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Parameter dependent Lyapunov functions for discrete time systems with time varying parametric uncertainties

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Abstract

In this paper, we consider discrete time systems with polytopic time varying uncertainty. We look for a class of parameter dependent Lyapunov functions which are quadratic on the system state and depend in a polytopic way on the uncertain parameter. We show that extending the new discrete time stability condition proposed by de Oliveira et al. (Systems Control Lett. 36 (1999) 135.) to the case of time varying uncertainty leads to a necessary and sufficient condition for the computation of such a Lyapunov function. This allows to check asymptotic stability of the system under study. The obtained linear matrix inequalities condition can also be used to cope with the control synthesis problem.

Keywords: Parameter dependent Lyapunov functions; Discrete time systems; Time varying parameters

1. Introduction

One of the most fundamental problems in systems theory is the construction of Lyapunov functions. The first application is stability analysis and in most of cases these functions are of great help in control synthesis problems. In particular, many robust control problems have been solved using the quadratic stability concept, that is looking for a single Lyapunov function, since there is a very large number of contributions in control literature on this topic, see [5] and references therein. The proposed conditions appear to be efficient due to the existence of powerful numerical tools to solve the resulting Riccati equations or linear matrix inequalities (LMI) [2].

In order to reduce the conservatism of the proposed solutions, looking for more general Lyapunov functions appears to be of great theoritical and practical interest. Among the first contributions to this problem, construction of parameter dependent Lyapunov functions (PDLF) have been proposed in [3,6,7]. The proposed Lyapunov functions are quadratic on the system state and depend affinely on the uncertain parameters. Construction of a class of nonquadratic Lyapunov functions is proposed in [1]. These functions are not PDLF and the result is based on "smoothing" a polyhedral function for which numerical

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algorithms are known to exist. All these contributions deal with problems formulated in the continuous time domain.

In the discrete time case, robust stability of a class of uncertain discrete time linear systems have been addressed using affine PDLF in [8]. Recently, nice results have been proposed for the construction of PDLF in the discrete time case [4]. The main advantage of the proposed condition, beside the fact that it leads to less conservative results than the quadratic stability based ones, consists in the introduction of an extra degree of freedom which allows to get a control law without an explicit dependence on the Lyapunov matrices. These works assume that the parameter uncertainty is time independent.

Unlike the continuous time case, there is a very few results concerning the construction of PDLF for discrete time uncertain systems with time varying uncertainty. To our knowledge, this problem has been investigated recently in [9]. Sufficient conditions are proposed to build Lyapunov function that depends quadratically on the uncertain parameters and quadratically on the state system. If such a Lyapunov function exists, the system is said to be bi-quadratically stable. In this paper, the uncertain parameters are supposed to have bounded rate of variation. At a time instant k, the parameter admissible values are assumed to depend only on the parameter value at the previous instant and on the rate of variation. Dependence on the instant time is not allowed.

Here, we consider discrete time systems with polytopic time varying uncertainty. We look for a class of PDLF which are quadratic on the system state and depend in a polytopic way on the uncertain parameter. Moreover, the rate of variation of the parameter uncertainty is not assumed bounded and the parameter admissible values may depend on the instant time. Extending the results in [4] to the time varying case, we obtain necessary and sufficient LMI conditions for the existence of such a Lyapunov function. The proposed result can be used for stability analysis and synthesis as well.

The paper is organized as follows. Section 2 gives a formulation of the problem under study. In Section 3, we state the main result of this contribution, namely a necessary and sufficient condition for the construction of a particular class of parameter dependent quadratic Lyapunov functions for discrete time parameter varying systems. This condition is applied to a control synthesis problem. We end the paper by an illustrative example and a conclusion.

2. Problem formulation

We consider dynamical uncertain discrete time systems:

$$x(k+1) = \mathscr{A}(\xi(k))x(k), \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $\xi \in \Xi \subset \mathbb{R}^p$ is a unknown but bounded time-varying parameter. The structure of the dynamical matrix $\mathscr{A}$ is assumed to be of the form

$$\begin{aligned} &\mathscr{A}(\xi(k)) = \sum_{i=1}^{N} \xi_i(k) A_i, \\ &\xi_i(k) \geqslant 0, \quad \sum_{i=1}^{N} \xi_i(k) = 1. \end{aligned} \tag{2}$$

The following theorem gives a standard result concerning the stability analysis problem [11,12].

Theorem 1. *The null solution of the system* (1) *is uniformly asymptotically stable if and only if there exists a Lyapunov function*

$$V(x(k), \xi(k)) = x(k)^{\mathrm{T}} \mathscr{P}(\xi(k)) x(k)$$

such that

$$\alpha_1(\|x\|) \leqslant V(x(k), \xi(k)) \leqslant \alpha_2(\|x\|) \tag{3}$$

and whose difference along the solution of (1) *is negative definite descrescent that is*

$$\begin{aligned} \mathscr{L} &= V(x(k+1), \xi(k+1)) - V(x(k), \xi(k)) \\ &\leqslant -\alpha_0(\|x\|) \end{aligned} \tag{4}$$

for all $x \in \mathbb{R}^n$ and $\xi \in \Xi$ and where $\alpha_0(.)$, $\alpha_1(.)$ and $\alpha_2(.)$ are κ_∞ functions. [1]

This result is quite general and practically it cannot be used in its present form since there is no systematic way to build the Lyapunov function $V(x(k), \xi(k))$ as a function of the uncertain time-varying parameter $\xi(k)$. Recall that quadratic stability refers to a particular, but conservative, choice of the Lyapunov function, namely $\mathscr{P}(\xi(k)) = P$ a constant matrix. Here, based on a similar structure uncertainty description, we look

[1] A function $\alpha : [0, \infty) \to [0, \infty)$ is a κ_∞ function if it is continuous, strictly increasing, zero at zero and unbounded ($\alpha(s) \to \infty$ as $s \to \infty$).

for PDLF of the form

$$V(x(k), \xi(k)) = x'(k)\mathscr{P}(\xi(k))x(k),$$

$$\text{with} \quad \mathscr{P}(\xi(k)) = \sum_{i=1}^{N} \xi_i(k) P_i, \tag{5}$$

where P_i are symmetric positive definite constant matrices of appropriate dimension. Such a PDLF satisfies condition (3) with $\alpha_2(\|x\|) = \sum_{i=1}^{N} \lambda_{\max}(P_i)\|x\|^2$ and $\alpha_1(\|x\|) = \varepsilon\|x\|^2$ with ε a sufficiently small positive scalar. Its difference along the solution of (1) is given by

$$\begin{aligned}\mathscr{L} &= V(x(k+1), \xi(k+1)) - V(x(k), \xi(k)) \\ &= x(k)^{\mathrm{T}}(\mathscr{A}^{\mathrm{T}}\mathscr{P}_{+}\mathscr{A} - \mathscr{P})x(k)\end{aligned} \tag{6}$$

with $\mathscr{A}$ given by (2) and

$$\mathscr{P} = \sum_{i=1}^{N} \xi_i(k) P_i,$$

$$\mathscr{P}_{+} = \sum_{i=1}^{N} \xi_i(k+1) P_i = \sum_{j=1}^{N} \xi_j(k) P_j. \tag{7}$$

To avoid confusion using the usual quadratic stability terminology, we define what we will denote in the sequel poly-quadratic stability.

Definition 2. *System* (1) *is said poly-quadratically stable if there exists a parameter dependent quadratic Lyapunov function* (5) *whose difference is negative definite decrescent.*

Poly-quadratic stability uses PDLF which are quadratic on the system state and depend in a polytopic way on the uncertain parameter to check that the system (1) is uniformly asymptotically stable. Hence, the problem investigated in this paper can be formulated as follows: *Find necessary and sufficient conditions to check if system* (1) *is poly-quadratically stable.*

3. Main result

The main result is given in the following theorem which states a necessary and sufficient condition to exhibit a PDLF of the form (5).

Theorem 3. *System* (1) *is poly-quadratically stable if and only if there exist symmetric positive definite matrices S_i, S_j and matrices G_i of appropriate dimensions such that*

$$\begin{bmatrix} G_i + G_i^{\mathrm{T}} - S_i & G_i^{\mathrm{T}} A_i^{\mathrm{T}} \\ A_i G_i & S_j \end{bmatrix} > \mathbf{0} \tag{8}$$

for all $i = 1, \ldots, N$ and $j = 1, \ldots, N$. In this case, the time varying PDLF is given by (2) *with*

$$\mathscr{P}(\xi(k)) = \sum_{i=1}^{N} \xi_i(k) S_i^{-1}.$$

Proof. To prove sufficiency, assume that condition (8) is feasible. Then

$$G_i + G_i^{\mathrm{T}} - S_i > \mathbf{0}.$$

Therefore, G_i is of full rank and as S_i is strictly positive definite, we have:

$$(S_i - G_i)^{\mathrm{T}} S_i^{-1} (S_i - G_i) \geqslant 0,$$

which is equivalent to

$$G_i^{\mathrm{T}} S_i^{-1} G_i \geqslant G_i^{\mathrm{T}} + G_i - S_i.$$

Then, satisfying (8) leads to

$$\begin{bmatrix} G_i^{\mathrm{T}} S_i^{-1} G_i & G_i^{\mathrm{T}} A_i^{\mathrm{T}} \\ A_i G_i & S_j \end{bmatrix} > \mathbf{0},$$

which is equivalent to

$$\begin{bmatrix} G_i^{\mathrm{T}} & \mathbf{0} \\ \mathbf{0} & S_j \end{bmatrix} \begin{bmatrix} S_i^{-1} & A_i^{\mathrm{T}} S_j^{-1} \\ S_j^{-1} A_i & S_j^{-1} \end{bmatrix} \begin{bmatrix} G_i & \mathbf{0} \\ \mathbf{0} & S_j \end{bmatrix} > \mathbf{0}.$$

Letting $P_i = S_i^{-1}$ and $P_j = S_j^{-1}$ one gets

$$\begin{bmatrix} P_i & A_i^{\mathrm{T}} P_j \\ P_j A_i & P_j \end{bmatrix} = Q_{ij} > \mathbf{0} \tag{9}$$

for all $i = 1, \ldots, N$ and $j = 1, \ldots, N$. For each i, multiply the corresponding $j = 1, \ldots, N$ inequalities by ξ_j and sum. Multiply the resulting $i = 1, \ldots, N$ inequalities by ξ_i and sum to get

$$\begin{bmatrix} \mathscr{P} & \mathscr{A}^{\mathrm{T}} \mathscr{P}_{+} \\ \mathscr{P}_{+} \mathscr{A} & \mathscr{P}_{+} \end{bmatrix} = \mathscr{Q} > \mathbf{0},$$

where $\mathscr{A}$ is given by (2) and $\mathscr{P}$ and $\mathscr{P}_{+}$ are defined by (7). The difference function is then given by

$$\mathscr{L} = -x(k)^{\mathrm{T}} \mathscr{Q}(k) x(k),$$

$$\mathscr{Q}(k) = \sum_{i=1}^{N} \xi_i(k) \left(\sum_{j=1}^{N} \xi_j(k) Q_{ij}(k) \right),$$

which is a negative definite decrescent quadratic form according to positive definiteness of the constant matrices Q_{ij} and to the particular admissible parameter values ($\xi_i \geqslant 0$, $\sum_{i=1}^{N} \xi_i = 1$).

To prove necessity, assume that the difference function $\mathscr{L}$ is negative definite decrescent function, then (6) is satisfied. Hence

$$P_i - A_i^{\mathrm{T}} P_j A_i > \mathbf{0}$$

for all $i = 1, \ldots, N$ and $j = 1, \ldots, N$. Letting $S_i = P_i^{-1}$ and $S_j = P_j^{-1}$ and using the Schur complement one gets

$$S_j - A_i S_i A_i^{\mathrm{T}} = T_{ij} > \mathbf{0}.$$

Let $G_i = S_i + g_i \mathbf{I}$ with g_i a positive scalar. There exists a sufficiently small g_i such that

$$g_i^{-2}(S_i + 2g_i\mathbf{I}) > A_i^{\mathrm{T}} T_{ij}^{-1} A_i,$$

which is equivalent, by Schur complement, to

$$\begin{bmatrix} S_i + 2g_i\mathbf{I} & -g_i A_i^{\mathrm{T}} \\ -A_i g_i & T_{ij} \end{bmatrix} > \mathbf{0},$$

which is nothing than

$$\begin{bmatrix} G_i + G_i^{\mathrm{T}} - S_i & S_i A_i^{\mathrm{T}} - G_i A_i^{\mathrm{T}} \\ A_i S_i - A_i G_i & S_j - A_i S_i A_i^{\mathrm{T}} \end{bmatrix} > \mathbf{0}.$$

To end the proof of Theorem 3, one can notice that the latest LMI is equivalent to

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -A_i & \mathbf{I} \end{bmatrix} \begin{bmatrix} G_i + G_i^{\mathrm{T}} - S_i & G_i^{\mathrm{T}} A_i^{\mathrm{T}} \\ A_i G_i & S_j \end{bmatrix} \begin{bmatrix} \mathbf{I} & -A_i^{\mathrm{T}} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} > \mathbf{0}.$$

□

The poly-quadratic stability condition proposed in the previous theorem involves several LMIs where some variables have to be positive definite. This is a classical convex problem which can be handled by means of efficient software. Being necessary and sufficient, this condition is different from the one used in [10] to cope with an H_∞ control problem. The main difference consists in the fact that the results in [10] are based on a sufficient condition of poly-quadratic stability. This condition can be recovered by imposing $G_i = G$ for all $i = 1, \ldots, N$ in the condition (8). This means that the necessity part fails in the proof of Theorem 3 and hence the results based on such a condition are conservative.

Condition (8) can be used to cope with the poly-quadratic stabilizability problem leading to less conservative control design conditions. A direct application could be the design of a particular parameter dependent control law. To this end, consider a discrete time linear parameter dependent system given by

$$x(k+1) = \sum_{i=1}^{N} \xi_i(k) A_i x(k) + Bu(k), \tag{10}$$

where A_i, B are given constant matrices. Assuming the parameter vector real-time measurable, a parameter dependent control law

$$u(k) = \sum_{i} \xi_i(k) K_i x(k) \tag{11}$$

such that the closed loop system

$$x(k+1) = \sum_{i}^{N} \xi_i(k)(A_i + BK_i)x(k) \tag{12}$$

is poly-quadratically stable can be obtained using the following result.

Theorem 4. *System* (10) *is poly-quadratically stabilizable by a parameter dependent control law* (11) *if and only if there exist symmetric positive definite matrices S_i, S_j and matrices G_i, R_i solution of the LMIs*

$$\begin{bmatrix} G_i + G_i^{\mathrm{T}} - S_i & (\bullet)^{\mathrm{T}} \\ A_i G_i + BR_i & S_j \end{bmatrix} > \mathbf{0} \tag{13}$$

for all $i = 1, \ldots, N$ and $j = 1, \ldots, N$. The parameter dependent state feedback control law is then given by (11) *with $K_i = R_i G_i^{-1}$.*

Proof. One can transpose directly the proof of Theorem 3 using the change of variable $R_i = K_i G_i$. Existence of G_i^{-1} is guaranteed by the following inequality $G_i + G_i^{\mathrm{T}} > S_i > \mathbf{0}$. □

As in [4], the determination of the control law does not depend explicitly on the Lyapunov matrices $P_i = S_i^{-1}$. The extra degree of freedom introduced by the matrices G_i, which are not constrained to be symmetric, are fully incorporated in the control variable.

4. Numerical example

In [4] the problem of finding the largest scalar γ such that the uncertain time invariant system

$$x(k+1) = A(\xi)x(k)$$

$$
= \left\{ \begin{bmatrix} 0.8 & -0.25 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0.03 \\ 0 & 0 & 1 & 0 \end{bmatrix} + \xi \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} [0.8 \quad -0.5 \quad 0 \quad 1] \right\} x(k)
$$

is asymptotically stable for all $|\xi| < \gamma$ has been considered. The exact maximum value of the time independent parameter ξ is $\gamma = 0.4620$ and has been found iteratively by root-locus. Using the condition proposed in [4], the authors were able to find this maximum value. To illustrate the condition proposed in our paper, we allow the parameter ξ to be time dependent, and we look for the largest scalar γ such that the corresponding uncertain time varying system is asymptotically stable. Table 1 gives the result obtained using the classical quadratic stability test and the necessary and sufficient condition of poly-quadratic stability (8). The exact maximum value γ has been recovered using the necessary and sufficient condition of Theorem 3. This allows to state that for this example the exact maximum value obtained in the time invariant case still is the maximum value preserving stability in the time varying case.

5. Conclusion

Extending the new stability condition proposed in [4] to the case of uncertain discrete time systems with time varying parametric uncertainty leads to a necessary and sufficient for the existence of a class of time varying PDLF. Based on a similar structure to the uncertainty description, these PDLF allow to check stability of the corresponding uncertain systems. A gain scheduling control synthesis problem has been considered as a first application of this condition for control design purpose. To evaluate deeply the improvement provided by this condition, one has to investigate new solutions based on the proposed condition for robust control analysis and synthesis problems. This will be addressed in a near future. Moreover, the result proposed in this paper does not depend on the "rate of variation" of the time varying parameter. This allows to think for a possible application of the proposed condition to some particular hybrid control problems. Such a possibility is currently under study.

Table 1
Maximum value of γ

Quadratic stability	$\gamma = 0.4279$
Condition of Theorem 3	$\gamma = 0.4620$

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