

# Discrete Mathematics Lecture Notes (WS18/19)

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## Lecture 1 (10.10.2018)

### Prelude: Motivation

#### What is the aim of the lecture?

Learn basic frameworks used in all areas of mathematics:

- Mathematicians deal with statements
- Usually the statements are about numbers
- The statements may be true or false
- To decide whether a statement is true or false requires a proof
- Use this framework to acquire some knowledge about principles of counting
- Graph theory has a direct application in real world problems
- The basic knowledge about algebraic methods will be used in coding theory

#### Example:

1. 15 is a multiple of 3
2. 20 is a multiple of 3

Theorem: 15 is a multiple of 3

Proof:  $15 = 3 * 5$

## Chapter 1: Principles of Counting

### Basic Counting Problems

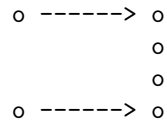
- Permutation:  $\frac{n!}{(n-r)!}$
- Combinations:  $\frac{n!}{(n-r)!r!}$

#### Definition 1.2.1

#### Remarks

Suppose that  $X$  and  $Y$  are sets. We say that we have a function/map from  $X$  to  $Y$  if for each  $x \in X$  we can specify a unique element in  $Y$ , which we denote by  $f(x)$ .

X                      Y



- $f(x)$  is defined  $\forall x \in X$
- these are just one such object  $\forall x \in X$

### Inverse Image Example

Given the function

$$f : \{1, 2, 3\}' \mapsto \{a, b, c, d\}$$

defined by

$$f(x) = \begin{cases} a, & \text{if } x = 1 \\ a, & \text{if } x = 2 \\ c, & \text{if } x = 3 \end{cases}$$

The produced map is:

$$\begin{array}{ll} a & \rightarrow 1 \\ a & \rightarrow 2 \\ b & \\ c & \rightarrow 3 \\ d & \end{array}$$

The image/inverse image of the following sets under  $f$  are:

1. set  $\{2, 3\}$ ; image:  $\{a, c\}$
2. set  $\{a\}$ ; inverse image:  $\{1, 2\}$
3. set  $\{a, b\}$ ; inverse image:  $\{1, 2\}$
4. set  $\{b, d\}$ ; inverse image:  $\emptyset$

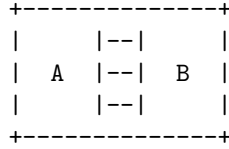
### Definition 1.2.2 Cardinality

A set  $A$  is finite if a bijective mapping  $A \mapsto \{1, \dots, n\}$  exists. (*This means that there is exactly  $n$  number of elements inside set  $A$ .*)

In this case  $n$  is called the **cardinality** of  $A$  and  $A$  has  $|A| := n$  elements.

Two sets  $A, B$  are defined to have the same cardinality if a bijective mapping  $A \mapsto B$  exists.

### Not Disjoint Sets



1.  $A = \{1, 2, 3, 4, 5\}, |A| = 5$
2.  $B = \{3, 4, 5, 6, 7\}, |B| = 5$

$$|A \cup B| \tag{1}$$

$$= |1, 2, 3, 4, 5, 6, 7| \tag{2}$$

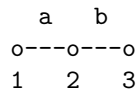
$$= 7 \tag{3}$$

$$\neq |A| + |B| \tag{4}$$

### Counting Sets

1.  $|X \cup Y| = |X| + |Y| - |X \cap Y|$
2.  $|X \cup Y \cup Z| = |X| + |Y| + |Z| + (|X \cap Y| + |X \cap Z| + |Y \cap Z|) - |X \cap Y \cap Z|$

### Double Counting Principle



1.  $N = 1, 2, 3$ , (nodes)
2.  $E = a, b$  (edges)
3.  $R =$  incidence

$$|R| \text{ (over the nodes)}$$

$$= |x \in E|1 \text{ is incident to } x| + |x \in E|2 \text{ is incident to } x| + |x \in E|3 \text{ is incident to } x| \tag{5}$$

$$= |a| + |a, b| + |b| \tag{6}$$

$$= 4 \tag{7}$$

$$|R| \text{ (over the edges)}$$

$$= |x \in N|x \text{ is incident to } a| + |x \in N|x \text{ is incident to } b| \tag{8}$$

$$= |1, 2| + |2, 3| \tag{9}$$

$$= 4 \tag{10}$$

## Lecture 2 (17.10.2018)

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#### Examples

- 1) The first person may choose among 100 seats, the second among 99 etc.  
So we have  $100 * 99 * \dots$

$$\frac{100!}{(100 - 95)!}, (n)_k$$

- 2) Let the pearls be enumerated by 1 to  $n$ . Then we cut the necklace at the part with number 1. So each assignment of pearls is bijectively mapped to an  $n$ -list, where the first element of the list always is the pearl with numbers 1. So there exists  $(n - 1)!$  possibilities

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#### Example

A card game consists of 52 cards:

- Each card has a suit out of {I, II, III, IV}
- Each card has a value out of {2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A}
- 2 cards form a pair, if they have the same value

How many possibilities exist so that we have among 5 arbitrary cards one pair and 3 cards with each the same value (but other than the pair)?

#### Solution

- 1) Choose the value of the pairs (13 possibilities)
- 2) Choose the value of the three cards (12 possibilities)
- 3) Choose the suit of the pair ( $4C2 = 6$  possibilities)
- 4) Choose the suit of the other three cards ( $4C3 = 4$  possibilities)

Therefore, we need the product rule:

$$p = 13 * 12 * C(4, 2) * C(4, 3) \quad (11)$$

$$= 3744 \quad (12)$$

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$$\begin{aligned} & C(n, m_1) * C(n - m_1, m_2) * \dots * C(m_k, m_k) \\ = & \frac{n!}{(n - m_1)!m_1!} * \frac{(n - m_1)!}{(n - m_1 - m_2)!m_2!} * \dots * \frac{m_k!}{m_k!(m_k - m_k!)} \\ & = \frac{n!}{m_1! * \dots * m_k!} \end{aligned}$$

## Lecture 3 (24.10.2018)

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The Stirling numbers  $s_{n,k}$  of first order is the number of permutation of a  $n$ -set with exactly  $k$  cycles

### Theorem 1.8.4

$S(n, 1) = 1$ ,  $S(n, n) = 1$  denotes from set  $n$ , choosing 1 partition or  $n$  partitions results in only one element

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Let  $f, g$  be the permutation

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 5 & 3 \end{pmatrix}$$

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 5 & 1 \end{pmatrix}$$

then

$$f \circ g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 5 & 3 & 2 \end{pmatrix}$$

$$g \circ f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 5 & 1 & 4 \end{pmatrix}$$

This example shows that in general the composition of permutation is not commutative since a permutation is bijective, also the inverse function is a permutation.

Let  $f$  be

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 5 & 3 \end{pmatrix}$$

First we invert, then we order the first row

$$f^{-1} = \begin{pmatrix} 2 & 1 & 4 & 5 & 3 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 3 & 4 \end{pmatrix}$$

$$f \circ f^{-1} = id, f^{-1} \circ f = id$$

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### Example

Let  $f$  be

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 8 & 1 & 5 & 9 & 3 & 7 & 6 \end{pmatrix}$$

We take the cycles without repetition:

- i. 1:  $1 \mapsto 2 \mapsto 4 \mapsto 1$ . The cycle is  $(1, 2, 4)$
- ii. 3:  $3 \mapsto 8 \mapsto 7 \mapsto 3$ . The cycle is  $(3, 8, 7)$
- iii. 5:  $5 \mapsto 5$ . The cycle is  $(5)$
- iv. 6:  $6 \mapsto 9 \mapsto 6$ . The cycle is  $(6, 9)$

The cycle representation of  $f$  is:

$$f = (1, 2, 4) \circ (3, 8, 7) \circ (5) \circ (6, 9)$$

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Let  $f$  be the permutation which describes the change of sorting:

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 1 & 5 & 9 & 2 & 6 & 10 & 3 & 7 & 11 & 4 & 8 & 12 \end{pmatrix} \quad (13)$$

$$= (2, 5, 6, 10, 4) \circ (3, 9, 11, 8, 7) \text{ (starting the cycle at 2)} \quad (14)$$

Since the two cycles have length 5, the cards are back to its original position after 5 procedures.

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Let  $f$  be a composition of transpositions:

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \quad (15)$$

$$= (1, 2, 3, 4) \quad (16)$$

$$= (1, 4) \circ (1, 3) \circ (1, 2) \quad (17)$$

But also adding  $(3, 4)$  and  $(4, 3)$  doesn't change the identity:

$$f = (1, 4) \circ (4, 3) \circ (3, 4) \circ (1, 3) \circ (1, 2) \quad (18)$$

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#### Further Remarks

Generalization of polynomials, where the number of terms is allowed to be infinite. The solution of a combinatorial problem can often be expressed as a sequence  $u_n$ . In such cases it is often appropriate to use methods based on the representation of  $u_n$  as a power series:

$$U(x) = u_0 + u_1x + u_2x^2 + \dots$$

where  $U(x)$  is called the generating function for the sequence  $u_n$

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#### Taylor Series

The n-th Taylor polynomial is defined as:

$$\begin{aligned} T_n f(x, a) &= \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \dots \end{aligned}$$

In the special case where  $a = 0$ , then the Taylor series is called McLaurin series.

$$f(x) = (1+x)^n \quad (19)$$

$$\Rightarrow f^{(k)}(x) \quad (20)$$

$$= (n(1+x)^{n-1})^{(k-1)} \quad (21)$$

$$= (n(n-1)(1+x)^{n-2})^{(k-2)} * \dots \quad (22)$$

$$= n(n-1) * (n(1+x)^{n-1})^{(k-1)} * \dots \quad (23)$$

$$\Rightarrow f^{(k)}(0) = n(n-1)(n-2) * \dots * (n-(k-1)) \quad (24)$$

This is to compute the sequence of coefficients from the generating function. The other way round, given a sequence and then compute the function is easy: sequence

$$\langle f_0, f_1, \dots \rangle = F(x) = f_0x^0 + f_1x^1 + f_2x^2 + \dots$$

$$F(x) = (1+x)^n = \binom{n}{0} + (n,1)x + (n,2)x^2 + \dots$$



can be regarded as saying the the generating function for the sequence defined by  $u_n = (n, k)$  for any given integer  $n$  is  $F(x) = (1 + x)^n$

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Convolution definition

$$c_k = a_0 b_k + a_1 b_{k-1} + \dots + a_k b_0$$

#### Example

Given

$$f(x) = 2 + 3x - 4x^2$$

$$g(x) = 5 - x + x^3$$

$$c_0 = a_0 b_0 = 2 * 5 = 10 \tag{25}$$

$$c_1 = a_0 b_1 + a_1 b_0 = (2 * -1) + (3 * 5) = 13 \tag{26}$$

$$c_2 = a_0 b_2 + a_1 b_1 + a_2 b_0 = (2 * 0) + (3 * -1) + (-4 * 5) = -23 \tag{27}$$

$$f(x) * g(x) = c_0 + c_1 x^1 + c_2 x^2 + \dots \tag{28}$$

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#### Example

Geometrical Series

$$(1 - x) \sum_{k=0}^{\infty} x^k \tag{29}$$

$$= \sum_{k=0}^{\infty} x^k - \sum_{k=0}^{\infty} x^{k+1} \tag{30}$$

$$= 1 + \sum_{k=1}^{\infty} x^k - \sum_{k=1}^{\infty} x^k = 1 \tag{31}$$

So  $(1 - x)$  is inverse to the geometrical series and we get  $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$