Methods for Linear Systems of Circuit Delay Differential Equations of Neutral Type

Alfredo Bellen, Nicola Guglielmi, and Albert E. Ruehli

***Abstract*—Delay differential equations (DDE's) occur in many different fields including circuit theory. Circuits which include delayed elements have become more important due to the increase in performance of VLSI systems. The two types of circuits which include elements with delay are transmission lines and partial element equivalent circuits. The solution of systems which include these circuit elements are performed with solvers similar to conventional ODE circuits simulators. Since DDE solvers are more fragile with respect to stability, we investigate the conditions for contractivity and determine sufficient conditions for the asymptotic stability of the zero solution by utilizing a suitable reformulation of the system.**

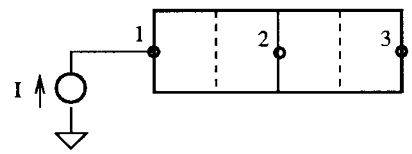
I. Introduction

Delay differential equations (DDE's) are assuming an increasingly important role in many disciplines like engineering, science and mathematics. A comprehensive introduction to the subject of DDE's and solvers is given in some review papers, e.g., [1]. Circuits which include delayed elements have become more important due to the increase in performance of VLSI systems. The two types of circuits which include elements with delay are transmission lines (TL) [2] and partial element equivalent circuits (PEEC's) [3]. Some of these problems result in so-called neutral DDE's or NDDE's. The solution of systems which include these circuit elements can be performed with circuit solvers similar to the ones for conventional ODE equations. However, DDE solvers require new features like history files and interpolation for the delayed elements. Since DDE solver are more fragile with respect to stability, we need to investigate the conditions for contractivity and determine sufficient conditions for the asymptotic stability of the zero solution. Here, we utilize a suitable reformulation of the system. Also, the numerical integration of these system requires the use of different numerical methods. In this paper, we attempt to give theoretical answers to some questions of importance like conditions on the system matrices for convergence and contractivity, interpolation accuracy and the suitability of certain integration methods.

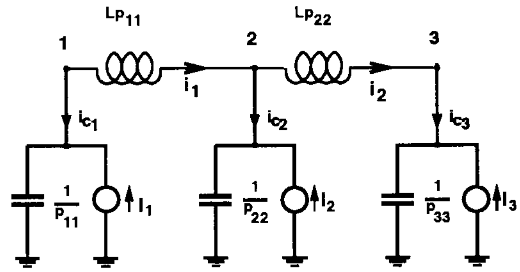
We motivate this work with a small test circuit example which consists of a PEEC circuit [4] shown in Fig. 1(b). This circuit represents a full wave equivalent circuit for the small metal strip which is discretized in two cells as shown in Fig. 1(a).

The PEEC model in Fig. 1(b) includes new circuit elements which involve retarded mutual coupling between the partial inductances of the form $Lp_{ij}i'_j(t-\tau)$ and retarded dependent current sources of the form $p_{ij}/p_{ii}i_{c_j}(t-\tau)$. The general form of the condensed MNA NDDE equations is given by

$$\begin{cases} C_0 y'(t) + G_0 y(t) + C_1 y'(t-\tau) \\ \quad + G_1 y(t-\tau) = Bu(t, t-\tau), & t \geq t_0 \\ y(t) = g(t), & t \leq t_0 \end{cases} \tag{1}$$



(a)



(b)

Fig. 1. (a) Metal strip with two L_p cells (three capacitive cells dashed) and (b) small PEEC model for metal strip.

where C_0 is a diagonal matrix. To be consistent with the mathematical notation we write the NDDE as

$$\begin{cases} y'(t) = Ly(t) + My(t-\tau) + Ny'(t-\tau), & t \geq t_0 \\ y(t) = g(t), & t \leq t_0 \end{cases} \tag{2}$$

where all the matrices L, M, and N as well as the initial vector g are real-valued. The delay τ is a positive constant and t_0 is the starting time. Due to the delay, we need to specify the initial conditions in terms of a history $g(t)$ where $(t_0-\tau) \leq t \leq t_0$. We will investigate the stability conditions on the system (2) such that $\lim_{t\to\infty} \|y(t)\| = 0$ for all initial functions $g(t)$, where $\|\cdot\|$ is a given norm in $\mathbb{R}^m$. We are also interested in the more restrictive property of contractivity of $y(t)$, which helps us in the stability analysis. In particular, in the first part of this paper, we examine sufficient conditions for contractivity and asymptotic stability of the true solutions of (2). In the second part, we investigate both the convergence of the numerical solution using Runge–Kutta methods and their contractivity and asymptotic stability properties. Several researchers have studied the asymptotic behavior of the exact solutions, e.g., Li [5] as well as the numerical solutions, e.g., Hu and Mitsui [6].

In this paper we use an approach based on the contractivity properties of the solutions of (2) and we extend the contractivity analysis to the numerical solution. Similar techniques were introduced for nonneutral nonlinear delay differential equations by Torelli [7] and have been further developed by Bellen and Zennaro [8]. Concerning the asymptotic properties of the solutions, sufficient conditions for the asymptotic stability were determined by techniques based on Lyapunov functionals, e.g., [9].

In this paper the basic solution approaches used are given in Sections II and III. Sections IV and V are devoted to numerical methods and issues. In Section VI we give a results for the small example systems. For brevity, we did not include the proofs of the new theorem. However, the proofs are given in [10].

II. Solution Methods of Delay Differential Equation

We consider two issues which are fundamental for the solution of the NDDE (2). The first one is based on the observation that the delay introduces a set of consecutive intervals, for which the solution

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A. Bellen is with the Dipartimento di Scienze Matematiche, Università di Trieste, I-34100 Trieste, Italy.

N. Guglielmi is with the Dipartimento di Matematica Pura Applicata, Università dell' Aquila, via Vetoio, L'Aquila, Italy.

A. E. Ruehli is with the IBM T. J. Watson Research Center, Yorktown Heights, NY 10598 USA.

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retains the smoothness of initial data. This points to the fact that the initial data $g(t)$ should be continuous. The second one consists of a suitable transformation of the system.

We first describe the step by step approach which is used for the integration of (2). To accomplish this, we have to determine the break points of the solution $y(t)$, which are points associated with discontinuities in the derivative of $y(t)$ due to the functional argument $\alpha(t) := t - \tau$. The break points for the constant delay case considered here are given by $\xi_n = t_0 + n\tau$, where $n = 0, 1 \cdots$. We use the break points to define the following intervals: the initial interval is $I_0 := (t_0 - \tau, \xi_0]$ and the consecutive intervals are $I_n := [\xi_{n-1}, \xi_n]$, $n \ge 1$. To treat the general case, we separately consider the solution on every interval I_n. Then we relate the solution in I_n to the previous interval I_{n-1} as will be shown below. Next, we consider the issue of the approximate numerical solution of the NDDE equation (2) by transforming it into a system of ordinary differential equations (ODE's) coupled with an algebraic recursive system. Then, we make use of the numerical solution techniques for ODE's. The classical idea for treating (2) is to set up the problem in the general form $y'(t) = F(t, y(t), \eta(t-\tau), \zeta(t-\tau))$, $t \in I_n$, where $\eta(s)$ and $\zeta(s)$ are approximations of $y(s)$ and $y'(s)$, respectively. These approximations are obtained with the same numerical method in the previous time interval I_{n-1}. Note that $\zeta(s)$ is the approximation of the derivative waveform $y'(t-\tau)$ which is a difficult task. For this reason we propose a formulation of the problem in the next section which does not require the approximation of the derivative of the numerical solution. After introducing the step by step integration we describe the transformation of the NDDE into an ODE. We will utilize this in both the mathematical analysis as well as in the practical numerical solution. We first rewrite the system (2) as follows:

$$\begin{cases} y'(t) - Ly(t) = N(y'(t-\tau) - Ly(t-\tau)) & \\ \quad +(M+NL)\, y(t-\tau), & t \ge t_0 \\ y(t) = g(t), & t \le t_0. \end{cases} \tag{3}$$

By setting $\Phi(t) := y'(t) - Ly(t)$, the system (3) is equivalent to

$$\begin{cases} y'(t) = Ly(t) + \Phi(t), \quad t \ge t_0, \\ y(t_0) = g(t_0) \end{cases} \tag{4}$$

where

$$\Phi(t) = \begin{cases} N\Phi(t-\tau) + (M+NL)\, y(t-\tau), & t \ge \xi_1, \\ Mg(t-\tau) + Ng'(t-\tau), & t_0 \le t < \xi_1 \end{cases} \tag{5}$$

which is in the form of an algebraic recursion. This transformed formulation has several advantages over the original form (2). It is evident that the new form is an ordinary differential equation (ODE) which involves an algebraic recursion. Specifically, the interesting aspect of the new system is the forcing term Φ, since the contractivity with respect to Φ is crucial for the stability of y.

III. Stability and Contractivity of the Solution

We are interested in the conditions for stability and contractivity of the model described by (2) since this is a fundamental issue which should be understood before attempting the numerical solution of the problem. Essentially, a stable numerical solution should be based on a stable model. Understanding which properties the matrices L, M and N have to fulfill is of key importance. We study the more restrictive set of conditions where stability is sought for the delay independent case. However, important properties are shared with the delay dependent stability case.

A. Stability with Respect to Forcing Term

In Section II, the NDDE has been tranformed to the system (4) and (5). This has the advantage that properties of the NDDE can now be studied by considering the corresponding ODE (4) for $g(t) = y_0$ in terms of the forcing term $\Phi(t)$. This problem has been analyzed before and we can utilize a recent result obtained by Zennaro [11], which is summarized by the following.

Theorem 3.1: Consider the system (4), where the components of the forcing term $\Phi(t)$ are assumed to be continuous functions. Given an inner product $\langle \cdot, \cdot \rangle$ in $\mathbb{R}^m$ and the corresponding norm $\| \cdot \|$, let $\mu[\cdot]$ be the logarithmic norm[1] induced by $\langle \cdot, \cdot \rangle$. If $\mu[L] < 0$, then the following inequality holds

$$\|y(t)\| \le E(t_0, t)\|y_0\| + (1 - E(t_0, t)) \sup_{t_0 \le x \le t} \left(\frac{\|\Phi(x)\|}{-\mu[L]} \right), \quad \forall\, t \ge t_0 \tag{6}$$

where $E(t_1, t_2) := \exp(-\mu[L](t_2 - t_1))$. With this, we are in a position to state the following theorem concerning the contractivity properties of the solutions of (2).

Theorem 3.2: For the linear constant coefficient system (2), the condition

$$\frac{\|M + NL\|}{-\mu[L]} + \|N\| \le 1 \tag{7}$$

implies the contractivity property:

$$\|y(t)\| \le \max\left\{ \kappa_0,\ \frac{\kappa_1}{-\mu[L]} \right\} \quad \forall\, t \ge t_0 \tag{8}$$

where $\kappa_0 = \sup_{s \le t_0} \|g(s)\|$, $\kappa_1 = \sup_{s \le t_0} \|Mg(s) + Ng'(s)\|$ for every initial function $g(t)$ and for every positive constant delay τ.

Another interesting result can be derived from this theorem by an additional condition on N. This is summarized in the remark.

Remark 3.1: Assume also that $\|N\| < 1$. As a consequence of Theorem 3.2, we have that

$$\|y'(t)\| \le \mathcal{A} + \|N\|\, \|y'(t-\tau)\|$$

where $\mathcal{A}$ is the following constant, depending on the initial data $g(s)$ and $g'(s)$, $s \le t_0$:

$$\mathcal{A} = (\|L\| + \|M\|) \max\left\{ \kappa_0,\ \frac{\kappa_1}{-\mu[L]} \right\}.$$

We define the $(|||\cdot|||)$ norm as $|||w|||_n := \max_{s \in [\zeta_{n-1}, \zeta_n]} \|w(s)\|$. Hence, for $n \ge 1$,

$$\begin{aligned} |||y'|||_{n+1} &\le \mathcal{A} + \|N\|\, |||y'|||_n, \\ &\le \|N\|^{n+1} \sup_{s \le t_0} \left\|g'(s)\right\| + \left(\sum_{j=0}^{n} \|N\|^j \right) \mathcal{A} \\ &\le \sup_{s \le t_0} \|g'(s)\| + \frac{1}{1 - \|N\|} \mathcal{A} \end{aligned}$$

which gives the boundedness of the derivative of the solution, with respect to the initial data g and g'.

By slightly strengthening the assumption (7), we can prove a theorem, which gives a better insight into the asymptotic behavior of the true solutions of (2).

[1] Some authors call $\mu[\cdot]$ the matrix measure (see, for example, [12]).

Theorem 3.3: For the linear constant coefficient model problem (2), the assumptions

$$\begin{cases} \|N\| < 1, \\ \dfrac{\|M + NL\|}{-\mu[L]} \le k(1 - \|N\|), \quad k < 1 \end{cases} \tag{9}$$

imply the exponential asymptotic stability of the solution $y(t)$.

Two other sufficient conditions for the asymptotic stability are available. The first one is due to Li and the second one is due to Hu and Mitsui.

Condition 3.1 [5]:

$$\begin{cases} \|N\| < 1, \\ \dfrac{\|M\| + \|N\|\|L\|}{-\mu[L]} < (1 - \|N\|). \end{cases} \tag{10}$$

Condition 3.2 [6]:

$$\mu[L] + \|M\| + \frac{\|NL\| + \|NM\|}{1 - \|N\|} < 0. \tag{11}$$

It is easy to prove that (9) is sharper than (10). However, the comparison with (11) is more difficult to interpret. In particular, we found examples for which (9) is sharper than (11), while for some cases the opposite is true. Specifically, if we restrict the problem to the *scalar* case, we found that (9) is better than (11). We need to emphasize that this analysis provides only sufficient conditions for asymptotic stability. We will end this section with a few observations. For some applications the condition $\mu[L] < 0$ is not fulfilled but the solutions are stable, at least for some values of the delay. Other interesting cases are characterized by the property that $\mu[L] = 0$. Determining the weakest conditions on L, M and N such that (2) is stable for the assigned delay τ is still an open problem.

IV. Numerical Solution

We introduce a class of integration methods appropriate for the numerical solution which are in the form of s-stages Runge–Kutta (RK) methods with continuous extensions. In general, functional delay equations are solved by an adaptation of the RK methods for ordinary differential equations, e.g., [13]. For the solution of the general ODE in the form $y'(t) = f(t, y(t))$, with initial condition $y(t_0) = y_0$ we consider the s-stage RK method

$$\begin{cases} Y_{k+1}^i = y_k + h_{k+1} \displaystyle\sum_{j=1}^{s} a_{ij} f(t_{k+1}^j, Y_{k+1}^j), \quad i = 1, \cdots, s, \\ y_{k+1} = y_k + h_{k+1} \displaystyle\sum_{i=1}^{s} w_i f(t_{k+1}^i, Y_{k+1}^i), \end{cases} \tag{12}$$

where $t_{k+1}^j := t_k + c_j h_{k+1}$, $c_i := \sum_{j=1}^{s} a_{ij}$, $i = 1, \cdots, s$ with $h_{k+1} := t_{k+1} - t_k$.

The s-stage RK method, characterized by the abscissas c_i, the weights w_i and the parameters a_{ij}, can be represented by a Butcher tableau which is given in its general form on the left.

$$\begin{array}{c|c} \mathbf{c} & \mathbf{A} \\ \hline & \mathbf{w}^T \end{array}, \quad \begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array}, \quad \begin{array}{c|cc} 0 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}, \quad \begin{array}{c|cc} 0 & \frac{1}{2} & -\frac{1}{2} \\ 1 & \frac{1}{2} & \frac{1}{2} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

We also give three specific Butcher tableaus for the methods which are of interest for ODE solvers and also for NDE's. The first specific tableau from the left is for the backward Euler (BE) method while the next tableaux in the middle and on the right are for the trapezoidal (TR) and the Lobatto III-C methods, respectively. To solve DDE's we need appropriate interpolations especially for the time history waveforms. A continuous extension of the s-stage RK method (12) is obtained by considering the interpolation

$$\eta(t_k + \theta h_{k+1}) = y_k + h_{k+1} \sum_{i=1}^{s} w_i(\theta)\, f(t_{k+1}^i, Y_{k+1}^i), \quad 0 \le \theta \le 1 \tag{13}$$

where $w_i(\theta)$, $i = 1, \cdots, s$, are polynomials of given degree d such that $w_i(0) = 0$ and $w_i(1) = w_i$. As an example, the continuous extension with $w_i(\theta) = \theta w_i$ turns out to be given by the linear interpolant $\eta(t_k + \theta h_{k+1}) = (1 - \theta) y_k + \theta y_{k+1}$ for $0 \le \theta \le 1$. An important class of methods, for which a continuous extension can be given in a natural way, is that of collocation methods, where $w_j(c_i) = a_{ij}$ and thus $\eta(t_n + c_j h_{n+1}) = Y_{n+1}^j$. In particular, Gaussian collocation methods for solving DDE's have been studied by Bellen [14], who proved nodal super-convergence properties on particularly constrained meshes where $t_n - \tau = t_{n-m}$, for m integer and for all $hm = \tau$.

Next the numerical solution of the system (2) is of interest. We will do this by utilizing the step by step integration of the transformations (4)–(5) given in Section II. The recursive step by step approach is applied to each interval $I_n = [\xi_{n-1}, \xi_n]$. The approximate solution $y(t)$, which is regular inside this interval is extended into the next interval after the numerical solution has been computed in the present interval. Specifically, after computing the continuous approximate solution $\eta(t)$ of $y(t)$, in the interval I_{n-1} $(n > 1)$, then the solution is computed in the next interval I_n by solving $y'(t) = Ly(t) + \overline{\Phi}(t)$ and $y(\xi_{j-1}) = \eta(\xi_{j-1})$, using a continuous RK method on the mesh $\Delta_n = t_{n,0} \equiv \xi_{n-1} < t_{n,1} < \cdots < t_{n,N_n} \equiv \xi_n$, with $N_n + 1$ points in I_n. During the process, the function $\overline{\Phi}(t)$ on the interval I_j is simultaneously evaluated by the following scheme which we call *Direct Evaluation* (DE) where have

$$\overline{\Phi}(t) = \begin{cases} N\overline{\Phi}(t - \tau) + (M + NL)\,\eta(t - \tau), \\ \qquad t \in [\xi_{n-1}, \xi_n], & n \ge 2 \\ Mg(t - \tau) + Ng'(t - \tau), & t < \xi_1. \end{cases} \tag{14}$$

We should note that this scheme requires, for the computation of $\overline{\Phi}$ at some point $t \in I_n$, to trace back the recursion until the initial interval. In the interval I_0, $\overline{\Phi} = \Phi$ which is known from the known initial condition functions g and g'.

We assume that the continuous Runge–Kutta method (12), (13) has uniform global order p on any finite interval $[a, b]$. Therefore, the initial value problem $y'(t) = f(t, y(t))$, $y(a) = y_a$, has a continuous numerical solution $\eta(t)$ satisfying

$$\max_{t \in [a, b]} \|y(t) - \eta(t)\| = \mathcal{O}(h^p). \tag{15}$$

Remark 4.1: RK methods of order $p = 1$ or 2 with linear interpolation provide a continuous RK method with uniform global order p.

For more details about this result and the attainable uniform order of a discrete RK method of order p, we refer to [15]. Here, we give the following result on the order of the proposed numerical schemes for the solution of the system (4) and (5).

Theorem 4.1: Consider the DE-scheme for the solution of (4) and (5). If we use a continuous RK method of uniform global order p for the numerical solution using the DE-scheme on each interval I_n, then the numerical solution has uniform global order p.

V. Stability and Contractivity Properties of the Numerical DE Scheme

In this section, we search for continuous RK methods which reproduce the behavior of the actual solutions. This is accomplished by investigating the contractivity properties of the DE-scheme. The above DE numerical scheme is based on the recursive solution of ODE's. Hence, we are interested in finding RK-methods which are contractive with respect to the usual test equation, $y'(t) = Ly(t) + F(t)$ with $y(t_k) = y_k$. In this section we utilize a definition of contractivity and cite results given in [8].

Definition 5.1: A continuous RK method is said to be *contractive* if the "continuous" numerical solution $\eta(t)$ satisfies

$$\max_{0\le\theta\le1} \|\eta(t_k+\theta h)\| \le \max\left\{\|y_k\|,\ \max_{1\le j\le s} \frac{\|F(t_k+c_jh)\|}{-\mu[L]}\right\} \tag{16}$$

whenever $\mu[L] < 0$ and for any step size $h > 0$.

Contractivity results have been obtained before in [8]. Specifically, it has been proven that the backward Euler ($p = 1$) and the 2-stage Lobatto III-C ($p = 2$) using linear interpolation are contractive according to Definition 5.1. The following theorem is relating to the contractivity for the DE scheme.

Theorem 5.1: Consider the DE-scheme for the numerical solution of (4) and (5) using a contractive continuous RK method. If (7) holds, then the numerical solution η satisfies the contractivity property

$$\|\eta(t)\| \le \max\left\{\kappa_0,\ \frac{\kappa_1}{-\mu[L]}\right\} \quad \forall t \ge t_0$$

for any mesh Δ (κ_0 and κ_1 being defined as in Theorem 3.2).

In the last theorem we consider the asymptotic stability of the numerical DE-scheme.

Theorem 5.2: We apply the DE-scheme to solve (4) and (5), and we use a contractive continuous RK method for the solution. Then, the numerical solution η asymptotically vanishes for any mesh Δ, if the conditions (9) hold.

We finally give the following summary for the Lobatto III-C integration method since it is particularly suitable for the solution of NDDE's.

Corollary 5.1: The Lobatto III-C method with linear interpolation for (13) provides a numerical solution which is accurate to order 2 for the DE scheme for (4) and (5). Further, the solution is contractive (asymptotically stable) if the contractivity condition (7) (asymptotic stability condition (9)) is fulfilled.

VI. Examples

We give an examples for the application of the theoretical results to a realistic NDDE problem which is motivated by the small PEEC model in Fig. 1. Of immediate interest for delay problems is the stability issue. The matrices for our example are

$$\frac{L}{100} = \begin{bmatrix} -7 & 1 & 2 \\ 3 & -9 & 0 \\ 1 & 2 & -6 \end{bmatrix}$$
$$\frac{M}{100} = \begin{bmatrix} 1 & 0 & -3 \\ -0.5 & -0.5 & -1 \\ -0.5 & -1.5 & 0 \end{bmatrix}$$
$$N = \frac{1}{72}\begin{bmatrix} -1 & 5 & 2 \\ 4 & 0 & 3 \\ -2 & 4 & 1 \end{bmatrix}. \tag{17}$$

and the continuous initial history is given by $g(t) = [\sin(t)\ \ \sin(2t)\ \ \sin(3t)]$. This example which is contractive as well as asymptotically stable is used for a comparison between numerical integration methods. Specifically, we compare the behavior of the contractive 2-stage Lobatto III-C method with the trapezoidal rule, which is one of the most popular methods used in conventional circuit solvers like Spice. It is well known that the solution for the scalar case can be stable even for positive values of $\mu[L]$. We choose $\tau = 1$ for the problem which is *moderately stiff.* It satisfies the conditions (9), which are $\mu_2[L] = -407.6 < 0$, $\|N\|_2 = 0.09 < 1$,

$$\frac{\|M+NL\|_2}{-\mu_2[L]} + \|N\|_2 = 0.992 < 1.$$

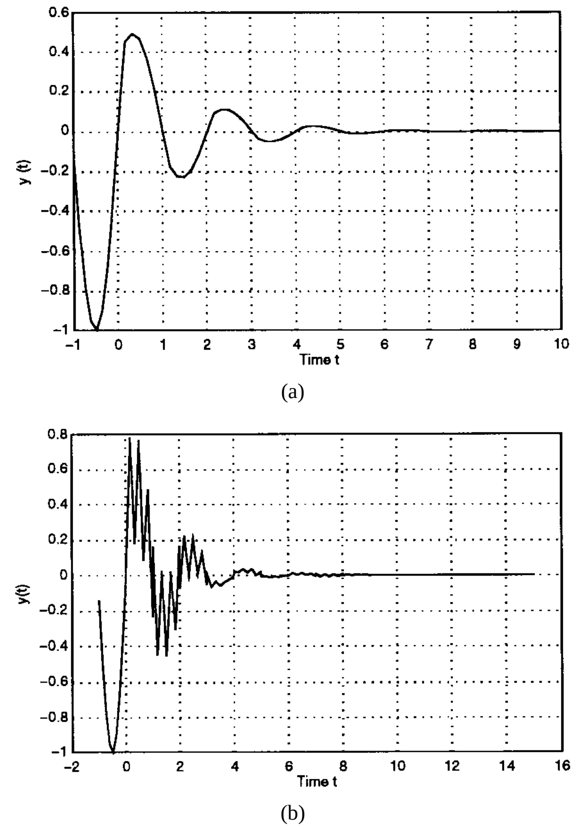

Fig. 2. Third component of the numerical solution computed by (a) the Lobatto III-C method and by (b) the trapezoidal rule.

Fig. 2 shows the numerical solution computed with the two-stage Lobatto III-C method (a) and also with the trapezoidal rule on a nonuniform mesh (b). We use the same mesh which consists of eight nonuniform steps on every interval I_n.

This example illustrates the smooth solution obtained with the Lobatto method. It also shows the lack of contractivity for the trapezoidal scheme which results in spurious oscillations of the solution even though the asymptotically stable global behavior is preserved. Good time domain solution for PEEC circuits have also been obtained using the back Euler method [4] and [16].

VII. Conclusions

In this work we determined easily verifiable conditions which imply asymptotic stability of the solutions of NDDE's. Other important aspects related to the numerical solution of NDDE's like interpolation and suitable numerical methods are carefully analyzed.

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A Fast Adaptive Algorithm for Image Restoration

Guo-Fang Xu, Tamal Bose, Wolfgang Kober, and John Thomas

***Abstract*— An adaptive filtering algorithm based on an Euclidean direction search (EDS) method is presented for image restoration. It is a fast algorithm and has a computational complexity of $O(N)$ for least squares optimization. Computer simulations illustrate that this algorithm is very effective in image restoration. The figures for signal-to-noise ratio improvement (SNRI) produced by this algorithm are comparable to those obtained by using the recently reported sample-based conjugate gradient (SCG) algorithm, which has a computational complexity of $O(N^2)$. This algorithm can also be extended to other applications in adaptive signal processing.**

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G.-F. Xu and T. Bose are with the Department of Electrical Engineering, University of Colorado, Denver, CO 80217-3364 USA.

W. Kober and J. Thomas are with the Data Fusion Corporation, Westminster, CO 80021 USA.

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I. Introduction

In the recent past, some efforts have been made to develop two-dimensional (2-D) adaptive filters. In [1], a 2-D adaptive algorithm, called the TDLMS, has been published, which is an extension of the popular least mean square (LMS) algorithm. This algorithm has been used for image restoration with some success. Some variations of the TDLMS have also been proposed [2]–[7]. Compared to other adaptive filters, the LMS type algorithm has received more attention, mainly because of its simplicity in computation and implementation. However, it is well known that the convergence of the LMS algorithm is very slow for colored inputs. Therefore, there is some interest in the use of algorithms which converge much faster, although at the expense of higher computational complexity. Due to the rapidly changing statistics of images, this faster convergence rate is desirable. For example, in [8], a quasi-Newton-based algorithm has been developed for 2-D adaptive finite impulse response (FIR) filters. Sample-based conjugate gradient (SCG) 2-D algorithms have also been developed [9], [10] and used for image restoration. Very recently, the principal authors of this paper have developed an adaptive algorithm (1-D) based on the direction set method. This algorithm was successfully used in several filtering applications such as noise cancellation, system identification, and channel equalization [11], [12].

In this paper, we develop a 2-D Euclidean direction search (EDS) algorithm for adaptive filtering and the algorithm is investigated for its performance in image restoration. The computational complexity of the algorithm is $O(N)$ for least squares minimization. Our computer simulations show that the EDS algorithm achieves a higher signal-to-noise ratio improvement (SNRI) than that of the 2DLMS algorithm and comparable to the sample-based CG algorithm.

The paper is organized as follows. In Section II, we present the direction set (DS) method which was introduced by Powell and later modified by Powell and Zangwill [13], [14]. In Section III, we show that by choosing a fixed coordinate direction set, the DS method can be simplified to an Euclidean direction search (EDS) method, and prove that the EDS method converges to the optimal solution asymptotically. We also intuitively show that the EDS algorithm has a faster convergence rate than the steepest descent method and less computational complexity than Newton's method. Section IV discuss the implementation of the sample-based EDS algorithm in detail. Computer simulation results are presented in Section V, and the concluding remarks are given in Section VI.

II. The Direction Set (DS) Based Methods

Direction set (DS) based methods are originally designed for solving unconstrained minimization problems of the form: $\min_{x\in R^N} f(x)$, without calculating the first partial derivatives of f for the cases where the derivatives either do not exist or are very expensive to evaluate. With an initial estimate $x_k^{(0)}$ of the solution x_* and a set of linearly independent directions $\{d_1, \cdots, d_N\}$, a DS method searches along each direction for a better estimate, i.e., $x_k^{(i)} = x_k^{(i-1)} + \alpha_i d_i$ where α_i is chosen so that

$$f\left(x_k^{(i)}\right) = \min_{\alpha_i \in R} f\left(x_k^{(i-1)} + \alpha_i d_i\right).$$

Searching through N directions is called one searching cycle. Before the next searching cycle, directions may be modified and a new starting estimate may be chosen.