow{ ilde{ text{tkat}} ilde{ text{tkat}} ilde{ text{tkat}} ilde{ text{tat}} ilde{ text{d}}} lpha_{j_1},lpha_{j_2},\ldots,lpha_{j_r}$  (极大无关组)

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$$\stackrel{\textstyle R}{\alpha_1, \alpha_2, \alpha_3, \alpha_4} \xrightarrow{\alpha_4 = 2\alpha_1 + 0\alpha_2 + 0\alpha_3} 
\stackrel{\textstyle \alpha_1, \alpha_2, \alpha_3} \xrightarrow{\alpha_3 = \alpha_1 + \alpha_2} 
\xrightarrow{\textstyle \operatorname{slip}(\alpha_4)}$$



逐个剔除

 $\alpha_1, \alpha_2, \ldots, \alpha_s \xrightarrow{\text{fiwtfshold fixed fixed$ 

例 求 
$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
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 $lpha_1,lpha_2,\ldots,lpha_s \xrightarrow{ ilde{ text{tkat}} ilde{ text{tkat}} ilde{ text{tkat}} ilde{ text{tat}} ilde{ text{dist}} i$ 

例 求 
$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
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 $egin{array}{c} \mathbf{p} \\ \alpha_1, \alpha_2, \alpha_3, \alpha_4 \end{array} \xrightarrow[]{\alpha_4 = 2\alpha_1 + 0\alpha_2 + 0\alpha_3} \quad \alpha_1, \alpha_2, \alpha_3 \xrightarrow[]{\alpha_3 = \alpha_1 + \alpha_2} \quad \alpha_1, \alpha_2 \xrightarrow[]{\mathrm{RK}} \quad \alpha_1, \alpha_2 \xrightarrow[]{\mathrm{RK}} \quad \alpha_2, \alpha_3 \xrightarrow[]{\mathrm{RK}} \quad \alpha_3, \alpha_4 \xrightarrow[]{\mathrm{RK}} \quad \alpha_4, \alpha_4 \xrightarrow[]{\mathrm{R$ 



逐个剔除

 $\alpha_1, \alpha_2, \ldots, \alpha_s \xrightarrow{\text{fiwtJsholightersholighter}} \alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r} \quad (极大无关组)$ 

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$$\alpha_1, \alpha_2, \alpha_3, \alpha_4$$



逐个剔除

 $lpha_1,lpha_2,\ldots,lpha_s \xrightarrow{ ilde{ t k被其余向量线性表示的向量}} lpha_{j_1},lpha_{j_2},\ldots,lpha_{j_r}$  (极大无关组)

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$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_1 = -\alpha_2 + \alpha_3 + 0\alpha_4}$$



诼个剔除

 $\alpha_1, \alpha_2, \ldots, \alpha_s \xrightarrow{\text{fiwtfsnold} \atop \text{直到不能再剔除为止}} \alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r} \quad (极大无关组)$ 

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 $egin{array}{c} \mathbf{p} \\ \alpha_1, \alpha_2, \alpha_3, \alpha_4 \end{array} \xrightarrow[]{\alpha_4 = 2\alpha_1 + 0\alpha_2 + 0\alpha_3} \quad \alpha_1, \alpha_2, \alpha_3 \xrightarrow[]{\alpha_3 = \alpha_1 + \alpha_2} \quad \alpha_1, \alpha_2 \xrightarrow[]{\mathrm{RK}} \quad \alpha_1, \alpha_2 \xrightarrow[]{\mathrm{RK}} \quad \alpha_2, \alpha_3 \xrightarrow[]{\mathrm{RK}} \quad \alpha_3, \alpha_4 \xrightarrow[]{\mathrm{RK}} \quad \alpha_4, \alpha_4 \xrightarrow[]{\mathrm{R$ 

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_1 = -\alpha_2 + \alpha_3 + 0\alpha_4}$$
 $\beta \otimes \alpha_1$ 



逐个剔除

 $lpha_1,lpha_2,\ldots,lpha_s \xrightarrow{ ilde{ text{tkat}} ilde{ text{tkat}} ilde{ text{tkat}} ilde{ text{tat}} ilde{ text{d}}} lpha_{j_1},lpha_{j_2},\ldots,lpha_{j_r}$  (极大无关组)

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 $egin{array}{c} \mathbf{p} \\ \boldsymbol{\alpha}_1, \, \boldsymbol{\alpha}_2, \, \boldsymbol{\alpha}_3, \, \boldsymbol{\alpha}_4 & \xrightarrow[]{\alpha_4 = 2\alpha_1 + 0\alpha_2 + 0\alpha_3} & \boldsymbol{\alpha}_1, \, \boldsymbol{\alpha}_2, \, \boldsymbol{\alpha}_3 & \xrightarrow[]{\alpha_3 = \alpha_1 + \alpha_2} & \boldsymbol{\alpha}_1, \, \boldsymbol{\alpha}_2 & \xrightarrow[]{\mathrm{MK}} \\ & & & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & \\ & & & \\ & &$ 

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_1 = -\alpha_2 + \alpha_3 + 0\alpha_4} \alpha_2, \alpha_3, \alpha_4$$



诼个剔除

 $lpha_1,lpha_2,\ldots,lpha_s \xrightarrow{ ilde{ t k被其余向量线性表示的向量}} lpha_{j_1},lpha_{j_2},\ldots,lpha_{j_r}$  (极大无关组)

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$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_1 = -\alpha_2 + \alpha_3 + 0\alpha_4} \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2 = \alpha_3 - \frac{1}{2}\alpha_4}$$



诼个剔除

 $lpha_1,lpha_2,\ldots,lpha_s \xrightarrow{ ilde{ text{tkat}} ilde{ text{tkat}} ilde{ text{tkat}} ilde{ text{tat}} ilde{ text{dist}} i$ 

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 $\xrightarrow{\text{slip}(\alpha_1)}$ 



逐个剔除

 $lpha_1,lpha_2,\ldots,lpha_s \xrightarrow{ ilde{ text{tkat}} ilde{ text{tkat}} ilde{ text{tkat}} ilde{ text{tat}} ilde{ text{d}}} lpha_{j_1},lpha_{j_2},\ldots,lpha_{j_r}$  (极大无关组)

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 $egin{array}{c} egin{array}{c} \mu \\ \alpha_1, \alpha_2, \alpha_3, \alpha_4 \end{array} \xrightarrow{lpha_4 = 2lpha_1 + 0lpha_2 + 0lpha_3} & lpha_1, lpha_2, lpha_3 \end{array} \xrightarrow{lpha_3 = lpha_1 + lpha_2} & lpha_1, lpha_2 \xrightarrow{\mathrm{REA}} & lpha_1, lpha_2 \xrightarrow{\mathrm{REA}} & lpha_2, lpha_3 \end{array}$ 

 $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_1 = -\alpha_2 + \alpha_3 + 0\alpha_4} \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2 = \alpha_3 - \frac{1}{2}\alpha_4} \alpha_3, \alpha_4$ 

逐个剔除

 $lpha_1,lpha_2,\ldots,lpha_s \xrightarrow{ ilde{ t k m j s nol 2} ilde{ t k m j s nol 2}} ilde{ alpha_{j_1},lpha_{j_2},\ldots,lpha_{j_r}}$  (极大无关组)

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 $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_1 = -\alpha_2 + \alpha_3 + 0\alpha_4} \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2 = \alpha_3 - \frac{1}{2}\alpha_4} \alpha_3, \alpha_4 \xrightarrow{\text{RX}} \alpha_5$ 

$$\alpha_1, \alpha_2, \ldots, \alpha_s \xrightarrow{\text{fixt it it fixed fixed$$

例 求 
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 $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ 



诼个剔除

$$\alpha_1, \alpha_2, \ldots, \alpha_s \xrightarrow{\text{fiwtfsnolightersnonly}} \alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r} \quad (极大无关组)$$
直到不能再剔除为止

例 求 
$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
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$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\beta \otimes \alpha_1} \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\beta \otimes \alpha_2} \alpha_3, \alpha_4 \xrightarrow{\beta \otimes \alpha_2} \alpha_3, \alpha_4 \xrightarrow{\beta \otimes \alpha_2} \alpha_3, \alpha_4 \xrightarrow{\delta \otimes \alpha_2} \alpha_3, \alpha_5 \xrightarrow{\delta \otimes \alpha_2} \alpha_5$$

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2=0\alpha_1+\alpha_3-\frac{1}{2}\alpha_4}$$



诼个剔除

 $lpha_1,lpha_2,\ldots,lpha_s \xrightarrow{ ilde{ t k被其余向量线性表示的向量}} lpha_{j_1},lpha_{j_2},\ldots,lpha_{j_r}$  (极大无关组)

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$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_1 = -\alpha_2 + \alpha_3 + 0\alpha_4} \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2 = \alpha_3 - \frac{1}{2}\alpha_4} \alpha_3, \alpha_4 \xrightarrow{\text{MX}} \alpha_3, \alpha_4 \xrightarrow{\text{MX}} \alpha_3, \alpha_4 \xrightarrow{\text{MX}} \alpha_4$$

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2 = 0\alpha_1 + \alpha_3 - \frac{1}{2}\alpha_4}$$
 $\emptyset$ 



$$\alpha_1, \alpha_2, \ldots, \alpha_s \xrightarrow{\text{fiwtfsnol} \text{states}} \alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}$$
 (极大无关组)

例 求 
$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
,  $\alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ 的一个极大无关组。

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_1 = -\alpha_2 + \alpha_3 + 0\alpha_4} \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2 = \alpha_3 - \frac{1}{2}\alpha_4} \alpha_3, \alpha_4 \xrightarrow{\text{Mb}} \alpha_2$$
 $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_1 = -\alpha_2 + \alpha_3 + 0\alpha_4} \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\text{Mb}} \alpha_2$ 

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2 = 0 \alpha_1 + \alpha_3 - \frac{1}{2} \alpha_4} \alpha_1, \alpha_3, \alpha_4$$



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 $\alpha_1, \alpha_2, \ldots, \alpha_s \xrightarrow{\text{fiwtJsho} = \text{states}} \text{fiwtJsho} \alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r} \quad (极大无关组)$ 

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$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
,  $\alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ 的一个极大无关组。

 $egin{array}{c} egin{array}{c} \mu \\ \alpha_1, \alpha_2, \alpha_3, \alpha_4 \end{array} \xrightarrow{lpha_4 = 2lpha_1 + 0lpha_2 + 0lpha_3} \quad lpha_1, lpha_2, lpha_3 \end{array} \xrightarrow{lpha_3 = lpha_1 + lpha_2} \quad lpha_1, lpha_2 \xrightarrow{\mathrm{RE}} \quad lpha_1, lpha_2 \xrightarrow{\mathrm{RE}} \quad lpha_2 \xrightarrow{\mathrm{RE}} \quad lpha_3 \xrightarrow{\mathrm{RE}} \quad lpha_3 \xrightarrow{\mathrm{RE}} \quad lpha_4 \xrightarrow{\mathrm{RE}} \quad$ 

 $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_1 = -\alpha_2 + \alpha_3 + 0\alpha_4} \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2 = \alpha_3 - \frac{1}{2}\alpha_4} \alpha_3, \alpha_4 \xrightarrow{\text{MX}} \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\text{MX}} \alpha_3, \alpha_4 \xrightarrow{\text{MX}} \alpha_4$ 

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2 = 0\alpha_1 + \alpha_3 - \frac{1}{2}\alpha_4} \alpha_1, \alpha_3, \alpha_4 \xrightarrow{\alpha_4 = 2\alpha_1 + 0\alpha_3}$$



逐个剧除

 $\alpha_1, \alpha_2, \ldots, \alpha_s \xrightarrow{\text{fiwtJsho} = \text{states}} \text{fiwtJsho} \alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r} \quad (极大无关组)$ 

例 求 
$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
,  $\alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ 的一个极大无关组。

 $egin{array}{c} egin{array}{c} \mu \\ \alpha_1, \alpha_2, \alpha_3, \alpha_4 \end{array} \xrightarrow{lpha_4 = 2lpha_1 + 0lpha_2 + 0lpha_3} & lpha_1, lpha_2, lpha_3 \end{array} \xrightarrow{lpha_3 = lpha_1 + lpha_2} & lpha_1, lpha_2 \xrightarrow{\mathrm{REA}} & lpha_1, lpha_2 \xrightarrow{\mathrm{REA}} & lpha_2, lpha_3 \end{array}$ 

 $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_1 = -\alpha_2 + \alpha_3 + 0\alpha_4} \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2 = \alpha_3 - \frac{1}{2}\alpha_4} \alpha_3, \alpha_4 \xrightarrow{\text{RX}} \alpha_3, \alpha_4 \xrightarrow{\text{RX}} \alpha_5$ 



 $\alpha_1, \alpha_2, \ldots, \alpha_s \xrightarrow{\text{fiwtJsholightersholighter}} \alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r} \quad (极大无关组)$ 

例 求 
$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
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 $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_1 = -\alpha_2 + \alpha_3 + 0\alpha_4} \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2 = \alpha_3 - \frac{1}{2}\alpha_4} \alpha_3, \alpha_4 \xrightarrow{\text{MX}}$ 

 $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2 = 0\alpha_1 + \alpha_3 - \frac{1}{2}\alpha_4} \alpha_1, \alpha_3, \alpha_4 \xrightarrow{\alpha_4 = 2\alpha_1 + 0\alpha_3} \alpha_1, \alpha_3$ 



 $\alpha_1, \alpha_2, \ldots, \alpha_s \xrightarrow{\text{fixting fixed fix$ 

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 $egin{array}{c} \mathbf{p} \\ \alpha_1, \alpha_2, \alpha_3, \alpha_4 & \xrightarrow{\alpha_4 = 2\alpha_1 + 0\alpha_2 + 0\alpha_3} \\ \hline \betalike \alpha_4 & \alpha_1, \alpha_2, \alpha_3 & \xrightarrow{\alpha_3 = \alpha_1 + \alpha_2} \\ \hline \betalike \alpha_3 & \alpha_1, \alpha_2 & \cot \alpha_2 \\ \hline \betalike \alpha_3 & \alpha_2, \alpha_3 & \cot \alpha_3 \\ \hline \betalike \alpha_3 & \alpha_3, \alpha_4 & \cot \alpha_4 \\ \hline \betalike \alpha_4 & \cot \alpha_2, \alpha_3 & \cot \alpha_3 \\ \hline \betalike \alpha_4 & \cot \alpha_4, \alpha_4 & \cot \alpha_4 \\ \hline \betalike \alpha_4 & \cot \alpha_4, \alpha_5 & \cot \alpha_4 \\ \hline \betalike \alpha_4 & \cot \alpha_5, \alpha_5 & \cot \alpha_5, \alpha_5 \\ \hline \betalike \alpha_5 & \cot \alpha_5, \alpha_5 & \cot \alpha_5, \alpha_5 & \cot \alpha_5, \alpha_5 \\ \hline \betalike \alpha_5 & \cot \alpha_5, \alpha_5 & \cot \alpha_5, \alpha_5 & \cot \alpha_5, \alpha_5 \\ \hline \betalike \alpha_5 & \cot \alpha_5, \alpha_5 & \cot \alpha_5, \alpha_5 & \cot \alpha_5, \alpha_5 \\ \hline \betalike \alpha_5 & \cot \alpha_5, \alpha_5 & \cot \alpha_5, \alpha_5 & \cot \alpha_5, \alpha_5 & \cot \alpha_5, \alpha_5 \\ \hline \betalike \alpha_5 & \cot \alpha_5, \alpha_5 & \cot \alpha_5, \alpha_5 & \cot \alpha_5, \alpha_5 & \cot \alpha_5, \alpha_5 \\ \hline \betalike \alpha_5 & \cot \alpha_5, \alpha_5 \\ \hline \betalike \alpha_5 & \cot \alpha_5, \alpha_5 & \cot \alpha$ 

 $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_1 = -\alpha_2 + \alpha_3 + 0\alpha_4} \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2 = \alpha_3 - \frac{1}{2}\alpha_4} \alpha_3, \alpha_4 \xrightarrow{\text{MX}} \alpha_5$ 

 $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2=0\alpha_1+\alpha_3-\frac{1}{2}\alpha_4} \alpha_1, \alpha_3, \alpha_4 \xrightarrow{\alpha_4=2\alpha_1+0\alpha_3} \alpha_1, \alpha_3 \xrightarrow{\text{AX}} 0$ 

还有其他极大无关组吗?



 $\alpha_1, \alpha_2, \ldots, \alpha_s \xrightarrow{\text{fixitation}} \text{fixing fixed f$ 

例 求 
$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
,  $\alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ 的一个极大无关组。

 $egin{array}{c} \mathbf{p} \\ \alpha_1, \alpha_2, \alpha_3, \alpha_4 & \xrightarrow{\alpha_4 = 2\alpha_1 + 0\alpha_2 + 0\alpha_3} \\ \hline \betalike \alpha_4 & \alpha_1, \alpha_2, \alpha_3 & \xrightarrow{\alpha_3 = \alpha_1 + \alpha_2} \\ \hline \betalike \alpha_3 & \alpha_1, \alpha_2 & \cot \alpha_2 \\ \hline \betalike \alpha_3 & \alpha_2, \alpha_3 & \cot \alpha_3 \\ \hline \betalike \alpha_3 & \alpha_3, \alpha_4 & \cot \alpha_4 \\ \hline \betalike \alpha_4 & \cot \alpha_2, \alpha_3 & \cot \alpha_3 \\ \hline \betalike \alpha_4 & \cot \alpha_4, \alpha_4 & \cot \alpha_4 \\ \hline \betalike \alpha_4 & \cot \alpha_4, \alpha_5 & \cot \alpha_4 \\ \hline \betalike \alpha_4 & \cot \alpha_5, \alpha_5 & \cot \alpha_5, \alpha_5 \\ \hline \betalike \alpha_5 & \cot \alpha_5, \alpha_5 & \cot \alpha_5, \alpha_5 & \cot \alpha_5, \alpha_5 \\ \hline \betalike \alpha_5 & \cot \alpha_5, \alpha_5 & \cot \alpha_5, \alpha_5 & \cot \alpha_5, \alpha_5 \\ \hline \betalike \alpha_5 & \cot \alpha_5, \alpha_5 & \cot \alpha_5, \alpha_5 & \cot \alpha_5, \alpha_5 \\ \hline \betalike \alpha_5 & \cot \alpha_5, \alpha_5 & \cot \alpha_5, \alpha_5 & \cot \alpha_5, \alpha_5 & \cot \alpha_5, \alpha_5 \\ \hline \betalike \alpha_5 & \cot \alpha_5, \alpha_5 & \cot \alpha_5, \alpha_5 & \cot \alpha_5, \alpha_5 & \cot \alpha_5, \alpha_5 \\ \hline \betalike \alpha_5 & \cot \alpha_5, \alpha_5 \\ \hline \betalike \alpha_5 & \cot \alpha_5, \alpha_5 & \cot \alpha$ 

 $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_1 = -\alpha_2 + \alpha_3 + 0\alpha_4} \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2 = \alpha_3 - \frac{1}{2}\alpha_4} \alpha_3, \alpha_4 \xrightarrow{\text{MX}} \alpha_5$ 

 $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \xrightarrow{\alpha_2 = 0\alpha_1 + \alpha_3 - \frac{1}{2}\alpha_4} \alpha_1, \alpha_3, \alpha_4 \xrightarrow{\alpha_4 = 2\alpha_1 + 0\alpha_3} \alpha_1, \alpha_3 \xrightarrow{\text{M} \times 1} \alpha_2$ 

还有其他极大无关组吗?

定理  $\alpha_{j_1}$ ,  $\alpha_{j_2}$ , ...,  $\alpha_{j_r}$  是  $\alpha_1$ ,  $\alpha_2$ , ...,  $\alpha_s$  的极大无关组,当且仅当

- $\alpha_1$ ,  $\alpha_2$ ,  $\cdots$ ,  $\alpha_s$  中每个向量都可由  $\alpha_{j_1}$ ,  $\alpha_{j_2}$ ,  $\ldots$ ,  $\alpha_{j_r}$  线性表示
- α<sub>j1</sub>, α<sub>j2</sub>, . . . , α<sub>jr</sub> 线性无关

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定理 极大无关组所包含向量的个数是唯一确定的。



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定理 极大无关组所包含向量的个数是唯一确定的。即:若

$$\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}; \qquad \beta_{k_1}, \beta_{k_2}, \ldots, \beta_{k_t}$$

都是  $\alpha_1$ ,  $\alpha_2$ , ...,  $\alpha_s$  的极大无关组, 则

定理  $\alpha_{j_1}$ ,  $\alpha_{j_2}$ , ...,  $\alpha_{j_r}$  是  $\alpha_1$ ,  $\alpha_2$ , ...,  $\alpha_s$  的极大无关组,当且仅当

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都是  $\alpha_1$ ,  $\alpha_2$ , ...,  $\alpha_s$  的极大无关组,则 r = t。

# 极大无关组的性质

定理  $\alpha_{j_1}$ ,  $\alpha_{j_2}$ , ...,  $\alpha_{j_r}$  是  $\alpha_1$ ,  $\alpha_2$ , ...,  $\alpha_s$  的极大无关组,当且仅当

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定理 极大无关组所包含向量的个数是唯一确定的。即: 若

$$\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}; \qquad \beta_{k_1}, \beta_{k_2}, \ldots, \beta_{k_t}$$

都是  $\alpha_1$ ,  $\alpha_2$ , ...,  $\alpha_s$  的极大无关组,则 r = t。

例设
$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
,  $\alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ , 则极大无关组是:  $\alpha_1, \alpha_2; \quad \alpha_1, \alpha_3; \quad \alpha_2, \alpha_3; \quad \alpha_2, \alpha_4; \quad \alpha_3, \alpha_4$ 



# 极大无关组的性质

定理  $\alpha_{j_1}$ ,  $\alpha_{j_2}$ , ...,  $\alpha_{j_r}$  是  $\alpha_1$ ,  $\alpha_2$ , ...,  $\alpha_s$  的极大无关组,当且仅当

- $\alpha_1$ ,  $\alpha_2$ , ···· ,  $\alpha_s$  中每个向量都可由  $\alpha_{i_1}$ ,  $\alpha_{i_2}$ , ... ,  $\alpha_{i_r}$  线性表示
- α<sub>j1</sub>, α<sub>j2</sub>, ..., α<sub>jr</sub> 线性无关

定理 极大无关组所包含向量的个数是唯一确定的。即:若

$$\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}; \qquad \beta_{k_1}, \beta_{k_2}, \ldots, \beta_{k_t}$$

都是  $\alpha_1$ ,  $\alpha_2$ , ...,  $\alpha_s$  的极大无关组, 则 r = t。

例设 
$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
,  $\alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ , 则极大无关组是:

 $\alpha_1, \alpha_2; \quad \alpha_1, \alpha_3; \quad \alpha_2, \alpha_3; \quad \alpha_2, \alpha_4; \quad \alpha_3, \alpha_4$ 

可见,每个极大无关组都由2个向量构成。



定义 向量组  $\alpha_1$ ,  $\alpha_2$ , ...,  $\alpha_s$  的极大无关组所包含向量的个数,称向量组的秩,记为:

$$r(\alpha_1, \alpha_2, \ldots, \alpha_s)$$

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例设
$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$

• r(A) = :

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$$r(A) = :$$

$$A = \left(\begin{array}{ccc} 1 & 3 & 5 \\ 2 & 4 & 6 \end{array}\right)$$

例设
$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$

$$r(A) = :$$

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} \xrightarrow{r_2 - 2r_1}$$



例设
$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$

• r(A) = :

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 3 & 5 \\ 0 & -2 & -4 \end{pmatrix}$$



例设
$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$

• r(A) = 2:

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 3 & 5 \\ 0 & -2 & -4 \end{pmatrix}$$



例设
$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$

 $\alpha_2$   $\alpha_3$ 

• r(A) = 2:

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 3 & 5 \\ 0 & -2 & -4 \end{pmatrix}$$



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•  $r(\alpha_1, \alpha_2, \alpha_3) = :$ 

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• 
$$r(\alpha_1, \alpha_2, \alpha_3) = :$$
  
 $\alpha_1, \alpha_2, \alpha_3$ 

 $\alpha_1, \alpha_2, \alpha_3$ 

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 $r(\alpha_1, \alpha_2, \alpha_3) = :$ 

$$\alpha_1, \alpha_2, \alpha_3 \xrightarrow{\alpha_3 = -\alpha_1 + 2\alpha_2}$$



例设
$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$

 $\alpha_2$   $\alpha_3$ 

• r(A) = 2:

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 3 & 5 \\ 0 & -2 & -4 \end{pmatrix}$$

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• r(A) = 2:

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•  $r(\alpha_1, \alpha_2, \alpha_3) = :$ 

$$\alpha_1, \alpha_2, \alpha_3 \xrightarrow{\alpha_3 = -\alpha_1 + 2\alpha_2} \alpha_1, \alpha_2$$

$$\emptyset \otimes \alpha_3$$



例设
$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$

• r(A) = 2:

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 3 & 5 \\ 0 & -2 & -4 \end{pmatrix}$$

 $r(\alpha_1, \alpha_2, \alpha_3) = :$ 

$$\alpha_1, \alpha_2, \alpha_3 \xrightarrow{\alpha_3 = -\alpha_1 + 2\alpha_2} \alpha_1, \alpha_2 \xrightarrow{\alpha_1, \alpha_2}$$
   
剔除 $\alpha_3$ 



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 $r(\alpha_1, \alpha_2, \alpha_3) = :$ 

$$\alpha_1, \alpha_2, \alpha_3 \xrightarrow{\alpha_3 = -\alpha_1 + 2\alpha_2} \alpha_1, \alpha_2 \xrightarrow{\alpha_1, \alpha_2 \text{线性无关}} \alpha_1, \alpha_2$$
为极大无关组



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•  $r(\alpha_1, \alpha_2, \alpha_3) = 2$ :

$$\alpha_1, \alpha_2, \alpha_3 \xrightarrow{\alpha_3 = -\alpha_1 + 2\alpha_2} \alpha_1, \alpha_2 \xrightarrow{\alpha_1, \alpha_2 \text{线性无关}} \alpha_1, \alpha_2$$
为极大无关组



例设
$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$
 $\beta_1$  $\beta_2$ 

• r(A) = 2:

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 3 & 5 \\ 0 & -2 & -4 \end{pmatrix}$$

•  $r(\alpha_1, \alpha_2, \alpha_3) = 2$ :

$$\alpha_1, \alpha_2, \alpha_3 \xrightarrow{\alpha_3 = -\alpha_1 + 2\alpha_2} \alpha_1, \alpha_2 \xrightarrow{\alpha_1, \alpha_2 \text{线性无关}} \alpha_1, \alpha_2$$
为极大无关组

 $r(\beta_1,\beta_2) = :$ 

例设
$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$
 $\beta_1$  $\beta_2$ 

• r(A) = 2:

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 3 & 5 \\ 0 & -2 & -4 \end{pmatrix}$$

•  $r(\alpha_1, \alpha_2, \alpha_3) = 2$ :

$$\alpha_1, \alpha_2, \alpha_3 \xrightarrow{\alpha_3 = -\alpha_1 + 2\alpha_2} \alpha_1, \alpha_2 \xrightarrow{\alpha_1, \alpha_2} \alpha_1, \alpha_2$$
为极大无关组

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$$\beta_1, \beta_2$$

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$$\beta_1, \beta_2 \xrightarrow{\beta_1, \beta_2} \text{ det } \Xi$$

例设
$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}_{\beta_2}^{\beta_1}$$

 $\alpha_2$   $\alpha_3$ 

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$$eta_1,eta_2 \stackrel{eta_1,eta_2$$
线性 $\mathbb{R}^{3}}{\longrightarrow} eta_1,eta_2$ 为极大无关组

例设
$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}_{\beta_2}^{\beta_1}$$

 $\alpha_2$   $\alpha_3$ 

• r(A) = 2:

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例设
$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$
 $\beta_2$ 

• r(A) = 2:

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可见,以上上个秩均相等,即  $r(A) = r(\alpha_1, \alpha_2, \alpha_3) = r(\beta_1, \beta_2)$ 。



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可见,以上上个秩均相等,即  $r(A) = r(\alpha_1, \alpha_2, \alpha_3) = r(\beta_1, \beta_2)$ 。

这不是巧合,而是恒成立!

$$A_{m \times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

$$A_{m \times n} = \begin{pmatrix} \alpha_{1} & \alpha_{2} & \alpha_{n} \\ \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix} \begin{pmatrix} \beta_{1} \\ \beta_{2} \\ \vdots \\ \beta_{m} \end{pmatrix}$$

# 秩

$$A_{m \times n} = \begin{pmatrix} \alpha_{1} & \alpha_{2} & \alpha_{n} \\ \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix} \begin{pmatrix} \beta_{1} \\ \beta_{2} \\ \vdots \\ \beta_{m} \end{pmatrix}$$

### 定义

- r(α<sub>1</sub>, α<sub>2</sub>,..., α<sub>n</sub>) 称为 A 的列秩;
- r(β<sub>1</sub>, β<sub>2</sub>,...,β<sub>m</sub>) 称为 A 的行秩;

# 秩

$$A_{m \times n} = \begin{pmatrix} \alpha_{1} & \alpha_{2} & \alpha_{n} \\ \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix}_{\beta_{m}}^{\beta_{1}}$$

### 定义

- $r(\alpha_1, \alpha_2, \ldots, \alpha_n)$  称为 A 的列秩;
- r(β<sub>1</sub>, β<sub>2</sub>, ..., β<sub>m</sub>) 称为 A 的行秩;

定理 
$$r(A) = r(\alpha_1, \alpha_2, \ldots, \alpha_n) = r(\beta_1, \beta_2, \ldots, \beta_m)$$



# 秩

$$A_{m \times n} = \begin{pmatrix} \alpha_{1} & \alpha_{2} & \alpha_{n} \\ \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix} \begin{pmatrix} \beta_{1} \\ \beta_{2} \\ \vdots \\ \beta_{m} \end{pmatrix}$$

#### 定义

- r(α<sub>1</sub>, α<sub>2</sub>, ..., α<sub>n</sub>) 称为 A 的列秩;
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$$r(A) = r(\alpha_1, \alpha_2, \ldots, \alpha_n) = r(\beta_1, \beta_2, \ldots, \beta_m)$$

应用计算向量组的秩可转化为计算矩阵的秩。

问题 给出 m 维的向量组  $\alpha_1$ ,  $\alpha_2$ ,  $\cdots$ ,  $\alpha_n$ , 如何求出其一组极大无关组?

步骤

问题 给出 m 维的向量组  $\alpha_1$ ,  $\alpha_2$ ,  $\cdots$ ,  $\alpha_n$ , 如何求出其一组极大无关组?

步骤
$$1. A_{m \times n} = \begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_n \\
\alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\
\alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn}
\end{pmatrix}$$

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简化的阶梯型矩阵

问题 给出 m 维的向量组  $lpha_1$  ,  $lpha_2$  ,  $\cdots$  ,  $lpha_n$  , 如何求出其一组极大无关组 ?

步骤
$$1. \ \, A_{m \times n} = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_n \\ \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix} \xrightarrow{\eta \not \approx f - g \not p}$$
简化的阶梯型矩阵

2. 通过简化的阶梯型矩阵, 求出 r(A)。

## 初等变换求极大无关组

问题 给出 m 维的向量组  $lpha_1$  ,  $lpha_2$  ,  $\cdots$  ,  $lpha_n$  , 如何求出其一组极大无关组 ?

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简化的阶梯型矩阵

2. 通过简化的阶梯型矩阵, 求出 *r(A)*。

利用  $r(\alpha_1, \alpha_2, ..., \alpha_n) = r(A)$ ,得出极大无关组所包含向量的个数



## 初等变换求极大无关组

问题 给出 m 维的向量组  $lpha_1$  ,  $lpha_2$  ,  $\cdots$  ,  $lpha_n$  , 如何求出其一组极大无关组 ?

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简化的阶梯型矩阵

- 2. 通过简化的阶梯型矩阵,求出 r(A)。 利用  $r(\alpha_1,\alpha_2,\ldots,\alpha_n)=r(A)$ ,得出极大无关组所包含向量的个数
- 3. 通过简化的阶梯型矩阵,容易看出线性无关的 r(A) 列,这就找到一组极大无关组



## 初等变换求极大无关组

问题 给出 m 维的向量组  $lpha_1$  ,  $lpha_2$  ,  $\cdots$  ,  $lpha_n$  , 如何求出其一组极大无关组 ?

步骤
$$1. \ A_{m \times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \xrightarrow{\eta \circledast \uparrow - \varphi \psi}$$
简化的阶梯型矩阵

- 2. 通过简化的阶梯型矩阵,求出 r(A)。 利用  $r(\alpha_1, \alpha_2, \ldots, \alpha_n) = r(A)$ ,得出极大无关组所包含向量的个数
- 3. 通过简化的阶梯型矩阵,容易看出线性无关的 r(A) 列,这就找到一组极大无关组
- 4. 通过简化的阶梯型矩阵,容易看出其余列如何用该选定极大无关组 线性表示

例 1 求向量组 
$$\alpha_1 = \begin{pmatrix} 2\\4\\2 \end{pmatrix}$$
,  $\alpha_2 = \begin{pmatrix} 1\\1\\0 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 2\\3\\1 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 3\\5\\2 \end{pmatrix}$ 的一个极

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$$\begin{pmatrix}
2 & 1 & 2 & 3 \\
4 & 1 & 3 & 5 \\
2 & 0 & 1 & 2
\end{pmatrix}
\xrightarrow[r_3-r_1]{r_2-2r_1}$$

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2 & 1 & 2 & 3 \\
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\xrightarrow[r_3-r_1]{r_2-2r_1}
\begin{pmatrix}
2 & 1 & 2 & 3 \\
0 & -1 & -1 & -1 \\
0 & -1 & -1 & -1
\end{pmatrix}
\longrightarrow$$

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\end{pmatrix}$$

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2 & 1 & 2 & 3 \\
4 & 1 & 3 & 5 \\
2 & 0 & 1 & 2
\end{pmatrix}
\xrightarrow[r_3-r_1]{r_2-2r_1}
\begin{pmatrix}
2 & 1 & 2 & 3 \\
0 & -1 & -1 & -1 \\
0 & -1 & -1 & -1
\end{pmatrix}
\longrightarrow
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2 & 1 & 2 & 3 \\
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\end{pmatrix}$$

$$r_1-r_2$$



例 1 求向量组 
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$$\begin{pmatrix}
2 & 1 & 2 & 3 \\
4 & 1 & 3 & 5 \\
2 & 0 & 1 & 2
\end{pmatrix}
\xrightarrow{r_2-2r_1}
\begin{pmatrix}
2 & 1 & 2 & 3 \\
0 & -1 & -1 & -1 \\
0 & -1 & -1 & -1
\end{pmatrix}
\longrightarrow
\begin{pmatrix}
2 & 1 & 2 & 3 \\
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

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2 & 1 & 2 & 3 \\
0 & -1 & -1 & -1 \\
0 & -1 & -1 & -1
\end{pmatrix}
\longrightarrow
\begin{pmatrix}
2 & 1 & 2 & 3 \\
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

$$\xrightarrow{r_1-r_2}
\begin{pmatrix}
2 & 0 & 1 & 2 \\
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\xrightarrow{\frac{1}{2}\times r_1}$$



例 1 求向量组 
$$\alpha_1 = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$$
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大元夫组;并把其余问重用域被人元夫组线性表示。
$$\begin{pmatrix} 2 & 1 & 2 & 3 \\ 4 & 1 & 3 & 5 \\ 2 & 0 & 1 & 2 \end{pmatrix} \xrightarrow{r_2-2r_1} \begin{pmatrix} 2 & 1 & 2 & 3 \\ 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{r_1-r_2} \begin{pmatrix} 2 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\frac{1}{2} \times r_1} \begin{pmatrix} 1 & 0 & \frac{1}{2} & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$



例 1 求向量组 
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解 
$$\alpha_1$$
  $\alpha_2$   $\alpha_3$   $\alpha_4$  
$$\begin{pmatrix} 2 & 1 & 2 & 3 \\ 4 & 1 & 3 & 5 \\ 2 & 0 & 1 & 2 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 2 & 1 & 2 & 3 \\ 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$\xrightarrow{r_1 - r_2} \begin{pmatrix} 2 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\frac{1}{2} \times r_1} \begin{pmatrix} 1 & 0 & \frac{1}{2} & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$



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\xrightarrow{r_2-2r_1}
\begin{pmatrix}
2 & 1 & 2 & 3 \\
0 & -1 & -1 & -1 \\
0 & -1 & -1 & -1
\end{pmatrix}
\longrightarrow
\begin{pmatrix}
2 & 1 & 2 & 3 \\
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

$$\xrightarrow{r_1-r_2} \left(\begin{array}{cccc} 2 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\frac{1}{2} \times r_1} \left(\begin{array}{cccc} 1 & 0 & \frac{1}{2} & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

所以

• 
$$r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2$$
;

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\end{pmatrix}$$

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所以

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$$r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2$$
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**整局大学** 

例 1 求向量组 
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,  $\alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$ 的一个极大无关组;并把其余向量用该极大无关组线性表示。

**ΔΤΙ** α- α α- α

$$\begin{pmatrix} 2 & 1 & 2 & 3 \\ 4 & 1 & 3 & 5 \\ 2 & 0 & 1 & 2 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 2 & 1 & 2 & 3 \\ 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{r_1-r_2} \left(\begin{array}{cccc} 2 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\frac{1}{2} \times r_1} \left(\begin{array}{cccc} 1 & 0 & \frac{1}{2} & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

所以

- $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2$ ;
- α<sub>1</sub>, α<sub>2</sub> 是极大无关组;

例 1 求向量组 
$$\alpha_1 = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$$
,  $\alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$ 的一个极大无关组;并把其余向量用该极大无关组线性表示。

人尤大组; 开龙共乐问里用

 $\begin{pmatrix}
2 & 1 & 2 & 3 \\
4 & 1 & 3 & 5 \\
2 & 0 & 1 & 2
\end{pmatrix}
\xrightarrow[r_3-r_1]{r_2-2r_1}
\begin{pmatrix}
2 & 1 & 2 & 3 \\
0 & -1 & -1 & -1 \\
0 & -1 & -1 & -1
\end{pmatrix}
\longrightarrow
\begin{pmatrix}
2 & 1 & 2 & 3 \\
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}$ 

$$\xrightarrow{r_1-r_2} \left(\begin{array}{cccc} 2 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\frac{1}{2} \times r_1} \left(\begin{array}{cccc} 1 & 0 & \frac{1}{2} & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

所以

- $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2$ ;
- α<sub>1</sub>, α<sub>2</sub> 是极大无关组;
- $\alpha_3 = \frac{1}{2}\alpha_1 + \alpha_2$ ,  $\alpha_4 = \alpha_1 + \alpha_2$

例 2 求向量组 
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$
,  $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一

例 2 求向量组 
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$
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$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 0 & 2 & -2 & 3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 - r_1}$$

例 2 求向量组 
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$
,  $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一

$$\begin{pmatrix}
1 & 0 & 1 & 2 \\
2 & 1 & 1 & 4 \\
1 & 1 & 0 & 3 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow[r_3-r_1]{r_2-2r_1}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 1 & -1 & 1 \\
0 & 2 & -2 & 3
\end{pmatrix}$$

例 2 求向量组 
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$
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$$\begin{pmatrix}
1 & 0 & 1 & 2 \\
2 & 1 & 1 & 4 \\
1 & 1 & 0 & 3 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow[r_3-r_1]{r_2-2r_1}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 1 & -1 & 1 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow[r_4-2r_2]{r_3-r_2}$$

例 2 求向量组 
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$
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1 & 0 & 1 & 2 \\
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1 & 1 & 0 & 3 \\
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\end{pmatrix}
\xrightarrow[r_3-r_1]{r_2-2r_1}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 1 & -1 & 1 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow[r_4-2r_2]{r_3-r_2}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
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\end{pmatrix}$$



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$$\begin{pmatrix}
1 & 0 & 1 & 2 \\
2 & 1 & 1 & 4 \\
1 & 1 & 0 & 3 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow[r_3-r_1]{r_2-2r_1}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 1 & -1 & 1 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow[r_4-2r_2]{r_3-r_2}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 3
\end{pmatrix}$$

$$\xrightarrow{-3r_3}$$



例 2 求向量组 
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$
,  $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一

$$\begin{pmatrix}
1 & 0 & 1 & 2 \\
2 & 1 & 1 & 4 \\
1 & 1 & 0 & 3 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow[r_3-r_1]{r_2-2r_1}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 1 & -1 & 1 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow[r_4-2r_2]{r_3-r_2}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 3
\end{pmatrix}$$

$$\frac{r_4 - 3r_3}{r_1 - 2r_3} \left( \begin{array}{cccc}
1 & 0 & 1 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{array} \right)$$

例 2 求向量组 
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$
,  $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 3 \end{pmatrix}$ 的一

$$\begin{pmatrix}
1 & 0 & 1 & 2 \\
2 & 1 & 1 & 4 \\
1 & 1 & 0 & 3 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow[r_3-r_1]{r_2-2r_1}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 1 & -1 & 1 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow[r_4-2r_2]{r_3-r_2}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 3
\end{pmatrix}$$

$$\frac{r_{4}-3r_{3}}{r_{1}-2r_{3}} \begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

例 2 求向量组 
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$$\begin{pmatrix}
1 & 0 & 1 & 2 \\
2 & 1 & 1 & 4 \\
1 & 1 & 0 & 3 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow[r_{3}-r_{1}]{r_{2}-2r_{1}}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 1 & -1 & 1 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow[r_{4}-3r_{3}]{r_{1}-2r_{3}}
\begin{pmatrix}
1 & 0 & 1 & 0 \\
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0 & 0 & 0 & 1
\end{pmatrix}$$



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$$\begin{pmatrix}
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\end{pmatrix}
\xrightarrow[r_3-r_1]{r_2-2r_1}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 1 & -1 & 1 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow[r_4-2r_2]{r_3-r_2}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 3
\end{pmatrix}$$

$$\frac{r_4 - 3r_3}{r_1 - 2r_3} \begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

所以
• 
$$r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 3$$
;

例 2 求向量组 
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$$\begin{pmatrix}
1 & 0 & 1 & 2 \\
2 & 1 & 1 & 4 \\
1 & 1 & 0 & 3 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow[r_3-r_1]{r_2-2r_1}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 1 & -1 & 1 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow[r_4-2r_2]{r_3-r_2}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 3
\end{pmatrix}$$

$$\frac{r_4 - 3r_3}{r_1 - 2r_3} \begin{pmatrix}
1 & 0 & 1 & 0 \\
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\end{pmatrix}$$

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所以
• 
$$r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 3$$
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α<sub>1</sub>, α<sub>2</sub>, α<sub>4</sub> 是极大无关组:

例 2 求向量组 
$$\alpha_1 = \begin{pmatrix} 1\\2\\1\\0 \end{pmatrix}$$
,  $\alpha_2 = \begin{pmatrix} 0\\1\\1\\2 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 1\\1\\0\\-2 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 2\\4\\3\\3 \end{pmatrix}$ 的一个极大无关组;并把其余向量用该极大无关组线性表示。

$$\begin{pmatrix}
1 & 0 & 1 & 2 \\
2 & 1 & 1 & 4 \\
1 & 1 & 0 & 3 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow[r_{3}-r_{1}]{r_{2}-2r_{1}}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 1 & -1 & 1 \\
0 & 2 & -2 & 3
\end{pmatrix}
\xrightarrow[r_{4}-2r_{3}]{r_{3}-r_{2}}
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}$$

$$\xrightarrow[r_{4}-3r_{3}]{r_{1}-2r_{3}}
\begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

所以
• 
$$r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 3$$
:

$$\alpha_3 = \alpha_1 - \alpha_2$$



例 3 求向量组 
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$
,  $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个

例 3 求向量组 
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$
,  $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个

例 3 求向量组 
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$
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$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \xrightarrow[r_4-4r_1]{r_2-2r_1} \xrightarrow[r_4-4r_1]{r_4-4r_1}$$

例 3 求向量组 
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$
,  $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个

$$\begin{pmatrix}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 5 \\
3 & 4 & 5 & 6 \\
4 & 5 & 6 & 7
\end{pmatrix}
\xrightarrow[r_4-4r_1]{r_2-2r_1}
\begin{pmatrix}
1 & 2 & 3 & 4 \\
0 & -1 & -2 & -3 \\
0 & -2 & -4 & -6 \\
0 & -3 & -6 & -9
\end{pmatrix}$$

例 3 求向量组 
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$
,  $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个

$$\begin{pmatrix}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 5 \\
3 & 4 & 5 & 6 \\
4 & 5 & 6 & 7
\end{pmatrix}
\xrightarrow[r_4-4r_1]{r_2-2r_1}
\begin{pmatrix}
1 & 2 & 3 & 4 \\
0 & -1 & -2 & -3 \\
0 & -2 & -4 & -6 \\
0 & -3 & -6 & -9
\end{pmatrix}$$

$$\frac{r_3-2r_2}{r_4-3r_2}$$

例 3 求向量组 
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$$\begin{pmatrix}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 5 \\
3 & 4 & 5 & 6 \\
4 & 5 & 6 & 7
\end{pmatrix}
\xrightarrow[r_{4}-4r_{1}]{r_{2}-2r_{1}}
\begin{pmatrix}
1 & 2 & 3 & 4 \\
0 & -1 & -2 & -3 \\
0 & -2 & -4 & -6 \\
0 & -3 & -6 & -9
\end{pmatrix}$$

$$\xrightarrow[r_{4}-3r_{2}]{r_{4}-3r_{2}}
\begin{pmatrix}
1 & 2 & 3 & 4 \\
0 & -1 & -2 & -3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

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$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$
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$$\begin{pmatrix}
1 & 2 & 3 & 4 \\
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4 & 5 & 6 & 7
\end{pmatrix}
\xrightarrow[r_4-3r_2]{r_2-2r_1}
\begin{pmatrix}
1 & 2 & 3 & 4 \\
0 & -1 & -2 & -3 \\
0 & -2 & -4 & -6 \\
0 & -3 & -6 & -9
\end{pmatrix}$$

$$\xrightarrow[r_4-3r_2]{r_4-3r_2}
\begin{pmatrix}
1 & 2 & 3 & 4 \\
0 & -1 & -2 & -3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\longrightarrow
\begin{pmatrix}
1 & 0 & -1 & -2 \\
0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}$$



例 3 求向量组 
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$
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4 & 5 & 6 & 7
\end{pmatrix}
\xrightarrow[r_{4}-4r_{1}]{r_{2}-2r_{1}}
\begin{pmatrix}
1 & 2 & 3 & 4 \\
0 & -1 & -2 & -3 \\
0 & -2 & -4 & -6 \\
0 & -3 & -6 & -9
\end{pmatrix}$$

$$\xrightarrow[r_{4}-3r_{2}]{r_{4}-3r_{2}}
\begin{pmatrix}
1 & 2 & 3 & 4 \\
0 & -1 & -2 & -3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\longrightarrow
\begin{pmatrix}
1 & 0 & -1 & -2 \\
0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}$$



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$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$
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$$\begin{pmatrix}
1 & 2 & 3 & 4 \\
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4 & 5 & 6 & 7
\end{pmatrix}
\xrightarrow[r_4-3r_2]{r_2-2r_1}
\begin{pmatrix}
1 & 2 & 3 & 4 \\
0 & -1 & -2 & -3 \\
0 & -2 & -4 & -6 \\
0 & -3 & -6 & -9
\end{pmatrix}$$

$$\xrightarrow[r_4-3r_2]{r_2-2r_2}
\begin{pmatrix}
1 & 2 & 3 & 4 \\
0 & -1 & -2 & -3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\longrightarrow
\begin{pmatrix}
1 & 0 & -1 & -2 \\
0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

所以
• 
$$r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2$$
;



例 3 求向量组 
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$
,  $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \xrightarrow[r_4-4r_1]{r_2-2r_1} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{pmatrix}$$

$$\xrightarrow[r_4-3r_2]{r_4-3r_2} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

所以
•  $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2$ ;

例 3 求向量组 
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$
,  $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \xrightarrow[r_4-4r_1]{r_3-3r_1} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{pmatrix}$$
$$\xrightarrow{r_3-2r_2} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2;$$

α<sub>1</sub>, α<sub>2</sub> 是极大无关组;

例 3 求向量组 
$$\alpha_1 = \begin{pmatrix} 1\\2\\3\\4 \end{pmatrix}$$
,  $\alpha_2 = \begin{pmatrix} 2\\3\\4\\5 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 3\\4\\5\\6 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 4\\5\\6\\7 \end{pmatrix}$ 的一个极大无关组;并把其余向量用该极大无关组线性表示。

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \xrightarrow[r_{3}-3r_{1}]{r_{3}-3r_{1}} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{pmatrix}$$

$$\xrightarrow[r_{3}-2r_{2}]{r_{4}-3r_{2}} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

• 
$$r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2;$$

$$\alpha_3 = -\alpha_1 + 2\alpha_2$$

所以

例 3 求向量组 
$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$
,  $\alpha_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$ ,  $\alpha_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$ ,  $\alpha_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$ 的一个极大无关组;并把其余向量用该极大无关组线性表示。

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

• 
$$r(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = r(A) = 2;$$

• 
$$\alpha_3 = -\alpha_1 + 2\alpha_2$$
,  $\alpha_4 = -2\alpha_1 + 3\alpha_2$ 



所以

证明设

$$r_1 = r(\alpha_1, \alpha_2, \dots, \alpha_s),$$
  

$$r_2 = r(\beta_1, \beta_2, \dots, \beta_t),$$

证明 设

$$r_1 = r(\alpha_1, \alpha_2, \ldots, \alpha_s), \quad \alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_{r_1}}$$
 是极大无关组  $r_2 = r(\beta_1, \beta_2, \ldots, \beta_t),$ 

例 假设向量组  $\alpha_1, \alpha_2, \ldots, \alpha_s$  可由  $\beta_1, \beta_2, \ldots, \beta_t$  线性表示,则  $r(\alpha_1, \alpha_2, \ldots, \alpha_s) \leq r(\beta_1, \beta_2, \ldots, \beta_t).$ 

## 证明 设

$$r_1 = r(\alpha_1, \alpha_2, ..., \alpha_s), \quad \alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_{r_1}}$$
 是极大无关组  $r_2 = r(\beta_1, \beta_2, ..., \beta_t), \quad \beta_{j_1}, \beta_{j_2}, ..., \beta_{j_{r_2}}$  是极大无关组

例 假设向量组  $\alpha_1, \alpha_2, \ldots, \alpha_s$  可由  $\beta_1, \beta_2, \ldots, \beta_t$  线性表示,则  $r(\alpha_1, \alpha_2, \ldots, \alpha_s) \leq r(\beta_1, \beta_2, \ldots, \beta_t).$ 

## 证明 设

$$r_1 = r(\alpha_1, \alpha_2, \ldots, \alpha_s), \quad \alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_{r_1}}$$
 是极大无关组  $r_2 = r(\beta_1, \beta_2, \ldots, \beta_t), \quad \beta_{j_1}, \beta_{j_2}, \ldots, \beta_{j_{r_2}}$  是极大无关组 注意到  $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_{r_1}}$  能由  $\beta_{j_1}, \beta_{j_2}, \ldots, \beta_{j_{r_2}}$  线性表示,

## 证明设

$$r_1 = r(\alpha_1, \alpha_2, ..., \alpha_s)$$
,  $\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_{r_1}}$  是极大无关组  $r_2 = r(\beta_1, \beta_2, ..., \beta_t)$ ,  $\beta_{j_1}, \beta_{j_2}, ..., \beta_{j_{r_2}}$  是极大无关组 注意到  $\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_{r_1}}$  能由  $\beta_{j_1}, \beta_{j_2}, ..., \beta_{j_{r_2}}$  线性表示,所以  $r_1 \leq r_2$ 。

证明设

$$r_1 = r(\alpha_1, \alpha_2, ..., \alpha_s), \quad \alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_{r_1}}$$
 是极大无关组  $r_2 = r(\beta_1, \beta_2, ..., \beta_t), \quad \beta_{j_1}, \beta_{j_2}, ..., \beta_{j_{r_2}}$  是极大无关组 注意到  $\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_{r_1}}$  能由  $\beta_{j_1}, \beta_{j_2}, ..., \beta_{j_{r_2}}$  线性表示,所以  $r_1 \leq r_2$  。

定理 设有向量组 (A):  $\alpha_1, \alpha_2, \ldots, \alpha_s$  (B):  $\beta_1, \beta_2, \ldots, \beta_t$ 

若它们等价.

证明设

$$r_1 = r(\alpha_1, \alpha_2, ..., \alpha_s)$$
,  $\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_{r_1}}$  是极大无关组  $r_2 = r(\beta_1, \beta_2, ..., \beta_t)$ ,  $\beta_{j_1}, \beta_{j_2}, ..., \beta_{j_{r_2}}$  是极大无关组 注意到  $\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_{r_1}}$  能由  $\beta_{j_1}, \beta_{j_2}, ..., \beta_{j_{r_2}}$  线性表示,所以  $r_1 \leq r_2$ 。

定理 设有向量组 
$$(A)$$
:  $\alpha_1, \alpha_2, \ldots, \alpha_s$   $(B)$ :  $\beta_1, \beta_2, \ldots, \beta_t$ 

若它们等价,则  $r(\alpha_1, \alpha_2, \ldots, \alpha_s) = r(\beta_1, \beta_2, \ldots, \beta_t)$ 。



证明 设 AB = Cmxs

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}$$

证明 设
$$AB = C_{m \times s}$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}$$

证明 设 
$$AB = C_{m \times s}$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}$$

证明 设 
$$AB = C_{m \times s}$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}$$

证明 设
$$AB = C_{m \times s}$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} \alpha_{11} & \alpha_{2} & \alpha_{n} \\ \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}$$

证明 设
$$AB = C_{m \times s}$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}$$

$$(\gamma_1 \ \gamma_2 \cdots \ \gamma_s) = (\alpha_1 \ \alpha_2 \cdots \ \alpha_n) \begin{pmatrix} b_{11} \ b_{12} \cdots b_{1s} \\ b_{21} \ b_{22} \cdots b_{2s} \\ \vdots \ \vdots \ \ddots \ \vdots \\ b_{n1} \ b_{n2} \cdots b_{ns} \end{pmatrix}$$



证明 设
$$AB = C_{m \times s}$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}$$

即

$$(\gamma_1 \ \gamma_2 \cdots \ \gamma_s) = (\alpha_1 \ \alpha_2 \cdots \ \alpha_n) \begin{pmatrix} b_{11} \ b_{12} \cdots b_{1s} \\ b_{21} \ b_{22} \cdots b_{2s} \\ \vdots \ \vdots \ \ddots \ \vdots \\ b_{n1} \ b_{n2} \cdots b_{ns} \end{pmatrix}$$

$$\Rightarrow \quad \gamma_1 = b_{11}\alpha_1 + b_{21}\alpha_2 + \dots + b_{n1}\alpha_n$$

证明 设
$$AB = C_{m \times s}$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}$$

即

$$(\gamma_1 \ \gamma_2 \cdots \ \gamma_s) = (\alpha_1 \ \alpha_2 \cdots \ \alpha_n) \begin{pmatrix} b_{11} \ b_{12} \cdots b_{1s} \\ b_{21} \ b_{22} \cdots b_{2s} \\ \vdots \ \vdots \ \ddots \ \vdots \\ b_{n1} \ b_{n2} \cdots b_{ns} \end{pmatrix}$$

$$\Rightarrow \gamma_1 = b_{11}\alpha_1 + b_{21}\alpha_2 + \cdots + b_{n1}\alpha_n$$
 等等

证明 设
$$AB = C_{m \times s}$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}$$

即

$$(\gamma_1 \ \gamma_2 \cdots \ \gamma_s) = (\alpha_1 \ \alpha_2 \cdots \ \alpha_n) \begin{pmatrix} b_{11} \ b_{12} \cdots b_{1s} \\ b_{21} \ b_{22} \cdots b_{2s} \\ \vdots \ \vdots \ \ddots \ \vdots \\ b_{n1} \ b_{n2} \cdots b_{ns} \end{pmatrix}$$

$$\Rightarrow \quad \gamma_1 = b_{11}\alpha_1 + b_{21}\alpha_2 + \dots + b_{n1}\alpha_n \quad 等等$$

可见  $\gamma_1, \ldots, \gamma_s$  由  $\alpha_1, \ldots, \alpha_n$  线性表示,

证明 设
$$AB = C_{m \times s}$$

$$\underbrace{\begin{pmatrix}
c_{11} & c_{12} & \cdots & c_{1s} \\
c_{21} & c_{22} & \cdots & c_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
c_{m1} & c_{m2} & \cdots & c_{ms}
\end{pmatrix}}_{C} = \underbrace{\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}}_{A} \begin{pmatrix}
b_{11} & b_{12} & \cdots & b_{1s} \\
b_{21} & b_{22} & \cdots & b_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
b_{n1} & b_{n2} & \cdots & b_{ns}
\end{pmatrix}}_{B}$$

即

$$(\gamma_1 \ \gamma_2 \cdots \ \gamma_s) = (\alpha_1 \ \alpha_2 \cdots \ \alpha_n) \begin{pmatrix} b_{11} \ b_{12} \cdots b_{1s} \\ b_{21} \ b_{22} \cdots b_{2s} \\ \vdots \ \vdots \ \ddots \ \vdots \\ b_{n1} \ b_{n2} \cdots b_{ns} \end{pmatrix}$$

可见 
$$\gamma_1, \ldots, \gamma_s$$
 由  $\alpha_1, \ldots, \alpha_n$  线性表示,所以

$$r(\gamma_1, \ldots, \gamma_s) \leq r(\alpha_1, \ldots, \alpha_n)$$

 $\gamma_1 = b_{11}\alpha_1 + b_{21}\alpha_2 + \cdots + b_{n1}\alpha_n$  等等



证明 设
$$AB = C_{m \times s}$$

$$\underbrace{\begin{pmatrix}
c_{11} & c_{12} & \cdots & c_{1s} \\
c_{21} & c_{22} & \cdots & c_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
c_{m1} & c_{m2} & \cdots & c_{ms}
\end{pmatrix}}_{C} = \underbrace{\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}}_{A} \begin{pmatrix}
b_{11} & b_{12} & \cdots & b_{1s} \\
b_{21} & b_{22} & \cdots & b_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
b_{n1} & b_{n2} & \cdots & b_{ns}
\end{pmatrix}}_{B}$$

即

$$(\gamma_1 \ \gamma_2 \cdots \ \gamma_s) = (\alpha_1 \ \alpha_2 \cdots \ \alpha_n) \begin{pmatrix} b_{11} \ b_{12} \cdots b_{1s} \\ b_{21} \ b_{22} \cdots b_{2s} \\ \vdots \ \vdots \ \ddots \ \vdots \\ b_{n1} \ b_{n2} \cdots b_{ns} \end{pmatrix}$$

$$\Rightarrow \gamma_1 = b_{11}\alpha_1 + b_{21}\alpha_2 + \cdots + b_{n1}\alpha_n \quad \mathfrak{F}\mathfrak{F}$$

可见 
$$\gamma_1, \ldots, \gamma_s$$
 由  $\alpha_1, \ldots, \alpha_n$  线性表示,所以

$$r(\gamma_1, \ldots, \gamma_s) \le r(\alpha_1, \ldots, \alpha_n) = r(A)$$



证明 设
$$AB = C_{m \times s}$$

$$\underbrace{\begin{pmatrix}
c_{11} & c_{12} & \cdots & c_{1s} \\
c_{21} & c_{22} & \cdots & c_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
c_{m1} & c_{m2} & \cdots & c_{ms}
\end{pmatrix}}_{C} = \underbrace{\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}}_{A} \begin{pmatrix}
b_{11} & b_{12} & \cdots & b_{1s} \\
b_{21} & b_{22} & \cdots & b_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
b_{n1} & b_{n2} & \cdots & b_{ns}
\end{pmatrix}}_{B}$$

即

$$(\gamma_1 \ \gamma_2 \cdots \gamma_s) = (\alpha_1 \ \alpha_2 \cdots \ \alpha_n) \begin{pmatrix} b_{11} \ b_{12} \cdots b_{1s} \\ b_{21} \ b_{22} \cdots b_{2s} \\ \vdots \ \vdots \ \ddots \ \vdots \\ b_{n1} \ b_{n2} \cdots b_{ns} \end{pmatrix}$$

$$\Rightarrow \gamma_1 = b_{11}\alpha_1 + b_{21}\alpha_2 + \cdots + b_{n1}\alpha_n \quad \text{\reff}$$

可见 
$$\gamma_1, \ldots, \gamma_s$$
 由  $\alpha_1, \ldots, \alpha_n$  线性表示,所以

$$r(AB) = r(\gamma_1, \ldots, \gamma_s) \le r(\alpha_1, \ldots, \alpha_n) = r(A)$$



$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}^{\beta_{1}}$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}^{\beta_{1}}$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}^{\beta_{1}}$$

$$\underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{\beta_{n}}^{\beta_{1}}$$

$$\underbrace{\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{bmatrix}}_{C} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{bmatrix}}_{B}^{\beta_{1}}$$

$$\underbrace{\begin{bmatrix}
c_{11} & c_{12} & \cdots & c_{1s} \\
c_{21} & c_{22} & \cdots & c_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
c_{m1} & c_{m2} & \cdots & c_{ms}
\end{bmatrix}}_{C} = \underbrace{\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}}_{A} \underbrace{\begin{bmatrix}
b_{11} & b_{12} & \cdots & b_{1s} \\
b_{21} & b_{22} & \cdots & b_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
b_{n1} & b_{n2} & \cdots & b_{ns}
\end{bmatrix}}_{\beta_{n}}^{\beta_{1}}$$

$$\frac{\delta_{1}}{\delta_{2}} \begin{pmatrix} c_{11} & c_{12} \cdots c_{1s} \\ c_{21} & c_{22} \cdots c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} \cdots c_{ms} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} \cdots a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \cdots b_{1s} \\ b_{21} & b_{22} \cdots b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} \cdots b_{ns} \end{pmatrix} \beta_{n}$$

$$\frac{\delta_{1}}{\delta_{2}} \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix} \stackrel{\beta_{1}}{\beta_{2}}$$

$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$$

$$\frac{\delta_{1}}{\delta_{2}} \underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{B}^{\beta_{1}}_{\beta_{2}}$$

$$\bigoplus_{C} \underbrace{\begin{pmatrix} \delta_{1} \\ \delta_{2} \\ \vdots \\ \delta_{n1} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n2} & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots &$$

$$\Rightarrow \quad \delta_1 = a_{11}\beta_1 + a_{12}\beta_2 + \dots + a_{1n}\beta_n$$

证明 设  $AB = C_{mxs}$ 

$$\frac{\delta_{1}}{\delta_{2}} \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix} \begin{pmatrix} \beta_{1} \\ \beta_{2} \end{pmatrix}$$

$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$$

$$\Rightarrow \quad \delta_1 = a_{11}\beta_1 + a_{12}\beta_2 + \dots + a_{1n}\beta_n \quad$$
 等等

证明 设  $AB = C_{m \times s}$ 

$$\frac{\delta_{1}}{\delta_{2}} \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix} \begin{pmatrix} \beta_{1} \\ \beta_{n} \\ \beta_$$

即

$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$$

$$\Rightarrow \quad \delta_1 = \alpha_{11}\beta_1 + \alpha_{12}\beta_2 + \dots + \alpha_{1n}\beta_n \quad \text{$\mathfrak{F}$}$$

可见  $\delta_1, \ldots, \delta_m$  由  $\beta_1, \ldots, \beta_n$  线性表示,

证明 设  $AB = C_{m \times s}$ 

$$\underbrace{ \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{bmatrix}}_{C} = \underbrace{ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{A} \underbrace{ \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{bmatrix}}_{B}^{\beta_{1}}$$

即

$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$$

$$\Rightarrow \quad \delta_1 = a_{11}\beta_1 + a_{12}\beta_2 + \dots + a_{1n}\beta_n \quad$$
\$\$

可见  $\delta_1, \ldots, \delta_m$  由  $\beta_1, \ldots, \beta_n$  线性表示,所以

$$r(\delta_1, \ldots, \delta_m) \leq r(\beta_1, \ldots, \beta_n)$$

证明 设  $AB = C_{m \times s}$ 

$$\frac{\delta_{1}}{\delta_{2}} \underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}}_{\beta_{n}}_{\beta_{n}}$$

即

$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$$

$$\Rightarrow \quad \delta_1 = a_{11}\beta_1 + a_{12}\beta_2 + \dots + a_{1n}\beta_n \quad \text{$\Re$}$$

可见  $\delta_1, \ldots, \delta_m$  由  $\beta_1, \ldots, \beta_n$  线性表示,所以

$$r(\delta_1, \ldots, \delta_m) \le r(\beta_1, \ldots, \beta_n) = r(B)$$

证明 设  $AB = C_{m \times s}$ 

$$\underbrace{ \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{ms} \end{bmatrix}}_{C} = \underbrace{ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{A} \underbrace{ \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{bmatrix}}_{B}^{\beta_{1}}$$

即

$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$$

$$\Rightarrow \quad \delta_1 = a_{11}\beta_1 + a_{12}\beta_2 + \cdots + a_{1n}\beta_n \quad \text{\$}$$

可见  $\delta_1, \ldots, \delta_m$  由  $\beta_1, \ldots, \beta_n$  线性表示,所以

 $r(AB) = r(\delta_1, \ldots, \delta_m) \le r(\beta_1, \ldots, \beta_n) = r(B)$