ESE 303 – Homework 11

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Problem 1

From the definition of WGN $W(t)$ and the property (1) of Gaussian process we know that $W(t_1)$ and $W(t_2)$ are jointly Gaussian. Jointly normal random variables are independent if and only if the correlation between $W(t_1)$ and $W(t_2)$ is 0. Therefore, it suffices to show that $W(t_1)$ and $W(t_2)$ are not correlated, for $t_1 \neq t_2$. This is true because the autocorrection function $R_W(t_1, t_2) = 0$ for $t_1 \neq t_2$. The reason is that the Dirac delta function $\delta(t) = 0, \forall t \neq 0$.

Problem 2

$X(t)$ is a Gaussian process since it is an integration of Gaussian processes, and because a linear functional of Gaussian processes is still a Gaussian process (property 1 of Gaussian processes).

The mean of $X(t)$:

$$\mu_X(t) = \mathbb{E}\left[\int_0^t W(\tau)d\tau\right] = \int_0^t \mathbb{E}[W(\tau)]d\tau = \int_0^t \mu_W(\tau)d\tau = 0$$

The reason is that $\delta(\tau) = 0$ for $\tau \neq 0$.

The autocorrelation function of $X(t)$:

$$\begin{aligned}
R_X(t_1, t_2) &= \mathbb{E}\left[\left(\int_0^{t_1} W(\tau_1)\, d\tau_1\right)\left(\int_0^{t_2} W(\tau_2)\, d\tau_2\right)\right] \\
&= \mathbb{E}\left[\int_0^{t_1}\int_0^{t_2} W(\tau_1)\, W(\tau_2)\, d\tau_2 d\tau_1\right] \\
&= \int_0^{t_1}\int_0^{t_2} \mathbb{E}\left[W(\tau_1)\, W(\tau_2)\right] d\tau_2 d\tau_1 \\
&= \int_0^{t_1}\int_0^{t_2} R_W(\tau_1, \tau_2)\, d\tau_2 d\tau_1 \\
&= \int_0^{t_1}\int_0^{t_2} \sigma^2 \delta(\tau_1 - \tau_2)\, d\tau_2 d\tau_1
\end{aligned}$$

Now, from the properties of Dirac delta function and what we learned in class:

$$\begin{aligned}
R_X(t_1, t_2) &= \begin{cases} \int_0^{t_1} \sigma^2 d\tau_1 = \sigma^2 t_1, & \text{for } t_1 < t_2 \\ \int_0^{t_2} \sigma^2 d\tau_2 = \sigma^2 t_2, & \text{for } t_1 > t_2 \end{cases} \\
&= \sigma^2 \min(t_1, t_2)
\end{aligned}$$

Therefore, the gaussian process at time t is a normal distribution $X(t) \sim \mathcal{N}\left(\mu_X(t), \sqrt{R_X(t,t)}\right) = \mathcal{N}\left(0, \sigma\sqrt{t}\right)$. Hence,

$$\mathbb{P}[X(t) > a] = 1 - \mathbb{P}[X(t) \leq a] = 1 - \Phi\left(\frac{a}{\sigma\sqrt{t}}\right)$$

where Φ is the cdf of the standard Gaussian.

Problem 3

Similar to Problem 2, $W_h(n)$ is still a Gaussian Process since integration is linear functional. The mean of $W_h(n)$:

$$\mu_{W_h}(n) = \mathbb{E}\left[W_h(n)\right] = \mathbb{E}\left[\int_{nh}^{(n+1)h} W(\tau)d\tau\right] = \int_{nh}^{(n+1)h} \mathbb{E}[W(\tau)]d\tau = \int_{nh}^{(n+1)h} 0d\tau = 0$$

The autocorrelation function of $W_h(n)$:

$$\begin{aligned}
R_{W_h}(n_1, n_2) &= \mathbb{E}\left[W_h(n_1) W_h(n_2)\right] \\
&= \mathbb{E}\left[\left(\int_{n_1h}^{(n_1+1)h} W(\tau_1)\, d\tau_1\right)\left(\int_{n_2h}^{(n_2+1)h} W(\tau_2)\, d\tau_2\right)\right] \\
&= \mathbb{E}\left[\int_{n_1h}^{(n_1+1)h} \int_{n_2h}^{(n_2+1)h} W(\tau_1)\, W(\tau_2)\, d\tau_2 d\tau_1\right] \\
&= \int_{n_1h}^{(n_1+1)h} \int_{n_2h}^{(n_2+1)h} \sigma^2 \delta(\tau_1 - \tau_2)\, d\tau_2 d\tau_1 \\
&= \begin{cases} 0, & n_1 \neq n_2 \\ \sigma^2 h, & n_1 = n_2 \end{cases}
\end{aligned}$$

Problem 4

The following MATLAB script simulates the process $X(t)$ using its discrete time version $X_h(n)$ obtained from $W_h(n)$ derived in Part C.

```
clear all
close all

h = 0.01;
variance = 1;
T = 10;


W = randn(ceil(T/h), 1)*sqrt(variance * h);
X = cumsum(W);


figure();
plot(h:h:T, X, 'Linewidth', 2);
xlabel('Time')
ylabel('X(t)');
xlim([0 T]);
grid on;

saveas('D_plot.jpg')
```

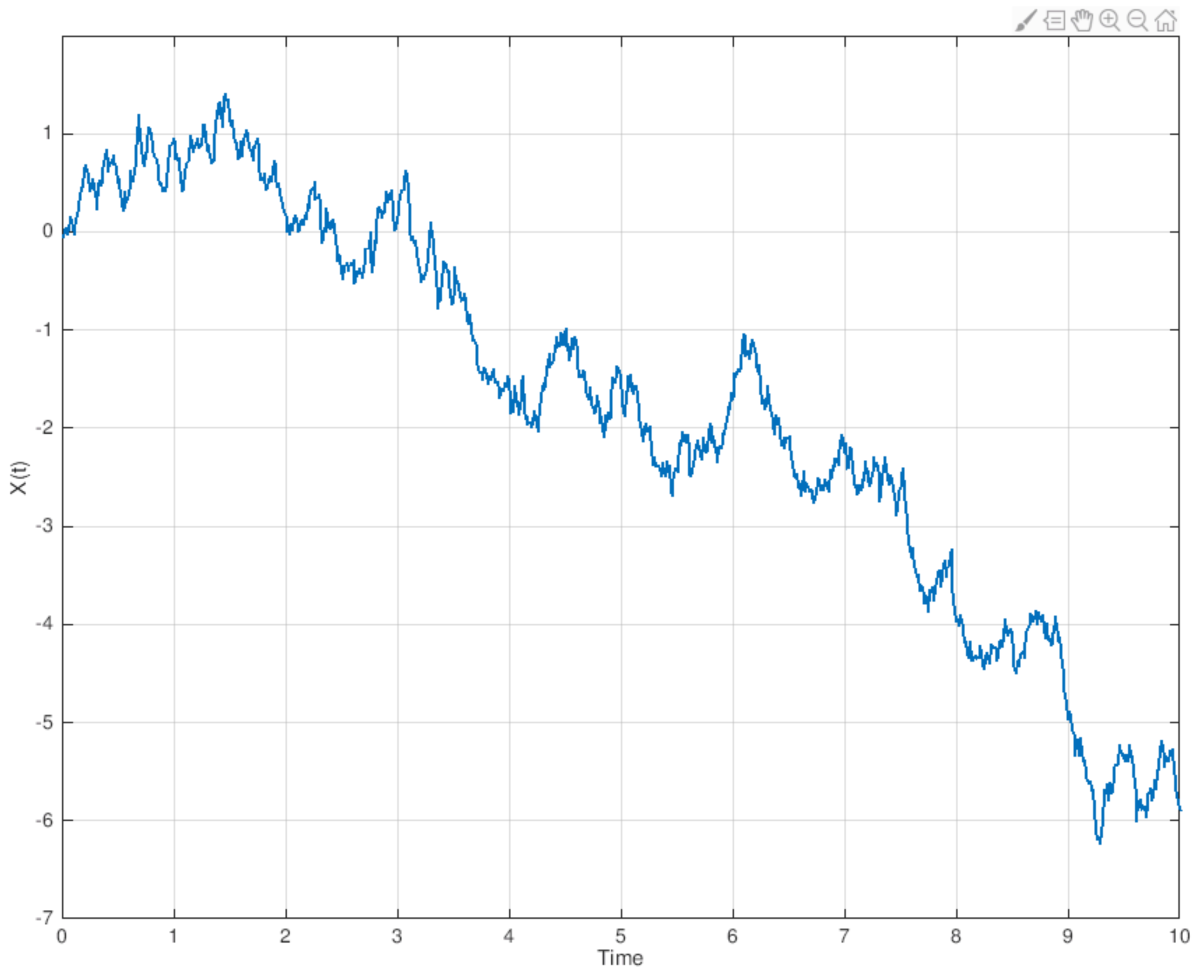

Figure 1: A sample path of the simulated Gaussian (Wiener) process $X(t)$ using a discrete approximation with step size $h = 0.01$ (part D).

A result of this simulation is shown in Figure 1.