ESE 303 – Homework 13

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Problem 1

The $Y_t(s)$ is a Gaussian process because it is stated in (1) that $Y_t(s)$ is normally distributed with mean μs and variance $\sigma^2 s$.

Since we assume that $Y_t(s)$ over disjoint time intervals are independent, we know that for any Y_{n_1} and $Y_{n_2}, n_1 \neq n_2$, the time intervals $(n_1, n_1 + h)$ and $(n_2, n_2 + h)$ are disjoint, we know that they are independent.

Using the above fact and from (2) that $Y_n = Y_{(n-1)h}(h)$, we plug (2) into (1) and get that Y_n is with mean μh and variance $\sigma^2 h$.

Therefore all Y_n are i.i.d, each with mean μh and variance $\sigma^2 h$.

Problem 2

We compute drift and volatility in MATLAB as follows:

```
cisco_stock_price;
h = 1;
close_price_l = log(close_price);
Y_n = close_price_l(2:252)-close_price_l(1:251);

L = size(Y_n, 1);


drift = sum(Y_n) /(h*L);
volatility = sum((Y_n- h *drift).^2)/((L-1)*h);
format short e
disp([drift, volatility])
```

And we get output Figure 1.

We obtain $\hat{\mu} = 1.7191 \times 10^{-3}$ and $\hat{\sigma}^2 = 5.9573 \times 10^{-4}$.

Problem 3

We use the following MATLAB code to compare the histogram with the computed PDFs and the CDFs:

```
clear, clc;

drift_volatility;
```

```
>> drift_volatility
   1.7191e-03   5.9573e-04
```

Figure 1: Drift and Volatility

```

range = -0.1:0.01:0.1;


n_elements = histcounts(Y_n, range);

range = range(2:end);
h1 = figure();
pdf = n_elements/L/0.01;
bar(range, pdf);
hold on;
plot(range, normpdf(range, drift*h, sqrt(volatility*h)), 'r', 'Linewidth', 2);
xlabel('x');
ylabel('pdf');
grid on;
legend('Estimated', 'Gaussian pdf');


h2 = figure();
cdf = cumsum(n_elements)/L;
stairs(range, cdf);
hold on;
plot(range, normcdf(range, drift*h, sqrt(volatility*h)), 'r', 'Linewidth', 2);
xlabel('x');
ylabel('cdf');
grid on;
legend('Estimated', 'Gaussian pdf');
```

The plots are shown in Figure 2 and Figure 3.
Therefore, the geometric brownian motion model is an acceptable approximation.

Problem 4

Apply slide 27 of *arbitrage_stock_and_option_pricing* on expected return and you can get the expected return for the stock:

$$\begin{aligned} E_r(t) &= e^{-\alpha t}\frac{\mathbb{E}[X(t)|X(0)]}{X(0)} \\ &= e^{\left(\hat{\mu}+\hat{\sigma}^2/2-\alpha\right)t} \end{aligned}$$

Plugging in $\alpha = 0.1\%$ and the computed $\hat{\mu}, \hat{\sigma^2}$, we get that

$$E_r = e^{(1.7191*10^{-3}+5.9573*10^{-4}/2-0.001)*365} = 1.44946166$$

. So the rate of return is 145%.

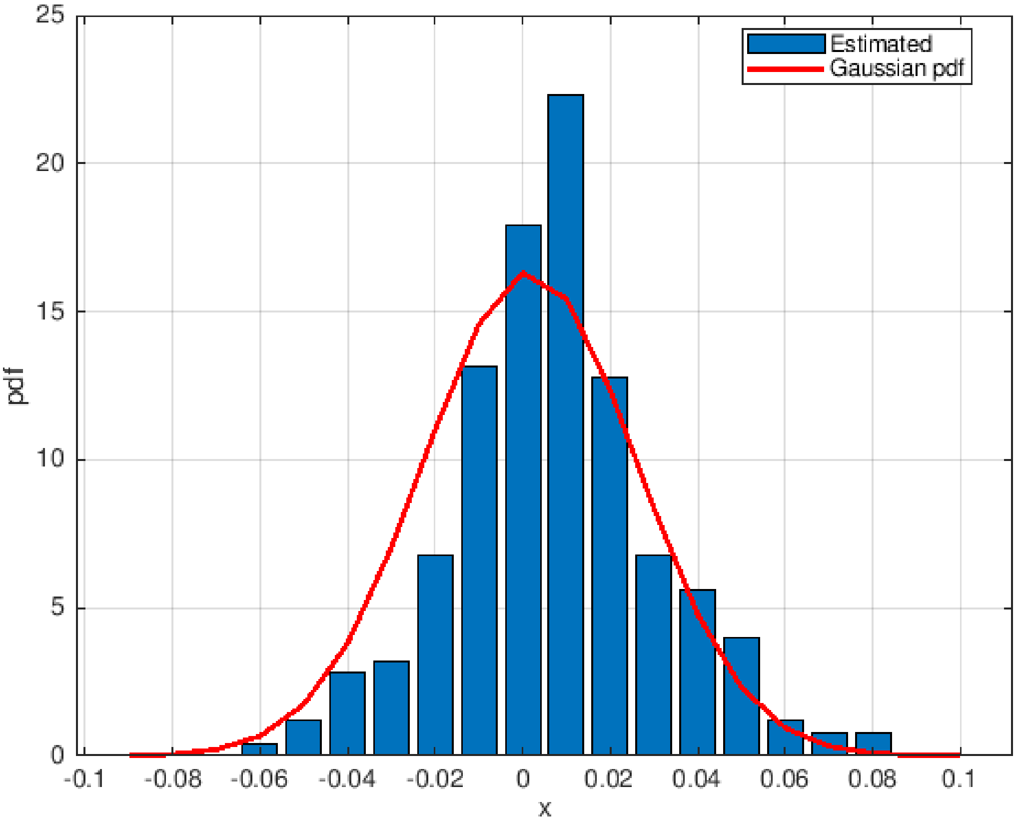

Figure 2: PDF of estimated Y_n to normal model

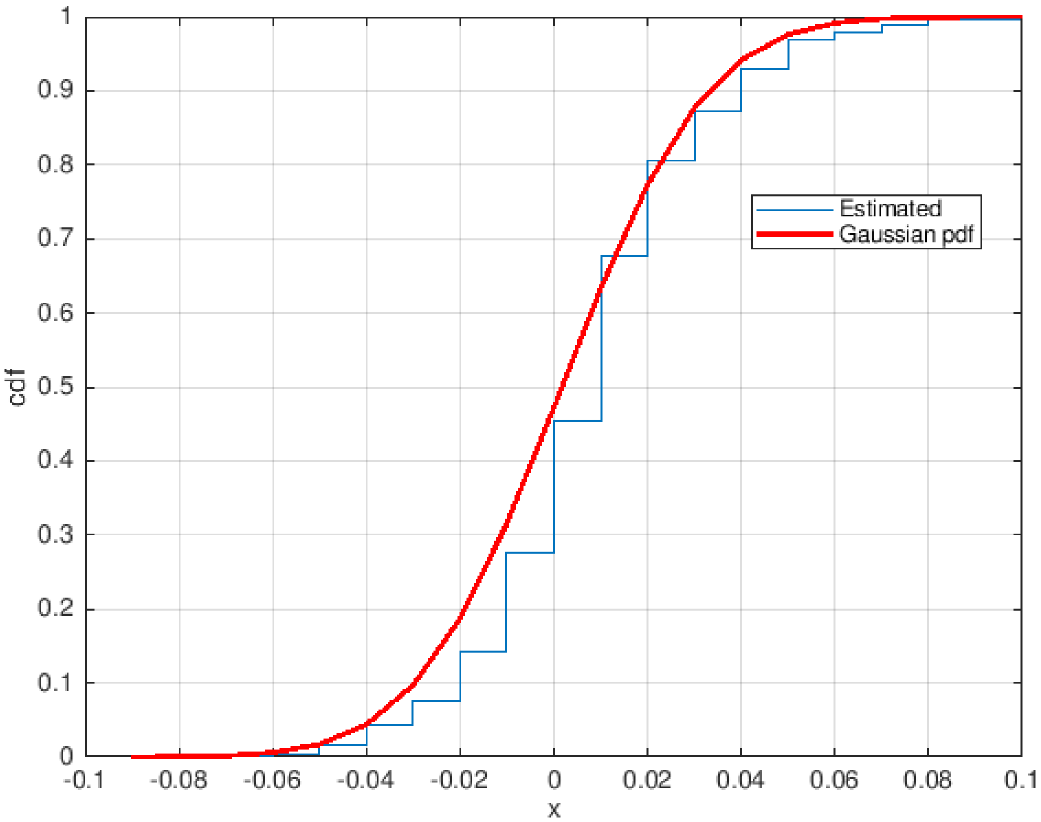

Figure 3: CDF of estimated Y_n to normal model

The logarithm of $\frac{X(t)}{X(0)}$ follows a normal distribution with mean μt and variance $\sigma^2 t$. Therefore,

$$\begin{aligned}\mathbb{P}\left[\frac{e^{-\alpha t}X(t)}{X(0)} \geq 1.05\right] &= \mathbb{P}\left[\log\left(\frac{e^{-\alpha t}X(t)}{X(0)}\right) \geq \log(1.05)\right] \\ &= \mathbb{P}\left[\log\left(\frac{X(t)}{X(0)}\right) \geq \log(1.05) + \alpha t\right], \\ &= 1 - \Phi\left(\frac{\log(1.05) + \alpha t - \mu t}{\sqrt{\sigma^2 t}}\right)\end{aligned}$$

where Φ represents the cdf of the standard normal. Evaluating the value, we get that the probability is 0.63.

Problem 5

E Risk neutral measure. From slide 26 of *arbitrage_stock and_option_pricing*, the risk neutral measure refers to a rescaled geometric Brownian motion whose drift parameter is $\alpha - \sigma^2/2$ and whose variance is σ^2. Plugging in the estimated values $\alpha = 0.001, \hat{\sigma}^2 = 5.9573 \times 10^{-4}$, we get the risk neutral measure for the CSCO stock is a geometric Brownian motion with mean $7.02 \times 10^{-4} \cdot t$ and variance $5.96 \times 10^{-4} \cdot t$.

Problem 6

From the definition of risk neural mea- sure, it is ready that the expected discounted rate of return in the alternative reality is one (its logarithm is 0). Thus, the non-discounted rate of return is α, i.e., the rate of return of the risk-free investment.

Problem 7

The discounted expected return with respect to the risk neutral measure is:

$$\mathbb{E}_q\left[e^{-\alpha t}[X(t) - K]_+\right]$$

From (7) and the fact that c is deterministic, we know that the net gain with respect to the risk neutral measure is 0. We may write as:

$$\mathbb{E}_q\left[e^{-\alpha t}[X(t) - K]_+ - c\right] = 0$$

which gives us $c(t)$:

$$c(t) = e^{-\alpha t}\mathbb{E}_q\left[\left[X(0)e^{Y(t)} - K\right]_+\right]$$

where $x(t) = x(0)e^{Y(t)}$ is a geometric brownian motion. This implies that $Y(t)$ is normally distributed with parameters $\left(\mu t, \sigma^2 t\right)$, and that:

$$c(t) = e^{-\alpha t}\int_{\log[K/X(0)]}^{\infty} \left(X(0)e^{y} - K\right)\exp\left[-\frac{(y-\mu t)^2}{2\sigma^2 t}\right] dy$$

We simplify this equation with the steps listed in the course slides, and obtain the closed form of c:

$$c = X(0)Q(b) - e^{-\alpha t}KQ(a)$$

where:

$$\begin{aligned}a &= \frac{\log(K) - \log[X(0)] - \left(\alpha - \sigma^2/2\right)t}{\sqrt{\sigma^2 t}} \\ b &= a - \sqrt{\sigma^2 t}\end{aligned}$$

Problem 8

We first write a MATLAB function to determined the option price c given a set of values of k.

```
function[c, mean, mean_up_bound, mean_low_bound] = get_prices(k, drift_n, ...
    volatility, close_price, t)

X0 = close_price(1,1);
alpha = 0.001;

mean = X0 * exp((drift_n + volatility/2)*t);
K = k*mean;

varX = X0^2 * exp((2*drift_n + volatility)*t) .* (exp(volatility*t) - 1);
mean_up_bound = mean + sqrt(varX);
mean_low_bound = mean - sqrt(varX);

c = zeros(size(K,1), length(t));
for K_i = 1:size(K,1)
    x = (log(K(K_i,:)/X0) - (alpha - volatility/2)*t) ./ sqrt(volatility*t);
    y = x - sqrt(volatility*t);
    N_a = 1 - normcdf(x, 0, 1);
    N_b = 1 - normcdf(y, 0, 1);
    c(K_i,:) = X0 * N_b - exp(-alpha*t) .* K(K_i,:) .* N_a;
end
end
```

The plotting code in MATLAB:

```
drift_volatility;
k = [0.8 ; 1 ; 1.2];
t = 1:251;
[c, mean, mean_up_bound, mean_low_bound] = get_prices(k, drift_n, volatility, ...
    close_price,t);
% Plot results
h1 = figure();
plot(t, c, 'LineWidth', 2);
xlabel('Strike time (days)');
ylabel('Option price');
grid on;
legend('K = 0.8 EX', 'K = EX', 'K = 1.2 EX', 'Location', 'Best');
xlim([1 250]);

h2 = figure();
fill([t fliplr(t)], [mean_up_bound fliplr(mean_low_bound)], 'r', 'Linestyle', ...
    'None', 'FaceAlpha', 0.2);
hold on
plot(t, mean, 'k', 'LineWidth', 2);
xlabel('Strike time (days)');
ylabel('Expected stock price');
grid on;
xlim([1 250]);
```

The plots are shown in Figure 4 and Figure 5. From the plots: Buying with $K = 1.2 * E[X(t)]$ corresponds to the situation where we would lose money by exercising our buying option. In this case, we would be better off purchasing the stock at the market price. However, buying at this price would still make sense if you expect volatility to favor us. Buying with $K = 0.8 * E[X(t)]$ we expect to be able to resell the stock at increased price and make profit.

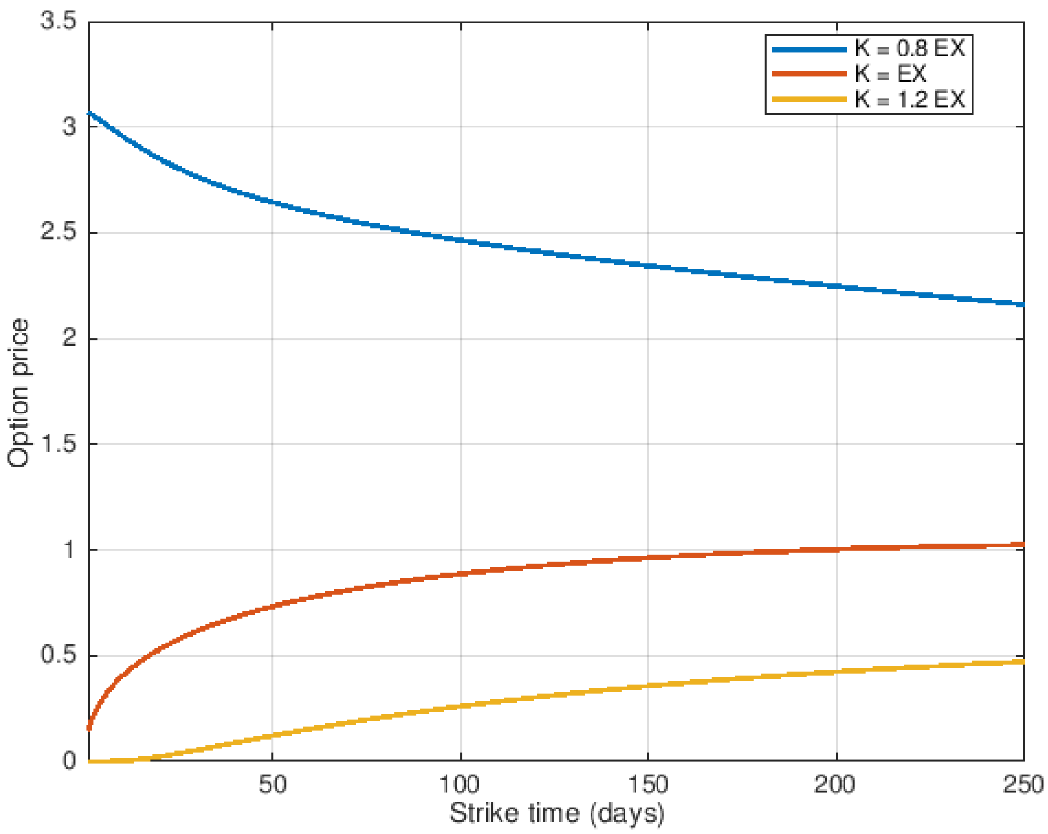

Figure 4: option price

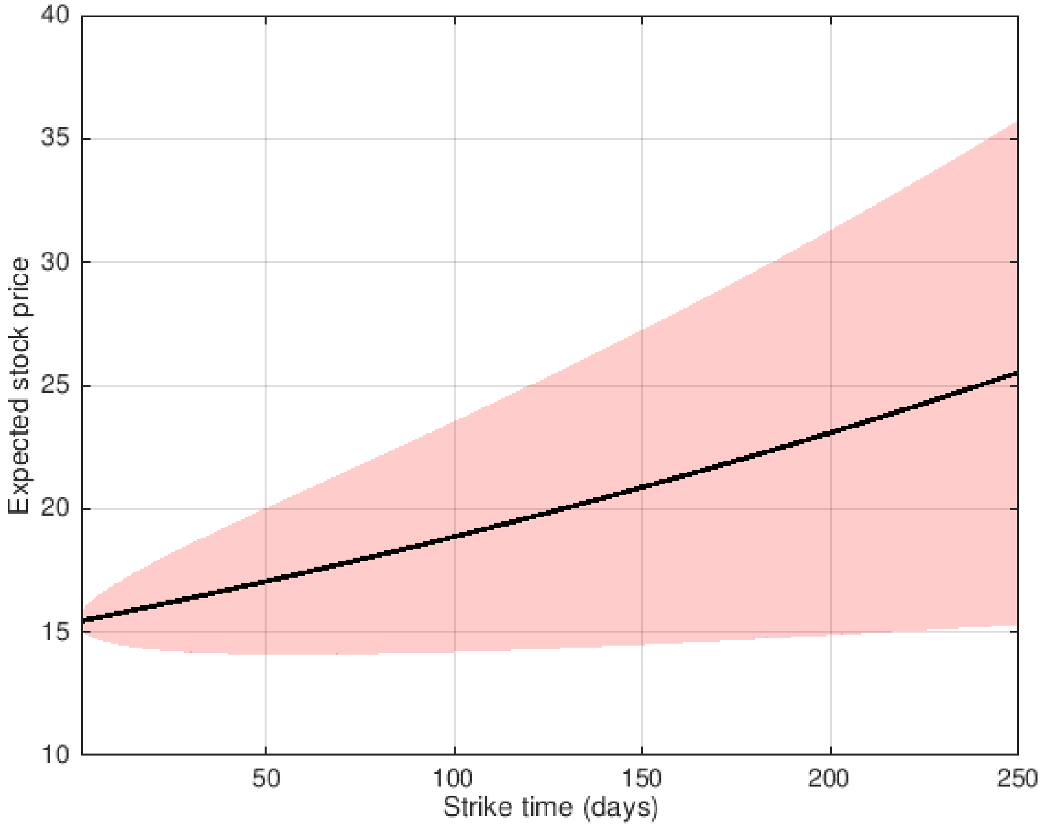

Figure 5: expected stock price with one standard deviation band