ESE 303 – Homework 11

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Problem 1

Jointly normal random variables (RVs) have the property of being independent if and only if they are uncorrelated. This is not true in general: it is very specific of Gaussian RVs (the only other case I know is RVs that take on only two different values, e.g., the Bernoulli RV). Note also that the RVs must be jointly normal: it is possible for X and Y to be normally distributed and for (X, Y) to not be a bivariate Gaussian. Nevertheless, the definition given in (1) from the exercise implies joint normality. Hence, suffices to show that $W(t_1)$ and $W(t_2)$ are not correlated for $t_1 \neq t_2$. Intuitively, we have that $R_W(t_1, t_2) = 0$ for $t_1 \neq t_2$. If you want to be formal, however, remember that we only defined $\delta(t)$ in terms of an integral [see (4)]. So we cannot say $\delta(t) = 0$ for $t \neq 0$ without proving it. This is actually fairly simple. Suppose $t \neq 0$ and write

$$\int_{t-\epsilon}^{t+\epsilon} \delta(\tau) d\tau = \int_{t-\epsilon}^{t+\epsilon} 1 \times \delta(\tau) d\tau = 0, \text{ for all } \epsilon > 0$$

where we used $f(t) = 1$ for all t in (4). since $f(\tau) > 0$ over the interval $[t - \epsilon, t + \epsilon]$, the integral vanishes if and only if $\delta(\tau) = 0$ for $\tau \in [t - \epsilon, t + \epsilon]$.

Problem 2

Recall that integration is a linear functional. Thus, $X(t)$ is a Gaussian process since it is defined as the linear functional of a Gaussian process.

Given that $\mu_W(t) = 0$, the mean function of $X(t)$ is

$$\mu_X(t) = \mathbb{E}\left[\int_0^t W(u) du\right] = \int_0^t \mathbb{E}[W(u)] du = \int_0^t \mu_W(t) du = 0$$

Switching the expected value and the integral like that should make you uneasy. After all, the expected value is an integral operator and it is not always the case that integrals can just be exchanged like that. In this case, however, we are justified (take a look at Fubinis theorem).

We are going to use the same result to derive the autocorrelation function of $X(t)$:

$$\begin{aligned}
R_X(t_1, t_2) &= \mathbb{E}\left[\left(\int_0^{t_1} W(u_1)\,du_1\right)\left(\int_0^{t_2} W(u_2)\,du_2\right)\right] \\
&= \mathbb{E}\left[\int_0^{t_1}\int_0^{t_2} W(u_1)\,W(u_2)\,du_2 du_1\right] \\
&= \int_0^{t_1}\int_0^{t_2} \mathbb{E}\left[W(u_1)\,W(u_2)\right] du_2 du_1 \\
&= \int_0^{t_1}\int_0^{t_2} R_W(u_1, u_2)\,du_2 du_1 \\
&= \int_0^{t_1}\int_0^{t_2} \sigma_W(u_1 - u_2)\,du_2 du_1
\end{aligned}$$

where we used the fact that $R_W(u_1, u_2) = \mathbb{E}\left[W(u_1)\,W(u_2)\right] = \sigma^2\delta(u_1 - u_2)$. Now, from the definition of the δ distribution in (4), we obtain

$$\begin{aligned}
R_X(t_1, t_2) &= \begin{cases} \int_0^{t_1}\sigma^2 du_1 = \sigma^2 t_1, & \text{for } t_1 < t_2 \\ \int_0^{t_2}\sigma^2 du_2 = \sigma^2 t_2, & \text{for } t_1 > t_2 \end{cases} \\
&= \sigma^2 \min(t_1, t_2)
\end{aligned}$$

Problem 3

This is an introduction to the the third problem.

a) This is the third problem, first question.