Learning and Money Adoption

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Abstract

We consider a dynamic monetary economy where agents gradually learn about the holding cost of a new asset and coordinate to adopt it as money or abandon it. We provide closed-form solutions for state-contingent asset prices and agents' adoption decision. The transactional benefits of using money are endogenous and can convexify or concavify the payoff structure. Thus, the arrival of new information can raise or reduce the asset's price, which is in sharp contrast to standard insights in experimentation models. Full disclosure of the asset type and an increase in the learning speed improve information but have different welfare implications.

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1 Introduction

When will a new money become a medium of exchange? This is a classic question in monetary theory, but has become timely again due to the growing popularity of crypto-currencies and digital monies. From ideas going back at least to Jevons (1876), the use of money depends on the intrinsic properties of the asset, such as storability, portability, and recognizability, and coordination among agents. In reality, it takes time for agents to learn about these properties and coordinate, and this learning process is especially relevant now because a digital money often associates with sophisticated new technologies. It is possible that agents abandon a high-quality new money during the learning process. Our goal is to study the dynamics of money adoption by modeling the learning and coordination process explicitly. We will characterize the life cycle of a new money — from its initial introduction to its eventual circulation or disappearance. We use the theory to study how information affects money adoption, prices, and welfare.

We introduce experimentation with two-armed bandits into the New Monetarist model Lagos and Wright (2005). There are a government money and a new asset that can be used as payment devices. In each period, with chance α^b a buyer can pay with both assets. With chance α^c only the new asset can be used. We interpret α^c as the trading opportunities created by the technologies associated with the new asset. Holding the government money is free but holding the new asset incurs a stochastic storage cost. The distribution of the storage cost is either H or L and is unknown to agents. Agents learn about the distribution by observing the realized storage costs and noisy public signals, and coordinate to use the new asset as money or abandon it. When the asset circulates, agents' belief about the storage cost rises if a good signal arrives and falls if a bad signal arrives. We focus on equilibria where the asset is abandoned when agents' belief falls below an endogenous cutoff. In the long run, the asset is either abandoned or permanently used as money. Our theory applies to various preferences, we use log utility in the baseline as it admits closed-form solutions of the asset price and the cutoff of disposal (Proposition 1). We solve the general case in Online Appendix.

The main message is that learning about money and learning about other real assets create very different conclusions regarding the impact of information. To represent the impact of information, we define the *information premium* as the expected change in the asset's price when a new signal arrives. We will show that the sign of this premium is crucial for predicting the positive and normative impact of information. In standard investment problems, more information usually raises the option value of the investment;

hence, the information premium is often positive in experimentation problems.

Money is a special asset as its dividends are the benefits of using it for transactions.[1] When agents are indifferent between using or abandoning money, the arrival of a good signal will slightly raise the value of money, while the arrival of a bad signal will induce agents to stop accepting money, and thus the transactional benefits vanish completely and instantly. Thus, the endogenous transactional benefit amplifies the negative impact of bad signals. This amplification can concavify or convexify the payoff structure and is the key determinant of the sign of the information premium. We show that *the information premium can be positive or negative* and is positive if and only if the information arrival chance or the nominal interest rate of government money is large (Proposition 2). As the information arrival chance rises, i.e., signals arrive more often, the cutoff of disposal falls. With a positive information premium, asset prices rise at all states (i.e., beliefs) as the information arrival chance rises. If the premium is negative, then asset prices rise for small beliefs and fall for all larger beliefs. (Proposition 3).

How does new information about the quality of money affect welfare? We define welfare as the expected discounted sum of trade surpluses minus the storage costs. Using the logic of a famous example in Hirshleifer (1971), Andolfatto and Martin (2013) argue that a government should suppress information regarding the means of payment. Their logic is that the surplus of monetary trade is concave in the value of money, so one can improve welfare by hiding information regarding the means of payment. We similarly show that the arrival of a signal reduces welfare on average, provided that the information premium is negative. But when the information premium is positive, the arrival of a signal can improve welfare, provided that agents are sufficiently optimistic. It is because the increase in asset prices raises the amount of output in trades and this effect dominates the Hirshleifer effect (Proposition 4).

Regulators can often force a private issuer to disclose information regarding the technology, stability, and reliability of a new digital money. We consider the welfare impact of two information disclosure polices — full disclosure of the asset's type and an increase in the information arrival chance. Full disclosure improves welfare if and only if agents' belief is low (Proposition 5). Intuitively, when agent's belief is high, hiding information is better as it raises the expected duration that the asset circulates. When agents' belief is low, they want to abandon the asset. Full disclosure can improve welfare

[1] This view is consistent with Tirole (1985): "the monetary market fundamental is not defined solely by the sequence of real interest rates. Its dividend depends on its price. [...] the market fundamental of money in general depends on the whole path of prices (to this extent money is a very special asset)."

as there is a small chance that agents will change their mind about money adoption.

As the information arrival chance rises, there are two welfare effects. First, the economy transitions more rapidly across states. Depending on the sign of the information premium, this effect can raise or reduce welfare. Second, the cutoff of disposal falls and the state-contingent asset prices change. The overall impact is ambiguous. Since the cutoff of disposal falls, agents are willing to hold the asset even when the belief is lower, by reveal preference they must be better off. Thus, for beliefs sufficiently close to the cutoff of disposal, welfare rises. For sufficiently high beliefs, welfare rises if and only if the asset price rises, which is equivalent to the information premium being positive (Proposition 6). A lesson is that although full disclosure and an increase in the information arrival chance are related their welfare impacts can be quite different.

Related Works. We build on the New Monetarist models surveyed in Lagos et al. (2017). Araujo and Guimaraes (2014, 2017) study coordination in the use of money in a model with indivisible money and indivisible goods, aka Kiyotaki and Wright (1989). They assume the intrinsic value of money evolves according to a stochastic process and prove there is a unique rationalizable equilibrium. Oberfield and Trachter (2012) also obtain an uniqueness result in the Kiyotaki and Wright model as the frequency of trade explodes. We complement them by analyzing a model with divisible money and goods. Araujo and Camargo (2015) study monetary exchanges as a dynamic game and show that money can fail to be essential even if public signals or monitoring is very limited. Lester et al. (2012) and Zhang (2014) study how the recognizability and usability of money affects its adoption. They study steady states while we study the dynamic process starting from the asset's issuance to its eventual abandonment or circulation.

We follow Geromichalos et al. (2007) and Lagos (2010) to introduce assets as payment devices. Lagos (2010) considers a monetary model with risky and safe assets and studies whether an asset's transactional role can explain the equity premium puzzle. We show that the endogenous transaction benefits of a liquid asset can concavify or convexify the payoff structure and hence information or risk can raise or reduce the asset's price.

We introduce experimentation with two-armed bandits, as exemplified by Rothschild (1974), Roberts and Weitzman (1981) and Keller and Rady (2015), into a monetary model. Balutel et al. (2021) use a related model to interpret Bitcoin adoption data. Compare to these works, we explicitly model the transactional role of money and show that the presence of the endogenous liquidity premium can lead to new, and sometimes opposite, insights regarding the role of information in bandit problems.

2 Model

The environment is similar to that of Lagos and Wright (2005). There are unit measure of buyers and unit measure of sellers. Time is discrete and continues forever. Each period has two subperiods: there is a decentralized market (DM) with payment frictions, followed by a frictionless centralized market (CM). There are two assets: a unit supply of a new asset, denoted by a, and M units of government fiat money, denoted by m.

In the DM, buyers and sellers meet randomly and bilaterally to trade a divisible good. Buyers enjoy utility $u(q)$ from consuming q units of the good and sellers' production cost is q. But buyers cannot borrow from sellers because they cannot commit to repay. Hence, buyers must pay with m or a. With probability α^b a buyer is in a type-b meeting where he can pay with both m and a. With probability α^c it is a type-c meeting and he can only pay with a.[2] We interpret α^b as the measure of trading opportunities available prior to the introduction of the new asset and α^c as the measure of trading opportunities created by the technologies associated with a cryptocurrency or digital money. The terms of trade is such that the buyer makes a take-it-leave-it offer to the seller.

In the CM, agents adjust their portfolio (a, m) by trading a numeraire good x and supply labor h. Utility from consumption and production in the CM is $U(x) - h$. Agents live forever and discount between the CM and DM at $\beta \in (0, 1)$. As usual in models following Lagos and Wright (2005), we assume $U(x)$ is quasi-linear and the production technology of the numeraire is linear in h such that agents' optimal portfolio choice does not depend on the current portfolio.[3]

The key novelty is that the new asset's type is either H or L and agents can only gradually learn about it. Holding the new asset incurs a per-unit storage cost which is stochastic and its realization is the same across all units of asset. This can be a cost to maintain a digital account and prevent theft or a loss due to the instability or insecurity of the record keeping system. In each period, with chance $1 - \chi$ the storage cost is zero. With chance χ, the cost is a random variable $C \in \{\underline{c}, \bar{c}\}$, where $\bar{c} > \underline{c} > 0$, and C follows a symmetric binary distribution: for H type asset, $P(C = \underline{c}|H) = s$ and $P(C = \bar{c}|H) = 1 - s$ where $s \in (0.5, 1)$. For L type asset, $P(C = \underline{c}|L) = 1 - s$ and

[2]One can also assume there is a measure of meetings where only m is accepted. We omit these meetings because they complicates the model and do not bring new insights.

[3]The presence of a CM with quasi-linear preferences conveniently reduces the distribution of money holding to a single value, but it might have non-trivial effect on money adoption — without a CM an agent may be willing to produce in exchange for an asset of inferior quality because she needs it as money in the future. Hence the presence of a CM is likely to reduce agents' incentive to adopt the new asset as money. We thank a referee for this observation.

$P(C = \bar{c}|L) = s$. Hence, the expected storage cost is higher when the asset type is L, i.e., $E[C|L] > E[C|H]$. When the realized storage cost is zero, agents do not learn from the realized cost, but with chance ζ all agents will receive a noisy public signal which has the same distribution as C. Hence, agents learn about the asset by observing the realizations of C or that of the public signals. Let $\eta \equiv \chi + \zeta < 1$ be the *information arrival chance*. When we study changes in information we will keep χ fixed and change ζ such that the distribution of storage cost is unchanged. We assume all agents receive the same information and thus there is no information asymmetry.

If no one holds the new asset for a period, then it disappears forever and there will be no trade in type-c meetings afterwards. The government money is free to hold and always circulates. We focus on stationary Markov equilibrium where current-period outcomes depend only on the current-period beliefs and not on histories or time. We assume $E[C|H]$ is small such that, if agents are sufficiently optimistic then they are willing to hold the asset, we will be more specific later. We study equilibria where agents abandon the new asset if and only if their beliefs about the asset type are sufficiently pessimistic.

3 Equilibrium

Beliefs Updating. Let $\pi \in [0, 1]$ be agents' belief that the asset type is H. It is the state variable of the economy. Agents update belief in the beginning of the CM based on the realized storage cost or public signal. The realization is either $C = \underline{c}$ (a good signal) or $C = \bar{c}$ (a bad signal). If $C = \underline{c}$, then by Bayes rule the posterior belief π' solves

$$\frac{\pi'}{1 - \pi'} = \frac{\pi s}{(1 - \pi)(1 - s)}, \tag{1}$$

and agents become more optimistic, i.e., $\pi' > \pi$. Since signals follow a symmetric binary distribution, if a good and a bad signal arrive consecutively, then the posterior remains unchanged. As a result, to keep track of beliefs, we only need to count the net number of good signals. Index π by j such that if the current belief is π_j and a good signal arrives, then the posterior is π_{j+1}. Let $\pi_0 \in (0, 1)$ be agents' initial prior at period $t = 0$. If the net number of good signals since period 0 is j, then the current-period belief is π_j.

Asset holding. In the CM agents choose their portfolio holding. Let ϕ_j be the CM price of the asset and ψ_j be the CM price of government money when the current-period belief is π_j. An agent with asset portfolio $(\tilde{m}, \tilde{a})$ in the CM solves

$$
\begin{aligned}
W(\tilde{m}, \tilde{a}, \pi_j) &= \max_{x,h,m,a} \{U(x) - h + \beta V(m, a, \pi_j)\} \\
\text{s.t.} \quad & x = \nu h + \tau + \phi_j(\tilde{a} - a) + \psi_j(\tilde{m} - m),
\end{aligned}
$$

where V is the DM value function, ν is the CM wage rate, and τ is a lump-sum transfer or tax from the government. The agent chooses the consumption of the numeraire, x, the labor supply, h, and the asset portfolio (m, a) to bring to the next DM subject to the budget constraint. The government controls the supply of government money via the lump-sum transfer or tax τ. We adopt a linear CM production technology, $\nu = 1$. Then, after eliminating h with the budget constraint, we have

$$
W(\tilde{m}, \tilde{a}, \pi_j) = \phi_j \tilde{a} + \psi_j \tilde{m} + \max_{x} \{U(x) - x\} + \max_{m,a} \{-\phi_j a - \psi_j m + \beta V(m, a, \pi_j)\}.
$$

We assume $U(0) = 0$, $U' > 0$, $U'' < 0$, and there exists an optimal CM consumption x^* that solves $U'(x^*) = 1$. For all agents, the choice of (m, a) is independent of the current portfolio $(\tilde{m}, \tilde{a})$. Also, the CM value function is linear in $\tilde{m}$ and $\tilde{a}$, namely $W_a(m, a, \pi_j) = \phi_j$ and $W_m(m, a, \pi_j) = \psi_j$. This linearity will be useful for simplifying the analysis of DM trades. For buyers, the first-order condition for m and a are

$$
\psi_j = \beta V_m(m, a, \pi_j), \tag{2}
$$

$$
\phi_j = \beta V_a(m, a, \pi_j). \tag{3}
$$

For sellers, $m = a = 0$ as they do not need asset for payment in the DM.

Decentralized Market Trades. In the DM, depending on the type of the meeting, the buyers can pay with either (a, m) or a. We define a buyer's *liquidity* ℓ as the expected value of the relevant assets in a DM meeting. To compute ℓ, let s_j be an agents' subjective probability that the next signal is a good one:

$$
s_j \equiv \bar{s}(\pi_j) \equiv \pi_j s + (1 - \pi_j)(1 - s). \tag{4}
$$

Let $c^H \equiv \chi[s\bar{c} + (1 - s)\underline{c}]$ and $c^L \equiv \chi[(1 - s)\bar{c} + s\underline{c}]$ be the expected storage cost when the asset quality is H and L, respectively. Then, the expected storage cost of the new asset at state π_j can be expressed as

$$
c_j \equiv \bar{c}(\pi_j) \equiv \pi_j c^H + (1 - \pi_j)c^L.
$$

For a seller, receiving assets in the DM means she will carry more assets to the next

CM. By the linearity of the CM value function, $W(\tilde{a}, \tilde{m}, \pi)$, the marginal benefit of receiving a units of asset is simply the product of a and the expected value of a unit of asset in the next CM. Therefore, the liquidity of a buyer in a c-meeting is given by

$$\ell_j^c \equiv a[\eta s_j \phi_{j+1} + \eta(1-s_j)\phi_{j-1} + (1-\eta)\phi_j - c_j]. \tag{5}$$

The expression in the square bracket is the expected value of one unit of asset in the next CM. If information arrives in the next CM, which has chance η, then the expected CM price of the asset is $s_j\phi_{j+1} + (1-s_j)\phi_{j-1}$, otherwise it is ϕ_j. For each unit of the asset, there is an expected storage cost of c_j.

In a b-meeting, buyers' liquidity is $\ell_j^b = \ell_j^m + \ell_j^c$, where the liquidity of m is

$$\ell_j^m \equiv m[\eta s_j \psi_{j+1} + \eta(1-s_j)\psi_{j-1} + (1-\eta)\psi_j]. \tag{6}$$

During a DM meeting, a buyer makes an offer that maximizes the DM trade surplus subject to the payment constraint, namely

$$\arg\max_q [u(q) - q] \text{ s.t. } q \leq \ell.$$

We assume the DM utility $u(q)$ satisfies $u' > 0$ and $u'' \leq 0$. If the efficient amount of trade, q^*, solution to $u'(q^*) = 1$, is finite, then the trade quantity is $q = \min\{\ell, q^*\}$ and otherwise it is $q = \ell$. The surplus is $S(\ell) \equiv u(q) - q$. Easily $S'(\ell) = u'(\ell) - 1 > 0$ for $\ell \in [0, q^*)$ and $S'(\ell) = 0$ for $\ell \geq q^*$.

Buyers' DM value function, $V(m, a, \pi_j)$, is the sum of the expected DM trade surplus and the expected continuation value to the next CM, namely

$$V(m, a, \pi_j) = \alpha^b S(\ell_j^b) + \alpha^c S(\ell_j^c) + E[W(m, a, \pi')|\pi_j]. \tag{7}$$

The expectation in the last term is taken over the realization of signals in the next CM given the current belief, π_j.

Asset Pricing. Due to the linearity of $W(m, a, \pi)$ in m and a, the slope of the DM value function V in (7) can be written as

$$V_a(m, a, \pi_j) = [\alpha^b S'(\ell_j^b) + \alpha^c S'(\ell_j^c) + 1]\frac{\ell_j^c}{a}, \tag{8}$$

$$V_m(m, a, \pi_j) = [\alpha^b S'(\ell_j^b) + 1]\frac{\ell_j^m}{m}. \tag{9}$$

Using (3), (8) and the market clearing $a = 1$, the asset price, ϕ_j, solves

$$\phi_j = \beta \ell_j^c [1 + \alpha^b S'(\ell_j^b) + \alpha^c S'(\ell_j^c)]. \tag{10}$$

The left side is the marginal cost of asset holding and the right side is the marginal benefit, which is the sum of the expected value of holding one more unit of asset in the next CM, $\beta \ell_j^c$, and the marginal increase in DM trade surplus, $\beta \ell_j^c [\alpha^b S'(\ell_j^b) + \alpha^c S'(\ell_j^c)]$.

Similarly, using (2), (9) and the market clearing $m = M$ for government money, the first-order condition of holding government money can be rewritten as

$$\psi_j = \beta \frac{\ell_j^m}{M} [1 + \alpha^b S'(\ell_j^b)]. \tag{11}$$

We assume the government targets the nominal interest rate of government money, i, by adjusting the total money supply, M, each period (In Online Appendix F we explore a version where the supply is fixed). As usual, i is defined by the Fisher equation

$$1 + i = \frac{1}{\beta} \left(\frac{\psi_j}{\ell_j^m / M} \right),$$

where $1/\beta$ is the equilibrium real interest rate and the fraction in parenthesis is the expected inflation rate because it is the value of money today divided the expected value of money next period. A higher i means the opportunity cost of carrying government money is higher. Using (11) and the definition of i, (10) can be rewritten as

$$\phi_j = \beta \ell_j^c [1 + i + \alpha^c S'(\ell_j^c)]. \tag{12}$$

This pricing equation holds as long as the new asset circulates. The transactional benefit in b-meetings is linear in ℓ_j^c and i, while that in c-meetings can be non-linear in ℓ_j^c.

Asset Abandonment and Off-Equilibrium-path Beliefs. We focus on equilibria where agents dispose the asset if and only if π_j is smaller than a certain cutoff. Let π_d be this cutoff. For $\pi_j \leq \pi_d$ the asset is disposed, so $\phi_j = 0$. For $\pi_j > \pi_d$, ϕ_j solves (12).

Since the asset price is 0 at state π_d, agents choose $a = 0$ only if the expected value of holding asset is negative, i.e., $\ell_d^c \leq 0$. When $\ell_d^c \leq 0$, not only the asset's return across CM is negative, but it also cannot be used as payment in DM because no seller would accept an asset with negative value. To characterize ℓ_d^c, we define $\tilde{\phi}_j$ as an agent's off-equilibrium-path belief about the price of the asset after other agents have abandoned it. Then, the condition $\ell_d^c \leq 0$ can be rewritten as

$$\eta s_d \tilde{\phi}_{d+1} + \eta(1 - s_d)\tilde{\phi}_{d-1} + (1 - \eta)\tilde{\phi}_d - c_d \leq 0. \tag{13}$$

We assume $\tilde{\phi}_j = \phi_j$, namely that the prices off and on equilibrium path are the same. Using this assumption and $\phi_d = \phi_{d-1} = 0$, the condition can be rewritten as

$$c_d \geq \eta s_d \phi_{d+1}. \tag{14}$$

Our assumption that $\tilde{\phi}_j = \phi_j$ can be justified by the following refinement: suppose when $\pi = \pi_d$ a fraction $1-\epsilon$ of agents abandon the asset and never trade it again. But a fraction $\epsilon > 0$ of agents hold on to the asset for one more period. If a good signal arrives in the next period, then they resume trading the asset as if they are on the equilibrium path. Otherwise they also abandon the asset. Given this assumption, if an agent holds the asset for one more period, then he could sell it at price ϕ_{d+1} if a good signal arrives. Hence, he believes that $\tilde{\phi}_{d+1} = \phi_{d+1}$ and $\tilde{\phi}_d = \tilde{\phi}_{d-1} = 0$. Our model is the limit $\epsilon \to 0$.[4]

Now we are ready to define an equilibrium.

Definition 1 *A stationary equilibrium is a list $\langle \{q_j, \phi_j\}_{j=d}^{\infty}, \pi_d \rangle$ that (i) q_j satisfies the equilibrium conditions in the DM market, (ii) $\phi_j = 0$ for $j \leq d$ and ϕ_j solves (12) for $j > d$, and (iii) the incentive condition (14) holds at $\pi_d \in (0, 1)$.*

Since asset holding is costly and the asset is intrinsically worthless, there always exists a nonmonetary equilibrium where the new asset is not used at all π_j. We focus on equilibrium where π_d is interior and the asset circulates when the agents are sufficiently optimistic. We present a schematic of the equilibrium price ϕ_j as a function of π_j in the left panel of Figure 1. In the long-run the asset is abandoned ($\pi = \pi_d$) or fully adopted ($\pi \to 1$). If the asset type is L, then the belief will eventually hit π_d. If the asset type is H, then with probability $1 - (1/s - 1)^{-d}$ the belief $\pi \to 1$ (shown in Online Appendix B) and the new asset will be adopted permanently. Naturally, this probability falls with the cutoff π_d, which is determined endogenously.

For tractability, we make two assumptions. We assume the DM utility is $u(y) = B\log(y) + y$ which implies the gains from trade, $u(y)–y$, is logarithmic. This assumption admits closed-form solutions for ϕ_j. We will discuss general preferences in Section 6.

We will also assume π_d is such that condition (14) binds and the reason is as follows. In Appendix, we show that the cutoff belief π_d must lie in an interval $[\underline{\pi}, \bar{\pi}]$ where $\underline{\pi}$ and $\bar{\pi}$ are only one signal apart, i.e., if the current belief is $\underline{\pi}$, then the posterior is $\bar{\pi}$ when one good signal arrives. It follows that given the initial prior π_0, there generically exists a unique cutoff $\pi_d \in [\underline{\pi}, \bar{\pi}]$. But given the discrete nature of the beliefs, a small

[4] If the off-equilibrium-path belief is $\tilde{\phi}_j = 0$ $\forall j$, then the left side of (13) is smaller and hence the inequality is easier to satisfy. Thus the equilibrium set is larger and multiple equilibria is possible.

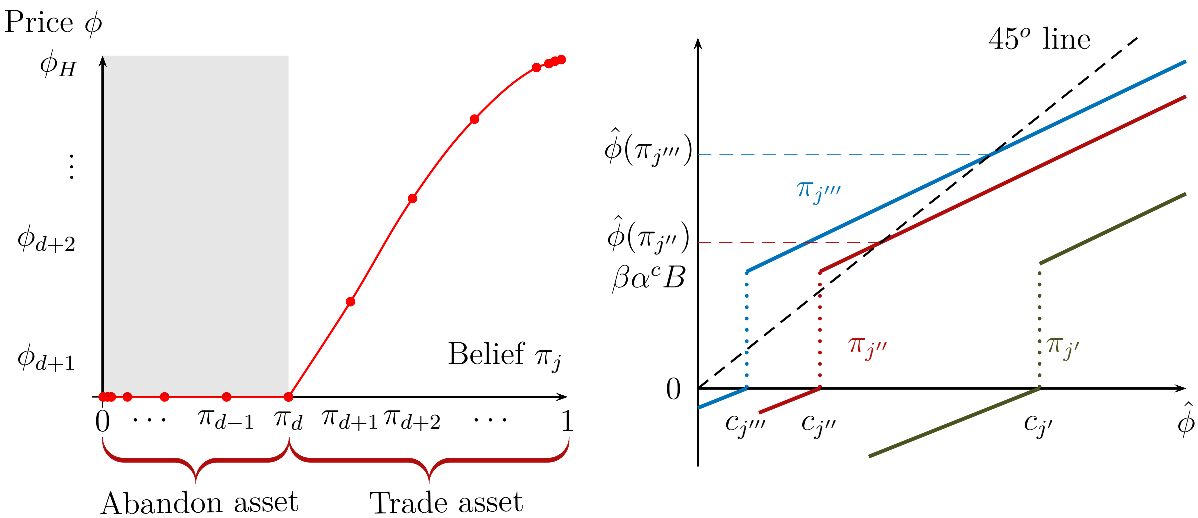

Figure 1: (Left) Equilibrium prices. (Right) Determination of $\hat{\phi}$ where $\pi_{j'''} > \pi_{j''} > \pi_{j'}$.

change in parameter value will lead to either no change or a discrete change in π_d, which makes the comparative statics clumsy. To simplify the presentation of the results, we characterize the equilibrium assuming π_d is such that (14) binds, which corresponds to $\pi_d = \bar{\pi}$. By doing so π_d varies continuously with parameters.

No Learning Benchmark. Before presenting the results of our dynamic model, it is useful to consider a benchmark where agents do not learn and π_j is fixed exogenously. Since beliefs are not changing, one can characterize the equilibrium by considering the steady state allocations. We denote $\hat{\phi}(\pi_j)$ as the price of the asset in such steady state, call it the *stationary price*. In a monetary equilibrium this price solves (12) with $\eta = 0$, namely

$$\hat{\phi}(\pi_j) = \beta[\hat{\phi}(\pi_j) - c_j] + \beta\{[\hat{\phi}(\pi_j) - c_j]i + \alpha^c B\}\mathbb{1}_{\{\hat{\phi}(\pi_j) > c_j\}}. \tag{15}$$

The left side is the marginal cost of asset holding. The first term on the right side is the marginal benefit of carrying asset across CMs minus the expected storage cost. The second term is the marginal transactional benefit in DM meetings. This benefit exists only when the value of the asset exceeds the storage cost, i.e., the indicator function $\mathbb{1}_{\{\hat{\phi}(\pi_j) > c_j\}} > 0$, for otherwise no seller would accept the asset. Since asset holding incurs a cost, there is always a non-monetary equilibrium where asset price is 0 and the marginal cost of asset holding strictly exceeds the marginal benefit. We illustrate the determination of asset price in monetary steady states in the right panel of Figure 1 where the right side of (15) is denoted by the colored lines. As π_j rises, the curve that represents the right side of (15) shifts to the left. For sufficiently large π_j, there is a positive solution of $\hat{\phi}(\pi_j)$ and a monetary steady state exists if and only if $\hat{\phi}(\pi_j) \geq c_j$.

Due to log utility, the monetary steady is unique when it exists.[5] By (15), $\hat{\phi}(\pi_j)$ can be written as

$$\hat{\phi}(\pi_j) = \frac{\beta[B\alpha^c - (1+i)c_j]}{1-\beta(1+i)}, \tag{16}$$

which naturally increases in $B\alpha^c$ and π_j. It rises in i because the asset and government money are competing medium of exchanges in b-meetings. Below we assume $\hat{\phi}(1) > c^H$ such that there exist a monetary steady state when agents think the asset type is H. We do not require assumptions on $\hat{\phi}(0)$ and it can be positive or negative.

4 Characterization and Impact of Information

4.1 Value of asset and the cutoff of disposing

We start by characterizing the asset prices when the asset circulates. By (5) and the market clearing condition $a = 1$, the asset pricing equation (12) can be rewritten as

$$\phi_j = \beta\{B\alpha^c - (1+i)c_j + (1+i)[\eta s_j \phi_{j+1} + \eta(1-s_j)\phi_{j-1} + (1-\eta)\phi_j]\}, \tag{17}$$

which is a second-order difference equation with variable coefficients. By exploiting the structure of Bayesian learning, we could solve for the state-contingent asset prices:

Proposition 1 (Characterization of Equilibrium)

1. ***(Existence.)*** *There exists a unique equilibrium with $\pi_d \in (0,1)$ if and only if*

$$\hat{\phi}(0)(1-s-s\Gamma) \geq c^L > c^H \geq \hat{\phi}(1)s(1-\Gamma), \tag{18}$$

where $\Gamma < (1-s)/s$ is the smaller root of the quadratic equation

$$\eta s\Gamma^2 + \left[(1-\eta) - \frac{1}{\beta(1+i)}\right]\Gamma + \eta(1-s) = 0.$$

2. ***(Cutoff of disposal.)*** *The cutoff $\pi_d \in (0,1)$ is given by*

$$\pi_d = \frac{c^L/\eta - \hat{\phi}(0)(1-s-\Gamma s)}{(c^L - c^H)/\eta + \hat{\phi}(1)(1-\Gamma)s - \hat{\phi}(0)(1-s-\Gamma s)}, \tag{19}$$

and it falls in B, α^c, i and η.

[5] In general, in this class of models if the expected dividend from an asset is negative, then there is a non-monetary steady state and an even number of monetary steady states. With log utility, one steady state is omitted as the right side of (15) jumps at c_j, as shown by the blue and red lines in the right panel of Figure 1. The omitted steady state would reappear if the right side of (15) is continuous in $\hat{\phi}$.

3. ***(Asset pricing.)*** *For $\pi_j \geq \pi_d$, the asset price ϕ_j is given by*

$$\phi_j = \phi(\pi_j) \equiv \hat{\phi}(\pi_j) - \hat{\phi}(\pi_d)\frac{\pi_j}{\pi_d}\left[\frac{(1-\pi_j)\pi_d}{\pi_j(1-\pi_d)}\right]^{\frac{\log(\Gamma)}{\log[(1-s)/s]}}. \tag{20}$$

As π_j rises from π_d to 1, ϕ_j rises from 0 to $\hat{\phi}(1)$.

Part 1 suggests that agents abandon the asset at some interior π_d if $\hat{\phi}(1)$ is not too high and $\hat{\phi}(0)$ is not too low. By Part 2 the cutoff naturally falls in B and α^c. It also falls in the nominal rate, i, as the government money competes with the new asset in type-b meetings. Thus, agents are more willing to hold the new asset when the opportunity cost of holding government money rises. At state π_d, the agents must choose to hold or abandon the asset and information is valuable for decision making. Hence, a higher η raises the option value of holding the asset and lowers π_d.

Part 3 states that the asset price can be written as a continuous function of π_j. It is the sum of two terms: The first term in the right side of (20), $\hat{\phi}(\pi_j)$, represents the transactional benefits minus the storage cost, holding the belief π_j constant. The second term is the price premium because of learning which can be positive or negative and it vanishes as $\pi_j \to 1$. Hence, the presence of learning can increase or reduce the asset value compared to the no learning benchmark. We will discuss the determinants of the sign of this price premium in details.

4.2 Information premium

Now we characterize the price impact of the arrival of a new signal and show that it is closely related to the sign of the price premium in (20). To represent the average impact of a new signal on asset prices, we define the *information premium* as

$$\lambda_j \equiv s_j\phi_{j+1} + (1-s_j)\phi_{j-1} - \phi_j, \tag{21}$$

which is the expected change of asset price when a signal arrives at state π_j. The next result argues that λ_j is proportional to the premium on the right side of (20) and characterizes how market fundamentals affect the sign of λ_j.

Proposition 2 (Information premium) *The information premium λ_j is given by*

$$\lambda_j = \lambda(\pi_j) \equiv -\frac{[1-\beta(1+i)]}{\beta(1+i)\eta}\hat{\phi}(\pi_d)\frac{\pi_j}{\pi_d}\left[\frac{(1-\pi_j)\pi_d}{\pi_j(1-\pi_d)}\right]^{\frac{\log(\Gamma)}{\log[(1-s)/s]}}. \tag{22}$$

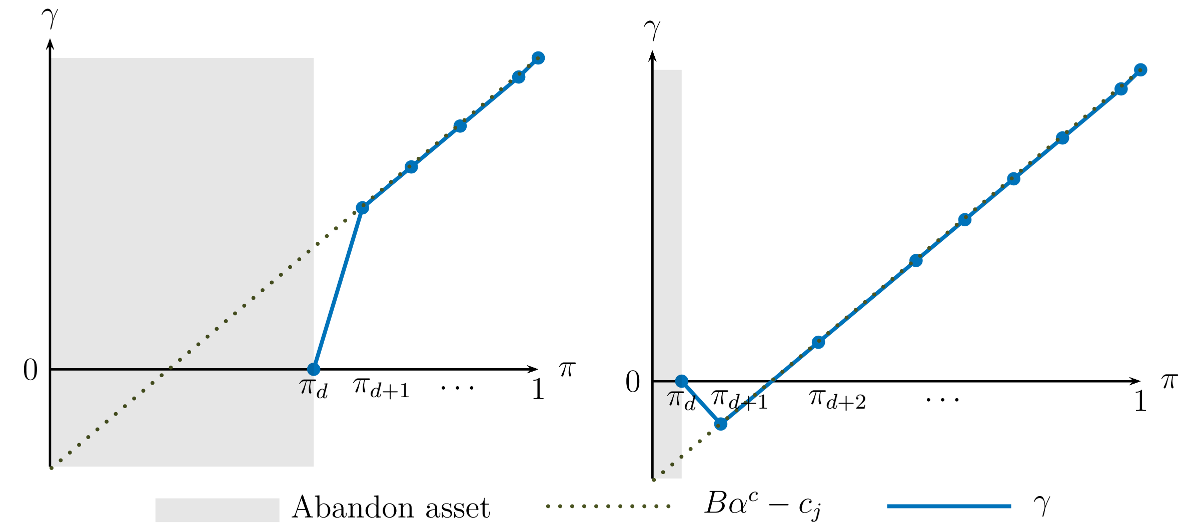

Figure 2: Marginal benefit of asset holding $\gamma(\pi_j)$. (Left) Large π_d. (Right) Small π_d.

As π_j rises, $|\lambda_j|$ falls and vanishes as $\pi_j \to 1$. If $\hat{\phi}(0) > 0$, then $\lambda_j < 0$. When $\hat{\phi}(0) \leq 0$:

1. ***(Change in information arrival chance.)*** *There exists a cutoff $\hat{\eta}$ such that if $\eta \geq \hat{\eta}$, then $\lambda_j \geq 0$ for $\pi_j \geq \pi_d$. If $\eta < \hat{\eta}$, then $\lambda_j < 0$ for $\pi_j \geq \pi_d$.*

2. ***(Change in monetary policy.)*** *There exists a cutoff $\hat{\imath}$ such that if $i > \hat{\imath}$, then $\lambda_j > 0$ for all $\pi_j \geq \pi_d$. If $i \leq \hat{\imath}$, then $\lambda_j \leq 0$ for all $\pi_j \geq \pi_d$.*

A key observation is that both $\lambda_j > 0$ and $\lambda_j < 0$ are possible, which implies that the arrival of information can raise or reduce the asset price on average. The reason is that the coordinated use of money can concavify or convexify the payoff structure. To wit, assume $i = 0$ and think of the asset as a standard Lucas tree that generates a state-contingent dividend $\gamma(\pi_j)$. For $\pi_j > \pi_d$, the dividend is $\gamma(\pi_j) = B\alpha^c - c_j$, which is the difference between the marginal transactional benefit and the expected storage cost. This difference is linear and increases in π_j, as illustrated by the dotted lines in Figure 2. At $\pi_j = \pi_d$, agents abandon the asset and $\gamma(\pi_d) = 0$. In Figure 2, we illustrate two possible shapes of γ. Depending on the value of π_d, the abandonment of money can concavify (left panel) or convexify $\gamma(\pi_j)$ (right panel). In the right panel, agents are willing to hold the asset at π_{d+1} even when $\gamma(\pi_{d+1}) < 0$ as a good signal may arrive in the future. In the left panel, at state π_d, the expected value of a unit of asset is negative, i.e., $\ell_d^c \leq 0$. As a result, no seller would accept it as payment and the transaction benefit $B\alpha^c$ is irrelevant. Hence, agents abandon the asset even if $B\alpha^c - c_d > 0$.

By standard logic of asset pricing, ϕ_j can be represented as an expected discounted sum of dividends, namely $\phi_j = E[\Sigma_{t=1}^{\infty}\beta^t\gamma(\pi^t)]$ where π^t is the belief t periods after.

If belief is fixed at π_j, then the discounted sum of dividend is simply $\beta\gamma(\pi_j)/(1-\beta)$ or equivalently $\hat{\phi}(\pi_j)$ by (16). Hence by Jensen's inequality, $\phi_j < \hat{\phi}(\pi_j)$ when $\gamma(\pi)$ is concave and $\phi_j > \hat{\phi}(\pi_j)$ when $\gamma(\pi)$ is convex. It follows that if π_d is large, then $\gamma(\pi)$ is concave and $\lambda_j < 0$, and otherwise $\gamma(\pi)$ is convex and $\lambda_j > 0$. Part 1 and 2 of Proposition 2 are true as π_d falls in η and i (Proposition 1). If $\hat{\phi}(0) > 0$, then the dotted line in Figure 2 is always above the x-axis and γ is concave. Hence $\lambda_j < 0$ if $\hat{\phi}(0) > 0$.

In Figure 3, we provide a numerical example of asset prices.[6] Asset price $\phi(\pi)$ is concave in π and smaller than the $\hat{\phi}$ when the information premium is negative (left panel). Otherwise, it is convex in π and always above the stationary price (right panel).

An insight from this analysis is that, in many investment problems with learning, information is useful for the investors to make decisions and thus the arrival of new information raises the value of the investment on average, which corresponds to $\lambda_j > 0$ in our model. But we show that the coordinated use of money may concavify payoffs and thus the arrival of new information can also reduce the value of the investment.

Another insight is that, by Part 2 of Proposition 2, a change in the nominal rate, i, affects the sign of λ_j and thus changes the qualitative impact of information on asset prices (we will show that this sign also affects the impact of information on welfare). This result highlights how currency competitions can affect the role of information.

4.3 Information and prices

Next we consider the impact of a change in the information arrival chance η on prices. Since λ_j has the same sign $\forall \pi_j \geq \pi_d$, we omit the subscript when referring to its sign.

Proposition 3 (Learning speed and asset prices)

1. *If $\lambda \geq 0$, then $\partial\phi(\pi)/\partial\eta > 0$ for $\pi \geq \pi_d$.*
2. *If $\lambda < 0$, then there is a cutoff $\pi^* > \pi_d$ such that $\partial\phi(\pi)/\partial\eta > 0$ for $\pi \in (\pi_d, \pi^*)$ and $\partial\phi(\pi)/\partial\eta \leq 0$ for $\pi \in [\pi^*, 1)$.*

Proposition 3 characterizes the impact of an increase of η on ϕ_j. Since $\hat{\phi}(\pi)$ is independent of η, an increase in η only affects ϕ via the information premium term. On the one hand, an increase in η raises the option value of asset holding, reducing π_d and raising $\phi(\pi)$. On the other hand, a larger η increases the magnitude of the information

[6] Parameter values are $\{i, \alpha^b, \alpha^c, B, \underline{c}, \chi, s\} = \{0.01, 0.1, 0.65, 0.0098, 0.0001, 0.07, 0.54\}$, $\{\bar{c}, \beta\} = \{0.15, 0.9894\}$ in the left and $\{0.19, 0.99\}$ in the right. High η is 0.99, and low η is 0.2 in the left panel and 0.4 is the right panel.

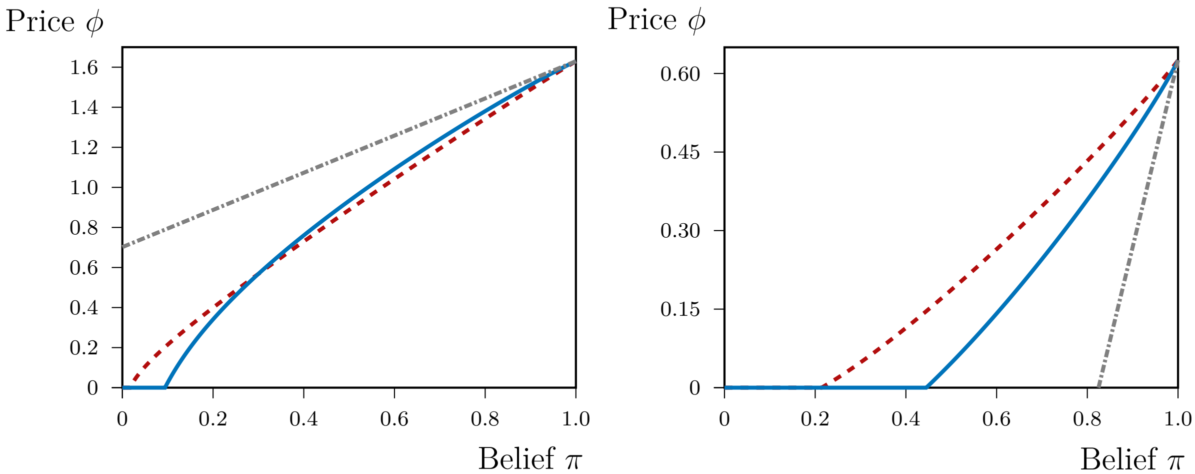

Figure 3: Asset prices. (Left) Blue solid lines represent a low η, red dashed lines represent a high η, and gray chain lines represent stationary price $\hat{\phi}$. $\lambda < 0$. (Right) $\lambda > 0$.

premium at all states. This effect is represented by the fact that Γ in (22) rises in η. If $\lambda > 0$ to begin with, then both effects inflate $\phi(\pi)$ at all $\pi \geq \pi_d$. We illustrate these results in Figure 3. If $\lambda < 0$, then the two effects oppose each other and $\phi(\pi)$ rises if and only if π is close to π_d. Hence, unlike standard experimentation problems, here faster learning can reduce the value of the asset.

5 Information and Welfare

Since the 2007 financial crisis, there has been a policy debate concerning whether the government should suppress information about assets that can serve as payment or saving device, see, for example, Andolfatto and Martin (2013) and Dang et al. (2013). We now use our model to address this question. In the context of digital currencies, information disclosure corresponds to requiring the issuer or the platform to reveal more information about the technology, design, and operation behind the new money.

We define welfare Ω_j as the expected discounted sum of surpluses at state π_j. Since sellers earn no profit, Ω_j equals the discounted sum of buyers' surpluses. The trade surplus in the CM is omitted as it is a constant in all states. To formulate Ω_j, define

$$\omega_j = \omega(\pi_j) \equiv \alpha^b S(\ell^b) + \alpha^c S(\ell^c_j) - c_j \tag{23}$$

as the expected DM trade surplus minus the expected storage cost at state π_j. Since the government stabilizes ℓ^b, only the last two terms vary across states. When the asset circulates, i.e., for $\pi_j > \pi_d$, $\Omega_j = \Omega(\pi_j)$ can be formulated recursively as

$$\Omega(\pi_j) = \omega(\pi_j) + \beta[(1-\eta)\Omega(\pi_j) + \eta s_j \Omega(\pi_{j+1}) + \eta(1-s_j)\Omega(\pi_{j-1})]. \tag{24}$$

The first term on the right is the expected DM surplus associated with state π_j, and the second term is the expected welfare in the next period. For $\pi_j \leq \pi_d$, there is no trade in the c-meetings and Ω_j is a constant.[7]

We first describe the welfare impact of a new signal and then discuss two disclosure polices: full disclosure of asset type and an increase in information arrival chance.

Arrival of a New Signal. Let Λ_j be the expected change in welfare when a new signal arrives, namely

$$\Lambda_j \equiv \eta s_j \Omega_{j+1} + \eta(1 - s_j)\Omega_{j-1} - \Omega_j.$$

Therefore, at state π_j, a new signal raises welfare on average if and only if $\Lambda_j \geq 0$.

Proposition 4 (Welfare impact of a signal) *Assume agents are sufficiently patient.*

1. *If the information premium is negative, i.e., $\lambda < 0$, then $\Lambda_j < 0$ for $\pi_j \geq \pi_d$.*
2. *If the information premium is positive, i.e., $\lambda \geq 0$, then there is a cutoff $\hat{\pi} \in (\pi_d, 1)$ such that $\Lambda_j < 0$ for $\pi_j \in (\pi_d, \hat{\pi})$ and $\Lambda_j \geq 0$ for $\pi_j \in [\hat{\pi}, 1)$.*

Part 1 of Proposition 4 states that if the information premium is negative, then the arrival of a new signal reduces welfare on expectation. This result is related to Andolfatto and Martin (2013), who compares full disclosure and full suppression of information regarding the means of payment. Their main insight is that if agents have concave utility, then welfare is a concave function of agents' beliefs and new information reduces expected welfare because it creates more fluctuations in beliefs. In our model this insight is true when $\lambda < 0$ — since the trade surplus in c-meetings, $S(\ell_j^c)$, is a concave function of liquidity ℓ_j^c, it would also be concave in π_j provided that ℓ_j^c is concave in π_j. The last condition is satisfied when the information premium is negative.

Part 2 of Proposition 4 shows that when the information premium is positive, the arrival of new information improves welfare on average, provided that π_j is large. The reason is that when the information premium is positive, new information raises the asset prices on average and thus creates more surplus during trade meetings. This result differs from that of Andolfatto and Martin (2013) because their parametric assumptions always imply a negative information premium.

Full Disclosure Now we compare the welfare impact of two disclosure policies. First, we consider fully revealing the asset's type. Let Ω_H and Ω_L be the welfare when $\pi = 1$

[7]Since we have assumed log-utility, the utility level is $-\infty$ when there is no trade. To resolve this technical issue, we assume $u(y) = B\log[\max(\ell, \underline{\ell})] - B\log(\underline{\ell})$ for some sufficiently small $\underline{\ell}$.

and $\pi = 0$, respectively. Given our definition of equilibrium with $\pi_d \in (0,1)$, when $\pi = 1$, agents always carry the new asset and the asset price is $\hat{\phi}(1)$. When $\pi = 0$, the asset is abandoned and there are trades only in the b-meetings. At state π_j, the expected welfare of fully disclosing the asset type is $\Omega_j^F \equiv \pi_j \Omega_H + (1-\pi_j)\Omega_L$. The next result compares Ω_j^F and Ω_j:

Proposition 5 (Full disclosure) *There is a cutoff $\pi^F \in (\pi_d, 1)$ such that $\Omega_j^F > \Omega_j$ if and only if $\pi_j < \pi^F$.*

Proposition 5 claims that full disclosure improves welfare if and only if the belief is low. For intuition, consider the cases when π_j is very low or very high. When π_j is low, agents refuse to hold the asset. It is optimal to fully reveal the asset type because there is a π_j chance that the type is H and agents will use asset for trade. When π_j is high, the asset is likely to be type H and the planner should encourage agents to use it. Without disclosure, agents use the asset to trade. With full disclosure, there is a $1 - \pi_j$ chance that agents will immediately abandon the asset. Therefore, it is optimal to hide information so that agents use the asset for a longer duration.

Increase in Learning Speed While full disclosure is a useful benchmark, it is often not a feasible policy tool. Another way to think about information disclosure policy is to consider an increase in information arrival chance, η. To wit, rewrite (24) as $\Omega_j = (\omega_j + \beta\eta\Lambda_j)/(1-\beta)$. As η rises, there are two welfare effects. First, the asset price, ϕ_j, changes as stated in Proposition 3, which affects the surplus, ω_j, in each state. Second, the economy transitions more rapidly across states. The welfare impact of more rapid transitions depends crucially on the sign of Λ_j. According to Proposition 4, the sign of Λ_j depends on the sign of the information premium, λ_j. The next result combines these insights to characterize the sign of $\partial\Omega(\pi)/\eta$.

Proposition 6 (Faster learning) *Assume agents are sufficiently patient.*

1. *There is a cutoff $\pi' \in [\pi_d, 1)$ such that $\partial\Omega(\pi)/\partial\eta \geq 0$ for $\pi \leq \pi'$.*
2. *If $\lambda < 0$, then there is a cutoff $\pi'' \in [\pi_d, 1)$ such that $\partial\Omega(\pi)/\partial\eta \leq 0$ for $\pi \geq \pi''$.*
3. *If $\lambda \geq 0$, then there is a cutoff $\pi'' \in [\pi_d, 1)$ such that $\partial\Omega(\pi)/\partial\eta \geq 0$ for $\pi \geq \pi''$.*

Part 1 of Proposition 6 states that welfare rises in η when π is small. Recall that an increase in η reduces the cutoff π_d and thus expands the set of states where the asset

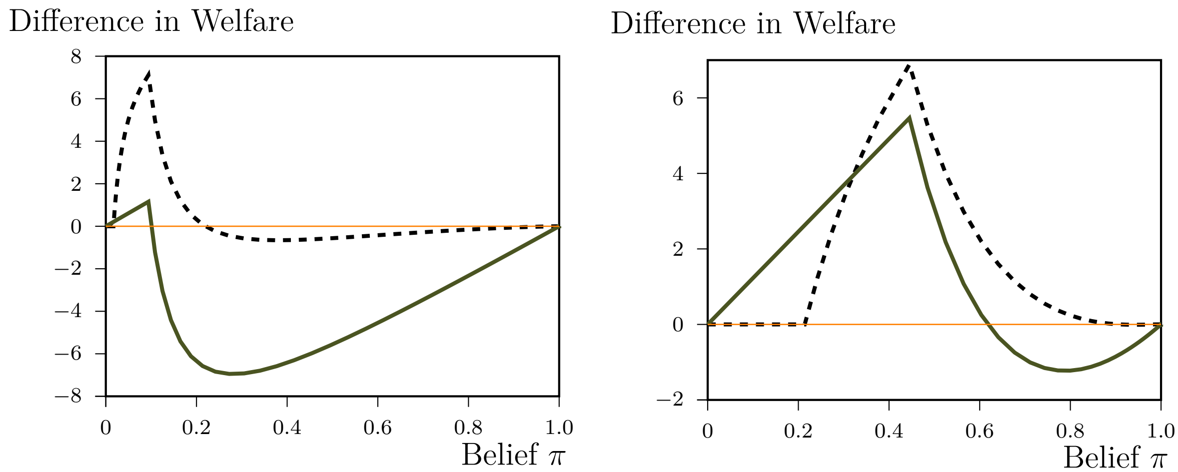

Figure 4: Welfare comparison. Green lines are the differences in welfare of full disclosure and low η, black dashed lines are that of high and low η. (Left) $\lambda_j < 0$. (Right) $\lambda_j > 0$.

circulates. In states where agents change from abandoning to holding the asset, the agents must be better off by doing so, thus $\Omega(\pi)$ rises.

Part 2 and 3 claim that, for large π, the welfare impact of disclosure depends on the sign of λ. When π is large, the asset price is close to $\hat{\phi}(1)$ and a change in η does not affect asset price much. In this case, the welfare effect of an increase in η is driven by the increase in the transition probability. By Proposition 4, if $\lambda < 0$, then so is Λ_j for large π_j. Thus, more fluctuations in π reduces welfare as stated in Part 2. For similar reasons, when $\lambda > 0$, welfare rises in η as shown by Part 3.

Proposition 5 and 6 highlight the differences between full disclosure and a marginal increase in η and the role of the information premium. For π close to 1, full disclosure always reduces welfare but an increase in η is desirable if and only if the information premium is positive. We illustrate these difference with a numerical example in Figure 4 where the parameters are the same as Figure 3. In the left panel, the welfare change due to full disclosure (green line) or an increase in η (black dashed line) are qualitatively similar — they both improve welfare if and only if π is small. In the right panel, an increase in η weakly raises welfare at all states while full disclosure reduces welfare when π is large. A lesson from this comparison is that although full disclosure and an increase in η both increase information, their welfare implications can be qualitatively different.

6 Generalizations and Extensions

Learning with general preferences The use of log utility simplified our analysis because the marginal benefit of holding a liquid asset, $\ell_j^c S'(\ell_j^c)$, is a constant. With general preferences, the marginal benefit will depend on the asset price and, because of this additional endogeneity, the asset price can be highly non-linear in π_j and the existence of multiple equilibria is possible. We study learning with general preferences in Online Appendix E and show that if the information arrival chance is sufficiently high, then the equilibrium is unique within the class of equilibria where $\pi_d \in (0,1)$ and prices can be derived by an iterative method.

With log utility, λ_j has the same sign at all $\pi_j \geq \pi_d$, but, with general preferences, λ_j can change sign as π_j rises from π_d. With log utility, as η rises, the change in ϕ_j is either positive at all π_j or is reverse single-crossing (i.e., positive and then negative) in π_j. In general, the change in ϕ_j can have multiple sign changes as π_j rises. We characterize these sign changes in Online Appendix.

In the baseline model the coordinated abandonment of money creates convexity or concavity in the payoff structure, which affects the welfare impact of information. In Online Appendix E we consider a version where storage costs are negative, i.e., $c^H, c^L < 0$ such that agents never abandon the asset. In this case the sign of the information premium is driven by the shape of the utility function and not by the abandonment of the asset. We provide a condition such that an increase in η can increase or decrease welfare at all states. This exercise highlights that the functional form of the preferences alone can play a significant role in shaping the optimal disclosure policy.

Two private monies Our baseline model considers competitions between a government money and a private money. In Online Appendix F, we consider two competing private monies. There is a unit supply of a new asset and a unit supply of safe asset with known quality and they are perfect substitutes for payment. We show that as agents' belief increases, the price of the new asset rises, the price of the safe asset falls, but the aggregate liquidity (i.e., sum of the value of the two assets) can be U-shaped. Hence, the arrival of a good signal about the new asset can surprisingly reduce aggregate liquidity and trade volume. Likewise the total welfare can be non-monotone in agents' belief and thus welfare can fall as a good signal arrives. The non-monotonicity of welfare is due to a coordination failure and pecuniary externality. A planner can improve welfare by banning the new asset when the beliefs are low but above π_d.

Learning about acceptability Sometimes the source of uncertainty regarding a new money lies in its acceptability and not on the storage cost. For example, users might not know how many trading opportunities are created by certain new technologies. In Online Appendix D. we study a version where the storage cost is known but the matching probability α^c is a random draw from a distribution. Agents can only gradually learn about this distribution by observing the realized measure of meetings and public signals. We show that Proposition 1-3 still hold and illustrate with numerical examples that the welfare impact of information is similar to that of the baseline. This extension shows that our main message is robust to the choice of the property that agents learn about.

7 Conclusion

We proposed and solved a tractable learning model to explain the dynamics of money adoption. In the model, agents gradually learn about the storage cost of a new money and coordinate to use or abandon it. We derived the state-contingent asset prices and the cutoff of disposing the asset and study the link between information and welfare.

A key insight is that the endogenous and coordinated use of money can concavify or convexify the marginal benefit of asset holding, resulting in a negative or positive information premium in the asset price. We characterize how fundamentals affect the sign of the information premium and illustrate that this sign is crucial for predicting the impact of information disclosure on allocations and welfare. Another insight is that although full disclosure and an increase in learning speed are intuitively similar, their welfare impact can be quite different. Therefore, in order to study information disclosure of liquid assets, it is important to be explicit about the micro-foundations.

A lot more can be done with this learning model and many interesting questions remain open. For example, one can build a learning theory of dominance currency and study why similar countries make different dollarization decisions. Another example is forward guidance, where a central bank tells the public what conditions will change or maintain the bank's policy actions. Our model provides a framework to think about how the public perceive and react to the evolving likelihood of these conditions.

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Omitted Proofs

Proof of Proposition 1. *Part 1:* Rewrite ϕ_j as $\phi_j = \hat{\phi}(\pi_j) + \pi_j\sigma_j$ where σ_j is a sequence that we will solve below. By the definition of s_j and π_j, we have $s_j = s\pi_j/\pi_{j+1}$ and $1 - s_j = (1-s)\pi_j/\pi_{j-1}$. Substitute these expressions into (17) then σ_j solves:

$$\sigma_j = \beta(1+i)[\eta s\sigma_{j+1} + \eta(1-s)\sigma_{j-1} + (1-\eta)\sigma_j]. \tag{25}$$

By standard results of difference equations, σ_j takes the form $\sigma_j = Cz_1^{j-d} + Dz_2^{j-d}$ where C and D are constants and z_1 and z_2 are the solution of the quadratic equation

$$\eta s z^2 + \left[(1-\eta) - \frac{1}{\beta(1+i)}\right] z + \eta(1-s) = 0. \tag{26}$$

Suppose $z_2 \geq z_1$. It is easy to verify that $z_2 > 1 > z_1$ and hence $D = 0$ must hold, for otherwise $\sigma_j \to \infty$ and $\phi_j \to \infty$ as $j \to \infty$. By (26), the smaller root z_1 is given by

$$z_1 = \Gamma(\eta, i, \beta, s) \equiv \frac{1-\beta(1+i)(1-\eta)}{2s\beta(1+i)\eta} - \sqrt{(\frac{1-\beta(1+i)(1-\eta)}{2s\beta(1+i)\eta})^2 - \frac{1-s}{s}} \tag{27}$$

which rises in η and i. The value of C is given by the condition $\phi(\pi_d) = 0$ and thus

$$\phi_j = \hat{\phi}(\pi_j) - \hat{\phi}(\pi_d)\frac{\pi_j}{\pi_d}\Gamma(\eta, i, \beta, s)^{j-d}.$$

To express the right side as a function of π_j, define

$$k(\eta, i, \beta, s) \equiv \frac{\log[\Gamma(\eta, i, \beta, s)]}{\log[(1-s)/s]} > 1. \tag{28}$$

Since the right side of (26) is negative at $z=(1-s)/s$, $\Gamma(\eta, i, \beta, s)<(1-s)/s$ and thus the inequality (28) holds. By (1),

$$\left(\frac{1-s}{s}\right)^{j-d} = \frac{(1-\pi_j)\pi_d}{\pi_j(1-\pi_d)}.$$

Using this equation and the definition of $k(\eta, i, \beta, s)$, ϕ_j can be rewritten as

$$\phi_j = \hat{\phi}(\pi_j) - \hat{\phi}(\pi_d)\frac{\pi_j}{\pi_d}\left[\frac{(1-\pi_j)\pi_d}{\pi_j(1-\pi_d)}\right]^{k(\eta,i,\beta,s)}.$$

Part 2: Uniqueness: Another way to compute the asset prices is by using the contraction mapping theorem. Rewrite (17) as

$$\phi_{j+d}=\max\left(0, \beta\{B\alpha^c+(i+1)[\eta s_{j+d}\phi_{j+1+d}+\eta(1-s_{j+d})\phi_{j-1+d}+(1-\eta)\phi_{j+d}-c_{j+d}]\}\right). \tag{29}$$

Given π_d and $\phi_d = 0$, the right side of (29) can be interpreted as a mapping from the set of weakly increasing sequence of $\phi_{j+d} \in [0, \hat{\phi}(1)]$ for $j \geq 1$ into itself. It is easy to check that this mapping satisfies the assumptions of the contraction mapping theorem and thus a unique fixed point exists. Since given π_d, the pricing formula (20) can be derived uniquely, it must also be the unique fixed point of (29). Hence ϕ_j rises in j. By a similar logic, one can show more strongly that the right side of (29) rises in π_j.

Next we discuss the uniqueness of π_d. As π_d rises, c_{j+d} falls and s_{j+d} rises at all $j \geq 0$. Therefore, by standard results, each element of the fixed-point sequence also increases. By (14), π_d satisfies

$$\phi_{d+1} \leq \frac{c_d}{\eta s_d} \tag{30}$$

in equilibrium. As π_d rises, the left side of (30)rises as argue above while the right side falls. Thus there is a unique solution of π_d such that (30) binds, call it $\bar{\pi}$. When $\pi_d = \bar{\pi}$, $\ell^c_d = 0$ and thus (12) binds, i.e. agents are indifferent between abandoning or holding the asset at π_d. All values of belief that satisfies (30) must satisfy $\pi_d \leq \bar{\pi}$.

Next, the smallest value of belief that generates a positive price sequence is such that $\phi_{d+1} = 0$, i.e. agents are indifferent between holding or abandoning at π_{d+1}. Since ϕ_{d+1} falls in π_d, we know there is a unique value of belief such that $\phi_{d+1} = 0$, call this threshold $\underline{\pi}$. Therefore, any equilibrium cutoff must be such that $\pi^d \in [\underline{\pi}, \bar{\pi}]$.

But $\underline{\pi}$ and $\bar{\pi}$ are exactly one signal away from each other, namely that if agents' belief is $\underline{\pi}$ and a good signal arrives, then their posterior belief is $\bar{\pi}$. It is because $\phi_{d+1} = 0$ when $\pi_d = \underline{\pi}$ and the first-order condition binds at π_{d+1}. But that is exactly the condition the defines $\bar{\pi}$, hence $\pi_{d+1} = \bar{\pi}$ when $\pi_d = \underline{\pi}$. Hence given π_0, there is at most one j such that π_j lies in the interval $[\underline{\pi}, \bar{\pi}]$. Therefore there is only one possible π_d. If π_0 is such that $\pi_j = \underline{\pi}$ and $\bar{\pi} = \pi_{j+1}$ at some j, then π_j and π_{j+1} are both equilibrium cutoff. But this case is non-generic and only holds for measure zero of parameter values.

Comparative statics with respect to B, α^c and i: Now we show how π_d varies with parameters, assuming π_d solves

$$\phi_{d+1} = \frac{c_d}{\eta s_d}. \tag{31}$$

Fixing π_d, the right side of (29) rises in B and α^c. Thus, fixing π_d, the left side of (31) rises in B and α^c and π_d must fall to balance the equation. Hence π_d falls in B and α^c.

Fixing π_d and $\{\phi_{j+d}\}_{j=1}^{\infty}$ at their equilibrium value and consider an increase in i. The right side of (29) rises because $\ell^c_{j+d} > 0$ for all $j \geq 1$ in equilibrium. Therefore, the sequence generated by the right side of (29) is strictly higher than the original equilibrium sequence. Since the mapping represented by (29) is monotone, i.e. every element of the resulting sequence is weakly higher if every element of the input sequence is weakly higher, the fixed point of the mapping must be higher than the original equilibrium sequence. Therefore, given π_d, an increase in i raises the left side of (31), thus π_d must fall to satisfy (31).

Formula for π_d: By (20) and $s_d = s\pi_d/\pi_{d+1}$, equation (31) can be written as

$$[1 - s - s\Gamma(\eta, i, \beta, s)]\hat{\phi}_d - \frac{c_d}{\eta} = -\hat{\phi}(1)(2s-1)\pi_d. \tag{32}$$

Since $1 - s - s\Gamma(\eta, i, \beta, s) > 0$ and $\hat{\phi}(1) > 0$, the left side rises in π_d and the right side falls in π_d. A solution of π_d exists if and only if (18) holds. By (32), π_d is given by (19).

Comparative statics of π_d with respect to η: Fixing π_d, the derivative of the left side of (32) with respect to η is

$$\begin{aligned} -s\frac{\partial \Gamma}{\partial \eta}\hat{\phi}_d + \frac{c_d}{\eta^2} =& \frac{-1}{\eta}\left\{[1 - s - s\Gamma(\eta, i, \beta, s)]\hat{\phi}_d - \frac{c_d}{\eta}\right\} \\ & - \frac{s\hat{\phi}_d}{\eta}\left[1 - \frac{1}{2s} - \frac{\frac{1}{\beta(1+i)} - 1 + \eta[1-4s(1-s)]}{4s^2\sqrt{(\frac{1-\beta(1+i)(1-\eta)}{2s\beta(1+i)\eta})^2 - \frac{1-s}{s}}}\right] \end{aligned}$$

When $\eta = 0$, the right side is strictly positive. When the entire derivative is 0, a further increase in η reduces the expression in the large square bracket and thus the entire expression rises. It follows that the derivative is positive at all η. Hence, π_d falls in η. ■

Proof of Proposition 2. Given ϕ_j, one can verify (22) by using (21). By (22), $\lambda_j < 0$ if and only if $\hat{\phi}(\pi_d) > 0$, or equivalently

$$\pi_d > \frac{(1+i)c^L - \alpha^c B}{(1+i)(c^L - c^H)}.$$

The numerator of the right side is proportional to $-\hat{\phi}(0)$ and thus $\lambda_j < 0$ when $\hat{\phi}(0) > 0$. When $\hat{\phi}(0) \leq 0$, by (19) the inequality can be rewritten as

$$\frac{\alpha^c B}{(1+i)c^L - \alpha^c B} > \frac{\beta\eta(2s-1)[B\alpha^c - (1+i)c^H]}{[1-\beta(1+i)](c^L - c^H)}. \tag{33}$$

Since $B\alpha^c - (1+i)c^H \propto \hat{\phi}(1) > 0$, the right side rises in η and the left side remains unchanged in η. When $\eta = 0$, the right side vanishes and hence the inequality holds. As $\eta \to 1$, the inequality (33) fails when β is sufficiently close to 1. Hence, if β is large, then there is a $\hat{\eta} \in (0,1)$ such that (33) holds if and if $\eta < \hat{\eta}$.

The left side of (33) falls while the right side rises in i, as $\beta B\alpha^c > c^H$. Hence there exists $\hat{\imath}$ such that the information premium is negative when $i < \hat{\imath}$. ■

Proof of Proposition 3. An increase in η only affects the second term in (20). Call this term Υ so that $\phi(\pi) = \hat{\phi}(\pi) - \Upsilon$. The derivative of Υ with respect to η is given by

$$\frac{\partial\Upsilon}{\partial\eta} = \Upsilon\frac{\partial\pi_d}{\partial\eta}\left[\frac{-\hat{\phi}(0)}{\hat{\phi}(\pi_d)\pi_d} + \frac{k(\eta,i,\beta,s)}{\pi_d(1-\pi_d)}\right] + \Upsilon\frac{\partial k(\eta,i,\beta,s)}{\partial\eta}\log\left[\frac{(1-\pi)\pi_d}{\pi(1-\pi_d)}\right], \tag{34}$$

where the derivative $\partial\pi_d/\partial\eta < 0$ by Proposition 3 and $\partial k/\partial\eta < 0$ by (27) and (28). Note that $\phi(\pi)$ rises in π for $\pi \geq \pi_d$. Therefore, as η rises, π_d falls, and $\phi(\pi)$ rises for $\pi \approx \pi_d$. Therefore, $\partial\Upsilon/\partial\eta > 0$ when $\pi \approx \pi_d$ or equivalently the first term in the right side of (34) is negative. Suppose $\hat{\phi}(\pi_d) > 0$, such that $\lambda < 0$ and $\Upsilon > 0$. Then the right side of (34) rises from a strictly negative value to $+\infty$ as π rises from π_d to 1. Therefore, fixing $\pi \geq \pi_d$, the right side of (20) rises in η if π is small and falls for all larger π.

Next, suppose $\hat{\phi}(\pi_d) \leq 0$, such that $\lambda \geq 0$ and $\Upsilon \geq 0$. Now the second term in (34) falls from 0 to $-\infty$ as π rises from π_d to 1. Therefore, the right side of (34) is negative at all $\pi \geq \pi_d$ and the right side of (20) rises in η at all $\pi \geq \pi_d$. ■

Proof of Proposition 4, 5 and 6 See Online Appendix.

Online Appendix for "Learning and Money Adoption"

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A Proof of Proposition 4-6

We say Ω_j is concave at π_j if $\Lambda_j < 0$ and is convex if $\Lambda_j > 0$. We use a similar definition of concavity for ω_j. The next lemma is useful for the proof of Proposition 4.

Lemma 1 (Curvature of the welfare function) *(i) If ω_j is concave at all $\pi_j \geq \pi_d$, then so is Ω_j. (ii) If ω_j is concave-convex, then so is Ω_j.*

Proof. (i) By (24), Ω_j is convex at π_j if $\Omega_j > \omega_j/(1-\beta)$ and otherwise it is concave. Since $\Omega_d = \omega_d/(1-\beta)$ at π_d, if $\Omega_{d+1} > \omega_{d+1}/(1-\beta)$, then Ω_j is convex at state π_{d+1} so $\Omega_{d+2} > \omega_{d+2}/(1-\beta)$ because ω_j is concave at all states. Applying this logic iteratively, Ω_j is convex at all states and cannot converge to the same limit as $\omega_j/(1-\beta)$. This leads to a contradiction and hence $\Omega_{d+1} \leq \omega_{d+1}/(1-\beta)$. By the same logic Ω_j cannot cut $\omega_j/(1-\beta)$ from below at all π_j and hence it must be concave at all states.

(ii) Suppose Ω_j changes from convex to concave at state $\pi_{j'}$, then it must cut $\omega_j/(1-\beta)$ from above at $j = j'$. Suppose ω_j is convex at $\pi_{j'}$, then for all $j > j'$, Ω_j is concave and ω_j is concave and Ω_j cannot converge to the same limit as $\omega_j/(1-\beta)$. Suppose ω_j is concave at $\pi_{j'}$. This leads to a contradiction because, by the logic of part (i), once Ω_j becomes convex it cannot intersect with ω_j at any state that ω_j is concave. ■

Proof of Proposition 4. We first characterize the curvature of ω_j. By (23), ω_j is an increasing concave function of the liquidity ℓ_j^c, and the latter can be written as $\ell_j^c = (\phi_j - \beta B \alpha^c)/[\beta(1+i)]$ by (17). If ϕ_j is concave in π_j for $\pi_j \geq \pi_d$, then so are ℓ_j^c and ω_j. Therefore ω_j is concave in π_j when the information premium is negative.

Next we argue ω_j is concave-convex when the information premium is positive. Given the solution of ϕ_j in (20), we can treat ω_j as a differentiable function of π_j, i.e. $\omega_j = \omega(\pi_j)$. The second derivative of $\omega(\pi)$ is proportional to

$$\frac{\partial^2 \omega(\pi)}{\partial \pi^2} \propto \frac{\partial}{\partial \pi}\left(\frac{\ell^{c'}(\pi)}{\ell^c(\pi)}\right) \propto \frac{\ell^{c''}(\pi)}{\ell^{c'}(\pi)} - \frac{\ell^{c'}(\pi)}{\ell^c(\pi)}. \tag{35}$$

Using the pricing formula in Proposition 1, we can derive

$$\ell^{c\prime}(\pi) = \frac{c^H - c^L}{1-\beta(1+i)} + \frac{\hat{\phi}(\pi_d)}{\beta(1+i)\pi_d}\left[\frac{(1-\pi)\pi_d}{\pi(1-\pi_d)}\right]^{k(\eta,i,\beta,s)}\left(\frac{k(\eta,i,\beta,s)}{1-\pi} - 1\right)$$

$$\ell^{c\prime\prime}(\pi) = -\frac{\hat{\phi}(\pi_d)}{\beta(1+i)\pi_d}\left[\frac{(1-\pi)\pi_d}{\pi(1-\pi_d)}\right]^{k(\eta,i,\beta,s)}\frac{k(\eta,i,\beta,s)[k(\eta,i,\beta,s)-1]}{(1-\pi)^2\pi}.$$

When $\pi \approx \pi_d$, $\ell^c(\pi) \approx 0$, and both $\ell^{c''}(\pi)$ and $\ell^{c'}(\pi)$ are bounded. Hence the right side of (35) is negative when $\pi \approx \pi_d$. Next we argue the fraction $\ell^{c''}(\pi)/\ell^{c'}(\pi)$ is U-shaped in π. The derivative is proportional to

$$\frac{\partial}{\partial \pi}\log\left(\frac{\ell^{c''}(\pi)}{\ell^{c'}(\pi)}\right) \propto \frac{\ell^{c\prime\prime\prime}(\pi)}{\ell^{c\prime\prime}(\pi)} - \frac{\ell^{c\prime\prime}(\pi)}{\ell^{c\prime}(\pi)} \tag{36}$$

where the fraction $\ell^{c'''}(\pi)/\ell^{c''}(\pi)$ can be written as

$$\frac{\ell^{c'''}(\pi)}{\ell^{c''}(\pi)} = -\frac{k(\eta,i,\beta,s)-2}{1-\pi} - \frac{k(\eta,i,\beta,s)+1}{\pi}.$$

As β rises to 1, $k(\eta,i,\beta,s)$ falls to 1 by (27) and (28). Assume $\bar{\beta} \in (0,1)$ is such that if $\beta > \bar{\beta}$ then $k(\eta,i,\bar{\beta},s) < 2$. If $\beta > \bar{\beta}$, then $\ell^{c'''}(\pi)/\ell^{c''}(\pi)$ rises in π, which implies $\ell^{c''}(\pi)/\ell^{c'}(\pi)$ is either decreasing, increasing or U-shaped in π.

Since $\ell^{c'}(\pi)/\ell^{c}(\pi)$ falls in π when $\pi \approx \pi_d$ and $\ell^{c''}(\pi)/\ell^{c'}(\pi)$ is either decreasing, increasing or U-shaped in π, by (35) $\ell^{c'}(\pi)/\ell^{c}(\pi)$ is either decreasing or U-shaped in π for $\pi \geq \pi_d$. This implies $\omega(\pi)$ is either concave or concave-convex in π. As $\pi \to 1$, $\ell^{c\prime}(\pi)$ converges to a positive constant and $\ell^{c\prime\prime}(\pi)$ explodes to $+\infty$. Hence, as $\pi \to 1$, $\partial^2\omega(\pi)/\partial\pi^2$ explodes to $+\infty$ by (35). Thus $\omega(\pi)$ is concave-convex.

Finally, by Lemma 1, if $\omega(\pi)$ is concave in π, then $\Omega(\pi)$ is concave. If $\omega(\pi)$ is concave-convex in π, then so is $\Omega(\pi)$. ■

Proof of Proposition 6. Suppose $\{\pi_d, \phi(\pi), \omega(\pi), \Omega(\pi)\}$ becomes $\{\hat{\pi}_d, \hat{\phi}(\pi), \hat{\omega}(\pi), \hat{\Omega}(\pi)\}$ as η rises to $\hat{\eta}$. By Proposition 3 $\hat{\pi}_d < \pi_d$ and thus $\hat{\phi}(\pi) > \phi(\pi)$ for states near π_d. It follows that $\hat{\Omega}(\pi) > \Omega(\pi)$ for states near π_d and thus the cutoff $\pi' \geq \pi_d$ exists.

Now we characterize the change in $\Omega(\pi)$ for large π. Suppose $\beta > \bar{\beta}$ where $\bar{\beta}$ is defined in the proof of Proposition 4. If $\lambda < 0$, then $\omega(\pi)$ and $\Omega(\pi)$ are concave by the proof of Proposition 4 and Lemma 1. If $\lambda > 0$, then $\omega(\pi)$ and $\Omega(\pi)$ are concave and then convex.

Part 1. Since we can write ϕ as a differentiable function of π, $\omega(\pi)$ is also differentiable in π. Therefore one can also denote $\Omega(\pi)$ as a differentiable function in π (see

Lemma 4 for details). We approximate $\Omega(\pi)$ by linearizing (24) around $\pi \approx 1$. Let $\omega'(1) \equiv \partial\omega(\pi)/\partial\pi|_{\pi=1} = B\ell^{c\prime}(1)/\ell^c(1) + c^H - c^L$. Linearizing (24) yields

$$\Omega_j = \omega(1) - \omega'(1)(1-\pi_j) + \beta[(1-\eta)\Omega_j + \eta s_j \Omega_{j+1} + \eta(1-s_j)\Omega_{j-1}]. \tag{37}$$

By the proof logic of Proposition 3, the solution $\tilde{\Omega}_j$ of (37) takes the form

$$\tilde{\Omega}(\pi) = \frac{\omega(1) - \omega'(1)(1-\pi)}{1-\beta} - D(\eta)\pi \left[\frac{(1-\pi)\pi_d}{\pi(1-\pi_d)}\right]^{k(\eta,0,\beta,s)}, \tag{38}$$

for some $D(\eta)$. The function $D(\eta)$ is strictly positive as $\tilde{\Omega}(\pi)$ is concave in π only if $D(\eta) > 0$, and $\Omega(\pi)$ is concave when $\lambda < 0$. As η rises, $\omega(1)$ and $\omega'(1)$ remain unchanged and $\partial k(\eta, 0, \beta, s)/\partial\eta < 0$. By a logic similar to the proof of Proposition 3, the change in $\tilde{\Omega}(\pi)$ is dominated by the change in $k(\eta, 0, \beta, s)$ when π is sufficiently close to 1. Therefore, $\tilde{\Omega}(\pi)$ will fall in η for sufficiently large π, even if $\partial D(\eta)/\partial\eta < 0$. Hence $\Omega(\pi)$ falls in η for sufficiently large π or equivalently there is $\pi'' < 1$ such that $\partial\Omega(\pi)/\partial\eta \leq 0$ for $\pi \geq \pi''$.

Part 2. If $\lambda > 0$, then $\Omega(\pi)$ is convex for large π. By the same proof logic of Part 1, there exists $\pi'' < 1$ such that $\partial\Omega(\pi)/\partial\eta \geq 0$ for $\pi \geq \pi''$. ■

Proof of Proposition 5. The expected welfare under full disclosure is $\Omega^F(\pi_j) = \pi_j\Omega_H + (1-\pi_j)\Omega_L$. Since $\Omega_j = \Omega_d = \Omega_L$ for all $\pi_j \leq \pi_d$, $\Omega^F(\pi_j) > \Omega_j$ for all $\pi_j \leq \pi_d$. At $\pi = 1$, the slope of $\Omega^F(\pi)$ is $\Omega_H - \Omega_L = \{\alpha^c S[\ell^c(1)] - c^L\}/(1-\beta)$. We compute the slope $\Omega'(1)$ by linearizing $\Omega(\pi)$ around $\pi = 1$, as in (38), which gives us $\Omega'(1) = \alpha^c S'[\ell^c(1)]\ell^{c\prime}(1)/(1-\beta)$. The difference in the slope is

$$\Omega^{F\prime}(1) - \Omega'(1) = \frac{1}{1-\beta}\left(\alpha^c\{\log[\ell^c(1)] - \log(\underline{\ell})\} - c^L - \alpha^c \frac{\ell^{c\prime}(1)}{\ell^c(1)}\right)$$

The right side is positive when $\underline{\ell}$ is sufficiently small. Therefore $\Omega^F_j > \Omega_j$ for small π_j and $\Omega^F_j < \Omega_j$ for $\pi_j \approx 1$. Since Ω^F_j is linear in π_j and Ω_j is either concave or concave-convex in π_j by Proposition 4, Ω^F_j must single-cross Ω_j from above. ■

B Formula for Long Run Adoption Chance

Assume the asset quality is H and let $\mathcal{A}_j$ be the probability that the asset will be adopted as money in the long run, namely the probability that $\pi \to 1$, provided that agent's current period prior is π_j. Clearly $\mathcal{A}_d = 0$ and $\lim_{j\to\infty} \mathcal{A}_j = 1$. At state π_j where $j \geq d+1$, $\mathcal{A}_j$ is given by

$$\mathcal{A}_j = s\mathcal{A}_{j+1} + (1-s)\mathcal{A}_{j-1} \iff s\mathcal{A}_{j+1} - \mathcal{A}_j + (1-s)\mathcal{A}_{j-1} = 0.$$

The characteristic equation for this second-order linear difference equation is $sz^2 - z + (1-s) = 0$. The roots are $(1-s)/s$ and 1. Therefore the solution of $\mathcal{A}_j$ takes the form $\mathcal{A}_j = C + D[(1-s)/s]^{j-d}$. By using $\mathcal{A}_d = 0$, we have $C + D = 0$. By $\mathcal{A}_\infty = 1$, we have $C = 1$. Therefore

$$\mathcal{A}_j = 1 - \left(\frac{1-s}{s}\right)^{j-d}.$$

Therefore if the asset quality is H and the agents' prior believe is π_0, then the asset will be eventually adopted with probability $1 - [s/(1-s)]^d$ which falls in d and rises in s.

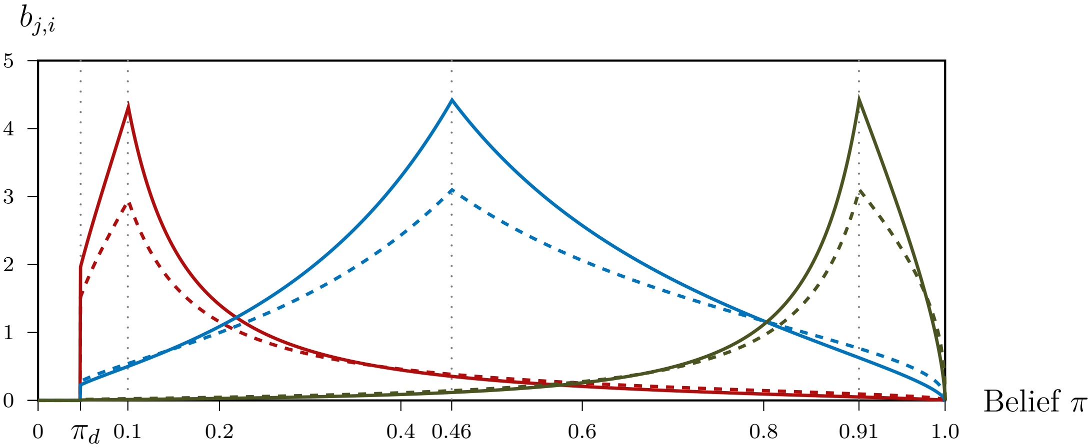

Figure 5: Three examples of $b_{j,i}$. Each solid colored line represents a $b_{j,i}$ when $\eta = 0.1$ and it shows how $b_{j,i}$ varies with π_i. The value of π_j for the red, blue and green lines are 0.1, 0.46 and 0.91 respectively. The dashed colored lines represent $b_{j,i}$ when $\eta = 0.9$.

C Formula for Welfare

The definition of Ω_j in equation (24) is a second-order non-homogeneous recurrence relation in Ω_j with variable coefficients and it is hard to solve in general. But here one can exploit the structure of Bayesian learning to decompose this equation into two second-order difference equations that are each solvable in closed-form. Therefore we can claim the following:

Lemma 2 *For $j > d$, we can represent welfare as a weighted sum of ω_i, namely $\Omega_j = \Omega_d + \Sigma_{i \geq d}^{\infty} b_{j,i}\omega_i$. The weight $b_{j,i} > 0$ are continuous and differentiable in (β, s, η) and can be solved in closed-form.*

The value of $b_{j,i}$ tells us how Ω_j depends on the DM trade surplus in each state. In Figure 5 we present three numerical examples of $b_{j,i}$ as a function of π_i (red, blue and green solid lines). It is single-peaked and grows more dispersed as η rises (colored dashed lines).

Lemma 3 *For $j \geq 0$, let Ψ_{d+j}^H be*

$$\Psi_{d+j}^H = \frac{1}{\beta\eta(1-s)} \left[\Sigma_{t=0}^{j-1} \frac{m_1^j}{m_2 - m_1} \left(\frac{m_2}{m_1^t} - \frac{m_1}{m_2^t} \right) \omega_{d+t+1} + \Sigma_{t=j}^{\infty} \frac{m_1(m_2^j - m_1^j)}{(m_2 - m_1)m_2^t} \omega_{d+t+1} \right]$$

where $m_2 > m_1$ are the roots of the characteristic equation

$$\frac{(1-s)}{s} - \frac{1-\beta(1-\eta)}{\beta\eta s} z + z^2 = 0.$$

Define Ψ^L_{d+j} similarly but with s replaced by $1-s$, namely

$$\Psi^L_{d+j} = \frac{1}{\beta\eta s}\left[\Sigma_{t=0}^{j-1} \frac{n_1^j}{n_2 - n_1}\left(\frac{n_2}{n_1^t} - \frac{n_1}{n_2^t}\right)\omega_{d+t+1} + \Sigma_{t=j}^{\infty} \frac{n_1(n_2^j - n_1^j)}{(n_2-n_1)n_2^t}\omega_{d+t+1}\right]$$

where $n_2 > n_1$ are the roots of

$$\frac{s}{1-s} - \frac{1-\beta(1-\eta)}{\beta\eta(1-s)} z + z^2 = 0.$$

For $j > 0$, the solution of Ω_{d+j} is equal to $\Omega_d + \pi_{d+j}\Psi^H_{d+j} + (1-\pi_{d+j})\Psi^L_{d+j}$.

Proof of Lemma 3. Suppose the modeler knows the true state of the world is H, denote the discounted sum of ω_j by Ψ^H_j which is the solution of the second-order difference equation

$$\Psi^H_j = \omega_j + \beta[\eta s \Psi^H_{j+1} + \eta(1-s)\Psi^H_{j-1} + (1-\eta)\Psi^H_j].$$

Similarly, define Ψ^L_j as the discounted sum of ω_j when the true state is known to be L. This sum solves the difference equation

$$\Psi^L_j = \omega_j + \beta[\eta(1-s)\Psi^L_{j+1} + \eta s \Psi^L_{j-1} + (1-\eta)\Psi^L_j].$$

Now, if the modeler does not know the true state, then the expected discounted sum of ω_j is $\Psi_j = \pi_j\Psi^H_j + (1-\pi_j)\Psi^L_j$. The welfare at state j is given by $\Omega_{d+j} = \Omega_d + \Psi_{d+j}$ where Ω_d is the welfare when the new money is abandoned. We can solve the second-order non-homogenous equation for Ψ^H_j and Ψ^L_j by using standard techniques. We will illustrate how to derive the solution for Ψ^H below.

The equation for Ψ^H can be rewritten as a linear nonhomogeneous second-order difference equation

$$\frac{1-s}{s}\Psi^H_j - \frac{1-\beta(1-\eta)}{\beta\eta s}\Psi^H_{j+1} + \Psi^H_{j+2} = \tilde{\omega}_j$$

where $\tilde{\omega}_j \equiv -\omega_{j+1}/(\beta\eta s)$. The characteristic equation is

$$\frac{1-s}{s} - \frac{1-\beta(1-\eta)}{\beta\eta s}z + z^2 = 0$$

and the solutions are

$$m_1 = \frac{1-\beta(1-\eta)}{2s\beta\eta} - \sqrt{\left(\frac{1-\beta(1-\eta)}{2s\beta\eta}\right)^2 - \frac{1-s}{s}}$$

and

$$m_2 = \frac{1-\beta(1-\eta)}{2s\beta\eta} + \sqrt{\left(\frac{1-\beta(1-\eta)}{2s\beta\eta}\right)^2 - \frac{1-s}{s}}.$$

It is easy to check that $m_2 > 1 > m_1 > 0$. Therefore the homogenous part of the solution of the difference equation takes the form $Am_1^j + Bm_2^j$. It can be verified that the overall solution takes the form

$$\Psi_{d+j}^H = \frac{s}{1-s}\left[\Sigma_{t=0}^j \frac{m_2}{m_1-m_2}(m_1^{j-t} - m_2^{j-t})\tilde{\omega}_{d+t} + \Sigma_{t=j}^{\infty} m_2^{j-t}\tilde{\omega}_{d+t} + Am_1^j + Bm_2^j\right].$$

The constant A is chosen such that $\Psi_d = 0$ and B is chosen such that $\lim_{j\to\infty}\Psi_{d+j}$ is finite. Since $m_2 > 1$, the only value of B that makes the limit of Ψ_j^H finite is

$$B = \frac{m_2}{m_1-m_2}\Sigma_{t=0}^{\infty}\frac{\tilde{\omega}_{d+t}}{m_2^t}.$$

To ensure $\Psi_d = 0$, the value of A is

$$A = -\frac{m_1}{m_1-m_2}\Sigma_{t=0}^{\infty}\frac{\tilde{\omega}_{d+t}}{m_2^t}.$$

Substituting the value of A and B into the solution of Ψ_{d+j}^H results in the desired formula. The method for solving Ψ_j^L is similar. Using the formula for Ω_j, it is immediate that the weight $b_{j,i}$ is differentiable in β, s and η. ■

Lemma 4 *The welfare can be written as*

$$\Omega_j = \Omega_d + \frac{\pi_j}{\beta\eta(1-s)}\left[\Sigma_{t=0}^{j-1}\frac{m_1^j}{m_2-m_1}\left(\frac{m_2}{m_1^t} - \frac{m_1}{m_2^t}\right)\frac{\omega_{d+t+1}}{\pi_{d+t+1}} + \Sigma_{t=j}^{\infty}\frac{m_1(m_2^j - m_1^j)}{(m_2-m_1)m_2^t}\frac{\omega_{d+t+1}}{\pi_{d+t+1}}\right].$$

Proof. By Lemma 3 one can check that $n_i = m_i s/(1-s)$ for $i = 1, 2$. Substitute this equation into Ψ^L_{d+j} then we have

$$\begin{aligned}
&\pi_{d+j}\Psi^H_{d+j} + (1-\pi_{d+j})\Psi^L_{d+j} \\
=&\frac{1}{\beta\eta}\Sigma_{t=0}^{j-1}\frac{m_1^j}{m_2-m_1}\left(\frac{m_2}{m_1^t}-\frac{m_1}{m_2^t}\right)\left[\left(\frac{1-s}{s}\right)^{j-t}\frac{(1-\pi_{d+j})}{s}+\frac{\pi_{d+j}}{1-s}\right]\omega_{d+t+1} \\
&+\frac{1}{\beta\eta}\Sigma_{t=j}^{\infty}\frac{m_1(m_2^j-m_1^j)}{(m_2-m_1)m_2^t}\left[\left(\frac{1-s}{s}\right)^{j-t}\frac{(1-\pi_{d+j})}{s}+\frac{\pi_{d+j}}{1-s}\right]\omega_{d+t+1}.
\end{aligned}$$

Using

$$\frac{\pi_{i+1}}{1-\pi_{i+1}} = \frac{s\pi_i}{(1-s)(1-\pi_i)},$$

we can rewrite

$$\left(\frac{1-s}{s}\right)^{j-t}\frac{(1-\pi_{d+j})}{s}+\frac{\pi_{d+j}}{1-s} = \frac{\pi_{d+j}}{(1-s)\pi_{d+t+1}}$$

and the desired result follows. ■

Lemma 5 *Suppose* $\Omega_j = \Omega_d + \Sigma_{i\geq d}^{\infty} b_{j,i}\omega_i$*. Then for* $i \leq j$*,*

$$b_{j,i} = \frac{\pi_j}{\beta\eta(1-s)}\frac{m_1^{j-d}}{(m_2-m_1)}\left(\frac{m_2}{m_1^{i-d-1}}-\frac{m_1}{m_2^{i-d-1}}\right)\frac{1}{\pi_i}.$$

and for $i > j$*,*

$$b_{j,i} = \frac{\pi_j}{\beta\eta(1-s)}\frac{m_1(m_2^{j-d}-m_1^{j-d})}{(m_2-m_1)m_2^{i-d-1}}\frac{1}{\pi_i}.$$

The weight $b_{j,i}$ *is log-supermodular in* (j,i)*.*

Proof. The expressions for $b_{j,i}$ follows immediately from Lemma 4. One can check that $b_{j,i_2}/b_{j,i_1}$ weakly rises in j for all $i_2 > i_1$. Hence $b_{j,i}$ is log-supermodular in (j,i). ■

D Learning about Usability

Usability is a key for cryptocurrencies adoption. A major selling point of crypto is that the technologies associated with it can create new trading opportunities, often by reducing transaction cost and increase the variety of goods and services. At the moment, it is unclear whether the number of trading opportunities created by these technologies are substantial. Hence a model where agents gradually learning about the usability of a money is relevant.

We assume the same market structure as in the baseline model, but make three changes. First, we study a single-currency economy by setting $\alpha^b = 0$. Second, the chance of entering a c-meeting, α^c, is a random variable, while the storage cost c is a constant. In each period, with chance $1 - \chi$, $\alpha^c = \hat{\alpha}^c$ where $\hat{\alpha}^c$ is a constant. With chance χ, the matching probability is a random variable $A \in \{\alpha_H^c, \alpha_L^c\}$ where $\alpha_H^c > \alpha_L^c$. The distribution of A is such that $P(A = \alpha_H^c|H) = s$, $P(A = \alpha_L^c|H) = 1 - s$, $P(A = \alpha_L^c|L) = s$ and $P(A = \alpha_H^c|L) = 1 - s$ where $s \in (0.5, 1)$. Therefore, the expected matching chance is higher when the asset type is H. When $\alpha^c = \hat{\alpha}^c$, with chance ζ the agents receive a noisy public signal about the asset's type and the signal has the same distribution as A. Again, we call $\eta = \chi + \zeta$ the information arrival chance.

The third modification concerns the timing of information. Instead of receiving information in the beginning of CM, we now assume agents receive the public signal or learn about the measure of DM meetings in the beginning of DM. The reason of this change is that, suppose agents do not know the aggregate measure of meetings in the DM, then they will use the fact that they are in a trade meeting to update their beliefs about the asset's type. This updating complicates the terms of trade, but it quickly becomes irrelevant because agents will learn about the aggregate measure of meetings in the following period. To eliminate this complication, we assume agents know about the measure of meetings in the beginning of the DM. As a result, the fact that an agent enters a trade meeting does not create new information about the asset's type.

Now we formula the asset pricing equations and the condition for abandoning the asset. Let $\bar{\alpha}_H^c \equiv (1 - \chi)\hat{\alpha}^c + \chi[s\alpha_H^c + (1 - s)\alpha_L^c]$ be the expected measure of c-type meetings when the asset quality is H. Define $\bar{\alpha}_L^c$ similarly. Then the expected measure of c-type meetings in state π_j is

$$\alpha_j^c \equiv \pi_j \bar{\alpha}_H^c + (1 - \pi_j)\bar{\alpha}_L^c.$$

The expected value of a units of asset in the next CM is

$$\ell_j^c \equiv a[\eta s_j \phi_{j+1} + \eta(1-s_j)\phi_{j-1} + (1-\eta)\phi_j - c]. \tag{39}$$

The key asset pricing equation, (10), now becomes

$$\begin{aligned} \phi_j =& \beta\ell_j^c + \beta a\{(1-\chi-\zeta)\hat{\alpha}^c(\phi_j - c)S'(\phi_j - c) \\ &+ \zeta\hat{\alpha}^c[s_j(\phi_{j+1}-c)S'(\phi_{j+1}-c) + (1-s_j)(\phi_{j-1}-c)S'(\phi_{j-1}-c)] \\ &+ \chi[s_j\alpha_H^c(\phi_{j+1}-c)S'(\phi_{j+1}-c) + (1-s_j)\alpha_L^c(\phi_{j-1}-c)S'(\phi_{j-1}-c)]\}. \end{aligned} \tag{40}$$

On the right side, the first term in the first line is the expected value of the asset in the following CM. The first in the braces represents the marginal transactional benefit when there is no public signal and the realization of the matching probability is $\hat{\alpha}^c$. This event occurs with chance $(1-\chi-\zeta)$. The second line is the marginal transactional benefit when the matching chance is $\hat{\alpha}^c$ and a public signal arrives, which has chance ζ. The third line is the marginal transactional benefit when the realization of matching probability is an informative signal of the asset's type, which occurs with chance χ.

Using market clearing $a = 1$ and assuming $u(y) = B\log(y)$, we can simplify (40) as

$$\phi_j = \beta(\ell_j^c + \alpha_j^c B). \tag{41}$$

By the logic leading to (15), the stationary price $\hat{\phi}(\pi_j)$ can be written as

$$\hat{\phi}(\pi_j) = \frac{\beta[B\alpha_j^c - c]}{1-\beta}. \tag{42}$$

As in the baseline model, we assume $\hat{\phi}(1) > c$ and $\hat{\phi}(0) < 0$ so that the agents are willing to abandon the asset at some interior cutoff. The condition that ensures agents abandon the asset at π_d, (14), is now given by

$$c \geq \eta s_d \phi_{d+1}. \tag{43}$$

As in the baseline model, we will assume π_d is such that (70) binds. Now we are ready to define an equilibrium.

Definition 2 *A stationary equilibrium is a list $\langle\{\phi_j\}_{j=d}^{\infty}, \pi_d\rangle$ that (i) $\phi_d = 0$, ϕ_j satisfies (10) for $\pi_j \geq \pi_{d+1}$ and $\lim_{j\to\infty}\phi_j = \hat{\phi}(1)$, and (ii) condition (70) binds at π_d.*

As in the baseline model, the price of the asset is the sum of the stationary price and a premium due to learning:

Proposition 7 (Pricing Formula and Cutoff of Disposal)

1. ***(Information premium.)*** *The information premium* λ_j *is given by*

$$\lambda_j = -\frac{(1-\beta)}{\beta\eta}\hat{\phi}(\pi_d)\frac{\pi_j}{\pi_d}\left[\frac{(1-\pi_j)\pi_d}{\pi_j(1-\pi_d)}\right]^{k(\eta,\beta,s)}. \tag{44}$$

   where $k(\eta,\beta,s)>1$*. As* π_j *rises, the magnitude of* λ_j *falls and vanishes as* $\pi_j \to 1$*.*

2. ***(Asset pricing.)*** *For* $j \geq d$*, the price* ϕ_j *is given by*

$$\phi_j = \hat{\phi}(\pi_j) + \frac{\beta\eta}{1-\beta}\lambda_j. \tag{45}$$

   The price ϕ_j *increases at all* $j \geq d$ *when* c *decreases or* B*,* $\bar{\alpha}_L^c$ *or* $\bar{\alpha}_H^c$*.*

3. ***(Asset abandonment.)*** *The cutoff* π_d *of disposing the new money rises in* c *and falls in* B*,* $\bar{\alpha}_L^c$ *and* $\bar{\alpha}_H^c$*.*

The intuition of these results are similar to that of Proposition 7. Next we show that the impact of a change in the information arrival chance η is similar to that of the baseline model:

Proposition 8 (Arrival of Information)

1. ***(Asset abandonment.)*** *The cutoff* π_d *of abandoning the asset falls in* η*.*

2. ***(Information premium.)*** *There is a cutoff* $\hat{\eta}$ *such that if* $\eta \geq \hat{\eta}$*, then* $\lambda_j \geq 0$ *for all* $j \geq d$*. If if* $\eta < \hat{\eta}$*, then* $\lambda_j < 0$ *for all* $j \geq d$*. If agents are sufficiently patient, then* $\hat{\eta} \in (0,1)$*.*

3. ***(Asset price.)*** *If* $\eta \geq \hat{\eta}$*, then* $\partial\phi_j/\partial\eta > 0$ *for all* $\pi_j \geq \pi_d$*. If* $\eta < \hat{\eta}$*, then there is a* $\pi^* > \pi_d$ *such that* $\partial\phi_j/\partial\eta > 0$ *if* $\pi_j \leq \pi^*$ *and* $\partial\phi_j/\partial\eta \leq 0$ *if* $\pi_j > \pi^*$*.*

We illustrate the welfare properties of this model by numerical examples.[8] For $\pi_j \geq \pi_d$, welfare Ω is again defined by (24), but now the expected surplus at state π_j is given

[8] Parameters used in this numerical example are: $\beta = 0.996, \alpha_L^c = 0.01, B = 0.97, c = 0.025, \chi = 0.07, s = 0.58$. High η represents $\eta = 0.99$. In the left panel $\alpha_H^c = 0.99$, $\hat{\alpha}^c = 0.005$ and low η is $\eta = 0.2$, and in the right panel $\alpha_H^c = 0.77$, $\hat{\alpha}^c = 0.0005$, and low η is 0.5.

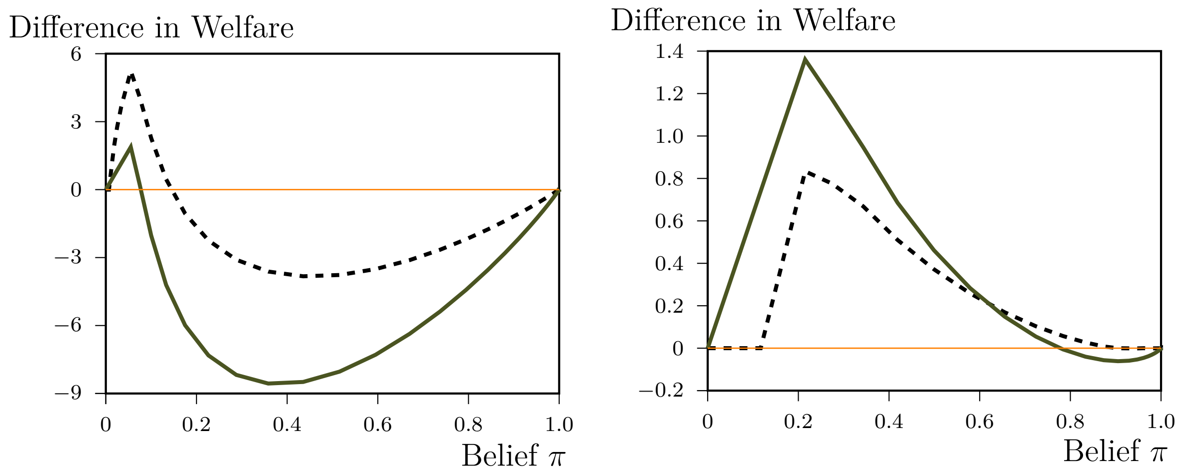

Figure 6: Welfare comparison. Green solid lines represent the differences in welfare of full information and low η, black dashed lines represent the differences in welfare of high and low η. (Left) $\lambda_j < 0$. (Right) $\lambda_j > 0$.

by

$$\begin{aligned}\omega_j =& (1-\chi-\zeta)\hat{\alpha}^c S(\ell_j^c) + \zeta\hat{\alpha}^c[s_j S(\ell_{j+1}^c) + (1-s_j)S(\ell_{j-1}^c)] \\ &+ \chi[s_j\alpha_H^c S(\ell_{j+1}^c) + (1-s_j)\alpha_L^c S(\ell_{j-1}^c)] - c.\end{aligned} \tag{46}$$

The first term on the right captures the DM trade surplus when there is no public signal and the realization of the matching probability is $\hat{\alpha}^c$. The second term is the DM surplus when the matching chance is $\hat{\alpha}^c$ and a public signal arrives. The third term is the expected trade surplus when the realization of matching probability is an informative signal of the asset's type. In the left panel of Figure 6, the information premium is negative and the change in welfare due to full disclosure is qualitatively similar to that of an increase in η. In the right panel, the information premium is positive and an increase in η raises welfare but full disclosure reduces welfare for π closes to 1. These results are consistent with the findings in Figure 4.

Proof of Proposition 7. The derivation of the pricing formula for ϕ_j is the same as that in the proof of Proposition 1. Given that the cutoff π_d solves $\phi(\pi_{d+1}) = c/(\eta\bar{s}_d)$, by (45), the solution of π_d is

$$\pi_d = \frac{\left\{\frac{1-\beta}{\beta\eta} + s[1-\Gamma(\eta,\beta,s)]\right\}c - sB\bar{\alpha}_L^c[1-\Gamma(\eta,\beta,s)] - (c-\bar{\alpha}_L^c B)(2s-1)}{sB(\bar{\alpha}_H^c - \bar{\alpha}_L^c)[1-\Gamma(\eta,\beta,s)] - (c-\bar{\alpha}_L^c B)(2s-1)} \tag{47}$$

where $\Gamma(\eta,\beta,s)$ is given by (27) by setting $i=0$. The comparative statics with respect to c, B, $\bar{\alpha}_L^c$ and $\bar{\alpha}_H^c$ follow the same argument in the proof of Proposition 1. ■

Proof of Proposition 8. Part 1. The formula (19) for the cutoff can be rewritten as

$$\pi_d = \frac{\left[\frac{1-\beta}{\beta s\eta(1-\Gamma)} + 1\right] c - B\bar{\alpha}_L^c - \frac{(c-\bar{\alpha}_L^c B)(2s-1)}{s(1-\Gamma)}}{B(\bar{\alpha}_H^c - \bar{\alpha}_L^c) - \frac{(c-\bar{\alpha}_L^c B)(2s-1)}{s(1-\Gamma)}}.$$

By (27), Γ and $\eta(1-\Gamma)$ rise in η. Therefore the right side fraction falls in η.

Part 2. The information premium is negative if and only if $\hat{\phi}(\pi_d) > 0$, or equivalently

$$\pi_d > \frac{c - \bar{\alpha}_L^c B}{B(\bar{\alpha}_H^c - \bar{\alpha}_L^c)}.$$

By (19) the inequality can be rewritten as

$$\frac{(1-\beta)c}{\beta(c - \bar{\alpha}_L^c B)} > (2s-1)s\eta[1-\Gamma(\eta, i, \beta, s)] \left[\frac{\bar{\alpha}_H^c B - c}{B(\bar{\alpha}_H^c - \bar{\alpha}_L^c)}\right]. \tag{48}$$

Since $\eta[1-\Gamma(\eta)]$ rises in η by (27), as η rise, the right side rises and the left side remains unchanged. When $\eta = 0$, the right side vanishes and hence the inequality holds. As $\eta \to 1$, the inequality (48) fails if β is sufficiently close to 1. Hence if β is large, then there is a $\hat{\eta} \in (0,1)$ such that (48) holds if and if $\eta < \hat{\eta}$.

Part 3: The comparative statics of ϕ_j with respect to η is the same as that in the proof of Proposition 3. ■

E Learning with General Preferences

In the baseline model we consider log utility, in this section we consider more general preferences and illustrate the importance of the assumption on preferences.

Here we assume there is only one asset and one type of DM trade meeting which occurs with chance α. The asset generates stochastic dividends which can be positive or negative. Specifically, each unit of asset generates a random payoff γ with chance χ. The realization of γ is the same across all units of asset and it has a symmetric binary distribution — if the asset quality is H, then $\gamma = \gamma^H > 0$ with chance $s \in (0.5, 1)$ and otherwise $\gamma = \gamma^L < 0$. If the asset quality is L, then $\gamma = \gamma^H$ with chance $1 - s$ and otherwise $\gamma = \gamma^L$. With chance ζ the asset creates no payoff in the current period but generates a noisy public signal which has the same distribution as γ.

One can interpret γ^H as dividends and γ^L as a unit cost of holding the asset, which can be a utility cost of managing or storing the asset. We make two restrictions on the value of γ^H and γ^L. We assume $s\gamma^L + (1-s)\gamma^H < 0$ such that if agents think the asset quality is L, namely $\pi = 0$, then the expected value $E[\gamma|\pi = 0] < 0$. We also assume $s\gamma^H + (1-s)\gamma^L = E[\gamma|\pi = 1] > 0$. These assumptions imply that when $\pi = 0$ there is a steady-state nonmonetary equilibrium where the asset is abandoned; when $\pi = 1$ there is a steady-state monetary equilibrium where the asset price is positive.[9] The expected dividend at state π_j is given by

$$\gamma_j \equiv \chi[s_j\gamma^H + (1-s_j)\gamma^L]. \tag{49}$$

For a buyer with a units of asset, the amount of liquidity is

$$\ell_j = a[\gamma_j + \eta s_j \phi_{j+1} + \eta(1-s_j)\phi_{j-1} + (1-\eta)\phi_j]. \tag{50}$$

The key asset pricing equation, (10), becomes

$$\phi_j = f(\ell_j) \equiv \beta\ell_j + \beta\alpha\ell S'(\ell_j). \tag{51}$$

The left side of (51) is the cost of carrying one more unit of asset from the CM. The right side is the marginal benefit of asset holding, which is the sum of the value of one

[9]If the expected dividend is positive, then the asset should be always be accepted as a mean of payment, as exemplified by Wallace and Zhu (2004). Gu and Wright (2016) further show that, even with minimal structure on the trading mechanism, there exists a unique monetary steady state.

more unit of asset in the next CM and the marginal increase in DM trade surplus. By the definition of ℓ in (50), we can rewrite (51) as

$$\phi_j = f[\gamma_j + \eta s_j \phi_{j+1} + \eta(1 - s_j)\phi_{j-1} + (1 - \eta)\phi_j]. \tag{52}$$

For $\ell \geq q^*$, buyers' liquidity needs are satisfied (i.e. $S'(\ell) = 0$) and the marginal benefit of asset holding is $f(\ell) = \beta\ell$. Therefore f is continuous at $\ell = q^*$ but there is an outward kink (i.e. f is convex at this point). For $\ell \in [0, q^*)$, we assume $f(0) = 0$ and $f'(\ell) > 0$. These are mild assumptions, for example, $f(0) = 0$ if $\lim_{q \to 0} qu'(q) = 0$, and $f'(\ell) > 0$ is satisfied when α is sufficiently small or the relative risk aversion of u is less than 1.

We will show that the curvature of f is a crucial determinant of the impact of information disclosure, so here we elaborate on our assumptions on it. In our analysis the relevant domain of $f(\ell)$ is $[0, \ell_H]$ where ℓ_H is the amount of liquidity that buyers carry in a world where the asset type is known to be H (we will derive ℓ_H later). Below we assume f is concave-convex (including the special cases that it is entirely concave or entirely convex) in the relevant domain and the slope of the convex part is less than 1. The latter assumption holds when the matching probability α is small because $f(\ell) = \beta\ell$ when $\alpha = 0$. We assume the former assumption for two reasons. First, for $\ell < q^*$, under various parametric forms of $u(q)$, f is often concave, convex, or concave and then convex. Lemma 6 provides several parametric examples to illustrate how different assumptions on u affects the shape of f:

Lemma 6 *Consider the curvature of $f(\ell)$ for $\ell \in [0, q^*)$.*

1. *Assume $u(q)$ has constant relative risk aversion (CRRA). If the relative risk aversion (RRA) $-qu''(q)/u'(q) < 1$, then f is concave. If RRA > 1, then f is convex.*

2. *If $u(q)$ has constant absolute risk aversion (CARA), then f is concave-convex.*

3. *If $u(q)$ has decreasing relative risk aversion (DRRA) and RRA ≤ 1, then f is concave.*

4. *If $u(q)$ has increasing relative risk aversion (IRRA) and RRA > 1, then f is convex.*

For $\ell \geq q^*$, f is linear and it has an outward kink at $\ell = q^*$, which makes f convex at q^*. So altogether f is often concave-convex.In Figure 7 we illustrate two schematics of f. In the left panel f is concave and then convex for $\ell \in [0, q^*]$ and then it becomes linear for $\ell \in [q^*, \ell^H]$. In the right panel $q^* > \ell^H$ and hence the liquidity premium does

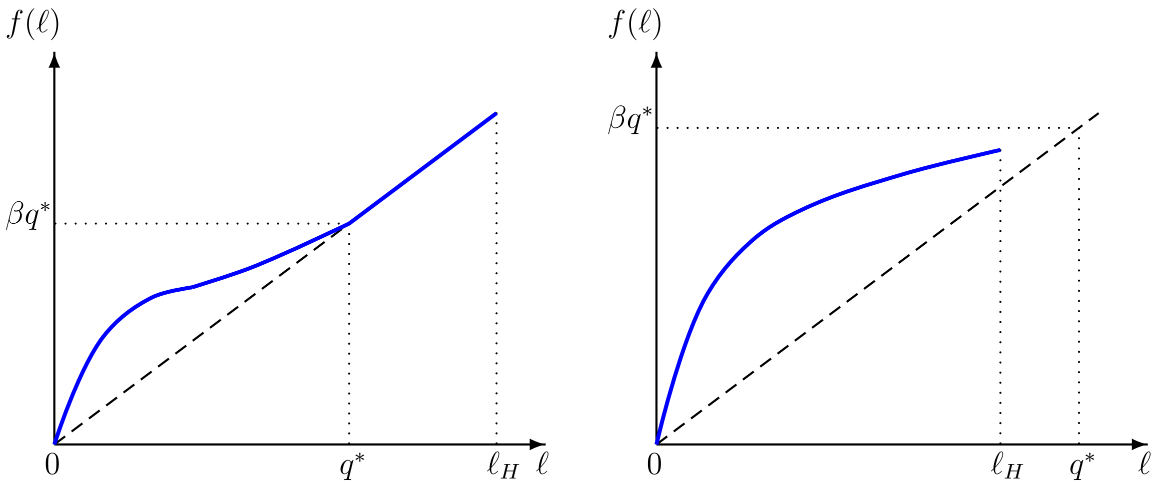

Figure 7: Marginal benefit of asset holding $f(\ell)$. (Left) Abundant liquidity (Right) Scare liquidity.

not vanish at $\ell = \ell_H$. In this case f can be entirely concave in the relevant domain. It is also easy to construct examples where f is entirely convex in the relevant region.

Since f is strictly increasing, it is invertible and we can rewrite (52) as a non-linear second-order difference equation:

$$\phi_{j+1} = \frac{1}{\eta s_j}[f^{-1}(\phi_j) - (1-\eta)\phi_j - \eta(1-s_j)\phi_{j-1} - \gamma_j]. \tag{53}$$

We assume the limit $\lim_{j\to\infty}\phi_j$ exists. Let $\phi_H \equiv \lim_{j\to\infty}\phi_j$, then by (52) ϕ_H must solve

$$\phi_H = f(\gamma_\infty + \phi_H). \tag{54}$$

The solution for ϕ_H is unique and strictly positive as we have assumed $\gamma_\infty \equiv \chi[s\gamma^H + (1-s)\gamma^L] > 0$, $f(0) = 0$, $f' > 0$ and $f' \leq 1$ when it is convex. The amount of liquidity that buyers carry is $\ell_H \equiv \gamma_\infty + \phi_H$.

By the logic leading to (14), at state π_d, the condition

$$\eta s_d \phi_{d+1} + \gamma_d \leq 0 \tag{55}$$

holds. In the baseline model, since we can solve for the asset prices in closed-form, it is relatively easy to establish uniqueness of equilibrium. To simplify the analysis, we assumed the initial prior is such that condition (55) binds in equilibrium. Under general preferences, it is possible that there are multiple equilibria for any given prior π_0. Hence

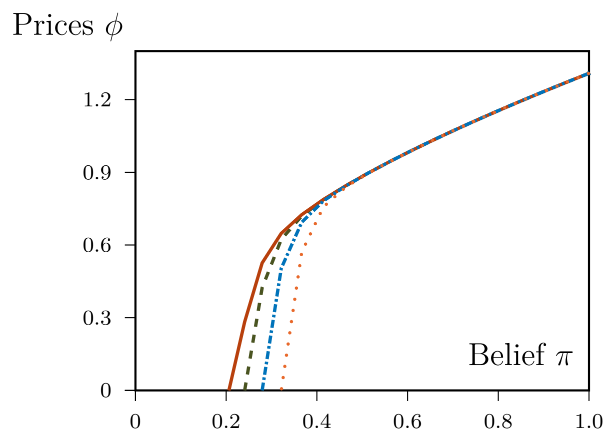

Figure 8: Example of multiple equilibria.

we assume π_0 is arbitrary and do not assume (55) binds at π_d.

Now we are ready to define an equilibrium.

Definition 3 *A stationary equilibrium is a list* $\langle\{\{\phi_j\}_{j=d}^{\infty}, \pi_d\rangle$ *that (i)* $\phi_d = 0$, ϕ_j *satisfies (53) for* $j \geq d+1$ *and* $\lim_{j\to\infty}\phi_j = \phi_H$, *and (ii) agents abandon the asset, i.e. (14) holds, at some* $\pi_d \in (0,1)$.

E.1 Existence and Uniqueness of Equilibrium

We present a numerical example of multiple equilibria in Figure 8.[10] In the example, across equilibria ϕ_j is higher when the cutoff π_d is smaller. The multiplicity is interesting but, as discussed in the introduction, to explain money adoption it is desirable to have a unique equilibrium. Below we provide a sufficient condition for equilibrium uniqueness and propose a simple iterative method to solve for (ϕ, π_d).

In models following Lucas (1978) a standard method to prove uniqueness is to assume the slope of f in (52) is less than 1 and then apply the contraction mapping theorem. But here the presence of DM trades induces non-linearity in f and its slope may exceed 1. Hence we need an alternative way to proceed and we find the assumption below useful.

Assumption 1 *Assume the slope of* $f(\ell)$ *is bounded above at all* ℓ, *namely*

$$f'(\ell) < \frac{1}{2\eta\sqrt{s(1-s)} + 1 - \eta}. \tag{56}$$

[10]The parameters used in all simulations are listed in Online Appendix H.

For example, when f is concave, the left side is bounded above by $f'(0)$ which may exceed 1. Specifically, by the definition of f in (51)

$$f'(0) = \lim_{q \to 0} \beta \left\{1 - \alpha + \alpha \left[u'(q) + u''(q)q\right]\right\}$$

and it is finite as long as $u''(0)$ and $u'(0)$ are finite. The right side of (56) can be interpreted as the rate at which agents' belief converges over time. To see this let $\mathcal{Z}$ be an indicator function that equals 1 if the asset's quality is H and 0 otherwise. When agents' belief is π, their expectation of $\mathcal{Z}$ is $E_\pi[\mathcal{Z}] = P(\mathcal{Z} = 1) = \pi$ and the standard deviation of $\mathcal{Z}$ is $\sigma_\pi \equiv \sqrt{\pi(1-\pi)}$. The standard deviation σ_π measures how uncertain the agents are about the asset's quality. Let π' be the next period belief. The convergence rate of beliefs can be measured by the ratio $\sigma_\pi / E_\pi[\sigma_{\pi'}]$ and it is equal to the right side of (56). If this ratio is ∞, then agents learn the asset's quality in 1 period. If it is 1, then signals never arrive or they are pure noise. Naturally the convergence rate increases in η or s and explodes as η and $s \to 1$.

Intuitively, Assumption 1 is satisfied when learning is sufficiently fast. Given Assumption 1 we show that the equilibrium is unique:

Proposition 9 (Uniqueness of Equilibrium) *Generically there is at most one equilibrium and the equilibrium price ϕ_j increases in j.*

Proposition 9 states that the equilibrium is unique under Assumption 1 except for measure zero of parameter values.[11] This results has two implications. First, there is no autarky equilibrium where money is not accepted at all states. The reason is that when π_j is sufficiently close to 1, the asset delivers strictly positive dividend and therefore sellers will accept the asset as payment. This idea is related to the commodity-money refinement used to reduce the equilibrium set of pure currency economies, see Wallace and Zhu (2004) or Garratt and Wallace (2018). The second implications is that, there is a unique equilibrium where agents abandon the asset when π is small and use the asset when π is large (i.e. Figure 8 is impossible). Intuitively, when the learning speed is sufficiently high, the asset prices are highly correlated across states and hence a small increase in ϕ_{d+1} strictly raises all subsequent ϕ_j by (53). This correlation makes sure there is a unique pair of ϕ_{d+1} and π_d that leads to $\lim_{j\to\infty} \phi_j = \phi_H$ and satisfies (55). If we drop the assumption that $\gamma_\infty > 0$ (as in the baseline model), then the first

[11] As shown in the proof of Proposition 9, for measure zero of parameter values, there are two equilibria and their cutoffs π_d are next to each other (i.e. they are one signal away from each other).

implication no longer holds, i.e. there can be an autarky equilibrium where the money is not accepted at all $\pi_j \in [0,1]$. But there will still be at most one equilibrium where $\lim_{j\to\infty} \phi_j = \phi_H > 0$.

Our uniqueness result is generalizable to other environments because Assumption 1 only imposes a restriction on the slope of f and does not depend on the exact micro-foundation (i.e. trading protocol, matching technology, entry, etc). For example, if agents bargain over the terms of trade, then the f function will take a different functional form but one can still write the asset pricing equation in the form of (52) and check whether (56) holds. In the working paper Choi and Liang (2021) we show how to apply Proposition 9 in settings with Walrasian pricing or entry of sellers.[12]

Now we solve for the equilibrium prices. Lucas (1978) shows that equilibrium asset prices can be computed by a simple iterative method which is an implication of the contraction mapping theorem. We claim that one can exploit the Tarski fixed-point theorem to derive a similar method for computing asset prices in our model. Let L be the set of bounded and weakly increasing sequences $\{\phi_j\}_{j=-\infty}^{\infty}$ where $\phi_j \in [0, \phi_H]$ at all integer $j \in \mathbb{Z}$. For any $\phi', \phi'' \in L$, we say ϕ' is larger than ϕ'', or $\phi' \geq \phi''$, if $\phi'_j \geq \phi''_j$ for all j. Define a mapping F by modifying the first-order condition (52):

$$F_j(\phi) \equiv \max\{f[\gamma_j + \eta s_j \phi_{j+1} + \eta(1-s_j)\phi_{j-1} + (1-\eta)\phi_j], 0\}. \tag{57}$$

We write $\phi' = F(\phi)$ if $\phi'_j = F_j(\phi)\ \forall j \in \mathbb{Z}$. Lemma 10 in the Appendix shows that F is order-preserving (i.e. $F(\phi)$ rises in ϕ) and maps $L \to L$. Denote $F^2(\phi) = F(F(\phi))$, $F^3(\phi) = F(F(F(\phi)))$ and so on. Proposition 10 shows that the equilibrium price sequence can be computed by iterating over F with an appropriate choice of initial sequence.

Proposition 10 (Equilibrium Asset Prices) *The equilibrium price sequence ϕ^* is a fixed point $\phi^* = F(\phi^*)$. Let ϕ^0 be a seed sequence where $\phi^0_j \equiv 0$ for all $j \in \mathbb{Z}$. Then $\phi^* = \lim_{n\to\infty} F^n(\phi^0)$ and the sequence $F^n(\phi^0)$ increases in n.*

Proposition 10 is an application of the Tarski's fixed-point theorem. Unlike Lucas's method, our iterative method requires a specific initial sequence ϕ^0 and otherwise the iterations might not converge. Note that not every fixed point of F is an equilibrium price sequence — there could exist a fixed point where $\phi_j > 0$ at all j and hence the

[12]If we assume buyers and sellers meet bilaterally and buyers make a take-it-leave-it offer to the sellers, then the f function will be identical to the one in our baseline model, provided that $c(q) = q$.

asset is never abandoned. If $\gamma_{-\infty}$ is sufficiently negative such that there is no positive solution to $\phi = f(\gamma_{-\infty} + \phi)$, then F has a unique fixed point and it corresponds to the equilibrium price sequence. As shown by the next corollary, Proposition 10 immediately yields several intuitive comparative statics:

Corollary 1 *The price ϕ^* weakly rises and the cutoff π_d weakly falls in α, γ^H and γ^L.*

Proof. Since the sequence $F(\phi)$ in (57) weakly rises in α and ϕ, the limit $\lim_{n\to\infty} F^n(\phi^0)$ weakly rises in α. Since $\phi^* = \lim_{n\to\infty} F^n(\phi^0)$ by Proposition 10, ϕ^* weakly rises in α. Since ϕ^* weakly rises, the cutoff π_d weakly fall. The proof for γ^H and γ^L are similar. ■

E.2 Impact of New Information

In the rest of this section we characterize the role of information in money adoption. Again we define the information premium at state π_j as $\lambda_j \equiv s_j\phi_{j+1} + (1-s_j)\phi_{j-1} - \phi_j$ which is the expected change in the asset's price when information arrives. Graphically it has the same sign as the curvature of ϕ_j, when ϕ_j is represented as a function of π_j. Indeed, if $\lambda_j > 0$, then ϕ is convex at π_j and it is concave if $\lambda_j < 0$, as shown in the left panel of Figure 9. The sign of λ_j is useful for understanding the impact of a new signal as well as for understanding the impact of a change in η. To see the latter claim, note that (52) can be rewritten as

$$\phi_j = f(\gamma_j + \eta\lambda_j + \phi_j). \tag{58}$$

The impact of a higher λ_j on ϕ_j is similar to that of a higher expected dividend γ_j. Therefore, fixing λ_j, the solution of ϕ_j rises in η if $\lambda_j > 0$. Clearly λ_j also changes in η in general equilibrium, but in the next subsection we will argue that the sign of λ_j is still a useful predictor of the overall impact of η.

The main result of this subsection is to characterize how the sign of λ_j depends on the curvature of f. We first illustrate the basic idea in a two-period example:

Two-period Example: Suppose there is a 1/2 ex-ante chance that the asset quality is H. For $\upsilon \in \{H, L\}$, the asset creates a per-period dividend γ^υ and hence υ is revealed perfectly by the dividend. Suppose when agents learn the state is υ, the second-period asset price is $\phi_\upsilon = f(\gamma^\upsilon + \phi_\upsilon)$. Then by (52) the first-period price is $\hat{\phi} = f[\bar{\gamma} + (\phi_H + \phi_L)/2]$ where $\bar{\gamma} \equiv (\gamma^H + \gamma^L)/2$. If $f(\ell)$ is convex, then the first-period information premium

$\lambda \equiv (\phi_H + \phi_L)/2 - \hat{\phi} > 0$ because $(\phi_H + \phi_L)/2 = [f(\gamma^H + \phi_H) + f(\gamma^L + \phi_L)]/2 > f[\bar{\gamma} + (\phi_H + \phi_L)/2] = \hat{\phi}$. By the same logic, $\lambda < 0$ if f is concave. ■

So λ and f'' tend to have the same sign. This example is slightly more subtle than a direct application of Jensen's inequality. It is because, due to the timing of the signals, our asset pricing equation takes the form $\phi_t = f(E[\gamma_{t+1} + \phi_{t+1}])$ instead of $\phi_t = E[f(\gamma_{t+1} + \phi_{t+1})]$.

Now we relate the information premium to the shape of f in the general model. Consider the solution of ϕ of the equation

$$\phi = f[\gamma(\pi_j) + \phi]. \tag{59}$$

This equation is analogous to condition (15) which defines that stationary price, $\hat{\phi}$. But equation (59) can have multiple solutions because f is non-linear and it has at most two solutions because we have assumed $f' < 1$ when it is convex. Let $\hat{\phi}(\pi_v)$ be the correspondence of the solutions. Let $\bar{\phi}_j(\pi)$ and $\underline{\phi}_j(\pi)$ be, respectively, the larger and smaller element of $\hat{\phi}(\pi)$. When there is only one solution then let $\bar{\phi}_j$ be the solution and set $\underline{\phi}_j = 0$. One can interpret $\bar{\phi}_j$ and $\underline{\phi}_j$ as the asset price in a steady-state equilibrium in a world with no learning and the asset generates a fixed dividend $\gamma(\pi_j)$. Since the correspondence $\hat{\phi}$ represents the intersection points between f and the 45 degree line, graphically $\hat{\phi}$ is equivalent to rotating the f function anti-clockwise by 45 degrees and then scale by a constant factor. In general $\hat{\phi}$ is a $\subset$-shaped curve as illustrated by the orange dashed line in the left panel of Figure 9.

Lemma 7 claims that ϕ is concave at state π_j (i.e. $\lambda_j < 0$) if and only if ϕ_j lies on the right side of $\hat{\phi}$. To illustrate this claim, we present a numerical example of $\hat{\phi}$ and ϕ in the left panel of Figure 9.

Lemma 7 *(i) $\bar{\phi}_j(\pi_j)$ rises in π_j and is concave-convex. $\underline{\phi}_j(\pi_j)$ falls in π_j and is convex. (ii) The price ϕ is weakly convex at π_d. For $j > d$, ϕ is concave at π_j if $\phi_j \in (\underline{\phi}_j, \bar{\phi}_j)$, it is linear if $\phi_j = \bar{\phi}_j$ or $\phi_j = \underline{\phi}_j$ and otherwise it is convex.*

Part (i) is immediate given the definition of $\bar{\phi}_j$ and $\underline{\phi}_j$ and our assumption on the shape of f. By this claim the correspondence $\tilde{\phi}$ forms a $\subset$-shaped curve. The first claim in Part (ii) is obvious because $\phi_{d-1} = \phi_d = 0$ and $\phi_{d+1} \geq 0$. The second claim in Part (ii) is true because if $\phi_j \in (\underline{\phi}_j, \bar{\phi}_j)$, then $\phi_j < f[\gamma(\pi_j) + \phi_j]$, and so λ_j must be negative to balance equation (58). The proof for the rest of the claim is similar.

In general ϕ_j is convex-concave-convex. If f is entirely concave or entirely convex, then one can sharpen the predictions:

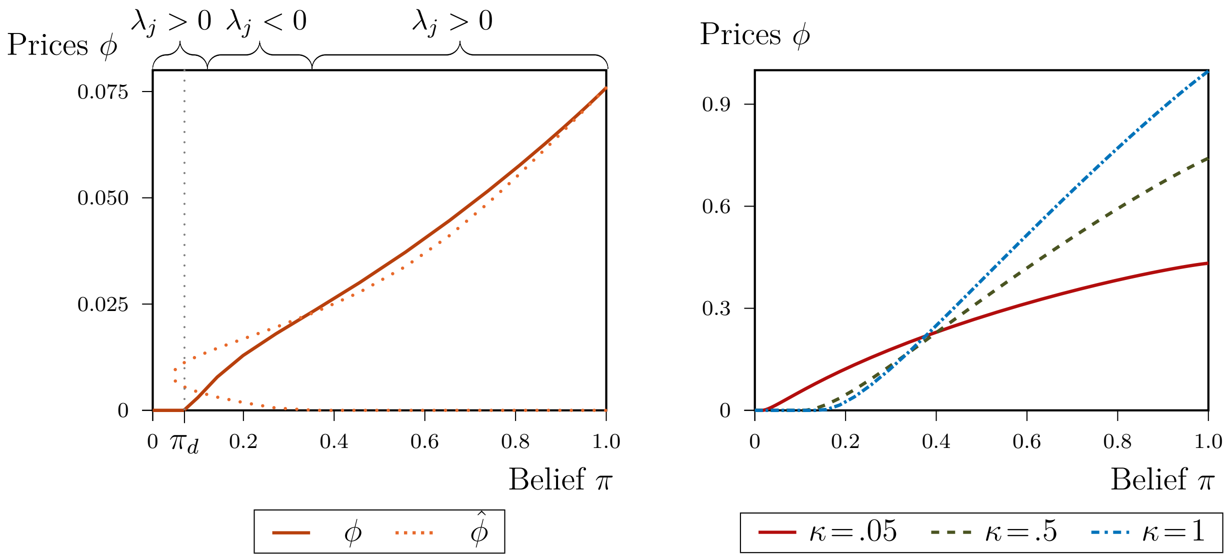

Figure 9: (Left) Example of non-monotone λ_j. (Right) Asset prices as γ^L and γ^H are scaled by a factor κ.

Proposition 11 (Information Premium)

1. *Assume $f(\ell)$ is concave-convex for $\ell \in [0, \ell_H]$. Then for $j > d$, the information premium λ_j is either (i) always positive, (ii) always negative (iii) positive and then negative as j rises (iv) negative and then positive as j rises or (v) positive, negative and then positive as j rises.*

2. *If $f(\ell)$ is concave for $\ell \in [0, \ell_H]$, then λ_j either satisfies case (i), (ii) or (iii).*

3. *If $f(\ell)$ is convex for $\ell \in [0, \ell_H]$, then λ_j satisfies case (i).*

So λ_j can at most have two sign changes — it is positive when π_j is very small or very large, and is negative otherwise. For intuition, note that when π_j is near π_d, agents are on the fence between holding or abandoning the asset. Information is useful for making decisions and hence the information premium $\lambda_j > 0$. When π_j is large, agents are unlikely to abandon the asset so new information is not useful for making investment decisions. But by Proposition 11 even in this case $\lambda_j > 0$ is possible, provided that part of the marginal benefit $f(\ell)$ of asset holding is convex. When the relevant region of f is convex, the arrival of new information on average increases the demand of the asset, and hence raising asset price on average. As discuss before, f can be convex for two reasons. First, the presence of the liquidity premium introduces convexity into $f(\ell)$ at $\ell = q^*$, as shown in the left panel of Figure 7. Intuitively, suppose the agents are optimistic enough such that the liquidity premium vanished and the asset is valued fundamentally.

If a good news arrives, then it further raises the asset's price. If a bad news arrives, then agents lower their expectation about the dividends, but the asset price might not drop much because it will start to include a liquidity premium. Hence new information raises the price of the asset on average. Another reason that f is convex is due to the functional form of u, as illustrated by Lemma 6. To conclude, the price impact of new information can be positive or negative and it depends crucially on the curvature of f and whether information is useful for decision making.

E.3 Increase in Information Disclosure

Now we consider an increase in the information arrival chance η. We raise η by raising the chance ζ that the asset generates a public signal and fixing the chance χ that it generates dividends. Proposition 12 explains the link between the curvature of f and the impact of a change in η:

Proposition 12 (Change in Information Arrival Chance) *Suppose as the information arrival chance rises from η^* to η', the price, cutoff and information premium change from $(\phi^*, \pi_{d^*}, \lambda^*)$ to $(\phi', \pi_{d'}, \lambda')$.*

1. *If $f(\ell)$ is concave-convex for $\ell \in [0, \ell_H]$, then there exist $\bar{j}$ and $\bar{\bar{j}}$ such that $\bar{j} \leq \bar{\bar{j}}$ and $\phi'_j \geq \phi^*_j$ if and only if $j \leq \bar{j}$ or $j \geq \bar{\bar{j}}$.*
2. *If $f(\ell)$ is concave for $\ell \in [0, \ell_H]$, then $\bar{\bar{j}} = \infty$.*
3. *If $f(\ell)$ is convex for $\ell \in [0, \ell_H]$, then $\phi'_j \geq \phi^*_j$ at all j.*

Corollary 2 *If $\lambda^*_j \geq 0\ \forall j > d^*$, then $\phi' \geq \phi^*$ and $\pi_{d'} \leq \pi_{d^*}$. If $\lambda^*_j \leq 0\ \forall j > d^*$ and $\bar{\gamma}_{d^*} + \eta' s_{d^*} \phi^*_{d^*+1} \leq 0$, then $\phi' \leq \phi^*$ and $\pi_{d'} \geq \pi_{d^*}$.*

A message of Corollary 2 is that the impact of a change in η depends crucially on the sign of the information premium λ_j. If $\lambda^*_j \geq 0$ at all $j \geq d$, then asset prices increase in η at all j. The cutoff π_d falls in η, meaning that the chance of adopting the asset in the long run is higher. If $\lambda^*_j \leq 0$ for all $j > d$, then prices fall in η, provided that $\bar{\gamma}_d + \eta' s_d \phi^*_{d+1} \leq 0$. This inequality is satisfied if the IC constraint is not binding when $\eta = \eta^*$, namely $\bar{\gamma}_{d^*} + \eta^* s_{d^*} \phi^*_{d^*+1} < 0$ and $\eta' - \eta^*$ is not too large. In this case, as η rises, the cutoff π_d rises and agents are more likely to abandon the asset in the long run.

We illustrate the impact of η in Figure 10. A lesson from this analysis is that the impact of more disclosure is ambiguous in general. It depends on the reputation of the

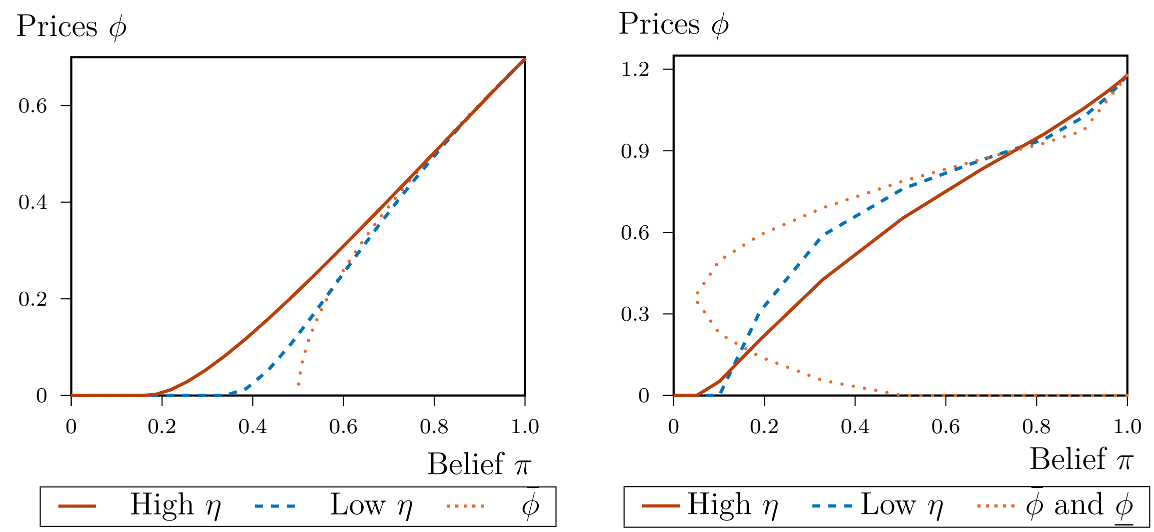

Figure 10: Numerical examples of an increase in η. (Left) Prices rise and π_d falls in η. (Right) Prices rise in η when π_j is small and otherwise fall. The cutoff π_d falls in η.

asset (i.e. the size of π_j) and the curvature of the price curve, which in turn depends on various model parameters. [13]

E.4 Welfare and Disclosure

Now we discuss the impact of a change in η on welfare. We assume the planner and the agents have the same belief and let Ω_j be the expected discounted sum of buyers' and sellers' payoffs when the current state is π_j. Again we omit the surplus due to the CM consumption because it is a constant. When the asset is abandoned, agents only trade in the CM. Hence for $\pi_j \leq \pi_d$, we have $\Omega_j = 0$. For $\pi_j > \pi_d$, Ω_j must take into account the changes in the asset prices and dividends, by the logic leading to (24)

$$\Omega_j = \omega_j + \beta[\eta s_j \Omega_{j+1} + \eta(1 - s_j)\Omega_{j-1} + (1 - \eta)\Omega_j]. \tag{60}$$

The first term $\omega_j \equiv \gamma_j + \beta\alpha S(q_j)$ is the expected dividends and DM surplus associated with state π_j, and the third term is the expected welfare in the next period.

In the baseline model, the coordinated abandonment of money convexifies or concav-

[13] Currently there is no disclosure policy for initial coin offerings (ICOs) and hence all disclosures are voluntary. Two recent empirical studies carry out cross-sectional analysis on a large number ICOs and the findings on the relationship between voluntary disclosure and ICO success (i.e. whether the coin circulates) are mixed: Howell et al. (2018) find positive correlation while Bourveau et al. (2019) find no consistent association between the two. Both studies find that ICOs with higher reputation ratings are more likely to succeed in raising funds, which is consistent with the main mechanism of our model.

ifies the payoff structure. Therefore an increase in η can raise or reduce welfare. Now we illustrate that, with general preferences, it is possible to create positive or negative change in welfare, even if agents never abandon the money. To do so we temporarily deviate from our assumptions and consider the case when $\gamma_{-\infty} > 0$ and so the asset is never abandoned. In this case the asset always serves as a means of payment and $\phi_{-\infty} = \phi_L > 0$ where $\phi_L = f(\gamma_{-\infty} + \phi_L)$. Since the asset is never abandoned, new information is payoff relevant but is not socially useful for investment. But even in this special case, welfare could go up or down in η:

Proposition 13 (Disclosure Could Raise or Reduce Welfare)

1. *If $f(\ell)$ is concave in ℓ for $\ell \in [0, \ell_H]$, then Ω_j weakly falls in η at all j.*

2. *If $u(q_j) - q_j$ is convex in π_j for $\pi_j \in [0, 1]$, then Ω_j weakly rises in η at all j.*

Part 1 is true for two reasons. First, when f is concave and the asset is never abandoned, the information premium is always negative and ϕ_j falls in η. Hence the surplus ω_j falls in η at each state. Second, since ϕ_j is concave in π_j, so is the liquidity ℓ_j. Since ω_j is concave in ℓ_j, ω_j is a concave function in π_j. As η rises, π_j transitions more frequently across states, and Ω_j falls in η in the spirit of Jensen's inequality. Altogether, welfare falls in η and information suppression is optimal.

But welfare can also rise in η, as suggested by part 2 of Proposition 13. First, since $u(q) - q$ is concave in liquidity ℓ, the sufficient condition of part 2 is satisfied only when ℓ_j is convex in π_j. This implies ϕ_j is also convex in π_j. Since the information premium is positive, by Corollary 2, ϕ_j rises in η at all j and hence the trade surplus is higher at each state. Second, since $u(q_j) - q_j$ is convex in π_j, more frequent transitions across states raises the expected traded surplus. This result is in sharp contrast with the conventional wisdom, e.g. Hirshleifer (1971), that information, which is not useful for making decisions, creates economic fluctuations that hurts the welfare of risk averse agents. Here the convexity of asset demand converts fluctuations in beliefs into higher asset prices which makes the asset more useful as a mean of payment or collateral. This effect improves welfare and outweighs the usual negative effect caused by uncertainties to risk averse agents.

Figure 11 presents two numerical examples where $\gamma_{-\infty} > 0$ and so the asset is never abandoned. The blue solid line represents an example where $f(\ell)$ is concave. As η rises, the asset prices fall and hence ω at each state drops (blue dashed line in the right panel). Since ω is concave in π, more rapid transitions reduce welfare, hence Ω falls in η at all

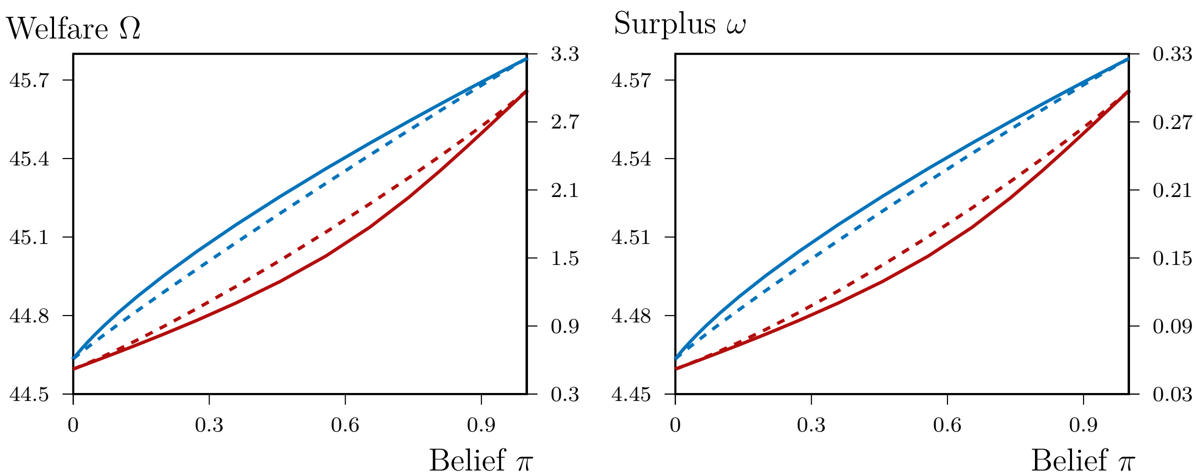

Figure 11: Two examples of a change in η. The blue lines (right axis) assume f is concave. As η rises from 0.1 (solid line) to 0.9 (dashed line), the welfare Ω and surplus ω fall. The red lines represent an example where the surplus ω is convex in π. As η rises rfom 0.1 to 0.9, Ω and ω increase as indicated by change from the red solid line to the red dashed line.

states (blue dashed line in right panel). The red solid line represents an example that corresponds to part 2 of Proposition 13. We choose a utility function such that ω is convex in π. As shown in the left panel, although new information is not useful for making the holding/abandoning decision, Ω rises in η at all states (red dashed line). Moreover Ω is convex in π and therefore the arrival of a new signal on average improves welfare.

Although the sufficient condition in part 2 of Proposition 13 is easy to understand, it is not obvious when will it be true. The proof of next corollary shows how to construct examples such that it holds. Intuitively, it requires that $u(q)$ is not too linear or too concave. If u is too linear, then $f(\ell)$ fails to be convex and hence ϕ will fall in η. If u is too concave, then the negative impact of economic fluctuations will dominate and hence welfare falls in η.

Corollary 3 *Suppose $\tilde{u}(q)$ is such that $q\tilde{u}'(q)$ is convex and $\lim_{q\to\infty} q\tilde{u}'(q) = 0$. Assume $u(q) = tAq + (1-t)\tilde{u}(q)$ where $t \in (0,1)$ and $A > 0$. There exists A, η γ^L and γ^H such that Ω_j rises in η at all j.*

F Competing Private Monies

We now study an economy with two competing private monies with general preferences as introduced in Section F. Then we will derive a necessary and sufficient condition such that the new money improves welfare.

F.1 Dual Asset Economy

We introduce a safe asset which generates a fixed dividend $\delta \geq 0$ in the beginning of each CM. Without loss of generality we normalize the supply of the safe asset to 1. One can interpret this asset as a government fiat money or a real asset with a known return. Below we call the asset in our baseline model as the risky asset. If the safe and risky assets both circulate, then buyers can use either one or a combination of them as a means of payment. We assume that if the risky asset is abandoned, then the economy will be in a monetary steady state where only the safe asset circulates. Let ψ^* be the price of the safe asset in this monetary steady state, by the logic leading to (54) it is the unique solution of $\psi^* = f(\delta + \psi^*)$.

Suppose both assets circulate. Let ψ_j be the price of the safe asset. An agent holding a units of risky asset and m units of safe asset in the DM has liquidity

$$\ell(a, m) = \delta m + \gamma_j a + \eta[s_j(\psi_{j+1} m + \phi_{j+1} a) + (1 - s_j)(\psi_{j-1} m + \phi_{j-1} a)] + (1 - \eta)(\psi_j m + \phi_j a).$$

When an agent enters the CM with a portfolio (a, m), his wealth is

$$\omega(a, m) = (\phi' + \gamma')a + (\psi' + \delta)m$$

where ϕ', γ', and ψ' are the realized value of the current period price of the risky asset, per unit dividend of the risky asset and price of the safe asset, respectively. The DM and CM value functions are similar to that of the baseline model, except now ℓ and ω are functions of the asset portfolio. The value functions are, respectively,

$$\begin{aligned} V(\ell, \pi_j) &= \alpha S(\ell) + \ell + E[W(0, \pi')|\pi_j] \quad \text{and} \\ W(\omega, \pi_j) &= \omega + \max_x \{U(x) - x\} + \max_{a,m} \{-\phi_j a - \psi_j m + \beta V(\ell, \pi_j)\}. \end{aligned}$$

By the first-order condition with respect to m and $dV(\ell, \pi_j)/dm = (d\ell/dm)[1 + \alpha S'(\ell)]$,

$$\psi_j = \beta[\delta + \eta s_j \psi_{j+1} + \eta(1 - s_j)\psi_{j-1} + (1 - \eta)\psi_j][1 + \alpha S'(\ell_j)]. \tag{61}$$

Similarly the first-order condition with respect to a can be written as

$$\phi_j = \beta[\gamma_j + \eta s_j \phi_{j+1} + \eta(1 - s_j)\phi_{j-1} + (1 - \eta)\phi_j][1 + \alpha S'(\ell_j)]. \tag{62}$$

In equilibrium all agents hold the same asset portfolio, hence $a = m = 1$ by market clearing. Define $\tau_j \equiv \psi_j + \phi_j$ as the value of the portfolio. Adding (61) and (62) yields

$$\tau_j = f(\ell_j) \equiv f[\gamma_j + \delta + \eta s_j \tau_{j+1} + \eta(1 - s_j)\tau_{j-1} + (1 - \eta)\tau_j]. \tag{63}$$

Again, we assume asset prices converge as $\pi_j \to 1$ and denote the limits by $\phi_H = \lim_{j\to\infty} \phi_j$ and $\psi_H = \lim_{j\to\infty} \psi_j$. By (61) and (62) these limits solve

$$\phi_H = \beta(\gamma_\infty + \phi_H)[1 + S'(\delta + \gamma_\infty + \tau_H)], \quad \psi_H = \beta(\delta + \psi_H)[1 + S'(\delta + \gamma_\infty + \tau_H)] \tag{64}$$

and $\tau_H \equiv \phi_H + \psi_H$ solves $\tau_H = f(\delta + \gamma_\infty + \tau_H)$.

As in the baseline model, agents are willing to abandon the risky asset at π_d if (14) holds. When the risky asset is abandoned $\phi_d = 0$, and hence $\tau_d = \psi_d = \psi^*$. A dual-asset equilibrium is a list $\langle \{q_j\}_{j=d}^\infty, \{\phi_j\}_{j=d}^\infty, \{\psi_j\}_{j=d}^\infty, \pi_d \rangle$ which solves the DM market problem, the first-order conditions (61) and (62), $\lim_{j\to\infty} \psi_j = \psi_H$ and $\lim_{j\to\infty} \phi_j = \phi_H$, and the incentive condition (14) holds at some $\pi_d \in (0, 1)$.

Our dual-asset equilibrium has a recursive structure. Given π_d, we can solve for τ_j by (63). Given π_d and τ_j, by (61) and (62) we can derive ϕ_j and ψ_j. Finally we check whether π_d and ϕ_j satisfy (14). The equilibrium is unique by an argument similar to, but also more complicated than, that of Proposition 9:

Proposition 14 (Dual-asset Equilibrium) *There is generically a unique equilibrium with dual assets. In equilibrium ϕ_j rises and ψ_j falls in j. The sum $\tau_j \equiv \phi_j + \psi_j$ is either rising or U-shaped in j and attains its maximum as $j \to \infty$.*

According to Proposition 14, as agents grow more optimistic about the risky asset, naturally ϕ_j rises, but ψ_j falls as the two assets are substitutable as a payment device. If $\delta = 0$, then the safe asset is intrinsically worthless and ψ_j vanishes as $\pi \to 1$ because

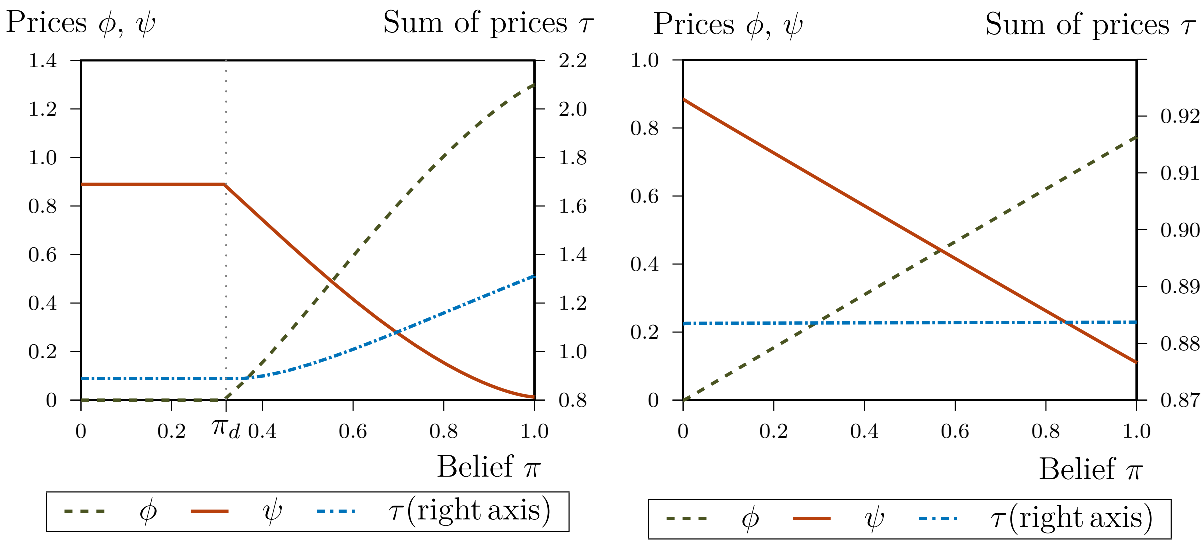

Figure 12: Asset prices in a dual-asset economy. (Left) Price of the asset portfolio τ rises in π. (Right) Both assets are close to fiat asset and τ is almost constant in π.

$\tau_H = \phi_H$ by (64). Consequently, as $\pi \to 1$, the economy permanently adopts the risky asset and stops using the safe asset as a medium of exchange.

The value of τ_j and ℓ_j comove by (63) and they are either rising or U-shaped in j by Proposition 14. A sufficient condition for them to be U-shaped is when γ^H is small. (proved by Proposition 15 below), namely when the dividends from an asset are small. The left panel of Figure 12 presents a numerical example where τ_j rises in π_j and the left panel of Figure 13 shows one where τ_j is U-shaped. When τ_j is U-shaped, a good news about the risky asset can reduce the aggregate liquidity and volume of trade.

Our dual-asset model provides a novel explanation of asset price volatility. As π_j rises from π_d to 1, the price of the safe asset drops and that of the risky asset rises, hence the exchange rate ϕ_j/ψ_j can fluctuate substantially over time. The sum τ_j, however, could be stable because the change in ψ_j and ϕ_j partially offset each other. Since the DM trade volume only depends on τ_j, the real economy could also be stable over time. We present a numerical example in the right panel of Figure 12 where $\delta = \kappa\tilde{\delta}, \gamma^L = \kappa\tilde{\gamma}^L$ and $\gamma^H = \kappa\tilde{\gamma}^H$. We assume κ is small such that both assets are almost fiat. As shown in the figure, τ_j is almost flat in π_j. This example explains why currency prices and exchange rates can be a lot more volatile than the underlying real economy.[14]

[14]This example is related to Kareken and Wallace (1981) and Garratt and Wallace (2018) who show that in an economy with competing fiat monies, the price of each currency is indetermined but the aggregate liquidity and real allocations are well-determined.

F.2 When will a New Money Improve Welfare?

Due to the development of the blockchain technology, it is easy to issue a new crypto-currency. From a regulator's perspective, sometimes the relevant choice is about issuing government money like CBDC, but some other time it is about whether to allow or ban the issuance a new private money (e.g., banning the issuance of new crypto-currency or banning merchants from accepting a crypto-currency). Now we study the condition under which a new private money can improve welfare.

Let Ω_j be the discounted sum of buyers' and sellers' payoffs at state π_j. For $j \leq d$, $\Omega_j = \Omega_d$ and is the expected discounted sum of payoffs in a monetary steady state with only the safe asset circulating, namely

$$\Omega_d = \frac{1}{1-\beta} \left\{ 2[U(x^*) - x^*] + \delta + \beta\alpha[u(q_d) - q_d] \right\}. \tag{65}$$

The first term in the braces is the sellers' and buyers' CM trade surplus. The second term is the dividends from the safe asset and the third term is the total trade surplus from DM trades. For $j \geq d+1$, Ω_j must take into account the changes in the asset prices and the risky asset's dividends, namely

$$\Omega_j = 2[U(x^*)-x^*]+\gamma_j+\delta+\beta\{\alpha[u(q_j)-q_j]+\eta s_j\Omega_{j+1}+\eta(1-s_j)\Omega_{j-1}+(1-\eta)\Omega_j\}. \tag{66}$$

The next proposition shows that the presence of the risky asset can reduce welfare:

Proposition 15 (Non-monotone Welfare) *In a dual-asset economy, there exists a cutoff $\pi^* \in [\pi_d, 1)$ such that $\Omega_j > \Omega_d$ if and only if $\pi_j > \pi^*$.*

(i) If τ_j rises in j for all $j \geq d$, then $\pi^* = \pi_d$ and Ω_j rises in j.

(ii) There exists $\tilde{\gamma}$ such that if $\gamma^H \geq \tilde{\gamma}$ then Ω_j rises in j for all $j \geq d$ and $\pi^* = \pi_d$. If $\gamma^H < \tilde{\gamma}$ then Ω_j is non-monotone in j and $\pi^* > \pi_d$. Moreover $\tilde{\gamma} > -(1-s)\gamma^L/s > 0$.

Proposition 15 states that the introduction of a new asset is welfare improving if and only if the prior belief is sufficiently high, namely $\pi_0 > \pi^*$. By Part (ii) of the proposition the planner should ban the use of the new asset as a medium of exchange when the benefit of using the asset is small (i.e. $\gamma^H < \tilde{\gamma}$) and the prior belief is low (i.e. $\pi_0 \in (\pi_d, \pi^*)$). Suppose only the safe asset circulates in periods $t<0$ and the risky asset is introduced at $t=0$. If the initial belief $\pi_0 \in (\pi_d, \pi^*)$, then the risky asset circulates

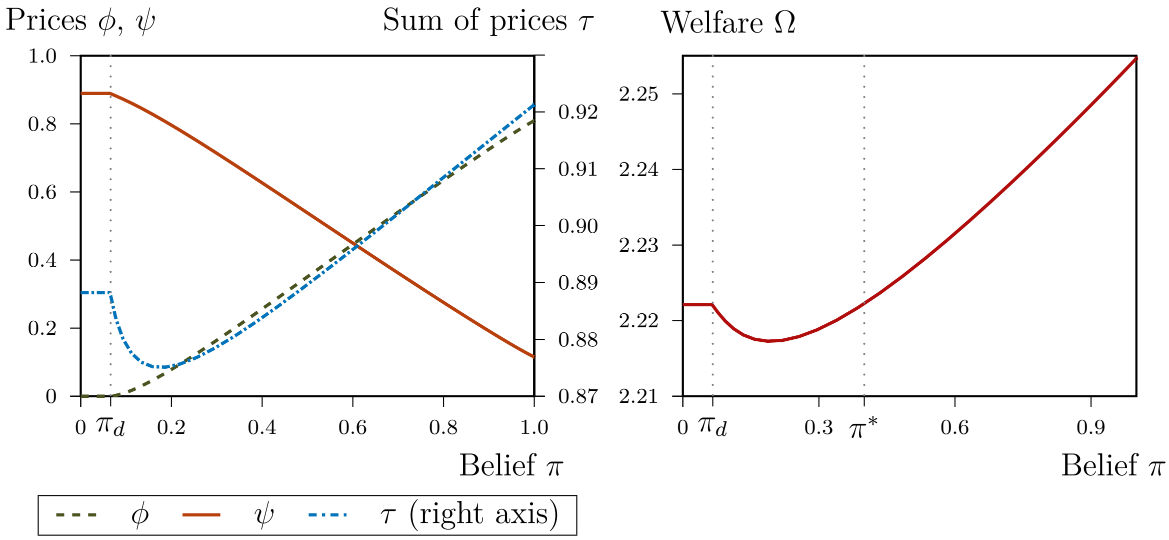

Figure 13: Non-monotone τ_j (Left) and welfare Ω_j (Right).

at $t = 0$ but the total surplus drops from Ω_d to Ω_0 by Proposition 15. This result is illustrated by a numerical example in the right panel of Figure 13 where $\gamma_\infty \approx 0$. In the example a social planner would introduce the risky asset if and only if $\pi_0 \geq \pi^*$. Moreover, since Ω_j is non-monotone in j, the arrival of a good news can reduce welfare. Hence a good news about the quality of a currency can be a bad news for the aggregate economy.

By Proposition 15 the equilibrium is inefficient when $\pi_0 \in (\pi_d, \pi^*)$. This inefficiency occurs because agents fail to abandon the risky asset even when the expected cost of asset holding is too high. When the risky asset circulates, its price ϕ_j includes an endogenous liquidity premium. This liquidity premium makes the asset a useful payment device and hence agents hold it even when the holding cost is high. If the risky asset is eliminated from the economy, then ϕ_j vanishes but ψ_j would rise, so τ_j might not change much. But since agents no longer pay the cost of holding the risky asset, the total welfare Ω_j rises. Naturally, this inefficiency occurs when the benefit γ^H of using the asset is small. By Proposition 15, the planner can raise welfare by banning a new private money when the potential benefit of using it is small and agents are pessimistic about its potential.

In principle a policy maker can achieve the first-best outcome by subsidizing the holding of the safe asset. Suppose the policy maker uses a lump sum tax to subsidize δ such that the safe asset alone can support the production level of q^* in the DM (i.e. implementing the Friedman rule, see Geromichalos et al. (2007)). In this case all agents would use the safe asset as a means of payment and only treat the risky asset as an investment. Since agents would invest optimally and efficiently, the planner does not need to intervene the agents' asset holding decision. Since the investment problem is isomorphic to a single-agent real-option problem, the more information disclosure, the more surplus the agents can extract from holding the risky asset. Therefore the planner would reveal maximal amount of information.

G Agents with Heterogenous Beliefs

Some people claim that crypto-currencies are not money because they are held by a small number of speculators and provide no transactional service for the general public.[15] To explain this phenomenon we introduce some optimistic agents, call hodlers, into the economy. Hodlers have a high initial prior about the quality of the asset but do not engage in DM trades. Among the buyers a fraction $\mu \in (0, 1)$ of them are hodlers and the rest are regular buyers. There is no hodler among sellers.

We denote π_0^h and π_0^r as the initial prior of hodlers and regular buyers, respectively, and assume $\pi_0^h > \pi_0^r$. Both types of buyers are rational and they agree to disagree on their beliefs. Hodlers and regular buyers receive the same information over time and both update beliefs according to Bayes rule. We denote regular buyers' belief π_j^r as the state of the economy and the asset is abandoned when $\pi_j^r \leq \pi_d^r$. At state π_j^r by Bayes rule a hodler's belief is given by

$$\pi_j^h = \Psi(\pi_j^r) \equiv \frac{\pi_j^r}{\pi_j^r + (1 - \pi_j^r)\frac{\pi_0^r(1-\pi_0^h)}{\pi_0^h(1-\pi_0^r)}}. \tag{67}$$

The difference $\pi_j^h - \pi_j^r$ is hump-shaped in j and it vanishes as $j \to +\infty$ or $-\infty$. We present a numerical example of $\Lambda(\cdot)$ in Figure 14.

By the logic leading to (50), a buyer (either a hodler or a regular buyer) with belief π thinks the expected value of one unit of asset in the next CM is

$$\bar{\ell}(\pi) \equiv \bar{\gamma}(\pi) + \eta\bar{s}(\pi)\phi_{j+1} + \eta[1 - \bar{s}(\pi)]\phi_{j-1} + (1 - \eta)\phi_j$$

where $\bar{\gamma}(\pi) = \chi[\bar{s}(\pi)\gamma^H + (1 - \bar{s}(\pi))\gamma^L]$. A regular buyer with a_j^r units of asset has liquidity $\bar{\ell}(\pi_j^r)a_j^r$ in a trade meeting. The value functions are the same as that in the

[15] The German federal government stated that "Crypto tokens are not real money" as "the volume of payments carried out using crypto is limited when compared to fiat currencies". According to Forbes "To be a true alternative [of money], a cryptocurrency must also be easy to use for day-to-day transactions".

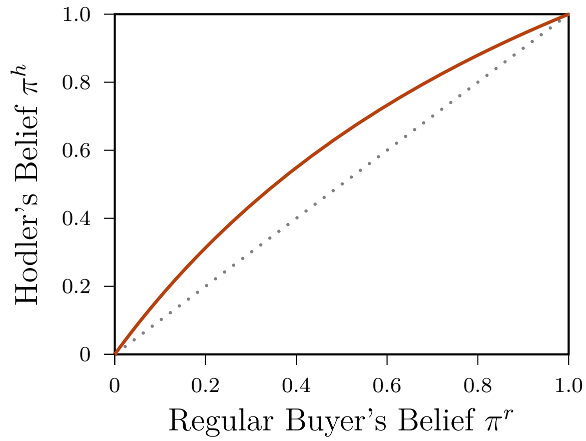

Figure 14: Beliefs of regular buyers and hodlers.

baseline model. Their asset holding is given by the first-order condition

$$\phi_j \leq \beta\bar{\ell}(\pi_j^r)\{1 + \alpha S'[\bar{\ell}(\pi_j^r)a_j^r]\}. \tag{68}$$

Since hodlers do not engage in DM trades, a_j^h is given by the first-order condition

$$\phi_j \leq \beta\bar{\ell}(\pi_j^h). \tag{69}$$

When compared to (68), the right side of (69) only includes the return of asset holding but not the benefit of using the asset as a means of payment. For both types of buyers, the first-order condition binds if the asset holding is strictly positive. Since hodlers are more optimistic than regular buyers, the asset is abandoned only when the hodlers are willing to do so. Therefore the IC constraint (14) becomes

$$0 \geq \bar{\gamma}(\pi_d^h) + \eta\bar{s}(\pi_d^h)\phi_{d+1}, \tag{70}$$

where $\pi_d^h \equiv \Psi(\pi_d^r)$ is the cutoff for a hodler to abandon the asset. An equilibrium is a list $\langle\{q_j\}_{j=d}^{\infty}, \{\phi_j\}_{j=d}^{\infty}, \{a_j^r\}_{j=d}^{\infty}, \{a_j^h\}_{j=d}^{\infty}, \pi_d^r\rangle$ that satisfies DM market equilibrium, (68), (69), the incentive constraint (70) and asset market clearing $1 = \mu a_j^h + (1-\mu)a_j^r$ for $j > d$.

For tractability we focus on the limit when the measure of hodlers vanishes. This limit is non-trivial because even measure zero of hodlers can potentially hold a positive fraction of the total asset supply, namely it is possible that $\lim_{\mu\to 0}\mu a_j^h > 0$ for some j. If $f[\bar{\ell}_j(\pi_j^r)] > \beta\bar{\ell}_j(\pi_j^h)$ at state π_j^r, then the marginal benefit for a regular buyer to hold the asset strictly exceeds that of the hodlers, thus $a_j^r = 1$, $a_j^h = 0$ and $\phi_j = f[\bar{\ell}_j(\pi_j^r)]$.

Alternatively, if $f[\bar{\ell}_j(\pi_j^r)] \leq \beta\bar{\ell}_j(\pi_j^h)$, then the hodlers are willing to hold the asset (i.e. $a_j^h > 0$ but it might or might not be finite) and hence $\phi_j = \beta\bar{\ell}_j(\pi_j^h)$. By (68) regular buyers' holding a_j^r solves $\bar{\ell}_j(\pi_j^r)\{1 + \alpha S'[\bar{\ell}_j(\pi_j^r)a_j^r]\} = \beta\bar{\ell}_j(\pi_j^h)$ and $a_j^r = 0$ if no positive solution exists. Combining the two cases, for $j > d$, the price of asset is given by

$$\phi_j = \max\{f[\bar{\ell}_j(\pi_j^r)], \beta\bar{\ell}_j(\pi_j^h)\}.$$

This condition is similar to (52) but now the unit price of the asset is determined by the maximum of the willingness-to-pay for a unit of asset among the regular buyers and the hodlers. Since this asset pricing equation does not depend on a_j^r, the equilibrium has a recursive structure — we can first solve for π_d^r and $\{\phi_j\}_{j=d}^{\infty}$, then solve for $\{a_j^r\}_{j=d}^{\infty}$.

To solve for ϕ_j we modify the mapping F in (57) as

$$F_j(\phi) \equiv \max\{f[\bar{\ell}_j(\pi_j^r)], \beta\bar{\ell}_j(\pi_j^h), 0\}. \tag{71}$$

The second term in the max operator is new and takes into account the hodlers' marginal benefit of asset holding. By the logic leading to Proposition 10 the equilibrium price sequence is a fixed point $\phi = F(\phi)$. As in the baseline model we focus on the case $\phi = \lim_{n\to\infty} F^n(\phi^0)$ which corresponds to a fixed point ϕ with $\pi_d^r \in (0, 1)$. In equilibrium, using (71), one can show that ϕ_j rises in j as in the baseline model. Moreover, if the hodlers' prior belief π_0^h goes up, then ϕ_j weakly increases at all j.

Right after the introduction of the asset, it is possible that hodlers hold all the assets and do not use it as money. As more good news arrive, the regular buyers might start to hold the asset and adopt it as a means of payment. Proposition 16 provides sufficient conditions for this process to happen. Let $\tilde{\pi}_d$ be the cutoff of disposing the asset in an economy without hodler and $\alpha = 0$. In an economy with hodlers and $\alpha > 0$, if hodler's initial belief $\pi_0^h > \tilde{\pi}_d$, then hodlers are definitely willing to hold the asset at $t = 0$.

Proposition 16 (Money Adoption) *Assume $\pi_0^h > \tilde{\pi}_d$. If α is sufficiently small, then hodlers initially hold all the assets, namely $a_0^h > a_0^r = 0$ at $t = 0$. As more good news arrive, regular buyers will eventually hold $a_j^r > 0$ and the asset will be used as money.*

Proposition 16 suggests that speculations by hodlers and the transactional role of the asset are not mutually exclusive. If α is sufficiently small, then initially hodlers have more incentive to hold the asset because their perceived return of asset holding is higher. As more good signals arrive, the difference $\pi_j^h - \pi_j^r$ vanishes (see Figure 14). Hodlers

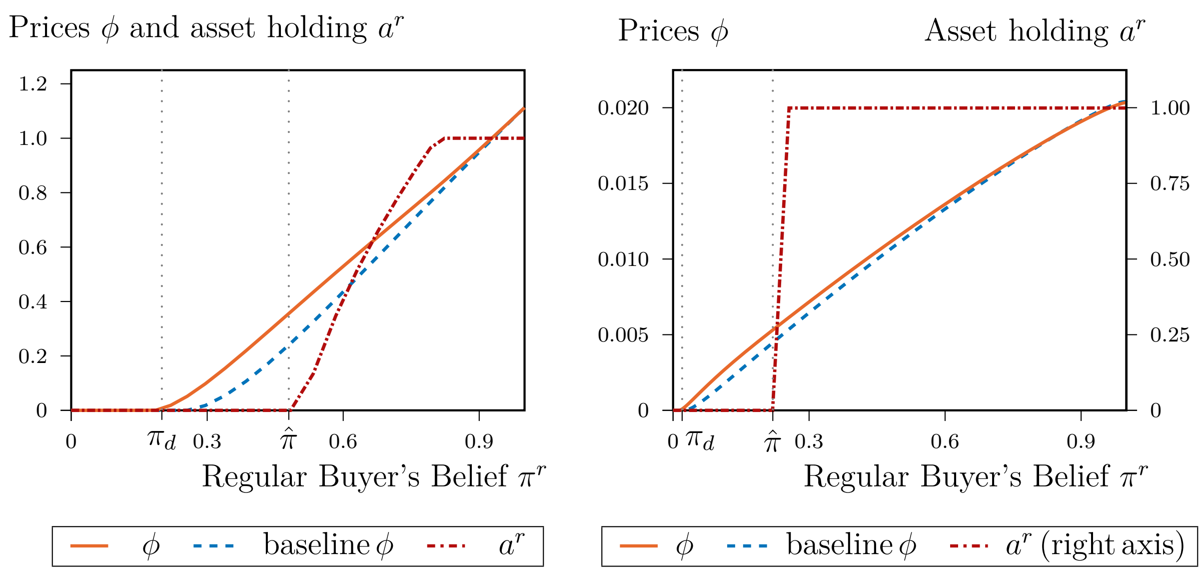

Figure 15: Asset prices and holding with hodlers. The parameters for both panels are the same except we take $\gamma_L, \gamma_H \to 0$ in the right panel (i.e. the asset becomes fiat).

and regular buyers almost have the same belief, but regular buyers have more incentive to hold the asset as they can use it as a means of payment. So as π_j^r rises, eventually hodlers will sell some assets to regular buyers and the economy will adopt the new asset as a means of payment. In Figure 15 we show the asset prices ϕ_j and asset holding a_j^r of regular buyers as functions of π_j^r. In order to highlight the impact of hodlers we also show the asset prices in an economy with no hodlers (i.e. our baseline model). In the left panel, the presence of hodlers reduces the cutoff π_d^r and raises ϕ_j. For $\pi_j^r \in (\pi_d^r, \hat{\pi})$, hodlers own all the assets. As π_j^r rises, regular buyers gradually start to hold the asset. In the right panel we show an example where γ_L and γ_H are close to 0 such that asset is close to a fiat money. The presence of hodlers has little impact on asset prices but has a large impact on asset holding behavior. When $\pi_j^r < \hat{\pi} \approx 0.2$ all assets are held by hodlers but regular buyers quickly purchase all asset when $\pi_j^r > 0.25$. A message of this example is that, even if the asset has no intrinsic value, which is the case for Bitcoin, some hodlers are still willing to invest in it. When the entire economy becomes sufficiently optimistic about the asset's quality, hodlers will sell the asset to other agents who will then use it as a medium of exchange.

Omitted Proof in Online Appendix

Proof of Lemma 6. Therefore the function f can be rewritten as $f(\ell) = \beta\ell + \alpha\ell[u'(\ell) - 1]$. It follows that the curvature of $f(\ell)$ has the properties as $\ell u'(\ell)$, or equivalently $f''(\ell) \propto 2u''(\ell) + \ell u'''(\ell)$.

Claim 1: When $u(q)$ has CRRA, then $u(\ell) = A\ell^{1-\eta}/(1-\eta)$. By direct differentiation one can check that if $\eta \in (0,1)$ then $\ell u'(\ell)$ is concave and if $\eta > 1$ then it is convex.

Claim 2: When $u(q)$ has CARA, then $u(\ell) = 1 - e^{-\eta\ell}$. Hence $f''(\ell) \propto \eta^2 e^{-\eta\ell}[-2+\ell\eta]$ is concave if and only if $\ell < 2/\eta$.

Claim 3: If $u(q)$ has DRRA, then

$$0 \leq \frac{d[\ell u''(\ell)/u'(\ell)]}{d\ell} = \frac{u''(\ell)}{u'(\ell)}\left[2 + \frac{\ell u'''(\ell)}{u''(\ell)} - 1 - \ell u''(\ell)/u'(\ell)\right].$$

If $-\ell u''(\ell)/u'(\ell) \leq 1$, then it must be the case that $2 + \ell u'''(\ell)/u''(\ell) > 0$ such that the inequality holds. Hence $f''(\ell) \propto 2u''(\ell) + \ell u'''(\ell) < 0$.

Claim 4: The proof of similar to the third claim and thus is omitted. ■

The existence and uniqueness proof is long and hence we provide an overview below. **Overview of the Existence and Uniqueness proof:** The proof has two steps. Given π_d and ϕ_{d+1}, one can derive $\{\phi_j\}_{j=d+1}^{\infty}$ by using (53) recursively. The first step is to show that given an arbitrary π_d, there is a unique choice of ϕ_{d+1} such that $\lim_{j\to\infty} \phi_j = \phi_H$. The second step is to show that generically (i.e. except for measure zero of parameter values) there is a unique π_d such that the incentive constraint (14) holds (see Lemma 9 and the proof of Proposition 9). Below we sketch the proof of step 1 and leave step 2 to the proof of Proposition 9.

Step 1: We first show a small increase in ϕ_{d+1} induces a strictly positive increase in all subsequent ϕ_j for $j > d+1$. Differentiate (53) with respect to ϕ_{d+1} and let $k_{j+1} \equiv d\phi_{j+1}/d\phi_j$ for $j \geq d+1$. Also let $\bar{f}' \equiv \max_\ell\{f'(\ell)\}$ be the upper bound of the slope of f. Then we have the inequality

$$k_{j+1} \geq \frac{1}{\eta s_j}[\frac{1}{\bar{f}'} - (1-\eta) - \eta(1-s_j)\frac{1}{k_j}]. \tag{72}$$

Define $\underline{k}_j \equiv (1-s_j)/\sqrt{s(1-s)} > 0$. By (72) and Assumption 1, one can check that if $k_j > \underline{k}_j$, then $k_{j+1} > \underline{k}_{j+1}$ $\forall j > d+1$. One can also check $k_{d+2} > \underline{k}_{d+2}$ (see Lemma 8). Altogether $k_j > \underline{k}_j > 0$ at all j which implies $d\phi_j/d\phi_{d+1} > 0$. Lemma 8 further shows that if the limit $\lim_{j\to\infty} \phi_j = \phi_H$, then the limit strictly increases in ϕ_{d+1}, and thus there can only be one choice of ϕ_{d+1} such that $\lim_{j\to\infty} \phi_j = \phi_H$. Hence given π_d, there is a unique sequence of ϕ_j converging to ϕ_H.

Lemma 8 *Given π_d and $\phi_d = 0$, one can pick a value of ϕ_{d+1} and then use (53) iteratively to derive a sequence of ϕ_j for $j \geq d+2$. (i) For each $j \geq d+2$, ϕ_j strictly increases in the initial guess ϕ_{d+1}. (ii) There is at most one ϕ_{d+1} such that $\lim_{j\to\infty} \phi_j = \phi_H$.*

Proof. *Part (i):* Consider an infinitesimal increase in ϕ_{d+1} and let $k_j = d\phi_j/d\phi_{j-1}$ for $j \geq d+2$. Therefore the change in any ϕ_j is $d\phi_j/d\phi_{d+1} = \Pi_{i=d+2}^{j} k_i$. Define $\underline{k}_j \equiv (1-s_j)/\sqrt{s(1-s)} > 0$, $\underline{k}_j$ falls in j by the definition of s_j in (49). We will show $k_{d+j} > \underline{k}_{d+j}$ for all $j \geq 1$. Let $B \equiv [1/f'(0) - (1-\eta)]/\eta$. By (53) and the concavity of f,

$$k_{j+1} \geq \frac{1}{s_j}[B - \frac{(1-s_j)}{k_j}].$$

Since the right side rises in k_j, if $k_j > \underline{k}_j$, then $k_{j+1} > \underline{k}_{j+1}$ provided that

$$\frac{1}{s_j}[B - \frac{(1-s_j)}{\underline{k}_j}] > \underline{k}_{j+1} \iff B > \sqrt{s(1-s)} + s_j \frac{(1-s_{j+1})}{\sqrt{s(1-s)}} = 2\sqrt{s(1-s)}.$$

The if and only if relationship uses the definition of $\underline{k}_j$ and $\underline{k}_{j+1}$. The last equation holds because by (1) and (49) $s_{j+1} = 1 - s(1-s)/s_j$. The right side is smaller than B if and only if Assumption 1 holds. Therefore if $k_j > \underline{k}_j$, then $k_{j+1} > \underline{k}_{j+1}$ under Assumption 1.

Finally we argue $k_{d+2} > \underline{k}_{d+2}$. By (53) and the definition of $\bar{f}'$

$$k_{d+2} \geq \frac{1}{\eta s_{d+1}} \left[\frac{1}{\bar{f}'} - (1-\eta)\right] = \frac{B}{s_{d+1}} \geq \frac{\sqrt{s(1-s)}}{s_{d+1}} + \frac{(1-s_{d+2})}{\sqrt{s(1-s)}} > \underline{k}_{d+2}.$$

The first equation uses the definition of B. The second inequality uses Assumption 1 and the last inequality uses the definition of $\underline{k}_{d+2}$.

Part (ii): Since $k_{d+j} > 0$ for all $j > 1$ by Part (i), the limit $\lim_{j\to\infty} \phi_{d+j}$ weakly rises in ϕ_{d+1}. By (53), if $\lim_{j\to\infty} \phi_j = \phi_H$, then for large j the difference equation for k_j becomes

$$k_{j+1} = \frac{1}{\eta s}[\frac{1}{f'(\phi_H)} - (1-\eta) - \frac{\eta(1-s)}{k_j}]. \tag{73}$$

In Figure 16 we show the 45 degree line as a blue line. The right side of (73) is increasing and concave in k_j and is shown as a red line in the figure. It is easy to check that the right side is smaller than k_j when $k_j \approx 0$ or when k_j is sufficiently large. When $k_j = 1$, the right-hand side exceeds 1 as $f'(\phi_H) < 1$. Therefore the red and blue line intersect twice and the larger intersection point in Figure 16, call it $\bar{k}$, strictly exceeds 1.

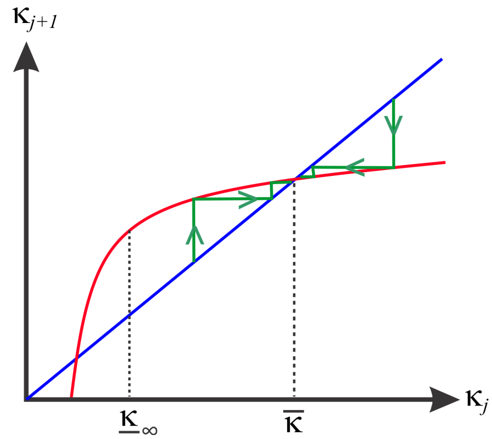

Figure 16: Convergence of k_j to $\bar{k}$.

Finally we argue $\lim_{j\to\infty} k_j = \bar{k} > 1$. The slope of the right-side of (73) is 1 at $k_j = \underline{k}_\infty \equiv \sqrt{(1-s)/s}$. Since the red line is concave and crosses the blue diagonal twice, it must lie above the blue at $k_j = \underline{k}_\infty$. Since $k_j > \underline{k}_\infty$ at all $j > d+1$ by Part (i), k_j must converge to $\bar{k}$ as $j \to \infty$, see the green lines in Figure 16. Therefore when $\lim_{j\to\infty} \phi_j = \phi_H$, we have $\lim_{j\to\infty} k_j > 1$, and hence $d[\lim_{j\to\infty} \phi_j]/d\phi_{d+1} = \Pi_{i=d+2}^{\infty} k_i > 0$. Since $\lim_{j\to\infty} \phi_j$ weakly increases in ϕ_{d+1} by Part (i) and strictly increases in ϕ_{d+1} when $\lim_{j\to\infty} \phi_j = \phi_H$, there is a unique ϕ_{d+1} such that $\lim_{j\to\infty} \phi_j = \phi_H$. ∎

Lemma 9 *Fixing an integer $z \in \mathbb{Z}$. Consider the set L_z of price sequence $\{\phi_j\}_{j=z}^{\infty}$ such that ϕ_j weakly rises in j and is bounded between $[0, \phi_H]$. Let T be a function that maps $\phi \in L_z$ into another sequence $\{\phi'_j\}_{j=z}^{\infty}$. For $j > z$*

$$\phi'_j = T_j(\phi) \equiv f[\gamma_j + \eta s_j \phi_{j+1} + \eta(1-s_j)\phi_{j-1} + (1-\eta)\phi_j] \tag{74}$$

and $\phi'_z = T_z(\phi) \equiv 0$. Let the seed sequence $\phi_j^0 = \phi_H$ for all $j > z$ and $\phi_z^0 = 0$. Suppose $\{\phi_j^*\}_{j=z}^{\infty}$ is a sequence that satisfies (53) with $\phi_z^* = 0$ and $\lim_{j\to\infty} \phi_j^* = \phi_H$. Then ϕ_j^* is given by $\phi^* = \lim_{n\to\infty} T^n(\phi^0)$ and increases in j.

Proof of Lemma 9. It is easy to check that T is order preserving, namely if $\phi'_j \geq \phi''_j$ for all $j \geq z$, then $T_j(\phi') \geq T_j(\phi'')$ for all $j \geq z$. It is also easy to verify that T maps bounded increasing sequences in $[0, \phi_H]$ back into itself, hence $T : L_z \to L_z$.

Since $\phi_H = f(\bar{\gamma}_\infty + \phi_H)$ by (54) and f is an increasing function, $\phi^0 \geq T(\phi^0)$. Since $\phi^0 \geq T(\phi^0)$ and T is monotone, $T^2(\phi^0) \equiv T[T(\phi^0)] \leq T(\phi^0) \leq \phi^0$ and by induction $T^n(\phi^0)$ falls in n. Let $\{\phi_j^*\}_{j=z}^{\infty}$ be a sequence that satisfies (53) with $\phi_z^* = 0$ and

$\lim_{j\to\infty}\phi^*_j = \phi_H$. By definition it is a fixed point of T, namely $T(\phi^*) = \phi^*$. Since T is monotone and the seed sequence $\phi^0_j \geq \phi^*_j$ at all $j \geq z$, $T(\phi^0) \geq \phi^*$. By the monotonicity of T, $T^n(\phi^0) \geq \phi^*$ for all $n > 0$. Since the sequence $T^n(\phi^0)$ weakly falls in n and is bounded below by ϕ^*, the limit $\phi^\infty \equiv \lim_{n\to\infty} T^n(\phi^0)$ exists, is weakly larger than ϕ^*, and ϕ^∞_j rises in j.

Since ϕ^∞_j is bounded above by ϕ_H and bounded below by ϕ^*_j, it must converge to ϕ_H as $j \to \infty$. Finally, by Lemma 8, for a given z, the sequence ϕ^* that converges to ϕ_H is unique. Therefore $\phi^\infty = \phi^*$, and thus ϕ^*_j increases in j. ■

Proof of Proposition 9. Let $\pi'_0 \in [0,1]$ be an arbitrarily chosen prior belief and let π'_j be the posterior belief after seeing $j \geq 0$ good signals. Rewrite (74) by (49) as

$$\hat{T}_j(\phi|\pi'_0) \equiv f[\bar{\gamma}(\pi'_j) + \eta\bar{s}(\pi'_j)\phi_{j+1} + \eta(1-\bar{s}(\pi'_j))\phi_{j-1} + (1-\eta)\phi_j] \tag{75}$$

for $j \geq 1$ and $\hat{T}_0 \equiv 0$. Let $\phi^{\pi'_0}$ be the fixed point $\phi^{\pi'_0} = \hat{T}(\phi^{\pi'_0}|\pi'_0)$, $\phi^{\pi'_0}$ is unique by Lemma 8 and 9. By Lemma 9 and the definition of an equilibrium, a belief π_d is an equilibrium cutoff if and only if the incentive constraint (13) holds at π_d, namely $\bar{\gamma}(\pi_d) + \eta\bar{s}(\pi_d)\phi^{\pi_d}_{d+1} \leq 0$. In STEP 1 below we argue that $\phi^{\pi'_0}_{d+j}$ strictly increases in π'_0 at all $j \geq 1$, hence the set of prior belief π'_0 that satisfies the IC constraint is a convex set. Then in STEP 2 we show that the set of π_d such that the incentive constraint holds is generically a singleton.

STEP 1: Suppose $\pi''_0 > \pi'_0$. By the definition of $\phi^{\pi'_0}$, $\phi^{\pi'_0} = \hat{T}(\phi^{\pi'_0}|\pi'_0)$. As π'_0 rises, π'_j strictly rises at all $j \geq 0$, and thus so do $\bar{s}(\pi'_j)$ and $\bar{\gamma}(\pi'_j)$. As $\bar{\gamma}(\pi'_j)$ rises, $\hat{T}_j(\phi|\pi'_0)$ rises strictly by (75). As $\bar{s}(\pi'_j)$ rises, $\hat{T}_j(\phi|\pi'_0)$ rises provided that ϕ_j is an increasing sequence. But as shown by Lemma 9, if $\phi^{\pi'_0}$ exists then $\phi^{\pi'_0}_j$ rises in j. Therefore $\hat{T}_j(\phi^{\pi'_0}|\pi''_0) > \hat{T}_j(\phi^{\pi'_0}|\pi'_0) = \phi^{\pi'_0}_j$ at all $j \geq 1$. Moreover, as discussed after (74), $\hat{T}_j(\phi|\pi''_0)$ is increasing in ϕ and maps increasing sequence of ϕ_j back into an increasing sequence. Hence $\hat{T}^n_j(\phi^{\pi'_0}|\pi''_0)$ rises in n at all $j \geq 1$. Since $\hat{T}^n_j(\phi^{\pi'_0}|\pi''_0)$ is bounded above by ϕ_H, the limit $\lim_{n\to\infty}\hat{T}^n_j(\phi^{\pi'_0}|\pi''_0)$ exists and strictly exceeds $\phi^{\pi'_0}_j$ at all $j \geq 1$. But $\lim_{n\to\infty}\hat{T}^n(\phi^{\pi'_0}|\pi''_0)$ is a fixed point of $\phi = \hat{T}(\phi|\pi''_0)$ and hence is equivalent to $\phi^{\pi''_0}$, so we have $\phi^{\pi''_0}_j > \phi^{\pi'_0}_j$ at all $j \geq 1$.

STEP 2: Consider the set of prior $\Pi \subseteq [0,1]$ such that if $\pi' \in \Pi$ then $\phi^{\pi'}_j$ rises in j and the incentive constraint (13) holds at π'. If π_d is an equilibrium cutoff, then $\pi_d \in \Pi$. The smallest element of Π, call it π^ℓ, is such that $\phi^{\pi^\ell}_0 = \phi^{\pi^\ell}_1 = 0$. It is unique because $\phi^{\pi'}_1$ strictly rises in π' by STEP 1. It is the smallest element of Π because any

$\pi' < \pi^\ell$ will result in $\phi_1^{\pi'} < 0 = \phi_0^{\pi'}$ by STEP 1, and hence $\phi_j^{\pi'}$ is not increasing. The largest element of Π, π^h, is such that the first-order condition (52) holds at $\phi_0^{\pi^h}$, namely $0 = \phi_0^{\pi^h} = f[\bar{\gamma}(\pi^h) + \eta\bar{s}(\pi^h)\phi_1^{\pi^h}]$, or equivalently $\bar{\gamma}(\pi^h) + \eta\bar{s}(\pi^h)\phi_1^{\pi^h} = 0$. Any cutoff $\pi' > \pi^h$ will lead to $\bar{\gamma}(\pi') > \bar{\gamma}(\pi^h)$, $\bar{s}(\pi') > \bar{s}(\pi^h)$ and $\phi_1^{\pi'} > \phi_1^{\pi^h}$ (by STEP 1), hence $\bar{\gamma}(\pi') + \eta\bar{s}(\pi')\phi_1^{\pi'} > 0$. This inequality violates the incentive condition (13) and hence $\pi' \notin \Pi$. Since $\bar{\gamma}(\pi') + \eta\bar{s}(\pi')\phi_1^{\pi'}$ rises strictly in π', the value of π^h is unique.

Given an initial prior π_0, any equilibrium cutoff π_d must satisfy $\pi_d \in [\pi^\ell, \pi^h]$. But π^ℓ and π^h are exactly one signal away from each other, namely that if agents' belief is π^ℓ and a good signal arrives, then their posterior belief is π^h. It is because $\phi_1^{\pi^\ell} = 0$ by the definition of π^ℓ and the first-order condition binds at $\phi_1^{\pi^\ell}$, and that is exactly the condition the defines π^h. Hence given π_0, there is at most one j such that π_j lies in the interval $[\pi^\ell, \pi^h]$. Therefore there is only one possible π_d. If π_0 is such that $\pi_j = \pi^\ell$ and $\pi^h = \pi_{j+1}$ at some j, then π_j and π_{j+1} are both equilibrium cutoff. But this case is non-generic and only holds for measure zero of parameter values. ■

Lemma 10 *The mapping F in (57) maps $L \to L$.*

Proof. First F maps weakly increasing sequences into weakly increasing sequences because γ_j and s_j increase in j and f is an increasing function. Next, F is order-preserving, namely if $\phi' \geq \phi''$ then $F(\phi') \geq F(\phi'')$, because f is an increasing function.

Suppose $\phi'_j = \phi_H$ for all j. Then $F_j(\phi') \leq \phi_H$ for all j by the monotonicity of f and (54). Since F is order-preserving, for any $\phi \in L$, $0 \leq F_j(\phi) \leq F_j(\phi') \leq \phi_H$ for all j. Hence F maps L back into L. ■

Proof of Proposition 10. By the definition of an equilibrium, the price sequence ϕ^* is clearly a fixed point of F. Now we argue that it can be derive by iterating over F. Let $\phi_j^0 = 0$ for all $j \in \mathbb{Z}$, it is clearly the smallest element in L. Since $F : L \to L$ is order-preserving and L is a complete lattice, the limit $\lim_{n\to\infty} F^n(\phi^0)$ converges to the lowest fixed point of F by Tarski's fixed-point theorem and $F^n(\phi^0)$ rises in n by Theorem 3.2 in Cousot and Cousot (1979). Finally we argue that this fixed point must have a finite d such that $\phi_j = 0$ for all $j \leq d$. Since $\gamma_{-\infty} < 0$, there exists j' such that $\gamma_j + \eta s_j \phi_H \leq 0 \; \forall j < j'$. Let $\hat{L}$ be a subset of L such that if $\phi \in \hat{L}$ then $\phi_j = 0 \; \forall j \leq j'$. We argue that F (57) maps $\hat{L} \to \hat{L}$ — if $\phi_j = 0$ for all $j \leq j'$, then $F_j(\phi) = 0$ for all $j \leq j'$ by (57). Since our seed sequence $\phi^0 \in \hat{L}$, the fixed point must have a $\pi_d \geq \pi_{j'} > 0$.
■

Proof of Lemma 7. Part (i): When $\bar{\phi}_j$ exists, by (59) its derivative with respect to γ_j is

$$\frac{d\bar{\phi}_j}{d\gamma_j} = \frac{f'(\gamma_j + \bar{\phi}_j)}{1 - f'(\gamma_j + \bar{\phi}_j)}.$$

Since $f'(\ell) < 1$ whenever f is convex and $\bar{\phi}_j$ is defined as the largest solution of (59), we have $1 - f'(\gamma_j + \bar{\phi}_j) > 0$ and hence $d\bar{\phi}_j/d\gamma_j > 0$. Since γ_j is linear in π_j by (49), $\bar{\phi}_j$ is increasing in π_j. When f is concave, the right side of the displayed equation falls in γ_j by the concavity of f and thus $\bar{\phi}_j$ is concave in π_j. Similarly, when f is convex, so is $\bar{\phi}_j$. Since f is concave-convex, so is $\bar{\phi}_j$

Since $f'(\ell) < 1$ whenever f is convex, if (59) has two solutions, then the smaller solution $\underline{\phi}_j$ must lie on the concave part of f. Using this observation, we can prove the desired claim on $\underline{\phi}_j$ by using a similar proof logic.

Part (ii): Easily ϕ_j is weakly convex at π_d because in equilibrium $\phi_{d+1} \geq \phi_d = \phi_{d-1} = 0$, and hence $\lambda_d \geq 0$. For the second claim, suppose $\phi_j \in (\underline{\phi}_j, \bar{\phi}_j)$. Then $f^{-1}(\phi_j) - \phi_j - \gamma_j < 0$ by the definition of $\bar{\phi}_j$ and $\underline{\phi}_j$. By (53)

$$s_j\phi_{j+1} + (1 - s_j)\phi_{j-1} - \phi_j = \frac{f^{-1}(\phi_j) - \phi_j - \gamma_j}{\eta} \tag{76}$$

and hence ϕ_j is convex at π_j. The last two claims are true by a similar proof logic. ■

Lemma 11 *If $\alpha = 0$, then the equilibrium price sequence ϕ_j is convex at all j.*

Proof. If $\alpha = 0$ then $f(\ell) = \beta\ell$. By (59) $\underline{\phi}_j = 0$ and $\bar{\phi}_j$ is an upward sloping straight connecting $(\pi^0, 0)$ and $(1, \phi_H)$ in the (π, ϕ) space where π^0 is defined as the belief where the expected dividend is zero, i.e. $\bar{\gamma}(\pi^0) = 0$. Since $\bar{\gamma}(\pi) > 0$ for all $\pi > \pi^0$, agents will not abandon the asset at any $\pi_j \geq \pi^0$. Hence $\pi_d < \pi^0$ and $\phi_j > \bar{\phi}_j$ for j close to d. The equilibrium price sequence ϕ_j cannot cut $\bar{\phi}_j$, for if ϕ_j cuts $\bar{\phi}_j$ from above then it must be concave afterwards by Lemma 7(b) and $\phi_j < \bar{\phi}_j$. Since ϕ_j becomes concave after the intersection, it cannot intersect with $\bar{\phi}_j$ again as $\bar{\phi}_j$ is a straight line. Hence ϕ_j cannot converge to ϕ_H as $\pi_j \to 1$, which leads to a contradiction. Therefore the price sequence must stay above $\bar{\phi}_j$ at all j and hence it is convex by Lemma 7(b). ■

Proof of Proposition 11.

Part (1): It suffices to prove that ϕ_j, as a function of π_j, can change from concave to convex at most once. Recall that $\bar{\phi}_j$ is increasing in π_j and is concave-convex. We first argue that ϕ_j cannot cut $\bar{\phi}_j$ from above when $\bar{\phi}_j$ is convex. Suppose it happens at some $\pi_{j'}$. After the intersection $\phi_{j'} < \bar{\phi}_{j'}$ and hence ϕ_j becomes concave by Lemma 7. Since $\bar{\phi}_j$ is convex and ϕ_j is concave, $\bar{\phi}_j > \phi_j$ for all $j \geq j'$ and the gap between them rises in j. Therefore $\bar{\phi}_j$ and ϕ_j cannot converge to the same limit as $\pi_j \to 1$.

Now suppose ϕ_j changes from concave to convex. By Lemma 7 it must be the case that ϕ_j cuts $\bar{\phi}_j$ from below. After the intersection, ϕ_j becomes convex. So if ϕ_j were to intersect with $\bar{\phi}_j$ again, then it must cut $\bar{\phi}_j$ at a point where $\bar{\phi}_j$ is convex. But this is impossible as argue above.

Part (2): Since ϕ_j rises and $\underline{\phi}_j$ falls in j, they can only cross once and if they cross then ϕ_j must cut $\underline{\phi}_j$ from below at some j'. For $j > j'$, we argue that $\bar{\phi}_j \geq \phi_j \geq \underline{\phi}_j$. Suppose not and $\phi_j > \bar{\phi}_j$ at some $j = j''$. Then the price sequence is convex at $\pi_{j''}$ by Lemma 7(b). By Lemma 7(a) $\bar{\phi}_j$ is concave in π when f is concave. Since ϕ_j is convex and $\bar{\phi}_j$ is concave, the gap $\phi_j - \bar{\phi}_j > 0$ is strictly increasing in j for $j \geq j''$ and hence ϕ_j and $\bar{\phi}_j$ cannot both converge to ϕ_H as $\pi \to 1$. This leads to a contradiction.

By a similar logic, if ϕ_j cuts $\bar{\phi}_j$ from above at some j' then $\bar{\phi}_j \geq \phi_j \geq \underline{\phi}_j$ $\forall j > j'$. Altogether ϕ_j can cross $\bar{\phi}_j$ or $\underline{\phi}_j$ at most once. By Lemma 7(b) ϕ_j is convex before the crossing and concave after the crossing, as shown in the left panel of Figure 9.

Part (3):Recall that we have assumed $f(0) = 0$ and $f'(\ell) \leq 1$ whenever f is convex. So if $f(\ell)$ is convex for $\ell \in [0, \ell_H]$, then $\bar{\phi}$ is increasing and convex and $\underline{\phi}$ does not exist. So the correspondence $\hat{\phi}$ is an increasing and convex curve. The price ϕ cannot cut $\bar{\phi}$ from above because after the intersection ϕ would be concave and cannot intersect with $\bar{\phi}$ again. This leads to a contradiction as $\bar{\phi}_j$ and ϕ_j cannot converge to the same limit as $\pi_j \to 1$. So ϕ_j is either entirely convex, or entirely concave or concave and then convex.

But ϕ_j is initially convex and hence it must be entirely convex. To see this note that $\bar{\phi}(\gamma)$ solves $f(\phi + \gamma) = \phi$ and hence if $\bar{\phi}(\gamma) = 0$, then $\gamma = 0$, as $f(0) = 0$. At $\pi = \pi_d$, the value of the dividend γ_d must be strictly negative because otherwise agents will not abandon the asset. So the curve ϕ_j cuts the x-axis at some γ_d that is strictly smaller than the point where $\bar{\phi}$ cuts the x-axis. Equivalently, ϕ_j is above $\bar{\phi}_j$ initially and hence it is initially convex. ■

The following definitions are useful for the proof of Proposition 12.

Definition 4 *(a) Let $\underline{F}_j(\phi'', \phi'_\ell, \eta)$ be a sequence defined over $j = \ell, \ell + 1, \ldots$. Assume $\underline{F}_\ell(\phi'', \phi'_\ell, \eta) = \phi'_\ell$,*

$$\underline{F}_{\ell+1}(\phi'', \phi'_\ell, \eta) = \max\{f[\bar{\gamma}_{\ell+1} + \eta s_{\ell+1}\phi''_{\ell+2} + \eta(1 - s_{\ell+1})\phi'_\ell + (1 - \eta)\phi''_{\ell+1}], 0\},$$

and $\underline{F}_j(\phi'', \phi'_\ell, \eta) = F_j(\phi'', \eta)$ *for* $j \geq \ell + 2$ *where* F *is defined in (57) and* η *is added as an argument.*

(b) Let $\bar{F}_j(\phi'', \phi'_\ell, \eta)$ *be a sequence defined over* $j = \ell, \ell - 1, \ldots$*. Assume* $\bar{F}_\ell(\phi'', \phi'_\ell, \eta) = \phi'_\ell$*,*

$$\bar{F}_{\ell-1}(\phi'', \phi'_\ell, \eta) = \max\{f[\bar{\gamma}_{\ell-1} + \eta s_{\ell-1}\phi'_\ell + \eta(1 - s_{\ell-1})\phi''_{\ell-2} + (1-\eta)\phi''_{\ell-1}], 0\}$$

and $\bar{F}_j(\phi'', \phi'_\ell, \eta) = F_j(\phi'', \eta)$ *for* $j \leq \ell - 2$*.*

We write $\underline{F}^2(\phi'', \phi'_\ell, \eta) = \underline{F}[\underline{F}(\phi'', \phi'_\ell, \eta), \phi'_\ell, \eta]$ and write $\underline{F}^n(\phi'', \phi'_\ell, \eta)$ if the mapping $\underline{F}$ is iterated for n times. We similarly define $\bar{F}^n(\phi'', \phi'_\ell, \eta)$.

Proof of Proposition 12.

Proof of Part (1): Since f is concave, by Proposition 11 the price sequence ϕ^*_j is convex-concave. Suppose ϕ^*_j changes from convex to concave at some $j = \rho$. First we claim ϕ'_j cannot cut ϕ^*_j from below at any $\nu \geq \rho$. Suppose not and assume $\phi'_{\nu-1} \leq \phi^*_{\nu-1}$ and $\phi'_\nu > \phi^*_\nu$. By definition $\underline{F}_j(\phi^*, \phi^*_{\nu-1}, \eta) = \phi^*_j$ for $j \geq \nu$. Since $\phi'_{\nu-1} \leq \phi^*_{\nu-1}$, $\underline{F}_j(\phi^*, \phi'_{\nu-1}, \eta) \leq \phi^*_j$ for $j \geq \nu$. As η rises $\underline{F}_j(\phi^*, \phi'_{\nu-1}, \eta)$ falls by (57) and because ϕ^*_j is concave for all $j \geq \nu - 1$ and $\phi'_{\nu-1} \leq \phi^*_{\nu-1}$. Therefore $\underline{F}_j(\phi^*, \phi'_{\nu-1}, \eta') \leq \phi^*_j$ for all $j \geq \nu$. Thus $\underline{F}^n_j(\phi^*, \phi'_{\nu-1}, \eta')$ falls in n as F is an order-preserving mapping. Since by definition $\underline{F}^n_j \geq 0$, the fixed point $\phi'' = \lim_{n\to\infty} \underline{F}^n(\phi^*, \phi'_{\nu-1}, \eta')$ exists and is lower than ϕ^*, namely $\phi'' \leq \phi^*$. But $\phi''_\nu \leq \phi^*_\nu < \phi'_\nu$ is impossible — this would imply ϕ''_j and ϕ'_j equals each other at $j = \nu - 1$, $\phi''_j < \phi'_j$ at $j = \nu$ and yet they both converge to ϕ_H as $j \to \infty$. By the proof logic of Lemma 8 if two distinct sequences intersect at some π_j, then they cannot both converge to ϕ_H. Hence ϕ'_j cannot cut ϕ^*_j from below at any belief larger than π_ρ.

Next ϕ' cannot cut ϕ^* from below at any $\nu < \rho$. Suppose not and assume $\phi'_{\nu-1} \leq \phi^*_{\nu-1}$ and $\phi'_\nu > \phi^*_\nu$. By a similar logic as the previous paragraph $\bar{F}_j(\phi^*, \phi'_\nu, \eta') \geq \phi^*_j$ for all $j < \nu$ and the inequality is strict at $j = \nu - 1$. Hence the fixed point $\phi'' = \lim_{n\to\infty} \bar{F}^n(\phi^*, \phi'_\nu, \eta') \geq \phi^*$ for all $j < \nu$ and $\phi''_{\nu-1} > \phi^*_{\nu-1}$. Therefore ϕ'' and ϕ' are two distinct sequences, i.e., $\phi''_{\nu-1} > \phi^*_{\nu-1} \geq \phi'_{\nu-1}$, but they intersect at $\phi'_\nu = \phi''_\nu$. This leads to a contradiction by Lemma 12 below. Altogether, since ϕ' cannot cut ϕ^* from below, it crosses ϕ^* at most once and must be from above.

Proof of Part (2): By Proposition 11, ϕ^*_j is convex-concave-convex. Suppose ϕ^*_j is concave for $j \in [j', j'']$ and otherwise convex. Using the proof logic of Part 1, one can prove the following claims.

1. ϕ'_j cannot cut ϕ^*_j from above at any $j > j''$.

2. ϕ'_j cannot cut ϕ^*_j from below at any $j < j'$.

3. If $\phi^*_i \geq \phi'_i$ and $\phi^*_k \geq \phi'_k$ where $j' \leq i < k \leq j''$, then $\phi^*_j \geq \phi'_j$ for all $j \in [i, k]$.

The third claim implies that, for $j \in [j', j'']$, the difference $\phi^*_j - \phi'_j$ has at most two sign changes with the sequence $-, +, -$ (including the special cases that the sign is always $-$, always $+$, $-, +$ and $+, -$). These three claims together prove the desired result. For example, if $\phi^*_j - \phi'_j \leq 0$ for all $j \in [j', j'']$, then ϕ' can only cut ϕ^*_j once and from above at some $j > j''$. So altogether $\phi^*_j - \phi'_j$ can change sign at most once from $+$ to $-$.

Proof of Part (3): The desired result follows immediately from part (3) of Proposition 11 and the first claim in Corollary 2. ∎

Proof of Corollary 2. *Proof of the first claim:* Suppose ϕ^*_j is convex at all $j > d$. Consider the mapping $F(\phi, \eta)$ in (57). By Proposition 10 $\phi^* = F(\phi^*, \eta)$. Since ϕ^*_j is convex and $\eta' > \eta$, $\phi'_j = F_j(\phi^*, \eta') \geq \phi^*_j$ at all $j \in \mathbb{Z}$ by (57). Since $F(\phi, \eta')$ increases in ϕ and $F(\phi^*, \eta') \geq \phi^*$, we have $F^n(\phi, \eta')$ rises in n and therefore the fixed point $\lim_{n\to\infty} F^n(\phi, \eta') \geq \phi^*$.
Proof of the second claim: If ϕ^*_j is concave at all $j > d$ and $\bar{\gamma}_d + \eta' s_d \phi^*_{d+1} \leq 0$, then $F_j(\phi^*, \eta') \leq \phi^*_j$ at all j. Since $F(\phi, \eta')$ increases in ϕ, $F^n(\phi^*, \eta')$ falls in n and therefore the fixed point $\lim_{n\to\infty} F^n(\phi, \eta') \leq \phi^*_j$. ∎

Lemma 12 *Suppose ϕ'_j is an equilibrium price sequence with cutoff π'_d. Let $\phi''_j \geq 0$ be another sequence that satisfies the incentive constraint at cutoff π''_d and satisfies (53) at all beliefs above π''_d, but not necessarily converges to ϕ_H as $j \to \infty$. Then it is impossible for ϕ'' to coincide with ϕ' from above, namely $\phi''_{\nu-1} > \phi'_{\nu-1}$ and $\phi''_\nu = \phi'_\nu$ at some ν.*

Proof. For any given cutoff π_d, one can choose a value for ϕ_{d+1} and then use (53) iteratively to derive a sequence ϕ_j for $j > d$. Let $\iota(\pi_d)$ be the value of ϕ_{d+1} such that $\lim_{j\to\infty} \phi_j \to \phi_H$. By Lemma 8, $\iota(\pi_d)$ is unique. By the last paragraph of the proof of Proposition 9, $\iota(\pi_d) \geq 0$ if and only if $\pi_d \geq \pi^\ell$.

Since $\phi''_{\nu-1} > \phi'_{\nu-1}$ and $\phi''_\nu = \phi'_\nu$, by (53) we have $\phi''_{\nu+1} < \phi'_{\nu+1}$. Since $\phi''_\nu = \phi'_\nu$, $\phi''_{\nu+1} < \phi'_{\nu+1}$ and $\lim_{j\to\infty} \phi'_j = \phi_H$, by the proof logic of Lemma 8, as $j \to \infty$ the sequence ϕ''_j is lower than ϕ_H. Therefore $\iota(\pi''_d) > \phi''_{d+1}$.

If $\pi''_d \in [\pi^\ell, \pi^h]$, then $\pi''_d = \pi'_d$. But this is impossible because if $\phi''_{d+1} = \phi'_{d+1}$, then ϕ'' and ϕ' are the same sequence which is a contradiction with $\phi''_{\nu-1} > \phi'_{\nu-1}$. If $\phi''_{d+1} \neq \phi'_{d+1}$, then by Lemma 8 it is impossible that $\phi''_\nu = \phi'_\nu$. If $\pi''_d < \pi^\ell$, then $0 > \iota(\pi''_d)$. This implies

$\iota(\pi''_d) < \phi''_{d+1}$ which is a contradiction. Finally if $\pi''_d > \pi^h$, then $\iota(\pi''_d) > 0$ and hence there is a sequence that starts from π''_d and converges to ϕ_H. This sequence does not intersect with ϕ' because if they cross each other then they cannot both converge to ϕ_H by the proof logic of Lemma 8. Since $\iota(\pi''_d) > \phi''_{d+1}$, by Lemma 8, ϕ'' is lower than the sequence that starts from π''_d and converges to ϕ_H. Hence $\phi''_j < \phi'_j$ at all j and it is impossible that $\phi'_\nu = \phi''_\nu$. ∎

Proof of Proposition 13. *Part (1)*: We first characterize the change in Ω_j induced by the change in ω_j. If $f(\ell)$ is concave in ℓ for $\ell \in [0, \ell_H]$ and $f(0) = 0$, then $f(\ell) > \ell$ in the relevant region. Therefore $f(\phi_j + \gamma_j) > \phi_j$ at each j. So the information premium must be negative to balance the first-order condition $f(\phi_j + \gamma_j + \eta\lambda_j) = \phi_j$. It follows that ϕ_j is concave. By the proof logic of Proposition 12 the price ϕ_j falls in η and hence ω_j falls in η. This effect reduces Ω_j as Ω_j is a weighted sum of ω_j by Lemma 4.

Next, we characterize the change in Ω_j induced by more rapid transitions across states. Note that (60) defines Ω_j in a way that is similar to how ϕ_j is defined in (52). Indeed we can rewrite (60) as

$$\Omega_j = \hat{f}\{(\omega_j)/\beta + \eta s_j \Omega_{j+1} + \eta(1 - s_j)\Omega_{j-1} + (1 - \eta)\Omega_j\}$$

where $\hat{f}(x) \equiv \beta x$. Therefore we can use our results in Proposition 11 and 12 to characterize Ω_j. Let $\bar{\Omega}_j$ be the counterpart of $\bar{\phi}_j$ such that it solves

$$\bar{\Omega}_j = \hat{f}\{\omega_j/\beta + \bar{\Omega}_j\} \iff \bar{\Omega}_j = \frac{\omega_j}{1 - \beta}.$$

Therefore the curvature of $\bar{\Omega}_j$ is determined by that of ω_j. Since ϕ_j is concave, one can show that ℓ_j is also concave in π_j. Therefore $u(q_j) - q_j$ is concave in ℓ_j. Since $\omega_j \equiv \gamma_j + \beta\alpha[u(q_j) - q_j]$ is a concave transform of ℓ_j, we know ω_j is concave in π_j, and thus so is $\bar{\Omega}_j$. Since $\bar{\Omega}_j$ is concave, by the proof logic of Proposition 11 Ω_j is also concave in π_j. Since Ω_j is concave in π_j, fixing ω_j, an increase in η reduces Ω_j by the proof logic of Proposition 12.

Part (2): Since $u(q_j) - q_j$ is an increasing and concave transform of ϕ_j, if $u(q_j) - q_j$ is convex in π_j, then ϕ_j must be convex in π_j. Therefore an increase in η raises ϕ_j by Corollary 2. Therefore ω_j increases in η and Ω_j increases.

By the same proof logic as part 1, if $u(q_j) - q_j$ is convex in π_j then ω_j and $\bar{\Omega}_j$ are convex in π_j. Therefore Ω_j is convex in π_j by the proof logic of Proposition 11. Hence, fixing ω_j, an increase in η raises Ω_j by the proof logic of Proposition 12. ■

Proof of Corollary 3. Since sellers make no surplus in trade, $S(\ell_j) = u(q_j) - q_j$. By part 2 of Proposition 13, it suffices to show that $S(\ell_j)$ is convex in π_j. Observe that when $\eta \approx 0$, $\ell_j \approx \gamma_j + \phi_j \approx \gamma_j + \bar{\phi}(\gamma_j)$. Hence if $S[\gamma_j + \bar{\phi}(\gamma_j)]$ is convex in π_j, then $S(\ell_j)$ is convex in π_j when η is small. Since γ_j is linear in π_j, it suffices to show $S[\gamma + \bar{\phi}(\gamma)]$ is convex in γ whenever $\gamma + \bar{\phi}(\gamma)$ is between ℓ_L and ℓ_H. But since we can choose γ^L and γ^H, we can effectively choose the relevant region of ℓ. So it suffices to show that there is an open interval such that if $\gamma + \bar{\phi}(\gamma)$ is in this interval then $S[\gamma + \bar{\phi}(\gamma)]$ is convex in γ.

Denote $\ell(\gamma) = \gamma + \bar{\phi}(\gamma)$. By (59) and the definition of f,

$$\frac{dS[\ell(\gamma)]}{d\gamma} = \frac{u'(\ell) - 1}{1 - f'(\ell)} = \frac{tA + (1-t)\tilde{u}'[\ell(\gamma)] - 1}{1 - \beta - \alpha[tA + (1-t)\{\tilde{u}'[\ell(\gamma)] + \ell(\gamma)\tilde{u}''[\ell(\gamma)]\} - 1]}.$$

Since $q\tilde{u}'(q)$ is convex, the expression $\tilde{u}'[\ell(\gamma)] + \ell(\gamma)\tilde{u}''[\ell(\gamma)]$ in the denominator rises in ℓ. By choosing A we can make the denominator positive but arbitrarily close to 0. In this case the entire expression rises in ℓ, implying that $S[\ell(\gamma)]$ is convex in γ. ■

Proof of Proposition 14. Given π_d and ψ^*, by the same proof logic of Lemma 8, there is at most one sequence $\{\hat{\tau}_j\}_{j=d}^{\infty}$ such that $\hat{\tau}_d = \psi^*$, $\lim_{j\to\infty} \hat{\tau}_j = \tau_H$ and satisfies (63) for all $j > d$. Now we argue $\hat{\tau}_j$, if it exists, must be either rising or U-shaped. Let $\pi_k > \pi_d$ be the smallest state such that $\hat{\tau}_j$ becomes concave. Note that since $\hat{\tau}_j$ is potentially non-monotone, we cannot use the argument in Proposition 11 to argue that it is convex-concave. But since $\hat{\tau}_j$ is convex at $j = d$, we can always find $k > d$ (k is potentially infinite) such that $\hat{\tau}_j$ becomes concave at state π_k.

For $j < k$, the sequence is convex and hence it is either rising or U-shaped in j. For $j \geq k$, we show below that $\hat{\tau}_j$ rises in j. Define a mapping $G_j : \mathbb{R}^{\infty} \to \mathbb{R}$ for $j = k, k+1, \ldots$. Assume

$$G_j(\tau) \equiv f(\delta + \gamma_j + \eta s_j \tau_{j+1} + \eta(1 - s_j)\tau_{j-1} + (1-\eta)\tau_j) \tag{77}$$

for $j > k$ and $G_k(\tau) = \hat{\tau}_k$. Let τ^0 be a seed sequence with $\tau_j^0 = \hat{\tau}_k$ for all $j \geq k$. Let L_k be the set of weakly increasing sequence $\{\tau_j\}_{j=k}^{\infty}$ that are bounded between $[\hat{\tau}_k, \tau_H]$. The set L_k is a complete lattice and by the proof logic of Lemma 10 the mapping G maps elements in L_k back into L_k. Since $\hat{\tau}_j$ is concave at π_k, $\hat{\tau}_k \leq f(\delta + \gamma_j + \hat{\tau}_k)$ for all $j \geq k$ because γ_j rises in j. Hence $\hat{\tau}_k \leq G_j(\tau^0)$ for all $j \geq k$ by (77), and thus $G^n(\tau^0)$ increases in n because G is order-preserving. Hence there is a fixed point

$\tau^* = G(\tau^*) = \lim_{n\to\infty} G^n(\tau^0)$ which is an weakly increasing sequence. But there can only be one sequence starting from $\hat{\tau}_k$ and converges to τ_H by Lemma 8. Hence $\tau^* = \hat{\tau}$ and $\hat{\tau}_j$ must be weakly rising for $j \geq k$. Given that $\hat{\tau}_j$ is weakly rising for $j \geq k$ and $\lim_{j\to\infty} \tau_j = \tau_H$, we can use the proof logic of Proposition 11 to argue that it is convex-concave for $j \geq k$. Therefore, for $j \geq d$, τ_j is convex-concave and it is weakly rising when it becomes concave.

Since τ_j is either rising or U-shaped in j for $j \geq d$, we can show it attains its global maximum at $\tau_H \equiv \lim_{j\to\infty} \tau_j$ by showing $\tau_H > \tau_d$. As explained after (64), τ_H is the unique solution of $\tau_H = f(\delta + \gamma_\infty + \tau_H)$. The total liquidity at the cutoff $\tau_d = \psi^*$ is the unique solution of $\psi^* = f(\delta + \psi^*)$. Since we have assumed $\gamma_\infty > 0$ and f is an increasing function, $\tau_H > \psi^*$.

Next we show ϕ_j rises and ψ_j falls in j. Let m be the smallest state where $\tau_j \equiv \phi_j + \psi_j$ turns from falling into rising. If τ_j is monotone in j then $m = -\infty$. Given π_d, $\phi_d = 0$ and the sequence τ, one can pick a ϕ_{d+1} and use (62) to generate a sequence $\{\phi_j\}_{j=d+1}^{\infty}$. There is a unique ϕ_{d+1} such that $\lim_{j\to\infty} \phi_j = \phi_H$ by a proof logic similar to that of Lemma 8. Since the sequence $\{\phi_j\}_{j=d+1}^{\infty}$ is unique, so is $\psi_j = \tau_j - \phi_j$. Since the sequence ψ is unique, it can be derived by an iterative method which we will now characterize and use it to show ψ_j falls in j. Define the right side of (61) as a mapping H_j for $j \geq m$ by

$$H_j(\psi|\tau, \pi_d) = \beta[\delta + \eta s_j \psi_{j+1} + \eta(1 - s_j)\psi_{j-1} + (1 - \eta)\psi_j](1 + S'(\ell_j)). \tag{78}$$

Note that ℓ_j only depends on τ via (63), namely $\ell_j = f^{-1}(\tau_j)$, and is independent of ψ. Let ψ^0 be a seed sequence with $\psi_j^0 = 0$ for all $j \geq m$. The equilibrium sequence of ψ can be computed by $\psi = \lim_{n\to\infty} H^n(\psi^0|\tau, \pi_d)$. Since $S'(\ell_j)$ falls in j for $j \geq m$, H_j maps any decreasing sequence of ψ into a decreasing sequence. Therefore the fixed point ψ_j falls in j for $j \geq m$. Since τ_j rises in j for $j \geq m$, $\phi_j = \tau_j - \psi_j$ rises in j for $j \geq m$.

For $j < m$, we compute ϕ by defining a mapping I_j by the right side of (62):

$$I_j(\phi|\tau, \pi_d) = \beta[\gamma_j + \eta s_j \phi_{j+1} + \eta(1 - s_j)\phi_{j-1} + (1 - \eta)\phi_j](1 + S'(\ell_j)).$$

Since $S'(\ell_j)$ rises in j for $j \geq m$, by a logic similar to the case with $j \geq m$ we can conclude that ϕ_j rises in j for $j \geq m$. It follows that $\psi_j = \tau_j - \phi_j$ falls in j for $j \geq m$.

So far we have shown that given π_d, one can uniquely solve for τ, ϕ and ψ. Finally we argue that there is a unique π_d that can satisfy the incentive condition (14). Given π_d, one can solve for the corresponding sequence of τ, ϕ and ψ. By STEP 1 in the proof of Proposition 9, τ_{d+j} rises in π_d for all $j \geq 1$, and hence $S'(\ell_{d+j})$ falls in π_d at each $j \geq 1$. Since $S'(\ell_{d+j})$ falls in π_d, $H_{d+j}(\psi|\tau, \pi_d)$ in (78) falls in π_d for any $j \geq 1$. Therefore the sequence ψ_{d+j} falls in π_d at each $j \geq 1$. Since τ_{d+j} rises and ψ_{d+j} falls in π_d, $\phi_{d+j} = \tau_{d+j} - \psi_{d+j}$ rises in π_d. Then by the argument in STEP 2 in the proof of Proposition 9, there is a unique π_d such that incentive constraint (14) is satisfied. ■

Lemma 13 is used to prove Proposition 15. Let $\Gamma_j \equiv u(q_j) - q_j$.

Lemma 13 *The sequence $\xi_j \equiv \gamma_j + \beta\alpha\Gamma_j$ single-crosses $\theta \equiv \beta\alpha\Gamma_d$ from below as j increases from $j = d$ to ∞ and ξ_j rises in j whenever $\xi_j > \theta$.*

Proof. By Proposition 14 and (63) $\ell_j = f^{-1}(\tau_j)$ is U-shaped in j in general (including the special case that τ_j rises in j). Therefore the trade surplus Γ_j is U-shaped in j for $j \geq d$. Since $\bar{\gamma}_\infty > 0$, we have $\ell_\infty > \ell_d$ and hence $\Gamma_\infty > \Gamma_d$. At π_d, the expected payoff $\bar{\gamma}_d < 0$ because otherwise the agents would not abandon the asset at π_d. By (49) γ_j rises in j and becomes strictly positive as j explodes.

Consider the region where $\gamma_j < 0$. If ξ_j crosses θ for the first time at some $\hat{j}$ (i.e. $\xi_j \leq \theta$ at $j = \hat{j} - 1$ and $\xi_j > \theta$ at $\hat{j}$), then $\Gamma_j > \Gamma_d$ and hence Γ_j must be increasing in j for $j \geq \hat{j}$. Therefore ξ_j must be increasing for $j \geq \hat{j}$ and hence it single-crosses θ and is increasing afterwards.

Next consider $\gamma_j \geq 0$. If τ_j is convex at some state π_j, then it must be the case that $\tau_j > \bar{\tau}$ where $\bar{\tau}$ is the unique solution of $\bar{\tau} = f(\delta + \gamma_j + \bar{\tau})$. But $\bar{\tau} > \psi^* = \tau_d$ because ψ^* by definition solves $\psi^* = f(\delta + \psi^*)$ and $\gamma_j \geq 0$. Since $\tau_j > \bar{\tau} \geq \tau_d$, we have $\Gamma_j > \Gamma_d$. Hence $\xi_j > \theta$ and no crossing can happen when τ_j is convex. If τ_j is linear or concave at j, then τ_j rises in j for all larger j because τ_j is convex-concave and it rises in j whenever it is concave (both claims are explained in the proof of Proposition 14). Therefore the sum ξ_j is increasing in j and hence ξ_j single-crosses θ. ■

Proof of Proposition 15. *Proof of the first claim*: By Lemma 13 ξ_j single-crosses θ from below as j increases from $j = d$ to ∞. Let $\pi_{j'}$ be the state where ξ_j single-crosses θ. Since ξ_j rises in j for $j \geq j'$, Ω_j rises in j for $j \geq j'$. To see this, one can define the right side of (66) as a function that maps a sequence of $\{\Omega_j\}_{j=d}^{\infty}$ into another sequence. Since ξ_j rises in j for $j \geq j'$, this function maps increasing sequences into increasing sequences. Hence by standard results (i.e. contraction mapping theorem), the fixed point of the mapping is also an increasing sequence for $j \geq j'$.

For $j < j'$, if Ω_j crosses Ω_d at $j = j''$ (i.e. $\Omega_{j''-1} \leq \Omega_d$ and $\Omega_{j''} > \Omega_d$), then Ω_j must be convex at j'' because by subtracting (65) from (66)

$$\Omega_{j''} - \Omega_d = \frac{1}{1-\beta}\left\{\xi_{j''} - \beta\alpha\Gamma_d + \eta[s_{j''}\Omega_{j''+1} + (1-s_{j''})\Omega_{j''-1} - \Omega_{j''})]\right\},$$

and $\xi_{j''} \le \theta \equiv \beta\alpha\Gamma_d$ as $j'' < j'$ and $\Omega_{j''} - \Omega_d > 0$ after Ω_j crosses Ω_d. Therefore Ω_j is convex and increasing at $j = j''$. By a similar logic, Ω_j is convex and increasing at all $j \in \{j'', j''+1, \ldots, j'-1\}$. Hence, altogether, for $j \ge d$, Ω_j can only single-cross Ω_d. Note that we have also shown that Ω_j rises in j whenever $\Omega_j \ge \Omega_d$.

Proof of Part (i): If τ_j rises in j for $j \ge d$, then by (66) and the contraction mapping theorem Ω_j also rises in j. Hence $\pi^* = \pi_d$.

Proof of Part (ii): We first show that $\Omega_{d+1} < \Omega_d$ when $\bar{\gamma}_\infty \approx 0$ (i.e. $\gamma^H \approx -(1-s)\gamma^L/s$). As $j \to \infty$, by (66)

$$\Omega_\infty = 2[U(x^*) - x^*] + \delta + \bar{\gamma}_\infty + \beta(\alpha\Gamma_j + \Omega_\infty).$$

By substituting $j = d+1$ into (66) and using $\Gamma_\infty \ge \Gamma_{d+1}$ and $\Omega_\infty \ge \Omega_{d+2}$,

$$\Omega_{d+1} \le 2[U(x^*) - x^*] + \delta + \bar{\gamma}_{d+1} + \beta[\alpha\Gamma_\infty) + \eta s_{d+1}\Omega_\infty + \eta(1-s_{d+1})\Omega_d + (1-\eta)\Omega_{d+1}]. \quad (79)$$

By (65) and (79), $\Omega_{d+1} < \Omega_d$ when

$$\begin{aligned} 0 &> (1-s_{d+1})\gamma^L + s_{d+1}\gamma^H + \beta\alpha(\Gamma_\infty - \Gamma_d) + \beta\eta s_{d+1}(\Omega_\infty - \Omega_d) \\ &= (s - s_{d+1})(\gamma^L - \gamma^H) + \bar{\gamma}_\infty + \beta\alpha(\Gamma_\infty - \Gamma_d) + \beta\eta s(\Omega_\infty - \Omega_d). \end{aligned} \quad (80)$$

We argue that (80) holds when $\bar{\gamma}_\infty$ is sufficiently small. Suppose $\bar{\gamma}_\infty \downarrow 0$ such that $\ell_\infty \to \ell_d$ and $\Omega_\infty \to \Omega_d$. Therefore the last three terms on the right side of (80) vanishes. In this case the cutoff π_d must be strictly smaller than 1. For suppose $\pi_d = 1$, then $\bar{\gamma}_\infty + \eta s\phi_H > 0$ and hence the agent will not abandon the asset at π_d which leads to a contradiction. Therefore $\pi_d < 1$ and $s - s_{d+1} \equiv \nu > 0$, and (80) holds strictly. By continuity there exists $\hat{\gamma}^H > -(1-s)\gamma^L/s$ such that if $\gamma^H \in (-(1-s)\gamma^L/s, \hat{\gamma}^H)$, then $\Omega_{d+1} < \Omega_d$ and Ω_j is non-monotone in j.

Finally, suppose $\Omega_j \ge \Omega_d$ for all $j \ge d$. By Lemma 14, an increase in γ^H raises all τ_j and thus Ω_j increases at each $j \ge d$. Hence $\Omega_j \ge \Omega_d$ for all $j \ge d$ and $\pi^* = \pi_d$. Since $\Omega_j \ge \Omega_d$ at all $j \ge d$, Ω_j rises in j for $j \ge d$ as mentioned before the proof of Part (i). Therefore there is a $\tilde{\gamma}$ such that Ω_j is increasing in j for all $\gamma^H > \tilde{\gamma}$. Since Ω_j is non-monotone when $\gamma^H \approx -(1-s)\gamma^L/s$, we know $\tilde{\gamma} > -(1-s)\gamma^L/s$. ■

Lemma 14 *In a dual-asset economy, as γ^H rises, π_d falls. The price ϕ_j and τ_j fall and ψ_j rises at all $j \geq d$.*

Proof. Given any arbitrary π_d, there is at most one sequence of τ_j such that $\lim_{j\to\infty} \tau_j = \tau_H$, $\tau_d = \psi^*$ and satisfies (63) at each $j > d$. As discussed in the proof of Proposition 14, this sequence is given by $\lim_{n\to\infty} G^n(\tau^0)$ where G is given by (77). This sequence increases in γ^H because the right side of (77) rises in γ^H. Since τ_j rises in γ^H for each $j \geq d$, so does ℓ_j by (63). By the concavity of S, $S'(\ell_j)$ falls in γ^H. Then by the mapping H in (78), the sequence of ψ_j falls in γ^H at each $j \geq d$. Since τ_j rises and ψ_j falls, $\phi_j = \tau_j - \psi_j$ rises at each $j \geq d$.

Since ϕ_{d+1} and $\bar{\gamma}_d$ rise in γ^H, the right side of the IC constraint (14) rises in γ^H. By the proof logic of STEP 2 in the proof of Proposition 9, the cutoff π_d must weakly falls in γ^H. Since the equilibrium π_d cutoff weakly falls in γ^H, the equilibrium sequence of τ_j rises in γ^H at all $j \geq d$ by STEP 1 in the proof of Proposition 9. ■

Proof of Proposition 16. Since $\pi_0^h > \tilde{\pi}_d$, hodlers are willing to hold some asset, therefore $\pi_0^r > \pi_d^r$. At any state $\pi_j^r > \pi_d^r$, hodlers hold all assets if and only if

$$\bar{\ell}_j(\pi_j^h) \geq \bar{\ell}_j(\pi_j^r)[1+\alpha\theta/(1-\theta)] \iff (\pi_j^h - \pi_j^r) \geq \frac{\alpha\theta\bar{\ell}_j(\pi_j^r)}{(1-\theta)(\gamma_H - \gamma_L + \phi_{j+1} - \phi_{j-1})}. \quad (81)$$

The gap $\pi_j^h - \pi_j^r > 0$ is hump-shaped in j (see Figure 14) and it vanishes as $j \to -\infty$ or $+\infty$. Since the denominator in the right side of (81) is bounded below by $(1-\theta)[\gamma_H - \gamma_L]$ and $\bar{\ell}_j(\pi_j^r)$ rises in α, there exists $\underline{\alpha} > 0$ such that (81) holds at π_0^r if $\alpha < \underline{\alpha}$. As $j \to +\infty$, the right side is $\alpha\theta(\bar{\gamma}_\infty + \phi_H)/[(1-\theta)(\gamma_H - \gamma_L)] > 0$ and the left side vanishes. Hence (81) fails as j explodes and regular buyers will hold a positive fraction of assets. ■

H Parameters for Numerical Examples in Online Appendix

We assume bilateral meetings and Kalai bargaining with buyer's bargaining power θ in our Online Appendix E - G. We use CM utility function $U(x) = x$. Other parameters are set as follows:

Table 1: Parameters for Numerical Examples

Fig	$u(q)$	β	α	s	γ_H	γ_L	θ	η	χ	others
5	$1-e^{-120*q}$	.96	.005	.6	1.44	-1.23	1	.1, .5, .9	.01	
8 R	$2q^{.5}$	.99	.05	.55	10	-10	.5	.01	.01	
9 L	$1-e^{-120*q}$	.96	.005	.6	1.44	-1.24	1	.9	.01	
9 R	$2q^{.5}$	.99	.02	.55	10	-10	.5	.99	.007	
10 L	$2q^{.5}$	.99	.01	.55	10	-10	.5	.1, .99	.005	
10 R	$2ln(1+q)$	.96	.3	.68	13	-12	1	.15, .95	.01	
11 blue	$1.14q+.8(10-e^{-2q^{2.2}})$	.9	.6	.6	14	-6	1	.01, .9	.01	
11 red	$1.19q^{.99}$	.9	.6	.6	9	-6	1	.01, .9	.01	
12 L	$2q^{.5}$	.99	.05	.55	10	-10	.5	.5	.01	$\delta=.0001$
12 R	$2q^{.5}$	.99	.05	.55	10	-10	.5	.5	.007	$\delta=.0001$, $\kappa=.009$
13	$2q^{.5}$	.99	.05	.55	10	-10	.5	.5	.007	$\delta=.0001$
14										$\pi_0=\tilde{\pi}_{12}$
15 L	$2q^{.5}$	.99	.01	.55	10	-10	.5	.5	.01	$\pi_0=\tilde{\pi}_{12}$
15 R	$2q^{.5}$	.99	.01	.55	10	-10	.5	.5	.01	$\pi_0=\tilde{\pi}_{12}$, κ=.001

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