Learning and Money Adoption

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Abstract

Hirshleifer (1971) famously argued that the public disclosure of socially useless information hurts welfare because it creates unwanted economic fluctuations. We show that this logic can fail if the disclosed information concerns the medium of exchange. We consider an economy where agents gradually learn about the quality of a new asset and coordinate to adopt it as a medium of exchange or abandon it. The demand of this money-like asset can be partially convex, and the convexity translates more economic fluctuations into higher asset prices, making the asset a more useful payment device. Therefore more information disclosure sometimes raises welfare, even when information is not socially useful, i.e. when new information does not affect agents' adoption decisions. When there are competing monies, the aggregate liquidity and welfare can be non-monotone in beliefs and hence a good news about a new money can be a bad news for the aggregate economy. In an extension with heterogenous agents we illustrate that the presence of some hodlers can change the allocation substantially.

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1 Introduction

In the history of economic and financial development, numerous governments, banks, and individuals have issued their own money. Some monies circulated for a long period of time and some disappeared quickly. When will a new asset become a medium of exchange? This is a classic question in monetary theory but has become timely again because of the growing popularity of various crypto-currencies and digital monies. From ideas going back at least to Jevons (1876), the use of money depends on the intrinsic properties of the asset (e.g., acceptability, storability and recognizability) and coordination among agents. But in practice it takes time for agents to learn about the properties of a new asset and coordinate to adopt it as money. Our goal is to explain the dynamics of money adoption by modeling this learning and coordination process explicitly. We will characterize the life cycle of a new money — from its initial introduction to its eventual disappearance or circulation in the long run. We apply the theory to address several substantive issues, including the impact of information disclosure, competing monies, and agents with heterogenous beliefs. Among other results, we show that more disclosure can increase asset prices and welfare, even when information is not useful for making adoption decisions.

We use the New Monetarist model Rocheteau and Wright (2005) with competitive pricing, which can be viewed as a variant of the Bewley-Aiyagari model (Bewley, 1986; Huggett, 1993; Aiyagari, 1994). Agents can hold a durable real asset and potentially use it as a means of payment or collateral. They can also freely abandon the asset in any period. The asset steadily generates stochastic payoffs which can be positive or negative. These payoffs can be interpreted as dividends or holding costs of the asset.[1] The distribution of the payoffs is either high or low and is unknown to agents. Agents gradually receive noisy signals about the distribution and based on the signal realizations they coordinate to abandon the asset or adopt it as a medium of exchange. The random payoffs can be arbitrarily close to zero and thus the asset can approximate fiat money.

To study the coordination in the use of money, we assume agents always use the asset as a medium of exchange when they are certain that the payoff distribution is high and they always abandon the asset if they are sure that the distribution is low. We look

[1] In monetary economics it is common to assume commodity monies, such as gold or silver coins, generate utility or disutility to the holders. Our assumption about the random payoffs also captures, in a reduce-form manner, learning about various intrinsic properties of the asset, such as its supply growth rate, acceptability and recognizability. Recently many crypto-currencies pay dividends, some even in daily frequency, see Coinsutra for some examples.

for equilibria such that the asset is abandoned if and only if agents' belief is lower than certain cutoff. When the asset is in circulation, its price and agents' belief rise if a good news arrives and fall if a bad new arrives. In the long run, agents either abandon the asset or permanently use it as money. Both events happen with strictly positive chances.

Models of money often have multiple equilibria — if agents believe no one would accept a currency tomorrow, then the currency does not circulate today; if agents think they can use the currency to buy goods and services tomorrow, then they accept it now. Hence the adoption of money is driven by self-fulfilling beliefs. But this approach cannot explain the actual onset of money adoption because the determination of self-fulfilling beliefs has to resort to forces outside the model. In our model the beliefs are determined endogenously because agents coordinate in the use of money based on the realizations of signals. We exploit the structure of symmetric binary signals to show that if the signal arrival chance and signal precision are sufficiently high, then equilibrium is unique. When the equilibrium is unique, asset prices can be computed by a simple iterative method. We believe this approach is generalizable and useful for studying non-linear asset pricing models.

Our learning model is catered to describe the impact of information disclosure on the price of money-like assets. Since the financial crisis in 2007, many regulators are interested in setting information disclosure policies for financial assets. This interest has become stronger due to the rise of crypto-currencies and initial coin offerings. In our environment one can interpret a more transparent disclosure policy as a higher arrival chance of information. We characterize (i) the impact of a new signal, (ii) the impact of a rise in the information arrival chance and (iii) the optimal choice of disclosure policy.

To characterize the impact of a new signal, we define the *information premium* as the expected change in the asset's price when a new signal arrives. This premium measures the price impact of new information. We show that this premium can be positive or negative and can have different signs across states. In particular it has at most two sign changes — it is positive when the beliefs are very low or very high, and is negative otherwise. When agents are pessimistic about the asset's quality, they are on the fence between abandoning the asset or not. In this case new information is useful for making investment decisions and hence the information premium is positive. When agents are very optimistic about asset quality, they are unlikely to abandon the asset and new information is not useful for making investment decision. But even in this case the information premium can be positive for subtle reasons — the key lies in the relationship between the marginal benefit of asset holding today and the expected asset price tomorrow. The latter affects the former via two channels. First, if tomorrow's price is higher, then naturally the benefit of owning a unit of asset is higher, so asset demand increases. The second channel is due to the asset's role as a means of payment or collateral. If the asset price is higher tomorrow, then the agents can carry less assets for transactional motive, hence asset demand falls. Since these two channels oppose each other, the marginal benefit of asset holding can be highly non-linear. If it is convex in the asset price, then new information on average raises the asset demand (intuitively via Jensen's inequality), and hence raises asset prices, resulting in a positive information premium.[2] Using our micro-founded model, we argue that the marginal benefit of asset holding is often concave-convex. The convex part is due to two reasons. First, convexity arises when the asset is valuable enough to pay for an efficient level of output. Second, certain functional forms of preferences can induce convexity in the marginal benefit function. Although we focus on fluctuations induced by learning, in principle this convexity can translate various i.i.d. or persistent shocks into higher asset prices. This result provides novel insights into how uncertainties affect the liquidity premium of money-like assets and to our knowledge it is new.

The impact of a change in the information arrival chance is similar to the impact of the arrival of one new signal. In general as the information arrival chance rises, the asset price increases if belief is very low or very high, and otherwise it decreases. If the marginal benefit of asset holding is entirely concave or entirely convex, then we can further sharpen the predictions. At the limit when the asset becomes fiat, new information becomes sunspot shocks. The information premium is always negative and the value of fiat money always falls in the information arrival chance.

We study the optimal disclosure policy from two perspectives. We first allow the information arrival chance to be state-contingent and let an asset issuer control these chances. We assume the issuer wants to maximize either the initial offering price of the asset or the long-run adoption chance. The optimal disclosure policy is characterized by two cutoffs — the issuer provides maximal amount of information when the beliefs

[2]This logic is related to how convexity of marginal utility (i.e. $u''' > 0$) can translate uncertainty into precautionary saving behavior, as shown by Leland (1968) and Sandmo (1970). See Fernández-Villaverde and Guerrón-Quintana (2020) for a recent review on risk and uncertainty. In our model the marginal benefit of holding money-like assets is endogenous and depends on the utility function, production cost, matching probability and the trading mechanism. Rocheteau and Wright (2013) and Gu et al. (2013) exploit the non-monotonicity of this marginal benefit function and show that it can lead to endogenous fluctuations such as chaotic dynamics or limit cycles in monetary or credit economies.

are very low and very high, and otherwise suppresses information. Intuitively the issuer would like to reveal information if and only if the information premium is positive.

We also consider a policy maker who wants to maximize welfare, namely the discounted sum of expected payoffs. An increase in the information arrival chance has two effects — it changes asset prices and hence the amount of trade surplus in each state; it also induces the economy to transition more rapidly across states. The welfare impact of each effect depends crucially on the curvature of the marginal benefit of asset holding. If the asset will never be abandoned and the marginal benefit of asset holding is concave, then welfare falls in the information arrival chance. But if the asset demand is convex and agents' utility functions are sufficiently linear, then welfare rises in the information arrival chance. Intuitively, a convex asset demand converts fluctuations in beliefs into higher asset prices, making the asset a more effective medium of exchange and raising the trade surplus at each state. Moreover the convexity of asset demand can also convexify the trade surplus as a function of agents' beliefs. Consequently more fluctuations in beliefs create a higher expected trade surplus. This finding is in sharp contrast to the famous example by Hirshleifer (1971), or the analysis by Andolfatto and Martin (2013) and Dang et al. (2013), who suggested that the government should suppress information about the quality of the medium of exchange, as long as it is not socially useful.

Our second application is to study money adoption in an economy with competing monies. We introduce a safe asset which can be a fiat money or a Lucas tree that generates deterministic dividends. Agents can use the new asset, the safe asset or a combination of them as means of payment. We show that as agents' belief increases the price of the new asset rises, the price of the safe asset falls, and the aggregate liquidity (i.e. sum of the value of the two assets) can be U-shaped. Hence the arrival of a good news about the new asset can ironically reduce aggregate liquidity and trade volume. Likewise the total welfare can be non-monotone in agents' belief and thus welfare can fall as a good news arrives. The non-monotonicity of welfare is due to a coordination failure: when agents adopt the new asset as money, its price includes an endogenous liquidity premium which makes the asset a useful payment device. Thus in equilibrium agents are willing to hold it even when the cost of holding asset is high. If a policy maker bans the use of the new asset, then the liquidity premium of the new asset vanishes but that of the safe asset would increase. The aggregate liquidity might not change much but the agents are better off as they no longer pay a high holding cost of the new asset. This coordination failure occurs if and only if agents' belief is low and the dividends from the new asset are sufficiently small. Therefore our model justifies the regulation of private money issuance and provides a necessary and sufficient condition such that it is efficient to ban the issuance of a new money. If the planner can subsidize the holding of the safe asset, then the first-best can be achieved by implementing the Friedman rule.

Some people think that crypto-currencies are not money because these currencies are only held by a small number of investors, often refer to as *hodlers*, due to speculative motives.[3] Our third application is to study this phenomenon by introducing some hodlers into the economy. Hodlers have a higher prior belief than other agents but do not use the asset as a means of payment. We consider the limit when the measure of hodlers converges to zero. We characterize how a measure zero of hodlers can substantially affect asset prices and the distribution of asset holdings. The heterogeneity in beliefs introduces an extensive marginal of money adoption — if hodlers are sufficiently more optimistic than non-hodlers, then initially hodlers own all assets. In this case the asset's price is positive but it provides no transactional service. As more good signals arrive the beliefs of hodlers and non-hodlers converge to each other and non-hodlers start to adopt the asset as a means of payment. Hence even if a crypto-currency does not provide transactional services early on, it might do so later. This result helps to explain the recent rise in the acceptability of Bitcoin as a medium of exchange.[4]

Two Potential Applications. A key mechanism of our theory is that agents coordinate in the use of money based on the arrival and realization of new information about the asset. We believe this mechanism is relevant for various financial assets, such as bank notes, crypto-currencies, stocks and bonds. We mention two potential applications below. The point is that even a binary signal structure can fit data reasonably well.

A. Bank notes. Users of bank notes must learn about the reliability of the issuing bank, the acceptability of its notes and the chance of receiving a counterfeit note. A famous experiment of adoption of bank notes is the Mississippi Bubble in France during early 18th century. The Banque Royale was one of the first financial institution to develop a monetary system with paper money. After the initial introduction, their bank notes became popular very quickly and at some point it was the main legal tender in France. But these notes disappeared shortly after the French government admitted that the value

[3]For example, the distribution of Bitcoin is very uneven, as in Feb 2019, the top 1% of Bitcoin addresses hold 90% of Bitcoin stock. Bitcoin addresses are used to represent where Bitcoins are sent to or from. We show the distribution of Bitcoin across addresses in Figure 16 in the Online Appendix.

[4]Bitcoin's acceptability has been rising recently. PayPal started to allow its users to trade Bitcoin in 2020 and in early 2021, it will allow users to use Bitcoin for payments to merchants. Coinbase introduced a Visa debit card that let users spend cryptocurrencies everywhere Visa cards are accepted.

of paper notes issued by the bank exceeded the value of the coinage that it held. We collect historical events related to Banque Royale and Mississippi Company and classify each event as a positive or a negative news.[5] A positive news about Banque Royale's bank note is coded as $+1$ and a negative news as -1, and the news index is the cumulative sum of the events.[6] In Figure 1 we plot the news index and market capitalization of the bank notes issued by Banque Royale. The introduction and disappearance of the bank notes took around two years and over time the value of these notes are correlated with the news index. The news index appears to lead the market capitalization, which suggests that it takes time for agents to react to new information. The fact that the value of bank notes is correlated with historical events may not be surprising, but it is notable that even a simple binary news index, which is consistent with our model's binary signal structure, can approximate the asset prices reasonably well.

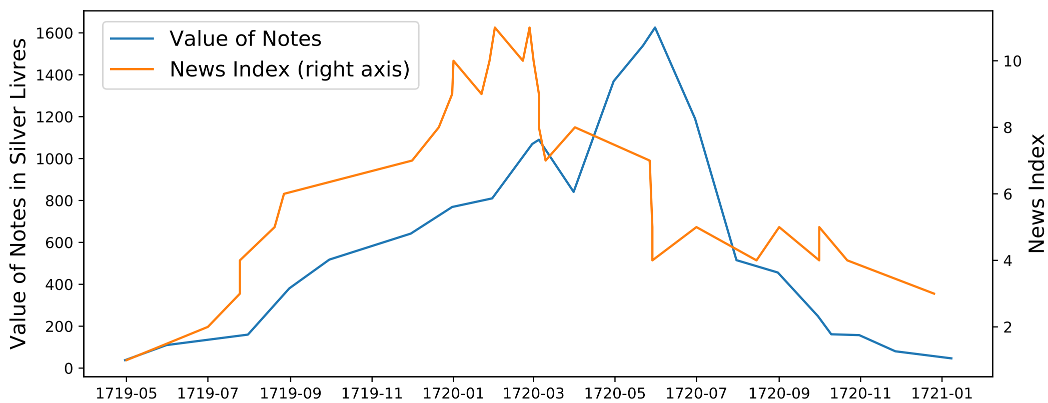


Figure 1: Value of Banque Royale's notes and news index

B. Crypto-currencies. When a new cryptoasset is issued, investors gradually learn about its rules and the quality of the underlying technology.[7] For example, the security of Bitcoin is an open question, from time to time some Bitcoins are stolen by hackers and investors change their beliefs about Bitcoin's reliability. To illustrate the relationship between crypto-currency prices and news arrival, we construct news indexes regarding two crypto-currencies: Bitcoin and BitConnect. Figure 2(a) shows Bitcoin's price and its

[5]The data on historical events and market capitalization are from Velde (2003) and Sandrock (2013).

[6]In Online Appendix A we explain the construction of all our news indexes in details.

[7]In a blog post on Alt-M: "The cryptoassets that have gained positive value competing against Bitcoin have done so not by cloning it but by offering new-and-better features. The most prominent improvements have been in four areas: greater speed in payment validation, greater privacy, greater security against 51 percent attacks, and better infrastructure for smart contracts and applications."

news index from Nov 2017 to Jul 2019. Bitcoin's price is volatile and its trend is highly correlated with the news index. For example Bitcoin encountered a huge crash in Jan 2018 as China effectively banned various Bitcoin exchanges. Around May 2019 Bitcoin's price experienced a boom after Facebook confirmed its intention to launch Libra, a global crypto-currency backed by a basket of financial assets. The news index Granger-causes the price of Bitcoin in a simple vector autoregression model at 1% significant level, which suggests the price changes could be triggered by the arrival of news (See Online Appendix A for details). This finding is consistent with our assumption that agents are learning about the profitability of Bitcoin via the news. As in the bank note example, even a simple binary news index is useful for predicting the movement of Bitcoin prices.

Investors' decision to abandon a crypto-currency is often triggered by the arrival of negative news. For instance, BitConnect was introduced in 2016 as a high-yield investment coin and during its heyday it had a market capitalization of over $2.6 billion. But later BitConnect was criticized for being a Ponzi scheme and in early 2018 a court order was granted to freeze the issuer's assets. After that BitConnect's price and trade volume plumped to zero. Figure 2(b) illustrates the price and news index of BitConnect.

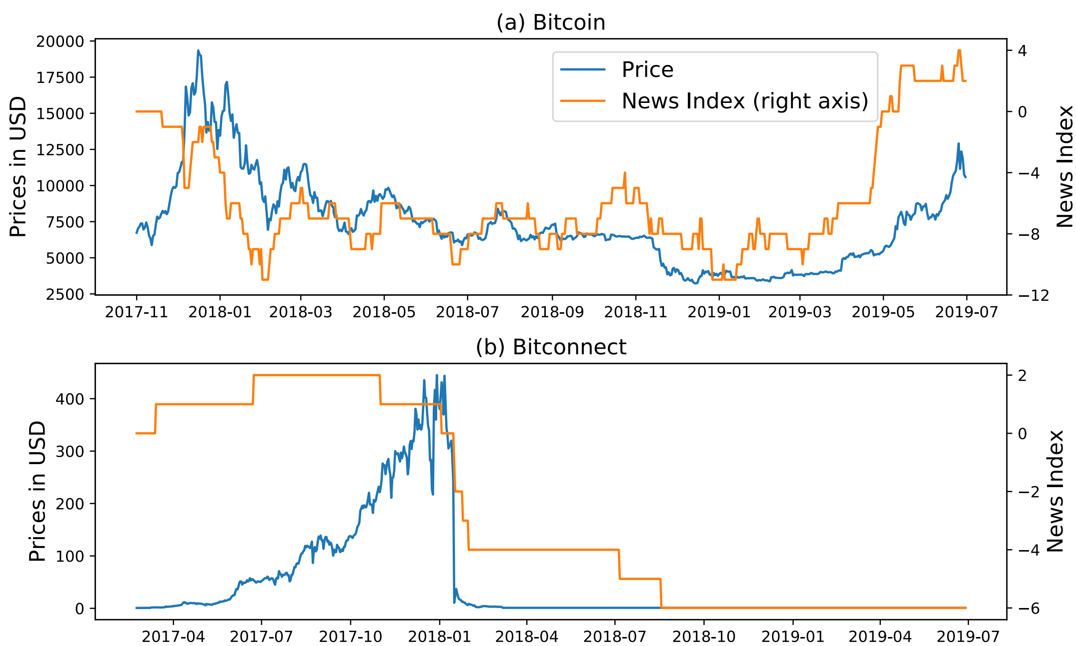


Figure 2: Crypto-currency prices and news index

Related Literature. We build on the New Monetarist literature surveyed in Lagos et al. (2017). The way we introduce a risky real asset as a medium of exchange is related to Geromichalos et al. (2007) and Lagos (2010, 2011).

Araujo and Guimaraes (2014, 2017) study coordination in the use of money in a model with indivisible money and indivisible goods (aka Kiyotaki and Wright, 1993). They also assume the intrinsic value of money evolves according to a stochastic process and prove there is a unique rationalizable equilibrium. We complement their work by analyzing a model with divisible money and divisible goods. Multiple equilibria can arise in our model and we derive a sufficient condition for uniqueness. Lester et al. (2012) and Zhang (2014) study how the recognizability and acceptability of a money affects its adoption. They study steady-state equilibria while we study the dynamic process starting from the issuance of the asset to its abandonment or circulation in the long run.

Our model is related to the currency attack models formalized by Obstfeld (1996). Morris and Shin (1998) use global game techniques to eliminate the multiplicity of equilibria and generate conclusions on the feasibility of fixed exchange rates. Chamley (1999, 2003) considers dynamic currency attack with recurrent incomplete information. Unlike global game models, our agents have the same expectation and know each other's actions. We derive a unique equilibrium by assuming a high learning speed and not by relaxing the common knowledge assumption. Another key difference is that we model the role of money as a medium of exchange and hence its liquidity premium is endogenous.

Lagos (2010) studies a related micro-founded monetary model with a risky and a safe asset. He studies theoretically and quantitatively the extend to which an asset's transactional role can explain the equity premium puzzle. In his calibration the marginal benefit of asset holding is concave and hence the presence of risk always makes an asset less valuable and less useful as a means of payment. We explore the case when the marginal benefit of asset holding is partially convex and show that new information/risk can make an asset a more effective means of payment and improve welfare.

Section 2 describes the model, Section 3 solves for the equilibrium, and Section 4 proves existence and uniqueness. In Section 5, 6, and 7, we study the role of information disclosure in money adoption, competing monies, and heterogenous agents, respectively. Section 8 concludes. An appendix contains the omitted proofs.

2 Model

The background environment is similar to that of Lagos and Wright (2005) and Rocheteau and Wright (2005). There is a unit measure of buyers, a unit measure of sellers, and a unit supply of a durable real asset. Time is discrete and continues forever. Each period has two subperiods: there is a decentralized market (DM) with payment frictions; then there is a frictionless centralized market (CM). A buyer may want a divisible good q from a seller in the DM, but his income accrues in the CM and he cannot borrow from the seller as he cannot commit to repay; so he must bring the asset from the previous CM to pay for the good. We can also interpret that the buyer uses the asset as a collateral to borrow from the seller in the DM and settles the debt in the following CM. In the CM everyone trades a numeraire good x and labor h and adjusts asset holding a. Period utility from consumption and production is $U(x) + u(q) - h$ for buyers and $U(x) - c(q) - h$ for sellers where q is the quantity of output in the DM. Agents live forever and discount between the CM and DM at $\beta \in (0, 1)$. In the DM, a fraction α of buyers want to consume and a fraction α of sellers are able to produce. These agents meet to trade assets and DM good q. The terms of trade are determined by competitive pricing. Each agent can freely abandon his asset anytime. If all agents abandon the asset, then it disappears forever and there will be no trade in the DM.

The asset's quality is either H or L and is unobservable to agents. Let $\pi \in [0, 1]$ be agents' subjective belief that the quality is H. Over time the asset generates random payoffs and noisy signals, and their distributions depend on the asset's quality. Specifically in the beginning of the CM each unit of asset generates a random payoff γ with chance χ. The realization of γ is the same across all units of asset and it has a symmetric binary distribution — if the asset quality is H, then $\gamma = \gamma_H > 0$ with chance $s \in (0.5, 1)$ and otherwise $\gamma = \gamma_L < 0$. If the asset quality is L, then $\gamma = \gamma_H$ with chance $1 - s$ and otherwise $\gamma = \gamma_L$. We call s the *precision* of information. With chance ζ the asset creates no payoff in the current period but generates a public signal. The distribution of the public signal is the same as that of γ. We call $\eta = \chi + \zeta$ the *information arrival chance*, namely the probability that a random payoff or a public signal arrives.

One can interpret γ_H as dividends and γ_L as a unit cost of holding the asset, which can be a utility cost of managing or storing the asset. We make two restrictions on the value of γ_H and γ_L. We assume $s\gamma_L + (1 - s)\gamma_H < 0$ such that if agents think the asset quality is L, namely $\pi = 0$, then the expected value $E[\gamma|\pi = 0] < 0$. We also assume $s\gamma_H + (1 - s)\gamma_L = E[\gamma|\pi = 1] > 0$. These assumptions imply that when $\pi = 0$ there is a steady-state nonmonetary equilibrium where the asset is abandoned; when $\pi = 1$ there is a steady-state monetary equilibrium where the asset price is positive.[8]

[8]In this class of monetary models if the expected dividend from an asset is negative, then there is a steady-state nonmonetary equilibrium. If the expected dividend is positive, then there is at least one

We focus on stationary Markov equilibria where trading outcomes depend only on the current-period belief π and not on histories or time.[9] We also restrict attention to equilibria where agents abandon the asset if and only if π is smaller than certain cutoff. To capture the notion of money adoption, we assume this cutoff is strictly between 0 and 1 such that the asset always circulates when $\pi \to 1$ and is abandoned when $\pi \to 0$.

The asset plays a dual-role as a financial investment and as a means of payment or a collateral. We illustrate these roles with two examples, which we will reference later.

Example 1 (Real Options Problem) *If $\alpha = 0$, then the asset is purely a financial investment that provides no transactional service. Agents' asset holding problem is the same as a standard single-agent real options problem.* □

Example 2 (Fiat Money) *Suppose $\gamma_L = \kappa\tilde{\gamma}_L$ and $\gamma_H = \kappa\tilde{\gamma}_H$ for some $\tilde{\gamma}_H > 0 > \tilde{\gamma}_L$ and $\kappa > 0$. As κ becomes arbitrarily small, the asset approaches fiat money. In this case asset holding generates little financial gains and it is driven by transaction motives.* □

3 Equilibrium

Beliefs Updating. The state variable of the economy is the current-period belief π. If a signal γ_H (i.e. a good news generated by a random payoff or a public signal) arrives, then by Bayes rule the posterior belief π' is the solution of

$$\frac{\pi'}{1-\pi'} = \frac{\pi s}{(1-\pi)(1-s)}. \tag{1}$$

Due to the symmetric binary signal structure, if a bad news γ_L arrives next period, then the posterior returns from π' to π. In other words a good and a bad signal cancel out each other; as a result, we only need to keep track of the net number of good signals. Index π by j such that if the current belief is π_j and a good signal arrives, then the posterior is π_{j+1}. Let $\pi_0 \in (0, 1)$ be agents' initial prior at period $t = 0$. If the net number of good signals since period 0 is j, then agents' current-period belief is π_j.

monetary steady-state and no nonmonetary equilibrium. See Wallace and Zhu (2004) for details.

[9]In principle agents can coordinate to use money based on various information, such as realizations of sunspot shocks or a deterministic time path of value of money, see, for example, Rocheteau and Wright (2013); Choi and Rocheteau (2019). Here we exclude these other channels and focus on coordination based on the belief about asset quality.

Let ϕ_j be the price of the asset when the current-period belief is π_j. Define

$$\bar{s}(\pi) \equiv \pi s + (1-\pi)(1-s) \quad \text{and} \quad \bar{\gamma}(\pi) \equiv \chi[\bar{s}(\pi)\gamma_H + (1-\bar{s}(\pi))\gamma_L]. \tag{2}$$

Denote $\bar{s}_j \equiv \bar{s}(\pi_j)$ as agents' subjective probability of receiving a good signal at state π_j. Let $\bar{\gamma}_j \equiv \bar{\gamma}(\pi_j)$ be the expected dividend generated by the asset in the CM.

Asset holding. An agent with wealth ω and belief π_j in the CM solves

$$\begin{aligned} &W(\omega, \pi_j) = \max_{x,h,a} \{U(x) - h + \beta V(\ell, \pi_j)\} \text{ st } x = \nu h - \phi_j a + \omega \\ &\text{where} \quad \ell \equiv a[\bar{\gamma}_j + \eta \bar{s}_j \phi_{j+1} + \eta(1-\bar{s}_j)\phi_{j-1} + (1-\eta)\phi_j] \end{aligned} \tag{3}$$

is liquidity taken out of the CM, ν is the CM wage rate, and V is the DM value function. The liquidity ℓ can be interpreted as the expected value of a units of assets in the next CM, measured in next CM's numeraire. We focus on stationary equilibrium, where W and V are independent of time, and to ease notation adopt a linear CM production technology $x = h$ so $\nu = 1$. Then, after eliminating h, we get

$$W(\omega, \pi_j) = \omega + \max_x \{U(x) - x\} + \max_a \{-\phi_j a + \beta V(\ell, \pi_j)\}.$$

We assume $U(0) = 0$, $U' > 0$, $U'' < 0$ and there exists x^* that solves $U'(x^*) = 1$. Hence the optimal CM consumption is x^*. For buyers the first-order condition for $a > 0$ is $\phi_j = \beta V_\ell(\ell, \pi_j)\ell/a$ where $V_\ell(\ell, \pi_j) \equiv dV(\ell, \pi_j)/d\ell$, and for sellers $a = 0$ as they do not need liquidity in the DM. For both, a is independent of ω and $W_\omega(\omega, \pi_j) \equiv dW(\omega, \pi_j)/d\omega = 1$, as usual in models following Lagos and Wright (2005).

Decentralized Market Trades. The problem that an active seller faces in the DM is to choose the quantity to supply q^s at cost $c(q^s)$, taking the unit price p as given. The seller receives income pq^s in the form of assets. By the linearity of W (i.e. $W_\omega(\omega, \pi_j) = 1$), the seller solves a standard profit maximization problem:

$$q^s = \arg\max_q [pq - c(q)].$$

We assume $c(0) = 0$, $c' > 0$, $c'' \leq 0$. The first-order condition is $p = c'(q^s)$. A buyer who wants to consume the DM good maximizes the surplus subject to the feasibility constraint, namely

$$q^b = \arg\max_q [u(q) - pq] \text{ st } pq \leq \ell.$$

We assume the utility $u(q)$ of consuming the DM good satisfies $u(0)=0$, $u'>0$, $u''\leq 0$ and there is a $q^* \in (0,\infty)$ where $u'(q^*)=c'(q^*)$. In equilibrium p clears the market and (p,q) is given by $p=c'(q)$ and $c'(q)q=\min\{\ell,\ell^*\}$ where $\ell^* \equiv c'(q^*)q^*$. Let

$$S(\ell_1,\ell_2) \equiv u[q(\ell_1)] - p(\ell_2)q(\ell_1)$$

be a buyer's trade surplus when he carries ℓ_1 units of liquidity and other buyers carry ℓ_2 units. To ease notation, we write $S(\ell) \equiv S(\ell,\ell)$ and $S'(\ell) \equiv dS(\ell_1,\ell_2)/d\ell_1|_{\ell_1=\ell_2=\ell}$. Easily $S(0)=0$, $S'(\ell)>0$ for $\ell \in [0,\ell^*)$ and $S'(\ell)=0$ otherwise.

Asset Pricing. Buyers' DM value is the sum of the expected trade surplus plus the expected CM continuation value, namely

$$V(\ell,\pi_j) = \alpha S(\ell) + E[W(\phi' a + \gamma' a, \pi')|\pi_j]$$

where $\alpha \in [0,1]$ is the probability of trading in the DM. The expectation operator in the second term is taken over the realization of next period's asset price ϕ', per unit dividend γ' and belief π', given the current period belief π_j. By $W_\omega(\omega,\pi_j) = 1$, we have

$$V_\ell(\ell,\pi_j) = \alpha S'(\ell) + 1. \tag{4}$$

Combining the first-order condition for asset holding in the CM, namely $\phi = \beta V_\ell(\ell,\pi_j)\,\ell/a$, and (4), and using the market clearing $a = 1$, we get

$$\phi_j = f(\ell) \equiv \beta\ell + \beta\alpha\ell S'(\ell). \tag{5}$$

The left side is the cost of carrying one more unit of asset from the CM. The right side is the marginal benefit of asset holding, which is the sum of the value of one more unit of asset in the next CM and the marginal increase in DM trade surplus. By the definition of ℓ in (3), we can rewrite (5) as

$$\phi_j = f[\bar{\gamma}_j + \eta\bar{s}_j\phi_{j+1} + \eta(1-\bar{s}_j)\phi_{j-1} + (1-\eta)\phi_j]. \tag{6}$$

For $\ell \geq \ell^*$, buyers' liquidity needs are satisfied (i.e. $S'(\ell) = 0$) and the marginal benefit of asset holding is $f(\ell) = \beta\ell$. Therefore f is continuous at $\ell = \ell^*$ but there is an outward kink (i.e. f is convex at this point). For $\ell \in [0, \ell^*)$, we assume $f(0) = 0$ and $f'(\ell) > 0$. These are mild assumptions, for example, $f(0) = 0$ if $\lim_{q\to 0} qu'(q) = 0$, and $f'(\ell) > 0$ is satisfied when α is sufficiently small or the relative risk aversion of u is less than 1.

We will show that the curvature of f is a crucial determinant of the impact of information disclosure, so here we elaborate on our assumptions on it. In our analysis the relevant domain of $f(\ell)$ is $[0, \ell_H]$ where ℓ_H is the amount of liquidity that buyers carry in a world where the asset type is known to be H (we will derive ℓ_H later). Below we assume f is concave-convex (including the special cases that it is entirely concave or entirely convex) in the relevant domain and the slope of the convex part is less than 1. The latter assumption holds when the matching probability α is small because $f(\ell) = \beta\ell$ when $\alpha = 0$. We assume the former assumption for two reasons. First, for $\ell < \ell^*$, under various parametric forms of $u(q)$, f is often concave, convex, or concave and then convex. Lemma 1 provides several parametric examples to illustrate how different assumptions on u affects the shape of f:

Lemma 1 *Assume $c(q) = q$. Consider the curvature of $f(\ell)$ for $\ell \in [0, \ell^*)$.*

1. *Assume $u(q)$ has constant relative risk aversion (CRRA). If the relative risk aversion (RRA) $-qu''(q)/u'(q) < 1$, then f is concave. If RRA > 1, then f is convex.*
2. *If $u(q)$ has constant absolute risk aversion (CARA), then f is concave-convex.*
3. *If $u(q)$ has decreasing relative risk aversion (DRRA) and RRA ≤ 1, then f is concave.*
4. *If $u(q)$ has increasing relative risk aversion (IRRA) and RRA > 1, then f is convex.*

For $\ell \geq \ell^*$, f is linear and it has an outward kink at $\ell = \ell^*$, which makes f convex at ℓ^*. So altogether f is often concave-convex.In Figure 3 we illustrate two schematics of f. In the left panel f is concave and then convex for $\ell \in [0, \ell^*]$ and then it becomes linear for $\ell \in [\ell^*, \ell^H]$. In the right panel $\ell^* > \ell^H$ and hence the liquidity premium does not vanish at $\ell = \ell_H$. In this case f can be entirely concave in the relevant domain. It is also easy to construct examples where f is entirely convex in the relevant region.

Since f is strictly increasing, it is invertible and we can rewrite (6) as a non-linear second-order difference equation:

$$\phi_{j+1} = \frac{1}{\eta\bar{s}_j}[f^{-1}(\phi_j) - (1-\eta)\phi_j - \eta(1-\bar{s}_j)\phi_{j-1} - \bar{\gamma}_j]. \tag{7}$$

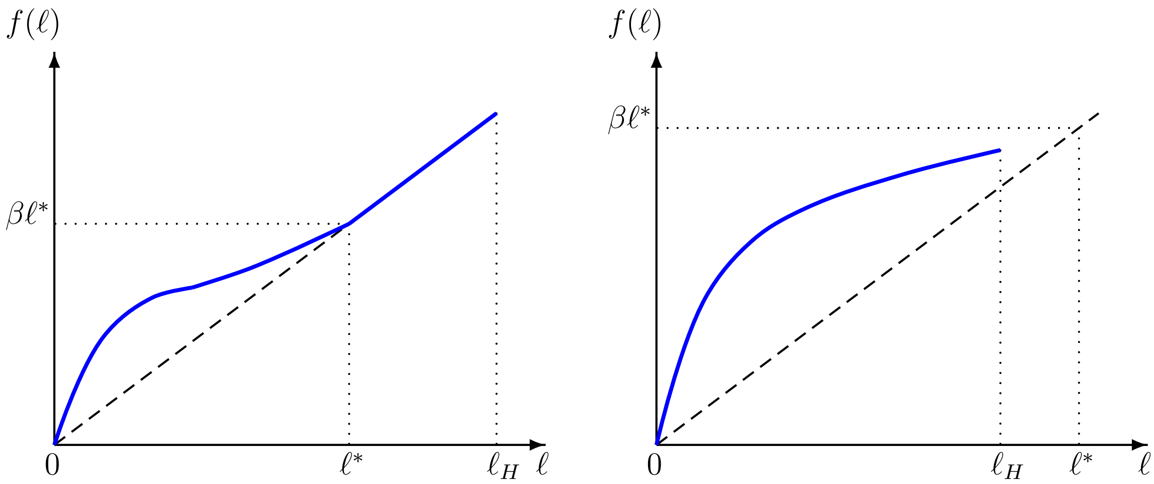


Figure 3: Marginal benefit of asset holding $f(\ell)$. (Left) Abundant liquidity (Right) Scare liquidity.

We assume the limit $\lim_{j\to\infty} \phi_j$ exists. Let $\phi_H \equiv \lim_{j\to\infty} \phi_j$, then by (6) ϕ_H must solve

$$\phi_H = f(\bar{\gamma}_\infty + \phi_H). \tag{8}$$

The solution for ϕ_H is unique and strictly positive as we have assumed $\bar{\gamma}_\infty \equiv \chi[s\gamma_H + (1-s)\gamma_L] > 0,\ f(0) = 0,\ f' > 0$ and $f' \leq 1$ when it is convex. The amount of liquidity that buyers carry is $\ell_H \equiv \bar{\gamma}_\infty + \phi_H$.

Asset Abandonment and Off-Equilibrium-path beliefs. We focus on equilibria where agents dispose the asset if and only if π_j is smaller than certain cutoff. Let π_d be this cutoff belief so that $\phi_j = 0$ for $j \leq d$. For $j > d$, ϕ_j is determined by (7).

Next we derive a condition such that agents abandon the asset. Let $\tilde{\phi}_j$ be an agent's off-equilibrium-path belief about the price of the asset after other agents have abandoned it. This agent is willing to abandon the asset at state π_d if the expected value of the asset holding is negative, i.e. the following incentive compatibility (IC) constraint holds:

$$0 \geq \bar{\gamma}_d + \eta\bar{s}_d\tilde{\phi}_{d+1} + \eta(1-\bar{s}_d)\tilde{\phi}_{d-1} + (1-\eta)\tilde{\phi}_d. \tag{9}$$

We assume $\tilde{\phi}_j = \phi_j$, namely that the prices off and on equilibrium path are the same. Using this assumption and $\phi_d = \phi_{d-1} = 0$, the IC constraint can be rewritten as

$$0 \geq \bar{\gamma}_d + \eta\bar{s}_d\phi_{d+1}. \tag{10}$$

Our assumption that $\tilde{\phi}_j = \phi_j$ can be justified by the following refinement: suppose when $\pi = \pi_d$ a fraction $1-\epsilon$ of agents abandon the asset and never trade it again. But a fraction $\epsilon > 0$ of agents hold on to the asset for one more period. If a good news arrives in the next period, then they resume trading the asset as if they are on the equilibrium path. Otherwise they also abandon the asset. Given this assumption, if an agent holds the asset for one more period, then he could sell it at price ϕ_{d+1} if a good news arrives. Hence his belief is $\tilde{\phi}_{d+1} = \phi_{d+1}$ and $\tilde{\phi}_d = \tilde{\phi}_{d-1} = 0$. Our model is the limit $\epsilon \to 0$.[10]

Now we are ready to define an equilibrium.

Definition 1 *A stationary equilibrium is a list $\langle \{q_j\}_{j=d}^{\infty}, \{\phi_j\}_{j=d}^{\infty}, \pi_d \rangle$ that (i) satisfies the equilibrium conditions in the DM market, (ii) $\phi_d = 0$, ϕ_j satisfies (7) for $j \geq d+1$ and $\lim_{j\to\infty} \phi_j = \phi_H$, and (iii) agents abandon the asset, i.e. (10) holds, at some $\pi_d \in (0,1)$.*

We present a schematic of the equilibrium price ϕ_j as a function of π_j in the left panel of Figure 4. Initially the asset circulates if and only if $\pi_0 > \pi_d$. In the long run either the asset is abandoned or $\pi \to 1$. Indeed, conditional on the asset's quality, over time the belief π converges to the truth at an exponential rate (see Lemma 2.1 in Chamley, 2004). Therefore if the asset quality is L, then given any $\pi_0 \in (\pi_d, 1)$, the belief will eventually hit π_d and the asset will be abandoned. If the asset quality is H, then at state π_0 the belief will eventually converge to 1 with ex-ante probability $1 - (1/s - 1)^{-d}$ (shown in Online Appendix E). We call this probability the *long-run adoption chance* because if $\pi \to 1$, then agents permanently and fully adopt the asset as a medium of exchange. Naturally, this chance falls with the cutoff π_d, which is determined endogenously.

In our baseline model the money adoption process takes place via the intensive margin — the value of money and the DM output level weakly increase in π for $\pi \in (0,1)$ (partial adoption) and they converge to a steady state value as $\pi \to 1$ (full adoption). We will consider the extensive margin in two different ways. In Section 7 we introduce heterogenous agents where different types of buyers adopt the asset at different times. In Online Appendix D we consider free entry of seller, namely sellers must pay a fixed cost per period to accept the new money. Naturally, more sellers accept the money when the value of the asset is higher. This extension does not change the basic model structure except it affects the functional form of f.

[10] If alternatively the off-equilibrium-path belief is $\tilde{\phi}_j = 0\ \forall j$, then the right side of the IC constraint (9) is smaller and hence the inequality is easier to satisfy. As a result the equilibrium set will be larger.

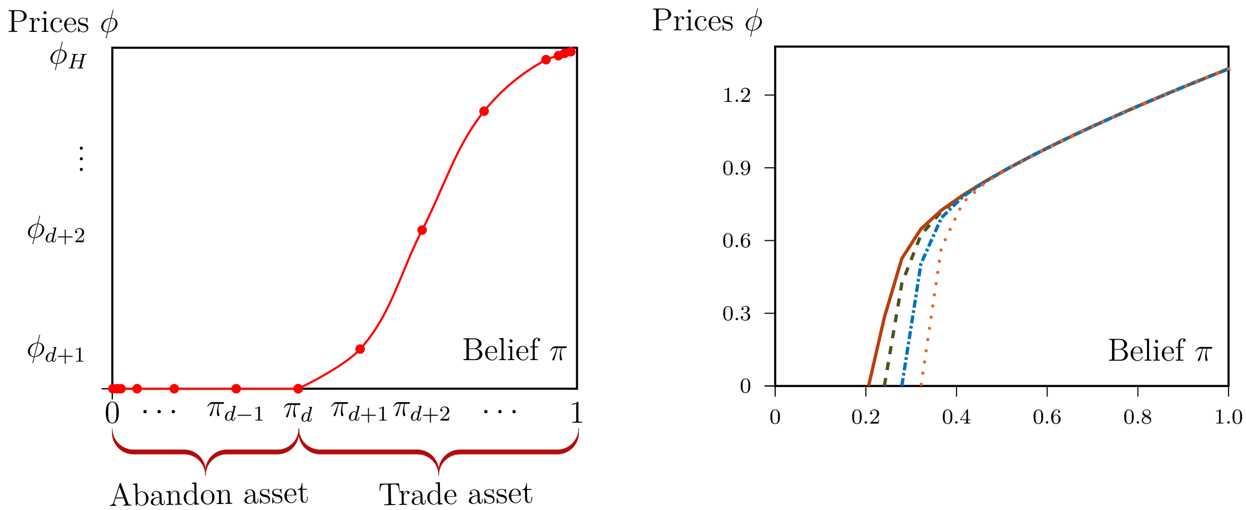


Figure 4: (Left) Schematic of equilibrium. (Right) Example of multiple equilibria.

4 Existence and Uniqueness of Equilibrium

We present a numerical example of multiple equilibria in the right panel of Figure 4.[11] In the example, across equilibria ϕ_j is higher when the cutoff π_d is smaller. The multiplicity is interesting but, as discussed in the introduction, to explain money adoption it is desirable to have a unique equilibrium. Below we provide a sufficient condition for equilibrium uniqueness and propose a simple iterative method to solve for (ϕ, π_d).

In models following Lucas (1978) a standard method to prove uniqueness is to assume the slope of f in (6) is less than 1 and then apply the contraction mapping theorem. But here the presence of DM trades induces non-linearity in f and its slope may exceed 1. Hence we need an alternative way to proceed and we find the assumption below useful.

Assumption 1 *Assume the slope of $f(\ell)$ is bounded above at all ℓ, namely*

$$f'(\ell) < \frac{1}{2\eta\sqrt{s(1-s)} + 1 - \eta}. \tag{11}$$

For example, when f is concave, the left side is bounded above by $f'(0)$ which may exceed 1. Specifically, by the definition of f in (5)

$$f'(0) = \lim_{q \to 0} \beta \left[1 - \alpha + \alpha \left(\frac{u'(q) + u''(q)q}{c'(q) + c''(q)q} \right)\right]$$

[11] The parameters used in all simulations are listed in Online Appendix F.

and it is finite as long as $u''(0)$ and $u'(0)/c'(0)$ are finite. The right side of (11) can be interpreted as the rate at which agents' belief converges over time. To see this let $\mathcal{Z}$ be an indicator function that equals 1 if the asset's quality is H and 0 otherwise. When agents' belief is π, their expectation of $\mathcal{Z}$ is $E_\pi[\mathcal{Z}] = P(\mathcal{Z} = 1) = \pi$ and the standard deviation of $\mathcal{Z}$ is $\sigma_\pi \equiv \sqrt{\pi(1-\pi)}$. The standard deviation σ_π measures how uncertain the agents are about the asset's quality. Let π' be the next period belief. The convergence rate of beliefs can be measured by the ratio $\sigma_\pi / E_\pi[\sigma_{\pi'}]$ and it is equal to the right side of (11). If this ratio is ∞, then agents learn the asset's quality in 1 period. If it is 1, then signals never arrive or they are pure noise. Naturally the convergence rate increases in η or s and explodes as η and $s \to 1$.

Intuitively, Assumption 1 is satisfied when learning is sufficiently fast. Given Assumption 1 we show that the equilibrium is unique:

Proposition 1 (Uniqueness of Equilibrium) *Generically there is at most one equilibrium and the equilibrium price ϕ_j increases in j.*

Proposition 1 states that the equilibrium is unique under Assumption 1 except for measure zero of parameter values.[12] Intuitively, when the learning speed is sufficiently high, the asset prices are highly correlated across states and hence a small increase in ϕ_{d+1} strictly raises all subsequent ϕ_j by (7). This correlation makes sure there is a unique pair of ϕ_{d+1} and π_d that leads to $\lim_{j\to\infty} \phi_j = \phi_H$ and satisfies the IC constraint.

Our uniqueness result is generalizable to other environments because Assumption 1 only imposes a restriction on the slope of f and does not depend on the exact microfoundation (i.e. trading protocol, matching technology, entry, etc). For example, if agents bargain over the terms of trade, then the f function will take a different functional form but one can still write the asset pricing equation in the form of (6) and check whether (11) holds. In Online Appendix D we show how to apply Proposition 1 in settings with bilateral bargaining, with and without entry of sellers.[13]

Now we solve for the equilibrium prices. Lucas (1978) shows that equilibrium asset prices can be computed by a simple iterative method which is an implication of the contraction mapping theorem. We claim that one can exploit the Tarski fixed-point theorem to derive a similar method for computing asset prices in our model. Let L be

[12] As shown in the proof of Proposition 1, for measure zero of parameter values, there are two equilibria and their cutoffs π_d are next to each other (i.e. they are one signal away from each other).

[13] If we assume buyers and sellers meet bilaterally and buyers make a take-it-leave-it offer to the sellers, then the f function will be identical to the one in our baseline model, provided that $c(q) = q$.

the set of bounded and weakly increasing sequences $\{\phi_j\}_{j=-\infty}^{\infty}$ where $\phi_j \in [0, \phi_H]$ at all integer $j \in \mathbb{Z}$. For any $\phi', \phi'' \in L$, we say ϕ' is larger than ϕ'', or $\phi' \geq \phi''$, if $\phi'_j \geq \phi''_j$ for all j. Define a mapping F by modifying the first-order condition (6):

$$F_j(\phi) \equiv \max\{f[\bar{\gamma}_j + \eta \bar{s}_j \phi_{j+1} + \eta(1-\bar{s}_j)\phi_{j-1} + (1-\eta)\phi_j], 0\}. \tag{12}$$

We write $\phi' = F(\phi)$ if $\phi'_j = F_j(\phi)$ $\forall j \in \mathbb{Z}$. Lemma 6 in the Appendix shows that F is order-preserving (i.e. $F(\phi)$ rises in ϕ) and maps $L \to L$. Denote $F^2(\phi) = F(F(\phi))$, $F^3(\phi) = F(F(F(\phi)))$ and so on. Proposition 2 shows that the equilibrium price sequence can be computed by iterating over F with an appropriate choice of initial sequence.

Proposition 2 (Equilibrium Asset Prices) *The equilibrium price sequence ϕ^* is a fixed point $\phi^* = F(\phi^*)$. Let ϕ^0 be a seed sequence where $\phi_j^0 \equiv 0$ for all $j \in \mathbb{Z}$. Then $\phi^* = \lim_{n\to\infty} F^n(\phi^0)$ and the sequence $F^n(\phi^0)$ increases in n.*

Proposition 2 is an application of the Tarski's fixed-point theorem. Unlike Lucas's method, our iterative method requires a specific initial sequence ϕ^0 and otherwise the iterations might not converge. Note that not every fixed point of F is an equilibrium price sequence — there could exist a fixed point where $\phi_j > 0$ at all j and hence the asset is never abandoned. If $\bar{\gamma}_{-\infty}$ is sufficiently negative such that there is no positive solution to $\phi = f(\bar{\gamma}_{-\infty} + \phi)$, then F has a unique fixed point and it corresponds to the equilibrium price sequence. As shown by the next corollary, Proposition 2 immediately yields several intuitive comparative statics:

Corollary 1 *The price ϕ^* weakly rises and the cutoff π_d weakly falls in α, γ_H and γ_L.*

Proof. Since the sequence $F(\phi)$ in (12) weakly rises in α and ϕ, the limit $\lim_{n\to\infty} F^n(\phi^0)$ weakly rises in α. Since $\phi^* = \lim_{n\to\infty} F^n(\phi^0)$ by Proposition 2, ϕ^* weakly rises in α. Since ϕ^* weakly rises, the cutoff π_d weakly fall. The proof for γ_H and γ_L are similar. ■

We end this section by considering the limit when the asset becomes fiat money. Consider the setup in Example 2. When κ is small the signals are almost payoff-irrelevant, but by Proposition 2 there always exists a unique equilibrium and ϕ_j rises in j. As $\kappa \to 0$ the cutoff $\pi_d \to 0$ because the asset is costless to hold. We present a numerical example in the right panel of Figure 5 where κ is vanishing. This example explains why seemingly irrelevant news, such as the introduction of Libra, can affect Bitcoin prices.

5 Information Disclosure

What is the impact of public disclosure about the quality of a money-like asset? This is a key topic in the debate concerning regulation of asset-backed securities or crypto-assets. We answer this question by interpreting a higher information arrival chance η as a more transparent environment. Below we characterize the impact of a single noisy signal as well as a change in η.

5.1 Impact of New Information

We define the *information premium* at state π_j as $\lambda_j \equiv \bar{s}_j \phi_{j+1} + (1 - \bar{s}_j)\phi_{j-1} - \phi_j$ which is the expected change in the asset's price when information arrives. This premium measures the price impact of new information. Graphically it has the same sign as the curvature of ϕ_j, when ϕ_j is represented as a function of π_j. Indeed, if $\lambda_j > 0$, then ϕ is convex at π_j and it is concave if $\lambda_j < 0$, as shown in the left panel of Figure 5. The sign of λ_j is useful for understanding the impact of a new signal as well as for understanding the impact of a change in η. To see the latter claim, note that (6) can be rewritten as

$$\phi_j = f(\bar{\gamma}_j + \eta\lambda_j + \phi_j). \tag{13}$$

The impact of a higher λ_j on ϕ_j is similar to that of a higher expected dividend $\bar{\gamma}_j$. Therefore, fixing λ_j, the solution of ϕ_j rises in η if $\lambda_j > 0$. Clearly λ_j also changes in η in general equilibrium, but in the next subsection we will argue that the sign of λ_j is still a useful predictor of the overall impact of η.

The main result of this subsection is to characterize how the sign of λ_j depends on the curvature of f. We first illustrate the basic idea in a two-period example:

Two-period Example: Suppose there is a 1/2 ex-ante chance that the state of the world is H. In state $i \in \{H, L\}$, the asset creates a per-period dividend γ_i and hence i is revealed perfectly by the dividend. Suppose when agents learn the state is i, the second-period asset price is $\phi_i = f(\gamma_i + \phi_i)$. Then by (6) the first-period price is $\hat{\phi} = f[\bar{\gamma} + (\phi_H + \phi_L)/2]$ where $\bar{\gamma} \equiv (\gamma_H + \gamma_L)/2$. If $f(\ell)$ is convex, then the first-period information premium $\lambda \equiv (\phi_H + \phi_L)/2 - \hat{\phi} > 0$ because $(\phi_H + \phi_L)/2 = [f(\gamma_H + \phi_H) + f(\gamma_L + \phi_L)]/2 > f[\bar{\gamma} + (\phi_H + \phi_L)/2] = \hat{\phi}$. By the same logic, $\lambda < 0$ if f is concave. $\square$
So λ and f'' tend to have the same sign. This example is slightly more subtle than a direct application of Jensen's inequality. It is because, due to the timing of the signals, our asset pricing equation takes the form $\phi_t = f(E[\gamma_{t+1} + \phi_{t+1}])$ instead of $\phi_t = E[f(\gamma_{t+1} + \phi_{t+1})]$.

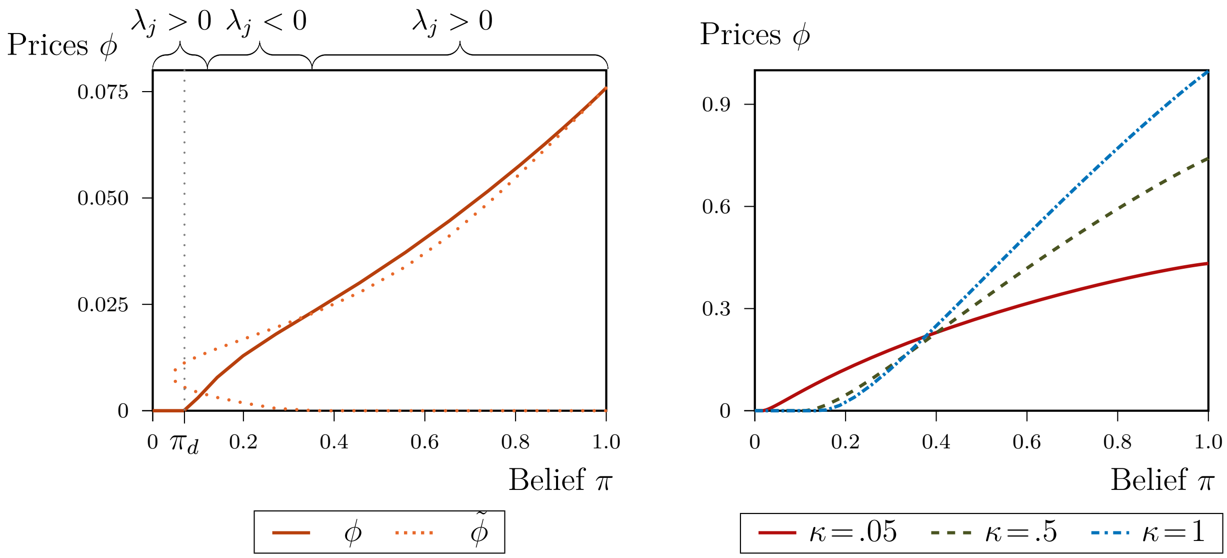


Figure 5: (Left) Example of non-monotone λ_j. (Right) Asset prices as γ_L and γ_H vanish.

Now we relate the information premium to the shape of f in the general model. Consider the solution of ϕ of the equation

$$\phi = f[\bar{\gamma}(\pi_j) + \phi]. \tag{14}$$

This equation has at most two solutions because we have assumed $f' < 1$ when it is convex. Let $\tilde{\phi}(\pi_i)$ be the correspondence of the solutions. Let $\bar{\phi}_j(\pi)$ and $\underline{\phi}_j(\pi)$ be, respectively, the larger and smaller element of $\tilde{\phi}(\pi)$. When there is only one solution then let $\bar{\phi}_j$ be the solution and set $\underline{\phi}_j = 0$. One can interpret $\bar{\phi}_j$ and $\underline{\phi}_j$ as the asset price in a steady-state equilibrium in a world with no learning and the asset generates a fixed dividend $\bar{\gamma}(\pi_j)$. Since the correspondence $\tilde{\phi}$ represents the intersection points between f and the 45 degree line, graphically $\tilde{\phi}$ is equivalent to rotating the f function anti-clockwise by 45 degrees and then scale by a constant factor. In general $\tilde{\phi}$ is a $\subset$-shaped curve as illustrated by the orange dashed line in the left panel of Figure 5.

Lemma 2 claims that ϕ is concave at state π_j (i.e. $\lambda_j < 0$) if and only if ϕ_j lies on the right side of $\tilde{\phi}$. To illustrate this claim, we present a numerical example of $\tilde{\phi}$ and ϕ in the left panel of Figure 5.

Lemma 2 *(i) $\bar{\phi}_j(\pi_j)$ rises in π_j and is concave-convex. $\underline{\phi}_j(\pi_j)$ falls in π_j and is convex. (ii) The price ϕ is weakly convex at π_d. For $j > d$, ϕ is concave at π_j if $\phi_j \in (\underline{\phi}_j, \bar{\phi}_j)$, it is linear if $\phi_j = \bar{\phi}_j$ or $\phi_j = \underline{\phi}_j$ and otherwise it is convex.*

Part (i) is immediate given the definition of $\bar{\phi}_j$ and $\underline{\phi}_j$ and our assumption on the shape of f. By this claim the correspondence $\tilde{\phi}$ forms a $\subset$-shaped curve. The first claim in Part (ii) is obvious because $\phi_{d-1} = \phi_d = 0$ and $\phi_{d+1} \geq 0$. The second claim in Part (ii) is true because if $\phi_j \in (\underline{\phi}_j, \bar{\phi}_j)$, then $\phi_j < f[\bar{\gamma}(\pi_j) + \phi_j]$, and so λ_j must be negative to balance equation (13). The proof for the rest of the claim is similar.

To see the economic content of the sign of λ_j, let us revisit Example 1 and 2. The first example shows that if information is useful for making decision, then $\lambda_j > 0$ even when f is linear. The second example shows that the intuition of the two-period example can easily be extended to case with fiat money and sunspot shocks.

Example 1': Information Makes Investment More Profitable. When $\alpha = 0$, i.e. in a pure real options problem, $f(\ell) = \beta\ell$ by (5). Therefore $\underline{\phi}_j = 0$ and $\bar{\phi}_j$ is an upward sloping straight line by (14). The equilibrium price sequence is to the left of $\bar{\phi}_j$ (we prove it in Lemma 7 in the appendix) and hence it is convex and $\lambda_j > 0$ at all $j > d$ by Lemma 2(b). Although marginal benefit f is linear, the information premium is positive because new information is useful for making the holding/abandoning decision. Hence the arrival of new information makes the asset more valuable on average. We present two numerical examples in the left panel of Figure 6 where ϕ_j is convex. $\square$

Example 2': Sunspot Shocks Reduce the Value of Fiat Money. When $\kappa \to 0$ the asset is costless to hold and hence agents never abandon the asset, namely $\pi_d \to 0$. As $\kappa \to 0$, $\bar{\phi}_j \to \phi_H$ and $\underline{\phi}_j \to 0$ by (14). In this case $\phi_j \in [\underline{\phi}_j, \bar{\phi}_j]$ at all j, and hence ϕ is concave and the information premium $\lambda_j < 0$ at all $j > d$ by Lemma 2(b). To see why the information premium is negative, note that when the asset is a fiat money, the steady-state value of money solves the equations $f(\phi) = \phi$. Given our assumptions on f, there are two solutions, namely 0 and ϕ_H, and $f(\phi) > \phi$ for $\phi \in (0, \phi_H)$. Therefore, in a sunspot equilibrium, for all $\phi_j \in (0, \phi_H)$, the information premium λ_j must be negative to balance the first-order condition $\phi_j = f(\phi_j + \eta\lambda_j)$.[14] In the right panel of Figure 5 we present numerical examples with different κ. As $\kappa \to 0$, the asset approaches fiat money and the price sequence becomes concave. $\square$

In general ϕ_j is convex-concave-convex. If f is entirely concave or entirely convex, then one can sharpen the predictions:

Proposition 3 (Information Premium)

1. *Assume $f(\ell)$ is concave-convex for $\ell \in [0, \ell_H]$. Then for $j > d$, the information premium λ_j is either (i) always positive, (ii) always negative (iii) positive and then*

[14] If we assume f is entirely convex and $f' > 1$, then $f(\phi) < \phi$ for $\phi \in [0, \phi_H]$ and thus sunspot shocks raise the price of fiat money.

negative as j rises (iv) negative and then positive as j rises or (v) positive, negative and then positive as j rises.

2. *If $f(\ell)$ is concave for $\ell \in [0, \ell_H]$, then λ_j either satisfies case (i), (ii) or (iii).*

3. *If $f(\ell)$ is convex for $\ell \in [0, \ell_H]$, then λ_j satisfies case (i).*

So λ_j can at most have two sign changes — it is positive when π_j is very small or very large, and is negative otherwise. For intuition, note that when π_j is near π_d, agents are on the fence between holding or abandoning the asset. Information is useful for making decisions and hence the information premium $\lambda_j > 0$. When π_j is large, agents are unlikely to abandon the asset so new information is not useful for making investment decisions. But by Proposition 3 even in this case $\lambda_j > 0$ is possible, provided that part of the marginal benefit $f(\ell)$ of asset holding is convex. When the relevant region of f is convex, the arrival of new information on average increases the demand of the asset, and hence raising asset price on average. As discuss before, f can be convex for two reasons. First, the presence of the liquidity premium introduces convexity into $f(\ell)$ at $\ell = \ell^*$, as shown in the left panel of Figure 3. Intuitively, suppose the agents are optimistic enough such that the liquidity premium vanished and the asset is valued fundamentally. If a good news arrives, then it further raises the asset's price. If a bad news arrives, then agents lower their expectation about the dividends, but the asset price might not drop much because it will start to include a liquidity premium. Hence new information raises the price of the asset on average. Another reason that f is convex is due to the functional form of u, as illustrated by Lemma 1. To conclude, the price impact of new information can be positive or negative and it depends crucially on the curvature of f and whether information is useful for decision making.

5.2 Increase in Information Disclosure

Now we consider an increase in the information arrival chance η. We raise η by raising the chance ζ that the asset generates a public signal and fixing the chance χ that it generates dividends. For example, the creator of a crypto-currency can raise η by disclosing more information about the technology and design of the currency system. Proposition 4 explains the link between the curvature of f and the impact of a change in η:

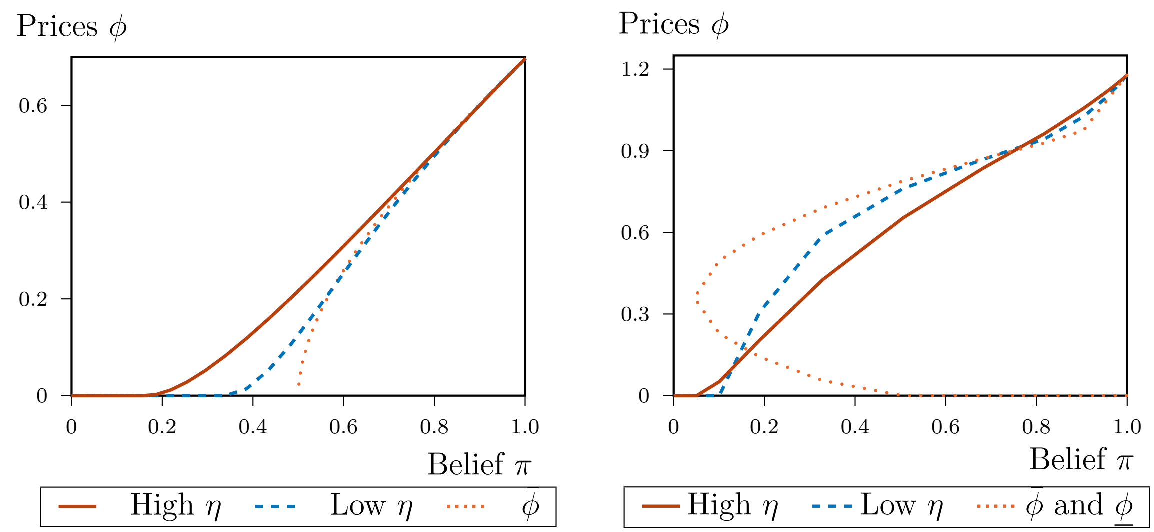


Figure 6: Numerical examples of an increase in η. (Left) Prices rise and π_d falls in η. (Right) Prices rise in η when π_j is small and otherwise fall. The cutoff π_d falls in η.

Proposition 4 (Change in Information Arrival Chance) *Suppose as the information arrival chance rises from η^* to η', the price, cutoff and information premium change from $(\phi^*, \pi_{d^*}, \lambda^*)$ to $(\phi', \pi_{d'}, \lambda')$.*

1. *If $f(\ell)$ is concave-convex for $\ell \in [0, \ell_H]$, then there exist $\bar{j}$ and $\bar{\bar{j}}$ such that $\bar{j} \leq \bar{\bar{j}}$ and $\phi'_j \geq \phi^*_j$ if and only if $j \leq \bar{j}$ or $j \geq \bar{\bar{j}}$.*

2. *If $f(\ell)$ is concave for $\ell \in [0, \ell_H]$, then $\bar{\bar{j}} = \infty$.*

3. *If $f(\ell)$ is convex for $\ell \in [0, \ell_H]$, then $\phi'_j \geq \phi^*_j$ at all j.*

Corollary 2 *If $\lambda^*_j \geq 0$ $\forall j > d^*$, then $\phi' \geq \phi^*$ and $\pi_{d'} \leq \pi_{d^*}$. If $\lambda^*_j \leq 0$ $\forall j > d^*$ and $\bar{\gamma}_{d^*} + \eta' \bar{s}_{d^*} \phi^*_{d^*+1} \leq 0$, then $\phi' \leq \phi^*$ and $\pi_{d'} \geq \pi_{d^*}$.*

A message of Corollary 2 is that the impact of a change in η depends crucially on the sign of the information premium λ_j. If $\lambda^*_j \geq 0$ at all $j \geq d$, then asset prices increase in η at all j. The cutoff π_d falls in η, meaning that the chance of adopting the asset in the long run is higher. If $\lambda^*_j \leq 0$ for all $j > d$, then prices fall in η, provided that $\bar{\gamma}_d + \eta' \bar{s}_d \phi^*_{d+1} \leq 0$. This inequality is satisfied if the IC constraint is not binding when $\eta = \eta^*$, namely $\bar{\gamma}_{d^*} + \eta^* \bar{s}_{d^*} \phi^*_{d^*+1} < 0$ and $\eta' - \eta^*$ is not too large. In this case, as η rises, the cutoff π_d rises and agents are more likely to abandon the asset in the long run.

The intuition for Proposition 4 is similar to that of Proposition 3. In a real options problem, information is always useful for making investment decisions, hence the asset becomes more valuable as η rises, see the left panel of Figure 6; when the asset is fiat money, prices fall in η because the information premium is negative. But even when the asset is never abandoned, asset prices can still increase in η in some states. It is, again, because monetary trades can introduce convexity to the marginal benefit of asset holding. As explain in the previous subsection, this convexity can make the information premium positive at some states, which in turn can make prices rises in η. We illustrate the non-monotone impact of η in the right panel of Figure 6.

A lesson from this analysis is that the impact of more disclosure is ambiguous in general. It depends on the reputation of the asset (i.e. the size of π_j) and the curvature of the price curve, which in turn depends on various model parameters. Currently there is no disclosure policy for initial coin offerings (ICOs) and hence all disclosures are voluntary. Two recent empirical studies carry out cross-sectional analysis on a large number ICOs and the findings on the relationship between voluntary disclosure and ICO success (i.e. whether the coin circulates) are mixed: Howell et al. (2018) find positive correlation while Bourveau et al. (2019) find no consistent association between the two.

5.3 Asset Issuer's Optimal Disclosure Policy

Now we endogenize η by assuming an asset issuer wants to maximize the initial offering price of the asset or the long-run adoption chance. We assume η is state-contingent and the issuer can choose $\eta_j \in [\underline{\eta}, \bar{\eta}]$ for all $j \in \mathbb{Z}$, where $1 > \bar{\eta} > \underline{\eta} > 0$. For example, suppose the asset issuer can withhold public signals from the agents, but cannot change the content of the signals. Then η_j is bounded below by the chance at which payoffs arrive and is bounded above by $\chi + \zeta$, namely $\underline{\eta} = \chi$ and $\bar{\eta} = \chi + \zeta$. We assume the issuer chooses $\{\eta_j\}_{j=-\infty}^{\infty}$ in period $t = 0$ and commits to it.

We first assume the asset issuer wants to maximize the initial offering price ϕ_0. The asset issuer's optimal policy η^I is a bang-bang solution characterized by two cutoffs. The issuer would like to reveal maximal amount of information when the belief is smaller than the lower cutoff or exceeds the higher cutoff, otherwise it is optimal to reveal minimal amount of information.

Proposition 5 (Asset Issuer's Optimal Disclosure Policy) *There are two cutoffs j' and j'' where $j' \leq j''$. The asset issuer's optimal policy η^I is such that $\eta^I_j = \bar{\eta}$ for $j \leq j'$ and $j \geq j''$ and $\eta^I_j = \underline{\eta}$ for $j \in (j', j'')$. The information premium λ_j is positive when $\eta^I_j = \bar{\eta}$ and it is negative when $\eta^I_j = \underline{\eta}$. If f is concave, then $j'' = \infty$. If f is convex, then $\eta^I_j = \bar{\eta}$ at all j.*

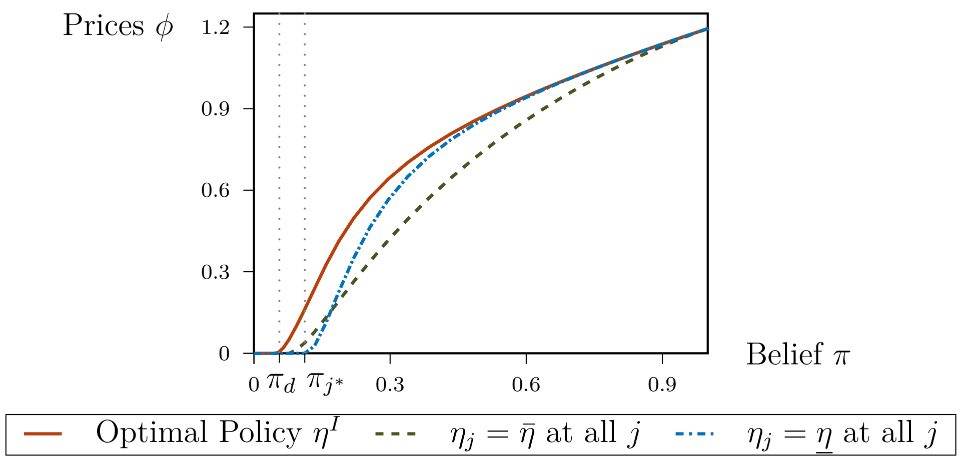


Figure 7: Asset prices under various disclosure policies.

Proposition 5 argues that it is optimal to reveal maximal amount of information when belief is very low or very high, otherwise hiding information is optimal. The intuition is related to that of Proposition 3 and 4 — to maximize the value of the asset it is optimal to reveal information if and only if the information premium λ_j is positive. By Proposition 5, if f is concave, then one can focus on policies with a single cutoff, hence the problem reduces to a simple one-dimensional optimization problem of solving j'. If f is convex, then it is optimal to always reveal maximal amount of information.

Suppose alternatively the asset issuer wants to maximize the long-run adoption chance. The proof of Proposition 5 implies the following corollary:

Corollary 3 (Adoption Chance Maximizing Policy) *The disclosure policy $\eta^{I'}$ that maximizes the long-run adoption chance is also a bang-bang solution with two cutoffs. If the equilibrium is unique under the policy $\eta^{I'}$ and η^I, then $\eta^{I'}$ is equivalent to η^I.*

Hence the asset issuer would still use a dual-cutoff disclosure policy (but not necessarily the same j' and j''). Note that our uniqueness result assumes a fixed η, so in principle there could be multiple equilibria under disclosure policy $\eta^{I'}$ or η^I. The second claim of Corollary 3 suggests that if the equilibrium is unique under $\eta^{I'}$ and η^I, then the two policies are the same. Intuitively, they are the same because both policies are about maximizing the value of the asset to the agents. We present a numerical example in Figure 7 where these policies coincide. We assume f is concave, $\bar{\eta}=0.97$ and $\underline{\eta}=0.17$ and the optimal policy is to set $\eta_j^{I'}=\bar{\eta}$ when $\pi_j > \pi_{j^*} = 0.16$ and otherwise $\eta_j^{I'} = \underline{\eta}$. We represent the prices associated with a fixed η_j by the green and blue lines. Under the optimal policy (orange line) the price ϕ_j is maximized at all states and π_d is minimized.

5.4 Welfare and Disclosure

Now we discuss the impact of a change in η on welfare. We assume the planner and the agents have the same belief and let Ω_j be the expected discounted sum of buyers' and sellers' payoffs when the current state is π_j. When the asset is abandoned, agents only trade in the CM. Hence for $j \leq d$, we have $\Omega_j = 2[U(x^*) - x^*]/(1-\beta)$ where $U(x^*) - x^*$ is each agent's CM trade surplus and $U'(x^*) = 1$. For $j > d$, Ω_j must take into account the changes in the asset prices and dividends, namely

$$\Omega_j = 2[U(x^*) - x^*] + \varepsilon_j + \beta[\eta \bar{s}_j \Omega_{j+1} + \eta(1-\bar{s}_j)\Omega_{j-1} + (1-\eta)\Omega_j]. \tag{15}$$

The first term is the CM trade surplus, the second term $\varepsilon_j \equiv \bar{\gamma}_j + \beta\alpha[u(q_j) - c(q_j)]$ is the expected dividends and DM surplus associated with state π_j, and the third term is the expected welfare in the next period. This equation is a second-order non-homogeneous recurrence relation in Ω_j with variable coefficients and it is hard to solve in general. But here one can exploit the structure of Bayesian learning to decompose this equation into two second-order difference equations that are each solvable in closed-form (see the proof of Lemma 3 for details). Therefore we can claim the following:

Lemma 3 *For $j > d$, we can represent welfare as a weighted sum of ε_i, namely $\Omega_j = \Omega_d + \Sigma_{i \geq d}^{\infty} b_{j,i}\varepsilon_i$. The weight $b_{j,i} > 0$ depends on (β, s, η) and can be solved in closed-form.*

The value of $b_{j,i}$ tells us how Ω_j depends on the dividends and DM trade surplus in each state. The closed-form formula for $b_{j,i}$ is tedious and it is available in the appendix. In Figure 8 we present three examples of $b_{j,i}$ as a function of π_i (red, blue and green solid lines). It is single-peaked and grows more dispersed as η rises (colored dashed lines). As η rises, ε_i moves in the same direction as ϕ_i, and hence in general it rises when π_i is very large and very small, and falls otherwise. Since $b_{j,i}$ and ε_i can rise or fall in η across states, the overall change in Ω_j is ambiguous in general. We illustrate this insight with a numerical example in Figure 9 where η rised from 0.1 (red solid line) to 0.9 (blue dashed line). For π near π_d, ϕ rises in η because information is useful for decision making, and hence ε rises in η as well. For large π, ε almost remains unchanged. Welfare Ω rises in η for small π because of the increase in ε. But it falls for large π as ε is concave in π and a more rapid transition across states reduces the expected surplus by Jensen's inequality.

Since the 2007 financial crisis, there has been a policy debate concerning whether the government should suppress information about assets that can serve as a means of

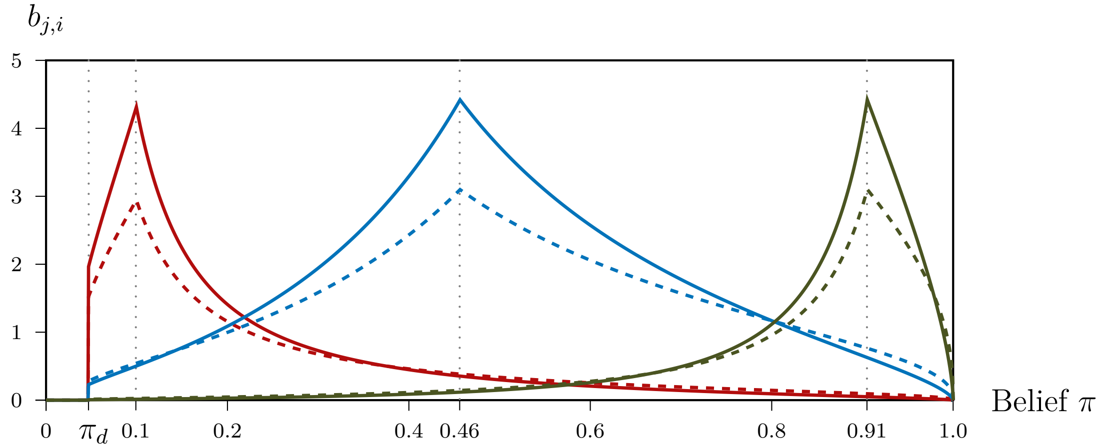


Figure 8: Three examples of $b_{j,i}$. Each solid colored line represents a $b_{j,i}$ when $\eta = 0.1$ and it shows how $b_{j,i}$ varies with π_i. The value of π_j for the red, blue and green lines are 0.1, 0.46 and 0.91 respectively. The dashed colored lines represent $b_{j,i}$ when $\eta = 0.9$.

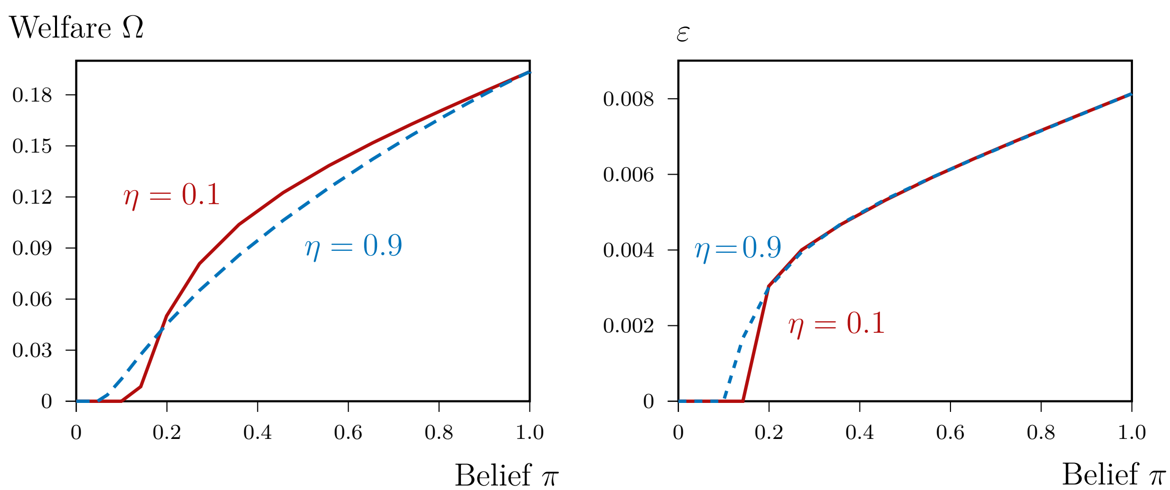


Figure 9: (Left) As η increases from 0.1 to 0.9, the surplus ε_j rises when π is small and (Right) the welfare Ω rises when π is small and falls when π is large.

payment or saving device, see, for example, Andolfatto and Martin (2013) and Dang et al. (2013). We now use our model to address this question. To do so we temporarily deviate from our assumptions and consider the case when $\bar{\gamma}_{-\infty} > 0$ and so the asset is never abandoned. In this case the asset always serves as a means of payment and $\phi_{-\infty} = \phi_L > 0$ where $\phi_L = f(\bar{\gamma}_{-\infty} + \phi_L)$. Since the asset is never abandoned, new information is payoff relevant but is not socially useful for investment. But even in this special case, welfare could go up or down in η:

Proposition 6 (Disclosing Socially Useless Information Could Raise Welfare) *Assume $c'(q)q$ is weakly convex.*

1. *If $f(\ell)$ is concave in ℓ for $\ell \in [0, \ell_H]$, then Ω_j weakly falls in η at all j.*

2. *If $u(q_j) - c(q_j)$ is convex in π_j for $\pi_j \in [0, 1]$, then Ω_j weakly rises in η at all j.*

The assumption on $c'(q)q$ is mild and it ensures the surplus $\varepsilon_j \equiv \bar{\gamma}_j + \beta\alpha[u(q) - c(q)]$ is concave in the amount of liquidity ℓ that buyers carry. Part 1 is true for two reasons. First, when f is concave and the asset is never abandoned, the information premium is always negative and ϕ_j falls in η. Hence the surplus ε_j falls in η at each state. Second, since ϕ_j is concave in π_j, so is the liquidity ℓ_j. Since ε_j is concave in ℓ_j, ε_j is a concave function in π_j. As η rises, π_j transitions more frequently across states, and Ω_j falls in η in the spirit of Jensen's inequality. Altogether, welfare falls in η and information suppression is optimal. This logic is consistent with the argument provided by Andolfatto and Martin (2013) to justify the suppression of information concerning the medium of exchange.

But welfare can also rise in η, as suggested by part 2 of Proposition 6. First, since $u(q) - c(q)$ is concave in liquidity ℓ, the sufficient condition of part 2 is satisfied only when ℓ_j is convex in π_j. This implies ϕ_j is also convex in π_j. Since the information premium is positive, by Corollary 2, ϕ_j rises in η at all j and hence the trade surplus is higher at each state. Second, since $u(q_j) - c(q_j)$ is convex in π_j, more frequent transitions across states raises the expected traded surplus. This result is in sharp contrast with the conventional wisdom, e.g. Hirshleifer (1971), that socially useless information creates economic fluctuations that hurts the welfare of risk averse agents. Here the convexity of asset demand converts fluctuations in beliefs into higher asset prices which makes the asset more useful as a mean of payment or collateral. This effect improves welfare and outweighs the usual negative effect caused by uncertainties to risk averse agents.

Figure 10 presents two numerical examples where $\gamma_{-\infty} > 0$ and so the asset is never abandoned. The blue solid line represents an example where $f(\ell)$ is concave. As η rises, the asset prices fall and hence ε at each state drops (blue dashed line in the right panel). Since ε is concave in π, a more dispersed $b_{j,i}$ reduces welfare, hence Ω falls in η at all states (blue dashed line in right panel). The red solid line represents an example that corresponds to part 2 of Proposition 6. We choose a utility function such that ε is convex in π. As shown in the left panel, although new information is not useful for making the holding/abandoning decision, Ω rises in η at all states (red dashed line). Moreover Ω is convex in π and therefore the arrival of a new signal on average improves welfare.

Although the sufficient condition in part 2 of Proposition 6 is easy to understand, it is not obvious when will it be true. The proof of next corollary shows how to construct examples such that it holds. Intuitively it requires $f(\ell)$ to be convex and $u(q)$ is sufficiently linear:

Corollary 4 *Suppose $\tilde{u}(q)$ is such that $q\tilde{u}'(q)$ is convex and $\lim_{q\to\infty} q\tilde{u}'(q) = 0$. Assume $u(q) = tAq + (1-t)\tilde{u}(q)$ and $c(q) = q$ where $t \in (0,1)$ and $A > 0$. There exists A, η γ_L and γ_H such that Ω_j rises in η at all j.*

6 Competing Monies

We now study the impact of introducing a new money by studying an economy with competing monies. We will derive a necessary and sufficient condition such that the new money improves welfare. We introduce a safe asset which generates a fixed dividend $\delta \geq 0$ in the beginning of each CM. Without loss of generality we normalize the supply of the safe asset to 1. One can interpret this asset as a government fiat money or a real asset with a known return. Below we call the asset in our baseline model as the risky asset. If the safe and risky assets both circulate, then buyers can use either one or a combination of them as a means of payment. We assume that if the risky asset is abandoned, then the economy will be in a monetary steady state where only the safe asset circulates. Let ψ^* be the price of the safe asset in this monetary steady state, by the logic leading to (8) it is the unique solution of $\psi^* = f(\delta + \psi^*)$.

Suppose both assets circulate. Let ψ_j be the price of the safe asset. An agent holding a units of risky asset and m units of safe asset in the DM has liquidity

$$\ell(a,m) = \delta m + \bar{\gamma}_j a + \eta[\bar{s}_j(\psi_{j+1}m + \phi_{j+1}a) + (1-\bar{s}_j)(\psi_{j-1}m + \phi_{j-1}a)] + (1-\eta)(\psi_j m + \phi_j a).$$

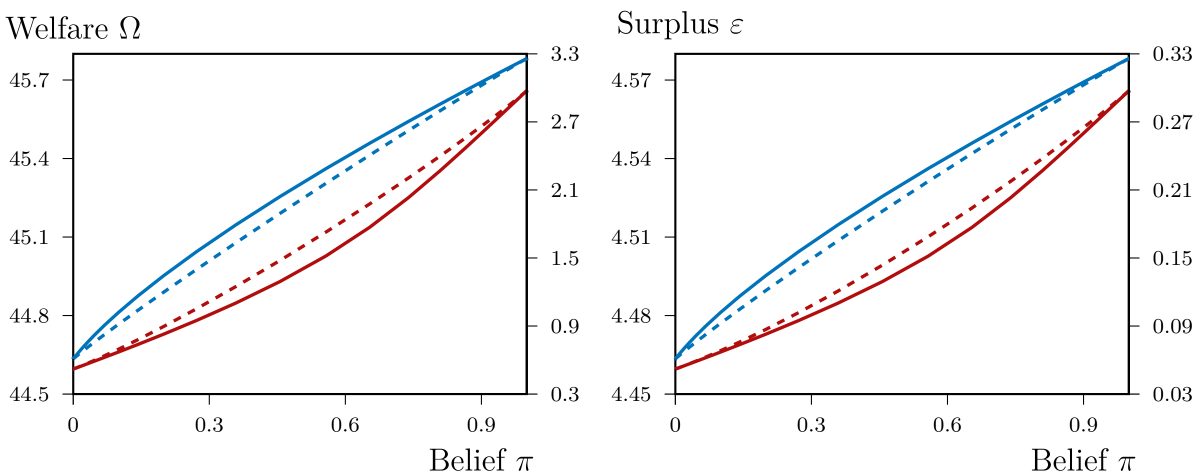


Figure 10: Two examples of a change in η. The blue lines (right axis) assume f is concave. As η rises from 0.1 (solid line) to 0.9 (dashed line), the welfare Ω and surplus ε fall. The red lines represent an example where the surplus ε is convex in π. As η rises rfom 0.1 to 0.9, Ω and ε increase as indicated by change from the red solid line to the red dashed line.

When an agent enters the CM with a portfolio (a, m), his wealth is

$$\omega(a, m) = (\phi' + \gamma')a + (\psi' + \delta)m$$

where ϕ', γ', and ψ' are the realized value of the current period price of the risky asset, per unit dividend of the risky asset and price of the safe asset, respectively. The DM and CM value functions are similar to that of the baseline model, except now ℓ and ω are functions of the asset portfolio. The value functions are, respectively,

$$\begin{aligned} V(\ell, \pi_j) &= \alpha S(\ell) + \ell + E[W(0, \pi')|\pi_j] \quad \text{and} \\ W(\omega, \pi_j) &= \omega + \max_x \{U(x) - x\} + \max_{a,m} \{-\phi_j a - \psi_j m + \beta V(\ell, \pi_j)\}. \end{aligned}$$

By the first-order condition with respect to m and $dV(\ell, \pi_j)/dm = (d\ell/dm)[1 + \alpha S'(\ell)]$,

$$\psi_j = \beta[\delta + \eta \bar{s}_j \psi_{j+1} + \eta(1 - \bar{s}_j)\psi_{j-1} + (1 - \eta)\psi_j][1 + \alpha S'(\ell_j)]. \tag{16}$$

Similarly the first-order condition with respect to a can be written as

$$\phi_j = \beta[\bar{\gamma}_j + \eta \bar{s}_j \phi_{j+1} + \eta(1 - \bar{s}_j)\phi_{j-1} + (1-\eta)\phi_j][1 + \alpha S'(\ell_j)]. \tag{17}$$

In equilibrium all agents hold the same asset portfolio, hence $a = m = 1$ by market clearing. Define $\tau_j \equiv \psi_j + \phi_j$ as the value of the portfolio. Adding (16) and (17) yields

$$\tau_j = f(\ell_j) \equiv f[\bar{\gamma}_j + \delta + \eta \bar{s}_j \tau_{j+1} + \eta(1 - \bar{s}_j)\tau_{j-1} + (1-\eta)\tau_j]. \tag{18}$$

Again, we assume asset prices converge as $\pi_j \to 1$ and denote the limits by $\phi_H = \lim_{j\to\infty} \phi_j$ and $\psi_H = \lim_{j\to\infty} \psi_j$. By (16) and (17) these limits solve

$$\phi_H = \beta(\bar{\gamma}_\infty + \phi_H)[1 + S'(\delta + \bar{\gamma}_\infty + \tau_H)], \quad \psi_H = \beta(\delta + \psi_H)[1 + S'(\delta + \bar{\gamma}_\infty + \tau_H)] \tag{19}$$

and $\tau_H \equiv \phi_H + \psi_H$ solves $\tau_H = f(\delta + \bar{\gamma}_\infty + \tau_H)$.

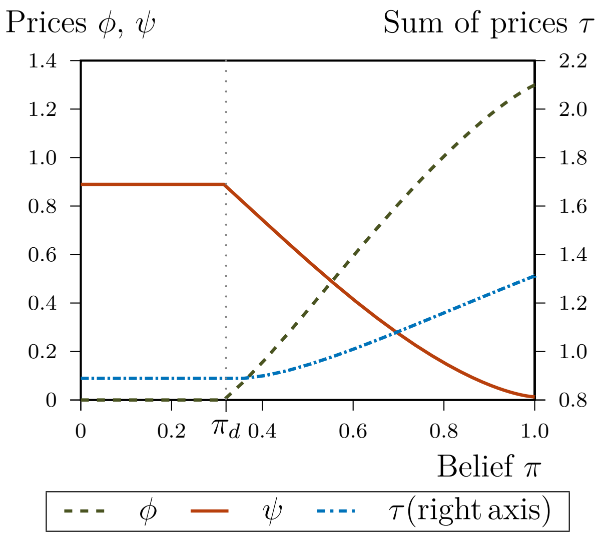
As in the baseline model, agents are willing to abandon the risky asset at π_d if (10) holds. When the risky asset is abandoned $\phi_d = 0$, and hence $\tau_d = \psi_d = \psi^*$. A dual-asset equilibrium is a list $\langle \{q_j\}_{j=d}^\infty, \{\phi_j\}_{j=d}^\infty, \{\psi_j\}_{j=d}^\infty, \pi_d \rangle$ which solves the DM market problem, the first-order conditions (16) and (17), $\lim_{j\to\infty} \psi_j = \psi_H$ and $\lim_{j\to\infty} \phi_j = \phi_H$, and the incentive condition (10) holds at some $\pi_d \in (0,1)$.

Our dual-asset equilibrium has a recursive structure. Given π_d, we can solve for τ_j by (18). Given π_d and τ_j, by (16) and (17) we can derive ϕ_j and ψ_j. Finally we check whether π_d and ϕ_j satisfy the IC constraint (10). The equilibrium is unique by an argument similar to, but also more complicated than, that of Proposition 1:

Proposition 7 (Dual-asset Equilibrium) *There is generically a unique equilibrium with dual assets. In equilibrium ϕ_j rises and ψ_j falls in j. The sum $\tau_j \equiv \phi_j + \psi_j$ is either rising or U-shaped in j and attains its maximum as $j \to \infty$.*

According to Proposition 7, as agents grow more optimistic about the risky asset, naturally ϕ_j rises, but ψ_j falls as the two assets are substitutable as a payment device. If $\delta = 0$, then the safe asset is intrinsically worthless and ψ_j vanishes as $\pi \to 1$ because $\tau_H = \phi_H$ by (19). Consequently, as $\pi \to 1$, the economy permanently adopts the risky asset and stops using the safe asset as a medium of exchange.

The value of τ_j and ℓ_j comove by (18) and they are either rising or U-shaped in j by Proposition 7. A sufficient condition for them to be U-shaped is when γ_H is small.



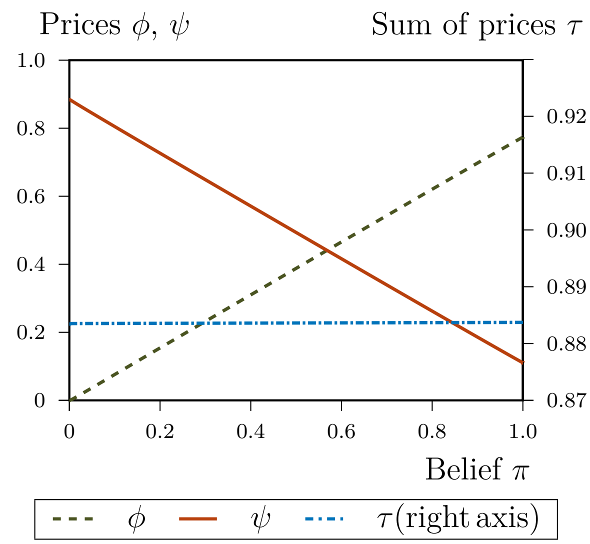


Figure 11: Asset prices in a dual-asset economy. (Left) Price of the asset portfolio τ rises in π. (Right) Both assets are close to fiat asset and τ is almost constant in π.

(proved by Proposition 8 below), namely when the dividends from an asset are small. The left panel of Figure 11 presents a numerical example where τ_j rises in π_j and the left panel of Figure 12 shows one where τ_j is U-shaped. When τ_j is U-shaped, a good news about the risky asset can reduce the aggregate liquidity and volume of trade.

Our dual-asset model provides a novel explanation of asset price volatility. As π_j rises from π_d to 1, the price of the safe asset drops and that of the risky asset rises, hence the exchange rate ϕ_j/ψ_j can fluctuate substantially over time. The sum τ_j, however, could be stable because the change in ψ_j and ϕ_j partially offset each other. Since the DM trade volume only depends on τ_j, the real economy could also be stable over time. We present a numerical example in the right panel of Figure 11 where $\delta = \kappa\tilde{\delta}, \gamma_L = \kappa\tilde{\gamma}_L$ and $\gamma_H = \kappa\tilde{\gamma}_H$ as in Example 2. We assume κ is small such that both assets are almost fiat. As shown in the figure, τ_j is almost flat in π_j. This example explains why currency prices and exchange rates can be a lot more volatile than the underlying real economy.[15]

Finally we consider whether the presence of the risky asset is welfare improving. Let Ω_j be the discounted sum of buyers' and sellers' payoffs at state π_j. For $j \leq d$, $\Omega_j = \Omega_d$ and is the expected discounted sum of payoffs in a monetary steady state with only the

[15] This example is related to Kareken and Wallace (1981) and Garratt and Wallace (2018) who show that in an economy with competing fiat monies, the price of each currency is indetermined but the aggregate liquidity and real allocations are well-determined.

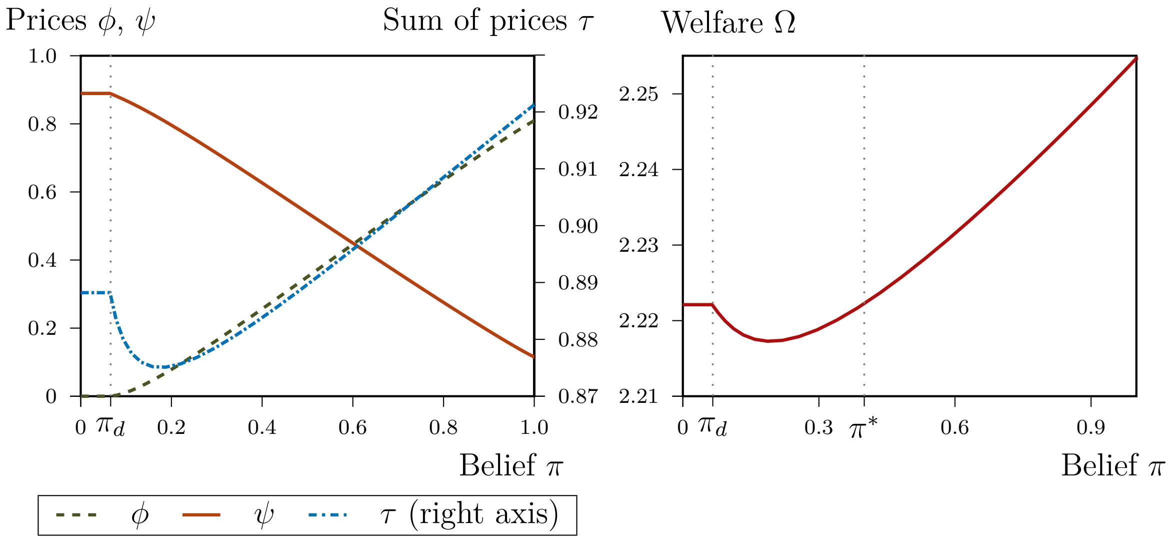


Figure 12: Non-monotone τ_j (Left) and welfare Ω_j (Right).

safe asset circulating, namely

$$\Omega_d = \frac{1}{1-\beta}\left\{2[U(x^*) - x^*] + \delta + \beta\alpha[u(q_d) - c(q_d)]\right\}. \tag{20}$$

The first term in the braces is the sellers' and buyers' CM trade surplus. The second term is the dividends from the safe asset and the third term is the total trade surplus from DM trades. For $j \geq d+1$, Ω_j must take into account the changes in the asset prices and the risky asset's dividends, namely

$$\Omega_j = 2[U(x^*)-x^*]+\bar{\gamma}_j+\delta+\beta\{\alpha[u(q_j)-c(q_j)]+\eta\bar{s}_j\Omega_{j+1}+\eta(1-\bar{s}_j)\Omega_{j-1}+(1-\eta)\Omega_j\}. \tag{21}$$

The next proposition shows that the presence of the risky asset can reduce welfare:

Proposition 8 (Non-monotone Welfare) *In a dual-asset economy, there exists a cutoff $\pi^* \in [\pi_d, 1)$ such that $\Omega_j > \Omega_d$ if and only if $\pi_j > \pi^*$.*

(i) If τ_j rises in j for all $j \geq d$, then $\pi^* = \pi_d$ and Ω_j rises in j.

(ii) There exists $\tilde{\gamma}$ such that if $\gamma_H \geq \tilde{\gamma}$ then Ω_j rises in j for all $j \geq d$ and $\pi^* = \pi_d$. If $\gamma_H < \tilde{\gamma}$ then Ω_j is non-monotone in j and $\pi^* > \pi_d$. Moreover $\tilde{\gamma} > -(1-s)\gamma_L/s > 0$.

Proposition 8 states that the introduction of a new asset is welfare improving if and only if the prior belief is sufficiently high, namely $\pi_0 > \pi^*$. By Part (ii) of the proposition the planner should ban the use of the new asset as a medium of exchange when the benefit of using the asset is small (i.e. $\gamma_H < \tilde{\gamma}$) and the prior belief is low (i.e. $\pi_0 \in (\pi_d, \pi^*)$). Suppose only the safe asset circulates in periods $t<0$ and the risky asset is introduced at $t=0$. If the initial belief $\pi_0 \in (\pi_d, \pi^*)$, then the risky asset circulates at $t=0$ but the total surplus drops from Ω_d to Ω_0 by Proposition 8. This result is illustrated by a numerical example in the right panel of Figure 12 where $\bar{\gamma}_\infty \approx 0$. In the example a social planner would introduce the risky asset if and only if $\pi_0 \geq \pi^*$. Moreover, since Ω_j is non-monotone in j, the arrival of a good news can reduce welfare. Hence a good news about the quality of a currency can be a bad news for the aggregate economy.

By Proposition 8 the equilibrium is inefficient when $\pi_0 \in (\pi_d, \pi^*)$. This inefficiency occurs because agents fail to abandon the risky asset even when the expected cost of asset holding is too high. When the risky asset circulates, its price ϕ_j includes an endogenous liquidity premium. This liquidity premium makes the asset a useful payment device and hence agents hold it even when the holding cost is high. If the risky asset is eliminated from the economy, then ϕ_j vanishes but ψ_j would rise, so τ_j might not change much. But since agents no longer pay the cost of holding the risky asset, the total welfare Ω_j rises. Naturally, this inefficiency occurs when the benefit γ_H of using the asset is small.

In principle a policy maker can achieve the first-best outcome by subsidizing the holding of the safe asset. Suppose the policy maker uses a lump sum tax to subsidize δ such that the safe asset alone can support the production level of q^* in the DM (i.e. implementing the Friedman rule, see Geromichalos et al. (2007)). In this case all agents would use the safe asset as a means of payment and only treat the risky asset as an investment. Since agents would invest optimally and efficiently, the planner does not need to intervene the agents' asset holding decision. Since the investment problem is isomorphic to a single-agent real-option problem, the more information disclosure, the more surplus the agents can extract from holding the risky asset. Therefore the planner would reveal maximal amount of information.

Due to the development of the blockchain technology, it is easy to issue a new crypto-currency. From a regulator's perspective, sometimes the relevant choice is not about subsidizing government money, but whether to allow or ban the issuance a new private money (e.g., banning the issuance of new crypto-currency or banning merchants from accepting a crypto-currency). By Proposition 8, the planner should ban a new private money when the potential benefit of using it is small and agents are pessimistic about its potential.

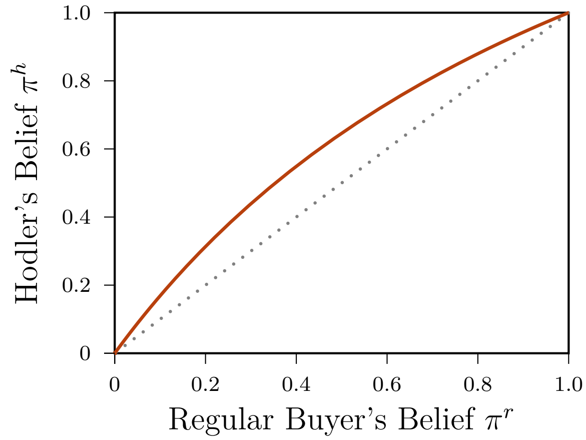


Figure 13: Beliefs of regular buyers and hodlers.

7 Heterogenous Beliefs

Some people claim that crypto-currencies are not money because they are held by a small number of speculators and provide no transactional service for the general public.[16] To explain this phenomenon we introduce some optimistic agents, call hodlers, into the economy. Hodlers have a high initial prior about the quality of the asset but do not engage in DM trades. Among the buyers a fraction $\mu \in (0,1)$ of them are hodlers and the rest are regular buyers. There is no hodler among sellers.

We denote π_0^h and π_0^r as the initial prior of hodlers and regular buyers, respectively, and assume $\pi_0^h > \pi_0^r$. Both types of buyers are rational and they agree to disagree on their beliefs. Hodlers and regular buyers receive the same information over time and both update beliefs according to Bayes rule. We denote regular buyers' belief π_j^r as the state of the economy and the asset is abandoned when $\pi_j^r \leq \pi_d^r$. At state π_j^r by Bayes rule a hodler's belief is given by

$$\pi_j^h = \Lambda(\pi_j^r) \equiv \frac{\pi_j^r}{\pi_j^r + (1-\pi_j^r)\frac{\pi_0^r(1-\pi_0^h)}{\pi_0^h(1-\pi_0^r)}}. \tag{22}$$

The difference $\pi_j^h - \pi_j^r$ is hump-shaped in j and it vanishes as $j \to +\infty$ or $-\infty$. We present a numerical example of $\Lambda(\cdot)$ in Figure 13.

By the logic leading to (3), a buyer (either a hodler or a regular buyer) with belief

[16] The German federal government stated that "Crypto tokens are not real money" as "the volume of payments carried out using crypto is limited when compared to fiat currencies". According to Forbes "To be a true alternative [of money], a cryptocurrency must also be easy to use for day-to-day transactions".

π thinks the expected value of one unit of asset in the next CM is

$$\bar{\ell}(\pi) \equiv \bar{\gamma}(\pi) + \eta\bar{s}(\pi)\phi_{j+1} + \eta[1-\bar{s}(\pi)]\phi_{j-1} + (1-\eta)\phi_j$$

where $\bar{s}(\pi)$ and $\bar{\gamma}(\pi)$ are defined in (2). A regular buyer with a_j^r units of asset has liquidity $\bar{\ell}(\pi_j^r)a_j^r$ in a trade meeting. The value functions are the same as that in the baseline model. Their asset holding is given by the first-order condition

$$\phi_j \leq \beta\bar{\ell}(\pi_j^r)\{1 + \alpha S'[\bar{\ell}(\pi_j^r)a_j^r]\}. \tag{23}$$

Since hodlers do not engage in DM trades, a_j^h is given by the first-order condition

$$\phi_j \leq \beta\bar{\ell}(\pi_j^h). \tag{24}$$

When compared to (23), the right side of (24) only includes the return of asset holding but not the benefit of using the asset as a means of payment. For both types of buyers, the first-order condition binds if the asset holding is strictly positive. Since hodlers are more optimistic than regular buyers, the asset is abandoned only when the hodlers are willing to do so. Therefore the IC constraint (10) becomes

$$0 \geq \bar{\gamma}(\pi_d^h) + \eta\bar{s}(\pi_d^h)\phi_{d+1}, \tag{25}$$

where $\pi_d^h \equiv \Lambda(\pi_d^r)$ is the cutoff for a hodler to abandon the asset. An equilibrium is a list $\langle \{q_j\}_{j=d}^{\infty}, \{\phi_j\}_{j=d}^{\infty}, \{a_j^r\}_{j=d}^{\infty}, \{a_j^h\}_{j=d}^{\infty}, \pi_d^r \rangle$ that satisfies DM market equilibrium, (23), (24), the incentive constraint (25) and asset market clearing $1 = \mu a_j^h + (1-\mu)a_j^r$ for $j > d$.

For tractability we focus on the limit when the measure of hodlers vanishes. This limit is non-trivial because even measure zero of hodlers can potentially hold a positive fraction of the total asset supply, namely it is possible that $\lim_{\mu \to 0} \mu a_j^h > 0$ for some j. If $f[\bar{\ell}_j(\pi_j^r)] > \beta\bar{\ell}_j(\pi_j^h)$ at state π_j^r, then the marginal benefit for a regular buyer to hold the asset strictly exceeds that of the hodlers, thus $a_j^r = 1$, $a_j^h = 0$ and $\phi_j = f[\bar{\ell}_j(\pi_j^r)]$. Alternatively, if $f[\bar{\ell}_j(\pi_j^r)] \leq \beta\bar{\ell}_j(\pi_j^h)$, then the hodlers are willing to hold the asset (i.e. $a_j^h > 0$ but it might or might not be finite) and hence $\phi_j = \beta\bar{\ell}_j(\pi_j^h)$. By (23) regular buyers' holding a_j^r solves $\bar{\ell}_j(\pi_j^r)\{1 + \alpha S'[\bar{\ell}_j(\pi_j^r)a_j^r]\} = \beta\bar{\ell}_j(\pi_j^h)$ and $a_j^r = 0$ if no positive solution exists. Combining the two cases, for $j > d$, the price of asset is given by

$$\phi_j = \max\{f[\bar{\ell}_j(\pi_j^r)], \beta\bar{\ell}_j(\pi_j^h)\}.$$

This condition is similar to (6) but now the unit price of the asset is determined by the maximum of the willingness-to-pay for a unit of asset among the regular buyers and the hodlers. Since this asset pricing equation does not depend on a_j^r, the equilibrium has a recursive structure — we can first solve for π_d^r and $\{\phi_j\}_{j=d}^{\infty}$, then solve for $\{a_j^r\}_{j=d}^{\infty}$.

To solve for ϕ_j we modify the mapping F in (12) as

$$F_j(\phi) \equiv \max\{f[\bar{\ell}_j(\pi_j^r)], \beta\bar{\ell}_j(\pi_j^h), 0\}. \tag{26}$$

The second term in the max operator is new and takes into account the hodlers' marginal benefit of asset holding. By the logic leading to Proposition 2 the equilibrium price sequence is a fixed point $\phi = F(\phi)$. As in the baseline model we focus on the case $\phi = \lim_{n\to\infty} F^n(\phi^0)$ which corresponds to a fixed point ϕ with $\pi_d^r \in (0,1)$. In equilibrium, using (26), one can show that ϕ_j rises in j as in the baseline model. Moreover, if the hodlers' prior belief π_0^h goes up, then ϕ_j weakly increases at all j.

Right after the introduction of the asset, it is possible that hodlers hold all the assets and do not use it as money. As more good news arrive, the regular buyers might start to hold the asset and adopt it as a means of payment. Proposition 9 provides sufficient conditions for this process to happen. Let $\tilde{\pi}_d$ be the cutoff of disposing the asset in an economy without hodler and $\alpha = 0$. In an economy with hodlers and $\alpha > 0$, if hodler's initial belief $\pi_0^h > \tilde{\pi}_d$, then hodlers are definitely willing to hold the asset at $t=0$.

Proposition 9 (Money Adoption) *Assume $\pi_0^h > \tilde{\pi}_d$. If α is sufficiently small, then hodlers initially hold all the assets, namely $a_0^h > a_0^r = 0$ at $t = 0$. As more good news arrive, regular buyers will eventually hold $a_j^r > 0$ and the asset will be used as money.*

Proposition 9 suggests that speculations by hodlers and the transactional role of the asset are not mutually exclusive. If α is sufficiently small, then initially hodlers have more incentive to hold the asset because their perceived return of asset holding is higher. As more good signals arrive, the difference $\pi_j^h - \pi_j^r$ vanishes (see Figure 13). Hodlers and regular buyers almost have the same belief, but regular buyers have more incentive to hold the asset as they can use it as a means of payment. So as π_j^r rises, eventually hodlers will sell some assets to regular buyers and the economy will adopt the new asset as a means of payment. In Figure 14 we show the asset prices ϕ_j and asset holding a_j^r of regular buyers as functions of π_j^r. In order to highlight the impact of hodlers we also show the asset prices in an economy with no hodlers (i.e. our baseline model). In the left panel, the presence of hodlers reduces the cutoff π_d^r and raises ϕ_j. For $\pi_j^r \in (\pi_d^r, \hat{\pi})$,

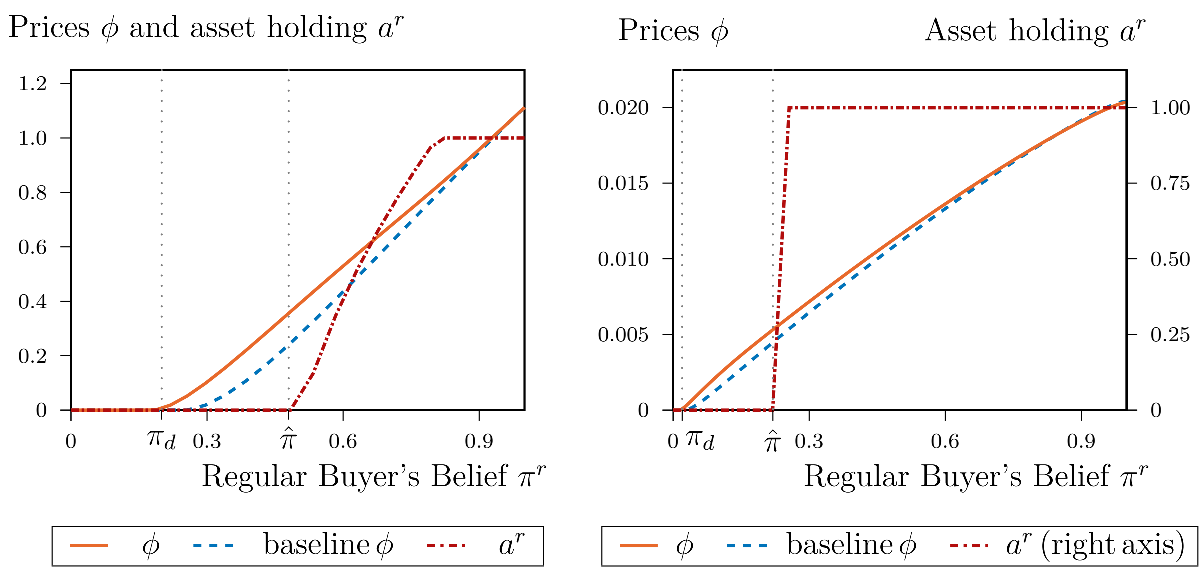


Figure 14: Asset prices and holding with hodlers. The parameters for both panels are the same except we take $\gamma_L, \gamma_H \to 0$ in the right panel (i.e. the asset becomes fiat).

hodlers own all the assets. As π_j^r rises, regular buyers gradually start to hold the asset. In the right panel we show an example where γ_L and γ_H are close to 0 such that asset is close to a fiat money. The presence of hodlers has little impact on asset prices but has a large impact on asset holding behavior. When $\pi_j^r < \hat{\pi} \approx 0.2$ all assets are held by hodlers but regular buyers quickly purchase all asset when $\pi_j^r > 0.25$. A message of this example is that, even if the asset has no intrinsic value, which is the case for Bitcoin, some hodlers are still willing to invest in it. When the entire economy becomes sufficiently optimistic about the asset's quality, hodlers will sell the asset to other agents who will then use it as a medium of exchange.

8 Conclusion

We propose a tractable learning model to explain the dynamics of money adoption. In the model agents gradually receive informative signals about the quality of a new asset and they coordinate to use or abandon the asset based on the realization of the signals. We provide a sufficient condition for equilibrium uniqueness and a simple procedure to compute asset prices. These results are new and we believe they are useful for studying other non-linear asset pricing models.

The model yields several new insights on disclosure policies. The information premium is positive if and only if agents are very pessimistic or very optimistic. The impact of more disclosure depends crucially on the curvature of the marginal benefit of asset holding. Contrary to the conventional wisdom, more disclosure sometimes can raise asset prices and improve welfare, even when information is socially useless.

When there are two competing monies, the aggregate liquidity and welfare can fall as agents receive good news about asset quality. In some cases it is efficient for a policy maker to ban a new money. Finally we study a version where agents have heterogenous beliefs and trading needs. Initially the new asset might be held by a small number of optimistic speculators and not used as a means of payment. As more good news arrive, the conservative agents start to hold the asset and use it for transactional purposes.

A lot more can be done with this model and many interesting questions remain open. For example we assume the asset supply is fixed but in reality the creator of a crypto-currency can commit to a time-varying or state-contingent asset supply. What is the implication of such supply? Another question concerns the asset's acceptability. If we assume accepting a currency is costly and requires an ex-ante investment (e.g., Lester et al., 2012), then the measure of sellers who accept the new asset will be path-dependent. This extension would be useful for understanding why similar countries can make very different dollarization decisions.

Appendix — Omitted Proofs

Proof of Lemma 1. When $c(q) = q$, we have $q = \ell$ because $qc'(q) = \ell$. Therefore the function f can be rewritten as $f(\ell) = \beta\ell + \alpha\ell[u'(\ell) - 1]$. It follows that the curvature of $f(\ell)$ has the properties as $\ell u'(\ell)$, or equivalently $f''(\ell) \propto 2u''(\ell) + \ell u'''(\ell)$.

Claim 1: When $u(q)$ has CRRA, then $u(\ell) = A\ell^{1-\eta}/(1-\eta)$. By direct differentiation one can check that if $\eta \in (0, 1)$ then $\ell u'(\ell)$ is concave and if $\eta > 1$ then it is convex.

Claim 2: When $u(q)$ has CARA, then $u(\ell) = 1 - e^{-\eta\ell}$. Hence $f''(\ell) \propto \eta^2 e^{-\eta\ell}[-2 + \ell\eta]$ is concave if and only if $\ell < 2/\eta$.

Claim 3: If $u(q)$ has DRRA, then

$$0 \leq \frac{d[\ell u''(\ell)/u'(\ell)]}{d\ell} = \frac{u''(\ell)}{u'(\ell)} \left[2 + \frac{\ell u'''(\ell)}{u''(\ell)} - 1 - \ell u''(\ell)/u'(\ell)\right].$$

If $-\ell u''(\ell)/u'(\ell) \leq 1$, then it must be the case that $2 + \ell u'''(\ell)/u''(\ell) > 0$ such that the inequality holds. Hence $f''(\ell) \propto 2u''(\ell) + \ell u'''(\ell) < 0$.

Claim 4: The proof of similar to the third claim and thus is omitted. ■

The existence and uniqueness proof is long and hence we provide an overview below.

Overview of the Existence and Uniqueness proof: The proof has two steps. Given π_d and ϕ_{d+1}, one can derive $\{\phi_j\}_{j=d+1}^{\infty}$ by using (7) recursively. The first step is to show that given an arbitrary π_d, there is a unique choice of ϕ_{d+1} such that $\lim_{j\to\infty} \phi_j = \phi_H$. The second step is to show that generically (i.e. except for measure zero of parameter values) there is a unique π_d such that the incentive constraint (10) holds (see Lemma 5 and the proof of Proposition 1). Below we sketch the proof of step 1 and leave step 2 to the proof of Proposition 1.

Step 1: We first show a small increase in ϕ_{d+1} induces a strictly positive increase in all subsequent ϕ_j for $j > d + 1$. Differentiate (7) with respect to ϕ_{d+1} and let $k_{j+1} \equiv d\phi_{j+1}/d\phi_j$ for $j \geq d + 1$. Also let $\bar{f}' \equiv \max_\ell\{f'(\ell)\}$ be the upper bound of the slope of f. Then we have the inequality

$$k_{j+1} \geq \frac{1}{\eta\bar{s}_j}[\frac{1}{\bar{f}'} - (1-\eta) - \eta(1-\bar{s}_j)\frac{1}{k_j}]. \tag{27}$$

Define $\underline{k}_j \equiv (1-\bar{s}_j)/\sqrt{s(1-s)} > 0$. By (27) and Assumption 1, one can check that if $k_j > \underline{k}_j$, then $k_{j+1} > \underline{k}_{j+1}$ $\forall j > d + 1$. One can also check $k_{d+2} > \underline{k}_{d+2}$ (see Lemma 4). Altogether $k_j > \underline{k}_j > 0$ at all j which implies $d\phi_j/d\phi_{d+1} > 0$. Lemma 4 further shows that if the limit $\lim_{j\to\infty}\phi_j = \phi_H$, then the limit strictly increases in ϕ_{d+1}, and thus there can only be one choice of ϕ_{d+1} such that $\lim_{j\to\infty}\phi_j = \phi_H$. Hence given π_d, there is a unique sequence of ϕ_j converging to ϕ_H.

Lemma 4 *Given π_d and $\phi_d = 0$, one can pick a value of ϕ_{d+1} and then use (7) iteratively to derive a sequence of ϕ_j for $j \geq d+2$. (i) For each $j \geq d+2$, ϕ_j strictly increases in the initial guess ϕ_{d+1}. (ii) There is at most one ϕ_{d+1} such that $\lim_{j\to\infty}\phi_j = \phi_H$.*

Proof. *Part (i):* Consider an infinitesimal increase in ϕ_{d+1} and let $k_j = d\phi_j/d\phi_{j-1}$ for $j \geq d+2$. Therefore the change in any ϕ_j is $d\phi_j/d\phi_{d+1} = \Pi_{i=d+2}^{j} k_i$. Define $\underline{k}_j \equiv (1-\bar{s}_j)/\sqrt{s(1-s)} > 0$, $\underline{k}_j$ falls in j by the definition of $\bar{s}_j$ in (2). We will show $k_{d+j} > \underline{k}_{d+j}$ for all $j \geq 1$. Let $B \equiv [1/f'(0) - (1-\eta)]/\eta$. By (7) and the concavity of f,

$$k_{j+1} \geq \frac{1}{\bar{s}_j}[B - \frac{(1-\bar{s}_j)}{k_j}].$$

Since the right side rises in k_j, if $k_j > \underline{k}_j$, then $k_{j+1} > \underline{k}_{j+1}$ provided that

$$\frac{1}{\bar{s}_j}[B - \frac{(1-\bar{s}_j)}{\underline{k}_j}] > \underline{k}_{j+1} \iff B > \sqrt{s(1-s)} + \bar{s}_j\frac{(1-\bar{s}_{j+1})}{\sqrt{s(1-s)}} = 2\sqrt{s(1-s)}.$$

The if and only if relationship uses the definition of $\underline{k}_j$ and $\underline{k}_{j+1}$. The last equation holds because by (1) and (2) $\bar{s}_{j+1} = 1 - s(1-s)/\bar{s}_j$. The right side is smaller than B if and only if Assumption 1 holds. Therefore if $k_j > \underline{k}_j$, then $k_{j+1} > \underline{k}_{j+1}$ under Assumption 1.

Finally we argue $k_{d+2} > \underline{k}_{d+2}$. By (7) and the definition of $\bar{f}'$

$$k_{d+2} \geq \frac{1}{\eta\bar{s}_{d+1}}\left[\frac{1}{\bar{f}'} - (1-\eta)\right] = \frac{B}{\bar{s}_{d+1}} \geq \frac{\sqrt{s(1-s)}}{\bar{s}_{d+1}} + \frac{(1-\bar{s}_{d+2})}{\sqrt{s(1-s)}} > \underline{k}_{d+2}.$$

The first equation uses the definition of B. The second inequality uses Assumption 1 and the last inequality uses the definition of $\underline{k}_{d+2}$.

Part (ii): Since $k_{d+j} > 0$ for all $j > 1$ by Part (i), the limit $\lim_{j\to\infty}\phi_{d+j}$ weakly rises in ϕ_{d+1}. By (7), if $\lim_{j\to\infty}\phi_j = \phi_H$, then for large j the difference equation for k_j becomes

$$k_{j+1} = \frac{1}{\eta s}[\frac{1}{f'(\phi_H)} - (1-\eta) - \frac{\eta(1-s)}{k_j}]. \tag{28}$$

In Figure 15 we show the 45 degree line as a blue line. The right side of (28) is increasing and concave in k_j and is shown as a red line in the figure. It is easy to check that the

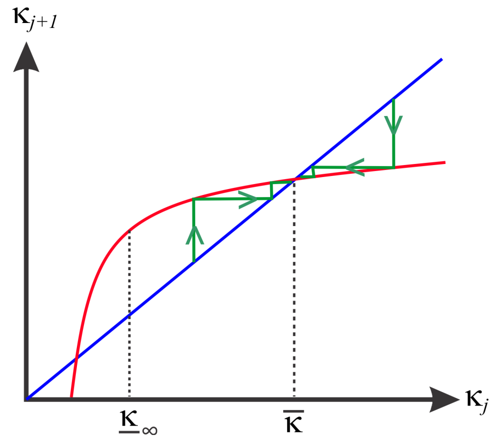


Figure 15: Convergence of k_j to $\bar{k}$.

right side is smaller than k_j when $k_j \approx 0$ or when k_j is sufficiently large. When $k_j = 1$, the right-hand side exceeds 1 as $f'(\phi_H) < 1$. Therefore the red and blue line intersect twice and the larger intersection point in Figure 15, call it $\bar{k}$, strictly exceeds 1.

Finally we argue $\lim_{j\to\infty} k_j = \bar{k} > 1$. The slope of the right-side of (28) is 1 at $k_j = \underline{k}_\infty \equiv \sqrt{(1-s)/s}$. Since the red line is concave and crosses the blue diagonal twice, it must lie above the blue at $k_j = \underline{k}_\infty$. Since $k_j > \underline{k}_\infty$ at all $j > d+1$ by Part (i), k_j must converge to $\bar{k}$ as $j \to \infty$, see the green lines in Figure 15. Therefore when $\lim_{j\to\infty} \phi_j = \phi_H$, we have $\lim_{j\to\infty} k_j > 1$, and hence $d[\lim_{j\to\infty} \phi_j]/d\phi_{d+1} = \Pi_{i=d+2}^{\infty} k_i > 0$. Since $\lim_{j\to\infty} \phi_j$ weakly increases in ϕ_{d+1} by Part (i) and strictly increases in ϕ_{d+1} when $\lim_{j\to\infty} \phi_j = \phi_H$, there is a unique ϕ_{d+1} such that $\lim_{j\to\infty} \phi_j = \phi_H$. ■

Lemma 5 *Fixing an integer $z \in \mathbb{Z}$. Consider the set L_z of price sequence $\{\phi_j\}_{j=z}^{\infty}$ such that ϕ_j weakly rises in j and is bounded between $[0, \phi_H]$. Let T be a function that maps $\phi \in L_z$ into another sequence $\{\phi'_j\}_{j=z}^{\infty}$. For $j > z$*

$$\phi'_j = T_j(\phi) \equiv f[\bar{\gamma}_j + \eta\bar{s}_j\phi_{j+1} + \eta(1-\bar{s}_j)\phi_{j-1} + (1-\eta)\phi_j] \tag{29}$$

and $\phi'_z = T_z(\phi) \equiv 0$. Let the seed sequence $\phi_j^0 = \phi_H$ for all $j > z$ and $\phi_z^0 = 0$. Suppose $\{\phi_j^*\}_{j=z}^{\infty}$ is a sequence that satisfies (7) with $\phi_z^* = 0$ and $\lim_{j\to\infty} \phi_j^* = \phi_H$. Then ϕ_j^* is given by $\phi^* = \lim_{n\to\infty} T^n(\phi^0)$ and increases in j.

Proof of Lemma 5. It is easy to check that T is order preserving, namely if $\phi'_j \geq \phi''_j$ for all $j \geq z$, then $T_j(\phi') \geq T_j(\phi'')$ for all $j \geq z$. It is also easy to verify that T maps bounded increasing sequences in $[0, \phi_H]$ back into itself, hence $T : L_z \to L_z$.

Since $\phi_H = f(\bar{\gamma}_\infty + \phi_H)$ by (8) and f is an increasing function, $\phi^0 \geq T(\phi^0)$. Since $\phi^0 \geq T(\phi^0)$ and T is monotone, $T^2(\phi^0) \equiv T[T(\phi^0)] \leq T(\phi^0) \leq \phi^0$ and by induction $T^n(\phi^0)$ falls in n. Let $\{\phi_j^*\}_{j=z}^{\infty}$ be a sequence that satisfies (7) with $\phi_z^* = 0$ and $\lim_{j\to\infty} \phi_j^* = \phi_H$. By definition it is a fixed point of T, namely $T(\phi^*) = \phi^*$. Since T is monotone and the seed sequence $\phi_j^0 \geq \phi_j^*$ at all $j \geq z$, $T(\phi^0) \geq \phi^*$. By the monotonicity of T, $T^n(\phi^0) \geq \phi^*$ for all $n > 0$. Since the sequence $T^n(\phi^0)$ weakly falls in n and is bounded below by ϕ^*, the limit $\phi^\infty \equiv \lim_{n\to\infty} T^n(\phi^0)$ exists, is weakly larger than ϕ^*, and ϕ_j^∞ rises in j.

Since ϕ_j^∞ is bounded above by ϕ_H and bounded below by ϕ_j^*, it must converge to ϕ_H as $j \to \infty$. Finally, by Lemma 4, for a given z, the sequence ϕ^* that converges to ϕ_H is unique. Therefore $\phi^\infty = \phi^*$, and thus ϕ_j^* increases in j. ■

Proof of Proposition 1. Let $\pi_0' \in [0,1]$ be an arbitrarily chosen prior belief and let π_j' be the posterior belief after seeing $j \geq 0$ good signals. Rewrite (29) by (2) as

$$\hat{T}_j(\phi|\pi_0') \equiv f[\bar{\gamma}(\pi_j') + \eta\bar{s}(\pi_j')\phi_{j+1} + \eta(1-\bar{s}(\pi_j'))\phi_{j-1} + (1-\eta)\phi_j] \tag{30}$$

for $j \geq 1$ and $\hat{T}_0 \equiv 0$. Let $\phi^{\pi_0'}$ be the fixed point $\phi^{\pi_0'} = \hat{T}(\phi^{\pi_0'}|\pi_0')$, $\phi^{\pi_0'}$ is unique by Lemma 4 and 5. By Lemma 5 and the definition of an equilibrium, a belief π_d is an equilibrium cutoff if and only if the incentive constraint (9) holds at π_d, namely $\bar{\gamma}(\pi_d) + \eta\bar{s}(\pi_d)\phi_{d+1}^{\pi_d} \leq 0$. In STEP 1 below we argue that $\phi_{d+j}^{\pi_0'}$ strictly increases in π_0' at all $j \geq 1$, hence the set of prior belief π_0' that satisfies the IC constraint is a convex set. Then in STEP 2 we show that the set of π_d such that the incentive constraint holds is generically a singleton.

STEP 1: Suppose $\pi_0'' > \pi_0'$. By the definition of $\phi^{\pi_0'}$, $\phi^{\pi_0'} = \hat{T}(\phi^{\pi_0'}|\pi_0')$. As π_0' rises, π_j' strictly rises at all $j \geq 0$, and thus so do $\bar{s}(\pi_j')$ and $\bar{\gamma}(\pi_j')$. As $\bar{\gamma}(\pi_j')$ rises, $\hat{T}_j(\phi|\pi_0')$ rises strictly by (30). As $\bar{s}(\pi_j')$ rises, $\hat{T}_j(\phi|\pi_0')$ rises provided that ϕ_j is an increasing sequence. But as shown by Lemma 5, if $\phi^{\pi_0'}$ exists then $\phi_j^{\pi_0'}$ rises in j. Therefore $\hat{T}_j(\phi^{\pi_0'}|\pi_0'') > \hat{T}_j(\phi^{\pi_0'}|\pi_0') = \phi_j^{\pi_0'}$ at all $j \geq 1$. Moreover, as discussed after (29), $\hat{T}_j(\phi|\pi_0'')$ is increasing in ϕ and maps increasing sequence of ϕ_j back into an increasing sequence. Hence $\hat{T}_j^n(\phi^{\pi_0'}|\pi_0'')$ rises in n at all $j \geq 1$. Since $\hat{T}_j^n(\phi^{\pi_0'}|\pi_0'')$ is bounded above by ϕ_H, the limit $\lim_{n\to\infty} \hat{T}_j^n(\phi^{\pi_0'}|\pi_0'')$ exists and strictly exceeds $\phi_j^{\pi_0'}$ at all $j \geq 1$. But $\lim_{n\to\infty} \hat{T}^n(\phi^{\pi_0'}|\pi_0'')$ is a fixed point of $\phi = \hat{T}(\phi|\pi_0'')$ and hence is equivalent to $\phi^{\pi_0''}$, so we have $\phi_j^{\pi_0''} > \phi_j^{\pi_0'}$ at all $j \geq 1$.

STEP 2: Consider the set of prior $\Pi \subseteq [0,1]$ such that if $\pi' \in \Pi$ then $\phi_j^{\pi'}$ rises in j and the incentive constraint (9) holds at π'. If π_d is an equilibrium cutoff, then

$\pi_d \in \Pi$. The smallest element of Π, call it π^ℓ, is such that $\phi_0^{\pi^\ell} = \phi_1^{\pi^\ell} = 0$. It is unique because $\phi_1^{\pi'}$ strictly rises in π' by STEP 1. It is the smallest element of Π because any $\pi' < \pi^\ell$ will result in $\phi_1^{\pi'} < 0 = \phi_0^{\pi'}$ by STEP 1, and hence $\phi_j^{\pi'}$ is not increasing. The largest element of Π, π^h, is such that the first-order condition (6) holds at $\phi_0^{\pi^h}$, namely $0 = \phi_0^{\pi^h} = f[\bar{\gamma}(\pi^h) + \eta\bar{s}(\pi^h)\phi_1^{\pi^h}]$, or equivalently $\bar{\gamma}(\pi^h) + \eta\bar{s}(\pi^h)\phi_1^{\pi^h} = 0$. Any cutoff $\pi' > \pi^h$ will lead to $\bar{\gamma}(\pi') > \bar{\gamma}(\pi^h)$, $\bar{s}(\pi') > \bar{s}(\pi^h)$ and $\phi_1^{\pi'} > \phi_1^{\pi^h}$ (by STEP 1), hence $\bar{\gamma}(\pi') + \eta\bar{s}(\pi')\phi_1^{\pi'} > 0$. This inequality violates the incentive condition (9) and hence $\pi' \notin \Pi$. Since $\bar{\gamma}(\pi') + \eta\bar{s}(\pi')\phi_1^{\pi'}$ rises strictly in π', the value of π^h is unique.

Given an initial prior π_0, any equilibrium cutoff π_d must satisfy $\pi_d \in [\pi^\ell, \pi^h]$. But π^ℓ and π^h are exactly one signal away from each other, namely that if agents' belief is π^ℓ and a good signal arrives, then their posterior belief is π^h. It is because $\phi_1^{\pi^\ell} = 0$ by the definition of π^ℓ and the first-order condition binds at $\phi_1^{\pi^\ell}$, and that is exactly the condition the defines π^h. Hence given π_0, there is at most one j such that π_j lies in the interval $[\pi^\ell, \pi^h]$. Therefore there is only one possible π_d. If π_0 is such that $\pi_j = \pi^\ell$ and $\pi^h = \pi_{j+1}$ at some j, then π_j and π_{j+1} are both equilibrium cutoff. But this case is non-generic and only holds for measure zero of parameter values. ■

Lemma 6 *The mapping F in (12) maps $L \to L$.*

Proof. First F maps weakly increasing sequences into weakly increasing sequences because $\bar{\gamma}_j$ and $\bar{s}_j$ increase in j and f is an increasing function. Next, F is order-preserving, namely if $\phi' \geq \phi''$ then $F(\phi') \geq F(\phi'')$, because f is an increasing function.

Suppose $\phi'_j = \phi_H$ for all j. Then $F_j(\phi') \leq \phi_H$ for all j by the monotonicity of f and (8). Since F is order-preserving, for any $\phi \in L$, $0 \leq F_j(\phi) \leq F_j(\phi') \leq \phi_H$ for all j. Hence F maps L back into L. ■

Proof of Proposition 2. By the definition of an equilibrium, the price sequence ϕ^* is clearly a fixed point of F. Now we argue that it can be derive by iterating over F. Let $\phi_j^0 = 0$ for all $j \in \mathbb{Z}$, it is clearly the smallest element in L. Since $F : L \to L$ is order-preserving and L is a complete lattice, the limit $\lim_{n\to\infty} F^n(\phi^0)$ converges to the lowest fixed point of F by Tarski's fixed-point theorem and $F^n(\phi^0)$ rises in n by Theorem 3.2 in Cousot and Cousot (1979). Finally we argue that this fixed point must have a finite d such that $\phi_j = 0$ for all $j \leq d$. Since $\bar{\gamma}_{-\infty} < 0$, there exists j' such that $\bar{\gamma}_j + \eta\bar{s}_j\phi_H \leq 0$ $\forall j < j'$. Let $\hat{L}$ be a subset of L such that if $\phi \in \hat{L}$ then $\phi_j = 0$ $\forall j \leq j'$. We argue that F (12) maps $\hat{L} \to \hat{L}$ — if $\phi_j = 0$ for all $j \leq j'$, then $F_j(\phi) = 0$ for all $j \leq j'$ by (12). Since our seed sequence $\phi^0 \in \hat{L}$, the fixed point must have a $\pi_d \geq \pi_{j'} > 0$. ■

Proof of Lemma 2. Part (i): When $\bar{\phi}_j$ exists, by (14) its derivative with respect to γ_j is

$$\frac{d\bar{\phi}_j}{d\gamma_j} = \frac{f'(\gamma_j + \bar{\phi}_j)}{1 - f'(\gamma_j + \bar{\phi}_j)}.$$

Since $f'(\ell) < 1$ whenever f is convex and $\bar{\phi}_j$ is defined as the largest solution of (14), we have $1 - f'(\gamma_j + \bar{\phi}_j) > 0$ and hence $d\bar{\phi}_j / d\gamma_j > 0$. Since γ_j is linear in π_j by (2), $\bar{\phi}_j$ is increasing in π_j. When f is concave, the right side of the displayed equation falls in γ_j by the concavity of f and thus $\bar{\phi}_j$ is concave in π_j. Similarly, when f is convex, so is $\bar{\phi}_j$. Since f is concave-convex, so is $\bar{\phi}_j$

Since $f'(\ell) < 1$ whenever f is convex, if (14) has two solutions, then the smaller solution $\underline{\phi}_j$ must lie on the concave part of f. Using this observation, we can prove the desired claim on $\underline{\phi}_j$ by using a similar proof logic.

Part (ii): Easily ϕ_j is weakly convex at π_d because in equilibrium $\phi_{d+1} \geq \phi_d = \phi_{d-1} = 0$, and hence $\lambda_d \geq 0$. For the second claim, suppose $\phi_j \in (\underline{\phi}_j, \bar{\phi}_j)$. Then $f^{-1}(\phi_j) - \phi_j - \bar{\gamma}_j < 0$ by the definition of $\bar{\phi}_j$ and $\underline{\phi}_j$. By (7)

$$\bar{s}_j \phi_{j+1} + (1 - \bar{s}_j)\phi_{j-1} - \phi_j = \frac{f^{-1}(\phi_j) - \phi_j - \bar{\gamma}_j}{\eta} \tag{31}$$

and hence ϕ_j is convex at π_j. The last two claims are true by a similar proof logic. ■

Lemma 7 *If $\alpha = 0$, then the equilibrium price sequence ϕ_j is convex at all j.*

Proof. If $\alpha = 0$ then $f(\ell) = \beta\ell$. By (14) $\underline{\phi}_j = 0$ and $\bar{\phi}_j$ is an upward sloping straight connecting $(\pi^0, 0)$ and $(1, \phi_H)$ in the (π, ϕ) space where π^0 is defined as the belief where the expected dividend is zero, i.e. $\bar{\gamma}(\pi^0) = 0$. Since $\bar{\gamma}(\pi) > 0$ for all $\pi > \pi^0$, agents will not abandon the asset at any $\pi_j \geq \pi^0$. Hence $\pi_d < \pi^0$ and $\phi_j > \bar{\phi}_j$ for j close to d. The equilibrium price sequence ϕ_j cannot cut $\bar{\phi}_j$, for if ϕ_j cuts $\bar{\phi}_j$ from above then it must be concave afterwards by Lemma 2(b) and $\phi_j < \bar{\phi}_j$. Since ϕ_j becomes concave after the intersection, it cannot intersect with $\bar{\phi}_j$ again as $\bar{\phi}_j$ is a straight line. Hence ϕ_j cannot converge to ϕ_H as $\pi_j \to 1$, which leads to a contradiction. Therefore the price sequence must stay above $\bar{\phi}_j$ at all j and hence it is convex by Lemma 2(b). ■

Proof of Proposition 3.

Part (1): It suffices to prove that ϕ_j, as a function of π_j, can change from concave to convex at most once. Recall that $\bar{\phi}_j$ is increasing in π_j and is concave-convex. We first argue that ϕ_j cannot cut $\bar{\phi}_j$ from above when $\bar{\phi}_j$ is convex. Suppose it happens at some $\pi_{j'}$. After the intersection $\phi_{j'} < \bar{\phi}_{j'}$ and hence ϕ_j becomes concave by Lemma 2. Since $\bar{\phi}_j$ is convex and ϕ_j is concave, $\bar{\phi}_j > \phi_j$ for all $j \geq j'$ and the gap between them rises in j. Therefore $\bar{\phi}_j$ and ϕ_j cannot converge to the same limit as $\pi_j \to 1$.

Now suppose ϕ_j changes from concave to convex. By Lemma 2 it must be the case that ϕ_j cuts $\bar{\phi}_j$ from below. After the intersection, ϕ_j becomes convex. So if ϕ_j were to intersect with $\bar{\phi}_j$ again, then it must cut $\bar{\phi}_j$ at a point where $\bar{\phi}_j$ is convex. But this is impossible as argue above.

Part (2): Since ϕ_j rises and $\underline{\phi}_j$ falls in j, they can only cross once and if they cross then ϕ_j must cut $\underline{\phi}_j$ from below at some j'. For $j > j'$, we argue that $\bar{\phi}_j \geq \phi_j \geq \underline{\phi}_j$. Suppose not and $\phi_j > \bar{\phi}_j$ at some $j = j''$. Then the price sequence is convex at $\pi_{j''}$ by Lemma 2(b). By Lemma 2(a) $\bar{\phi}_j$ is concave in π when f is concave. Since ϕ_j is convex and $\bar{\phi}_j$ is concave, the gap $\phi_j - \bar{\phi}_j > 0$ is strictly increasing in j for $j \geq j''$ and hence ϕ_j and $\bar{\phi}_j$ cannot both converge to ϕ_H as $\pi \to 1$. This leads to a contradiction.

By a similar logic, if ϕ_j cuts $\bar{\phi}_j$ from above at some j' then $\bar{\phi}_j \geq \phi_j \geq \underline{\phi}_j$ $\forall j > j'$. Altogether ϕ_j can cross $\bar{\phi}_j$ or $\underline{\phi}_j$ at most once. By Lemma 2(b) ϕ_j is convex before the crossing and concave after the crossing, as shown in the left panel of Figure 5.

Part (3):Recall that we have assumed $f(0) = 0$ and $f'(\ell) \leq 1$ whenever f is convex. So if $f(\ell)$ is convex for $\ell \in [0, \ell_H]$, then $\bar{\phi}$ is increasing and convex and $\underline{\phi}$ does not exist. So the correspondence $\tilde{\phi}$ is an increasing and convex curve. The price ϕ cannot cut $\bar{\phi}$ from above because after the intersection ϕ would be concave and cannot intersect with $\bar{\phi}$ again. This leads to a contradiction as $\bar{\phi}_j$ and ϕ_j cannot converge to the same limit as $\pi_j \to 1$. So ϕ_j is either entirely convex, or entirely concave or concave and then convex.

But ϕ_j is initially convex and hence it must be entirely convex. To see this note that $\bar{\phi}(\gamma)$ solves $f(\phi + \gamma) = \phi$ and hence if $\bar{\phi}(\gamma) = 0$, then $\gamma = 0$, as $f(0) = 0$. At $\pi = \pi_d$, the value of the dividend γ_d must be strictly negative because otherwise agents will not abandon the asset. So the curve ϕ_j cuts the x-axis at some γ_d that is strictly smaller than the point where $\bar{\phi}$ cuts the x-axis. Equivalently, ϕ_j is above $\bar{\phi}_j$ initially and hence it is initially convex. ■

The following definitions are useful for the proof of Proposition 4.

Definition 2 *(a) Let $\underline{F}_j(\phi'', \phi'_\ell, \eta)$ be a sequence defined over $j = \ell, \ell+1, \ldots$. Assume $\underline{F}_\ell(\phi'', \phi'_\ell, \eta) = \phi'_\ell$,*

$$\underline{F}_{\ell+1}(\phi'', \phi'_\ell, \eta) = \max\{f[\bar{\gamma}_{\ell+1} + \eta \bar{s}_{\ell+1}\phi''_{\ell+2} + \eta(1 - \bar{s}_{\ell+1})\phi'_\ell + (1-\eta)\phi''_{\ell+1}], 0\},$$

and $\underline{F}_j(\phi'', \phi'_\ell, \eta) = F_j(\phi'', \eta)$ *for* $j \geq \ell + 2$ *where* F *is defined in (12) and* η *is added as an argument.*
(b) Let $\bar{F}_j(\phi'', \phi'_\ell, \eta)$ *be a sequence defined over* $j = \ell, \ell-1, \ldots$. *Assume* $\bar{F}_\ell(\phi'', \phi'_\ell, \eta) = \phi'_\ell$,

$$\bar{F}_{\ell-1}(\phi'', \phi'_\ell, \eta) = \max\{f[\bar{\gamma}_{\ell-1} + \eta \bar{s}_{\ell-1}\phi'_\ell + \eta(1 - \bar{s}_{\ell-1})\phi''_{\ell-2} + (1-\eta)\phi''_{\ell-1}], 0\}$$

and $\bar{F}_j(\phi'', \phi'_\ell, \eta) = F_j(\phi'', \eta)$ *for* $j \leq \ell - 2$.

We write $\underline{F}^2(\phi'', \phi'_\ell, \eta) = \underline{F}[\underline{F}(\phi'', \phi'_\ell, \eta), \phi'_\ell, \eta]$ and write $\underline{F}^n(\phi'', \phi'_\ell, \eta)$ if the mapping $\underline{F}$ is iterated for n times. We similarly define $\bar{F}^n(\phi'', \phi'_\ell, \eta)$.

Proof of Proposition 4.
Proof of Part (1): Since f is concave, by Proposition 3 the price sequence ϕ^*_j is convex-concave. Suppose ϕ^*_j changes from convex to concave at some $j = \rho$. First we claim ϕ'_j cannot cut ϕ^*_j from below at any $\nu \geq \rho$. Suppose not and assume $\phi'_{\nu-1} \leq \phi^*_{\nu-1}$ and $\phi'_\nu > \phi^*_\nu$. By definition $\underline{F}_j(\phi^*, \phi^*_{\nu-1}, \eta) = \phi^*_j$ for $j \geq \nu$. Since $\phi'_{\nu-1} \leq \phi^*_{\nu-1}$, $\underline{F}_j(\phi^*, \phi'_{\nu-1}, \eta) \leq \phi^*_j$ for $j \geq \nu$. As η rises $\underline{F}_j(\phi^*, \phi'_{\nu-1}, \eta)$ falls by (12) and because ϕ^*_j is concave for all $j \geq \nu - 1$ and $\phi'_{\nu-1} \leq \phi^*_{\nu-1}$. Therefore $\underline{F}_j(\phi^*, \phi'_{\nu-1}, \eta') \leq \phi^*_j$ for all $j \geq \nu$. Thus $\underline{F}^n_j(\phi^*, \phi'_{\nu-1}, \eta')$ falls in n as F is an order-preserving mapping. Since by definition $\underline{F}^n_j \geq 0$, the fixed point $\phi'' = \lim_{n\to\infty} \underline{F}^n(\phi^*, \phi'_{\nu-1}, \eta')$ exists and is lower than ϕ^*, namely $\phi'' \leq \phi^*$. But $\phi''_\nu \leq \phi^*_\nu < \phi'_\nu$ is impossible — this would imply ϕ''_j and ϕ'_j equals each other at $j = \nu - 1$, $\phi''_j < \phi'_j$ at $j = \nu$ and yet they both converge to ϕ_H as $j \to \infty$. By the proof logic of Lemma 4 if two distinct sequences intersect at some π_j, then they cannot both converge to ϕ_H. Hence ϕ'_j cannot cut ϕ^*_j from below at any belief larger than π_ρ.

Next ϕ' cannot cut ϕ^* from below at any $\nu < \rho$. Suppose not and assume $\phi'_{\nu-1} \leq \phi^*_{\nu-1}$ and $\phi'_\nu > \phi^*_\nu$. By a similar logic as the previous paragraph $\bar{F}_j(\phi^*, \phi'_\nu, \eta') \geq \phi^*_j$ for all $j < \nu$ and the inequality is strict at $j = \nu - 1$. Hence the fixed point $\phi'' = \lim_{n\to\infty} \bar{F}^n(\phi^*, \phi'_\nu, \eta') \geq \phi^*$ for all $j < \nu$ and $\phi''_{\nu-1} > \phi^*_{\nu-1}$. Therefore ϕ'' and ϕ' are two distinct sequences, i.e., $\phi''_{\nu-1} > \phi^*_{\nu-1} \geq \phi'_{\nu-1}$, but they intersect at $\phi'_\nu = \phi''_\nu$. This leads to a contradiction by Lemma 8 below. Altogether, since ϕ' cannot cut ϕ^* from below, it crosses ϕ^* at most once and must be from above.

Proof of Part (2): By Proposition 3, ϕ^*_j is convex-concave-convex. Suppose ϕ^*_j is concave for $j \in [j', j'']$ and otherwise convex. Using the proof logic of Part 1, one can prove the following claims.

1. ϕ'_j cannot cut ϕ^*_j from above at any $j > j''$.

2. ϕ'_j cannot cut ϕ^*_j from below at any $j < j'$.

3. If $\phi^*_i \geq \phi'_i$ and $\phi^*_k \geq \phi'_k$ where $j' \leq i < k \leq j''$, then $\phi^*_j \geq \phi'_j$ for all $j \in [i, k]$.

The third claim implies that, for $j \in [j', j'']$, the difference $\phi^*_j - \phi'_j$ has at most two sign changes with the sequence $-, +, -$ (including the special cases that the sign is always $-$, always $+$, $-, +$ and $+, -$). These three claims together prove the desired result. For example, if $\phi^*_j - \phi'_j \leq 0$ for all $j \in [j', j'']$, then ϕ' can only cut ϕ^*_j once and from above at some $j > j''$. So altogether $\phi^*_j - \phi'_j$ can change sign at most once from $+$ to $-$.

Proof of Part (3): The desired result follows immediately from part (3) of Proposition 3 and the first claim in Corollary 2. ■

Proof of Corollary 2. *Proof of the first claim:* Suppose ϕ^*_j is convex at all $j > d$. Consider the mapping $F(\phi, \eta)$ in (12). By Proposition 2 $\phi^* = F(\phi^*, \eta)$. Since ϕ^*_j is convex and $\eta' > \eta$, $\phi'_j = F_j(\phi^*, \eta') \geq \phi^*_j$ at all $j \in \mathbb{Z}$ by (12). Since $F(\phi, \eta')$ increases in ϕ and $F(\phi^*, \eta') \geq \phi^*$, we have $F^n(\phi, \eta')$ rises in n and therefore the fixed point $\lim_{n\to\infty} F^n(\phi, \eta') \geq \phi^*$.
Proof of the second claim: If ϕ^*_j is concave at all $j > d$ and $\bar{\gamma}_d + \eta' \bar{s}_d \phi^*_{d+1} \leq 0$, then $F_j(\phi^*, \eta') \leq \phi^*_j$ at all j. Since $F(\phi, \eta')$ increases in ϕ, $F^n(\phi^*, \eta')$ falls in n and therefore the fixed point $\lim_{n\to\infty} F^n(\phi, \eta') \leq \phi^*_j$. ■

Lemma 8 *Suppose ϕ'_j is an equilibrium price sequence with cutoff π'_d. Let $\phi''_j \geq 0$ be another sequence that satisfies the incentive constraint at cutoff π''_d and satisfies (7) at all beliefs above π''_d, but not necessarily converges to ϕ_H as $j \to \infty$. Then it is impossible for ϕ'' to coincide with ϕ' from above, namely $\phi''_{\nu-1} > \phi'_{\nu-1}$ and $\phi''_\nu = \phi'_\nu$ at some ν.*

Proof. For any given cutoff π_d, one can choose a value for ϕ_{d+1} and then use (7) iteratively to derive a sequence ϕ_j for $j > d$. Let $\iota(\pi_d)$ be the value of ϕ_{d+1} such that $\lim_{j\to\infty} \phi_j \to \phi_H$. By Lemma 4, $\iota(\pi_d)$ is unique. By the last paragraph of the proof of Proposition 1, $\iota(\pi_d) \geq 0$ if and only if $\pi_d \geq \pi^\ell$.

Since $\phi''_{\nu-1} > \phi'_{\nu-1}$ and $\phi''_\nu = \phi'_\nu$, by (7) we have $\phi''_{\nu+1} < \phi'_{\nu+1}$. Since $\phi''_\nu = \phi'_\nu$, $\phi''_{\nu+1} < \phi'_{\nu+1}$ and $\lim_{j\to\infty} \phi'_j = \phi_H$, by the proof logic of Lemma 4, as $j \to \infty$ the sequence ϕ''_j is lower than ϕ_H. Therefore $\iota(\pi''_d) > \phi''_{d+1}$.

If $\pi''_d \in [\pi^\ell, \pi^h]$, then $\pi''_d = \pi'_d$. But this is impossible because if $\phi''_{d+1} = \phi'_{d+1}$, then ϕ'' and ϕ' are the same sequence which is a contradiction with $\phi''_{\nu-1} > \phi'_{\nu-1}$. If $\phi''_{d+1} \neq \phi'_{d+1}$, then by Lemma 4 it is impossible that $\phi''_\nu = \phi'_\nu$. If $\pi''_d < \pi^\ell$, then $0 > \iota(\pi''_d)$. This implies

$\iota(\pi''_d) < \phi''_{d+1}$ which is a contradiction. Finally if $\pi''_d > \pi^h$, then $\iota(\pi''_d) > 0$ and hence there is a sequence that starts from π''_d and converges to ϕ_H. This sequence does not intersect with ϕ' because if they cross each other then they cannot both converge to ϕ_H by the proof logic of Lemma 4. Since $\iota(\pi''_d) > \phi''_{d+1}$, by Lemma 4, ϕ'' is lower than the sequence that starts from π''_d and converges to ϕ_H. Hence $\phi''_j < \phi'_j$ at all j and it is impossible that $\phi'_\nu = \phi''_\nu$. ■

Proof of Proposition 5. We prove the claim under the special case that f is concave, the proof logic for the general case is similar. When f is concave, the disclosure policy is a bang-bang solution — there is a cutoff j' such that the optimal policy is to reveal maximal amount of information for $j \leq j'$ and suppress information when $j > j'$. We will show that given any disclosure policy that is not a bang-bang solution, we can find a bang-bang policy that leads to higher prices at all states. This implies a bang-bang policy can generate a higher ϕ_0 or a smaller π_d.

Consider a mapping $\tilde{F}$ that allows η to be state-contingent:

$$\tilde{F}_j(\phi) \equiv \max\{f[\bar{\gamma}_j + \eta_j \bar{s}_j \phi_{j+1} + \eta_j(1-\bar{s}_j)\phi_{j-1} + (1-\eta_j)\phi_j], 0\}. \tag{32}$$

By the logic leading to Proposition 2, an equilibrium price sequence is a fixed point of $\tilde{F}$. Let ϕ^* be the equilibrium price solving $\phi^* = \tilde{F}(\phi^*)$ where the choice of $\{\eta_j\}_{j=-\infty}^{\infty}$ is optimal. If ϕ^* is convex at π_i and $\eta_i < \bar{\eta}$, then an increase in η_i strictly raises $\tilde{F}_i(\phi^*)$ by (32). Since the mapping $\tilde{F}$ is order-preserving, $\tilde{F}^n_k(\phi^*)$ weakly rises as η_i rises to $\bar{\eta}$, at all $k \in \mathbb{Z}$ and $n \geq 1$. Since prices are bounded above by ϕ_H, the limit $\phi' = \lim_{n\to\infty} \tilde{F}^n(\phi^*)$ exists and is a fixed point of the new mapping with $\eta_i = \bar{\eta}$. Since $\phi'_i \leq \phi_H$, $\phi'_i > \phi^*_i$ and $\lim_{j\to\infty} \phi^*_j = \phi_H$, we have $\lim_{j\to\infty} \phi'_j = \phi_H$. The fixed point ϕ' has a cutoff $\pi'_d \in (0,1)$ by the same proof logic of Proposition 2. Therefore ϕ' is an equilibrium price sequence. Since $\phi'_j > \phi^*_j$, the original choice of $\{\eta_j\}_{j=-\infty}^{\infty}$ cannot be an optimal disclosure policy. It follows that if ϕ^* is convex at π_j then the optimal policy requires $\eta_j = \bar{\eta}$. Similarly, if the price sequence is concave at π_j, then a decrease in η_j to $\underline{\eta}$ raises the entire price sequence. Since any equilibrium price sequence is convex-concave by Proposition 3 (the proof of Proposition 3 does not require η_j to be constant in j), the optimal policy must be a bang-bang solution — $\eta_j = \bar{\eta}$ for large j and $\eta_j = \underline{\eta}$ for all smaller j.

Finally, if f is convex, then by the proof logic of part (3) of Proposition 3 the information premium is always weakly positive. Therefore revealing information always raises the asset price ϕ_j.

■

Proof of Corollary 3. The first claim follows directly from the proof of Proposition 5. Hence η^P and η^I are both bang-bang policies. Assume these two policies are different. Let ϕ^P be the price under the disclosure policy η^P and let ϕ^I be that under η^I. Let π_d be the cutoff for abandoning the asset under policy η^P. By the definition of η^P and η^I, $\phi^P_{d+1} > \phi^I_{d+1} = 0$ and $\phi^P_0 < \phi^I_0$. Hence ϕ^P and ϕ^I are not rank ordered. Let F_I be the mapping in (12) under the disclosure policy η^I. Recall that η^P and η^I are not the same and ϕ^P_j is concave if and only if $j > j^*$. Hence $F_I(\phi^P) \leq \phi^P$. Since F_I is order-preserving, $F^n_I(\phi^P)$ falls in n, the limit $\lim_{n\to\infty} F^n_I(\phi^P)$ exists and is smaller than ϕ^P. Since the equilibrium price sequence is unique, it must be $\phi^I = \lim_{n\to\infty} F^n_I(\phi^P)$. But this implies $\phi^P \geq \phi^I$ which leads to a contradiction. ■

Proof of Lemma 3. We will prove the lemma by showing the following claim: For $j \geq 0$, let Ψ^H_{d+j} be

$$\Psi^H_{d+j} = \frac{1}{\beta\eta(1-s)}\left[\Sigma_{t=0}^{j-1}\frac{m_1^j}{m_2-m_1}\left(\frac{m_2}{m_1^t}-\frac{m_1}{m_2^t}\right)\varepsilon_{d+t+1} + \Sigma_{t=j}^{\infty}\frac{m_1(m_2^j-m_1^j)}{(m_2-m_1)m_2^t}\varepsilon_{d+t+1}\right]$$

where $m_2 > m_1$ are the roots of the characteristic equation

$$\frac{(1-s)}{s} - \frac{1-\beta(1-\eta)}{\beta\eta s}z + z^2 = 0.$$

Define Ψ^L_{d+j} similarly but with s replaced by $1-s$. For $j > 0$, the solution of Ω_{d+j} is equal to $\Omega_d + \pi_{d+j}\Psi^H_{d+j} + (1-\pi_{d+j})\Psi^L_{d+j}$.

To see the above, suppose the modeler knows the true state of the world is H, denote the discounted sum of ε_j by Ψ^H_j which is the solution of the second-order difference equation

$$\Psi^H_j = \varepsilon_j + \beta[\eta s\Psi^H_{j+1} + \eta(1-s)\Psi^H_{j-1} + (1-\eta)\Psi^H_j].$$

Similarly, define Ψ^L_j as the discounted sum of ε_j when the true state is known to be L. This sum solves the difference equation

$$\Psi^L_j = \varepsilon_j + \beta[\eta(1-s)\Psi^L_{j+1} + \eta s\Psi^L_{j-1} + (1-\eta)\Psi^L_j].$$

Now, if the modeler does not know the true state, then the expected discounted sum of ε_j is $\Psi_j = \pi_j\Psi^H_j + (1-\pi_j)\Psi^L_j$. The welfare at state j is given by $\Omega_{d+j} = \Omega_d + \Psi_{d+j}$ where

Ω_d captures the CM trade surpluses. We can solve the second-order non-homogenous equation for Ψ_j^H and Ψ_j^L by using standard techniques. We will illustrate how to derive the solution for Ψ^H below.

The equation for Ψ^H can be rewritten as a linear nonhomogeneous second-order difference equation

$$\frac{1-s}{s}\Psi_j^H - \frac{1-\beta(1-\eta)}{\beta\eta s}\Psi_{j+1}^H + \Psi_{j+2}^H = \tilde{\varepsilon}_j$$

where $\tilde{\varepsilon}_j \equiv -\varepsilon_{j+1}/(\beta\eta s)$. The characteristic equation is $(1-s)/s - [1-\beta(1-\eta)]/(\beta\eta s)z + z^2 = 0$ and the solutions are

$$m_1 = \frac{1-\beta(1-\eta)}{2s\beta\eta} - \sqrt{\left(\frac{1-\beta(1-\eta)}{2s\beta\eta}\right)^2 - \frac{1-s}{s}}$$

and

$$m_2 = \frac{1-\beta(1-\eta)}{2s\beta\eta} + \sqrt{\left(\frac{1-\beta(1-\eta)}{2s\beta\eta}\right)^2 - \frac{1-s}{s}}.$$

It is easy to check that $m_2 > 1 > m_1 > 0$. Therefore the homogenous part of the solution of the difference equation takes the form $Am_1^j + Bm_2^j$. It can be verified that the overall solution takes the form

$$\Psi_{d+j}^H = \frac{s}{1-s}\left[\Sigma_{t=0}^{j}\frac{m_2}{m_1-m_2}(m_1^{j-t} - m_2^{j-t})\tilde{\varepsilon}_{d+t} + \Sigma_{t=j}^{\infty}m_2^{j-t}\tilde{\varepsilon}_{d+t} + Am_1^j + Bm_2^j\right].$$

The constant A is chosen such that $\Psi_d = 0$ and B is chosen such that $\lim_{j\to\infty}\Psi_{d+j}$ is finite. Since $m_2 > 1$, the only value of B that makes the limit of Ψ_j finite is

$$B = \frac{m_2}{m_1-m_2}\Sigma_{t=0}^{\infty}\frac{\tilde{\varepsilon}_{d+t}}{m_2^t}.$$

To ensure $\Psi_d = 0$, the value of A is

$$A = -\frac{m_1}{m_1-m_2}\Sigma_{t=0}^{\infty}\frac{\tilde{\varepsilon}_{d+t}}{m_2^t}.$$

Substituting the value of A and B into the solution of Ψ_{d+j}^H results in the desired formula. The method for solving Ψ_j^L is similar. ■

Proof of Proposition 6. *Part (1)*: We first characterize the change in Ω_j induced by the change in ε_j. If $f(\ell)$ is concave in ℓ for $\ell \in [0, \ell_H]$ and $f(0) = 0$, then $f(\ell) > \ell$ in the relevant region. Therefore $f(\phi_j + \bar{\gamma}_j) > \phi_j$ at each j. So the information premium must be negative to balance the first-order condition $f(\phi_j + \bar{\gamma}_j + \eta\lambda_j) = \phi_j$. It follows that ϕ_j is concave. By the proof logic of Proposition 4 the price ϕ_j falls in η and hence ε_j falls in η. This effect reduces Ω_j as Ω_j is a weighted sum of ε_j by Lemma 3.

Next, we characterize the change in Ω_j induced by more rapid transitions across states. Note that (15) defines Ω_j in a way that is similar to how ϕ_j is defined in (6). Indeed we can rewrite (15) as

$$\Omega_j = \hat{f}\{(2[U(x^*) - x^*] + \varepsilon_j)/\beta + \eta\bar{s}_j\Omega_{j+1} + \eta(1-\bar{s}_j)\Omega_{j-1} + (1-\eta)\Omega_j\}$$

where $\hat{f}(x) \equiv \beta x$. Therefore we can use our results in Proposition 3 and 4 to characterize Ω_j. Let $\bar{\Omega}_j$ be the counterpart of $\bar{\phi}_j$ such that it solves

$$\bar{\Omega}_j = \hat{f}\{(2[U(x^*) - x^*] + \varepsilon_j)/\beta + \bar{\Omega}_j\} \iff \bar{\Omega}_j = \frac{2[U(x^*) - x^*] + \varepsilon_j}{1-\beta}.$$

Therefore the curvature of $\bar{\Omega}_j$ is determined by that of ε_j. Since ϕ_j is concave, one can show that ℓ_j is also concave in π_j. Since $c'(q_j)q_j = \ell_j$, we can show $u(q_j) - c(q_j)$ is concave in ℓ_j by using the convexity of $c'(q_j)q_j$. Since $\varepsilon_j \equiv \bar{\gamma}_j + \beta\alpha[u(q_j) - c(q_j)]$ is a concave transform of ℓ_j, we know ε_j is concave in π_j, and thus so is $\bar{\Omega}_j$. Since $\bar{\Omega}_j$ is concave, by the proof logic of Proposition 3 Ω_j is also concave in π_j. Since Ω_j is concave in π_j, fixing ε_j, an increase in η reduces Ω_j by the proof logic of Proposition 4.

Part (2): Since $u(q_j) - c(q_j)$ is an increasing and concave transform of ϕ_j, if $u(q_j) - c(q_j)$ is convex in π_j, then ϕ_j must be convex in π_j. Therefore an increase in η raises ϕ_j by Corollary 2. Therefore ε_j increases in η and Ω_j increases.

By the same proof logic as part 1, if $u(q_j) - c(q_j)$ is convex in π_j then ε_j and $\bar{\Omega}_j$ are convex in π_j. Therefore Ω_j is convex in π_j by the proof logic of Proposition 3. Hence, fixing ε_j, an increase in η raises Ω_j by the proof logic of Proposition 4. ∎

Proof of Corollary 4. When $c(q) = q$, sellers make no surplus in trade and hence $u(q_j) - c(q_j) = S(\ell_j)$. By part 2 of Proposition 6, it suffices to show that $S(\ell_j)$ is convex in π_j. Observe that when $\eta \approx 0$, $\ell_j \approx \bar{\gamma}_j + \phi_j \approx \bar{\gamma}_j + \bar{\phi}(\bar{\gamma}_j)$. Hence if $S[\bar{\gamma}_j + \bar{\phi}(\bar{\gamma}_j)]$ is convex in π_j, then $S(\ell_j)$ is convex in π_j when η is small. Since $\bar{\gamma}_j$ is linear in π_j, it suffices to show $S[\gamma + \bar{\phi}(\gamma)]$ is convex in γ whenever $\gamma + \bar{\phi}(\gamma)$ is between ℓ_L and ℓ_H. But since we can choose γ_L and γ_H, we can effectively choose the relevant region of ℓ. So it suffices to show that there is an open interval such that if $\gamma + \bar{\phi}(\gamma)$ is in this interval then $S[\gamma + \bar{\phi}(\gamma)]$ is convex in γ.

Denote $\ell(\gamma) = \gamma + \bar{\phi}(\gamma)$. By (14) and the definition of f,

$$\frac{dS[\ell(\gamma)]}{d\gamma} = \frac{u'(\ell) - 1}{1 - f'(\ell)} = \frac{tA + (1-t)\tilde{u}'[\ell(\gamma)] - 1}{1 - \beta - \alpha[tA + (1-t)\{\tilde{u}'[\ell(\gamma)] + \ell(\gamma)\tilde{u}''[\ell(\gamma)]\} - 1]}.$$

Since $q\tilde{u}'(q)$ is convex, the expression $\tilde{u}'[\ell(\gamma)] + \ell(\gamma)\tilde{u}''[\ell(\gamma)]$ in the denominator rises in ℓ. By choosing A we can make the denominator positive but arbitrarily close to 0. In this case the entire expression rises in ℓ, implying that $S[\ell(\gamma)]$ is convex in γ. ■

Proof of Proposition 7. Given π_d and ψ^*, by the same proof logic of Lemma 4, there is at most one sequence $\{\hat{\tau}_j\}_{j=d}^{\infty}$ such that $\hat{\tau}_d = \psi^*$, $\lim_{j\to\infty} \hat{\tau}_j = \tau_H$ and satisfies (18) for all $j > d$. Now we argue $\hat{\tau}_j$, if it exists, must be either rising or U-shaped. Let $\pi_k > \pi_d$ be the smallest state such that $\hat{\tau}_j$ becomes concave. Note that since $\hat{\tau}_j$ is potentially non-monotone, we cannot use the argument in Proposition 3 to argue that it is convex-concave. But since $\hat{\tau}_j$ is convex at $j = d$, we can always find $k > d$ (k is potentially infinite) such that $\hat{\tau}_j$ becomes concave at state π_k.

For $j < k$, the sequence is convex and hence it is either rising or U-shaped in j. For $j \geq k$, we show below that $\hat{\tau}_j$ rises in j. Define a mapping $G_j : \mathbb{R}^\infty \to \mathbb{R}$ for $j = k, k+1, \ldots$. Assume

$$G_j(\tau) \equiv f(\delta + \bar{\gamma}_j + \eta \bar{s}_j \tau_{j+1} + \eta(1 - \bar{s}_j)\tau_{j-1} + (1-\eta)\tau_j) \tag{33}$$

for $j > k$ and $G_k(\tau) = \hat{\tau}_k$. Let τ^0 be a seed sequence with $\tau_j^0 = \hat{\tau}_k$ for all $j \geq k$. Let L_k be the set of weakly increasing sequence $\{\tau_j\}_{j=k}^{\infty}$ that are bounded between $[\hat{\tau}_k, \tau_H]$. The set L_k is a complete lattice and by the proof logic of Lemma 6 the mapping G maps elements in L_k back into L_k. Since $\hat{\tau}_j$ is concave at π_k, $\hat{\tau}_k \leq f(\delta + \bar{\gamma}_j + \hat{\tau}_k)$ for all $j \geq k$ because $\bar{\gamma}_j$ rises in j. Hence $\hat{\tau}_k \leq G_j(\tau^0)$ for all $j \geq k$ by (33), and thus $G^n(\tau^0)$ increases in n because G is order-preserving. Hence there is a fixed point $\tau^* = G(\tau^*) = \lim_{n\to\infty} G^n(\tau^0)$ which is an weakly increasing sequence. But there can only be one sequence starting from $\hat{\tau}_k$ and converges to τ_H by Lemma 4. Hence $\tau^* = \hat{\tau}$ and $\hat{\tau}_j$ must be weakly rising for $j \geq k$. Given that $\hat{\tau}_j$ is weakly rising for $j \geq k$ and $\lim_{j\to\infty} \tau_j = \tau_H$, we can use the proof logic of Proposition 3 to argue that it is convex-concave for $j \geq k$. Therefore, for $j \geq d$, τ_j is convex-concave and it is weakly rising when it becomes concave.

Since τ_j is either rising or U-shaped in j for $j \geq d$, we can show it attains its global maximum at $\tau_H \equiv \lim_{\to\infty} \tau_j$ by showing $\tau_H > \tau_d$. As explained after (19), τ_H is the unique solution of $\tau_H = f(\delta + \bar{\gamma}_\infty + \tau_H)$. The total liquidity at the cutoff $\tau_d = \psi^*$ is the unique solution of $\psi^* = f(\delta + \psi^*)$. Since we have assumed $\bar{\gamma}_\infty > 0$ and f is an increasing function, $\tau_H > \psi^*$.

Next we show ϕ_j rises and ψ_j falls in j. Let m be the smallest state where $\tau_j \equiv \phi_j + \psi_j$ turns from falling into rising. If τ_j is monotone in j then $m = -\infty$. Given π_d, $\phi_d = 0$ and the sequence τ, one can pick a ϕ_{d+1} and use (17) to generate a sequence $\{\phi_j\}_{j=d+1}^{\infty}$. There is a unique ϕ_{d+1} such that $\lim_{j\to\infty} \phi_j = \phi_H$ by a proof logic similar to that of Lemma 4. Since the sequence $\{\phi_j\}_{j=d+1}^{\infty}$ is unique, so is $\psi_j = \tau_j - \phi_j$. Since the sequence ψ is unique, it can be derived by an iterative method which we will now characterize and use it to show ψ_j falls in j. Define the right side of (16) as a mapping H_j for $j \geq m$ by

$$H_j(\psi|\tau, \pi_d) = \beta[\delta + \eta\bar{s}_j\psi_{j+1} + \eta(1-\bar{s}_j)\psi_{j-1} + (1-\eta)\psi_j](1 + S'(\ell_j)). \tag{34}$$

Note that ℓ_j only depends on τ via (18), namely $\ell_j = f^{-1}(\tau_j)$, and is independent of ψ. Let ψ^0 be a seed sequence with $\psi_j^0 = 0$ for all $j \geq m$. The equilibrium sequence of ψ can be computed by $\psi = \lim_{n\to\infty} H^n(\psi^0|\tau, \pi_d)$. Since $S'(\ell_j)$ falls in j for $j \geq m$, H_j maps any decreasing sequence of ψ into a decreasing sequence. Therefore the fixed point ψ_j falls in j for $j \geq m$. Since τ_j rises in j for $j \geq m$, $\phi_j = \tau_j - \psi_j$ rises in j for $j \geq m$.

For $j < m$, we compute ϕ by defining a mapping I_j by the right side of (17):

$$I_j(\phi|\tau, \pi_d) = \beta[\bar{\gamma}_j + \eta\bar{s}_j\phi_{j+1} + \eta(1-\bar{s}_j)\phi_{j-1} + (1-\eta)\phi_j](1 + S'(\ell_j)).$$

Since $S'(\ell_j)$ rises in j for $j \geq m$, by a logic similar to the case with $j \geq m$ we can conclude that ϕ_j rises in j for $j \geq m$. It follows that $\psi_j = \tau_j - \phi_j$ falls in j for $j \geq m$.

So far we have shown that given π_d, one can uniquely solve for τ, ϕ and ψ. Finally we argue that there is a unique π_d that can satisfy the incentive condition (10). Given π_d, one can solve for the corresponding sequence of τ, ϕ and ψ. By STEP 1 in the proof of Proposition 1, τ_{d+j} rises in π_d for all $j \geq 1$, and hence $S'(\ell_{d+j})$ falls in π_d at each $j \geq 1$. Since $S'(\ell_{d+j})$ falls in π_d, $H_{d+j}(\psi|\tau, \pi_d)$ in (34) falls in π_d for any $j \geq 1$. Therefore the sequence ψ_{d+j} falls in π_d at each $j \geq 1$. Since τ_{d+j} rises and ψ_{d+j} falls in π_d, $\phi_{d+j} = \tau_{d+j} - \psi_{d+j}$ rises in π_d. Then by the argument in STEP 2 in the proof of Proposition 1, there is a unique π_d such that incentive constraint (10) is satisfied. ■

Lemma 9 is used to prove Proposition 8. Let $\Gamma_j \equiv u(q_j) - c(q_j)$.

Lemma 9 *The sequence* $\xi_j \equiv \bar{\gamma}_j + \beta\alpha\Gamma_j$ *single-crosses* $\theta \equiv \beta\alpha\Gamma_d$ *from below as* j *increases from* $j = d$ *to* ∞ *and* ξ_j *rises in* j *whenever* $\xi_j > \theta$.

Proof. By Proposition 7 and (18) $\ell_j = f^{-1}(\tau_j)$ is U-shaped in j in general (including the special case that τ_j rises in j). Therefore the trade surplus Γ_j is U-shaped in j for $j \geq d$. Since $\bar{\gamma}_\infty > 0$, we have $\ell_\infty > \ell_d$ and hence $\Gamma_\infty > \Gamma_d$. At π_d, the expected payoff $\bar{\gamma}_d < 0$ because otherwise the agents would not abandon the asset at π_d. By (2) $\bar{\gamma}_j$ rises in j and becomes strictly positive as j explodes.

Consider the region where $\bar{\gamma}_j < 0$. If ξ_j crosses θ for the first time at some $\hat{j}$ (i.e. $\xi_j \leq \theta$ at $j = \hat{j} - 1$ and $\xi_j > \theta$ at $\hat{j}$), then $\Gamma_j > \Gamma_d$ and hence Γ_j must be increasing in j for $j \geq \hat{j}$. Therefore ξ_j must be increasing for $j \geq \hat{j}$ and hence it single-crosses θ and is increasing afterwards.

Next consider $\bar{\gamma}_j \geq 0$. If τ_j is convex at some state π_j, then it must be the case that $\tau_j > \bar{\tau}$ where $\bar{\tau}$ is the unique solution of $\bar{\tau} = f(\delta + \bar{\gamma}_j + \bar{\tau})$. But $\bar{\tau} > \psi^* = \tau_d$ because ψ^* by definition solves $\psi^* = f(\delta + \psi^*)$ and $\bar{\gamma}_j \geq 0$. Since $\tau_j > \bar{\tau} \geq \tau_d$, we have $\Gamma_j > \Gamma_d$. Hence $\xi_j > \theta$ and no crossing can happen when τ_j is convex. If τ_j is linear or concave at j, then τ_j rises in j for all larger j because τ_j is convex-concave and it rises in j whenever it is concave (both claims are explained in the proof of Proposition 7). Therefore the sum ξ_j is increasing in j and hence ξ_j single-crosses θ. ■

Proof of Proposition 8. *Proof of the first claim*: By Lemma 9 ξ_j single-crosses θ from below as j increases from $j = d$ to ∞. Let $\pi_{j'}$ be the state where ξ_j single-crosses θ. Since ξ_j rises in j for $j \geq j'$, Ω_j rises in j for $j \geq j'$. To see this, one can define the right side of (21) as a function that maps a sequence of $\{\Omega_j\}_{j=d}^{\infty}$ into another sequence. Since ξ_j rises in j for $j \geq j'$, this function maps increasing sequences into increasing sequences. Hence by standard results (i.e. contraction mapping theorem), the fixed point of the mapping is also an increasing sequence for $j \geq j'$.

For $j < j'$, if Ω_j crosses Ω_d at $j = j''$ (i.e. $\Omega_{j''-1} \leq \Omega_d$ and $\Omega_{j''} > \Omega_d$), then Ω_j must be convex at j'' because by subtracting (20) from (21)

$$\Omega_{j''} - \Omega_d = \frac{1}{1-\beta}\left\{\xi_{j''} - \beta\alpha\Gamma_d + \eta[\bar{s}_{j''}\Omega_{j''+1} + (1 - \bar{s}_{j''})\Omega_{j''-1} - \Omega_{j''})]\right\},$$

and $\xi_{j''} \leq \theta \equiv \beta\alpha\Gamma_d$ as $j'' < j'$ and $\Omega_{j''} - \Omega_d > 0$ after Ω_j crosses Ω_d. Therefore Ω_j is convex and increasing at $j = j''$. By a similar logic, Ω_j is convex and increasing at all $j \in \{j'', j'' + 1, \ldots, j' - 1\}$. Hence, altogether, for $j \geq d$, Ω_j can only single-cross Ω_d. Note that we have also shown that Ω_j rises in j whenever $\Omega_j \geq \Omega_d$.

Proof of Part (i): If τ_j rises in j for $j \geq d$, then by (21) and the contraction mapping theorem Ω_j also rises in j. Hence $\pi^* = \pi_d$.

Proof of Part (ii): We first show that $\Omega_{d+1} < \Omega_d$ when $\bar{\gamma}_\infty \approx 0$ (i.e. $\gamma_H \approx -(1-s)\gamma_L/s$). As $j \to \infty$, by (21)

$$\Omega_\infty = 2[U(x^*) - x^*] + \delta + \bar{\gamma}_\infty + \beta(\alpha\Gamma_j + \Omega_\infty).$$

By substituting $j = d+1$ into (21) and using $\Gamma_\infty \geq \Gamma_{d+1}$ and $\Omega_\infty \geq \Omega_{d+2}$,

$$\Omega_{d+1} \leq 2[U(x^*)-x^*]+\delta+\bar{\gamma}_{d+1}+\beta[\alpha\Gamma_\infty)+\eta\bar{s}_{d+1}\Omega_\infty+\eta(1-\bar{s}_{d+1})\Omega_d+(1-\eta)\Omega_{d+1}]. \quad (35)$$

By (20) and (35), $\Omega_{d+1} < \Omega_d$ when

$$\begin{aligned} 0 &> (1-\bar{s}_{d+1})\gamma_L + \bar{s}_{d+1}\gamma_H + \beta\alpha(\Gamma_\infty - \Gamma_d) + \beta\eta\bar{s}_{d+1}(\Omega_\infty - \Omega_d) \\ &= (s-\bar{s}_{d+1})(\gamma_L - \gamma_H) + \bar{\gamma}_\infty + \beta\alpha(\Gamma_\infty - \Gamma_d) + \beta\eta s(\Omega_\infty - \Omega_d). \end{aligned} \quad (36)$$

We argue that (36) holds when $\bar{\gamma}_\infty$ is sufficiently small. Suppose $\bar{\gamma}_\infty \downarrow 0$ such that $\ell_\infty \to \ell_d$ and $\Omega_\infty \to \Omega_d$. Therefore the last three terms on the right side of (36) vanishes. In this case the cutoff π_d must be strictly smaller than 1. For suppose $\pi_d = 1$, then $\bar{\gamma}_\infty + \eta s\phi_H > 0$ and hence the agent will not abandon the asset at π_d which leads to a contradiction. Therefore $\pi_d < 1$ and $s - \bar{s}_{d+1} \equiv \nu > 0$, and (36) holds strictly. By continuity there exists $\hat{\gamma}_H > -(1-s)\gamma_L/s$ such that if $\gamma_H \in (-(1-s)\gamma_L/s, \hat{\gamma}_H)$, then $\Omega_{d+1} < \Omega_d$ and Ω_j is non-monotone in j.

Finally, suppose $\Omega_j \geq \Omega_d$ for all $j \geq d$. By Lemma 10, an increase in γ_H raises all τ_j and thus Ω_j increases at each $j \geq d$. Hence $\Omega_j \geq \Omega_d$ for all $j \geq d$ and $\pi^* = \pi_d$. Since $\Omega_j \geq \Omega_d$ at all $j \geq d$, Ω_j rises in j for $j \geq d$ as mentioned before the proof of Part (i). Therefore there is a $\tilde{\gamma}$ such that Ω_j is increasing in j for all $\gamma_H > \tilde{\gamma}$. Since Ω_j is non-monotone when $\gamma_H \approx -(1-s)\gamma_L/s$, we know $\tilde{\gamma} > -(1-s)\gamma_L/s$. ■

Lemma 10 *In a dual-asset economy, as γ_H rises, π_d falls. The price ϕ_j and τ_j fall and ψ_j rises at all $j \geq d$.*

Proof. Given any arbitrary π_d, there is at most one sequence of τ_j such that $\lim_{j\to\infty} \tau_j = \tau_H$, $\tau_d = \psi^*$ and satisfies (18) at each $j > d$. As discussed in the proof of Proposition 7, this sequence is given by $\lim_{n\to\infty} G^n(\tau^0)$ where G is given by (33). This sequence increases in γ_H because the right side of (33) rises in γ_H. Since τ_j rises in γ_H for each $j \geq d$, so does ℓ_j by (18). By the concavity of S, $S'(\ell_j)$ falls in γ_H. Then by the mapping H in

(34), the sequence of ψ_j falls in γ_H at each $j \geq d$. Since τ_j rises and ψ_j falls, $\phi_j = \tau_j - \psi_j$ rises at each $j \geq d$.

Since ϕ_{d+1} and $\bar{\gamma}_d$ rise in γ_H, the right side of the IC constraint (10) rises in γ_H. By the proof logic of STEP 2 in the proof of Proposition 1, the cutoff π_d must weakly falls in γ_H. Since the equilibrium π_d cutoff weakly falls in γ_H, the equilibrium sequence of τ_j rises in γ_H at all $j \geq d$ by STEP 1 in the proof of Proposition 1. ■

Proof of Proposition 9. Since $\pi_0^h > \tilde{\pi}_d$, hodlers are willing to hold some asset, therefore $\pi_0^r > \pi_d^r$. At any state $\pi_j^r > \pi_d^r$, hodlers hold all assets if and only if

$$\bar{\ell}_j(\pi_j^h) \geq \bar{\ell}_j(\pi_j^r)[1+\alpha\theta/(1-\theta)] \iff (\pi_j^h - \pi_j^r) \geq \frac{\alpha\theta\bar{\ell}_j(\pi_j^r)}{(1-\theta)(\gamma_H - \gamma_L + \phi_{j+1} - \phi_{j-1})}. \quad (37)$$

The gap $\pi_j^h - \pi_j^r > 0$ is hump-shaped in j (see Figure 13) and it vanishes as $j \to -\infty$ or $+\infty$. Since the denominator in the right side of (37) is bounded below by $(1-\theta)[\gamma_H - \gamma_L]$ and $\bar{\ell}_j(\pi_j^r)$ rises in α, there exists $\underline{\alpha} > 0$ such that (37) holds at π_0^r if $\alpha < \underline{\alpha}$. As $j \to +\infty$, the right side is $\alpha\theta(\bar{\gamma}_\infty + \phi_H)/[(1-\theta)(\gamma_H - \gamma_L)] > 0$ and the left side vanishes. Hence (37) fails as j explodes and regular buyers will hold a positive fraction of assets. ■

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Online Appendix for "Learning and Money Adoption"

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A News Indexes

We follow Biais et al. (2018)'s approach to construct news indexes for bank notes and crypto-currencies. In this section we explain the details of the construcition. Our list of news is available upon request.

Banque Royale Notes News Indexes

We collect news concerning Banque Royale and Mississippi Company's merge and acquisition activities, business and operation, legal tender status of various currencies in France and policy changes. We manually classify each of them as a positive or a negative news. A positive news is coded as $+1$ and a negative news as -1, and the news index is the cumulative sum of all events over time. For example, Banque Royale's notes were given legal tender throughout France on Jan 28, 1720. We consider this as a positive news and hence news index increased by 1. From May 1719 to Jan 1721, there are 31 historical events obtained from Velde (2003) and Sandrock (2013). Market capitalization of total notes issued is taken from Table 4 of Velde (2003).

Crypto-currency News Indexes

For crypto-currencies we collect news concerning three types of events – (1) the usefulness of the crypto-currency as a means of payment, (2) the ease with which it can be exchanged, and (3) the security of it. We manually collect news about two crypto-currencies: Bitcoin and BitConnect. Bitcoin events were collected from reddit Bitcoin forum, 99Bitcoins's this week in Bitcoin on Youtube, Legality of Bitcoin by country or territory and ones documented in Table 3-5 of Biais et al. (2018). In total we identified 224 Bitcoin events from Sep 2010 to Jul 2019. We use the daily Bitcoin adjusted closing price from Yahoo Finance. Since most of our self-collected events occurred after Oct

2017, we focus on the sample period from Nov 2017 to Jul 2019 in Figure 2. In contrast to the success of Bitcoin, BitConnect was abandoned by investors, i.e. it has very low trading volume. We collected 10 major BitConnect events by searching Cointelegraph. BitConnect prices are from Investing.com.

Granger Test

We test whether the news index of Bitcoin affects Bitcoin prices. We use daily Bitcoin prices and news index from Nov 2017 to July 2019. Consider the two-variable VAR

$$\begin{bmatrix}\text{news index}_t\\ \text{log of price}_t\end{bmatrix} = \mathbf{b_0} + \mathbf{B_1}\begin{bmatrix}\text{news index}_{t-1}\\ \text{log of price}_{t-1}\end{bmatrix} + \cdots + \mathbf{B_k}\begin{bmatrix}\text{news index}_{t-k}\\ \text{log of price}_{t-k}\end{bmatrix} + \begin{bmatrix}\varepsilon_{1,t}\\ \varepsilon_{2,t}\end{bmatrix}$$

where $\mathbf{b_0}$ is a vector of intercept terms and each of $\mathbf{B_1}$ to $\mathbf{B_k}$ is a 2×2 matrix of coefficients. The lag length $k = 1$ is recommended by the likelihood ratio test, final prediction error, Akaike's information criterion, Hannan-Quinn information criterion and Schwarz's Bayesian information criterion.

We test the null hypothesis that all coefficients on lags of news index in the price equation are equal to zero, against the alternative that at least one is not zero. The p-value is 0.0004 and thus the news index Granger-causes the log of Bitcoin's price at 1% significant level.

B Distribution of Bitcoin Across Bitcoin Addresses

A Bitcoin address is a number for representing where Bitcoins are sent to or from. We retrieved Bitcoin addresses and balances from the Bitcoin blockchain using BlockSci (see Kalodner et al. (2017) for detailed explanations of BlockSci). In Feb 20, 2019, there were 24,457,100 addresses and the total unspent balance of all addresses was 11,223,926, which was worth 44.9 billion US dollars. We plot the distribution of Bitcoin in Figure 16. The wealthiest address had a balance of 69,370 coins, and for presentation we excluded this address in the figure.

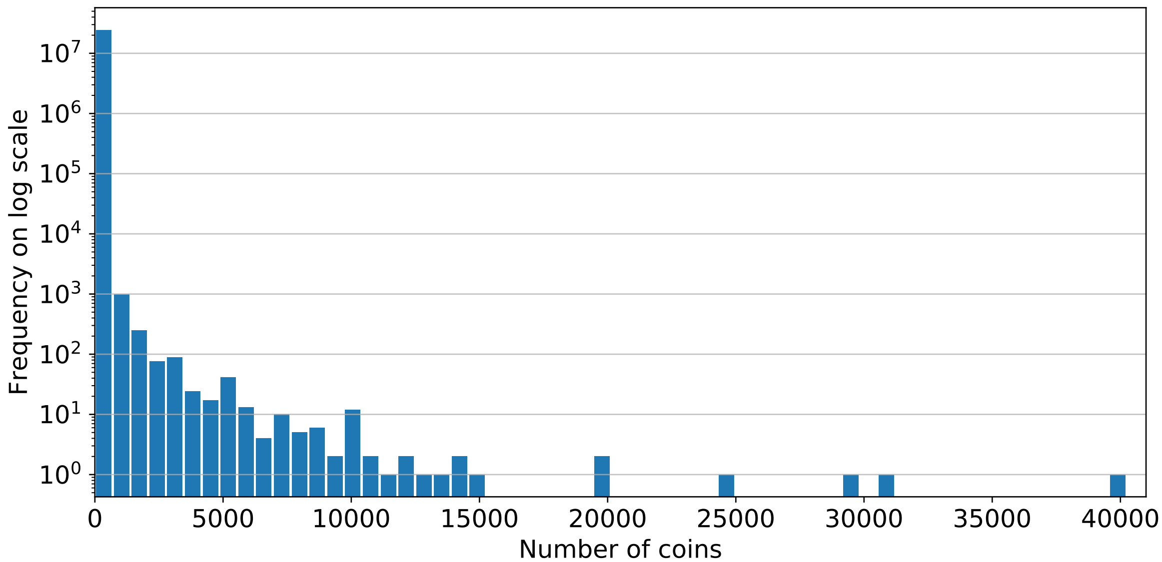

Figure 16: Distribution of Bitcoin across addresses.

C Concavity of Marginal Benefit of Asset Holding

Now we provide sufficient conditions such that $f(\ell)$ in (5) satisfies $f(0) = 0$, $f'(\ell) > 0$ and $f''(\ell) \leq 0$ for $\ell \in [0, \ell^*]$. Since we have assumed $\ell \leq \ell^*$, the DM market outcome is given by $p = c'(q)$ and $c'(q)q = \ell$. Using these equations we can derive $S'(\ell) \equiv u'(q)/c'(q) - 1$. By (5),

$$f(\ell) = \beta q[\alpha u'(q) + (1-\alpha)c'(q)] \text{ where } c'(q)q = \ell.$$

Thus $f(0) = 0$ if $\lim_{q \to 0} qu'(q) = 0$. But since u is concave, we have $qu'(q) < u(q)$, and hence $\lim_{q \to 0} qu'(q) = 0$ by $u(0) = 0$.

Next by differentiating $f(\ell)$ with respect to ℓ we have

$$f'(\ell) = \beta \left[1 - \alpha + \alpha \left(\frac{u'(q)[1 + u''(q)q/u'(q)]}{c'(q)[1 + c''(q)q/c'(q)]} \right) \right] \text{ where } c'(q)q = \ell.$$

The right side is positive provided that $-u''(q)q/u'(q) < 1 \; \forall q \leq q^*$, namely that the coefficient of relative risk aversion is smaller than 1 at all $q \leq q^*$. The right side falls in q if $-u''(q)q/u'(q)$ and $c''(q)q/c'(q)$ rise in q, namely that $u(q)$ has non-decreasing relative risk aversion and $c(q)$ has non-increasing relative risk aversion. We summarize these results with the following lemma.

Lemma 11 *The function $f(\ell)$ in (5) satisfies $f(0) = 0$. Moreover $f'(\ell) > 0$ and $f''(\ell) \leq 0$ for $\ell \in [0, \ell^*]$ if $-u''(q)q/u'(q) < 1$ $\forall q \leq q^*$, $u(q)$ has non-decreasing relative risk aversion and $c(q)$ has non-increasing relative risk aversion.*

D Alternative Micro-foundations

In this section we consider an alternative micro-foundation for the DM market and show that Proposition 1 and 2 are still applicable.

D.1 Bilateral Bargaining

Suppose the active sellers and the buyers who want to consume the DM good are matched randomly and bilaterally to bargain over the terms of trade. We assume the cost of producing the DM good is $c(q) = q$. We use Kalai (1977) bargaining solution to determine the terms of trade because it has several advantages over Nash bargaining in models with payment frictions (see Hu and Rocheteau, 2019). Kalai's solution is found by maximizing the buyer's surplus subject to the feasibility constraint $p \leq \ell$ and the seller getting a share $1 - \theta$ of the total surplus:

$$\max_{p,q} \{u(q) - p\} \text{ st } p - q = (1 - \theta)[u(q) - q] \text{ and } p \leq \ell. \tag{38}$$

The solution of (38) is $p = \min\{\ell, \ell^*\}$ where $\ell^* \equiv (1 - \theta)u(q^*) + \theta q^*$. The output quantity q solves $(1-\theta)u(q)+\theta q = \ell$ if $p = \ell$, and otherwise $q = q^*$. Let $S(\ell) \equiv u(q)-p$ be the buyer's trade surplus in a bilateral meeting when he carries ℓ units of liquidity. Using the bargaining solution $S(0) = 0$, $S'(\ell) > 0$ for $\ell \in [0, \ell^*)$ and $S'(\ell) = 0$ otherwise.

D.2 Concavity of f under Kalai bargaining

Now we provide parametric examples such that $f(\ell)$ in (5) is increasing and concave in ℓ. Since we have assumed $\ell \leq \ell^*$, the bargaining solution implies ℓ and q are linked by the equation $\ell = (1 - \theta)u(q) + \theta q$. Using this equation and the definition $S(\ell) \equiv \theta[u(q) - q]$, we can show that the derivative of $f(\ell)$ is

$$f'(\ell) = \beta \left[1 + \alpha \left(\frac{\theta(u'(q) - 1)}{(1 - \theta)u'(q) + \theta} + \frac{\theta u''(q)\ell}{[(1 - \theta)u'(q) + \theta]^3} \right)\right].$$

Since q is a function of ℓ and does not depend on α, $f'(\ell) > 0$ if α is not too large.

Next, the sign of the second derivative of $f(\ell)$ is

$$f''(\ell) \propto 2u''(q)[(1-\theta)u'(q)+\theta] + \ell\left[u'''(q) - \frac{3(1-\theta)u''(q)^2}{(1-\theta)u'(q)+\theta}\right]. \tag{39}$$

Lemma 12 *(i) If $u(q) = 1 - e^{-bq}$, then $f(\ell)$ is concave for $\ell \leq \ell^*$ provided that $b < 2$. (ii) If $u(q) = Aq^b$ for $b < 1$, then $f(\ell)$ is concave for $\ell \leq \ell^*$ provided that $b \in [1/2, 1)$.*

Proof. Part (i): Since $u'(q) \geq 1$ and $u(q) > q$ for $q < q^*$, the right side of (39) is smaller than $2u''(q) + \ell u'''(q)$. If $u(q) = 1 - e^{-bq}$, then $u''' > 0$ and hence the right side of (39) is smaller than $2u''(q) + u(q)u'''(q)$. Using the functional form of u one can show this expression is negative provided that $b < 2$.
Part(ii): By (39) and $u(q) = Aq^b$

$$\begin{aligned} f''(\ell) &\propto -2[(1-\theta)u'(q)+\theta] + [(1-\theta)u(q)+\theta q]\left[\frac{2-b}{q} - \frac{3(1-\theta)Aq^{b-2}b(1-b)}{(1-\theta)u'(q)+\theta}\right] \\ &= 2(1-\theta)(1-b)u'(q)/b + \frac{(1-\theta)u(q)+\theta q}{(1-\theta)bu(q)+\theta q}\left[(2b-3)(1-\theta)u'(q) - b\theta\right]. \end{aligned}$$

The fraction in the second term exceeds 1 as $b < 1$. Hence the right side is negative if

$$[2(1-b) + (2b-3)b](1-\theta)u'(q) - b^2\theta < 0.$$

The square bracket is negative if and only if $b \in [1/2, 2]$. Thus $f'' < 0$ if $b \in [1/2, 1)$. ■

Since f is concave, the left side of (11) is bounded above by $f'(0) = \beta[1 + \alpha\theta/(1-\theta)]$. Intuitively Assumption 1 is satisfied when the matching probability α and buyer's bargaining power θ are not too large, or learning is fast, namely η and s are close to 1.

D.3 Free Entry of Sellers

Suppose there is free entry of sellers in the DM (e.g., Rocheteau and Wright (2005)) or the sellers pay a per period cost to accept the asset. Let n be the seller-to-buyer ratio, k be the entry/adoption cost and $\alpha(n)$ be each buyer's matching probably. We assume $\alpha' > 0$ and $\alpha'' < 0$ and $\alpha'(n)n/\alpha < 1$. The free entry condition is given by $\frac{\alpha(n)}{n}[p-q] = k$. Assume the liquidity constraint is binding, namely $\ell < \ell^*$. By the Kalai bargaining solution, the free entry condition can be rewritten as

$$\frac{\alpha(n)}{n}\frac{1-\theta}{\theta}S(\ell) = k$$

where $S(\ell)$ is the buyer's trade surplus and θ is buyer's bargaining power. This condition defines n as an implicit function of ℓ and $n(\ell)$ rises in ℓ with $n(0) = 0$ and $n(\ell^*) < \infty$.

The first order condition for asset holding can still be written as (6) but now f becomes

$$f(\ell) \equiv \beta\ell[1 + \alpha(n)S'(\ell)].$$

The derivative f' is

$$f'(\ell) \equiv \beta\{1 + \alpha'(n)\ell S'(\ell) + \alpha(n)[\ell S''(\ell) + S'(\ell)]\}.$$

If we assume $\alpha(n) = An^\rho$ where $\rho < 1$, then $f'(\ell)$ can be reexpressed as

$$f'(\ell) \equiv \beta\{1 + \frac{(1-\theta)A^2 n(\ell)^{2\rho-1} S'(\ell)^2 \ell}{\theta k(1-\rho)} + \alpha[n(\ell)][\ell S''(\ell) + S'(\ell)]\}.$$

If $\rho > 1/2$, then the right side is a continuous function of ℓ and is finite for all $\ell \in [0, \ell^*]$. Hence there is a constant $\bar{f}'$ such that $f'(\ell) \leq \bar{f}'$ for all $\ell \in [0, \ell^*]$. When s and η are sufficiently large, the right side of (11) exceeds $\bar{f}'$. In this case Proposition 1 and 2 hold and hence there is a unique equilibrium.

E Long Run Adoption Chance

Assume the asset quality is H and let $\mathcal{A}_j$ be the probability that the asset will be adopted as money in the long run, namely the probability that $\pi \to 1$, provided that agent's current period prior is π_j. Clearly $\mathcal{A}_d = 0$ and $\lim_{j\to\infty} \mathcal{A}_j = 1$. At state π_j where $j \geq d+1$, $\mathcal{A}_j$ is given by

$$\mathcal{A}_j = s\mathcal{A}_{j+1} + (1-s)\mathcal{A}_{j-1} \iff s\mathcal{A}_{j+1} - \mathcal{A}_j + (1-s)\mathcal{A}_{j-1} = 0.$$

The characteristic equation for this second-order linear difference equation is $sz^2 - z + (1-s) = 0$. The roots are $(1-s)/s$ and 1. Therefore the solution of $\mathcal{A}_j$ takes the form $\mathcal{A}_j = C + D[(1-s)/s]^{j-d}$. By using $\mathcal{A}_d = 0$, we have $C + D = 0$. By $\mathcal{A}_\infty = 1$, we have $C = 1$. Therefore

$$\mathcal{A}_j = 1 - \left(\frac{1-s}{s}\right)^{j-d}.$$

Therefore if the asset quality is H and the agents' prior believe is π_0, then the asset will be eventually adopted with probability $1 - [s/(1-s)]^d$ which falls in d and rises in s.

F Parameters for Numerical Examples

In all simulations we assume bilateral meetings and Kalai bargaining as described in Online Appendix D. We use CM utility function $U(x) = x$ and DM cost function $c(q) = q$ for all numerical examples. Other parameters are set as follows:

Table 1: Parameters for Numerical Examples

Fig	$u(q)$	β	α	s	γ_H	γ_L	θ	η	χ	others
4 R	$2q^{.5}$	.99	.05	.55	10	-10	.5	.01	.01	
5 L	$1-e^{-120*q}$	.96	.005	.6	1.44	-1.24	1	.9	.01	
5 R	$2q^{.5}$	.99	.02	.55	10	-10	.5	.99	.007	
6 L	$2q^{.5}$	.99	.01	.55	10	-10	.5	.1, .99	.005	
6 R	$2ln(1+q)$	.96	.3	.68	13	-12	1	.15, .95	.01	
7	$2q^{.5}$	.99	.5	.55	10	-10	.5	.17, .97	.007	
8, 9	$1-e^{-120*q}$	.96	.005	.6	1.44	-1.23	1	.1, .5, .9	.01	
10 blue	$1.14q + .8(10 - e^{-2q^{2.2}})$	.9	.6	.6	14	-6	1	.01, .9	.01	
10 red	$1.19q^{.99}$	.9	.6	.6	9	-6	1	.01, .9	.01	
11 L	$2q^{.5}$	.99	.05	.55	10	-10	.5	.5	.01	$\delta = .0001$
11 R	$2q^{.5}$	.99	.05	.55	10	-10	.5	.5	.007	$\delta = .0001$, κ=.009
12	$2q^{.5}$	.99	.05	.55	10	-10	.5	.5	.007	$\delta = .0001$
13										$\pi_0 = \tilde{\pi}_{12}$
14 L	$2q^{.5}$	.99	.01	.55	10	-10	.5	.5	.01	$\pi_0 = \tilde{\pi}_{12}$
14 R	$2q^{.5}$	.99	.01	.55	10	-10	.5	.5	.01	$\pi_0 = \tilde{\pi}_{12}$, κ=.001