

Analysis of a System of Linear Delay Differential Equations

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A new analytic approach to obtain the complete solution for systems of delay differential equations (DDE) based on the concept of Lambert functions is presented. The similarity with the concept of the state transition matrix in linear ordinary differential equations enables the approach to be used for general classes of linear delay differential equations using the matrix form of DDEs. The solution is in the form of an infinite series of modes written in terms of Lambert functions. Stability criteria for the individual modes, free response, and forced response for delay equations in different examples are studied, and the results are presented. The new approach is applied to obtain the stability regions for the individual modes of the linearized chatter problem in turning. The results present a necessary condition to the stability in chatter for the whole system, since it only enables the study of the individual modes, and there are an infinite number of them that contribute to the stability of the system. [DOI: 10.1115/1.1568121]

Introduction

Delays are inherent in many physical and engineering systems. In particular, pure delays are often used to ideally represent the effects of transmission, transportation, and inertial phenomena. Delay differential equations (DDEs) constitute basic mathematical models for real phenomena, for instance in engineering, mechanics, and economics. This paper presents a new analytic approach, based on Lambert functions, for the solution of a system of non-homogeneous linear DDEs and its applications in the stability analysis of the linear chatter problem.

There are applications of delay in active vibration and noise control [1]. Some other applications of delay differential systems in modeling and analysis are reported in [2] for conveyor belts, metal rolling system, population models, economic systems, remote control, urban traffic, electric transmission line, heat exchangers, control systems for nuclear reactors with time delay [3], artificial neural networks, manufacturing systems [4], and capacity management [5].

Delay differential equations, also known as *difference-differential equations*, are a special class of differential equations called functional differential equations. Delay differential equations were initially introduced in the 18th century by Laplace and Condorcet [2]. However, the rapid development of the theory and applications of those equations did not come until after the Second World War, and continues today. The basic theory concerning the stability of systems described by equations of this type was developed by Pontryagin in 1942. Important works have been written by Bellman and Cooke in 1963 [6], Smith in 1957 [7], Pinney in 1958, Halanay in 1966, El'sgol'ts and Norkin in 1971, Myshkis in 1972, Yanushevski in 1978, Marshal in 1979, and Hale in 1977. The reader is referred to the detailed review in [2].

An important application of delay equations is the reduction of chatter in machining processes. Machine tool chatter is one of the major constraints that limits productivity of the machining processes. Chatter is a self-excited vibration, which is a result of the interaction between the tool structure and the cutting process dynamics [8] and [9]. Extensive research has been done on understanding the mechanisms behind the chatter problem in machining. The fundamentals of chatter are outlined in [10], and a survey of the previous research can be found in [11]. A root-locus-based approach to the stability problem of delay differential equations, and a discussion on regenerative machine tool chatter is presented

in [12], and an exact methodology, which reveals stability regions for a system of linear delay differential equations in the time delay domain and indicates the number of unstable characteristic roots in any given stability region, is presented in [13].

The delay operator (1) can be expressed in the form of an infinite series. The principal difficulty in studying delay differential equations lies in its special transcendental character. Delay problems always lead to an infinite spectrum of frequencies. The determination of this spectrum requires a corresponding determination of zeros of certain analytic functions. If one is interested only in connections between the theory and problems of oscillations, then most of these difficulties can be set aside, since it might not be the entire infinite spectrum of frequencies, but only a few of them that are actually responsible for the appearance of these oscillations. This can simplify the problem considerably. One of the approximation methods of this kind is the well-known Padé approximation, which results in a shortened repeating fraction for the approximation of the characteristic equation of the delay [14–16].

Another approach to the delay problem is not to approximate the delay, but to look at the problem with the entire delay spectrum. Delay differential equations are often solved using numerical methods, asymptotic solutions, and graphical tools. Applying Runge-Kutta methods to obtain a $\pi(0)$ -stable numerical algorithm to solve homogeneous linear DDEs is studied in [17]. Many research studies have attempted to find asymptotic solutions using perturbation methods for the chatter problem. For example, stability analysis of machine tool chatter using perturbation methods can be found in [11]. Another stability analysis of a similar chatter problem using complex analysis can be found in [9]. One of the early approaches which deals with the design of control systems with delay is the so called *Smith predictor* presented by O.J. Smith in his seminal paper in 1957 [7]. The Smith predictor converts the problem into a delay free one. This simplifies the control design by alleviating the rational time delay approximation problem.

Several attempts have been made to find an analytical solution for delay differential equations by solving its characteristic equation under different conditions. A recent related study on analytic solution of linear DDEs can be found in [18]. A Fourier-like analysis of the existence of the solution and its properties for the nonlinear DDEs is studied by Wright [19]. Similar approaches to linear and nonlinear DDEs are also reported by Bellman [6]. The uniqueness of the solution and its properties for the linear DDEs with varying coefficients is studied by Wright [19]. Solution properties for the linear DDEs with asymptotically constant coefficients are also studied by Wright in [20].

In this paper, a new approach to solve the transcendental characteristic equation of a system of DDEs, based upon a class of

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functions called Lambert functions, is presented. The advantage of this approach compared to other available analytical solution techniques lies in the fact that the form of the solution obtained is analogous to the general solution form of ordinary differential equations, and the concept of the state transition matrix in ODEs can be generalized to DDEs using the concept of Lambert functions. The solution technique is first presented for a scalar first-order linear DDE, then generalized to find the complete solution to general classes of linear delay differential equations. The method is then used to study a machine tool linear chatter problem.

First-Order Problem

A pure time delay, an essential element in the modeling and description of delay systems, has the property that input and output are identical in form, and the only difference is that of translation along the time axis. The transfer function representing a delay operator with delay time T can be represented by,

$$G(s) = \frac{X(s)}{U(s)} = e^{-sT} \quad (1)$$

where the input $u(t)$ results in a response $x(t)$ equal to $u(t-T)$. In many complex systems, such as feedback control systems, there will be more than one delay. These delays may be, for example, in measurement at the output, in control at the input or in the feedback path.

It was stated earlier that DDEs are a special case of a class of differential equations called functional differential equations. A general homogeneous first order functional differential equation is given by:

$$\dot{y}(t) = f(t, y(t), y(\zeta(t))) \quad (2)$$

This type of equation is useful in modeling various complex phenomena in which the change in a quantity is dependent upon how the quantity is affected by some other mechanism. Linearizing the time-invariant form of Eq. (2) with respect to y and defining the function $\zeta(t) = t - T$ results in a simple form of a homogeneous first order delay differential equation as represented by the boundary value problem:

$$\begin{aligned} \dot{y}(t) + \alpha y(t-T) + \beta y(t) &= 0 \quad T > 0 \\ y(t) &= \phi(t) \quad t \in [0, T] \end{aligned} \quad (3)$$

for all cases of the constants α , β , and T . The interval $[0, T]$ is called the *pre-interval* and $\phi(t)$ is called the *preshape function*, which corresponds to the initial configuration of $y(t)$ on the pre-interval. As expected from first-order differential equations, the solution exhibits exponential growth or decay. What is, perhaps, unexpected is that this first-order problem may also have solutions that are oscillatory or even unstable. The nature of each solution depends heavily upon the relationship between the values of α , β , and T as well as the preshape function $\phi(t)$.

The theorems of existence and uniqueness of the solution to Eq. (3) for the special case with $\beta=0$ are given in [18], and for the general case in [6]. Assuming a candidate solution in the form of $y(t) = Ce^{st}$, and substituting it into Eq. (3), we get the nonlinear, transcendental characteristic equation;

$$F(s) = (s + \beta)e^{sT} + \alpha = 0 \quad (4)$$

The difficulty in the analysis of the DDEs lies in finding an analytical method to solve this transcendental equation.

To solve Eq. (4), a new approach based upon a class of functions, $W(s)$, called Lambert functions, is presented. By a definition presented in 1758 by Lambert [21] and Euler [22], every function $W(s)$ that satisfies

$$W(s)e^{W(s)} = s \quad (5)$$

is called a Lambert function. This concept is applied to solve the transcendental characteristic equation of the delay differential equation. Equation (4) can be written as:

$$(s + \beta)e^{sT} = -\alpha \quad (6)$$

Multiplying both sides of Eq. (6) by $Te^{\beta T}$, yields

$$T(s + \beta)e^{(s+\beta)T} = -\alpha Te^{\beta T} \quad (7)$$

Based on the definition of the Lambert function in Eq. (5), it is clear that

$$W(-\alpha Te^{\beta T})e^{W(-\alpha Te^{\beta T})} = -\alpha Te^{\beta T} \quad (8)$$

Comparing Eqs. (7) and (8)

$$T(s + \beta) = W(-\alpha Te^{\beta T}) \quad (9)$$

Thus, the solutions of the equation which describe the characteristic spectrum of Eq. (3) can be written as

$$s = \frac{1}{T}W(-\alpha Te^{\beta T}) - \beta \quad (10)$$

Similarly, the solution of the characteristic equation for the simplest form of linear DDE where $\beta=0$ is

$$s = \frac{1}{T}W(-\alpha T) \quad (11)$$

In the most general form, the Lambert function is a complex function with infinite branches. Calculation of the principal branch is presented in series form by Caratheodory [23] and given below:

$$W_0(s) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} s^n \quad (12)$$

Calculation of other branches of the Lambert function, for $k = -\infty, \dots, \infty$, is presented by [24]

$$W_k(s) = \ln_k(s) - \ln(\ln_k(s)) + \sum_{l=0}^{\infty} \sum_{m=1}^{\infty} C_{lm} \frac{(\ln(\ln_k(s)))^m}{(\ln_k(s))^{l+m}} \quad (13)$$

where $\ln_k(s) = \ln(s) + 2\pi i k$ indicates the k -th logarithm branch, and the coefficients C_{lm} can be expressed in terms of Stirling Cycle numbers [25]:

$$C_{lm} = \frac{1}{m!} (-1)^l \begin{bmatrix} l+m \\ l+1 \end{bmatrix} \quad (14)$$

A detailed discussion of the convergence of these doubly infinite series and algorithms to calculate them are presented in [26].

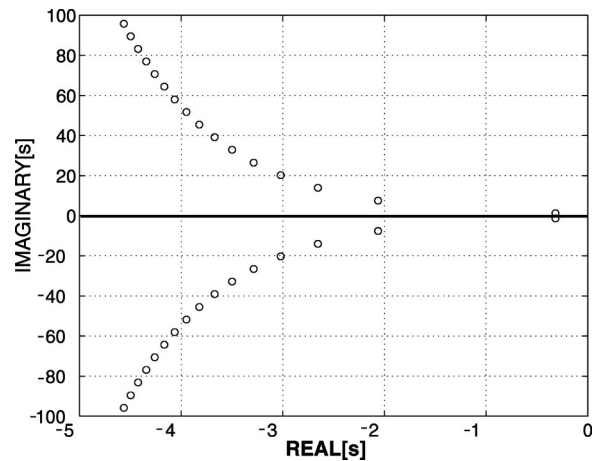


Fig. 1 Roots of the characteristic equation in Eq. (11) when $\alpha = T = 1$, and $\beta = 0$

For Eq. (3) with the values of $T=1$, $\alpha=1$, and $\beta=0$, the first 30 roots of the characteristic equation are shown on the complex plane in Fig. 1. Each pair of complex roots are derived using the corresponding branches of the Lambert function. As shown, the critical poles closest to the imaginary axis, which determine the stability of the system, correspond to the principal branch in Eq. (12). Using Eqs. (11) and (12), it can be observed that instability occurs whenever $\alpha T > \pi/2$. This relationship leads us to exactly determine the stability condition for the first order DDE with respect to the variation of its parameters α and T . This stability criteria is associated with the stability of the principal root of the characteristic equation (11), and it is shown in Fig. 2. Note that similar curves for other roots of this equation can also be obtained. However, they lie above the curve shown in Fig. 2 for the principal root and do not affect the shape or size of the stable region in the $\alpha-T$ plane. However, as it will be seen subsequently, this is not necessarily true for the systems with higher-order dynamics; more steps need to be taken to study the stability of these systems. Considering the behavior of the principal root for the characteristic equation (10), the stability condition for this delay differential equation with respect to the variation of all its parameters α , β , and T is shown in Fig. 3. It should be noticed that small values of α , and large values of β , move the equation toward stable region, and higher delay time T is required to move to the unstable region.

The complete homogeneous solution for Eq. (3), using the results in Eq. (10), can be written as:

$$y(t) = \sum_{k=-\infty}^{\infty} C_k e^{(1/T W_k(-\alpha T e^{\beta T}) - \beta)t} \quad (15)$$

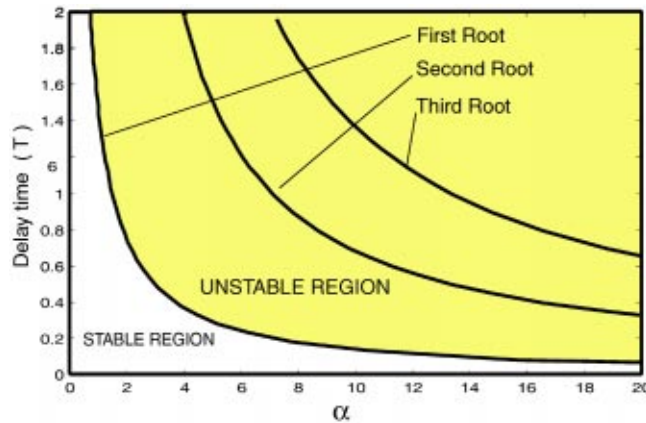


Fig. 2 Stability criteria for DDE in Eq. (3) with $\beta=0$

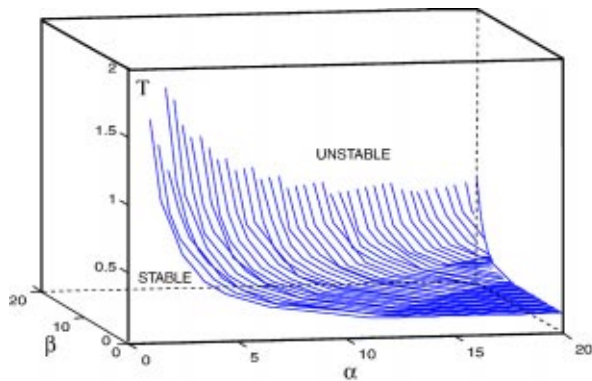


Fig. 3 Stability criteria for the generalized first order DDE in Eq. (3)

for any acceptable choice of C_k corresponding to a specified preshape function. The values of C_k can be obtained using the method presented in the Appendix and considering $\xi_k(t) = e^{(1/T W_k(-\alpha T e^{\beta T}) - \beta)t}$, which yields:

$$y(t) = \sum_{k=-\infty}^{\infty} \lim_{N \rightarrow \infty} \{ \Omega^{-1}(T, N) \Phi \}_k e^{(1/T W_k(-\alpha T e^{\beta T}) - \beta)t} \quad (16)$$

where $\Omega(T, N)$ is defined in the Appendix as a matrix with the functions $\xi_k(t)$ as its elements, and $\{\cdot\}_k$ represents the k^{th} element of the corresponding vector. The vector Φ contains the preshape function evaluated in $t \in [0, T]$ as its elements.

As a numerical example, the methodology represented in the Appendix is used to find the general solution for this delay differential equation with given parameter values. The behavior of the modes ($\xi_1(t)$, $\xi_2(t)$, and $\xi_3(t)$) for this solution for the parameter values of $\alpha=1$, $\beta=1$, and $T=1$ is shown in Fig. 4. The preshape function is assumed to be

$$\phi(t) = 1 \quad t \in [0, T] \quad (17)$$

where $T=1$. The general homogeneous solution for this equation considering $\phi(t)=1$ is shown in Fig. 5. As shown in the figure, using the methodology presented in the Appendix, $y(t)$ converges for large values of N . Fig. 6 also shows the convergence of the solution and the required minimum value for the number of iterations N needed for convergence. Different solution responses for this equation for different values of the time delay T are shown in Fig. 7.

Similarly, the homogeneous solution to the simplest first order homogeneous DDE for $\beta=0$ can be written as,

$$y(t) = \sum_{k=-\infty}^{\infty} C_k e^{1/T W_k(-\alpha T)t} \quad (18)$$

for any acceptable choice of C_k which are derived based on the preshape function in Eq. (3) (see Appendix),

$$y(t) = \sum_{k=-\infty}^{\infty} \lim_{N \rightarrow \infty} \{ \Omega^{-1}(T, N) \Phi \}_k e^{1/T W_k(-\alpha T)t} \quad (19)$$

where again $\Omega(T, N)$ is defined in the Appendix as a matrix with the functions $\xi_k(t) = e^{1/T W_k(-\alpha T)t}$ as its elements.

It can be observed that the form of the solution obtained in Eq. (18) can be compared to the general solution form of the first-order linear ordinary differential equation. The solution $y(t)$ shown in Eq. (18) is a linear combination of the infinite numbers of modes $e^{1/T W_k(-\alpha T)t}$, which characterize the delay equation. Note that the modes, based on the Lambert functions,

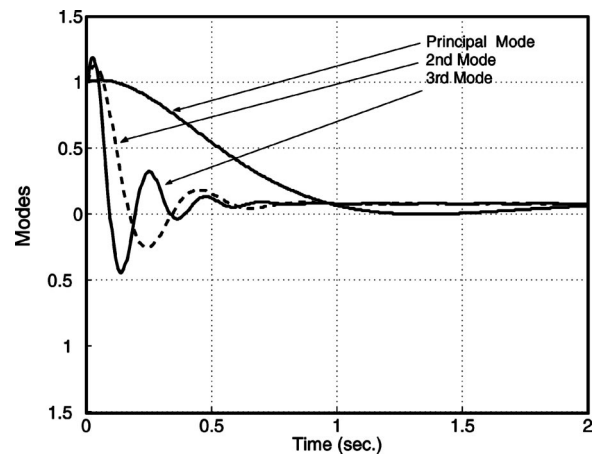


Fig. 4 Principal ($k=0$), second ($k=1$), and third ($k=2$) modes in the free solution form for Eq. (15)

$W_k(-\alpha T)$, play a role similar to the state transition matrix for linear ODEs. This property can then be used to extend the solution to the more general vector-matrix cases of systems of linear DDEs. This will be presented in a subsequent section.

The non-homogeneous version of the delay differential equation (3) can be written as

$$\begin{aligned} \dot{y} + \alpha y(t-T) + \beta y(t) &= u(t) \\ y(t) &= \phi(t) \quad t \in [0, T] \end{aligned} \quad (20)$$

where $u(t)$ is a continuous function representing the external excitation to the delay equation, and it can be written as,

$$u(t) = \sum_{k=-\infty}^{\infty} f_k \psi_k(t) \quad (21)$$

where $\psi_k(t)$ are any given orthonormal functions, and the coefficients f_k can be obtained using Fourier integrals. In the case that $\psi_k(t) = e^{(1/T) W_k(-\alpha T e^{\beta T}) - \beta} t$, which are not orthonormal functions, the corresponding coefficients can be obtained using the methodology given in the Appendix,

$$f_k = \lim_{\substack{N \rightarrow \infty \\ b \rightarrow \infty}} \{ \Omega^{-1}(b, N) \mathbf{u} \}_k \quad (22)$$

where \mathbf{u} is a vector with $u(t)$ as its elements for $t \in [0, b]$ (see Appendix). The Lambert mode of the Eq. (3) is $\xi_k(t) = e^{(1/T) W_k(-\alpha T e^{\beta T}) - \beta} t$ with properties:

$$\begin{aligned} \dot{\xi}_k(t) &= \left(\frac{1}{T} W_k(-\alpha T e^{\beta T}) - \beta \right) \xi_k(t) \\ \xi_k(t-T) &= e^{-(1/T) W_k(-\alpha T e^{\beta T}) - \beta} T \xi_k(t) \end{aligned} \quad (23)$$

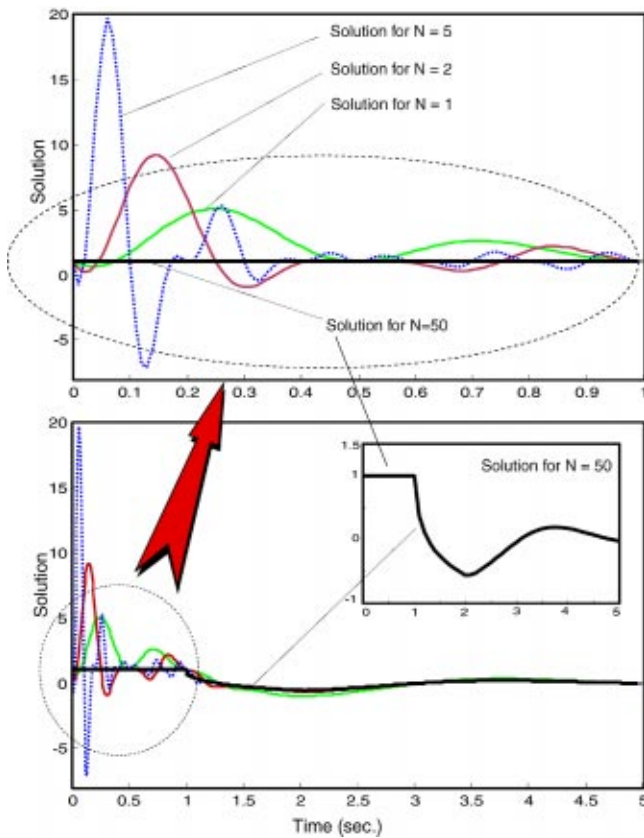


Fig. 5 Free solution for Eq. (15) for large values of N and parameter values $\alpha=1$, $\beta=1$, and $T=1$

To obtain the particular solution to Eq. (20), assume the solution form

$$y_p(t) = \sum_{k=-\infty}^{\infty} p_k(t) e^{(1/T) W_k(-\alpha T e^{\beta T}) - \beta} t \quad (24)$$

Differentiating $y_p(t)$ with respect to time yields

$$\dot{y}_p(t) = \sum_{k=-\infty}^{\infty} \dot{p}_k(t) \xi_k(t) + \sum_{k=-\infty}^{\infty} p_k(t) \left[\frac{1}{T} W_k(-\alpha T e^{\beta T}) - \beta \right] \xi_k(t) \quad (25)$$

Multiplying the terms inside the summation in the second term on the right hand side of Eq. (25) by $(e^{-W_k(-\alpha T e^{\beta T}) + \beta T}) / (e^{-W_k(-\alpha T e^{\beta T}) + \beta T})$ and using the properties in Eq. (23) yields

$$\sum_{k=-\infty}^{\infty} p_k(t) \xi_k(t-T) \left[\frac{1}{T} W_k(-\alpha T e^{\beta T}) - \beta \right] e^{W_k(-\alpha T e^{\beta T})} e^{-\beta T} \quad (26)$$

or

$$\begin{aligned} \sum_{k=-\infty}^{\infty} p_k(t) \xi_k(t-T) \\ \times \left[\underbrace{\frac{1}{T} W_k(-\alpha T e^{\beta T}) e^{W_k(-\alpha T e^{\beta T})} e^{-\beta T} - \beta e^{W_k(-\alpha T e^{\beta T})} e^{-\beta T}}_{=-\alpha} \right] \end{aligned} \quad (27)$$

and therefore,

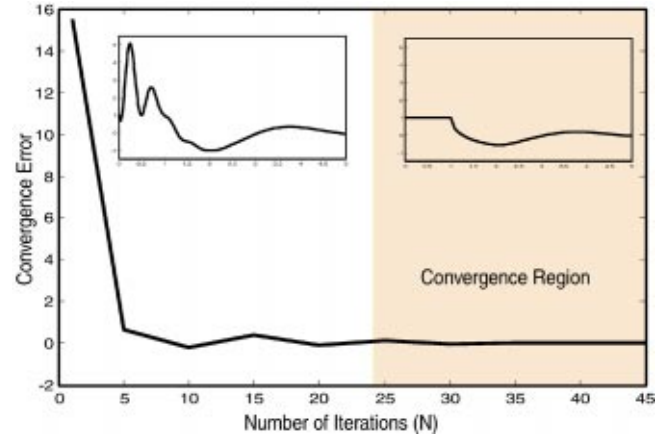


Fig. 6 Convergence of the free solution for Eq. (15) for large values of N and parameter values $\alpha=1$, $\beta=1$, and $T=1$

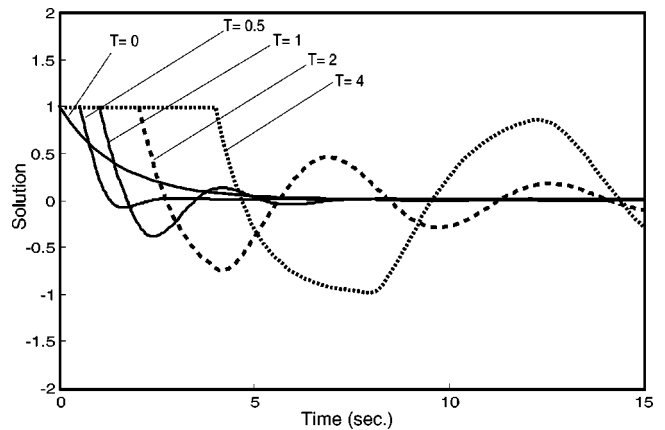


Fig. 7 Free solution for Eq. (15) for different values of time delay with $\alpha=1$, $\beta=1$, and $T=0, 0.5, 1, 2, 4$

$$-\alpha \sum_{k=-\infty}^{\infty} p_k(t) \xi_k(t-T) - \beta \sum_{k=-\infty}^{\infty} p_k(t) \underbrace{\xi_k(t-T) e^{W_k(-\alpha T e^{\beta T}) - \beta T}}_{=\xi_k(t)} \quad (28)$$

Assuming the $p_k(t)$, have the property

$$p_k(t-T) = p_k(t) \quad (29)$$

reduces Eq. (28) to the form

$$-\alpha y_p(t-T) - \beta y_p(t) \quad (30)$$

Substituting this back into the second term on the right hand side of Eq. (25) yields

$$\dot{y}_p(t) = \sum_{k=-\infty}^{\infty} \dot{p}_k(t) \xi_k(t) - \alpha y_p(t-T) - \beta y_p(t) \quad (31)$$

and this leads to

$$u(t) = \sum_{k=-\infty}^{\infty} \dot{p}_k(t) e^{(1/T W_k(-\alpha T e^{\beta T}) - \beta)t} \quad (32)$$

and from Eqs. (21), (22), and (32) one obtains

$$p_k(t) = \int_0^t \lim_{\substack{0 \rightarrow \infty \\ N \rightarrow \infty \\ b \rightarrow \infty}} \{\Omega^{-1}(b_\tau, N) \mathbf{u}\}_k d\tau \quad (33)$$

where $\Omega(b_\tau, N)$ is defined in the Appendix as a matrix with the functions $\xi_k(t) = e^{(1/T W_k(-\alpha T e^{\beta T}) - \beta)t}$ as its elements. The particular solution can then be written as

$$y_p(t) = \sum_{k=-\infty}^{\infty} \left(\int_0^t \lim_{\substack{0 \rightarrow \infty \\ N \rightarrow \infty \\ b \rightarrow \infty}} \{\Omega^{-1}(b_\tau, N) \mathbf{u}\}_k d\tau \right) \xi_k(t) \quad (34)$$

and the total solution is given by:

$$y(t) = \left(\sum_{k=-\infty}^{\infty} C_k \xi_k(t) \right) + \sum_{k=-\infty}^{\infty} \left(\int_0^t \lim_{\substack{0 \rightarrow \infty \\ N \rightarrow \infty \\ b \rightarrow \infty}} \{\Omega^{-1}(b_\tau, N) \mathbf{u}\}_k d\tau \right) \xi_k(t) \quad (35)$$

Similar results can be obtained for the simplest form of linear DDE where $\beta=0$ in Eq. (20). For brevity, only the results are presented here:

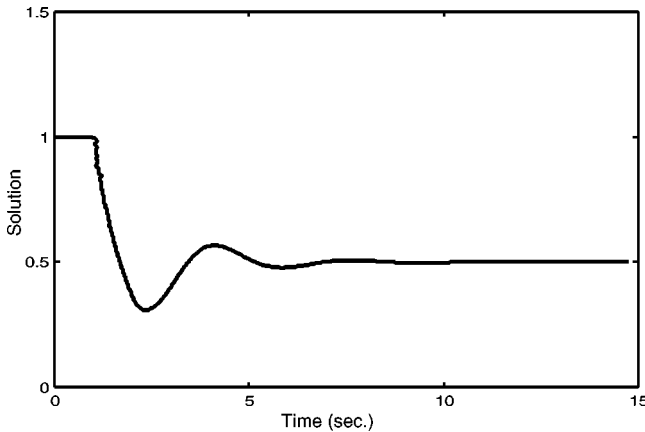


Fig. 8 Complete solution for Eq. (20) for a unit step input and parameter values $\alpha=1$, $\beta=1$, and $T=1$

$$y(t) = \left(\sum_{k=-\infty}^{\infty} C_k \xi_k(t) \right) + \sum_{k=-\infty}^{\infty} \left(\int_0^t \lim_{\substack{0 \rightarrow \infty \\ N \rightarrow \infty \\ b \rightarrow \infty}} \{\Omega^{-1}(b_\tau, N) \mathbf{u}\}_k d\tau \right) \xi_k(t) \quad (36)$$

or

$$y(t) = \sum_{k=-\infty}^{\infty} \left[C_k + \int_0^t \lim_{\substack{0 \rightarrow \infty \\ N \rightarrow \infty \\ b \rightarrow \infty}} \{\Omega^{-1}(b_\tau, N) \mathbf{u}\}_k d\tau \right] e^{1/T W_k(-\alpha T)t} \quad (37)$$

Please note that Eq. (35) is identical to Eq. (36) except for their corresponding Lambert modes, $\xi_k(t)$, as elements of the matrix $\Omega(b_\tau, N)$. The complete solution for Eq. (20) with a unit step input for parameter values $\alpha=1$, $\beta=1$, and $T=1$, is shown in Fig. 8.

Generalization to Systems of DDEs. A further generalization of Eq. (3) is the matrix form

$$\dot{\mathbf{y}}(t) + \mathbf{A}\mathbf{y}(t-T) + \mathbf{B}\mathbf{y}(t) = 0 \quad (38)$$

where \mathbf{A} and \mathbf{B} are $n \times n$ matrices, and $\mathbf{y}(t)$ is an $n \times 1$ vector. The characteristic equation can be written in matrix form as

$$s\mathbf{I} + \mathbf{A}e^{-sT} + \mathbf{B} = 0 \quad (39)$$

or

$$s\mathbf{I}e^{sT} = -\mathbf{A} - \mathbf{B}e^{sT} \quad (40)$$

where \mathbf{I} is the identity matrix. Multiplying both sides of the equation by T ,

$$(sT)\mathbf{I}e^{sT} = (-\mathbf{A})T - \mathbf{B}Te^{sT} \quad (41)$$

It can be shown that

$$\mathbf{I}(e^{sT}) = e^{(sT)\mathbf{I}} \quad (42)$$

and

$$\mathbf{A}(e^{sT}) = \mathbf{A}\mathbf{I}e^{(sT)} = \mathbf{A}e^{(sT)\mathbf{I}} \quad (43)$$

substituting Eqs. (42) and (43) into Eq. (41),

$$(sT)\mathbf{I}e^{(sT)\mathbf{I}} = (-\mathbf{A})T - \mathbf{B}Te^{sT\mathbf{I}} \quad (44)$$

and

$$(s\mathbf{I} + \mathbf{B})Te^{(sT)\mathbf{I}} = (-\mathbf{A})T \quad (45)$$

Multiplying both sides by $e^{\mathbf{B}T}$,

$$(s\mathbf{I} + \mathbf{B})Te^{(s\mathbf{I} + \mathbf{B})T} = (-\mathbf{A})Te^{\mathbf{B}T} \quad (46)$$

One can define the generalized matrix form of the Lambert function as

$$\mathbf{W}(-\mathbf{A})e^{\mathbf{W}(-\mathbf{A})} = -\mathbf{A} \quad (47)$$

where $\mathbf{W}(-\mathbf{A})$ is a square $n \times n$ matrix and the branches for this generalized matrix form of the Lambert function are obtained as

$$\begin{aligned} \mathbf{W}_k(-\mathbf{A}) &= \ln(-\mathbf{A}) + 2\pi i k \mathbf{I} - \ln(\ln(-\mathbf{A}) + 2\pi i k \mathbf{I}) \\ &+ \sum_{l=0}^{\infty} \sum_{n=1}^{\infty} \mathbf{C}_{ln} \ln^n(\ln(-\mathbf{A}) + 2\pi i k \mathbf{I}) (\ln(-\mathbf{A}) \\ &+ 2\pi i k \mathbf{I})^{-l-n} \end{aligned} \quad (48)$$

for $k = -\infty \dots \infty$.

Considering Eq. (46), and using the definition of the generalized Lambert function in Eq. (47) yields

$$(s\mathbf{I} + \mathbf{B})T = \mathbf{W}(-\mathbf{A}Te^{\mathbf{B}T}) \quad (49)$$

and the characteristics roots are

$$s\mathbf{I} = \frac{1}{T} \mathbf{W}(-\mathbf{A}T e^{\mathbf{B}T}) - \mathbf{B} \quad (50)$$

In the special case where $\mathbf{B}=0$, the result becomes

$$s\mathbf{I} = \frac{1}{T} \mathbf{W}(-\mathbf{A}T) \quad (51)$$

The roots of the characteristic equation (Eq. (39)) are the eigenvalues of the matrix $1/T \mathbf{W}(-\mathbf{A}T e^{\mathbf{B}T}) - \mathbf{B}$, and describe the stability of the vector delay differential equation. The roots from the general equation (50) can be obtained using the same approach by diagonalizing the matrix $-\mathbf{A}T e^{\mathbf{B}T}$. The free solution $\mathbf{y}(t)$ can then be described as;

$$\mathbf{y}(t) = \sum_{k=-\infty}^{\infty} e^{(1/T \mathbf{W}_k(-\mathbf{A}T e^{\mathbf{B}T}) - \mathbf{B})t} \mathbf{C}_k \quad (52)$$

where the \mathbf{C}_k values are $n \times 1$ vectors determined from the preshape-function states.

As a specific example, consider a simple 2×2 vector DDE:

$$\dot{y}_1(t) + y_1(t-T) = 0 \quad (53)$$

$$\dot{y}_2(t) + y_1(t-T) + 3y_2(t-T) = 0$$

or

$$\dot{\mathbf{y}}(t) + \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} \mathbf{y}(t-T) = 0 \quad (54)$$

The diagonalized and modal matrices for matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}$$

are

$$\Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$V = \begin{bmatrix} 0 & 0.89 \\ 1 & -0.44 \end{bmatrix}.$$

The matrix Lambert function $\mathbf{W}(\mathbf{A})$ can be computed from

$$\mathbf{W}(\mathbf{A}) = V \mathbf{W}(\Lambda) V^{-1} \quad (55)$$

where for $k=0$

$$\mathbf{W}(\Lambda) = \begin{bmatrix} W(3) & 0 \\ 0 & W(1) \end{bmatrix} = \begin{bmatrix} 1.04 & 0 \\ 0 & 0.56 \end{bmatrix} \quad (56)$$

and therefore,

$$\mathbf{W}(\mathbf{A}) = V \begin{bmatrix} 1.04 & 0 \\ 0 & 0.56 \end{bmatrix} V^{-1} = \begin{bmatrix} 0.56 & 0 \\ 0.24 & 1.04 \end{bmatrix} \quad (57)$$

It is easy to verify that $\mathbf{W}(\mathbf{A}) e^{\mathbf{W}(\mathbf{A})} = \mathbf{A}$.

The homogeneous solution $\mathbf{y}(t)$ can then be described, in a form similar to the free solution in terms of the state transition matrix in ODEs, as

$$\mathbf{y}(t) = \sum_{k=-\infty}^{\infty} e^{1/T \mathbf{W}_k(-\mathbf{A}T) t} \mathbf{C}_k \quad (58)$$

where the \mathbf{C}_k values are $n \times 1$ vectors determined from the preshape function states. For the example vector DDE in Eq. (53), choosing $T=0.1$, the solution

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

can be obtained:

$$\mathbf{y}(t) = \dots + e^{1/T \mathbf{W}_{-1}(-\mathbf{A}T) t} \mathbf{C}_{-1} + e^{1/T \mathbf{W}_0(-\mathbf{A}T) t} \mathbf{C}_0 + e^{1/T \mathbf{W}_1(-\mathbf{A}T) t} \mathbf{C}_1 + \dots \quad (59)$$

or,

$$\mathbf{y}(t) = \dots + e^{\begin{bmatrix} -44.49-73.07i & 0 \\ 5.74-0.64i & -33-74.36i \end{bmatrix} t} \begin{bmatrix} C_{-1}^1 \\ C_{-1}^2 \end{bmatrix} + e^{\begin{bmatrix} -1.11 & 0 \\ -1.88 & -4.89 \end{bmatrix} t} \begin{bmatrix} C_0^1 \\ C_0^2 \end{bmatrix} + e^{\begin{bmatrix} -44.49+73.07i & 0 \\ 5.74+0.64i & -33+74.36i \end{bmatrix} t} \begin{bmatrix} C_1^1 \\ C_1^2 \end{bmatrix} + \dots \quad (60)$$

where the dominant modes of the equation's behavior are $e^{-1.11t}$ and $e^{-4.89t}$, which are stable modes associated with the principal branch (i.e., $\mathbf{W}_0(-\mathbf{A}T)$). The non-complex form of the first mode ($k=\pm 1$) is a linear combination of $e^{-44.49t} \sin(73.07t)$, $e^{-44.49t} \cos(73.07t)$, $e^{-33t} \sin(74.36t)$, and $e^{-33t} \cos(74.36t)$.

The non-homogeneous matrix form of the delay differential equation can be written as

$$\dot{\mathbf{y}}(t) + \mathbf{A}\mathbf{y}(t-T) + \mathbf{B}\mathbf{y}(t) = \tilde{\mathbf{B}}\mathbf{u}(t) \quad (61)$$

where $\tilde{\mathbf{B}}$ is an $n \times r$ matrix, and $\mathbf{u}(t)$ is a $r \times 1$ vector. The particular solution to Eq. (61), similar to the scalar case, can be written as:

$$\mathbf{y}_p(t) = \sum_{k=-\infty}^{\infty} [e^{(1/T \mathbf{W}_k(-\mathbf{A}T e^{\mathbf{B}T}) - \mathbf{B})t}] \times \left(\int_0^t \lim_{\substack{N \rightarrow \infty \\ b \rightarrow \infty}} \{ \mathbf{\Omega}^{-1}(b, N) \tilde{\mathbf{B}}\mathbf{u} \}_k d\tau \right) \quad (62)$$

where $\mathbf{\Omega}$ is a square compound matrix containing $\xi_k(t) = e^{(1/T \mathbf{W}_k(-\mathbf{A}T e^{\mathbf{B}T}) - \mathbf{B})t}$ as its elements, and $\tilde{\mathbf{B}}\mathbf{u}$ is a compound vector with $\tilde{\mathbf{B}}\mathbf{u}(t)$ as its elements.

The complete solution is given by the superposition of Eq. (62) and the homogeneous solution presented in Eq. (52),

$$\mathbf{y}(t) = \left[\sum_{k=-\infty}^{\infty} e^{(1/T \mathbf{W}_k(-\mathbf{A}T e^{\mathbf{B}T}) - \mathbf{B})t} \mathbf{C}_k \right] + \sum_{k=-\infty}^{\infty} [e^{(1/T \mathbf{W}_k(-\mathbf{A}T e^{\mathbf{B}T}) - \mathbf{B})t}] \times \left(\int_0^t \lim_{\substack{N \rightarrow \infty \\ b \rightarrow \infty}} \{ \mathbf{\Omega}^{-1}(b, N) \tilde{\mathbf{B}}\mathbf{u} \}_k d\tau \right) \quad (63)$$

The general results given in Eqs. (52) and (63) can be used in a wide variety of applications. One of the interesting engineering applications in which these analytical tools can provide a better understanding of the system is the chatter problem described in the next section.

Machining Chatter Example

As an example of applying the presented approach to a practical problem, consider a simple turning problem resulting in a single degree of freedom linear chatter equation linearized about an equilibrium point such as a steady cutting condition. The linearized model is expressed by a set of linear time invariant delay differential equations [9]. The stability of such a linearized model can be used to determine the conditions for the onset of chatter, but does not capture the amplitude limiting nonlinearities associated with chatter vibrations. The linearized chatter equation can be expressed in state space form as represented in [9]:

$$\frac{d(\delta x)}{dt} = \mathbf{B}\delta x(t) + \mathbf{A}\delta x(t-T) \quad (64)$$

where T is the time needed for one spindle revolution in the turning process, \mathbf{B} and \mathbf{A} are the linearized coefficient matrices of the process model and are functions of the machine-tool/workpiece structural parameters such as natural frequency, damping, and stiffness. Comparing Eq. (64) with Eq. (38) demonstrates the similarity of the two equations and this suggests the opportunity to apply the results presented in solving chatter equations. Considering a one-degree-of-freedom dynamic system with the parameters provided in [9], and defining $\delta x = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$, Eq. (64) can be written as

$$\begin{aligned} \frac{d(\delta x)}{dt} = & \begin{bmatrix} 0 & 1 \\ -\left(1 + \frac{K_c}{K_m}\right)\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \delta x(t) \\ & + \begin{bmatrix} 0 & 0 \\ \mu \frac{K_c}{K_m}\omega_n^2 & 0 \end{bmatrix} \delta x(t-T) \end{aligned} \quad (65)$$

where \mathbf{B} and \mathbf{A} are

$$\begin{aligned} \mathbf{B} &= \begin{bmatrix} 0 & 1 \\ -\left(1 + \frac{K_c}{K_m}\right)\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \\ \mathbf{A} &= \begin{bmatrix} 0 & 0 \\ \mu \frac{K_c}{K_m}\omega_n^2 & 0 \end{bmatrix} \end{aligned} \quad (66)$$

Applying the results obtained in the previous sections, the roots of the characteristic equations for Eq. (65) can be written using Eq. (50) as

$$s\mathbf{I} = \frac{1}{T}\mathbf{W}(\mathbf{A}Te^{-\mathbf{B}T}) + \mathbf{B} \quad (67)$$

The stability conditions are obtained by finding the roots of the characteristic equation:

$$\det\left\{s\mathbf{I} - \left[\frac{1}{T}\mathbf{W}(\mathbf{A}Te^{-\mathbf{B}T}) + \mathbf{B}\right]\right\} = 0 \quad (68)$$

This equation, for each branch of Lambert function ($k = -\infty, \dots, -1, 0, 1, \dots, \infty$), has two roots. It is shown in [12] that there are two distinct pairs of characteristic roots dictating the

stability of the system. Stability lobes for the roots associated with the three branches of the Lambert function ($k = -1, 0, 1$) given by the equations above where, $\zeta = 0.05$, $\omega_n = 150$, and $\mu = 1$, are obtained numerically using Eq. (68) and shown in Fig. 9. It needs to be emphasized that these lobes represent the stability conditions for the six related roots associated with $k = -1, 0, 1$ only, and the stability of the whole system depends on all the roots of the infinite spectrum. However, considering the results in [12], the simulation results shown in Fig. 9 indicate that the stability of the system is dominated by the two distinct roots associated with the principal branch of the Lambert function ($k = 0$).

Furthermore, the methodology presented here can yield the closed form solution to the linearized chatter delay differential equation (65). The results presented in Eq. (52) can be used to express the closed form solution to the homogeneous chatter equation (64) as;

$$\delta \mathbf{x}(t) = \sum_{k=-\infty}^{\infty} e^{(1/T)\mathbf{W}_k(\mathbf{A}Te^{-\mathbf{B}T}) + \mathbf{B})t} \mathbf{C}_k \quad (69)$$

where \mathbf{B} and \mathbf{A} are shown in Eq. (66) and the \mathbf{C}_k values are $n \times 1$ vectors representing the preshape function states.

Summary and Conclusions

In this paper, a new approach for the stability analysis, free solution, and forced solution of systems of linear delay differential equations has been presented. The solution obtained using Lambert functions is in a form analogous to the state transition matrix in systems of linear ordinary differential equations. A generalized matrix Lambert function is then introduced to solve systems of linear delay differential equation in vector-matrix form. Stability analysis, free response and forced response for several cases of DDEs based on their parameters is presented in the paper based on this new solution approach. As an example, The new approach is applied to find an analytic solution to a linear machine tool chatter problem, and its stability lobe for the three branches of the Lambert function is obtained.

The main advantage of the presented analytical approach is its ability to provide a closed form solution to systems of homogeneous linear delay differential equations in a compact form similar to systems of ordinary differential equations. The solution is in the form of an infinite series of modes, which are expressed in terms of Lambert functions. It provides a tool to study the behavior of the individual modes of the equation. The solution, however, is presented by a linear combination of all the modes in the form of an infinite series. A numerical approximation to this infinite series can be computed by the methodology presented in the Appendix.

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Appendix

Any given continuous function $\theta(t)$ can be represented as an infinite series using the Lambert coefficients, L_k , and the Lambert modes, $\xi_k(t)$

$$\theta(t) = \sum_{k=-\infty}^{\infty} L_k \xi_k(t) \quad t \in [0, b] \quad (70)$$

where the Lambert modes, $\xi_k(t)$ are the modes generated by the solutions of the delay differential equations, and b can go to infinity ($b \rightarrow \infty$).

To find the values of the coefficients L_k , assume that the most dominant modes are the first N modes where N is a large number ($N \rightarrow \infty$), and that the function can be approximated by,

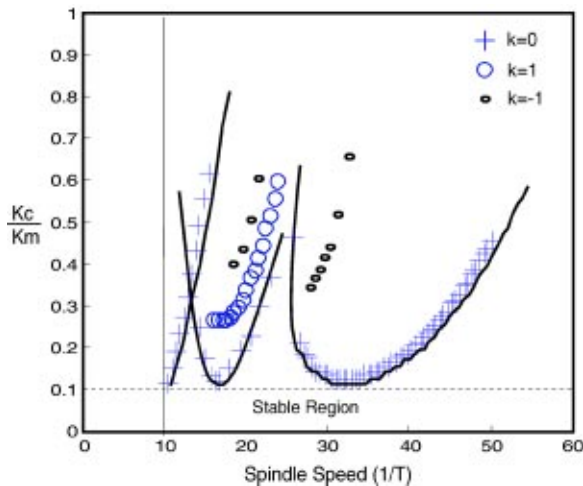


Fig. 9 Stability lobes for the chatter equation

$$\theta(t) \approx \sum_{k=-N}^N L_k \xi_k(t) \quad t \in [0, b] \quad (71)$$

By dividing the interval $[0, b]$ into $2N$ divisions, Eq. (71) can be written as,

$$\underbrace{\begin{bmatrix} \theta(b) \\ \theta\left(b - \frac{b}{2N}\right) \\ \theta\left(b - \frac{2b}{2N}\right) \\ \vdots \\ \theta(0) \end{bmatrix}}_{\Theta} = \underbrace{\begin{bmatrix} \xi_{-N}(b) & \cdots & \xi_N(b) \\ \xi_{-N}\left(b - \frac{b}{2N}\right) & \cdots & \xi_N\left(b - \frac{b}{2N}\right) \\ \xi_{-N}\left(b - \frac{2b}{2N}\right) & \cdots & \xi_N\left(b - \frac{2b}{2N}\right) \\ \vdots & \ddots & \vdots \\ \xi_{-N}(0) & \cdots & \xi_N(0) \end{bmatrix}}_{\Omega(b, N)} \underbrace{\begin{bmatrix} L_{-N} \\ L_{-N+1} \\ L_{-N+2} \\ \vdots \\ L_N \end{bmatrix}}_{\mathbf{L}} \quad (72)$$

where the vector \mathbf{L} represents an approximation for the coefficients L_k for large values of N . The function $\theta(t)$ can then be written as

$$\theta(t) = \lim_{N \rightarrow \infty} \sum_{k=-N}^N L_k \xi_k(t) \quad t \in [0, b] \quad (73)$$

and assuming that the $(2N+1) \times (2N+1)$ matrix Ω is invertible, \mathbf{L} can be obtained as

$$\mathbf{L} = \Omega^{-1}(b, N) \Theta \quad (74)$$

and L_k can also be represented as

$$L_k = \lim_{N \rightarrow \infty} \{\Omega^{-1}(b, N) \Theta\}_k \quad (75)$$

where $\{\cdot\}_k$ represents the k^{th} element of the corresponding vector. Finally, Eq. (73) can be written as

$$\theta(t) = \lim_{N \rightarrow \infty} \sum_{k=-N}^N \{\Omega^{-1}(b, N) \Theta\}_k \xi_k(t) \quad t \in [0, b] \quad (76)$$

Coefficients of homogeneous and particular solutions of the delay differential equations can be evaluated using the approach presented. For the homogeneous solution, the preshape function $\phi(t)$ can be written as

$$\phi(t) = \sum_{k=-\infty}^{\infty} C_k \xi_k(t) \quad t \in [0, T] \quad (77)$$

where $\xi_k(t) = e^{(1/T) W_k(-\alpha T e^{\beta T}) - \beta t}$ for the delay equation in Eq. (3), and $\xi_k(t) = e^{1/T W_k(-\alpha T) t}$ for the delay equation in Eq. (3) with $\beta=0$. For the homogeneous solution Eq. (72) can be written as

$$\underbrace{\begin{bmatrix} \phi(T) \\ \phi\left(T - \frac{T}{2N}\right) \\ \phi\left(T - \frac{2T}{2N}\right) \\ \vdots \\ \phi(0) \end{bmatrix}}_{\Phi} = \underbrace{\begin{bmatrix} \xi_{-N}(T) & \cdots & \xi_N(T) \\ \xi_{-N}\left(T - \frac{T}{2N}\right) & \cdots & \xi_N\left(T - \frac{T}{2N}\right) \\ \xi_{-N}\left(T - \frac{2T}{2N}\right) & \cdots & \xi_N\left(T - \frac{2T}{2N}\right) \\ \vdots & \ddots & \vdots \\ \xi_{-N}(0) & \cdots & \xi_N(0) \end{bmatrix}}_{\Omega(T, N)} \underbrace{\begin{bmatrix} C_{-N} \\ C_{-N+1} \\ C_{-N+2} \\ \vdots \\ C_N \end{bmatrix}}_{\mathbf{C}} \quad (78)$$

where the vector \mathbf{C} represents an N -term approximation of C_k , and consequently C_k can be represented as

$$C_k = \lim_{N \rightarrow \infty} \{\Omega^{-1}(T, N) \Phi\}_k \quad (79)$$

The particular solution for the delay differential equation is obtained assuming that the external excitation function $u(t)$ can be represented as

$$u(t) = \sum_{k=-\infty}^{\infty} f_k \xi_k(t) \quad t \in [0, b]_{b \rightarrow \infty} \quad (80)$$

For the particular solution, Eq. (72) can be written as

$$\underbrace{\begin{bmatrix} u(b) \\ u\left(b - \frac{b}{2N}\right) \\ u\left(b - \frac{2b}{2N}\right) \\ \vdots \\ u(0) \end{bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} \xi_{-N}(b) & \cdots & \xi_N(b) \\ \xi_{-N}\left(b - \frac{b}{2N}\right) & \cdots & \xi_N\left(b - \frac{b}{2N}\right) \\ \xi_{-N}\left(b - \frac{2b}{2N}\right) & \cdots & \xi_N\left(b - \frac{2b}{2N}\right) \\ \vdots & \ddots & \vdots \\ \xi_{-N}(0) & \cdots & \xi_N(0) \end{bmatrix}}_{\Omega(b, N)} \underbrace{\begin{bmatrix} f_{-N} \\ f_{-N+1} \\ f_{-N+2} \\ \vdots \\ f_N \end{bmatrix}}_{\mathbf{f}} \quad (81)$$

where vector \mathbf{f} represents an N -term approximation of f_k , and consequently f_k can be represented as

$$f_k = \lim_{\substack{N \rightarrow \infty \\ b \rightarrow \infty}} \{\Omega^{-1}(b, N) \mathbf{u}\}_k \quad (82)$$

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