Cournot Game

Model with one market, finite number of competitors and no information asymmetry.

Starting with 2 oligopoly firms, and expand to n-firms problem.

Consider following constraint model :

Model Setups

- Market Demand

$$\begin{aligned} P &= 1200 - 2Q \\ Q &= \sum_i q_i, \forall i \in \{1,2\} \end{aligned} \tag{1}$$

- **Firm setups**

  Each firm faces its own capacity constraint, which means it cannot produce greater than this upper bound.

$$\begin{aligned} \bar{q}_1 &= 300 \\ \bar{q}_2 &= 100 \end{aligned} \tag{2}$$

  Firms' costs were described by following marginal cost function.

$$\begin{aligned} MC_1 &= 100 \\ MC_2 &= 200 \end{aligned} \tag{3}$$

Optimization Problem

Consider each firm's profit, could formulate 2 profit optimization problem.

$$\begin{gathered} \max_{q_i} \quad \Pi_i = (1200 - 2 \cdot \sum_i q_i)q_i - MC_i \cdot q_i, \qquad \forall i \in \{1,2\} \\ s.t. \\ q_i \leq \overline{q_i} \\ q_i \geq 0 \end{gathered} \tag{4}$$

Could get the First Order Conditon (**FOC**):

$$\partial_{q_i} : 1200 - 4 \cdot q_i - 2 \cdot q_{j \neq i} - MC_i = 0, \quad \forall i \tag{5}$$

But it could not be formulate into *Lagrangian* form due to having two quadratic optimization objective functions.

(*P.S.* CPLEX micqp could dealing with quadratic form of objective function)

Complementarity (**)

Using the same method as Koichiro Ito in *"Sequential Market, Market Power and Arbitrage"*.

FOC could be expressed as complementarity problem.

Then FOC will be a constraint and in this complementarity problem, there is no objective funciton like above. Hence no need to

$$\begin{array}{rlrr}
\text{[FOC 1]} & P - \sum_J q_{ij}/b - \alpha_{ij} - \beta_{ij} q_{ij} - \psi_{ij} \leq 0 & \forall i, j, & \text{(B.1)} \\
\text{[FOC 2]} & P - \sum_J q_{ij}/b - \alpha_{ij} - \beta_{ij} q_{ij} - \psi_{ij} \geq M\underline{u}_{ij} - M & \forall i, j, & \text{(B.2)} \\
\text{[Complementarity]} & \psi_{ij} - M\overline{u}_{ij} \leq 0 & \forall i, j, & \text{(B.3)} \\
\text{[Definition } \underline{\mathbf{u}}\text{]} & q_{ij} - \overline{q}_i \underline{u}_{ij} \leq 0 & \forall i, j, & \text{(B.4)} \\
\text{[Definition } \overline{\mathbf{u}}\text{]} & \overline{q}_i \overline{u}_{ij} - q_{ij} \leq 0 & \forall i, j, & \text{(B.5)} \\
\text{[Sorting 1]} & \overline{u}_{ij} - \underline{u}_{ij} \leq 0 & \forall i, j, & \text{(B.6)} \\
\text{[Sorting 2]} & \underline{u}_{ij} - \underline{u}_{ij-1} \leq 0 & \forall i, j = 2 \ldots J, & \text{(B.7)} \\
\text{[Sorting 3]} & \overline{u}_{ij} - \overline{u}_{ij-1} \leq 0 & \forall i, j = 2 \ldots J, & \text{(B.8)}
\end{array}$$

Upper & Lower Bound Indicators

Define upper and lower bound dummies

$\overline{u}_i = 1$ when reaching maximum capacity

$\underline{u}_i = 1$ when firm is producing

Following are the constraint of bounds under some operations :

$$q_i \leq \overline{q}_i \cdot \underline{u}_i \tag{6}$$

$$\overline{q}_i \overline{u}_i \leq q_i \tag{7}$$

Since when producing lower bound indicator should be 1, otherwise will be 0. Then it follows the Equation (6). Similarily, when reaching capacity q at most equal to capacity. If not reaching capacity q still need to be at least greater than 0.

Hence (6) and (7) provide a decent way to formulate the constraint of q_i

Shadow Value

ψ_i denotes the **shadow value** , and it should be **finite**.

A shadow value is the **profit gap** between **optimal** and **constrained solution**.

Hence a shadow price should satisify following equation.

First define a **M** which is sufficiently large. *(In practice set to 1e6)*

$$\psi_i \leq M \cdot \overline{u}_i \tag{8}$$

Since shadow price only has positive number when capacity is reached.

With this shadow price we can rewrite the first order conditions as above.

CPLEX milp function

A OP problem :

$$\begin{gathered} \min_{q_i} \qquad f'x \\ s.t. \\ Aineq'x \leq bineq \\ Aenq'x = benq \\ \ell b \leq x \\ x \leq ub \end{gathered} \tag{9}$$

$$x = (q_1, q_2, \psi_1, \psi_2, \overline{u}_1, \underline{u}_1, \overline{u}_2, \underline{u}_2)' \tag{10}$$

This function takes following arguments and yeilds the optimal solution vector of X .

```
x = cplexmilp(f, Aineq, bineq, Aeq, beq, sostype, sosind, soswt, lb, ub, ctype)
```

P.S. If there exists no ineq. then set Aineq, bineq to empty column vector [] .

(The manual of this function provided on https://www.ibm.com/support/knowledgecenter/SSSA5P_12.7.1/ilog.odms.cplex.help/refmatlabcplex/html/cplexmilp-m.html)

MATLAB code

Modifying the code written by Pei-Hsuan, Hsiao.

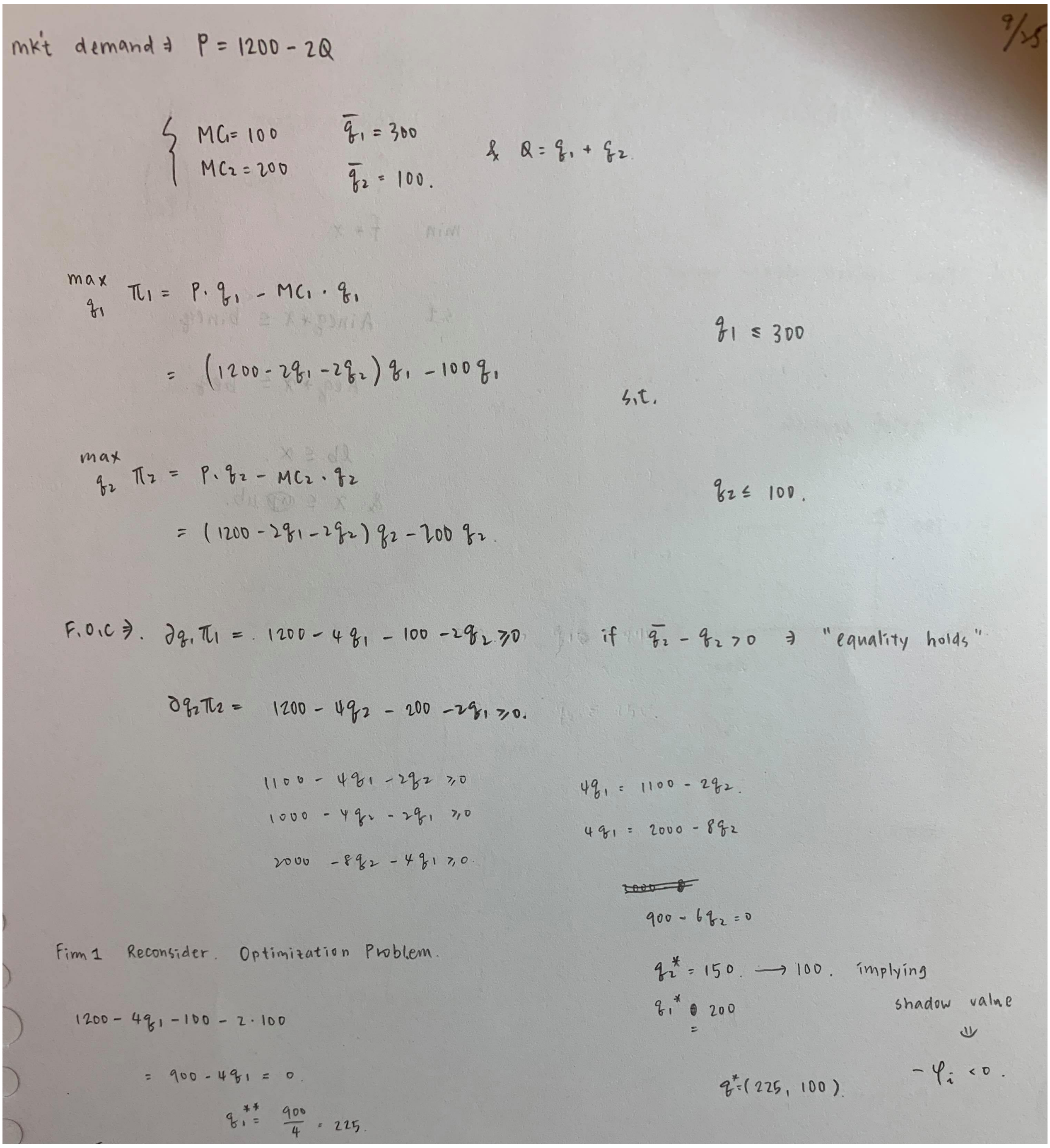
```
function [sol] = getOptimalSolution(P, Nfirm, qbar, MC)
% cplex optimization studio
addpath('/Applications/CPLEX_Studio_Community129/cplex/examples/src/matlab');
addpath('/Applications/CPLEX_Studio_Community129/cplex/matlab/x86-64_osx');
M = 10^6
%% bounds
%lb = zeros(Nvar,1);
lb = rep([0],4*Nfirm);  % lower bounds = 0
%ub = 10000*ones(Nvar,1);
ub = rep([M],4*Nfirm);  % upper bounds < inf

%% types
 % type of the solutions q's and phi's are not restricted, so types are 'C'.
 % u's are resticted to be integers,so type 'I' is needed.
 % so there are (2*number of firms) 'C' and (2* number of firms) of 'I'
ctype = [repmat('C',1,2*Nfirm) repmat('I',1,2*Nfirm)];


%% objective
f = zeros(4*Nfirm,1);  % Since we have no objective function, set all coefficients to 0

%% Matrix of the coefficients of the inequalities
Shape = rep([1],Nfirm);
Aineq1 = kron(Shape,rep([-2 0 0 0],Nfirm)');
for i=1:Nfirm
    Aineq1(i,i) = -4;
    Aineq1(i,i+Nfirm) = -1;
end
Aineq2 = kron(Shape,rep([2 0 0 0],Nfirm)');
for i=1:Nfirm
    Aineq2(i,i) = 4;
```

```
    Aineq2(i,i+Nfirm) = 1;
    Aineq2(i,2*i+2*Nfirm) = M;
end
Aineq3 = kron(Shape,rep([0 0 0 0],Nfirm)');
for i=1:Nfirm
    Aineq3(i,i) = -1;
    Aineq3(i,2*i+2*Nfirm-1) = qbar(i);
end
Aineq4 = kron(Shape,rep([0 0 0 0],Nfirm)');
for i=1:Nfirm
    Aineq4(i,i) = 1;
    Aineq4(i,2*i+2*Nfirm) = -qbar(i);
end
Aineq5 = kron(Shape,rep([0 0 0 0],Nfirm)');
for i=1:Nfirm
    Aineq5(i,i+Nfirm) = 1;
    Aineq5(i,2*i+2*Nfirm-1) = -M;
end
Aineq = [Aineq1; Aineq2; Aineq3; Aineq4; Aineq5];

%% Matrix of the right side of the inequalities,
% note that the inequalities are all of the directions are <=
bineq = rep([0], 5*Nfirm);
for i=1:Nfirm
    bineq(i) = -(P-MC(i));
    bineq(i+Nfirm) = M+(P-MC(i));
end


% equalities - price
Aeq = [];     % NO equations needed
beq = [];     % NO equations needed

%% solution
sol = cplexmilp(f, Aineq, bineq, Aeq, beq, [], [], [], lb, ub, ctype);
end
```

Example

Back to the original 2 firms setups

```
sol = getOptimalSolution(1200, 2, [300, 100], [100, 200])

% output
(225, 100, 0, 150, 0, 1, 1, 1)
```



mk't demand ∋ $P = 1200 - 2Q$ 9/25

$$\begin{cases} MC_1 = 100 & \bar{q}_1 = 300 \\ MC_2 = 200 & \bar{q}_2 = 100. \end{cases} \quad \& \quad Q = q_1 + q_2.$$

$$\max_{q_1} \pi_1 = P \cdot q_1 - MC_1 \cdot q_1 = (1200 - 2q_1 - 2q_2) q_1 - 100 q_1 \quad \text{s.t.} \quad q_1 \le 300$$

$$\max_{q_2} \pi_2 = P \cdot q_2 - MC_2 \cdot q_2 = (1200 - 2q_1 - 2q_2) q_2 - 200 q_2. \quad q_2 \le 100.$$

F.O.C ∋. $\partial_{q_1} \pi_1 = 1200 - 4q_1 - 100 - 2q_2 \ge 0$ if $\bar{q}_2 - q_2 > 0$ ∋ "equality holds"

$\partial_{q_2} \pi_2 = 1200 - 4q_2 - 200 - 2q_1 \ge 0.$

$$1100 - 4q_1 - 2q_2 \ge 0$$
$$1000 - 4q_2 - 2q_1 \ge 0$$
$$2000 - 8q_2 - 4q_1 \ge 0.$$

$$4q_1 = 1100 - 2q_2.$$
$$4q_1 = 2000 - 8q_2$$
$$\cancel{1000 - 8}$$
$$900 - 6q_2 = 0$$

$q_2^* = 150. \longrightarrow 100.$ implying shadow value $\Downarrow$ $-\varphi_i < 0.$

$q_1^* = 200$

$q^* = (225, 100).$

Firm 1 Reconsider. Optimization Problem.

$$1200 - 4q_1 - 100 - 2 \cdot 100 = 900 - 4q_1 = 0.$$

$$q_1^{**} = \frac{900}{4} = 225.$$

Example 2 :

Expand to N = 11, but there are 2 kinds of firms.

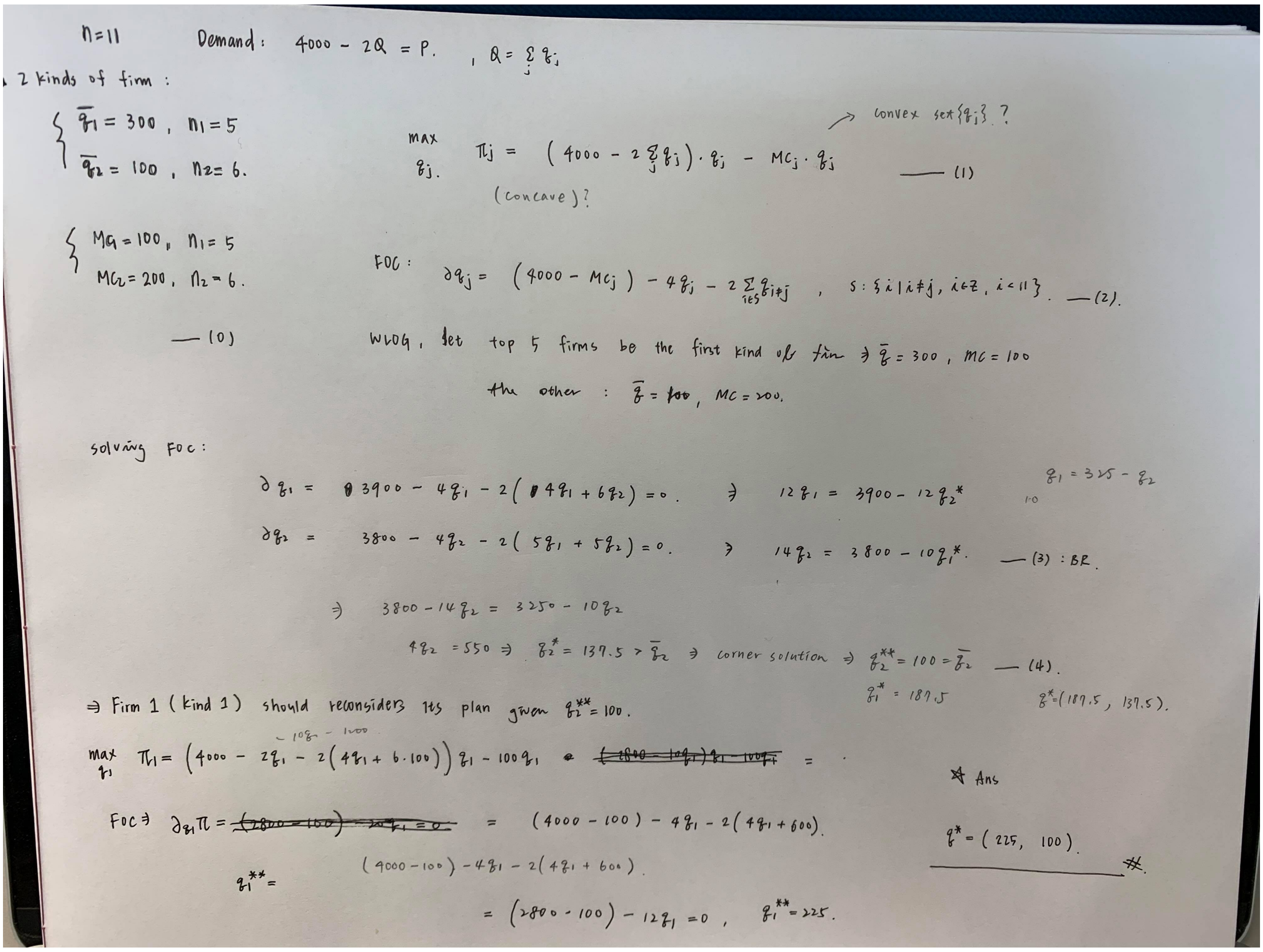
- **Market Demand** : $P = 4000 - 2Q$

- **Kind 1** :

$\overline{q} = 300, \quad mc = 100$

- **Kind 2** :

$\overline{q} = 100, \quad mc = 200$



$n = 11$ Demand: $4000 - 2Q = P$, $Q = \sum_j q_j$

2 kinds of firm:

$\begin{cases} \overline{q}_1 = 300, & n_1 = 5 \\ \overline{q}_2 = 100, & n_2 = 6. \end{cases}$

$\begin{cases} MC_1 = 100, & n_1 = 5 \\ MC_2 = 200, & n_2 = 6. \end{cases}$ — (0)

$\max_{q_j} \; \pi_j = \left(4000 - 2\sum_j q_j\right) \cdot q_j - MC_j \cdot q_j$ — (1)

(concave)? convex set $\{q_j\}$?

FOC: $\partial q_j = (4000 - MC_j) - 4q_j - 2\sum_{i \in S} q_{i \neq j}$, $S: \{i \mid i \neq j, i \in \mathbb{Z}, i < 11\}$. — (2).

WLOG, let top 5 firms be the first kind of firm $\Rightarrow \overline{q} = 300$, $MC = 100$

the other: $\overline{q} = \cancel{100}$, $MC = 200$.

solving FOC:

$\partial q_1 = 3900 - 4q_1 - 2(4q_1 + 6q_2) = 0. \Rightarrow 12q_1 = 3900 - 12q_2^*$ $\quad q_1 = 325 - q_2$

$\partial q_2 = 3800 - 4q_2 - 2(5q_1 + 5q_2) = 0. \Rightarrow 14q_2 = 3800 - 10q_1^*.$ — (3) : BR.

$\Rightarrow 3800 - 14q_2 = 3250 - 10q_2$

$4q_2 = 550 \Rightarrow q_2^* = 137.5 > \overline{q}_2 \Rightarrow$ corner solution $\Rightarrow q_2^{**} = 100 = \overline{q}_2$ — (4).

$q_1^* = 187.5$ $\quad q^* = (187.5, 137.5)$.

$\Rightarrow$ Firm 1 (kind 1) should reconsiders its plan given $q_2^{**} = 100$.

$\max_{q_1} \; \pi_1 = \left(4000 - 2q_1 - 2(4q_1 + 6 \cdot 100)\right) q_1 - 100q_1$ $\cancel{(2800 - 10q_1)q_1 - 100q_1}$ =

FOC $\Rightarrow \partial_{q_1}\pi = \cancel{(2800 - 100) - 20q_1 = 0}$ $= (4000 - 100) - 4q_1 - 2(4q_1 + 600).$

$q_1^{**} =$ $(4000 - 100) - 4q_1 - 2(4q_1 + 600).$

$= (2800 - 100) - 12q_1 = 0, \quad q_1^{**} = 225.$

☆ Ans

$q^* = (225, 100).$ #.

```
qbar = [300 100 300 100 300 100 300 100 100 300 100];
MC   = [100 200 100 200 100 200 100 200 200 100 200];

sol = getOptimalSolution(4000, 11, qbar, MC)

% ouput
sol =
    225
    100
```

```
    225
    100
    225
    100
    225
    100
    100
    225
    100 ... omitted ...
```