Discrete Mathematics Exercise Sheet Answers (WS18/19)

Created by Liang Chun

Exercise Sheet 1

Exercise 1

1. $\{n \in N \mid \text{n is a multiple of 2}\}$
2. $\{n \in N \mid n^2 < 21\}$
3. a. True
   b. False, because $3 * 7 * 13 = 273$
   c. False, because $\sqrt{11279} \notin \mathbb{N}$

Exercise 2

Two things are to prove:

$$C \subseteq D \tag{1}$$

$$D \subseteq C \tag{2}$$

1. Given $c \in C$, we have $c = 6m$ for $m \in \mathbb{N}$.
   i. Rewrite this as $c = 2 * (3m)$
      - $\Rightarrow c$ is a multiple of 2
   ii. Rewrite this as $c = 3 * (2m)$
      - $\Rightarrow c$ is a multiple of 3
   iii. $\therefore c \in D, C \subseteq D$
2. Given $d \in D$, we have $d = 2r$ and $d = 3s$ for $r, s \in \mathbb{N}$. Thus $3s$ is equal to the even number $2r$. If s is odd and $3s$ is odd, then s must be even:
   i. Let $s = 2t \implies d = 3s = 2 * 2 * t = 6 * t$
   ii. $\therefore d \in C, D \subseteq C$

Exercise 3

1. f is surjective since each element occurs at least once as a value of f. f is not injective since the element 4 occurs twice as a value of f. f is not bijective.
2. g is surjective and injective therefore g is also bijective.
3. h is not surjective since the value $d \in X$ is not a value of h. g is injective. h is not bijective.

Exercise 4

Since $j(n)$ is an even number $2n$, an odd integer such as 3 cannot be a value of j. Therefore j is not surjective.

In order to prove that j is injective, we have to show that there cannot be distinct numbers n and n' such that $j(n) = j(n')$.

This would imply that $2n = 2n'$, and cancelling the factor 2, we obtain $n = n'$

Thus there is at most one number n such that $j(n)$ has a given value $\implies$ j is injective.

Exercise 5

Using the pigeon hole principle, suppose that the man has two fictitious boxes, one for grey and one for brown socks. If three socks are drawn, then one such box must contain two socks of the relevant type while drawing fewer does not ensure this.

Exercise 6

1.
   a. Reflexive: No, because $x * x = x^2$ is not always equal to 24
   b. Symmetric: Yes, because $x * y = 24 \Leftrightarrow y * x = 24$
   c. Transitive: No, because $x * y = 24$ and $y * z = 24$, and this does not imply $x * z = 24$
      i. For example: $x = 3$, $y = 8$, $z = 3$, $x*y = 24$, $y*z = 24$, $x*y \neq 24$
2.
   a. Reflexive: Yes, because $x + x = 2x$ is always even
   b. Symmetric: Yes, because $x + y \Leftrightarrow y + x$ holds
   c. Transitive: Yes, because if $x + y = 2a$, $y + z = 2b$, then $x + z = 2(a + b - y)$ is even

Exercise 7

$$R = \{(1, 18), (2, 9), (3, 6), (6, 3), (9, 2), (18, 1)\} \tag{3}$$

Exercise 8

First find out what is important and what is to ignore. The year is irrelevant and January could be replaced by any other month with 31 days. If i and $j \in X$, then evidently i and j represent the same day of the week if and only if $j - i$ is divisible by 7.

1. Reflexive: Yes, because i certainly represents the same day as itself

$$i - i = 0 \text{ is divisible by } 7 \tag{4}$$

2. Symmetric: Yes, if i and j represent the same day of the week, then j and i will do as well

$$\text{if } j - i \text{ is divisible by } 7 \tag{5}$$

$$\text{so } j - i = 7 * m \tag{6}$$

$$\text{and also } i - j = -7m \text{ is divisible by } 7 \tag{7}$$

3. Transitive: Yes if i and j represent the same day, and if j and k represent the same day, then i and k is true

   Let:

$$j - i = 7m \tag{8}$$

$$k - j = 7n \tag{9}$$

   then,

$$\begin{aligned} k - i &= 7n + j + 7m - j && (10) \\ &= 7(n + m) && (11) \end{aligned}$$

Exercise 9

Double counting rule: summarizing over rows = summarizing over columns

	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1
2		1		1		1		1
3			1			1		
4				1				1
5					1			
6						1		
7							1	
8								1

Exercise 10

Following the double counting principle $R \subseteq R * G$ is the relation defined by bRg if boy b knows girl g:

$$
\begin{aligned}
|R| &= 32 * 5 && (12)\\
&= |G| * 8 && (13)\\
&= 20 && (14)
\end{aligned}
$$

Exercise 11

- Without any restrictions: $26^4 = 456976$ possibilities
- Without "b": $25^4 = 390652$ possibilities

Gaining a feeling for properties of numbers: A change from 26 to 25, $\frac{1}{25} \approx 4\%$ causes a decrease of roughly 14.5%

Exercise 12

- $i, j, k > 0$ otherwise a pasty would hold an absolute majority
- It holds $k = 151 - i - j$, so in fact we only have two variables
- If $i = 1 \Rightarrow (j, k) = (75, 75)$
- If $i = 2 \Rightarrow (j, k) = (75, 74), (74, 75)$
- If $i = 3 \Rightarrow (j, k) = (75, 73), (74, 74), (73, 75)$
- If $i = 75 \Rightarrow (j, k) = (75, 1), (74, 2), ..., (1, 75)$

The combined probability using the sum rule:

$$
\sum_{i=1}^{n} i = \frac{n(n-1)}{2}, \quad n \in \mathbb{N} \tag{15}
$$

$$
\sum_{i=1}^{75} i = \frac{75 * 76}{2} = 2850 \tag{16}
$$

Exercise 13

a. Initially invite the 3 particular friends. This can only be done in 1 way. Now you need to invite $3 = 6 - 3$ friend from the remaining $7 = 10 - 3$. This can be done in $C(7, 3) = 35$ ways.
b. Remove 3 friends which leaves 7 friends remaining, then choose 6 out of the 7 ($C(7, 6) = C(7, 1) = 7$ ways)

Exercise 14

a. We have in total $n = 10 + 4 = 14$ people, because we have a round table, there are 13! possible arrangements.

b. Since all the 4 girls sit together, we can group them into together and consider them as a single girl. Then $n = 10 + 1 = 11$ is the total number of people. The people can be seated at a round table in 10! ways. The girls can be arranged among themselves in 4! ways. In total we have $10!4! = 87091200$ possibilities.

c. Hence, let's initially arrange the 10 boys: 9! possibilities. Now there are 10 positions left to place the 4 girls so that no two girls sit together. In addition, we have to take into consideration the possibilities to order the girls. $\Rightarrow 9! * C(10, 4) * 4!$

Exercise 15

If the elements of X are listed $X = \{x_1, ..., x_m\}$ then there are two choices for $f(x_1)$, two choices for $f(x_2)$ and so on. Moreover the choices are independent in the sense that for $i \neq j$, the choice for x_i does not affect and is not affected by the choice for x_j. $\Rightarrow |F| = 2^m$

Exercise 16

Let Y be a set with 8 elements. The set X of subsets of Y has $2^8 = 256$ elements due to lemma 1.3.2. The set W of subsets of X has $2^{256} \approx 1.157 \times 10^7$ elements.

Exercise 17

The expression on the left hand side is the sum of the numbers of r-subsets of an n-set taken over all possible values of r, hence the left hand side is equal to the total number of subsets of an n-set according to lemma 1.3.2, this is 2^n.

Exercise 18

$$A \hat{=} \text{ students that can read French (47)} \tag{17}$$
$$B \hat{=} \text{ students that can read German (35)} \tag{18}$$
$$C \hat{=} \text{ students that can read Russian (20)} \tag{19}$$

a. By using corollary 1.2.6, we know the number of students that can read French or German or both as $|A \cup B| = |A| + |B| - |A \cap B| = 47 + 35 - 23 = 59$.
$\Rightarrow$ Number of students that can read neither language: $67 - 59 = 8$

b.

$$
\begin{aligned}
|A \cup B \cup C| &= |A| + |B| + |C| + (|A \cap B| + |B \cap C| + |B \cap C|) + |A \cap B \cap C| && (20)\\
&= 47 + 35 + 20 - 23 - 12 - 11 + 5 && (21)\\
&= 61 && (22)
\end{aligned}
$$

Probelm 22

The Exercise can be formulated as : find the number of the surjections from a set of 11 places to the set $\{M, I, S, P\}$ such that 4 of the places are mapped to I, 4 to S, 2 to P, and 1 to M.

Multinomial number:

$$\left(\begin{array}{c} 11 \\ 4{,}4{,}2{,}1 \end{array} \right) = \frac{11!}{4! \times 4! \times 2! \times 1!} \tag{23}$$

Exercise 23

$$
\begin{aligned}
\sigma &= (1, 6, 3, 4, 2, 5) && (24)\\
&= (1, 5) \circ (1, 2) \circ (1, 4) \circ (1, 3) \circ (1, 6) && (25)\\
&= (1, 6) \circ (6, 3) \circ (3, 4) \circ (4, 2) \circ (2, 5) && (26)
\end{aligned}
$$

Counters Example:

$$(1, 5) \circ (1, 2) \circ (1, 4) \circ (1, 6) \circ (1, 3), \quad 1 \mapsto 3 \tag{27}$$

Disjoint: τ only moves elements that σ fixed and vice versa.

$\Rightarrow$ The used permutations are not disjoint in both representations, so they must not commute (commutative).

Not disjoint, but commutative:

$$\sigma = \left(\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{array} \right) \quad \tau = \left(\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{array} \right) \tag{28}$$

To check if they are commutative ($\sigma \circ \tau = \tau \circ \sigma$):

$$\sigma \circ \tau = \left(\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{array} \right) \quad \tau \circ \sigma = \left(\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{array} \right) \tag{29}$$

Examples for commutative permutations:

$$\sigma \circ id = id \circ \sigma = \sigma \tag{30}$$
$$\sigma^{-1} \circ \sigma = \sigma \circ \sigma^{-1} = id \tag{31}$$

Exercise 24

Compute a partial fraction decomposition of the following rational function:

$$\frac{x+2}{x^6 + x^4 - x^2 - 1} \tag{32}$$

The decomposition of the denominator: Use 1 as a 0.

```
 (x^6 + x^4 - x^2 - 1):(x - 1) = x^5 + x^4 + 2x^3 + 2x^2 + 1
-(x^6 - x^5)
        x^5 + x^4
       -x^5 + x^4
             2x^4
            -2x^4 + 2x^3
                   2x^3 - x^2
                  -2x^3 + 2x^2
                        x^2
                       -x^2 + x
                              x - 1
                             -x + 1
                                  0
```

-1 is a zero of the remaining the polynomial:

```
(x^5 + x^4 + 2x^3 + 2x^2 + x + 1):(x + 1) = x^4 + 2x^2 + 1
-x^5 - x^4
             2x^3 + 2x^2
            -2x^3 - 2x^2
                          x + 1
                         -x - 1
                              0
```

We can rewrite:

$$x^4 + 2x^2 + 1 = (x^2+1)^2 \tag{33}$$

into:

$$\Rightarrow x^6 + x^4 - x^2 - 1 = (x-1)(x+1)(x^2+1)^2 \tag{34}$$
$$\Rightarrow r(x) = \frac{x+2}{x^6 + x^4 - x^2 - 1} = \frac{A}{(x-1)} + \frac{A}{(x+1)} + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2} \tag{35}$$
$$\begin{aligned}\Leftrightarrow x + 2 = {} & A(x+1)(x^2+1)^2 + B(x-1)(x^2+1)^2 \\ & + (Cx+D)(x-1)(x+1)(x^2+1) + (Ex+F)(x-1)(x+1)\end{aligned} \tag{36}$$

Comparision of coeffecients:

$$x^0 : 1 = A - B - D - F \tag{37}$$
$$x^1 : 0 = A + B - C - E \tag{38}$$
$$x^2 : 0 = 2A - 2B + F \tag{39}$$
$$x^3 : 0 = 2A + 2B + E \tag{40}$$
$$x^4 : 0 = A - B + D \tag{41}$$
$$x^5 : 0 = A + B + C \tag{42}$$

Use Gaussian Elimination to compuite $A - F$

$$r(x) = \frac{\frac{3}{8}}{x-1} + \frac{-\frac{3}{8}}{x+1} + \frac{-\frac{1}{4}x - \frac{1}{2}}{x^2+1} + \frac{-\frac{1}{2}}{(x^2+1)^2} \tag{43}$$

Exercise 25

Let $U(x)$ be the generating function for the sequence u_n. Using the initial values of u_0, u_1, and the equation for u_{n+2}, $n \geq 0$, we have:

$$U(x) = u_0 + u_1x^1 + u_2x^2 + u_3x^3 + ... \tag{44}$$
$$= 0 + x + \sum_{n=2}^{\infty} u_n x^n \tag{45}$$
$$= x + \sum_{n=0}^{\infty} u_{n+2} x^{n+2} \tag{46}$$
$$= x + \sum_{n=0}^{\infty} (5u_{n+1} - 6u_n) x^{n+2} \tag{47}$$
$$= x + 5x\left(\sum_{n=0}^{\infty} u_n x^n - u_0x^0\right) - 6x^2\left(\sum_{n=0}^{\infty} u_n x^n\right) \tag{48}$$
$$U(x) = x + 5xU(x) - 6x^2U(x) \tag{49}$$
$$\Rightarrow U(x) = \frac{x}{1 - 5x + 6x^2} \tag{50}$$

(46) We need to split the sums so that the indexes are equal to it's exponents
(47) Move $5x$ out, and subtract by u_0x^0 since we decreased the index

The main idea now is to expand the right hand side as a formal power series.. To do this we factorize the denominators:

$$1-5x+6x^2 = (1-\alpha x)(1-\beta x) \tag{51}$$
$$1-5\frac{1}{y}+6\frac{1}{y^2} = (1-\frac{\alpha}{y})(1-\frac{\beta}{y}) \quad (\text{sub } x=\frac{1}{y}) \tag{52}$$
$$y^2-5y+6 = (y-\alpha)(y-\beta) \quad (\text{mult } y^2) \tag{53}$$
$$y_{1,2} = \frac{5}{2} \pm \sqrt{\left(\frac{5}{2}\right)^2-6} = \frac{5}{2} \pm \frac{1}{2} \tag{54}$$
$$y_1 = \alpha = 3, \quad y_2 = \beta = 2 \tag{55}$$
$$\Rightarrow 1-5x+6x^2 = (1-3x)(1-2x) \tag{56}$$

Decompose into partial fractions:

$$\frac{x}{1-5x+6x^2} = \frac{A}{1-3x} + \frac{B}{1-2x} \tag{57}$$
$$x = A(1-2x)+B(1-3x) \tag{58}$$

Now we need to compare the coefficients:

$$0 = A+B, \quad 1 = -2A-3B \tag{59}$$
$$A = -B, \quad 1 = 2B-3B = -1B \tag{60}$$
$$A = 1, \quad B = -1 \tag{61}$$

Substituting A, and B into U(x):

$$U(x) = \frac{1}{1-3x} - \frac{1}{1-2x} \tag{62}$$
$$= \sum_{n=0}^{\infty}(3x)^n - \sum_{n=0}^{\infty}(2x)^n \tag{63}$$
$$= \sum_{n=0}^{\infty}(3^n-2^n)x^n \tag{64}$$
$$\tag{65}$$

Therefore, the generating function is:

$$\Rightarrow u_n = 3^n - 2^n \tag{66}$$

Exercise 26

a. $p_n = 5a_n$

$$P(x) = p_0 + p_1x + p_2x^2 + ... \tag{67}$$
$$= 5a_0 + 5a_1x + 5a_2x^2 + ... \tag{68}$$
$$= 5A(x) \tag{69}$$

b. $q_n = 5 + a_n$

$$P(x) = q_0 + q_1x + q_2x^2 + ... \tag{70}$$
$$= (a_0 + 5) + (a_1 + 5)x + (a_2 + 5)x^2 + ... \tag{71}$$
$$= a_0 + a_1 + a_2x^2 + ... + 5 + 5x + 5x^2 + ... \tag{72}$$
$$= A(x) + \frac{5}{1-x} \tag{73}$$

c. $r_n = a_{n+5}$

$$R(x) = r_0 + r_1x + r_2x^2 + ... \tag{74}$$
$$= a_5 + r_6x + r_7x^2 + ... \tag{75}$$
$$= \frac{\overbrace{(a_0 + a_1x + a_2x^2 + ...)}^{A(x)} - (a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4)}{x^5} \tag{76}$$

d. $h_n = \begin{cases} a_{n-5}, & \text{if } n \geq 5 \\ 0, & \text{else} \end{cases}$

$$H(x) = h_0 + h_1x + h_2x^2 + ... \tag{77}$$
$$= 0 + 0 + 0 + 0 + 0 + a_0x^5 + a_1x^6 + ... \tag{78}$$
$$= x^5(a_0 + a_1x + a_2x^2 + ..) \tag{79}$$
$$= x^5A(x) \tag{80}$$

Exercise 27

1. We already know that $(1, 1, 1, 1, 1, 1, ...)$ has the generating function

$$\frac{1}{1-x} \tag{81}$$

2. Use the substitution $2x$ for $x \Rightarrow (1, 2, 4, 8, ...)$ with the generating function

$$\frac{1}{1-2x} \tag{82}$$

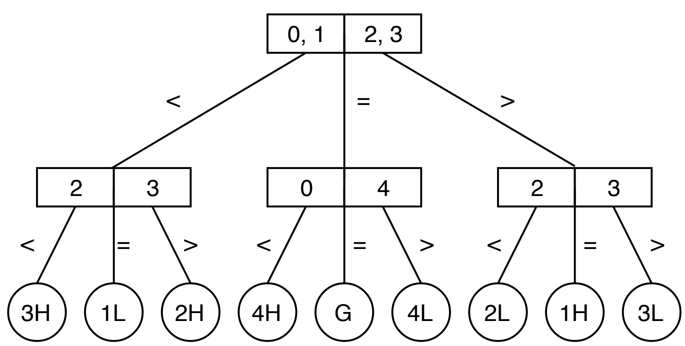
3. Use the substitution x^2 for $x \Rightarrow (1, 0, 2, 0, 4, 0, ...)$ with the generating function
$$\frac{1}{1 - 2x^2} \tag{83}$$

4. Now we need to shift all the elements in the sequence one place to the right $\Rightarrow (0, 1, 0, 2, 0, 4, ...)$ by multiplying it by x:
$$\frac{x}{1 - 2x^2} \tag{84}$$

5. Now we add the two sequences from (3) and (4) to get our generating function
$$\frac{1}{1 - 2x^2} + \frac{x}{1 - 2x^2} \tag{85}$$

Exercise 28

1. There are $2 * 4 + 1$ final outcomes: $G, 1H, 1L, 2H, 2L, 3H, 3L, 5H, 4L$
2. The decision tree is ternary, since these are three possible results of each decision.



Exercise Sheet 2

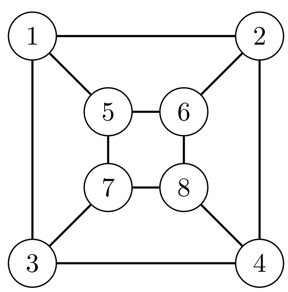
Exercise 1

The graphs are not isomorphic, although:

- they have the same number of vertices
- all nodes have degree 3
- they have the same number of edges

The first graph has two 3-cycles. If it was isomorphic to the second graph, that would have 3-cycles too, but there are none.

Exercise 2



$V = \{000, 001, 010, 011, 100, 101, 110, 111\}$ maps to $\{1, 2, 3, 4, 5, 6, 7, 8\}$

Exercise 3

Both graphs are isomorphic. TODO:

Exercise 4

Prove that:

$$\forall n \in \mathbb{N}, \quad \sum_{r=1}^{n} 4r + 3 = 2n^2 + 5n \tag{86}$$

Induction basis: The statement is true when $n = 1$

$$\sum_{r=1}^{1} 4r + 3 = 4 * 1 + 3 = 7 = 2 * 1^2 + 5 * 1 \tag{87}$$

Induction assumption: Suppose it is true for the value k,

$$\sum_{r=1}^{k} 4r + 3 = 2r^2 + 5r \tag{88}$$

Induction step: We have to deduce that for the value $k + 1$

$$\sum_{r=1}^{k+1} 4r + 3 = \sum_{r=1}^{k} 4r + 3 + [4(k + 1) + 3] \tag{89}$$

Using the assumption this is:

$$2k^2 + 5k + 4(k + 1) + 3 = 2k^2 + 9k + 7 = 2k^2 + 4k + 2 + 5k + 5 \tag{90}$$
$$= 2(k + 1)^2 + 5(k + 1) \tag{91}$$

$\Rightarrow$ Induction basis and the induction step are true

$\Rightarrow$ This means the formula is true $\forall n \in \mathbb{N}$

Exercise 5

$n =$	1	2	3	4	5	6
$S_n =$	1	9	36	100	225	441

Conjecture:

$$\sum_{r=1}^{n} r^3 = \left(\frac{n(n + 1)}{2}\right)^2 \tag{92}$$

Induction basis:

$$\sum_{r=1}^{1} r^3 = \left(\frac{1(1 + 1)}{2}\right)^2 \tag{93}$$

Induction assumption:

$$\sum_{r=1}^{k} r^3 = \left(\frac{k(k + 1)}{2}\right)^2 \tag{94}$$

Induction step:

$$\sum_{r=1}^{k+1} r^3 = \sum_{r=1}^{k} r^3 + (k+1)^3 \tag{95}$$

$$= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 \tag{96}$$

$$= (k+1)^2 \left(\frac{k^2}{4} + (k+1)\right) \tag{97}$$

$$= (k+1)^2 \left(\frac{k^2+4k+4}{4}\right) \tag{98}$$

$$= \left(\frac{(k+1)^2(k+2)^2}{2^2}\right) \tag{99}$$

$$= \left(\frac{(k+1)(k+2)}{2}\right) \tag{100}$$

$\Rightarrow$ Induction basis and the induction step are true

$\Rightarrow$ This means the formula is true $\forall n \in \mathbb{N}$