Chap1-2: Outline

Distinguish between two kinds of infinite sets, the enumerable and the nonenumerable (可枚举集和不可枚举集)

- The concepts
- Examples of enumerable sets
- Examples of nonenumerable sets

An intuitive definition

An enumerable, or countable set is one whose elements can be enumerated: arranged in a single list.

Example

- The set $\mathbf{Z}^+$ of positive integers: $1, 2, 3, 4, \ldots$
- The set of integers: $0, -1, 1, -2, 2, \ldots$

Some remarks on the intuitive definition

- The list may be empty, finite or unending
- The following list of integers is not acceptable

$$-1, -2, -3, \ldots, 0, 1, 2, 3, \ldots$$

  So each element of the set must appear as the nth entry, for some $n \in \mathbf{Z}^+$
- It is ok if some members show up more than once on the list, *e.g.*,

$$\mathbf{Z}^+ : 1, 1, 2, 2, 3, 3, \ldots$$

- It is also ok if no element shows up on some positions of the list, *e.g.*

$$\mathbf{Z}^+ : 1, -, 2, -, 3, -, \ldots$$

Terminology regarding functions

- Let A and B be nonempty sets. A function from A to B is an assignment of exactly one element of B to each element of A.
- A partial function from A to B is an assignment of at most one element of B to each element of A. For $a \in A$, if no element of B is assigned to a, we say that $f(a)$ is undefined, denoted by $f(a) = \perp$.
- The domain of f is the set of elements a such that $f(a)$ is defined.
- If $f(a)$ is defined and $f(a) = b$, we say that b is the image of a. The range of f is the set of all images of elements of A.

Terminology regarding functions (2)

- A function f is said to be one-to-one, or injective if $\forall a \forall b(f(a) = f(b) \to a = b)$. A function is said to be an injection if it is one-to-one.
- A function f from A to B is called onto, or surjective if $\forall b \in B \exists a \in A(f(a) = b)$. A function is said to be an surjection if it is onto.
- A function f is a one-to-one correspondence, or a bijection if it is both one-to-one and onto.
- Two sets A and B are equinumerous (等势的) if there is a bijection between A and B

The formal definition

Definition

A set is enumerable if it is the range of some partial function from $\mathbf{Z}^+$.

Proposition.

The following are equivalent:

- A set A is enumerable
- There exists a surjection from $\mathbf{Z}^+$ to A
- There exists an injection from A to $\mathbf{Z}^+$
- There exists a bijection between A and a subset of $\mathbf{Z}^+$.

Proof: Exercise

Examples

- The empty set, any finite set
- The set of negative integers
- The set of even positive integers
- Any subset of positive integers
- The set of integers

Example: $\mathbf{Z}^+ \times \mathbf{Z}^+$

Different ways of proving that it is enumerable

A pairing function is an injection from $\mathbf{Z}^+ \times \mathbf{Z}^+$ to $\mathbf{Z}^+$

Method 1: Cantor's pairing function

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

$$J(m,n) = \frac{(m+n-2)(m+n-1)}{2} + m$$

Example: $\mathbf{Z}^+ \times \mathbf{Z}^+$ (2)

Method 2

- Imagine we have a hotel with an enumerable infinity of rooms
- Each day we have an enumerable infinity of guests
- How should we accommodate all the guests?
  - In Day 1, we put each guest in every other room
  - In Day 2, we put each guest in every other remaining room
  - ...
- What is the pairing function?
  - $j(1, n) = 2n - 1$, $2n$'s remain
  - $j(2, n) = 2(2n - 1)$, $4n$'s remain,
  - $j(3, n) = 4(2n - 1)$, $8n$'s remain
  - $j(m, n) = 2^{m-1}(2n - 1)$

Example: $\mathbf{Z}^+ \times \mathbf{Z}^+$ (3)

Method 3

- $f(m,n) = 2^m 3^n$

More examples

- The set of positive rational numbers
- The set of ordered k-triples of positive integers, for any fixed k
- The set of finite sequences of positive integers
  - Method 1: $f((a_1, \ldots, a_n)) = J(n, J_n((a_1, \ldots, a_n))$
  - Method 2: $f((a_1, \ldots, a_n)) = p_1^{a_1} \cdot \ldots \cdot p_n^{a_n}$
- The set of finite sets of positive integers
- any subset of an enumerable set
- the union of any two enumerable sets
- The set of finite strings from an enumerable alphabet of symbols

Examples of nonenumerable sets

Cantor's diagonal argument

Theorem

The set of functions from $\mathbf{Z}^+$ to the set $\{0, 1\}$ is not enumerable.

	1	2	3	4	$\dots$
s_1	$s_1(1)$	$s_1(2)$	$s_1(3)$	$s_1(4)$	$\dots$
s_2	$s_2(1)$	$s_2(2)$	$s_2(3)$	$s_2(4)$	$\dots$
s_3	$s_3(1)$	$s_3(2)$	$s_3(3)$	$s_3(4)$	$\dots$
s_4	$s_4(1)$	$s_4(2)$	$s_4(3)$	$s_4(4)$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Examples of nonenumerable sets

Cantor's Theorem

The set of all sets of positive integers is not enumerable.

Proof: diagonal argument

Corollary

The set of real numbers is not enumerable.

Proof: We construct an injection from $P(\mathbf{Z}^+)$ to $(0,1)$:
$f(S) = 0.a_1 a_2 \ldots a_n \ldots$, where $a_n = 1$ if $n \in S$, and 2 otherwise.

A more general result

Theorem

For any set S, there does not exist a surjection from S to $P(S)$.

Some terminology

- Theorem: important general result
- Lemmas: less important results on the way to a theorem
- Corollaries: directly follow from a theorem
- Propositions: free-standing less important results