§3.1 线性方程组的消元解法

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记号

考虑 n 个未知量 m 个方程的线性方程组:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

可以,等价地,改写成矩阵形式

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m \times n} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

可以, 等价地, 改写成矩阵形式

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m \times n}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}_{b}$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

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$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m \times n}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}_{b} \quad \Rightarrow \quad Ax = b$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

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$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m \times n}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}_{b} \quad \Rightarrow \quad Ax = b$$

整个方程组的信息包含在:

$$(A \vdots b) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_{mn} \end{pmatrix}$$



$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

可以,等价地,改写成矩阵形式

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m \times n}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}_{b} \quad \Rightarrow \quad Ax = b$$

整个方程组的信息包含在:

增广矩阵
$$(A \vdots b) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix}$$

例解方程组:
$$\begin{cases} x_1+ & x_2- & x_3=2\\ x_1+2x_2 & =-1\\ 4x_1+7x_2- & x_3=-1\\ -3x_1-4x_2+2x_3=-3 \end{cases}$$

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$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases}$$

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消元法步骤示例:

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases}$$

$$\begin{cases} 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases}$$

(2)-(1)

例解方程组:
$$\begin{cases} x_1+ & x_2- & x_3=2\\ x_1+2x_2 & =-1\\ 4x_1+7x_2- & x_3=-1\\ -3x_1-4x_2+2x_3=-3 \end{cases}$$

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例解方程组:
$$\begin{cases} x_1+ & x_2- & x_3=2\\ x_1+2x_2 & =-1\\ 4x_1+7x_2- & x_3=-1\\ -3x_1-4x_2+2x_3=-3 \end{cases}$$

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$$\downarrow (A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 - 4 & 2 & -3 \end{pmatrix}$$

例解方程组:
$$\begin{cases} x_1+ & x_2- & x_3=2\\ x_1+2x_2 & =-1\\ 4x_1+7x_2- & x_3=-1\\ -3x_1-4x_2+2x_3=-3 \end{cases}$$

消元法步骤示例:

$$x_1 + x_2 - x_3 = 2$$

 $x_1 + 2x_2 = -1$
 $4x_1 + 7x_2 - x_3 = -1$ \rightarrow $\begin{cases} x_1 + x_2 + 2x_3 - 3 \\ x_1 + x_2 + 2x_3 - 3 \\ x_1 + x_2 + x_3 - 3 \\ x_2 + x_3 + x_3 - 3 \\ x_1 + x_2 + x_3 - 3 \\ x_2 + x_3 + x_3 - 3 \\ x_1 + x_2 + x_3 - 3 \\ x_2 + x_3 + x_3 - 3 \\ x_3 + x_3 +$

$$= -1$$

$$x_3 = -1$$

$$x_2 - x_3 = -1$$

 $x_2 + 2x_3 = -3$

$$x_1 + 7x_2 - x_3 = -1$$

 $x_1 - 4x_2 + 2x_3 = -3$

 $(A \vdots b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & 2 \end{pmatrix}$

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases} \rightarrow \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases}$$
(2)-(1)

 $r_2 - r_1$

例解方程组:
$$\begin{cases} x_1+ & x_2- & x_3=2\\ x_1+2x_2 & =-1\\ 4x_1+7x_2- & x_3=-1\\ -3x_1-4x_2+2x_3=-3 \end{cases}$$

消元法步骤示例:

$$\begin{cases}
x_1 + x_2 - x_3 = 2 \\
x_1 + 2x_2 = -1 \\
4x_1 + 7x_2 - x_3 = -1 \\
-3x_1 - 4x_2 + 2x_3 = -3
\end{cases}$$

$$\rightarrow \begin{cases}
x_1 + x_2 - x_3 = 2 \\
x_2 + x_3 = -3 \\
4x_1 + 7x_2 - x_3 = -1 \\
-3x_1 - 4x_2 + 2x_3 = -3
\end{cases}$$
(2)-(1)

 $(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{pmatrix}$ $r_2 - r_1$



例解方程组:
$$\begin{cases} x_1+ & x_2- & x_3=2\\ x_1+2x_2 & =-1\\ 4x_1+7x_2- & x_3=-1\\ -3x_1-4x_2+2x_3=-3 \end{cases}$$

$$\begin{array}{c|c}
4x_1 + 7x_2 - \\
-3x_1 - 4x_2 + \\
\end{array}$$

$$\begin{cases} x_{1} + x_{2} - x_{3} = 2 \\ x_{1} + 2x_{2} = -1 \\ 4x_{1} + 7x_{2} - x_{3} = -1 \\ -3x_{1} - 4x_{2} + 2x_{3} = -3 \end{cases} \rightarrow \begin{cases} x_{1} + x_{2} - x_{3} = 2 \\ x_{2} + x_{3} = -3 \\ 4x_{1} + 7x_{2} - x_{3} = -1 \\ -3x_{1} - 4x_{2} + 2x_{3} = -3 \end{cases}$$
(2)-(1)

 $(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{pmatrix}$ $r_2 - r_1$

例解方程组:
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases}$$

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 $(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{pmatrix}$ $r_2 - r_1$

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$$\begin{array}{ccc}
x_3 = 2 \\
= -1 \\
x_3 = -1
\end{array}
\longrightarrow$$

例:

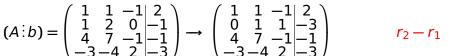
$$- x_3 = 2$$

$$= -1$$

例解方程组: $\begin{cases} x_1+ x_2- x_3=2\\ x_1+2x_2=-1\\ 4x_1+7x_2- x_3=-1\\ -3x_1-4x_2+2x_3=-3 \end{cases}$









例解方程组:
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases}$$

消元法步骤示例:

$$\begin{cases}
x_1 + x_2 - x_3 = 2 \\
x_1 + 2x_2 = -1 \\
4x_1 + 7x_2 - x_3 = -1 \\
-3x_1 - 4x_2 + 2x_3 = -3
\end{cases}$$

$$\begin{cases}
x_1 + x_2 - x_3 = 2 \\
x_2 + x_3 = -3 \\
3x_2 + 3x_3 = -9 \\
-3x_1 - 4x_2 + 2x_3 = -3
\end{cases}$$
(3) - 4(1)

例:
$$- x_3 = 2$$

$$= -1$$

 $(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{pmatrix}$ $r_3 - 4r_1$

例解方程组:
$$\begin{cases} x_1 + x_2 - x_3 = 2\\ x_1 + 2x_2 = -1\\ 4x_1 + 7x_2 - x_3 = -1\\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases}$$

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$$x_3 = 2$$

$$= -1$$

$$x_3 = -1$$

$$=-1$$

$$-x_3 = -1$$

$$-2x_3 = -3$$



例解方程组:
$$\begin{cases} x_1+ & x_2- & x_3=2\\ x_1+2x_2 & =-1\\ 4x_1+7x_2- & x_3=-1\\ -3x_1-4x_2+2x_3=-3 \end{cases}$$

(-3
$$x_1$$
-4 x_2 +2 x_3 =-3
消元法步骤示例:
$$\begin{cases} x_1 + x_2 - x_3 = 2\\ x_1 + 2x_2 = -1\\ 4x_1 + 7x_2 - x_3 = -1\\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases}$$
 \Rightarrow
$$\begin{cases} x_1 + x_2 - x_3 = 2\\ x_2 + x_3 = -3\\ 3x_2 + 3x_3 = -9\\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases}$$
 (4) + 3(1)

$$\begin{array}{ll}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_7 \\
x_7$$

$$7x_2 - x_3 = -1$$

 $4x_2 + 2x_3 = -3$

$$(A : b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 3 & -9 \\ -3 & -4 & 2 & -3 \end{pmatrix}$$

例解方程组:
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 $(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 3 & -9 \\ -3 & -4 & 2 & -3 \end{pmatrix} \qquad r_3 - 4r_1$



例解方程组:
$$\begin{cases} x_1+ & x_2- & x_3=2\\ x_1+2x_2 & =-1\\ 4x_1+7x_2- & x_3=-1\\ -3x_1-4x_2+2x_3=-3 \end{cases}$$

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(4) + 3(1)

$$(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 3 & -9 \\ -3 & -4 & 2 & -3 \end{pmatrix} \qquad r_4 + 3r_1$$



例解方程组:
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$$(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 3 & -9 \\ 0 & -1 & -1 & 3 \end{pmatrix}$$

$$r_4 + 3r_1$$





例解方程组:
$$\begin{cases} x_1+ & x_2- & x_3=2\\ x_1+2x_2 & =-1\\ 4x_1+7x_2- & x_3=-1\\ -3x_1-4x_2+2x_3=-3 \end{cases}$$

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(3) - 3(2)

$$\begin{pmatrix}
1 & 1 & -1 & 2 \\
1 & 2 & 0 & -1 \\
4 & 7 & -1 & -1 \\
-3 & -4 & 2 & -3
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 1 & -1 & 2 \\
0 & 1 & 1 & -3 \\
0 & 3 & 3 & -9 \\
0 & -1 & -1 & 3
\end{pmatrix}$$

$$r_4 + 3r_1$$



例解方程组:
$$\begin{cases} x_1+ & x_2- & x_3=2\\ x_1+2x_2 & =-1\\ 4x_1+7x_2- & x_3=-1\\ -3x_1-4x_2+2x_3=-3 \end{cases}$$

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 (3) - 3(2)

$$(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 3 & -9 \\ 0 & -1 & -1 & 3 \end{pmatrix} \qquad r_4 + 3r_1$$



例解方程组:
$$\begin{cases} x_1+ & x_2- & x_3=2\\ x_1+2x_2 & =-1\\ 4x_1+7x_2- & x_3=-1\\ -3x_1-4x_2+2x_3=-3 \end{cases}$$

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases} \rightarrow \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \\ 0 = 0 \\ -x_2 - x_3 = 3 \end{cases}$$
 (3) - 3(2)

$$(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 3 & -9 \\ 0 & -1 & -1 & 3 \end{pmatrix} \qquad r_3 - 2r_2$$



例解方程组:
$$\begin{cases} x_1+ & x_2- & x_3=2\\ x_1+2x_2 & =-1\\ 4x_1+7x_2- & x_3=-1\\ -3x_1-4x_2+2x_3=-3 \end{cases}$$

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases} \rightarrow \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \\ 0 = 0 \\ -x_2 - x_3 = 3 \end{cases}$$
 (3) - 3(2)

$$(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 3 \end{pmatrix} \qquad r_3 - 2r_2$$



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例解方程组:
$$\begin{cases} x_1+ & x_2- & x_3=2\\ x_1+2x_2 & =-1\\ 4x_1+7x_2- & x_3=-1\\ -3x_1-4x_2+2x_3=-3 \end{cases}$$

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases} \rightarrow \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \\ 0 = 0 \\ -x_2 - x_3 = 3 \end{cases}$$
 (4) + (2)

$$\begin{cases} 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases} \xrightarrow{0 = 0} \begin{cases} 0 = 0 \\ -x_2 - x_3 = 3 \end{cases}$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$



例解方程组:
$$\begin{cases} x_1+ & x_2- & x_3=2\\ x_1+2x_2 & =-1\\ 4x_1+7x_2- & x_3=-1\\ -3x_1-4x_2+2x_3=-3 \end{cases}$$

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases} \rightarrow \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \\ 0 = 0 \end{cases} \tag{4} + (2)$$

 $(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 3 \end{pmatrix} \qquad r_3 - 3r_2$



例解方程组:
$$\begin{cases} x_1+ & x_2- & x_3=2\\ x_1+2x_2 & =-1\\ 4x_1+7x_2- & x_3=-1\\ -3x_1-4x_2+2x_3=-3 \end{cases}$$

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases} \rightarrow \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \\ 0 = 0 \end{cases} \tag{4} + (2)$$

 $(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 3 \end{pmatrix} \qquad r_4 + r_2$

例解方程组:
$$\begin{cases} x_1+ & x_2- & x_3=2\\ x_1+2x_2 & =-1\\ 4x_1+7x_2- & x_3=-1\\ -3x_1-4x_2+2x_3=-3 \end{cases}$$

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases} \rightarrow \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \\ 0 = 0 \\ 0 = 0 \end{cases} \tag{4} + (2)$$

 $(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad r_4 + r_2$





例解方程组:
$$\begin{cases} x_1+ & x_2- & x_3=2\\ x_1+2x_2 & =-1\\ 4x_1+7x_2- & x_3=-1\\ -3x_1-4x_2+2x_3=-3 \end{cases}$$

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases} \rightarrow \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \\ 0 = 0 \\ 0 = 0 \end{cases}$$
(1)-(2)

$$(A : b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad r_4 + r_2$$



例解方程组:
$$\begin{cases} x_1+ & x_2- & x_3=2\\ x_1+2x_2 & =-1\\ 4x_1+7x_2- & x_3=-1\\ -3x_1-4x_2+2x_3=-3 \end{cases}$$

$$\begin{cases} x_{1} + x_{2} - x_{3} = 2 \\ x_{1} + 2x_{2} = -1 \\ 4x_{1} + 7x_{2} - x_{3} = -1 \\ -3x_{1} - 4x_{2} + 2x_{3} = -3 \end{cases} \rightarrow \begin{cases} x_{1} - 2x_{3} = 5 \\ x_{2} + x_{3} = -3 \\ 0 = 0 \end{cases}$$
(1) - (2)

$$(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad r_4 + r_2$$



例解方程组:
$$\begin{cases} x_1+ & x_2- & x_3=2\\ x_1+2x_2 & =-1\\ 4x_1+7x_2- & x_3=-1\\ -3x_1-4x_2+2x_3=-3 \end{cases}$$

$$\begin{cases} x_{1} + x_{2} - x_{3} = 2 \\ x_{1} + 2x_{2} = -1 \\ 4x_{1} + 7x_{2} - x_{3} = -1 \\ -3x_{1} - 4x_{2} + 2x_{3} = -3 \end{cases} \rightarrow \begin{cases} x_{1} - 2x_{3} = 5 \\ x_{2} + x_{3} = -3 \\ 0 = 0 \end{cases}$$
(1) - (2)

$$(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad r_1 - r_2$$



例解方程组:
$$\begin{cases} x_1+ & x_2- & x_3=2\\ x_1+2x_2 & =-1\\ 4x_1+7x_2- & x_3=-1\\ -3x_1-4x_2+2x_3=-3 \end{cases}$$

$$\begin{cases} x_{1} + x_{2} - x_{3} = 2 \\ x_{1} + 2x_{2} = -1 \\ 4x_{1} + 7x_{2} - x_{3} = -1 \\ -3x_{1} - 4x_{2} + 2x_{3} = -3 \end{cases} \rightarrow \begin{cases} x_{1} - 2x_{3} = 5 \\ x_{2} + x_{3} = -3 \\ 0 = 0 \end{cases}$$
(1) - (2)

$$(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad r_1 - r_2$$



例解方程组:
$$\begin{cases} x_1+ & x_2- & x_3=2\\ x_1+2x_2 & =-1\\ 4x_1+7x_2- & x_3=-1\\ -3x_1-4x_2+2x_3=-3 \end{cases}$$

$$\begin{cases} x_{1} + x_{2} - x_{3} = 2 \\ x_{1} + 2x_{2} = -1 \\ 4x_{1} + 7x_{2} - x_{3} = -1 \\ -3x_{1} - 4x_{2} + 2x_{3} = -3 \end{cases} \rightarrow \begin{cases} x_{1} - 2x_{3} = 5 \\ x_{2} + x_{3} = -3 \\ 0 = 0 \end{cases} \Rightarrow \begin{cases} x_{1} = 5 + 2x_{3} \\ x_{2} = -3 - x_{3} \end{cases}$$

$$0 = 0$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad$$



例解方程组:
$$\begin{cases} x_1+ x_2- x_3=2\\ x_1+2x_2=-1\\ 4x_1+7x_2- x_3=-1\\ -3x_1-4x_2+2x_3=-3 \end{cases}$$

消元法步骤示例:

$$\begin{cases}
x_1 + x_2 - x_3 = 2 \\
x_1 + 2x_2 = -1 \\
4x_1 + 7x_2 - x_3 = -1 \\
-3x_1 - 4x_2 + 2x_3 = -3
\end{cases}$$

$$\begin{cases}
x_1 - 2x_3 = 5 \\
x_2 + x_3 = -3 \\
0 = 0
\end{cases}$$

$$\begin{cases}
x_1 = 5 + 2x_3 \\
x_2 = -3 - x_3 \\
0 = 0
\end{cases}$$

示例:

$$\begin{array}{ccc}
- & x_3 = 2 \\
 & = -1
\end{array}$$

$$\begin{array}{ccc}
 & x_1 & - \\
 & x_2 + \\
\end{array}$$

 $(A : b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 = 5 + 2x_3 \\ x_2 = -3 - x_3 \end{cases}$

例解方程组:
$$\begin{cases} x_1+ x_2- x_3=2\\ x_1+2x_2=-1\\ 4x_1+7x_2- x_3=-1\\ -3x_1-4x_2+2x_3=-3 \end{cases}$$

$$(-3x_1 - 4x_2 + 2x_3 = -1)$$

消元法步骤示例:
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \end{cases} \begin{cases} x_1 \\ x_2 \end{cases}$$

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases} \rightarrow \begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \\ 0 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 5 + 2x_3 \\ x_2 = -3 - x_3 \end{cases}$$

 $(A : b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 = 5 + 2x_3 \\ x_2 = -3 - x_3 \end{cases}$

初等行变换求解线性方程组

$$Ax = b \implies (A : b) \xrightarrow{M = f(b)}$$
 简化的阶梯型矩阵

回忆: 阶梯形矩阵

形如:

的矩阵, 其中 $b_1, b_2, \ldots, b_r \neq 0$, 称为阶梯型矩阵。

简化的阶梯型矩阵

形如:

简化的阶梯型矩阵

形如:

简化的阶梯型矩阵

形如:

称为简化的阶梯型矩阵。



$$\begin{pmatrix}
2x_1 + 3x_2 + x_3 = -4 \\
-2x_1 - 3x_2 + x_3 = -4 \\
1 & 2 & 0 & -1 \\
2 & 5 & 1 & -5
\end{pmatrix}$$

$$(A:b) = \begin{pmatrix}
1 & 1 & -1 & 2 \\
1 & 2 & 0 & -1 \\
2 & 5 & 1 & -5
\end{pmatrix}$$

$$(A:b) = \begin{pmatrix} \frac{1}{2} & \frac{2}{5} & 0 \\ -2 & -3 & 1 \\ -1 & -1 \end{pmatrix}$$

例 1 解方程组: $\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$ $\begin{pmatrix} 1 & 1 & -1 \\ \frac{1}{2} & 2 & 0 \\ -1 & -5 \\ -2 & -3 & 1 & -1 \end{pmatrix} \xrightarrow[r_3 - 2r_1]{r_3 - 2r_1} \xrightarrow[r_4 + 2r_1]{r_4 + 2r_1}$

$$\mathbf{R} \quad (A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{pmatrix} \frac{r_2 - r_3 - r_4}{r_4 + r_4}$$

$$\begin{cases} 2x_1 + 5x_2 + x_3 = -1 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$

$$(A : b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ \frac{1}{2} & \frac{2}{2} & 0 & -\frac{1}{2} \end{pmatrix} \xrightarrow{r_2 - r_1}$$

例 1 解方程组:
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{pmatrix} \xrightarrow[r_3 - 2r_1]{r_3 - 2r_1}{r_4 + 2r_1} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \end{pmatrix}$$

例 1 解方程组:
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{pmatrix} \xrightarrow[r_3 - 2r_1]{r_3 - 2r_1} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 3 & -9 \end{pmatrix}$$

$$\begin{cases} -2x_1 - 3x_2 + x_3 = -1 \\ 1 & 2 & 0 \\ -1 & \frac{r_2 - r_1}{r_1} \end{cases}$$

$$\mathbf{A}:b) = \begin{pmatrix}
1 & 1 & -1 & 2 \\
1 & 2 & 0 & -1 \\
2 & 5 & 1 & -5 \\
-2 & -3 & 1 & -1
\end{pmatrix} \xrightarrow[r_4+2r_1]{r_2-r_1} \begin{pmatrix}
1 & 1 & -1 & 2 \\
0 & 1 & 1 & -1 \\
0 & 3 & 3 & -9 \\
0 & -1 & -1 & 3
\end{pmatrix}$$

$$(A:b) = \begin{pmatrix} -2x_1 - 3x_2 + x_3 = -1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{r_2 - r_1}{r_1} & \frac{r_2 - r_1}{r_2} \end{pmatrix}$$

$$\mathbf{H} \quad (A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{pmatrix} \xrightarrow[r_4+2r_1]{r_2-r_1} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 3 & -9 \\ 0 & -1 & -1 & 3 \end{pmatrix}$$

$$r_3 - 3r_2$$



$$\begin{pmatrix} -2 & -3 & 1 & | & -3 & 1 \\ -2 & -3 & 1 & | & -1 & | & -3 & | \\ \hline r_{3}-3r_{2} & \begin{pmatrix} 1 & 1 & -1 & | & 2 & | \\ 0 & 1 & 1 & | & -3 & | & -3 & | \\ \hline r_{3}-3r_{2} & & & & & & & & & & \\ \end{pmatrix}$$

$$\xrightarrow[r_4+r_2]{r_3-3r_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \end{array} \right)$$



$$\begin{array}{c|ccccc}
 & -2 & -3 & 1 & |-1 & 7 & 7 \\
\hline
 & r_{3}-3r_{2} & \begin{pmatrix} 1 & 1 & -1 & | & -3 & | \\ 0 & 1 & 1 & | & -3 & | \\ 0 & 0 & 0 & | & 0 & |
\end{array}$$

$$\begin{pmatrix} -2x_1 - 3x_2 + x_3 = -1 \\ 1 & 1 & -1 \\ 2 & 1 & 1 \end{pmatrix}$$

$$\mathbf{H} \quad (A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{pmatrix} \xrightarrow[r_4+2r_1]{r_2-r_1} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 3 & -9 \\ 0 & -1 & -1 & 3 \end{pmatrix}$$

$$\xrightarrow[r_4+r_2]{} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$



$$\begin{pmatrix}
-2x_1 - 3x_2 + x_3 = -1 \\
1 & 1 & -1 & 2 \\
1 & 2 & 0 & -1 \\
2 & 5 & 1 & -5 \\
-2 & -3 & 1 & -1
\end{pmatrix}
\xrightarrow[r_4+2r_1]{r_2-r_1}
\begin{pmatrix}
1 & 1 & -1 & 2 \\
0 & 1 & 1 & -1 & 2 \\
0 & 3 & 3 & -9 \\
0 & -1 & -1 & 3
\end{pmatrix}$$

$$\begin{array}{c|ccccc}
 & 2 & 5 & 1 & -5 \\
 & -2 & -3 & 1 & -1
\end{array}$$

$$\begin{array}{c|ccccc}
 & r_{3}-2r_{1} \\
 & r_{4}+2r_{1}
\end{array}$$

$$\begin{array}{c|cccc}
 & 1 & 1 & -1 \\
 & 0 & 1 & 1 \\
 & 0 & 0 & 0
\end{array}$$

$$\begin{array}{c|cccc}
 & r_{1}-r_{2} \\
 & 0 & 0 & 0
\end{array}$$



$$\begin{cases} 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$



$$\begin{array}{cccc}
2x_1 + & 5x_2 + & x_3 & = -5 \\
-2x_1 - & 3x_2 + & x_3 & = -1
\end{array}$$

$$(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{pmatrix} \xrightarrow[r_4+2r_1]{r_2-r_1} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 3 & -9 \\ 0 & -1 & -1 & 3 \end{pmatrix}$$

$$\xrightarrow[r_4+r_2]{r_3-3r_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \xrightarrow[r_1-r_2]{r_1-r_2} \left(\begin{array}{ccc|c} 1 & 0 & -2 & -5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

所以原方程组等价于



$$\mathbf{H} \quad (A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{pmatrix} \xrightarrow[r_4+2r_1]{r_2-r_1} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -1 & -3 \\ 0 & -1 & -1 & -3 \end{pmatrix}$$

$$\xrightarrow[r_4+r_2]{r_3-3r_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \xrightarrow[r_1-r_2]{r_1-r_2} \left(\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

所以原方程组等价于

$$\begin{cases} x_1 + -2x_3 = 5 \\ x_2 + x_3 = -3 \end{cases}$$



$$\mathbf{H} \quad (A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{pmatrix} \xrightarrow[r_4+2r_1]{r_2-r_1} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & -1 & -1 & 3 \end{pmatrix}$$

$$r_{3}-3r_{2} \quad \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ -3 & 1 & -3 \end{pmatrix} \xrightarrow[r_{1}-r_{2}]{r_{1}-r_{2}} \begin{pmatrix} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -3 \end{pmatrix}$$

$$\xrightarrow{r_3-3r_2} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{r_1-r_2} \begin{pmatrix} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
所以原方积组等价于

所以原方程组等价于

$$\begin{cases} x_1 + & -2x_3 = 5 \\ x_2 + & x_3 = -3 \end{cases} \iff \begin{cases} x_1 + & = 5 + 2x_3 \\ x_2 = -3 - x_3 \end{cases}$$

例 1 解方程组:
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{pmatrix} \xrightarrow[r_3 - 2r_1]{r_3 - 2r_1} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -1 & 2 \\ 0 & 3 & 3 & -9 \\ 0 & -1 & -1 & 3 \end{pmatrix}$$

$$\frac{r_3 - 3r_2}{r_4 + r_2} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{r_1 - r_2} \begin{pmatrix} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
所以原方程组等价于

$$\begin{cases} x_1 + & -2x_3 = 5 \\ & x_2 + & x_3 = -3 \end{cases} \Leftrightarrow \begin{cases} x_1 + & = 5 + 2x_3 \\ & x_2 = -3 - x_3 \end{cases}$$
所以通解是:
$$\begin{cases} (c_1 为任意常数) \end{cases}$$

所以通解是: $\begin{cases} x_1 = 5 + 2c_1 \\ (c_1 为任意常数) \\ x_3 = c_1 \end{cases}$

$$r_{3}-3r_{2}$$

 $r_{4}+r_{2}$
所以原方程组等价于
 $\begin{cases} x_{1}+ & -2 \\ & x_{2}+ \end{cases}$
所以通解是:

例 1 解方程组: $\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$ $\mathbf{H} \quad (A:b) = \begin{pmatrix} 1 & 1 & -1 & | & -2 \\ 1 & 2 & 0 & | & -1 \\ 2 & 5 & 1 & | & -5 \\ -2 & -3 & 1 & | & -1 \end{pmatrix} \xrightarrow[r_1+2r_2]{r_2-r_1} \begin{pmatrix} 1 & 1 & -1 & | & -3 \\ 0 & 1 & 1 & | & -3 \\ 0 & -1 & -1 & | & -3 \end{pmatrix}$ $\begin{cases} x_1 + & -2x_3 = 5 \\ & x_2 + & x_3 = -3 \end{cases} \iff \begin{cases} x_1 + & = 5 + 2x_3 \\ & x_2 = -3 - x_3 \end{cases}$

所以通解是: $\begin{cases} x_1 = 5 + 2c_1 \\ x_2 = -3 - c_1 \end{cases} (c_1$ (c_1 为任意常数) $x_3 = c_1$

 $\mathbf{H} \quad (A:b) = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ -2 & 5 & 1 \\ -2 & -3 & 1 \end{pmatrix} \xrightarrow{r_2 - r_1} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix} \xrightarrow{r_3} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 3 & 3 \\ 0 & -1 & -1 \end{pmatrix} \xrightarrow{r_3} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 3 & 3 \\ 0 & -1 & -1 \end{pmatrix}$ $\xrightarrow{r_3 - 3r_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & -2 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \xrightarrow{r_1 - r_2} \left(\begin{array}{ccc|c} 1 & 0 & -2 & -5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array} \right)$ 所以原方程组等价于 $\begin{cases} x_1 + & -2x_3 = 5 \\ & x_2 + & x_3 = -3 \end{cases} \iff \begin{cases} x_1 + & = 5 + 2x_3 \\ & x_2 = -3 - x_3 \end{cases}$ 所以通解是: $\begin{cases} x_1 = 5 + 2c_1 \\ x_2 = -3 - c_1 \end{cases} (c_1$ (c_1 为任意常数) $x_3 = c_1$

例 2 解方程组:
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

例 2 解方程组: $\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$ $\begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix}$

例 2 解方程组: $\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$ $\begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow[r_3 - 5r_1]{r_3 - 3r_1}{r_3 - 3r_1}$

$$\begin{cases}
-2x_1 - 3x_2 - 9x_3 = -53 \\
3x_1 + 6x_2 + 13x_3 = 88 \\
5x_1 + 9x_2 + 22x_3 = 141
\end{cases}$$

$$\begin{pmatrix}
1 & 2 & 4 & 28 \\
2 & 3 & 2 & 53
\end{pmatrix}$$

例 2 解方程组:
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$\begin{pmatrix} A:b \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow[r_3 - 3r_1]{r_3 - 3r_1}{r_4 - 5r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 1 & 2 & 4 & 28 \\ 1 & 2 & 4 & 28 \end{pmatrix}$$



$$\begin{cases} 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

例 2 解方程组:
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$\begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow[r_3 - 3r_1]{r_3 - 3r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \end{pmatrix}$$



$$\begin{cases} 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 143 \end{cases}$$

例 2 解方程组:
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$\begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow[r_2 + 2r_1]{r_3 - 3r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

$$3x_1 + 6x_2 + 13x_3 = 88$$

$$5x_1 + 9x_2 + 22x_3 = 141$$

例 2 解方程组:
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$\begin{pmatrix} A:b \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & 25 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow[r_3 - 3r_1]{r_3 - 3r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

例 2 解方程组:
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$\mathbf{H} \quad (A : b) = \begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow[r_4-5r_1]{r_2+2r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

$$r_4+r_2$$



例 2 解方程组:
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$\mathbf{H} \quad (A:b) = \begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow{r_2 + 2r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

$$\xrightarrow{r_4+r_2} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

例 2 解方程组:
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$\mathbf{H} \quad (A:b) = \begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow[r_4-5r_1]{r_2+2r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

$$\xrightarrow{r_4+r_2} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{pmatrix} \xrightarrow{r_4-r_3}$$

例 2 解方程组:
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$\mathbf{H} \quad (A:b) = \begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow[r_4-5r_1]{r_2+2r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

$$\xrightarrow{r_4+r_2} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{pmatrix} \xrightarrow{r_4-r_3} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$



例 2 解方程组:
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$\mathbf{H} \quad (A:b) = \begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow[r_4-5r_1]{r_2+2r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

$$\xrightarrow{r_4+r_2} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{pmatrix} \xrightarrow{r_4-r_3} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\frac{r_2+r_3}{r_1-4r_2}$$



例 2 解方程组:
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$\mathbf{H} \quad (A:b) = \begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow[r_4-5r_1]{r_2+2r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

$$\xrightarrow{r_4+r_2} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{array} \right) \xrightarrow{r_4-r_3} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$\begin{array}{c|c}
r_2+r_3 \\
\hline
r_1-4r_3
\end{array}
\left(\begin{array}{ccc|c}
0 & 0 & 1 & 4 \\
0 & 0 & 0 & 0
\end{array}\right)$$

例 2 解方程组:
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$\mathbf{H} \quad (A:b) = \begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow[r_4-5r_1]{r_2+2r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

$$\xrightarrow{r_4+r_2} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{pmatrix} \xrightarrow{r_4-r_3} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\frac{r_2 + r_3}{r_1 - 4r_3} \left(\begin{array}{ccc|c} 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

例 2 解方程组:
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$\mathbf{H} \quad (A:b) = \begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow[r_4-5r_1]{r_2+2r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

$$\xrightarrow{r_4+r_2} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{pmatrix} \xrightarrow{r_4-r_3} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{c|cccc} & \xrightarrow{r_2+r_3} & \begin{pmatrix} 1 & 2 & 0 & 12 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$



例 2 解方程组:
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$\mathbf{H} \quad (A:b) = \begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow[r_4-5r_1]{r_2+2r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

$$\xrightarrow{r_4+r_2} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{pmatrix} \xrightarrow{r_4-r_3} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[r_1-4r_3]{r_2+r_3} \left(\begin{array}{ccc} 1 & 2 & 0 & 12 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{r_1-2r_2}$$

例 2 解方程组:
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$\mathbf{H} \quad (A:b) = \begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow[r_4-5r_1]{r_2+2r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

$$\xrightarrow{r_4+r_2} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{pmatrix} \xrightarrow{r_4-r_3} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[r_1-4r_3]{\left(\begin{array}{ccc|c}1&2&0&12\\0&1&0&7\\0&0&1&4\\0&0&0&0\end{array}\right)}\xrightarrow[r_1-2r_2]{\left(\begin{array}{ccc|c}1&0&0&-2\\0&1&0&7\\0&0&1&4\\0&0&0&0\end{array}\right)}$$

例 2 解方程组: $\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$

$$\xrightarrow{r_4+r_2} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{pmatrix} \xrightarrow{r_4-r_3} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\frac{r_{2}+r_{3}}{r_{1}-4r_{3}}\begin{pmatrix} 1 & 2 & 0 & | & 12 \\ 0 & 1 & 0 & | & 7 \\ 0 & 0 & 1 & | & 4 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \xrightarrow{r_{1}-2r_{2}} \begin{pmatrix} 1 & 0 & 0 & | & -2 \\ 0 & 1 & 0 & | & 7 \\ 0 & 0 & 1 & | & 4 \\ 0 & 0 & 0 & | & 4 \end{pmatrix}$$
所以原方程组等价于
$$\begin{cases} x_{1} & = -2 \\ x_{2} & = 7 \\ x_{3} & = 4 \end{cases}$$



例 2 解方程组: $\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$

$$\text{(A:b)} = \begin{pmatrix}
 5x_1 + 9x_2 + 22x_3 = 141 \\
 -2 -3 -9 \\
 3 & 6 & 13 \\
 5 & 9 & 22
 \end{vmatrix}
 \xrightarrow{\begin{array}{c}
 1 & 2 & 4 \\
 -2 & 3 & -9 \\
 3 & 6 & 13 \\
 4 & 141
 \end{array}
 \xrightarrow{\begin{array}{c}
 r_2 + 2r_1 \\
 r_3 - 3r_1 \\
 r_4 - 5r_1
 \end{array}
 \xrightarrow{\begin{array}{c}
 1 & 2 & 4 \\
 0 & 1 & -1 \\
 0 & 0 & 1 \\
 0 & -1 & 2
 \end{array}
 \xrightarrow{\begin{array}{c}
 4 \\
 3 \\
 4
 \end{array}$$

$$\xrightarrow{r_4+r_2} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{pmatrix} \xrightarrow{r_4-r_3} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\frac{r_{2}+r_{3}}{r_{1}-4r_{3}}\begin{pmatrix} 1 & 2 & 0 & 12 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{r_{1}-2r_{2}} \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
所以原方程组等价于
$$\begin{cases} x_{1} & = -2 \\ x_{2} & = 7 \\ x_{3} & = 4 \end{cases}$$

所以原方程组等价于
$$\begin{cases} x_1 &= -x_2 \\ x_2 &= 7 \end{cases}$$



例 3 解方程组:
$$\begin{cases} 4x_1 + 2x_2 - 7x_3 = -3\\ 2x_1 + x_2 - 4x_3 = -1\\ 5x_1 + 3x_2 - 11x_3 = 2\\ x_1 + x_2 - 4x_3 = 2 \end{cases}$$

$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & -3 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 1 & 1 & -4 & 2 \end{pmatrix}$$

$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & -3 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 1 & 1 & -4 & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4}$$

$$(A \vdots b) = \begin{pmatrix} 4 & 2 & -7 & -3 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 1 & 1 & -4 & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & 2 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 4 & 2 & -7 & -3 \end{pmatrix}$$

$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 - 5r_1} \xrightarrow{r_4 - 4r_1}$$



$$(A : b) = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 - 5r_1} \xrightarrow{r_4 - 4r_1}$$

$$\begin{pmatrix} 1 & 1-4 & 2 \\ & & \end{pmatrix}$$



$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & -3 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 1 & 1 & -4 & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & 2 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 4 & 2 & -7 & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{bmatrix} r_2 - 2r_1 & r_3 - 5r_1 & r_4 - 4r_1 \\ r_4 - 4r_1 & r_4 - 4r_1 \end{bmatrix}$$

$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 - 5r_1} \xrightarrow{r_3 - 4r_3}$$

$$\begin{pmatrix} 1 & 1 - 4 & 2 \\ 0 - 1 & 4 & -5 \\ 0 - 2 & 9 & -8 \end{pmatrix}$$

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例 3 解方程组:
$$\begin{cases} 4x_1 + 2x_2 - 7x_3 = -3\\ 2x_1 + x_2 - 4x_3 = -1\\ 5x_1 + 3x_2 - 11x_3 = 2\\ x_1 + x_2 - 4x_3 = 2 \end{cases}$$

$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 - 5r_1} \xrightarrow{r_4 - 4r_1}$$

$$\begin{pmatrix} 1 & 1 - 4 & 2 \\ 0 - 1 & 4 & -5 \\ 0 - 2 & 9 & -8 \\ 0 - 2 & 9 & -11 \end{pmatrix}$$



$$(A : b) = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \frac{r_2 - 2r_1}{r_3 - 5r_1}$$

$$\begin{pmatrix}
1 & 1 - 4 & 2 \\
0 & -1 & 4 & -5 \\
0 & -2 & 9 & -8 \\
0 & -2 & 9 & -11
\end{pmatrix}
\xrightarrow[r_4 - 2r_2]{r_3 - 2r_2}$$

$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 - 5r_1} \xrightarrow{r_4 - 4r_1}$$

$$\begin{pmatrix} 1 & 1 - 4 \\ 0 & -1 & 4 \\ 0 & -2 & 9 \\ 0 & -2 & 9 \\ -11 \end{pmatrix} \xrightarrow[r_4 - 2r_2]{r_3 - 2r_2} \begin{pmatrix} 1 & 1 - 4 \\ 0 - 1 & 4 \\ -5 \end{pmatrix}$$

$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & -3 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 1 & 1 & -4 & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & 2 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 4 & 2 & -7 & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 - 5r_1} \xrightarrow{r_4 - 4r_1}$$

$$\begin{pmatrix} 1 & 1 - 4 \\ 0 & -1 & 4 \\ 0 & -2 & 9 \\ 0 & -2 & 9 \\ -11 \end{pmatrix} \xrightarrow[r_4 - 2r_2]{r_3 - 2r_2} \begin{pmatrix} 1 & 1 - 4 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \\ -5 \\ 2 \end{pmatrix}$$

$$(A : b) = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \frac{r_2 - 2r_1}{r_4 - 4r_1}$$

$$\begin{pmatrix} 1 & 1-4 \\ 0-1 & 4 \\ 0-2 & 9 \\ 0-2 & 9 \\ -11 \end{pmatrix} \xrightarrow[r_4-2r_2]{r_3-2r_2} \begin{pmatrix} 1 & 1-4 \\ 0-1 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 \end{pmatrix}$$

$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & -3 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 1 & 1 & -4 & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & 2 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 4 & 2 & -7 & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 - 5r_1} \xrightarrow{r_4 - 4r_1}$$

$$\begin{pmatrix} 1 & 1 & -4 \\ 0 & -1 & 4 \\ 0 & -2 & 9 \\ 0 & -2 & 9 \\ -11 \end{pmatrix} \xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 & 1 & -4 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 \end{pmatrix} \xrightarrow{r_4 - r_3}$$

$$(A : b) = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \frac{r_2 - 2r_1}{r_4 - 4r_1}$$

$$\begin{pmatrix} 1 & 1-4 & 2 \\ 0-1 & 4 & -5 \\ 0-2 & 9 & -8 \\ 0-2 & 9 & -11 \end{pmatrix} \xrightarrow{r_3-2r_2} \begin{pmatrix} 1 & 1-4 & 2 \\ 0-1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{pmatrix} \xrightarrow{r_4-r_3} \begin{pmatrix} 1 & 1-4 & 2 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$



例 3 解方程组:
$$\begin{cases} 4x_1 + 2x_2 - 7x_3 = -3\\ 2x_1 + x_2 - 4x_3 = -1\\ 5x_1 + 3x_2 - 11x_3 = 2\\ x_1 + x_2 - 4x_3 = 2 \end{cases}$$

$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \frac{r_2 - 2r_1}{r_3 - 5r_1}$$

$$\begin{pmatrix} 1 & 1 - 4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & -2 & 9 & -8 \\ 0 & -2 & 9 & -11 \end{pmatrix} \xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 & 1 - 4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{pmatrix} \xrightarrow{r_4 - r_3} \begin{pmatrix} 1 & 1 - 4 & 2 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

所以原方程组等价于



$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \frac{r_2 - 2r_1}{r_3 - 5r_1}$$

$$\begin{pmatrix}
1 & 1 & -4 & 2
\end{pmatrix}
\begin{pmatrix}
1 & 1 & -4 & 2
\end{pmatrix}
\begin{pmatrix}
1 & 1 & -4 & 2
\end{pmatrix}
\begin{pmatrix}
1 & -1 & 4 & -5 \\
0 & -2 & 9 & -8 \\
0 & -2 & 9 & -11
\end{pmatrix}
\begin{pmatrix}
1 & 1 & -4 & 2
\end{pmatrix}
\begin{pmatrix}
1 & 1 & -4 & 2
\end{pmatrix}
\begin{pmatrix}
1 & 1 & -4 & 2
\end{pmatrix}
\begin{pmatrix}
1 & -1 & 4 & -5 \\
0 & 0 & 1 & -1
\end{pmatrix}
\begin{pmatrix}
1 & 1 & -4 & 2
\end{pmatrix}
\begin{pmatrix}
1 & -1 & 4 & -5 \\
0 & 0 & 1 & -5
\end{pmatrix}$$

所以原方程组等价于
$$\begin{cases} x_1 + x_2 - 4x_3 = 2\\ -x_2 + 4x_3 = -5\\ x_3 = 2\\ 0 = -3 \end{cases}$$



$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \frac{r_2 - 2r_1}{r_3 - 5r_1}$$

$$\begin{pmatrix} 1 & 1 & -4 \\ 0 & -1 & 4 \\ 0 & -2 & 9 \\ 0 & -2 & 9 \\ 0 & -2 & 9 \\ -11 \end{pmatrix} \xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 & -1 & -4 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 \end{pmatrix} \xrightarrow{r_4 - r_3} \begin{pmatrix} 1 & -1 & -4 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \\ -5 \\ 0 & 0 & 1 \\ -3 \end{pmatrix}$$
所以原方程组等价于
$$\begin{cases} x_1 + & x_2 - 4x_3 = 2 \\ -x_2 + 4x_3 = -5 \\ x_3 = 2 \\ 0 = -3 \end{cases} \Rightarrow \mathcal{E}\text{M!}$$

● 暨南大²

§3.1 线性方程组的消元解法

$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 - 5r_1} \xrightarrow{r_4 - 4r_1}$$

$$\begin{pmatrix}
1 & 1 & -4 & 2 \\
0 & -1 & 4 & -5 \\
0 & -2 & 9 & -8 \\
0 & -2 & 9 & -11
\end{pmatrix}
\xrightarrow{r_3 - 2r_2}
\begin{pmatrix}
1 & 1 - 4 & 2 \\
0 & -1 & 4 & -5 \\
0 & 0 & 1 & 2 \\
0 & 0 & 1 & -1
\end{pmatrix}
\xrightarrow{r_4 - r_3}
\begin{pmatrix}
1 & 1 - 4 & 2 \\
0 & -1 & 4 & -5 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & -3
\end{pmatrix}$$

所以原方程组等价于
$$\begin{cases} x_1 + x_2 - 4x_3 = 2 \\ -x_2 + 4x_3 = -5 \\ x_3 = 2 \\ 0 = -3 \end{cases} \Rightarrow$$
 无解!

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$$\begin{cases} x_{1} + & x_{2} - x_{3} = 2 \\ x_{1} + & 2x_{2} = -1 \\ 2x_{1} + & 5x_{2} + & x_{3} = -5 \\ -2x_{1} - & 3x_{2} + & x_{3} = -1 \end{cases} \begin{cases} x_{1} + & 2x_{2} + & 4x_{3} = & 28 \\ -2x_{1} - & 3x_{2} - & 9x_{3} = & -53 \\ 3x_{1} + & 6x_{2} + & 13x_{3} = & 88 \\ 5x_{1} + & 9x_{2} + & 22x_{3} = & 141 \end{cases} \begin{cases} 4x_{1} + & 2x_{2} - & 7x_{3} = & -3 \\ 2x_{1} + & x_{2} - & 4x_{3} = & -1 \\ 5x_{1} + & 3x_{2} - & 11x_{3} = & 2 \\ x_{1} + & x_{2} - & 4x_{3} = & 2 \end{cases}$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad$$



$$\begin{cases} x_{1+} & x_{2-} & x_{3} = 2 \\ x_{1+} & 2x_{2} & = -1 \\ 2x_{1+} & 5x_{2+} & x_{3} = -5 \\ -2x_{1-} & 3x_{2+} & 3x_{3} = -1 \end{cases}$$

$$\begin{cases} x_{1+} & 2x_{2} & 4x_{3} = 28 \\ -2x_{1-} & 3x_{2-} & 9x_{3} = -53 \\ 3x_{1+} & 6x_{2+} & 13x_{3} = 88 \\ 5x_{1+} & 9x_{2+} & 22x_{3} = 141 \end{cases}$$

$$\begin{cases} 4x_{1+} & 2x_{2-} & 7x_{3} = -3 \\ 2x_{1+} & x_{2-} & 4x_{3} = -1 \\ 5x_{1+} & 3x_{2-} & 11x_{3} = 2 \\ x_{1+} & x_{2-} & 4x_{3} = 2 \end{cases}$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$(A \vdots b) \qquad \qquad (A \vdots$$















总结 定理 方程组 $\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$

 $\Leftrightarrow Ax = b$ 的

总结
定理 方程组
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

$$⇔ Ax = b$$
 的

$$1. r(A : b) = r(A)$$

2.
$$r(A) \neq r(A : b)$$



总结
定理 方程组
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

 $\Leftrightarrow Ax = b$ 的

$$1. r(A : b) = r(A)$$

$$r(A) \neq r(A : b) \Leftrightarrow r(A) < r(A : b)$$

$$⇔ Ax = b$$
 的

1.
$$r(A : b) = r(A)$$
$$r(A) = r(A : b) < n$$
$$r(A) = r(A : b) = n$$

2.
$$r(A) \neq r(A : b) \Leftrightarrow r(A) < r(A : b)$$

 $\Leftrightarrow Ax = b$ 的

1.
$$r(A : b) = r(A)$$

$$r(A) = r(A : b) < n$$

• 只有唯一解
$$\Leftrightarrow$$
 $r(A) = r(A : b) = n$

2.
$$r(A) \neq r(A : b) \Leftrightarrow r(A) < r(A : b)$$

 $\Leftrightarrow Ax = b$ 的

$$1. r(A : b) = r(A)$$

- 有无穷多解 \Leftrightarrow r(A) = r(A : b) < n
- 只有唯一解 \Leftrightarrow r(A) = r(A : b) = n
- $r(A) \neq r(A : b) \Leftrightarrow r(A) < r(A : b)$ 2.

$$⇔ Ax = b$$
 的

- 1. 有解 \Leftrightarrow r(A:b) = r(A)
 - 有无穷多解 \Leftrightarrow r(A) = r(A : b) < n
 - 只有唯一解 \Leftrightarrow r(A) = r(A : b) = n
- $r(A) \neq r(A : b) \Leftrightarrow r(A) < r(A : b)$ 2.

$$\Leftrightarrow Ax = b$$
 的

- 1. 有解 \Leftrightarrow r(A:b) = r(A)
 - 有无穷多解 \Leftrightarrow r(A) = r(A : b) < n
 - 只有唯一解 \Leftrightarrow r(A) = r(A : b) = n
- 2. 无解 \Leftrightarrow $r(A) \neq r(A : b) \Leftrightarrow r(A) < r(A : b)$

 $\Leftrightarrow Ax = b$ 的

解有如下情形:

- 1. 有解 \Leftrightarrow r(A:b) = r(A)
 - 有无穷多解 \Leftrightarrow r(A) = r(A : b) < n
 - 只有唯一解 \Leftrightarrow r(A) = r(A : b) = n
- 2. 无解 \Leftrightarrow $r(A) \neq r(A : b) \Leftrightarrow r(A) < r(A : b)$

注

• r(A:b) = r(A) 的值,相当于方程组中"独立"方程个数,此时



 $\Leftrightarrow Ax = b$ 的

解有如下情形:

- 1. 有解 \Leftrightarrow r(A:b) = r(A)
 - 有无穷多解 ⇔ r(A) = r(A:b) < n
 - 只有唯一解 \Leftrightarrow r(A) = r(A : b) = n
- \mathcal{E} 无解 \Leftrightarrow $r(A) \neq r(A : b) \Leftrightarrow r(A) < r(A : b)$

注

- r(A:b) = r(A) 的值,相当于方程组中"独立"方程个数,此时
 - n − r(A) 为自由变量的个数



练习求解
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1\\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3\\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7\\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$

练习求解
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1\\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3\\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7\\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$



练习求解
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1\\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3\\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7\\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} r_3 + r_1$$

练习求解
$$\begin{cases} x_1 + & 2x_2 + & x_3 + & x_4 + & x_5 = 1 \\ 2x_1 + & 4x_2 + & 3x_3 + & x_4 + & x_5 = 3 \\ -x_1 - & 2x_2 + & x_3 + & 3x_4 - & 3x_5 = 7 \\ & & & 2x_3 + & 5x_4 - & 2x_5 = 9 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$



练习求解
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1\\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3\\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7\\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$

$$(A \vdots b) = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$

$$r_3 - 2r_2$$



练习求解
$$\begin{cases} x_1 + & 2x_2 + & x_3 + & x_4 + & x_5 = 1 \\ 2x_1 + & 4x_2 + & 3x_3 + & x_4 + & x_5 = 3 \\ -x_1 - & 2x_2 + & x_3 + & 3x_4 - & 3x_5 = 7 \\ & & 2x_3 + & 5x_4 - & 2x_5 = 9 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$

$$\xrightarrow[r_4-2r_2]{r_4-2r_2}
\left(\begin{array}{cccccccccc}
1 & 2 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & -1 & -1 & 1 \\
0 & 0 & 0 & 6 & 0 & 6 \\
0 & 0 & 0 & 7 & 0 & 7
\end{array}\right)$$



练习求解
$$\begin{cases} x_1 + & 2x_2 + & x_3 + & x_4 + & x_5 = 1 \\ 2x_1 + & 4x_2 + & 3x_3 + & x_4 + & x_5 = 3 \\ -x_1 - & 2x_2 + & x_3 + & 3x_4 - & 3x_5 = 7 \\ & & 2x_3 + & 5x_4 - & 2x_5 = 9 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$

$$\xrightarrow[r_4-2r_2]{ \begin{array}{c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 6 & 0 & 6 \\ 0 & 0 & 0 & 7 & 0 & 7 \end{array} } \xrightarrow[\bar{\tau}\times r_4]{\frac{1}{6}\times r_3}$$



练习求解
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1\\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3\\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7\\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$

$$\frac{r_{3-2r_{2}}}{r_{4-2r_{2}}} \left(\begin{array}{cccccc}
1 & 2 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & -1 & -1 & 1 \\
0 & 0 & 0 & 6 & 0 & 6 \\
0 & 0 & 0 & 7 & 0 & 7
\end{array} \right) \xrightarrow{\frac{1}{6} \times r_{3}} \left(\begin{array}{ccccccc}
1 & 2 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & -1 & -1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1
\end{array} \right)$$



练习求解
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1\\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3\\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7\\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$

$$r_4-r_3$$



练习求解
$$\begin{cases} x_1 + & 2x_2 + & x_3 + & x_4 + & x_5 = 1 \\ 2x_1 + & 4x_2 + & 3x_3 + & x_4 + & x_5 = 3 \\ -x_1 - & 2x_2 + & x_3 + & 3x_4 - & 3x_5 = 7 \\ & & 2x_3 + & 5x_4 - & 2x_5 = 9 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$

$$\frac{r_{3}-2r_{2}}{r_{4}-2r_{2}} \xrightarrow{\begin{cases}
1 & 2 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & -1 & -1 & 1 \\
0 & 0 & 0 & 6 & 0 & 6 \\
0 & 0 & 0 & 7 & 0 & 7
\end{cases}
\xrightarrow{\frac{1}{6}\times r_{3}} \xrightarrow{\frac{1}{6}\times r_{3}} \xrightarrow{\frac{1}{7}\times r_{4}} \xrightarrow{\begin{pmatrix}
1 & 2 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & -1 & -1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1
\end{pmatrix}$$

$$\xrightarrow{r_{4}-r_{3}} \xrightarrow{\begin{pmatrix}
1 & 2 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & -1 & -1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$



练习求解
$$\begin{cases} x_1 + & 2x_2 + & x_3 + & x_4 + & x_5 = 1 \\ 2x_1 + & 4x_2 + & 3x_3 + & x_4 + & x_5 = 3 \\ -x_1 - & 2x_2 + & x_3 + & 3x_4 - & 3x_5 = 7 \\ & & 2x_3 + & 5x_4 - & 2x_5 = 9 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$

$$\frac{r_{3}-2r_{2}}{r_{4}-2r_{2}} \left(\begin{array}{cccccc} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 6 & 0 & 6 \\ 0 & 0 & 0 & 7 & 0 & 7 \end{array} \right) \xrightarrow{\frac{1}{6}\times r_{3}} \left(\begin{array}{ccccccc} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array} \right) \xrightarrow{\frac{1}{7}\times r_{4}} \left(\begin{array}{ccccccc} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array} \right) \xrightarrow{r_{2}+r_{3}} \left(\begin{array}{cccccc} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \xrightarrow{r_{1}+r_{3}} \left(\begin{array}{ccccc} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$



练习求解
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1\\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3\\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7\\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$

$$\frac{r_{3}-2r_{2}}{r_{4}-2r_{2}} \left(\begin{array}{cccccc} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 6 & 0 & 6 \\ 0 & 0 & 0 & 7 & 0 & 7 \end{array} \right) \frac{1}{6} \times r_{3} \xrightarrow{\frac{1}{6}} \left(\begin{array}{ccccccc} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array} \right) \frac{1}{7} \times r_{4} \left(\begin{array}{cccccc} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array} \right) \frac{r_{2}+r_{3}}{r_{1}-r_{3}} \left(\begin{array}{cccccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

练习求解
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1\\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3\\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7\\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$

$$r_1-r_2$$



练习求解
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1\\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3\\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7\\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 6 & 0 & 6 \\ 0 & 0 & 0 & 7 & 0 & 7 \end{pmatrix} \xrightarrow{\frac{1}{5} \times r_3} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{r_4 - r_3} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{r_2 + r_3} \begin{pmatrix} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{r_1-r_2} \begin{pmatrix}
0 & 0 & 1 & -1 & -1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$\xrightarrow{r_1-r_2} \begin{pmatrix}
1 & 2 & 0 & 0 & 2 & | -2 \\
0 & 0 & 1 & 0 & -1 & | 2 \\
0 & 0 & 0 & 1 & 0 & | 1 \\
0 & 0 & 0 & 0 & 0 & | 0
\end{pmatrix}$$



练习 求解
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1\\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3\\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7\\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$
解

$$(A:b) = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 6 & 0 & 6 \\ 0 & 0 & 0 & 7 & 0 & 7 \end{pmatrix} \xrightarrow{\frac{1}{6} \times r_3} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{r_4 - r_3} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{r_2 + r_3} \begin{pmatrix} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$



练习求解
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1 \\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3 \\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7 \\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

练习求解
$$\begin{cases} x_1 + \ 2x_2 + \ x_3 + \ x_4 + \ x_5 = \ 1 \\ 2x_1 + \ 4x_2 + \ 3x_3 + \ x_4 + \ x_5 = \ 3 \\ -x_1 - \ 2x_2 + \ x_3 + \ 3x_4 - \ 3x_5 = \ 7 \\ 2x_3 + \ 5x_4 - \ 2x_5 = \ 9 \end{cases}$$

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

• 可见 r(A) = r(A : b) = 3 < 5,有无穷多的解,

练习求解
$$\begin{cases} x_1 + & 2x_2 + & x_3 + & x_4 + & x_5 = 1 \\ 2x_1 + & 4x_2 + & 3x_3 + & x_4 + & x_5 = 3 \\ -x_1 - & 2x_2 + & x_3 + & 3x_4 - & 3x_5 = 7 \\ & & 2x_3 + & 5x_4 - & 2x_5 = 9 \end{cases}$$

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

• 可见 r(A) = r(A : b) = 3 < 5,有无穷多的解,含 5 - 3 = 2 个自由变量



练习求解
$$\begin{cases} x_1 + & 2x_2 + & x_3 + & x_4 + & x_5 = 1 \\ 2x_1 + & 4x_2 + & 3x_3 + & x_4 + & x_5 = 3 \\ -x_1 - & 2x_2 + & x_3 + & 3x_4 - & 3x_5 = 7 \\ & & 2x_3 + & 5x_4 - & 2x_5 = 9 \end{cases}$$

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- 可见 r(A) = r(A : b) = 3 < 5,有无穷多的解,含 5 3 = 2 个自由变量
- 由既约阶梯形矩阵可知,原方程组等价于

$$\begin{cases} x_1 + 2x_2 & + 2x_5 = -2 \\ x_3 & - x_5 = 2 \\ x_4 & = 1 \end{cases}$$

练习求解
$$\begin{cases} x_{1}+&2x_{2}+&x_{3}+&x_{4}+&x_{5}=&1\\ 2x_{1}+&4x_{2}+&3x_{3}+&x_{4}+&x_{5}=&3\\ -x_{1}-&2x_{2}+&x_{3}+&3x_{4}-&3x_{5}=&7\\ &&2x_{3}+&5x_{4}-&2x_{5}=&9 \end{cases}$$

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- 可见 r(A) = r(A : b) = 3 < 5,有无穷多的解,含 5 3 = 2 个自由变量
- 由既约阶梯形矩阵可知,原方程组等价于

$$\begin{cases} x_1 + 2x_2 & + 2x_5 = -2 \\ x_3 & - x_5 = 2 \\ x_4 & = 1 \end{cases} \Rightarrow \begin{cases} x_1 = -2 - 2x_2 - 2x_5 \\ x_3 = 2 + x_5 \\ x_4 = 1 \end{cases}$$

练习求解 $\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1 \\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3 \\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7 \\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$

解

$$(A:b) \longrightarrow \begin{pmatrix} \begin{array}{c|cccc} & 2 & 2 & 0 & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- 可见 r(A) = r(A : b) = 3 < 5,有无穷多的解,含 5 3 = 2 个自由变量
- 由既约阶梯形矩阵可知,原方程组等价于

$$\begin{cases} x_1 + 2x_2 & + 2x_5 = -2 \\ x_3 & - x_5 = 2 \\ x_4 & = 1 \end{cases} \Rightarrow \begin{cases} x_1 = -2 - 2x_2 - 2x_5 \\ x_3 = 2 + x_5 \\ x_4 = 1 \end{cases}$$

所以通解是

$$\begin{cases} x_1 = \\ x_2 = \\ x_3 = \\ x_4 = \\ x_5 = \end{cases}$$



练习求解 $\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1 \\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3 \\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7 \\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$

解

$$(A:b) \longrightarrow \begin{pmatrix} \begin{array}{c|cccc} & 2 & 2 & 0 & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- 可见 r(A) = r(A : b) = 3 < 5,有无穷多的解,含 5 3 = 2 个自由变量
- 由既约阶梯形矩阵可知,原方程组等价于

$$\begin{cases} x_1 + 2x_2 & + 2x_5 = -2 \\ x_3 & - x_5 = 2 \\ x_4 & = 1 \end{cases} \Rightarrow \begin{cases} x_1 = -2 - 2x_2 - 2x_5 \\ x_3 = 2 + x_5 \\ x_4 = 1 \end{cases}$$

$$egin{cases} x_1 = \ x_2 = c_1 \ x_3 = \ x_4 = \ x_5 = c_2 \end{cases}$$
 $(c_1, c_2$ 为任意常数)

练习求解 $\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1 \\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3 \\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7 \\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$

解

$$(A:b) \longrightarrow \begin{pmatrix} \begin{array}{c|cccc} 1 & 2 & 0 & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- 可见 r(A) = r(A : b) = 3 < 5,有无穷多的解,含 5 3 = 2 个自由变量
- 由既约阶梯形矩阵可知,原方程组等价于

$$\begin{cases} x_1 + 2x_2 & + 2x_5 = -2 \\ x_3 & - x_5 = 2 \\ x_4 & = 1 \end{cases} \Rightarrow \begin{cases} x_1 = -2 - 2x_2 - 2x_5 \\ x_3 = 2 + x_5 \\ x_4 = 1 \end{cases}$$

章
$$\begin{cases} x_1 = -2 - 2c_1 - 2c_2 \\ x_2 = c_1 \\ x_3 = \\ x_4 = \\ x_5 = c_2 \end{cases}$$
 $(c_1, c_2$ 为任意常数)

练习求解 $\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1 \\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3 \\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7 \\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$

解

- 可见 r(A) = r(A : b) = 3 < 5,有无穷多的解,含 5 3 = 2 个自由变量
- 由既约阶梯形矩阵可知,原方程组等价于

$$\begin{cases} x_1 + 2x_2 & + 2x_5 = -2 \\ x_3 & - x_5 = 2 \\ x_4 & = 1 \end{cases} \Rightarrow \begin{cases} x_1 = -2 - 2x_2 - 2x_5 \\ x_3 = 2 + x_5 \\ x_4 = 1 \end{cases}$$

是
$$\begin{cases} x_1 = -2 - 2c_1 - 2c_2 \\ x_2 = c_1 \\ x_3 = 2 + c_2 \\ x_4 = \\ x_5 = c_2 \end{cases} \qquad (c_1, c_2 为任意常数)$$

练习求解 $\begin{cases} x_{1}+&2x_{2}+&x_{3}+&x_{4}+&x_{5}=&1\\ 2x_{1}+&4x_{2}+&3x_{3}+&x_{4}+&x_{5}=&3\\ -x_{1}-&2x_{2}+&x_{3}+&3x_{4}-&3x_{5}=&7\\ &&2x_{3}+&5x_{4}-&2x_{5}=&9 \end{cases}$

解

- 可见 r(A) = r(A : b) = 3 < 5,有无穷多的解,含 5 3 = 2 个自由变量
- 由既约阶梯形矩阵可知。原方程组等价于

$$\begin{cases} x_1 + 2x_2 & + 2x_5 = -2 \\ x_3 & - x_5 = 2 \\ x_4 & = 1 \end{cases} \Rightarrow \begin{cases} x_1 = -2 - 2x_2 - 2x_5 \\ x_3 = 2 + x_5 \\ x_4 = 1 \end{cases}$$

以通解是
$$\begin{cases} x_1 = -2 - 2c_1 - 2c_2 \\ x_2 = c_1 \\ x_3 = 2 + c_2 \\ x_4 = 1 \\ x_5 = c_2 \end{cases} \quad (c_1, c_2 为任意常数)$$



$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 2x_4 = 1 \\ -x_2 + (a-3)x_3 - 2x_4 = b \\ 3x_1 + 2x_2 + x_3 + ax_4 = -1 \end{cases}$$
 有无穷解、唯一解,及无解?

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 2x_4 = 1 \\ -x_2 + (\alpha - 3)x_3 - 2x_4 = b \\ 3x_1 + 2x_2 + x_3 + \alpha x_4 = -1 \end{cases}$$
有无穷解、唯一解,及无解?

$$(A \vdots b) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & a - 2 & 2 & 1 \\ 0 & -1 & a - 3 & -2 & b \\ 3 & 2 & 1 & a - 1 \end{pmatrix}$$

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 2x_4 = 1 \\ -x_2 + (a-3)x_3 - 2x_4 = b \\ 3x_1 + 2x_2 + x_3 + ax_4 = -1 \end{cases}$$
 有无穷解、唯一解,及无解?

$$(A \vdots b) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & a - 3 & -2 & b \\ 3 & 2 & 1 & a - 1 \end{pmatrix} \xrightarrow{r_4 - 3r_1}$$

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 2x_4 = 1 \\ -x_2 + (a-3)x_3 - 2x_4 = b \\ 3x_1 + 2x_2 + x_3 + ax_4 = -1 \end{cases}$$
 有无穷解、唯一解,及无解?

$$(A \vdots b) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & a - \frac{2}{3} & -\frac{2}{3} & | & 1 \\ 0 & -\frac{1}{3} & a - \frac{2}{3} & -\frac{2}{3} & | & b \\ 3 & 2 & a - \frac{2}{3} & -\frac{2}{3} & | & b \end{pmatrix} \xrightarrow{r_4 - 3r_1} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & a - \frac{2}{3} & -\frac{2}{3} & | & 1 \\ 0 & -1 & a - \frac{3}{3} & -\frac{2}{3} & | & b \\ 0 & -1 & a - \frac{3}{3} & -\frac{2}{3} & | & b \end{pmatrix}$$



$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 2x_4 = 1 \\ -x_2 + (a-3)x_3 - 2x_4 = b \\ 3x_1 + 2x_2 + x_3 + ax_4 = -1 \end{cases}$$
 有无穷解、唯一解,及无解?

$$(A \vdots b) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & a - \frac{2}{3} & -\frac{2}{3} & | & b \\ 0 & -1 & a & | & -1 \end{pmatrix} \xrightarrow{r_4 - 3r_1} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 2 & 1 \\ 0 & -1 & a - \frac{3}{3} & -\frac{2}{3} & | & b \\ 0 & -1 & -2 & a - 3 & | & -1 \end{pmatrix}$$

$$r_3+r_2$$



$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 2x_4 = 1 \\ -x_2 + (a-3)x_3 - 2x_4 = b \\ 3x_1 + 2x_2 + x_3 + ax_4 = -1 \end{cases}$$
 有无穷解、唯一解,及无解?

$$(A \vdots b) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & -1 & a - 3 & -2 & b \\ 3 & 2 & a - 1 & a - 1 \end{pmatrix} \xrightarrow{r_4 - 3r_1} \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & -1 & a - 3 & -2 & b \\ 0 & -1 & a - 3 & -2 & a - 3 & -1 \end{pmatrix}$$

$$\xrightarrow[r_4+r_2]{r_3+r_2} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 2 & 1 \\ 0 & 0 & a-1 & 0 & a-1 & b+1 \\ 0 & 0 & 0 & a-1 & 0 \end{pmatrix}$$



$$\begin{cases} x_{1}+&x_{2}+&x_{3}+&x_{4}=&0\\ &x_{2}+&2x_{3}+&2x_{4}=&1\\ &-x_{2}+&(\alpha-3)x_{3}-&2x_{4}=&b\\ 3x_{1}+&2x_{2}+&x_{3}+&\alpha x_{4}=&-1 \end{cases}$$
 有无穷解、唯一解,及无



$$\begin{cases} x_1 + & x_2 + & x_3 + & x_4 = 0 \\ & x_2 + & 2x_3 + & 2x_4 = 1 \\ & -x_2 + & (\alpha - 3)x_3 - & 2x_4 = b \\ 3x_1 + & 2x_2 + & x_3 + & ax_4 = -1 \end{cases}$$
 有无穷解、唯一解,及无

解?

- 当 a ≠ 1 时
- 当 a = 1 时

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 2x_4 = 1 \\ -x_2 + (a-3)x_3 - 2x_4 = b \end{cases}$$
 有无穷解、唯一解,及无 $3x_1 + 2x_2 + x_3 + ax_4 = -1$

$$(A:b) \longrightarrow \begin{pmatrix} \begin{vmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix} b + \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

- 当 a ≠ 1 时
- 当 a = 1 时

$$\begin{cases} x_1 + & x_2 + & x_3 + & x_4 = & 0 \\ & x_2 + & 2x_3 + & 2x_4 = & 1 \\ & -x_2 + & (\alpha - 3)x_3 - & 2x_4 = & b \\ 3x_1 + & 2x_2 + & x_3 + & ax_4 = & -1 \end{cases}$$
 有无穷解、唯一解,及无

解?

$$(A:b) \longrightarrow \begin{pmatrix} \begin{vmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 2 & 1 \\ 0 & 0 & 2a - 1 & 0 & b + 1 \\ 0 & 0 & 0 & 2a - 1 & 0 \end{pmatrix}$$

- 当 $\alpha \neq 1$ 时(b为任意数), r(A) = r(A : b) = 4,
- 当 a = 1 时

$$\begin{cases} x_1 + & x_2 + & x_3 + & x_4 = & 0 \\ & x_2 + & 2x_3 + & 2x_4 = & 1 \\ & -x_2 + & (\alpha - 3)x_3 - & 2x_4 = & b \\ 3x_1 + & 2x_2 + & x_3 + & ax_4 = & -1 \end{cases}$$
 有无穷解、唯一解,及无

$$(A : b) \longrightarrow \begin{pmatrix} \begin{vmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 2 & 1 \\ 0 & 0 & 2a - 1 & 0 & b + 1 \\ 0 & 0 & 0 & 2a - 1 & 0 \end{pmatrix}$$

- 当 α ≠ 1 时(b 为任意数), r(A) = r(A · b) = 4, 有唯一解;
- 当 a = 1 时

$$\begin{cases} x_1 + & x_2 + & x_3 + & x_4 = & 0 \\ & x_2 + & 2x_3 + & 2x_4 = & 1 \\ & -x_2 + & (\alpha - 3)x_3 - & 2x_4 = & b \\ 3x_1 + & 2x_2 + & x_3 + & \alpha x_4 = & -1 \end{cases}$$
 有无穷解、唯一解,及无

解?

- 当 $\alpha \neq 1$ 时(b 为任意数), $r(A) = r(A \cdot b) = 4$,有唯一解;
- 当 a = 1 时

$$\begin{cases} x_1 + & x_2 + & x_3 + & x_4 = & 0 \\ & x_2 + & 2x_3 + & 2x_4 = & 1 \\ & -x_2 + & (\alpha - 3)x_3 - & 2x_4 = & b \\ 3x_1 + & 2x_2 + & x_3 + & ax_4 = & -1 \end{cases}$$
 有无穷解、唯一解,及无

解?

- 当 $\alpha \neq 1$ 时(b 为任意数),r(A) = r(A : b) = 4,有唯一解;
- 当 a = 1 时

$$\begin{cases} x_{1}+ & x_{2}+ & x_{3}+ & x_{4}=0\\ & x_{2}+ & 2x_{3}+ & 2x_{4}=1\\ & -x_{2}+ & (a-3)x_{3}- & 2x_{4}=b \end{cases}$$

$$\begin{cases} 3x_{1}+ & 2x_{2}+ & x_{3}+ & ax_{4}=-1\\ & 0 & 1 & 1 & 1\\ 0 & 0 & a-1 & 0 & a-1\\ 0 & 0 & 0 & a-1 & 0 \end{cases}$$

$$\begin{cases} 1 & 1 & 1 & 1 & 0\\ 0 & 1 & 1 & 1\\ 0 & 0 & a-1 & 0 & a-1\\ 0 & 0 & 0 & a-1 & 0 & a-1\\ 0 & 0 & 0 & 0 & a-1\\ 0 & 0 & 0 & 0 & a-1\\ 0 & 0 & 0 & 0 & a-1\\ 0 & 0 & 0 & 0 & a-1\\ 0 & 0 & 0 & 0 & a-1\\ 0 & 0 & 0 & a-1\\ 0 & 0 & 0 & 0 &$$

$$\begin{pmatrix} 0 & 0 & 0 & a-1 & 0 \end{pmatrix}$$

• 当 $a \neq 1$ 时(b 为任意数), $r(A) = r(A \cdot b) = 4$,有唯一解;

- 当 a = 1 时 $(A:b) \longrightarrow \begin{pmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & b+1 \end{pmatrix}$
- - -a=1, b=-1 时
 - $a = 1, b \neq -1$ 时



$$\begin{cases} x_{1}+ & x_{2}+ & x_{3}+ & x_{4}=0\\ & x_{2}+ & 2x_{3}+ & 2x_{4}=1\\ & -x_{2}+ & (a-3)x_{3}- & 2x_{4}=b \end{cases}$$

$$\begin{cases} 3x_{1}+ & 2x_{2}+ & x_{3}+ & ax_{4}=-1\\ & 0 & 1 & 1 & 1\\ 0 & 0 & a-1 & 0 & a-1\\ 0 & 0 & 0 & a-1 & 0 \end{cases}$$

$$\begin{cases} 1 & 1 & 1 & 1 & 0\\ 0 & 1 & 1 & 1\\ 0 & 0 & a-1 & 0 & a-1\\ 0 & 0 & 0 & a-1 & 0 & a-1\\ 0 & 0 & 0 & 0 & a-1\\ 0 & 0 & 0 & 0 & a-1\\ 0 & 0 & 0 & 0 & a-1\\ 0 & 0 & 0 & 0 & a-1\\ 0 & 0 & 0 & 0 & a-1\\ 0 & 0 & 0 & a-1\\ 0 & 0 & 0 & 0 &$$

$$\begin{pmatrix} 0 & 0 & 0 & a-1 & 0 \end{pmatrix}$$

• 当 $a \neq 1$ 时(b 为任意数), $r(A) = r(A \cdot b) = 4$,有唯一解;

- 当 a = 1 时 $(A:b) \longrightarrow \begin{pmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & b+1 \end{pmatrix}$
- - -a=1, b=-1 时
 - $a = 1, b \neq -1$ 时



$$\begin{cases} x_{1}+ & x_{2}+ & x_{3}+ & x_{4}=0\\ & x_{2}+ & 2x_{3}+ & 2x_{4}=1\\ & -x_{2}+ & (a-3)x_{3}- & 2x_{4}=b \end{cases}$$

$$\begin{cases} -x_{2}+ & (a-3)x_{3}- & 2x_{4}=b\\ 3x_{1}+ & 2x_{2}+ & x_{3}+ & ax_{4}=-1 \end{cases}$$

$$R$$

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 0\\ 0 & 1 & 2 & 0\\ 0 & 0 & a-1 & 0\\ 0 & 0 & a-1 & 0 \end{cases}$$

$$\begin{cases} 1 & 1 & 1 & 0\\ 0 & 1 & a-1\\ 0 & 0 & a-1 & 0\\ 0 & 0 & a-1 & 0 \end{cases}$$

• 当
$$\alpha \neq 1$$
时(b 为任意数), $r(A) = r(A : b) = 4$, 有唯一解;

-
$$a = 1$$
, $b = -1$ 时, $r(A) = r(A : b) = 2 < 4$,

-
$$a = 1, b \neq -1$$
 时



- 当 α ≠ 1 时(b 为任意数), r(A) = r(A · b) = 4, 有唯一解;
- 当 a = 1 时

•
$$\exists a = 1 \text{ FT}$$

$$(A : b) \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & b+1 \\ 0 & 0 & 0 & 0 & b+1 \end{pmatrix}$$

- $-\alpha = 1, b = -1$ 时, r(A) = r(A : b) = 2 < 4, 有无穷多解
- $a = 1, b \neq -1$ 时



$$\begin{cases} x_{1}+ & x_{2}+ & x_{3}+ & x_{4}=0\\ & x_{2}+ & 2x_{3}+ & 2x_{4}=1\\ & -x_{2}+ & (a-3)x_{3}- & 2x_{4}=b \end{cases}$$

$$\begin{cases} -x_{2}+ & (a-3)x_{3}- & 2x_{4}=b\\ 3x_{1}+ & 2x_{2}+ & x_{3}+ & ax_{4}=-1 \end{cases}$$

$$R$$

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 0\\ 0 & 1 & 2 & 0\\ 0 & 0 & a-1 & 0\\ 0 & 0 & a-1 & 0 \end{cases}$$

$$\begin{cases} 1 & 1 & 1 & 0\\ 0 & 1 & a-1\\ 0 & 0 & a-1 & 0\\ 0 & 0 & a-1 & 0 \end{cases}$$

● 当
$$\alpha \neq 1$$
 时 (b 为任意数), $r(A) = r(A : b) = 4$, 有唯一解;

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & b+1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

-
$$a = 1$$
, $b = -1$ 时, $r(A) = r(A : b) = 2 < 4$, 有无穷多解

-
$$a = 1, b \neq -1$$
 时



例 讨论 a, b 取何值时,方程组 $\begin{cases} x_1 + & x_2 + & x_3 + & x_4 = 0 \\ & x_2 + & 2x_3 + & 2x_4 = 1 \\ & -x_2 + & (\alpha - 3)x_3 - & 2x_4 = b \\ 3x_1 + & 2x_2 + & x_3 + & ax_4 = -1 \end{cases}$ 有无穷解、唯一解,及无

- 当 α ≠ 1 时(b 为任意数), r(A) = r(A · b) = 4, 有唯一解;

• 当
$$a = 1$$
 时
$$(A:b) \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & b+1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- $-\alpha = 1, b = -1$ 时, r(A) = r(A : b) = 2 < 4, 有无穷多解
- a = 1, $b \neq -1$ 时, r(A) = 2 < 3 = r(A : b),



$$\begin{cases} x_{1}+ & x_{2}+ & x_{3}+ & x_{4}=0\\ & x_{2}+ & 2x_{3}+ & 2x_{4}=1\\ & -x_{2}+ & (a-3)x_{3}- & 2x_{4}=b \end{cases}$$

$$\begin{cases} 3x_{1}+ & 2x_{2}+ & x_{3}+ & ax_{4}=-1\\ & 0 & 1 & 2 & 2\\ 0 & 0 & a-1 & 0 & a-1\\ 0 & 0 & 0 & a-1 & 0 \end{cases}$$

$$\begin{cases} 1 & 1 & 1 & 1 & 0\\ 0 & 1 & 2 & 2\\ 0 & 0 & a-1 & 0 & a-1\\ 0 & 0 & a-1 & 0 & a-1 \end{cases}$$

● 当
$$\alpha \neq 1$$
时(b 为任意数), $r(A) = r(A : b) = 4$,有唯一解;

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & b+1 \\ 0 & 0 & 0 & 0 & b+1 \end{pmatrix}$$

$$-\alpha = 1$$
, $b = -1$ 时, $r(A) = r(A : b) = 2 < 4$, 有无穷多解

-
$$a = 1$$
, $b \neq -1$ 时, $r(A) = 2 < 3 = r(A : b)$, 无解



例 讨论 a, b 取何值时,方程组 $\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \end{cases}$ 有无穷 $3x_1 + 4x_2 + ax_3 = b$

解、唯一解,及无解?

$$(A \vdots b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \frac{1}{b}$$



$$(A:b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \frac{1}{b} \frac{r_2 - 2r_1}{r_3 - 3r_1}$$



$$\mathbf{(A:b)} = \begin{pmatrix} 1 & 2 & 3 & | & -1 \\ 2 & 3 & 5 & | & -1 \\ 3 & 4 & a & | & -b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 3 & | & 1 \\ & & & & & & & \\ & & & & & & & \\ \end{pmatrix}$$



解、唯一解.及无解?

$$(A:b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \begin{pmatrix} 1 \\ b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$



例 讨论 a, b 取何值时,方程组 $\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \end{cases}$ 有无穷 $3x_1 + 4x_2 + ax_3 = b$

解、唯一解,及无解?

$$(A:b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \begin{pmatrix} 1 \\ b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -1 \\ 0 & -2 & a - 9 \end{pmatrix} \begin{pmatrix} 1 \\ b - 3 \end{pmatrix}$$



解、唯一解.及无解?

$$(A:b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 \\ 0 \\ -2 \\ a - 9 \end{pmatrix} \begin{pmatrix} 3 \\ b - 3 \end{pmatrix}$$

$$r_3 - 2r_2$$

解、唯一解,及无解?

$$(A : b) = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 3 & 5 & -1 \\ 3 & 4 & a & b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & -2 & -3 & -3 \\ 0 & -2 & a - 9 & b - 3 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 & 2 & 3 & -1 \\ 0 & 0 & a - 7 & b + 3 \end{pmatrix}$$

解、唯一解,及无解?

$$(A : b) = \begin{pmatrix} \frac{1}{2} & \frac{2}{3} & \frac{3}{5} & -\frac{1}{b} \\ \frac{2}{3} & \frac{3}{4} & \frac{5}{a} & -\frac{1}{b} \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} \frac{1}{0} & -\frac{2}{1} & -\frac{3}{1} \\ 0 & -\frac{2}{1} & a - \frac{9}{1} & b - \frac{3}{3} \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} \frac{1}{0} & -\frac{2}{1} & -\frac{3}{1} \\ 0 & 0 & a - \frac{7}{1} & b + \frac{3}{3} \end{pmatrix}$$

- 当 a ≠ 7 时
- 当 a = 7 时

例 讨论
$$a$$
, b 取何值时,方程组
$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \\ 3x_1 + 4x_2 + ax_3 = b \end{cases}$$

解、唯一解,及无解?

$$(A:b) = \begin{pmatrix} 1 & 2 & 3 & | & -1 & | & \frac{r_2 - 2r_1}{r_3 - 3r_1} & \begin{pmatrix} 1 & -2 & | & -1 & | & -\frac{3}{3} & | & -\frac{3}{3$$

- 当 α ≠ 7 时
- 当 α = 7 时

例 讨论
$$a$$
, b 取何值时,方程组
$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \\ 3x_1 + 4x_2 + ax_3 = b \end{cases}$$

解、唯一解,及无解?

$$(A:b) = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 3 & 5 & -1 \\ 3 & 4 & a & -1 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & -2 & -3 \\ 0 & -2 & a - 9 \\ -3 & -3 & -3 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -3 \\ 0 & 0 & a - 7 \\ 0 & 0 & a - 7 \end{pmatrix} \xrightarrow{b+3}$$

- 当 α ≠ 7 时(b 为任意数), r(A · b) = r(A) = 3,
- 当 α = 7 时

解、唯一解,及无解?

$$(A : b) = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 3 & 5 & -1 \\ 3 & 4 & a & -1 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & -2 & -3 & -3 \\ 0 & -2 & a - 9 & b - 3 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & a - 7 & b + 3 \end{pmatrix}$$

- 当 $\alpha \neq 7$ 时 (b 为任意数), r(A : b) = r(A) = 3, 有唯一解;
- 当 α = 7 时

例 讨论 a, b 取何值时,方程组 $\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \\ 3x_1 + 4x_2 + ax_3 = b \end{cases}$

解、唯一解,及无解?

$$(A:b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \begin{pmatrix} 1 \\ b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \xrightarrow{r_3 - 3r_1} \begin{pmatrix} 1 \\ 0 \\ -2 \\ a - 9 \end{pmatrix} \begin{pmatrix} -3 \\ b - 3 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ -1 \\ a - 7 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ b + 3 \end{pmatrix}$$

- 当 $\alpha \neq 7$ 时(b为任意数), r(A : b) = r(A) = 3, 有唯一解;
- 当 a = 7 时

例 讨论 a, b 取何值时,方程组 $\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \\ 3x_1 + 4x_2 + ax_3 = b \end{cases}$

解、唯一解,及无解?

$$(A:b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \begin{pmatrix} 1 \\ b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \xrightarrow{r_3 - 3r_1} \begin{pmatrix} 1 \\ 0 \\ -2 \\ a - 9 \end{pmatrix} \begin{pmatrix} -3 \\ b - 3 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ -1 \\ a - 7 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ b + 3 \end{pmatrix}$$

- 当 $\alpha \neq 7$ 时 (b 为任意数), r(A:b) = r(A) = 3, 有唯一解;
- 当 a = 7 时 $(A:b) \longrightarrow \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & b+3 \end{pmatrix}$



例 讨论 a, b 取何值时,方程组 $\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \end{cases}$ 有无穷 $3x_1 + 4x_2 + \alpha x_3 = b$ 解、唯一解. 及无解? $(A:b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & 6 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 - 2 \end{pmatrix} \xrightarrow{r_2 - 2} \begin{pmatrix} 3 \\ 0 \\ -1 \\ 0 - 3 \end{pmatrix}$

$$\xrightarrow{r_3-2r_2} \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & a-7 & b+3 \end{pmatrix}$$
• 当 $a \neq 7$ 时(b 为任意数), $r(A:b) = r(A) = 3$,有唯一解;

- 当 α = 7 时
- $(A:b) \longrightarrow \left(\begin{array}{cc|c} 1 & -\frac{2}{1} & -\frac{3}{1} & -\frac{1}{3} \\ 0 & -\frac{1}{1} & -\frac{1}{3} \end{array} \right)$

-
$$a = 7$$
, $b = -3$ 时

-
$$a = 7, b \neq -3$$
 时



例 讨论 a, b 取何值时,方程组 $\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \\ 3x_1 + 4x_2 + ax_3 = b \end{cases}$

解、唯一解,及无解?

$$(A:b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 \\ 0 \\ -1 \\ -2 \\ a - 9 \end{pmatrix} \begin{pmatrix} 3 \\ b - 3 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ -1 \\ a - 7 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ b + 3 \end{pmatrix}$$

- 当 $a \neq 7$ 时 (b 为任意数), $r(A \cdot b) = r(A) = 3$, 有唯一解;
- $\exists a = 7 \text{ Pf}$ $(A : b) \longrightarrow \begin{pmatrix} \begin{vmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & b + 3 \end{pmatrix}$
 - a = 7, b = -3 时
 - a = 7, $b \neq -3$ 时



例 讨论 a, b 取何值时,方程组 $\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \end{cases}$ 有无穷 $3x_1 + 4x_2 + ax_3 = b$

 $(A:b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 \\ 0 & -1 \\ 0 & -2 & a-9 \\ b-3 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \\ b-3 \end{pmatrix}$

$$\frac{r_3 - 2r_2}{0} \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & a - 7 & b + 3 \end{pmatrix}$$
• 当 $a \neq 7$ 时(b 为任意数), $r(A:b) = r(A) = 3$,有唯一解;

- 当 α = 7 时
- $(A:b) \longrightarrow \begin{pmatrix} 1 & 2 & 3 & -\frac{1}{3} \\ 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{pmatrix}$
 - -a = 7, b = -3 时, r(A : b) = r(A) = 2 < 3,
 - a = 7, $b \neq -3$ 时



例 讨论 a, b 取何值时,方程组 $\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \\ 3x_1 + 4x_2 + ax_3 = b \end{cases}$

解、唯一解,及无解?

$$\mathbf{(A:b)} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \frac{1}{b} \begin{pmatrix} \frac{r_2 - 2r_1}{r_3 - 3r_1} \end{pmatrix} \begin{pmatrix} 1 & -\frac{2}{a} & -\frac{3}{b} & -\frac{3}{a} \\ 0 & -\frac{2}{a} & a - 9 & b - 3 \end{pmatrix} \\
\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 & 2 & 3 & -\frac{3}{b} & -\frac{3}{a} \\ 0 & 0 & a - 7 & b + 3 \end{pmatrix}$$

- 当 $a \neq 7$ 时 (b 为任意数), r(A : b) = r(A) = 3, 有唯一解;
- - -a = 7, b = -3 时, r(A : b) = r(A) = 2 < 3, 有无穷多解
 - $a = 7, b \neq -3$ 时



例 讨论 a, b 取何值时,方程组 $\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \end{cases}$ 有无穷 $3x_1 + 4x_2 + \alpha x_3 = b$ 解、唯一解. 及无解? $(A:b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 \\ 0 & -1 \\ 0 & -2 & a-9 \\ b-3 \end{pmatrix}$

$$\xrightarrow{r_3-2r_2} \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & a-7 & b+3 \end{pmatrix}$$
• 当 $a \neq 7$ 时(b 为任意数), $r(A : b) = r(A) = 3$,有唯一解;

当 a = 7 时

•
$$\exists a = 7 \text{ ft}$$

$$(A \vdots b) \longrightarrow \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & b+3 \end{pmatrix}$$

 $-\alpha = 7, b = -3$ 时,r(A : b) = r(A) = 2 < 3,有无穷多解

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解、唯一解,及无解?

- 当 α ≠ 7 时(b 为任意数), r(A · b) = r(A) = 3, 有唯一解;
- $\stackrel{\text{def}}{=} a = 7 \text{ pt}$ $(A \stackrel{\text{def}}{=} b) \longrightarrow \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & b+3 \end{pmatrix}$
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例 讨论 a, b 取何值时,方程组 $\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \end{cases}$ 有无穷 $3x_1 + 4x_2 + ax_3 = b$ 解、唯一解,及无解?

群、唯一解,及尤解

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例 讨论 a, b 取何值时,方程组 $\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \end{cases}$ 有无穷 $3x_1 + 4x_2 + ax_3 = b$ 解、唯一解,及无解?

胖、唯一胖,及兀胜

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 - $\alpha = 7$, $b \neq -3$ 时, $r(A : b) = 3 \neq 2 = r(A)$, 无解



• 一般线性方程组 $A_{m \times n} x = b$ (m 个方程, n 个未知量)

Ax = b	有无穷解	有唯一解	无解
	r(A) = r(A : b) < n	r(A) = r(A : b) = n	r(A) < r(A : b)

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• 一般线性方程组 $A_{m \times n} x = b$ (m 个方程, n 个未知量)

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	r(A) = r(A : b) < n	r(A) = r(A : b) = n	r(A) < r(A : b)

Ax = 0	有无穷解	有唯一解(零解)
	r(A) < n	r(A) = n

例解齐次线性方程组
$$\begin{cases} x_1-&x_2+&5x_3-&x_4&=0\\ x_1+&x_2-&2x_3+&3x_4&=0\\ 3x_1-&x_2+&8x_3+&x_4&=0\\ x_1+&3x_2-&9x_3+&7x_4&=0 \end{cases}$$

例解齐次线性方程组
$$\begin{cases} x_1- & x_2+ 5x_3- & x_4=0\\ x_1+ & x_2- 2x_3+ 3x_4=0\\ 3x_1- & x_2+ 8x_3+ x_4=0\\ x_1+ 3x_2- 9x_3+ 7x_4=0 \end{cases}$$

$$(A \vdots b) = \begin{pmatrix} 1 & -1 & 3 & -1 & 0 \\ 1 & 1 & -2 & 3 & 0 \\ 3 & -1 & 8 & 1 & 0 \\ 1 & 3 & -9 & 7 & 0 \end{pmatrix}$$



例解齐次线性方程组 $\begin{cases} x_1- & x_2+ 5x_3- & x_4=0\\ x_1+ & x_2- 2x_3+ 3x_4=0\\ 3x_1- & x_2+ 8x_3+ x_4=0\\ x_1+ 3x_2- 9x_3+ 7x_4=0 \end{cases}$

$$(A:b) = \begin{pmatrix} 1 & -1 & 3 & -1 & 0 \\ 1 & 1 & -2 & 3 & 0 \\ 3 & -1 & 8 & 1 & 0 \\ 1 & 3 & -9 & 7 & 0 \end{pmatrix} \xrightarrow[r_3 - 3r_1]{r_3 - 3r_1} \xrightarrow[r_4 - r_1]{r_4 - r_1}$$

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例解齐次线性方程组
$$\begin{cases} x_1- & x_2+ 5x_3- & x_4=0\\ x_1+ & x_2- 2x_3+ 3x_4=0\\ 3x_1- & x_2+ 8x_3+ & x_4=0\\ x_1+ 3x_2- 9x_3+ 7x_4=0 \end{cases}$$



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$$(A:b) = \begin{pmatrix} 1 & -1 & 5 & -1 & 0 \\ 1 & 1 & -2 & 3 & 0 \\ 3 & -1 & 8 & 1 & 0 \\ 1 & 3 & -9 & 7 & 0 \end{pmatrix} \xrightarrow[r_4-r_1]{r_2-r_1} \begin{pmatrix} 1 & -1 & 5 & -1 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 2 & -7 & 4 & 0 \end{pmatrix}$$



例解齐次线性方程组
$$\begin{cases} x_1- & x_2+ 5x_3- x_4=0\\ x_1+ & x_2- 2x_3+ 3x_4=0\\ 3x_1- & x_2+ 8x_3+ x_4=0\\ x_1+ 3x_2- 9x_3+ 7x_4=0 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & -1 & 5 & -1 & 0 \\ 1 & 1 & -2 & 3 & 0 \\ 3 & -1 & 8 & 1 & 0 \\ 1 & 3 & -9 & 7 & 0 \end{pmatrix} \xrightarrow[r_3 - 3r_1]{r_3 - 3r_1} \begin{pmatrix} 1 & -1 & 5 & -1 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 4 & -14 & 8 & 0 \end{pmatrix}$$



例解齐次线性方程组
$$\begin{cases} x_1-&x_2+&5x_3-&x_4&=0\\ x_1+&x_2-&2x_3+&3x_4&=0\\ 3x_1-&x_2+&8x_3+&x_4&=0\\ x_1+&3x_2-&9x_3+&7x_4&=0 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & -1 & 5 & -1 & 0 \\ 1 & 1 & -2 & 3 & 0 \\ 3 & -1 & 8 & 1 & 0 \\ 1 & 3 & -9 & 7 & 0 \end{pmatrix} \xrightarrow[r_4-r_1]{r_2-r_1} \begin{pmatrix} 1 & -1 & 5 & -1 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 4 & -14 & 8 & 0 \end{pmatrix}$$

$$\frac{r_3 - r_2}{r_4 - 2r_2}$$



例解齐次线性方程组
$$\begin{cases} x_1- & x_2+ 5x_3- & x_4=0\\ x_1+ & x_2- 2x_3+ 3x_4=0\\ 3x_1- & x_2+ 8x_3+ & x_4=0\\ x_1+ & 3x_2- 9x_3+ 7x_4=0 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & -1 & 5 & -1 & 0 \\ 1 & 1 & -2 & 3 & 0 \\ 3 & -1 & 8 & 1 & 0 \\ 1 & 3 & -9 & 7 & 0 \end{pmatrix} \xrightarrow[r_4-r_1]{r_2-r_1} \begin{pmatrix} 1 & -1 & 5 & -1 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 4 & -14 & 8 & 0 \end{pmatrix}$$

$$\xrightarrow{r_3-r_2} \begin{pmatrix} 1 & -1 & 5 & -1 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 2 & -7 & 4 & 0 \end{pmatrix}$$

$$\xrightarrow[r_4-2r_2]{r_4-2r_2}
\left(
\begin{array}{ccccc}
1 & -1 & 5 & -1 & 0 \\
0 & 2 & -7 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{array}
\right)$$

例解齐次线性方程组
$$\begin{cases} x_1- & x_2+ 5x_3- & x_4=0\\ x_1+ & x_2- 2x_3+ 3x_4=0\\ 3x_1- & x_2+ 8x_3+ x_4=0\\ x_1+ 3x_2- 9x_3+ 7x_4=0 \end{cases}$$

$$\frac{1}{2} \times r_2$$



例解齐次线性方程组 $\begin{cases} x_1- & x_2+ 5x_3- & x_4=0\\ x_1+ & x_2- 2x_3+ 3x_4=0\\ 3x_1- & x_2+ 8x_3+ x_4=0\\ x_1+ 3x_2- 9x_3+ 7x_4=0 \end{cases}$

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$$\begin{array}{c}
r_{3}-r_{2} \\
\hline
r_{4}-2r_{2}
\end{array}
\begin{pmatrix}
1 & -1 & 5 & -1 & 0 \\
0 & 2 & -7 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$\begin{array}{c}
\frac{1}{2} \times r_{2} \\
\hline
0 & 1 & -7/2 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}$$



例解齐次线性方程组 $\begin{cases} x_1-&x_2+&5x_3-&x_4&=0\\ x_1+&x_2-&2x_3+&3x_4&=0\\ 3x_1-&x_2+&8x_3+&x_4&=0\\ x_1+&3x_2-&9x_3+&7x_4&=0 \end{cases}$

$$(A:b) = \begin{pmatrix} 1 & -1 & 5 & -1 & 0 \\ 1 & 1 & -2 & 3 & 0 \\ 3 & -1 & 8 & 1 & 0 \\ 1 & 3 & -9 & 7 & 0 \end{pmatrix} \xrightarrow[r_4-r_1]{r_2-r_1} \begin{pmatrix} 1 & -1 & 5 & -1 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 4 & -14 & 8 & 0 \end{pmatrix}$$



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例解齐次线性方程组 $\begin{cases} x_1- & x_2+ 5x_3- & x_4=0\\ x_1+ & x_2- 2x_3+ 3x_4=0\\ 3x_1- & x_2+ 8x_3+ & x_4=0\\ x_1+ & 3x_2- 9x_3+ 7x_4=0 \end{cases}$

解

所以原方程组等价于

例解齐次线性方程组 $\begin{cases} x_1-&x_2+&5x_3-&x_4&=0\\ x_1+&x_2-&2x_3+&3x_4&=0\\ 3x_1-&x_2+&8x_3+&x_4&=0\\ x_1+&3x_2-&9x_3+&7x_4&=0 \end{cases}$

解

所以原方程组等价于

$$\begin{cases} x_1 + & \frac{3}{2}x_3 + x_4 = 0 \\ x_2 - \frac{7}{2}x_3 + 2x_4 = 0 \end{cases}$$

例解齐次线性方程组 $\begin{cases} x_1-&x_2+5x_3-x_4=0\\ x_1+&x_2-2x_3+3x_4=0\\ 3x_1-&x_2+8x_3+x_4=0\\ x_1+3x_2-9x_3+7x_4=0 \end{cases}$

解

所以原方程组等价于

$$\begin{cases} x_1 + \frac{3}{2}x_3 + x_4 = 0 \\ x_2 - \frac{7}{2}x_3 + 2x_4 = 0 \end{cases} \iff \begin{cases} x_1 + \frac{3}{2}x_3 - x_4 \\ x_2 = \frac{7}{2}x_3 - 2x_4 \end{cases}$$



例解齐次线性方程组 $\begin{cases} x_1-&x_2+5x_3-x_4=0\\ x_1+&x_2-2x_3+3x_4=0\\ 3x_1-&x_2+8x_3+x_4=0\\ x_1+3x_2-9x_3+7x_4=0 \end{cases}$

所以原方程组等价于
$$\begin{cases} x_1 + \frac{3}{2}x_3 + x_4 = 0 \\ x_2 - \frac{7}{2}x_3 + 2x_4 = 0 \end{cases} \iff \begin{cases} x_1 + \frac{3}{2}x_3 - x_4 \\ x_2 = \frac{7}{2}x_3 - 2x_4 \end{cases}$$

所以 $\begin{cases} x_3 = c_1 \\ x_4 = c_2 \end{cases}$

例解齐次线性方程组 $\begin{cases} x_1-&x_2+&5x_3-&x_4&=0\\ x_1+&x_2-&2x_3+&3x_4&=0\\ 3x_1-&x_2+&8x_3+&x_4&=0\\ x_1+&3x_2-&9x_3+&7x_4&=0 \end{cases}$ 解

所以原方程组等价于
$$\begin{cases} x_1 + \frac{3}{2}x_3 + x_4 = 0 \\ x_2 - \frac{7}{2}x_3 + 2x_4 = 0 \end{cases} \iff \begin{cases} x_1 + \frac{3}{2}x_3 - x_4 \\ x_2 = \frac{7}{2}x_3 - 2x_4 \end{cases}$$

所以 $\begin{cases} x_1 = -\frac{3}{2}c_1 - c_2 \\ x_3 = c_1 \\ x_4 = c_2 \end{cases}$

例解齐次线性方程组 $\begin{cases} x_1-&x_2+&5x_3-&x_4&=0\\ x_1+&x_2-&2x_3+&3x_4&=0\\ 3x_1-&x_2+&8x_3+&x_4&=0\\ x_1+&3x_2-&9x_3+&7x_4&=0 \end{cases}$ 解

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$$\begin{cases} x_1 + \frac{3}{2}x_3 + x_4 = 0 \\ x_2 - \frac{7}{2}x_3 + 2x_4 = 0 \end{cases} \iff \begin{cases} x_1 + \frac{3}{2}x_3 - x_4 \\ x_2 = \frac{7}{2}x_3 - 2x_4 \end{cases}$$

所以 $\begin{cases} x_1 = -\frac{3}{2}c_1 - c_2 \\ x_2 = \frac{7}{2}c_1 - 2c_2 \\ x_3 = c_1 \\ x_4 = c_2 \end{cases}$

例解齐次线性方程组 $\begin{cases} x_1-&x_2+&5x_3-&x_4&=0\\ x_1+&x_2-&2x_3+&3x_4&=0\\ 3x_1-&x_2+&8x_3+&x_4&=0\\ x_1+&3x_2-&9x_3+&7x_4&=0 \end{cases}$ 解

所以原方程组等价于
$$\begin{cases} x_1 + \frac{3}{2}x_3 + x_4 = 0 \\ \Leftrightarrow \end{cases} \begin{cases} x_1 + x_4 = 0 \end{cases}$$

所以原方程组等价于

 $\begin{cases} x_1 + \frac{3}{2}x_3 + x_4 = 0 \\ x_2 - \frac{7}{2}x_3 + 2x_4 = 0 \end{cases} \iff \begin{cases} x_1 + \frac{3}{2}x_3 - x_4 \\ x_2 = \frac{7}{2}x_3 - 2x_4 \end{cases}$

所以 $\begin{cases} x_1 = -\frac{3}{2}c_1 - c_2 \\ x_2 = \frac{7}{2}c_1 - 2c_2 \\ x_3 = c_1 \\ x_4 = c_2 \end{cases}$ (注自由变量个数 = 2 = 4 - r(A))