MAT1865/APM466 Assignment 1
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Fundamental Questions - 25 points

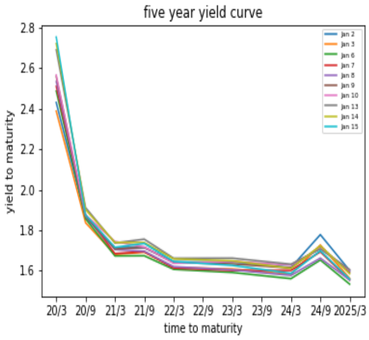
1. (a) The governments issue bonds to raise funds in order to finance some projects.
   (b) The yield curve indicates the health state of the market, if the yield curve is way too high, companies would have difficulties to finance projects, and a low yield curve indicates the financial market is inactive.
   (c) The government can issue government bonds such as pensions or corporate bonds to the market in order to reduce the amount of money circuiting in the market.

2. Chosen bonds: "CAN 1.5 2020-3-1", "CAN 3.5 2020-6-1", "CAN 0.75 2020-9-1", "CAN 0.75 2021-3-1", "CAN 0.75 2021-9-1", "CAN 0.5 2022-3-1", "CAN 1.75 2023-3-1", "CAN 2.5 2024-3-1", "CAN 1.5 2024-9-1", "CAN 1.25 2025-3-1" **Explanation**: Since we start to trace the bond information from Jan 2 and all bonds are paid semi-annully, ideally the first bond matures in 2020-7-1, which is almost half a year from the starting date, then the second one matures in 2021-1-1, and the third one matures in 2021-7-1, etc. But we notice that in these 32 bonds, none of them follow such pattern, but we still need to choose bonds mature half a year in between, so we choose the first bond matures in 2020-3-1, the second one matures in 2020-9-1, and the third one matures in 2021-3-1 etc. If there are two bonds mature in the same date, we choose the one with closer issue date to limit time influence. Since there are no bonds maturing in 2022-9-1 and 2023-9-1, we can only choose nine bonds following the semi-annual maturing pattern, and we still need to add one more bond. We choose the bond which matures in 2020-6-1, simply because such bond has zero coupon.

3. The eigenvalues associated with the covariance matrix of those stochastic processes tell us the characteristics of the yield curve. The eigenvalues tell you how much variance can b e explained by its associated eigenvector. The largest eigenvalue represents the largest variance, and the second largest eigenvalue corresponds to the second largest variance, etc.The largest variance comes from a parallel shift in the curve, the second largest variance comes from a tilt of the curve, and the third largest variance comes from the flexing of the curve. Eigenvectors can help us identify the shifts, tilts, flexing and so on. For example for a yield curve we usually have that the first eigenvector has all components positive (parallel level shift), the second eigenvector has the first half of the components positive and the second half negative (slope tilt), the third eigenvector has the first third of the components positive, second third negative, and the last third positive (flexing). **[1][2]**

Empirical Questions - 75 points

4. (a) **declare variable:** 1)**X**: number of days until maturity as a fraction of a year. 2)**Y**: ytm at X, dependent variable with respect to X. 3)**N**: number of coupon payments remaining. 4) $\mathbf{t_n}$: number of days until n-th coupon as a fraction of a year. 5)F: face value 6)**cp**: clean price 7) **cr**: coupon rate 8)**C**: coupon payment, $C = cr/2$ 9)**dc** = days since last coupon payment as a fraction of a year 10)**ac**: accrued interest 11)**PV**: present value or dirty price of the bond 12) **F**: face value
   Equations: 1) $C = cr/2$. 2)$ac = dc * cr/365$. 3)$PV = cp + ac$. 4) $F = 100 + C$
   5) $PV = \sum_{n=1}^{N} Ce^{-Yt_n} + Fe^{-YX}$
   In **apm466a1.ipynb** file we use *generate_ytm_per_day* function to generate the a list of ytm_n where each ytm_n represent the ytm for bond n during ten days. Since we could calulate PV by adding the clean price and accrued interest as equation 3 illustrates, we could calculate the value of Y on the right hand side in equation 5. We first set $Y = C/100$, then constantly increase or decrease Y until the equilibrium is reached, then such Y value is the ytm value wanted. According to the ytms calculated for the bonds mature in 22/3, 23/3, 24/3, we could linearly estimate the ytms for 22/9, 23/9. After deriving ten *ytm_list* where each list containing the data for each day, we plot a 5-year yield curve superimposed on-top of each other. Below is the plotted graph. [3]



(b) **declare variables:** Similar to the variables in **(a)**, with some extra variables: 1) $\mathbf{Y_{t_n}}$: spot rate at t_n 2) $\mathbf{Y_X}$: spot rate at X, dependent variable with respect to X.

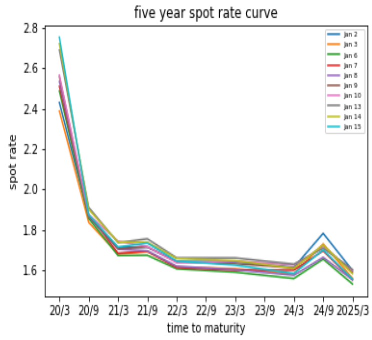
Equations: 1) $C = cr/2$. 2)$ac = dc * cr/365$. 3)$PV = cp + ac$. 4) $F = 100 + C$ 5) $Y_{t_N} = Y_X$

5) $PV = \sum_{n=1}^{N} Ce^{-Y_{t_n} t_n} + Fe^{-Y_X X}$ if the bond matures half year from the previous one, 6)$PV = \sum_{n=1}^{N-1} Ce^{-Y_{t_n} t_n} + Ce^{-\frac{Y_{t_{N-1}}+Y_X}{2} \cdot t_N} + Fe^{-Y_X X}$ if the bond matures one year from the previous one, and we need to do the linear estimation.

We use bootstrapping technique to calculate spot rates. Note for two consecutive bonds if their maturity dates have over half year time difference, use linear estimation. Below is the pseudo code, and the graph for spot rate. We could notice that the spot rate curves are very close to ytm curves only with some minor differences.

Algorithm *calculate_spot_rate(bonds)*

procedure *calculate_spot_rate(bonds)*
  assign empty list as spot_rate_list
  for i in length(bonds): **do**
    if i is 0 **then**
      $Y_X := -\frac{1}{X} \ln \frac{P}{C+F}$
      spot_rate_list.append(Y_X)
    time_difference := bonds[i].maturity_date - bonds[i-1].maturity_date
    if i > 0 **then**
      if time_difference > 0.5 year **then** solve for Y_X according to the equation
        $PV = \sum_{n=1}^{N-1} Ce^{-Y_{t_n} t_n} + Ce^{-\frac{Y_{t_{N-1}}+Y_X}{2} \cdot t_N} + Fe^{-Y_X X}$
        spot_rate_list.append(Y_X)
      if time_difference $\leqslant$ 0.5 year **then** solve for Y_X according to the equation
        $PV = \sum_{n=1}^{N} Ce^{-Y_{t_n} t_n} + Fe^{-Y_X X}$
        spot_rate_list.append($\frac{t_{t_{N-1}}+Y_X}{2}$)
        spot_rate_list.append(Y_X)
  return spot_rate_list



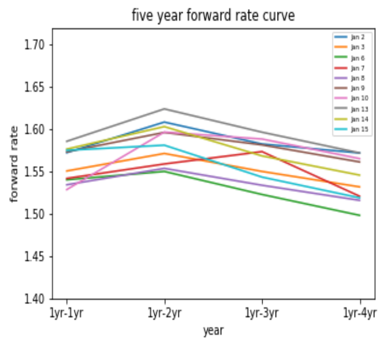
(c) **declare variables:** 1) $\mathbf{r_{0n}}$: the spot rate from year 0 to year n. 2) $\mathbf{f_{1n}}$: the forward rate from year 1 to year n.

Equations: 1)$e^{r_{01} \cdot 1} \cdot e^{f_{1t} \cdot (t-1)} = e^{r_{0t} \cdot t} \to f_{1t} = \frac{r_{0t} \cdot t - r_{01}}{t-1}$.

Use the spot rates we calculated in (b), we can then derive the 1-year forward rate according to the equation calculated above. Below is the pseudo code and labeled graph:

Algorithm *calculate_forward_rate*

procedure *calculate_forward_rate(spot_rates_in_march)*
  assign empty list as forward_rates_list
  r_{01} := spot_rates_in_march[1]
  j := 2
  for spot_rate in spot_rates_in_march[2:] **do**
    forward_rate := (spot_rate * j - r_{01}) / (j-1)
    forward_rate_list.append(forward_rate)
    j := j + 1
  return forward_rate_list



5. **YTM**: We first set up a 10×5 matrix where the each row representing each day, and each column representing each year, as the left matrix represents.

   Next we construct another 9×5 matrix X where each random variable X_i has a time series Xi, j given by $X_{i,j} = \log(r_{i,j+1}/r_{i,j})$, and the result would be like the middle matrix.
   After deriving matrix X, we could calculate the 5×5 covariance matrix by $(X - 1 \cdot \bar{X})^T (X - 1 \cdot \bar{X})/8$. The result is like the right matrix.

$$\begin{bmatrix} 1.8618 & 1.8366 & 1.856 & 1.8768 & 1.8669 & 1.8694 & 1.875 & 1.9112 & 1.9012 & 1.8749 \\ 1.7138 & 1.6911 & 1.6734 & 1.6937 & 1.6956 & 1.715 & 1.7169 & 1.7552 & 1.7384 & 1.7341 \\ 1.64275 & 1.6115 & 1.59865 & 1.60675 & 1.6116 & 1.6366 & 1.6409 & 1.6612 & 1.6531 & 1.63535 \\ 1.62905 & 1.5951 & 1.5756 & 1.60085 & 1.59015 & 1.62295 & 1.6364 & 1.6464 & 1.62975 & 1.606 \\ 1.7777 & 1.7262 & 1.6524 & 1.6928 & 1.6617 & 1.7201 & 1.711 & 1.7067 & 1.7223 & 1.6978 \end{bmatrix} \begin{bmatrix} 0.01362 & -0.0105 & -0.0111 & 0.0052 & -0.0013 & -0.0029 & -0.0191 & 0.0052 & 0.0139 \\ 0.0133 & 0.0105 & -0.012 & -0.0011 & -0.0113 & -0.0011 & -0.0220 & 0.0096 & 0.0024 \\ 0.0192 & 0.0080 & -0.0050 & -0.0030 & -0.0153 & -0.0026 & -0.0122 & 0.0048 & 0.0107 \\ 0.0210 & 0.0123 & -0.0158 & 0.0067 & -0.0204 & -0.0082 & -0.0060 & 0.0101 & 0.0146 \\ 0.0293 & 0.0436 & -0.0241 & 0.0185 & -0.0345 & 0.0053 & 0.0025 & -0.0090 & 0.0143 \end{bmatrix} \begin{bmatrix} 1.30101739e-04 & 8.94766906e-05 & 8.26383412e-05 & 1.00980301e-04 & 5.98365287e-05 \\ 8.94766906e-05 & 1.41972539e-04 & 1.16699723e-04 & 1.40376444e-04 & 1.82737952e-04 \\ 8.26383412e-05 & 1.16699723e-04 & 1.25098642e-04 & 1.46871748e-04 & 1.94325104e-04 \\ 1.00980301e-04 & 1.40376444e-04 & 1.46871748e-04 & 2.13866657e-04 & 2.94514938e-04 \\ 5.98365287e-05 & 1.82737952e-04 & 1.94325104e-04 & 2.94514938e-04 & 6.22666831e-04 \end{bmatrix}$$

forward rate: Similar to the analysis for YTM, first set up a 10×4 matrix as the left matrix, then the next 9×5 time series matrix X as middle, and the covariance matrix on the right.

$$\begin{bmatrix} 1.5723 & 1.5507 & 1.5403 & 1.5419 & 1.5344 & 1.5733 & 1.5287 & 1.5857 & 1.5765 & 1.5751 \\ 1.60845 & 1.5714 & 1.55015 & 1.559 & 1.5538 & 1.5962 & 1.5966 & 1.624 & 1.6031 & 1.58115 \\ 1.5825 & 1.5501 & 1.5229 & 1.5735 & 1.5338 & 1.5815 & 1.5884 & 1.5965 & 1.5685 & 1.5435 \\ 1.5719 & 1.5319 & 1.4983 & 1.5208 & 1.5160 & 1.5613 & 1.5652 & 1.572 & 1.5458 & 1.5189 \end{bmatrix} \begin{bmatrix} 0.0138 & 0.0067 & -0.0010 & 0.0048 & -0.0250 & 0.0287 & -0.0366 & 0.0058 & 0.0008 \\ 0.0233 & 0.0136 & -0.0056 & 0.0033 & -0.0269 & -0.0002 & -0.0170 & 0.0129 & 0.0137 \\ 0.0206 & 0.0177 & -0.0326 & 0.0255 & -0.0305 & -0.0043 & -0.0050 & 0.0176 & 0.0160 \\ 0.0257 & 0.0221 & -0.0149 & 0.0031 & -0.0294 & -0.0025 & -0.0043 & 0.0168 & 0.0175 \end{bmatrix} \begin{bmatrix} 0.00038622 & 0.00022733 & 0.00018891 & 0.00018493 \\ 0.00022733 & 0.00026418 & 0.00028864 & 0.00028249 \\ 0.00018891 & 0.00028864 & 0.00049305 & 0.00036401 \\ 0.00018493 & 0.00028249 & 0.00036401 & 0.00034258 \end{bmatrix}$$

6. **YTM:** For YTM the eigenvalues and their associated eigenvectors are listed as below:
   $9.62056725e^{-04}, [-0.17298505\ -0.64521975\ 0.58958315\ 0.44157945\ 0.10567382]^T$
   $2.04082203e^{-04}, [-0.30906617\ -0.37307159\ -0.71145934\ 0.36791372\ -0.35179652]^T$
   $3.41210452e^{-05}, [-0.31462237\ -0.29324061\ -0.2813533\ -0.36216164\ 0.77762586]^T$
   $2.24011734e^{-05}, [-0.44275925\ -0.22978847\ 0.2130021\ -0.67338569\ -0.50233785]^T$
   $1.10452618e^{-05}, [-0.76126952\ 0.55291615\ 0.1472676\ 0.29161232\ 0.0895933]^T$
   forward rates: For forward rates the eigenvalues and associated eigenvectors are listed as below:
   $1.01908859e^{-03}, [0.13372948\ 0.96477788\ 0.15766259\ -0.16267323]^T$
   $3.85193843e^{-04}, [0.48158546\ 0.13388859\ -0.6962447\ 0.5151627]^T$
   $5.54239753e^{-06}, [0.65938866\ -0.19406148\ -0.11219439\ -0.71760654]^T$
   $6.86986408e^{-05}, [0.56160347\ -0.11669462\ 0.69122951\ 0.43952891]^T$
   The eigenvector that is associated with the largest eigenvalue tell you how much variance can be explained by its associated eigenvector, so it represents the data's largest variation after orthogonal decomposition is in that eigenvector's direction. Therefore, the highest eigenvalue indicates the highest variance in the data was observed in the direction of its eigenvector, and that direction can explain data's $100 * x$ percent of total variation, where x could be calculated as $x = \frac{\lambda_i}{\sum_{i=1}^{n} \lambda_i}$, which is the result of largest associated eigenvalue divided by sum of total eigenvalues.

References and GitHub Link to Code

References

[1] Principal Component Analysis of yield curve change [online]. Available: https://quant.stackexchange.com/questions/36844/principal-component-analysis-of-yield-curve-change [Accessed Jan 2020]
[2] What is percentage of variance in PCA? [online]. Available: https://stats.stackexchange.com/questions/31908/what-is-percentage-of-variance-in-pca [Accessed Jan 2020]
[3] Continuous Compound Interest. [Nov 2019] [online]. Available: https://www.investopedia.com/articles/07/continuously_compound.asp [Accessed Jan 2020]

GitHub Link to Code:
https://github.com/liangc40/APM466Assignments.git

Link to Overleaf Project:
https://www.overleaf.com/read/jzdbxqwyfbpg
https://www.overleaf.com/7283883988brhhkngqhkxb