Model-Protected Multi-Task Learning
Supplementary Material

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APPENDIX B
WISHART DISTRIBUTION

Definition 8 (Gupta and Nagar [4]). *A $d \times d$ random symmetric positive definite matrix $\mathbf{E}$ is said to have a Wishart distribution $\mathbf{E} \sim W_d(\mathbf{V}, \nu)$ if its probability density function is*

$$p(\mathbf{E}) = \frac{|\mathbf{E}|^{(\nu-d-1)/2} \exp(-\mathrm{tr}(\mathbf{V}^{-1}\mathbf{E})/2)}{2^{\frac{\nu d}{2}} |\mathbf{V}|^{1/2} \Gamma_d(\nu/2)},$$

where $\nu > d - 1$ and $\mathbf{V}$ is a $d \times d$ positive definite matrix.

APPENDIX C
MODEL-DECOMPOSED MP-MTL METHODS

In this section, we consider the extension of our MP-MTL framework for MTL methods using the decomposed parameter/model matrix. Specifically, we focus on the following problem, where the trace norm is used for knowledge sharing across tasks and the $\|\cdot\|_1$ norm (sum of the ℓ_1 norm for vectors) is used for entry-wise outlier detection, as described in Algorithm 4.

$$\min_{\mathbf{W}} \ \sum_{i=1}^{m} \mathcal{L}_i(\mathbf{X}_i \mathbf{w}_i, \mathbf{y}_i) + \lambda_1 \|\mathbf{P}\|_* + \lambda_2 \|\mathbf{Q}\|_1 \quad (24)$$
$$s.t. \ \mathbf{W} = \mathbf{P} + \mathbf{Q},$$

where $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{d \times m}$.

We note that in Algorithm 4, the role of $\mathbf{P}$ is the same as the role of $\mathbf{W}$ in Algorithm 2, and the additional procedures introduced to update $\mathbf{Q}$ are still STL algorithms. As such, we have the result in Corollary 2.

Corollary 2. *Algorithm 4 is an (ϵ, δ) - Iterative MP-MTL algorithm.*

Remark 3. *Based on Algorithm 4, this will result in a similar procedure and identical theoretical results with respect to privacy by replacing the trace norm with the $\ell_{2,1}$ norm to force group sparsity in $\mathbf{P}$ or by replacing the $\|\cdot\|_1$ norm with the $\ell_{1,2}$ norm (sum of the ℓ_2 norm of column vectors) or $\|\cdot\|_F^2$ (square of the Frobenius norm).*

APPENDIX D
MP-MTL FRAMEWORK WITH SECURE MULTI-PARTY COMPUTATION

Pathak et al. [10] considered the demand for secure multi-party computation (SMC): protecting data instances from leaking to the curator and leaking between tasks during joint learning. However, by Proposition 3, the method of Pathak et al. [10] may introduce excess noise to protect both the data instances and the models simultaneously. To avoid unnecessary noise, we consider a divide-and-conquer strategy to ensure privacy for a single data instance and the model separately. Specifically, in each iteration of the Iterative

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Algorithm 4 Model-Protected Low-Rank and SParse (MP-LR-SP) Estimator

Input: Datasets $(\mathbf{X}^m, \mathbf{y}^m) = \{(\mathbf{X}_1, \mathbf{y}_1), \ldots, (\mathbf{X}_m, \mathbf{y}_m)\}$, where $\forall i \in [m]$, $\mathbf{X}_i \in \mathbb{R}^{n_i \times d}$ and $\mathbf{y}_i \in \mathbb{R}^{n_i \times 1}$. Privacy loss $\epsilon, \delta \geq 0$. Number of iterations T. Step size η. Regularization parameter $\lambda_1, \lambda_2 > 0$. Norm clipping parameter $K > 0$. Acceleration parameters $\{\beta_t\}$. Initial models of tasks $\mathbf{W}^{(0)}$.

Output: $\widehat{\mathbf{W}}^{(1:T)}$.

1: For $t = 1, \ldots, T$, set ϵ_t such that $\tilde{\epsilon} \leq \epsilon$, where $\tilde{\epsilon}$ is defined in (7).
2: Let $\mathbf{P}^{(0)} = \mathbf{Q}^{(0)} = \widehat{\mathbf{Q}}^{(0)} = \mathbf{W}^{(0)}$.
3: **for** $t = 1 : T$ **do**
4:   Norm clipping: $\tilde{\mathbf{p}}_i^{(t-1)} = \mathbf{p}_i^{(t-1)} / \max(1, \frac{\|\mathbf{p}_i^{(t-1)}\|_2}{K})$ for all $i \in [m]$. Let $\widehat{\mathbf{P}}^{(0)} = \widetilde{\mathbf{P}}^{(0)}$.
5:   Compute sensitivity: $s_i^{(t-1)} = 2$ for all $i \in [m]$.
6:   $\widetilde{\mathbf{\Sigma}}^{(t)} = \widetilde{\mathbf{P}}^{(t-1)} (\widetilde{\mathbf{P}}^{(t-1)})^{\mathrm{T}}$.
7:   $\mathbf{\Sigma}^{(t)} = \widetilde{\mathbf{\Sigma}}^{(t-1)} + \mathbf{E}$, where $\mathbf{E} \sim W_d(\frac{\max_i s_i^{(t-1)}}{2\epsilon_t} \mathbf{I}_d, d+1)$ is a sample of the Wishart distribution.
8:   Perform SVD decomposition: $\mathbf{U}\mathbf{\Lambda}\mathbf{U}^{\mathrm{T}} = \mathbf{\Sigma}^{(t)}$.
9:   Let $\mathbf{S}_{\eta\lambda_1}$ be a diagonal matrix and let $\mathbf{S}_{\eta\lambda_1,ii} = \max\{0, 1 - \eta\lambda_1/\sqrt{\mathbf{\Lambda}_{ii}}\}$ for $i = 1, \ldots, \min\{d, m\}$.
10:   Let $\hat{\mathbf{p}}_i^{(t)} = \mathbf{U}\mathbf{S}_{\eta\lambda_1}\mathbf{U}^{\mathrm{T}}\mathbf{p}_i^{(t-1)}$ for all $i \in [m]$.
11:   Let $\hat{\mathbf{q}}_i^{(t)} = \mathrm{sign}(\mathbf{q}_i^{(t-1)}) \circ \max\{0, |\mathbf{q}_i^{(t-1)}| - \eta\lambda_2\}$ for all $i \in [m]$, where $\circ$ denotes the entry-wise product.
12:   Let $\widetilde{\mathbf{W}}^{(t)} = \widehat{\mathbf{P}}^{(t)} + \widehat{\mathbf{Q}}^{(t)}$.
13:   Let $\mathbf{z}_{i,p}^{(t)} = \hat{\mathbf{p}}_i^{(t)} + \beta_t(\hat{\mathbf{p}}_i^{(t)} - \hat{\mathbf{p}}_i^{(t-1)})$ for all $i \in [m]$.
14:   Let $\mathbf{z}_{i,q}^{(t)} = \hat{\mathbf{q}}_i^{(t)} + \beta_t(\hat{\mathbf{q}}_i^{(t)} - \hat{\mathbf{q}}_i^{(t-1)})$ for all $i \in [m]$.
15:   $\mathbf{p}_i^{(t)} = \mathbf{z}_{i,p}^{(t)} - \eta \frac{\partial \mathcal{L}_i(\mathbf{X}_i(\mathbf{z}_{i,p}^{(t)} + \mathbf{z}_{i,q}^{(t)}), \mathbf{y}_i)}{\partial \hat{\mathbf{p}}_i^{(t)}}$ for all $i \in [m]$.
16:   $\mathbf{q}_i^{(t)} = \mathbf{z}_{i,q}^{(t)} - \eta \frac{\partial \mathcal{L}_i(\mathbf{X}_i(\mathbf{z}_{i,p}^{(t)} + \mathbf{z}_{i,q}^{(t)}), \mathbf{y}_i)}{\partial \hat{\mathbf{q}}_i^{(t)}}$ for all $i \in [m]$.
17: **end for**

MP-MTL algorithms, we perform private sharing after introducing the perturbation to the parameter matrix to protect a single data instance, as described in Algorithm 5, where a noise vector is added in Step 5 to the model vector based on sensitivity of replacing a single data instance.

The results in Proposition 4 show that we can simultaneously protect a single data instance and the model using such a divide-and-conquer strategy. Because it is not necessary to protect all the data instances in each task using data-protected algorithms, the perturbation for data-instance protection can be reduced.

Proposition 4. *Use Lemma 4 and Theorem 1. Algorithm 5 is an $(\epsilon_{mp}, \delta_{mp})$ - Iterative MP-MTL algorithm and an $(\epsilon_{dp}, \delta_{dp})$ - Iterative DP-MTL algorithm.*

Algorithm 5 MP-MTL framework with Secure Multi-party Computation (SMC)

Input: Datasets $(\mathbf{X}^m, \mathbf{y}^m) = \{(\mathbf{X}_1, \mathbf{y}_1), \ldots, (\mathbf{X}_m, \mathbf{y}_m)\}$, where $\forall i \in [m]$, $\mathbf{X}_i \in \mathbb{R}^{n_i \times d}$ and $\mathbf{y}_i \in \mathbb{R}^{n_i \times 1}$. Privacy loss for model protection $\epsilon_{\mathrm{mp}}, \delta_{\mathrm{mp}} \geq 0$. Privacy loss for single data instance protection $\epsilon_{\mathrm{dp}} \geq 0$. Number of iterations T. Shared information matrices $\mathbf{M}^{(0)}$. Initial models of tasks $\mathbf{W}^{(0)}$.

Output: $\widehat{\mathbf{W}}^{(1:T)}$.

1: For $t = 1, \ldots, T$, set $\epsilon_{\mathrm{mp},t}$ such that $\tilde{\epsilon}_{\mathrm{mp}} \leq \epsilon_{\mathrm{mp}}$, where $\tilde{\epsilon}_{\mathrm{mp}}$ is defined in (7), taking $\epsilon_t = \epsilon_{\mathrm{mp},t}, \epsilon = \epsilon_{\mathrm{mp}}, \delta = \delta_{\mathrm{mp}}$.

2: For $t = 1, \ldots, T$, set $\epsilon_{\mathrm{dp},t}$ such that $\tilde{\epsilon}_{\mathrm{dp}} \leq \epsilon_{\mathrm{dp}}$, where $\tilde{\epsilon}_{\mathrm{dp}}$ is defined in (7), taking $\epsilon_t = \epsilon_{\mathrm{dp},t}, \epsilon = \epsilon_{\mathrm{dp}}, \delta = \delta_{\mathrm{dp}}$.

3: **for** $t = 1 : T$ **do**

4:     Compute the sensitivity vector $\tilde{\mathbf{s}}^{(t-1)} = [\tilde{s}_1^{(t-1)}, \ldots, \tilde{s}_m^{(t-1)}]^{\mathrm{T}}$, which is defined for all $i \in [m]$,

$$\tilde{s}_i^{(t-1)} = \max_{(\mathbf{w}_i')^{(t-1)}} \|\mathbf{w}_i^{(t-1)} - (\mathbf{w}_i')^{(t-1)}\|_2,$$

    where $(\mathbf{w}_i')^{(t-1)}$ is assumed to be generated using $\mathcal{D}_i'$, which differ with $\mathcal{D}_i$ in a single data instance.

5:     $\tilde{\mathbf{w}}_i^{(t-1)} = \mathbf{w}_i^{(t-1)} + \mathbf{b}_i$, where $\mathbf{b}_i$ is a sample with the density function of

$$p(\mathbf{b}_i) \propto \exp\left(-\frac{\tilde{s}_i^{(t-1)}}{\epsilon_{\mathrm{dp},t}} \|\mathbf{b}_i\|_2\right),$$

    for all $i \in [m]$.

6:     Compute the sensitivity vector $\mathbf{s}^{(t-1)} = [s_1^{(t-1)}, \ldots, s_m^{(t-1)}]^{\mathrm{T}}$, which is defined for all $i \in [m]$,

$$s_i^{(t-1)} = \max_{(\mathbf{w}_i')^{(t-1)}} |\|\tilde{\mathbf{w}}_i^{(t-1)}\|_2^2 - \|(\tilde{\mathbf{w}}_i')^{(t-1)}\|_2^2|,$$

    where $(\tilde{\mathbf{w}}_i')^{(t-1)}$ is assumed to be generated arbitrarily.

7:     $\widetilde{\boldsymbol{\Sigma}}^{(t)} = \widetilde{\mathbf{W}}^{(t-1)}(\widetilde{\mathbf{W}}^{(t-1)})^{\mathrm{T}}$ (or $\widetilde{\boldsymbol{\Sigma}}^{(t)} = (\widetilde{\mathbf{W}}^{(t-1)})^{\mathrm{T}}\widetilde{\mathbf{W}}^{(t-1)}$).

8:     $\boldsymbol{\Sigma}^{(t)} = \widetilde{\boldsymbol{\Sigma}}^{(t)} + \mathbf{E}$, where $\mathbf{E} \sim W_d(\frac{\max_i s_i^{(t-1)}}{2\epsilon_{\mathrm{mp},t}}\mathbf{I}_d, d+1)$ (or $\mathbf{E} \sim W_m(\mathrm{diag}(\mathbf{s}^{(t-1)}/2\epsilon_{\mathrm{mp},t}), m+1)$) is a sample of the Wishart distribution.

9:     Perform an arbitrary mapping $f : \boldsymbol{\Sigma}^{(1:t)} \to \mathbf{M}^{(t)}$.

10:     $\hat{\mathbf{w}}_i^{(t)} = \mathcal{A}_{\mathrm{st},i}(\mathbf{M}^{(t)}, \tilde{\mathbf{w}}_i^{(0:t-1)}, \mathbf{X}_i, \mathbf{y}_i)$ for all $i \in [m]$, where $\mathbf{w}_i^{(0:t-1)}$ are for the initialization.

11:     Set the input for the next iteration: $\mathbf{W}^{(t)} = \widehat{\mathbf{W}}^{(t)}$.

12: **end for**

APPENDIX E
RESULTS OF UTILITY ANALYSES UNDER OTHER TWO SETTINGS

Here we consider the other two settings of $\{\epsilon_t\}$.

A. Setting No.1

In this setting, we have

$$\epsilon = \sum_{t=1}^{T} \epsilon_t.$$

Theorem 7 (Low rank - Convexity - Setting No.1). *Consider Algorithm 2. For an index $k \leq q$ that suffices the definition in Lemma 2 for all $t \in [T]$, $\eta = 1/L$, $\lambda = \Theta(LK\sqrt{m})$, assume $\epsilon_t \leq 4Kk^2d(\log d)/q^2$ for $t \in [T]$.*
No acceleration: *If we set $\beta_t = 0$ for $t \in [m]$, then setting*

$$T = \Theta\left(\left[\frac{(\alpha/2-1)^2|\alpha+1|\sqrt{m}\epsilon}{kd\log d}\right]^{\phi(\alpha)}\right)$$

for $\mathcal{E} = f(\frac{1}{T}\sum_{t=1}^{T}\widehat{\mathbf{W}}^{(t)}) - f(\mathbf{W}_*)$, we have with high probability,

$$\mathcal{E} = O\left(K^2L\left[\frac{kd\log d}{(\alpha/2-1)^2|\alpha+1|\sqrt{m}\epsilon}\right]^{\phi(\alpha)}\right), \quad (25)$$

where

$$\phi(\alpha) = \begin{cases} 1/(\alpha+1), & \alpha > 2; \\ 1/3, & -1 < \alpha < 2; \\ 1/(2-\alpha), & \alpha < -1. \end{cases} \quad (26)$$

Accelerated: *If we set $\beta_t = (t-1)/(t+2)$ for $t \in [m]$, then setting*

$$T = \Theta\left(\left[\frac{(\alpha/2-2)^2|\alpha+1|\sqrt{m}\epsilon}{kd\log d}\right]^{\phi(\alpha)/2}\right)$$

for $\mathcal{E} = f(\widehat{\mathbf{W}}^{(T)}) - f(\mathbf{W}_*)$, we have with high probability,

$$\mathcal{E} = O\left(K^2L\left[\frac{kd\log d}{(\alpha/2-2)^2|\alpha+1|\sqrt{m}\epsilon}\right]^{\phi(\alpha)}\right), \quad (27)$$

where

$$\phi(\alpha) = \begin{cases} 2/(\alpha+1), & \alpha > 4; \\ 2/5, & -1 < \alpha < 4; \\ 2/(4-\alpha), & \alpha < -1. \end{cases} \quad (28)$$

Theorem 8 (Group sparse - Convexity - Setting No.1). *Consider Algorithm 3. For an index $k \leq d$ that suffices the definition in Lemma 3 for all $t \in [T]$, $\eta = 1/L$, $\lambda = \Theta(LKd\sqrt{m})$, assume $\epsilon_t \leq k^2\log(d)/4Kd(d-k)^2m$ for $t \in [T]$.*
No acceleration: *If we set $\beta_t = 0$ for $t \in [m]$, then setting*

$$T = \Theta\left(\left[\frac{(\alpha/2-1)^2|\alpha+1|Km\epsilon}{k\log d}\right]^{\phi(\alpha)}\right).$$

for $\mathcal{E} = f(\frac{1}{T}\sum_{t=1}^{T}\widehat{\mathbf{W}}^{(t)}) - f(\mathbf{W}_*)$, we have with high probability,

$$\mathcal{E} = O\left(K^2L\left[\frac{k\log d}{(\alpha/2-1)^2|\alpha+1|Km\epsilon}\right]^{\phi(\alpha)}\right), \quad (29)$$

where $\phi(\alpha)$ is defined in (26).
Accelerated: *If we set $\beta_t = (t-1)/(t+2)$ for $t \in [m]$, then setting*

$$T = \Theta\left(\left[\frac{(\alpha/2-2)^2|\alpha+1|Km\epsilon}{k\log d}\right]^{\phi(\alpha)/2}\right).$$

for $\mathcal{E} = f(\widehat{\mathbf{W}}^{(T)}) - f(\mathbf{W}_*)$, we have with high probability,

$$\mathcal{E} = O\left(K^2L\left[\frac{k\log d}{(\alpha/2-2)^2|\alpha+1|Km\epsilon}\right]^{\phi(\alpha)}\right), \quad (30)$$

where $\phi(\alpha)$ is defined in (28).

Now we further assume that $mf(\mathbf{W})$ is μ-strongly convex and has L-Lipschitz-continuous gradient, where $\mu < L$. We set $\epsilon_t = \Theta(Q^{-t})$ for $Q > 0$ and $t \in [T]$ for this case.

Theorem 9 (Low rank - Strong convexity - Setting No.1). *Consider Algorithm 2. For an index $k \leq q$ that suffices the definition in Lemma 2 for all $t \in [T]$, $\eta = 1/L$, $\lambda = \Theta(LK\sqrt{m})$, assume $\epsilon_t \leq 4Kk^2d(\log d)/q^2$ for $t \in [T]$.*
No acceleration: *If we set $\beta_t = 0$ for $t \in [m]$, then denoting $Q_0 = 1 - \mu/L$ and setting*

$$T = \Theta\left(\log_{1/\psi(Q,Q_0^2)}\left[\frac{(Q_0/\sqrt{Q}-1)^2|1-Q|\sqrt{m}\epsilon}{kd\log d}\right]\right)$$

for $\mathcal{E} = \frac{1}{\sqrt{m}}\|\widehat{\mathbf{W}}^{(T)} - \mathbf{W}_*\|_F$, we have with high probability,

$$\mathcal{E} = O\left(K\left[\frac{kd\log d}{(Q_0/\sqrt{Q}-1)^2|1-Q|\sqrt{m}\epsilon}\right]^{\log_{\psi(Q,Q_0^2)}Q_0}\right), \quad (31)$$

where $\psi(\cdot, \cdot)$ *is defined in* (19).

Accelerated*: If we set* $\beta_t = (1 - \sqrt{\mu/L})/(1 + \sqrt{\mu/L})$ *for* $t \in [m]$, *then denoting* $Q'_0 = 1 - \sqrt{\mu/L}$ *and setting*

$$T = \Theta\left(\log_{1/\psi(Q,Q'_0)}\left[\frac{(\sqrt{Q'_0}/\sqrt{Q} - 1)^2|1 - Q|\sqrt{m}\epsilon}{kd\log d}\right]\right)$$

for $\mathcal{E} = f(\widehat{\mathbf{W}}^{(T)}) - f(\mathbf{W}_*)$, *we have with high probability,*

$$\mathcal{E} = O\left(K\left[\frac{kd\log d}{(\sqrt{Q'_0}/\sqrt{Q} - 1)^2|1 - Q|\sqrt{m}\epsilon}\right]^{\log_{\psi(Q,Q'_0)} Q'_0}\right), \quad (32)$$

where $\psi(\cdot, \cdot)$ *is defined in* (19).

Theorem 10 (Group sparse - Strong convexity - Setting No.1)**.** *Consider Algorithm 3. For an index* $k \le d$ *that suffices the definition in Lemma 3 for all* $t \in [T]$, $\eta = 1/L$, $\lambda = \Theta(LKd\sqrt{m})$, *assume* $\epsilon_t \le k^2\log(d)/4Kd(d - k)^2m$ *for* $t \in [T]$.

No acceleration*: If we set* $\beta_t = 0$ *for* $t \in [m]$, *then denoting* $Q_0 = 1 - \mu/L$ *and setting*

$$T = \Theta\left(\log_{1/\psi(Q,Q_0^2)}\left[\frac{(Q_0/\sqrt{Q} - 1)^2|1 - Q|Km\epsilon}{k\log d}\right]\right)$$

for $\mathcal{E} = \frac{1}{\sqrt{m}}\|\widehat{\mathbf{W}}^{(T)} - \mathbf{W}_*\|_F$, *we have with high probability,*

$$\mathcal{E} = O\left(K\left[\frac{k\log d}{(Q_0/\sqrt{Q} - 1)^2|1 - Q|Km\epsilon}\right]^{\log_{\psi(Q,Q_0^2)} Q_0}\right), \quad (33)$$

where $\psi(\cdot, \cdot)$ *is defined in* (19).

Accelerated*: If we set* $\beta_t = (1 - \sqrt{\mu/L})/(1 + \sqrt{\mu/L})$ *for* $t \in [m]$, *then denoting* $Q'_0 = 1 - \sqrt{\mu/L}$ *and setting*

$$T = \Theta\left(\log_{1/\psi(Q,Q'_0)}\left[\frac{(\sqrt{Q'_0}/\sqrt{Q} - 1)^2|1 - Q|Km\epsilon}{k\log d}\right]\right)$$

for $\mathcal{E} = f(\widehat{\mathbf{W}}^{(T)}) - f(\mathbf{W}_*)$, *we have with high probability,*

$$\mathcal{E} = O\left(K\left[\frac{k\log d}{(\sqrt{Q'_0}/\sqrt{Q} - 1)^2|1 - Q|Km\epsilon}\right]^{\log_{\psi(Q,Q'_0)} Q'_0}\right), \quad (34)$$

where $\psi(\cdot, \cdot)$ *is defined in* (19).

Then we optimize the utility bounds with respect to the respective budget allocation strategies.

Theorem 11 (Budget allocation - Setting No.1)**.** *Consider Algorithm 2 and Algorithm 3.*

For convex f, *use Theorem 7 and Theorem 8.*

(1) No acceleration: Both the bounds in (25) and (29) achieve their respective minimums w.r.t. α *at* $\alpha = 0$. *Meanwhile,* $\phi(\alpha) = 1/3$.

(2) Accelerated: Both the bounds in (27) and (30) achieve their respective minimums w.r.t. α *at* $\alpha = 2/3$. *Meanwhile,* $\phi(\alpha) = 2/5$.

For strongly convex f, *use Theorem 9 and Theorem 10.*

(1) No acceleration: Both the bounds in (31) and (33) achieve their respective minimums w.r.t. Q *at* $Q = Q_0^{2/3}$. *Meanwhile,* $\log_{\psi(Q,Q_0^2)} Q_0 = 1/2$.

(2) Accelerated: Both the bounds in (32) and (34) achieve their respective minimums w.r.t. Q *at* $Q = (Q'_0)^{1/3}$. *Meanwhile,* $\log_{\psi(Q,Q'_0)} Q'_0 = 1$.

B. Setting No.2

In this setting, we have

$$\epsilon = \sum_{t=1}^{T}\frac{(e^{\epsilon_t} - 1)\epsilon_t}{(e^{\epsilon_t} + 1)} + \sqrt{\sum_{t=1}^{T}2\epsilon_t^2\log\left(\frac{1}{\delta}\right)}.$$

Theorem 12 (Low rank - Convexity - Setting No.2)**.** *Consider Algorithm 2. For an index* $k \le q$ *that suffices the definition in*

Lemma 2 for all $t \in [T]$, $\eta = 1/L$, $\lambda = \Theta(LK\sqrt{m})$, *assume* $\epsilon_t \le 4Kk^2d(\log d)/q^2$ *for* $t \in [T]$.

No acceleration*: If we set* $\beta_t = 0$ *for* $t \in [m]$, *then setting*

$$T = \Theta\left(\left[\frac{(\alpha/2 - 1)^2\sqrt{|2\alpha + 1|}\sqrt{m}\epsilon}{kd\log d\sqrt{\log(1/\delta) + 2\epsilon}}\right]^{\phi(\alpha)}\right)$$

for $\mathcal{E} = f(\frac{1}{T}\sum_{t=1}^{T}\widehat{\mathbf{W}}^{(t)}) - f(\mathbf{W}_*)$, *we have with high probability,*

$$\mathcal{E} = O\left(K^2L\left[\frac{kd\log d\sqrt{\log(1/\delta) + 2\epsilon}}{(\alpha/2 - 1)^2\sqrt{|2\alpha + 1|}\sqrt{m}\epsilon}\right]^{\phi(\alpha)}\right), \quad (35)$$

where

$$\phi(\alpha) = \begin{cases} 2/(2\alpha + 1), & \alpha > 2; \\ 2/5, & -1/2 < \alpha < 2; \\ 1/(2 - \alpha), & \alpha < -1/2. \end{cases} \quad (36)$$

Accelerated*: If we set* $\beta_t = (t - 1)/(t + 2)$ *for* $t \in [m]$, *then setting*

$$T = \Theta\left(\left[\frac{(\alpha/2 - 2)^2\sqrt{|2\alpha + 1|}\sqrt{m}\epsilon}{kd\log d\sqrt{\log(1/\delta) + 2\epsilon}}\right]^{\phi(\alpha)/2}\right)$$

for $\mathcal{E} = f(\widehat{\mathbf{W}}^{(T)}) - f(\mathbf{W}_*)$, *we have with high probability,*

$$\mathcal{E} = O\left(K^2L\left[\frac{kd\log d\sqrt{\log(1/\delta) + 2\epsilon}}{(\alpha/2 - 2)^2\sqrt{|2\alpha + 1|}\sqrt{m}\epsilon}\right]^{\phi(\alpha)}\right), \quad (37)$$

where

$$\phi(\alpha) = \begin{cases} 4/(2\alpha + 1), & \alpha > 4; \\ 4/9, & -1/2 < \alpha < 4; \\ 2/(4 - \alpha), & \alpha < -1/2. \end{cases} \quad (38)$$

Theorem 13 (Group sparse - Convexity - Setting No.2)**.** *Consider Algorithm 3. For an index* $k \le d$ *that suffices the definition in Lemma 3 for all* $t \in [T]$, $\eta = 1/L$, $\lambda = \Theta(LKd\sqrt{m})$, *assume* $\epsilon_t \le k^2\log(d)/4Kd(d - k)^2m$ *for* $t \in [T]$.

No acceleration*: If we set* $\beta_t = 0$ *for* $t \in [m]$, *then setting*

$$T = \Theta\left(\left[\frac{(\alpha/2 - 1)^2\sqrt{|2\alpha + 1|}Km\epsilon}{k\log d\sqrt{\log(1/\delta) + 2\epsilon}}\right]^{\phi(\alpha)}\right).$$

for $\mathcal{E} = f(\frac{1}{T}\sum_{t=1}^{T}\widehat{\mathbf{W}}^{(t)}) - f(\mathbf{W}_*)$, *we have with high probability,*

$$\mathcal{E} = O\left(K^2L\left[\frac{k\log d\sqrt{\log(1/\delta) + 2\epsilon}}{(\alpha/2 - 1)^2\sqrt{|2\alpha + 1|}Km\epsilon}\right]^{\phi(\alpha)}\right), \quad (39)$$

where $\phi(\alpha)$ *is defined in* (36).

Accelerated*: If we set* $\beta_t = (t - 1)/(t + 2)$ *for* $t \in [m]$, *then setting*

$$T = \Theta\left(\left[\frac{(\alpha/2 - 2)^2\sqrt{|2\alpha + 1|}Km\epsilon}{k\log d\sqrt{\log(1/\delta) + 2\epsilon}}\right]^{\phi(\alpha)/2}\right).$$

for $\mathcal{E} = f(\widehat{\mathbf{W}}^{(T)}) - f(\mathbf{W}_*)$, *we have with high probability,*

$$\mathcal{E} = O\left(K^2L\left[\frac{k\log d\sqrt{\log(1/\delta) + 2\epsilon}}{(\alpha/2 - 2)^2\sqrt{|2\alpha + 1|}Km\epsilon}\right]^{\phi(\alpha)}\right), \quad (40)$$

where $\phi(\alpha)$ *is defined in* (38).

Now we further assume that $mf(\mathbf{W})$ is μ-strongly convex and has L-Lipschitz-continuous gradient, where $\mu < L$. We set $\epsilon_t = \Theta(Q^{-t})$ for $Q > 0$ and $t \in [T]$ for this case.

Theorem 14 (Low rank - Strong convexity - Setting No.2)**.** *Consider Algorithm 2. For an index* $k \le q$ *that suffices the definition in Lemma 2 for all* $t \in [T]$, $\eta = 1/L$, $\lambda = \Theta(LK\sqrt{m})$, *assume* $\epsilon_t \le 4Kk^2d(\log d)/q^2$ *for* $t \in [T]$.

No acceleration: *If we set $\beta_t = 0$ for $t \in [m]$, then denoting $Q_0 = 1 - \mu/L$ and setting*

$$T = \Theta\left(\log_{1/\psi(Q,Q_0^2)}\left[\frac{(Q_0/\sqrt{Q}-1)^2\sqrt{|1-Q^2|}\sqrt{m}\epsilon}{kd\log d\sqrt{\log(1/\delta)+2\epsilon}}\right]\right)$$ (41)

for $\mathcal{E} = \frac{1}{\sqrt{m}}\|\widehat{\mathbf{W}}^{(T)} - \mathbf{W}_*\|_F$, we have with high probability,

$$\mathcal{E} = O\left(K\left[\frac{kd\log d\sqrt{\log(1/\delta)+2\epsilon}}{(Q_0/\sqrt{Q}-1)^2\sqrt{|1-Q^2|}\sqrt{m}\epsilon}\right]^{\log_{\psi(Q,Q_0^2)}Q_0}\right),$$ (41)

where $\psi(\cdot,\cdot)$ is defined in (19).
Accelerated: *If we set $\beta_t = (1-\sqrt{\mu/L})/(1+\sqrt{\mu/L})$ for $t \in [m]$, then denoting $Q_0' = 1 - \sqrt{\mu/L}$ and setting*

$$T = \Theta\left(\log_{1/\psi(Q,Q_0')}\left[\frac{(\sqrt{Q_0'}/\sqrt{Q}-1)^2\sqrt{|1-Q^2|}\sqrt{m}\epsilon}{kd\log d\sqrt{\log(1/\delta)+2\epsilon}}\right]\right)$$ (42)

for $\mathcal{E} = f(\widehat{\mathbf{W}}^{(T)}) - f(\mathbf{W}_*)$, we have with high probability,

$$\mathcal{E} = O\left(K\left[\frac{kd\log d\sqrt{\log(1/\delta)+2\epsilon}}{(\sqrt{Q_0'}/\sqrt{Q}-1)^2\sqrt{|1-Q^2|}\sqrt{m}\epsilon}\right]^{\log_{\psi(Q,Q_0')}Q_0'}\right),$$ (42)

where $\psi(\cdot,\cdot)$ is defined in (19).

Theorem 15 (Group sparse - Strong convexity - Setting No.2)**.** *Consider Algorithm 3. For an index $k \le d$ that suffices the definition in Lemma 3 for all $t \in [T]$, $\eta = 1/L$, $\lambda = \Theta(LKd\sqrt{m})$, assume $\epsilon_t \le k^2\log(d)/4Kd(d-k)^2m$ for $t \in [T]$.*
No acceleration: *If we set $\beta_t = 0$ for $t \in [m]$, then denoting $Q_0 = 1 - \mu/L$ and setting*

$$T = \Theta\left(\log_{1/\psi(Q,Q_0^2)}\left[\frac{(Q_0/\sqrt{Q}-1)^2\sqrt{|1-Q^2|}Km\epsilon}{k\log d\sqrt{\log(1/\delta)+2\epsilon}}\right]\right)$$ (43)

for $\mathcal{E} = \frac{1}{\sqrt{m}}\|\widehat{\mathbf{W}}^{(T)} - \mathbf{W}_*\|_F$, we have with high probability,

$$\mathcal{E} = O\left(K\left[\frac{k\log d\sqrt{\log(1/\delta)+2\epsilon}}{(Q_0/\sqrt{Q}-1)^2\sqrt{|1-Q^2|}Km\epsilon}\right]^{\log_{\psi(Q,Q_0^2)}Q_0}\right),$$ (43)

where $\psi(\cdot,\cdot)$ is defined in (19).
Accelerated: *If we set $\beta_t = (1-\sqrt{\mu/L})/(1+\sqrt{\mu/L})$ for $t \in [m]$, then denoting $Q_0' = 1 - \sqrt{\mu/L}$ and setting*

$$T = \Theta\left(\log_{1/\psi(Q,Q_0')}\left[\frac{(\sqrt{Q_0'}/\sqrt{Q}-1)^2\sqrt{|1-Q^2|}Km\epsilon}{k\log d\sqrt{\log(1/\delta)+2\epsilon}}\right]\right)$$ (44)

for $\mathcal{E} = f(\widehat{\mathbf{W}}^{(T)}) - f(\mathbf{W}_*)$, we have with high probability,

$$\mathcal{E} = O\left(K\left[\frac{k\log d\sqrt{\log(1/\delta)+2\epsilon}}{(\sqrt{Q_0'}/\sqrt{Q}-1)^2\sqrt{|1-Q^2|}Km\epsilon}\right]^{\log_{\psi(Q,Q_0')}Q_0'}\right),$$ (44)

where $\psi(\cdot,\cdot)$ is defined in (19).

Then we optimize the utility bounds with respect to the respective budget allocation strategies.

Theorem 16 (Budget allocation - Setting No.2)**.** *Consider Algorithm 2 and Algorithm 3.*
For convex f, use Theorem 12 and Theorem 13.
(1) No acceleration: *Both the bounds in* (35) *and* (39) *achieve their respective minimums w.r.t. α at $\alpha = 0$. Meanwhile, $\phi(\alpha) = 2/5$.*
(2) Accelerated: *Both the bounds in* (37) *and* (40) *achieve their respective minimums w.r.t. α at $\alpha = 2/5$. Meanwhile, $\phi(\alpha) = 4/9$.*
For strongly convex f, use Theorem 14 and Theorem 15.

(1) No acceleration: *Both the bounds in* (41) *and* (43) *achieve their respective minimums w.r.t. Q at $Q = Q_0^{2/5}$. Meanwhile, $\log_{\psi(Q,Q_0^2)}Q_0 = 1/2$.*
(2) Accelerated: *Both the bounds in* (42) *and* (44) *achieve their respective minimums w.r.t. Q at $Q = (Q_0')^{1/5}$. Meanwhile, $\log_{\psi(Q,Q_0')}Q_0' = 1$.*

APPENDIX F
TRADEOFF BETWEEN ACCURACY AND TRAINING-DATA PERCENTAGE

Since the MTL behavior changes significantly when the training-data percentage (the size of the training data divided by the size of the entire dataset) changes, we evaluated the methods on both real-world datasets at different training-data percentages. We use the unified normalized score defined below to evaluate all methods relative to the performances of the non-private STL and non-private MTL methods in each setting:

$$s_{\text{norm}} = \frac{s_{\text{priv}} - s_{\text{STL}}}{s_{\text{MTL}} - s_{\text{STL}}},$$ (45)

where s_{STL} denotes the average performance of the non-private STL methods, s_{MTL} denotes the average performance of the non-private MTL methods, and s_{priv} denotes the average performance of the private method under evaluation, which may be either of our MP-MTL algorithms or the DP-MTRL method. The performance measure used may be either the average AUC or the nMSE. A larger value of the normalized score implies better performance of the evaluated method. Such a normalized score describes the relative prediction performance of the evaluated method.

Here, we present the results only for our low-rank algorithm (denoted by MP-MTL-LR) because it always outperforms our group-sparse algorithm (MP-MTL-GS) in the above experiments.

From Fig. 1, we can observe that on the School dataset, our MP-MTL method behaves similarly at different training-data percentages, whereas on the Patient Survey dataset, the performances are best when the training-data percentage is equal to 30% and worst when the training-data percentage is equal to 90%. These findings can be attributed to the following factors: (1) the relative advantage of MTL over STL is more significant when the training-data percentage is small, and (2) the average data size in the Patient Survey dataset is much larger than that in the School dataset, causing the advantage of MTL over STL to be more likely to saturate when the training-data percentage is large. At each training-data percentage, our MP-MTL method outperforms DP-MTRL, especially when ϵ is small.

APPENDIX G
## LEMMAS OF DIFFERENTIAL PRIVACY

- Post-Processing immunity. This property helps us safely use the output of a differentially private algorithm without additional information leaking, as long as we do not touch the dataset $\mathcal{D}$ again.

  Lemma 4 (Post-Processing immunity. Proposition 2.1 in Dwork et al. [3])**.** *Let algorithm $\mathcal{A}_1(\mathcal{B}_1) : \mathcal{D} \to \theta_1 \in \mathcal{C}_1$ be an (ϵ, δ) - differential privacy algorithm, and let $f : \mathcal{C}_1 \to \mathcal{C}_2$ be an arbitrary mapping. Then, algorithm $\mathcal{A}_2(\mathcal{B}_2) : \mathcal{D} \to \theta_2 \in \mathcal{C}_2$ is still (ϵ, δ) - differentially private, i.e., for any set $\mathcal{S} \subseteq \mathcal{C}_2$,*
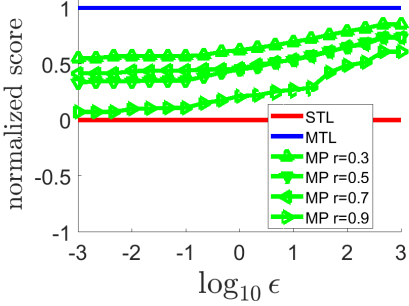
  $$\mathbb{P}(\theta_2 \in \mathcal{S} \mid \mathcal{B}_2 = \mathcal{D}) \le e^\epsilon \mathbb{P}(\theta_2 \in \mathcal{S} \mid \mathcal{B}_2 = \mathcal{D}') + \delta.$$

- Group privacy. This property guarantees the graceful increment of the privacy budget when more output variables need differentially private protection.

  Lemma 5 (Group privacy. Lemma 2.2 in Vadhan [12])**.** *Let algorithm $\mathcal{A}(\mathcal{B}) : \mathcal{D} \to \theta \in \mathcal{C}$ be an (ϵ, δ) - differential privacy algorithm. Then, considering two neighboring datasets $\mathcal{D}$ and*
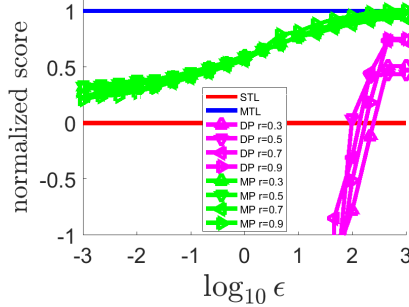
(a) School Data

(b) Patient Survey Data

Figure 1. Algorithm performance on real-world datasets at different training-data percentages. The normalized score is defined in (45). MTL denotes the non-private MTL method proposed by Ji and Ye [6]. MP denotes our low-rank method MP-MTL-LR, and DP denotes DP-MTRL. r denotes the training-data percentage.

$\mathcal{D}'$ that differ in k entries, the algorithm satisfies for any set $\mathcal{S} \subseteq \mathcal{C}$

$$\mathbb{P}(\theta \in \mathcal{S} \mid \mathcal{B} = \mathcal{D}) \le e^{k\epsilon}\mathbb{P}(\theta \in \mathcal{S} \mid \mathcal{B} = \mathcal{D}') + ke^{k\epsilon}\delta.$$

- Combination. This property guarantees the linear incrementing of the privacy budget when the dataset $\mathcal{D}$ is repeatedly used.

  Lemma 6 (Combination. Theorem 3.16 in Dwork et al. [3])**.** *Let algorithm $\mathcal{A}_i : \mathcal{D} \to \theta_i \in \mathcal{C}_i$ be an (ϵ_i, δ_i) - differential privacy algorithm for all $i \in [k]$. Then, for $\mathcal{A}_{[k]} : \mathcal{D} \to (\theta_1, \theta_2, \cdots, \theta_k) \in \bigotimes_{j=1}^{k} \mathcal{C}_j$ is a $\left(\sum_i \epsilon_i, \sum_i \delta_i\right)$ - differentially private algorithm.*

- Adaptive composition. This property guarantees privacy when an iterative algorithm is adopted on *different* datasets that may nevertheless contain information relating to the same individual.

  Lemma 7 (Adaptive composition. Directly taken Theorem 3.5 in Kairouz et al. [8])**.** *Let algorithm $\mathcal{A}_1(\mathcal{B}_1) : \mathcal{D}_1 \to \theta_1$ be an (ϵ_1, δ_1) - differential privacy algorithm, and for $t = 2, \ldots, T$, let $\mathcal{A}_t(\mathcal{B}_t) : (\mathcal{D}_t, \theta_1, \theta_2, \cdots, \theta_{t-1}) \to \theta_t \in \mathcal{C}_t$ be (ϵ_t, δ_t) - differentially private for all given $(\theta_1, \theta_2, \cdots, \theta_{t-1}) \in \bigotimes_{t'=1}^{t-1} \mathcal{C}_{t'}$. Then, for all neighboring datasets $\mathcal{D}_t$ and $\mathcal{D}_t'$ that differ in a single entry relating to the same individual and for any set $\mathcal{S} \subseteq \bigotimes_{t=1}^{T} \mathcal{C}_t,$*

$$\mathbb{P}\left((\theta_1, \cdots, \theta_T) \in \mathcal{S} \mid \bigcap_{t=1}^{T}(\mathcal{B}_t = (\mathcal{D}_t, \boldsymbol{\theta}_{1:t-1}))\right)$$

$$\le e^{\epsilon}\mathbb{P}\left((\theta_1, \cdots, \theta_T) \in \mathcal{S} \mid \bigcap_{t=1}^{T}(\mathcal{B}_t = (\mathcal{D}_t', \boldsymbol{\theta}_{1:t-1}))\right) \quad (46)$$

$$+ 1 - (1-\delta)\prod_{t=1}^{T}(1-\delta_t),$$

where

$$\boldsymbol{\theta}_{1:t-1} = \begin{cases} \emptyset, & t = 1 \\ \theta_1, \theta_2, \cdots, \theta_{t-1}, & t \ge 2, \end{cases}$$

and

$$\epsilon = \min\left\{\sum_{t=1}^{T} \epsilon_t, \sum_{t=1}^{T} \frac{(e^{\epsilon_t} - 1)\epsilon_t}{(e^{\epsilon_t} + 1)} + \sqrt{\sum_{t=1}^{T} 2\epsilon_t^2 \log\left(\frac{1}{\delta}\right)}, \right.$$

$$\left. \sum_{t=1}^{T} \frac{(e^{\epsilon_t} - 1)\epsilon_t}{(e^{\epsilon_t} + 1)} + \sqrt{\sum_{t=1}^{T} 2\epsilon_t^2 \log\left(e + \frac{\sqrt{\sum_{t=1}^{T} \epsilon_t^2}}{\delta}\right)}\right\}.$$

APPENDIX H
LEMMAS FOR PRIVACY GUARANTEES

The following lemma shows that STL algorithms do not increase the privacy budget when they are concatenated with an MTL algorithm.

Lemma 8. *For an (ϵ, δ) - Non-iterative MP-MTL algorithm $\mathcal{A}_{mp}$: $(\mathbf{W} \in \mathbb{R}^{d \times m}, \mathcal{D}^m) \to \widetilde{\mathbf{W}} \in \mathbb{R}^{d \times m}$ and any STL algorithm $\mathcal{A}_{st} : (\widetilde{\mathbf{W}}, \mathcal{D}^m) \to \widehat{\mathbf{W}} \in \mathbb{R}^{d \times m}$, an algorithm $\mathcal{A}_{mp+st} : (\mathbf{W} \in \mathbb{R}^{d \times m}, \mathcal{D}^m) \to \widehat{\mathbf{W}} \in \mathbb{R}^{d \times m}$ that first uses $\mathcal{A}_{mp}$ before applying $\mathcal{A}_{st}$ is still an (ϵ, δ) - Non-iterative MP-MTL algorithm. Moreover, an algorithm $\mathcal{A}_{st+mp}$ that first uses a deterministic STL algorithm before applying an (ϵ, δ) - Non-iterative MP-MTL algorithm is also an (ϵ, δ) - Non-iterative MP-MTL algorithm.*

The following result shows that adopting a series of Non-iterative MP-MTL algorithms defined in Definition 5 iteratively, we can develop an Iterative MP-MTL algorithm, as described in Algorithm 6.

Algorithm 6 Iterative MP-MTL build by Non-iterative MP-MTL

Input: Datasets $(\mathbf{X}^m, \mathbf{y}^m) = \{(\mathbf{X}_1, \mathbf{y}_1), \ldots, (\mathbf{X}_m, \mathbf{y}_m)\}$, where $\forall i \in [m]$, $\mathbf{X}_i \in \mathbb{R}^{n_i \times d}$ and $\mathbf{y}_i \in \mathbb{R}^{n_i \times 1}$. Number of iterations T. Privacy loss $\{\epsilon_t, \delta_t\}_{t=1,\ldots,T}, \delta \ge 0$. Initial models of tasks $\mathbf{W}^{(0)}$.

Output: $\widehat{\mathbf{W}}^{(1:T)}$.

1: **for** $t = 1 : T$ **do**
2: $\quad \widehat{\mathbf{W}}^{(t)} = \mathcal{A}_{mp}(\mathbf{W}^{(t-1)}, \mathbf{X}^m, \mathbf{y}^m)$, where $\mathcal{A}_{mp}$ denotes an (ϵ_t, δ_t)-Non-iterative MP-MTL algorithm.
3: $\quad \mathbf{w}_i^{(t)} = \mathcal{A}_{st,i}(\hat{\mathbf{w}}_i^{(t)}, \mathbf{X}_i, \mathbf{y}_i)$, for $i = 1, \ldots, m$, where $\mathcal{A}_{st,i}$ denotes a deterministic STL algorithm for the i-th task.
4: **end for**

Lemma 9. *Use Lemmas 7 and 8. Algorithm 6 is an $(\epsilon, 1 - (1 - \delta)\prod_{t=1}^{T}(1-\delta_t))$ - Iterative MP-MTL algorithm, where ϵ is defined in Lemma 7.*

APPENDIX I
LEMMAS FOR UTILITY ANALYSIS

Lemma 10. *For a integer $T \ge 1$, a constant $\alpha \in \mathbb{R}$, by EulerMaclaurin formula [1], we have*

$$\sum_{t=1}^{T} t^\alpha = \begin{cases} O(T^{\alpha+1}/(\alpha+1)), & \alpha > -1; \\ O(1/(-\alpha-1)), & \alpha < -1. \end{cases}$$

Proof. This is the direct result of EulerMaclaurin formula [1]. $\square$

Lemma 11. *For a integer $T \ge 1$ and a constant $Q > 0$, we have*

$$\sum_{t=1}^{T} Q^{-t} \le \begin{cases} \frac{1}{Q-1}, & Q > 1; \\ \frac{Q^{-T}}{1-Q}, & Q < 1. \end{cases}$$

Proof. Because $\sum_{t=1}^{T} Q^{-t} = Q^{-1}\frac{1-Q^{-T}}{1-Q^{-1}}$, we complete the proof. $\qquad\square$

Lemma 12. *For a constant $c_1, c_2 > 0$, a constant $\epsilon_0 > 0$, a integer $T \geq 1$, a mapping $s : t \in [T] \to s(t) > 0$, a mapping $S_1 : T \to S_1(T) > 0$ and a mapping $S_2 : T \to S_2(T) > 0$, then if $\sum_{t=1}^{T} \epsilon_0 s(t) \geq c_1$ and $\sum_{t=1}^{T} s(t) \leq S_1(T)$, we have*

$$1/\epsilon_0 \leq S_1(T)/c_1.$$

On the other hand, if $\sqrt{\sum_{t=1}^{T} \epsilon_0^2 s^2(t)} \geq c_2$ and $\sum_{t=1}^{T} s^2(t) \leq S_2(T)$, we have

$$1/\epsilon_0 \leq \sqrt{S_2(T)}/c_2.$$

Proof. If $\sum_{t=1}^{T} \epsilon_0 s(t) \geq c_1$, $1/\epsilon_0 \leq \sum_{t=1}^{T} s(t)/c_1 \leq S_1(T)/c_1$.

On the other hand, if $\sqrt{\sum_{t=1}^{T} \epsilon_0^2 s^2(t)} \geq c_2$, $1/\epsilon_0 \leq \sqrt{\sum_{t=1}^{T} s^2(t)}/c_2 \leq \sqrt{S_2(T)}/c_2$. $\qquad\square$

Lemma 13. *Consider Algorithm 2. For an index $k \leq q$ that suffices the definition in Lemma 2 for all $t \in [T]$, $\eta = 1/L$, $\lambda = \Theta(LK\sqrt{m})$, set $\epsilon_t \leq 4Kk^2d(\log d)/q^2$ for $t \in [T]$. Assume in each iteration, $\mathbf{E}$ is the defined Wishart random matrix. We have with probability at least $1 - d^{-c}$ for some constant $c > 1$ that*

$$\varepsilon_t = \frac{1}{2\eta}\|\widehat{\mathbf{W}}^{(t)} - \mathbf{C}\|_F^2 + \lambda\|\widehat{\mathbf{W}}^{(t)}\|_*$$
$$- \left\{ \min_{\mathbf{W}} \frac{1}{2\eta}\|\mathbf{W} - \mathbf{C}\|_F^2 + \lambda\|\mathbf{W}\|_* \right\} \qquad (47)$$
$$= O\left(\frac{K^2\sqrt{m}kd\log d}{\eta\epsilon_t} \right).$$

Proof. First, using Lemma 1 of Jiang et al. [7], we have in the t-th step, with probability at least $1 - d^{-c}$ for some constant $c > 1$,

$$\sigma_1(\mathbf{E}) = O\left(d(\log d)\sigma_1\left(\frac{\max_i s_i^{(t-1)}}{2\epsilon_t}\mathbf{I}_d \right) \right) = O(d(\log d)K/\epsilon_t).$$

We also have $\sigma_1(\mathbf{C}) \leq \|\mathbf{C}\|_F \leq \sqrt{m}\max_i \|\mathbf{C}_i\|_2 \leq K\sqrt{m}$, where $\mathbf{C}_i$ is the i-th column of $\mathbf{C}$.

As such, by Lemma 2, in the t-th iteration, for $\epsilon_t \leq 4Kk^2d(\log d)/q^2$, where $q = \min\{d, m\}$, we have

$$\varepsilon_t = \frac{1}{2\eta}\|\widehat{\mathbf{W}}^{(t)} - \mathbf{C}\|_F^2 + \lambda\|\widehat{\mathbf{W}}^{(t)}\|_*$$
$$- \left\{ \min_{\mathbf{W}} \frac{1}{2\eta}\|\mathbf{W} - \mathbf{C}\|_F^2 + \lambda\|\mathbf{W}\|_* \right\}$$
$$\leq \frac{1}{\eta}\left(\frac{\sigma_1^2(\mathbf{C})}{\eta\lambda} + \sigma_1(\mathbf{C}) \right)\left[k\frac{\sigma_1(\mathbf{E})}{2\eta\lambda} \right.$$
$$+ (r_c - k)I(r_c > k)\sqrt{\sigma_1(\mathbf{E})} + \left(\frac{k(k-1)}{\eta\lambda} + 2k \right)\sigma_1(\mathbf{E}) \right]$$
$$\leq \frac{1}{\eta}\left(\frac{K^2m}{\eta\lambda} + K\sqrt{m} \right)\left[k\frac{\sigma_1(\mathbf{E})}{2\eta\lambda} \right.$$
$$+ q\sqrt{\sigma_1(\mathbf{E})} + \left(\frac{k(k-1)}{\eta\lambda} + 2k \right)\sigma_1(\mathbf{E}) \right]$$
$$= O\left(\frac{1}{\eta}\left(\frac{K^2m}{\eta\lambda} + K\sqrt{m} \right)\left(\frac{k^2}{\eta\lambda} + 2k \right)\frac{d(\log d)K}{\epsilon_t} \right),$$

where in the second inequality, the terms with $\sigma_1(\mathbf{E})$ nominate due to the condition on ϵ_t.

Further assuming $\eta = 1/L$ and $\lambda = \Theta(LK\sqrt{m})$, we complete the proof. $\qquad\square$

Lemma 14. *Consider Algorithm 3. For an index $k \leq d$ that suffices the definition in Lemma 3 for all $t \in [T]$, $\eta = 1/L$, $\lambda = \Theta(LKd\sqrt{m})$, set $\epsilon_t \leq k^2\log(d)/4Kd(d - k)^2m$ for $t \in [T]$. Assume in each iteration, $\mathbf{E}$ is the defined Wishart random matrix.*

We have with probability at least $1 - d^{-c}$ for some constant $c > 1$ that

$$\varepsilon_t = \frac{1}{2\eta}\|\widehat{\mathbf{W}}^{(t)} - \mathbf{C}\|_F^2 + \lambda\|\widehat{\mathbf{W}}^{(t)}\|_{2,1}$$
$$- \left\{ \min_{\mathbf{W}} \frac{1}{2\eta}\|\mathbf{W} - \mathbf{C}\|_F^2 + \lambda\|\mathbf{W}\|_{2,1} \right\} \qquad (48)$$
$$= O\left(\frac{Kk\log d}{\eta\epsilon_t} \right).$$

Proof. Similarly as in proof for Lemma 13, by Lemma 3, in the t-th iteration, we have

$$\varepsilon_t = \frac{1}{2\eta}\|\widehat{\mathbf{W}}^{(t)} - \mathbf{C}\|_F^2 + \lambda\|\widehat{\mathbf{W}}^{(t)}\|_{2,1}$$
$$- \left\{ \min_{\mathbf{W}} \frac{1}{2\eta}\|\mathbf{W} - \mathbf{C}\|_F^2 + \lambda\|\mathbf{W}\|_{2,1} \right\}$$
$$\leq \frac{1}{\eta}\left[\frac{r_{c,s}}{\eta\lambda}\left(\max_{j\in[d]}\|\mathbf{C}^j\|_2 \right)^2 + \left(\max_{j\in[d]}\|\mathbf{C}^j\|_2 \right) \right]$$
$$\cdot \left[\frac{k}{2\eta\lambda}\max_{j:\eta^2\lambda^2\leq\boldsymbol{\Sigma}_{jj,0}}|\mathbf{E}_{jj}| \right.$$
$$+ (r_{c,s} - k)I(r_{c,s} > k)\max_{j:\eta^2\lambda^2>\boldsymbol{\Sigma}_{jj,0}}\sqrt{|\mathbf{E}_{jj}|} \right]$$
$$\leq \frac{1}{\eta}\left[\frac{r_{c,s}}{\eta\lambda}\|\mathbf{C}\|_F^2 + \|\mathbf{C}\|_F \right]$$
$$\cdot \left[\frac{k}{2\eta\lambda}\sigma_1(\mathbf{E}) + (r_{c,s} - k)I(r_{c,s} > k)\sqrt{\sigma_1(\mathbf{E})} \right]$$
$$\leq \frac{1}{\eta}\left(\frac{dK^2m}{\eta\lambda} + K\sqrt{m} \right)\left[k\frac{\sigma_1(\mathbf{E})}{2\eta\lambda} + (d - k)\sqrt{\sigma_1(\mathbf{E})} \right].$$

Further setting $\eta = 1/L$ and $\lambda = \Theta(LKd\sqrt{m})$, assuming $\epsilon_t \leq k^2\log(d)/4Kd(d - k)^2m$, we have

$$\varepsilon_t = O\left(\frac{1}{\eta}\left(\frac{dK^2m}{\eta\lambda} + K\sqrt{m} \right)\frac{k}{\eta\lambda}\frac{d(\log d)K}{\epsilon_t} \right)$$
$$= O\left(\frac{Kk\log d}{\eta\epsilon_t} \right). \qquad (49)$$

$\square$

Lemma 15. *For matrices $\mathbf{W}_1, \mathbf{W}_2 \in \mathcal{W} \subset \mathbb{R}^{d\times m}$, we have*

$$\|\mathbf{W}_1 - \mathbf{W}_2\|_F = O(K\sqrt{m}).$$

Proof. Because $\mathbf{W}_1, \mathbf{W}_2 \in \mathcal{W}$, $\max_{i\in[m]}\|\mathbf{w}_{i,1}\|_2 \leq K$. Therefore,

$$\|\mathbf{W}_1 - \mathbf{W}_2\|_F \leq 2\|\mathbf{W}_1\|_F \leq 2\sqrt{m}\max_{i\in[m]}\|\mathbf{w}_{i,1}\|_2 \leq 2K\sqrt{m}.$$

$\square$

Lemma 16. *For constants $\epsilon, \delta \geq 0$, a integer $T \geq 1$, a series constants $\epsilon_t > 0$ for $t \in [T]$, then if*

$$\epsilon = \sum_{t=1}^{T}\frac{(e^{\epsilon_t} - 1)\epsilon_t}{(e^{\epsilon_t} + 1)} + \sqrt{\sum_{t=1}^{T}2\epsilon_t^2\log\left(\frac{1}{\delta} \right)},$$

we have

$$\sqrt{\sum_{t=1}^{T}\epsilon_t^2} \geq \frac{\sqrt{2}\epsilon}{2\sqrt{\log(1/\delta) + 2\epsilon}}.$$

On the other hand, if

$$\epsilon = \sum_{t=1}^{T}\frac{(e^{\epsilon_t} - 1)\epsilon_t}{(e^{\epsilon_t} + 1)} + \sqrt{\sum_{t=1}^{T}2\epsilon_t^2\log\left(e + \frac{\sqrt{\sum_{t=1}^{T}\epsilon_t^2}}{\delta} \right)},$$

we have

$$\sqrt{\sum_{t=1}^{T} \epsilon_t^2} \geq \max\left\{ \sqrt{\frac{\epsilon}{1 + \sqrt{2}/(e\delta)}}, \frac{\sqrt{2}\epsilon}{2\sqrt{\log(e + \epsilon/\sqrt{2}\delta) + 2\epsilon}} \right\}.$$

Proof. If $\epsilon = \sum_{t=1}^{T} \frac{(e^{\epsilon_t} - 1)\epsilon_t}{(e^{\epsilon_t} + 1)} + \sqrt{\sum_{t=1}^{T} 2\epsilon_t^2 \log\left(\frac{1}{\delta}\right)}$,

Because $(e^x - 1)/(e^x + 1) \leq x$ for $x \geq 0$, then

$$\epsilon \leq \sum_{t=1}^{T} \epsilon_t^2 + \sqrt{\sum_{t=1}^{T} 2\epsilon_t^2 \log\left(\frac{1}{\delta}\right)}.$$

Solving the inequality with respect to $\sqrt{\sum_{t=1}^{T} \epsilon_t^2}$, we get

$$\sqrt{\sum_{t=1}^{T} \epsilon_t^2} \geq \frac{\sqrt{2}\epsilon}{\sqrt{\log(1/\delta) + 2\epsilon} + \sqrt{\log(1/\delta)}}$$

$$\geq \frac{\sqrt{2}\epsilon}{2\sqrt{\log(1/\delta) + 2\epsilon}}.$$

If we have

$$\epsilon = \sum_{t=1}^{T} \frac{(e^{\epsilon_t} - 1)\epsilon_t}{(e^{\epsilon_t} + 1)} + \sqrt{\sum_{t=1}^{T} 2\epsilon_t^2 \log\left(e + \frac{\sqrt{\sum_{t=1}^{T} \epsilon_t^2}}{\delta}\right)}. \quad (50)$$

Because $(e^x - 1)/(e^x + 1) \leq x$ for $x \geq 0$, then

$$\epsilon \leq \sum_{t=1}^{T} \epsilon_t^2 + \sqrt{\sum_{t=1}^{T} 2\epsilon_t^2 \log\left(e + \frac{\sqrt{\sum_{t=1}^{T} \epsilon_t^2}}{\delta}\right)}$$

$$\leq \sum_{t=1}^{T} \epsilon_t^2 + \frac{\sqrt{2}\sum_{t=1}^{T} \epsilon_t^2}{e\delta},$$

where the second inequality is because $\log(e + x) \leq x/e + 1$ for $x \geq 0$.

As such,

$$\sqrt{\sum_{t=1}^{T} \epsilon_t^2} \geq \sqrt{\frac{\epsilon}{1 + \sqrt{2}/(e\delta)}}.$$

On the other hand, by (50), it also holds that $\sqrt{2}\sqrt{\sum_{t=1}^{T} \epsilon_t^2} \leq \epsilon$. Then we have

$$\epsilon \leq \sum_{t=1}^{T} \epsilon_t^2 + \sqrt{\sum_{t=1}^{T} 2\epsilon_t^2 \log\left(e + \frac{\sqrt{\sum_{t=1}^{T} \epsilon_t^2}}{\delta}\right)}$$

$$\leq \sum_{t=1}^{T} \epsilon_t^2 + \sqrt{\sum_{t=1}^{T} 2\epsilon_t^2 \log\left(e + \frac{\epsilon}{\sqrt{2}\delta}\right)}.$$

Solving the inequality with respect to $\sqrt{\sum_{t=1}^{T} \epsilon_t^2}$, we also get

$$\sqrt{\sum_{t=1}^{T} \epsilon_t^2} \geq \frac{\sqrt{2}\epsilon}{\sqrt{\log(e + \epsilon/\sqrt{2}\delta) + 2\epsilon} + \sqrt{\log(e + \epsilon/\sqrt{2}\delta)}}$$

$$\geq \frac{\sqrt{2}\epsilon}{2\sqrt{\log(e + \epsilon/\sqrt{2}\delta) + 2\epsilon}}.$$

$\square$

Lemma 17. *For constants $\kappa, \epsilon_0 > 0$, $c_1, c_2 > 0$, $\alpha \in \mathbb{R}$, a integer $T \geq 1$, assuming $\epsilon_t = \epsilon_0 t^\alpha$, $\varepsilon_t = O(\kappa/\epsilon_t)$ for $t \in [T]$, if $\sum_{t=1}^{T} \epsilon_t \geq c_1$, we have*

$$\sum_{t=1}^{T} \sqrt{\varepsilon_t} = \begin{cases} O\left(\sqrt{\frac{\kappa T^{\alpha+1}}{c_1(\alpha/2-1)^2(\alpha+1)}}\right), & \alpha > 2; \\ O\left(\sqrt{\frac{\kappa T^3}{c_1(\alpha/2-1)^2(\alpha+1)}}\right), & -1 < \alpha < 2; \\ O\left(\sqrt{\frac{\kappa T^{2-\alpha}}{c_1(\alpha/2-1)^2(-\alpha-1)}}\right), & \alpha < -1, \end{cases}$$

and

$$\sum_{t=1}^{T} t\sqrt{\varepsilon_t} = \begin{cases} O\left(\sqrt{\frac{\kappa T^{\alpha+1}}{c_1(\alpha/2-2)^2(\alpha+1)}}\right), & \alpha > 4; \\ O\left(\sqrt{\frac{\kappa T^5}{c_1(\alpha/2-2)^2(\alpha+1)}}\right), & -1 < \alpha < 4; \\ O\left(\sqrt{\frac{\kappa T^{4-\alpha}}{c_1(\alpha/2-2)^2(-\alpha-1)}}\right), & \alpha < -1. \end{cases}$$

If $\sqrt{\sum_{t=1}^{T} \epsilon_t^2} \geq c_2$, we have

$$\sum_{t=1}^{T} \sqrt{\varepsilon_t} = \begin{cases} O\left(\sqrt{\frac{\kappa T^{\alpha+1/2}}{c_2(\alpha/2-1)^2\sqrt{2\alpha+1}}}\right), & \alpha > 2; \\ O\left(\sqrt{\frac{\kappa T^{5/2}}{c_2(\alpha/2-1)^2\sqrt{2\alpha+1}}}\right), & -1/2 < \alpha < 2; \\ O\left(\sqrt{\frac{\kappa T^{2-\alpha}}{c_2(\alpha/2-1)^2\sqrt{-2\alpha-1}}}\right), & \alpha < -1/2, \end{cases}$$

and

$$\sum_{t=1}^{T} t\sqrt{\varepsilon_t} = \begin{cases} O\left(\sqrt{\frac{\kappa T^{\alpha+1/2}}{c_2(\alpha/2-2)^2\sqrt{2\alpha+1}}}\right), & \alpha > 4; \\ O\left(\sqrt{\frac{\kappa T^{9/2}}{c_2(\alpha/2-2)^2\sqrt{2\alpha+1}}}\right), & -1/2 < \alpha < 4; \\ O\left(\sqrt{\frac{\kappa T^{4-\alpha}}{c_2(\alpha/2-2)^2\sqrt{-2\alpha-1}}}\right), & \alpha < -1/2. \end{cases}$$

Proof. If $\sum_{t=1}^{T} \epsilon_t = \sum_{t=1}^{T} \epsilon_0 t^\alpha \geq c_1$. We have

$$\sum_{t=1}^{T} \sqrt{\varepsilon_t} = O\left(\sum_{t=1}^{T} \sqrt{\kappa/\epsilon_t}\right) = O\left(\sum_{t=1}^{T} \sqrt{\frac{\kappa}{\epsilon_0 t^\alpha}}\right)$$

$$= O\left(\sum_{t=1}^{T} t^{-\alpha/2}\sqrt{\frac{\kappa}{\epsilon_0}}\right).$$

Using Lemma 12, we have

$$\sum_{t=1}^{T} \sqrt{\varepsilon_t} = O\left(\sum_{t=1}^{T} t^{-\alpha/2}\sqrt{\frac{\kappa}{c_1}\sum_{t=1}^{T} t^\alpha}\right).$$

Then using Lemma 10, if $\alpha > 2$, i.e., $-\alpha/2 < -1$, we have

$$\sum_{t=1}^{T} \sqrt{\varepsilon_t} = O\left(\frac{1}{-(-\alpha/2) - 1}\sqrt{\frac{\kappa}{c_1}\frac{T^{\alpha+1}}{\alpha+1}}\right)$$

$$= O\left(\sqrt{\frac{\kappa}{c_1}\frac{T^{\alpha+1}}{(\alpha/2-1)^2(\alpha+1)}}\right).$$

Results under other conditions can be proved similarly. $\square$

Lemma 18. *For constants $\kappa, \epsilon_0 > 0$, $c_1, c_2 > 0$, $Q_0 \in (0, 1)$, $Q > 0$, a integer $T \geq 1$, assuming $\epsilon_t = \epsilon_0 Q^{-t}$, $\varepsilon_t = O(\kappa/\epsilon_t)$ for $t \in [T]$, if $\sum_{t=1}^{T} \epsilon_t \geq c_1$, we have*

$$\sum_{t=1}^{T} Q_0^{-t}\sqrt{\varepsilon_t} = \begin{cases} O\left(\sqrt{\frac{\kappa Q^{-T}}{c_1(Q_0/\sqrt{Q}-1)^2(1-Q)}}\right), & 0 < Q < Q_0^2; \\ O\left(\sqrt{\frac{\kappa(Q_0^2)^{-T}}{c_1(Q_0/\sqrt{Q}-1)^2(1-Q)}}\right), & Q_0^2 < Q < 1; \\ O\left(\sqrt{\frac{\kappa(Q_0^2/Q)^{-T}}{c_1(Q_0/\sqrt{Q}-1)^2(Q-1)}}\right), & Q > 1, \end{cases}$$

and

$$\sum_{t=1}^{T}\sqrt{\varepsilon_t Q_0^{-t}} = \begin{cases} O\left(\sqrt{\dfrac{\kappa Q^{-T}}{c_1(\sqrt{Q_0}/\sqrt{Q}-1)^2(1-Q)}}\right), & 0 < Q < Q_0; \\[2.2ex] O\left(\sqrt{\dfrac{\kappa Q_0^{-T}}{c_1(\sqrt{Q_0}/\sqrt{Q}-1)^2(1-Q)}}\right), & Q_0 < Q < 1; \\[2.2ex] O\left(\sqrt{\dfrac{\kappa (Q_0/Q)^{-T}}{c_1(\sqrt{Q_0}/\sqrt{Q}-1)^2(Q-1)}}\right), & Q > 1. \end{cases}$$

If $\sqrt{\sum_{t=1}^{T}\epsilon_t^2} \geq c_2$, we have

$$\sum_{t=1}^{T} Q_0^{-t}\sqrt{\varepsilon_t} = \begin{cases} O\left(\sqrt{\dfrac{\kappa Q^{-T}}{c_2(Q_0/\sqrt{Q}-1)^2\sqrt{1-Q^2}}}\right), & 0 < Q < Q_0^2; \\[2.2ex] O\left(\sqrt{\dfrac{\kappa (Q_0^2)^{-T}}{c_2(Q_0/\sqrt{Q}-1)^2\sqrt{1-Q^2}}}\right), & Q_0^2 < Q < 1; \\[2.2ex] O\left(\sqrt{\dfrac{\kappa (Q_0^2/Q)^{-T}}{c_2(Q_0/\sqrt{Q}-1)^2\sqrt{Q^2-1}}}\right), & Q > 1, \end{cases}$$

and

$$\sum_{t=1}^{T} \sqrt{\varepsilon_t Q_0^{-t}} = \begin{cases} O\left(\sqrt{\dfrac{\kappa Q^{-T}}{c_2(\sqrt{Q_0}/\sqrt{Q}-1)^2\sqrt{1-Q^2}}}\right), & 0 < Q < Q_0; \\[2.2ex] O\left(\sqrt{\dfrac{\kappa Q_0^{-T}}{c_2(\sqrt{Q_0}/\sqrt{Q}-1)^2\sqrt{1-Q^2}}}\right), & Q_0 < Q < 1; \\[2.2ex] O\left(\sqrt{\dfrac{\kappa (Q_0/Q)^{-T}}{c_2(\sqrt{Q_0}/\sqrt{Q}-1)^2\sqrt{Q^2-1}}}\right), & Q > 1. \end{cases}$$

Proof. If $\sum_{t=1}^{T}\epsilon_t = \sum_{t=1}^{T}\epsilon_0 Q^{-t} \geq c_1$. We have

$$\sum_{t=1}^{T} Q_0^{-t}\sqrt{\varepsilon_t} = O\left(\sum_{t=1}^{T} Q_0^{-t}\sqrt{\kappa/\epsilon_t}\right) = O\left(\sum_{t=1}^{T} Q_0^{-t}\sqrt{\frac{\kappa}{\epsilon_0 Q^{-t}}}\right)$$
$$= O\left(\sum_{t=1}^{T}(Q_0/\sqrt{Q})^{-t}\sqrt{\frac{\kappa}{\epsilon_0}}\right).$$

Using Lemma 12, we have

$$\sum_{t=1}^{T}\sqrt{\varepsilon_t} = O\left(\sum_{t=1}^{T}(Q_0/\sqrt{Q})^{-t}\sqrt{\frac{\kappa}{c_1}\sum_{t=1}^{T}Q^{-t}}\right).$$

Then using Lemma 11, if $Q < Q_0^2 < 1$, i.e., $Q_0/\sqrt{Q} > 1$, we have

$$\sum_{t=1}^{T}\sqrt{\varepsilon_t} = O\left(\frac{1}{Q_0/\sqrt{Q}-1}\sqrt{\frac{\kappa}{c_1}\frac{Q^{-T}}{1-Q}}\right)$$
$$= O\left(\sqrt{\frac{\kappa Q^{-T}}{c_1(Q_0/\sqrt{Q}-1)^2(1-Q)}}\right).$$

Results under other conditions can be proved similarly. □

Lemma 19. *For constants* $L, c_3, c_4 > 0$, *a integer* $T \geq 1$, *matrices* $\widetilde{\mathbf{W}}^{(0)}, \mathbf{W}_* \in \mathcal{W} \subset \mathbb{R}^{d\times m}$, *if it holds for a series of positive constants* $\{\varepsilon_t\}$ *that* $\sum_{t=1}^{T}\sqrt{\varepsilon_t} = O(\sqrt{c_4 T^{c_3}})$, *setting* $T = \Theta((K^2 Lm/c_4)^{1/c_3})$, *we have*

$$\mathcal{E} = \frac{L}{2mT}\left(\|\widetilde{\mathbf{W}}^{(0)} - \mathbf{W}_*\|_F + 2\sum_{t=1}^{T}\sqrt{\frac{2\varepsilon_t}{L}} + \sqrt{2\sum_{t=1}^{T}\frac{\varepsilon_t}{L}}\right)^2$$
$$= O\left(K^2 L\left[\frac{c_4}{K^2 Lm}\right]^{1/c_3}\right).$$

Proof. First, because $\varepsilon_t > 0$ for $t \in [T]$, we have

$$\sqrt{\sum_{t=1}^{T}\varepsilon_t} \leq \sum_{t=1}^{T}\sqrt{\varepsilon_t}.$$

Then combining Lemma 15, it suffices that

$$\mathcal{E} = O\left(\frac{L}{mT}\left[K\sqrt{m} + \frac{1}{\sqrt{L}}\sum_{t=1}^{T}\sqrt{\varepsilon_t}\right]^2\right)$$
$$= O\left(\left[K\sqrt{\frac{L}{T}} + \frac{1}{\sqrt{mT}}\sum_{t=1}^{T}\sqrt{\varepsilon_t}\right]^2\right)$$
$$= O\left(\left[K\sqrt{\frac{L}{T}} + \frac{1}{\sqrt{mT}}\sqrt{c_4 T^{c_3}}\right]^2\right).$$

Then setting $T = \Theta((K^2 Lm/c_4)^{1/c_3})$, we complete the proof. □

Lemma 20. *For constants* $L, c_3, c_4 > 0$, *a integer* $T \geq 1$, *matrices* $\widetilde{\mathbf{W}}^{(0)}, \mathbf{W}_* \in \mathcal{W} \subset \mathbb{R}^{d\times m}$, *if it holds for a series of positive constants* $\{\varepsilon_t\}$ *that* $\sum_{t=1}^{T}\sqrt{\varepsilon_t} = O(\sqrt{c_4 T^{c_3}})$, *setting* $T = \Theta((K^2 Lm/c_4)^{1/c_3})$, *we have*

$$\mathcal{E} = \frac{2L}{m(T+1)^2}\left(\|\widetilde{\mathbf{W}}^{(0)} - \mathbf{W}_*\|_F\right.$$
$$\left. + 2\sum_{t=1}^{T}t\sqrt{\frac{2\varepsilon_t}{L}} + \sqrt{2\sum_{t=1}^{T}t^2\frac{\varepsilon_t}{L}}\right)^2$$
$$= O\left(K^2 L\left[\frac{c_4}{K^2 Lm}\right]^{2/c_3}\right).$$

Proof. First, because $\varepsilon_t > 0$ for $t \in [T]$, we have

$$\sqrt{\sum_{t=1}^{T}t^2\varepsilon_t} \leq \sum_{t=1}^{T}\sqrt{t^2\varepsilon_t} = \sum_{t=1}^{T}t\sqrt{\varepsilon_t}.$$

Then combining Lemma 15, it suffices that

$$\mathcal{E} = O\left(\frac{L}{mT^2}\left[K\sqrt{m} + \frac{1}{\sqrt{L}}\sum_{t=1}^{T}t\sqrt{\varepsilon_t}\right]^2\right)$$
$$= O\left(\left[K\frac{\sqrt{L}}{T} + \frac{1}{\sqrt{mT}}\sum_{t=1}^{T}t\sqrt{\varepsilon_t}\right]^2\right)$$
$$= O\left(\left[K\frac{\sqrt{L}}{T} + \frac{1}{\sqrt{mT}}\sqrt{c_4 T^{c_3}}\right]^2\right).$$

Then setting $T = \Theta((K^2 Lm/c_4)^{1/c_3})$, we complete the proof. □

Lemma 21. *For constants* $L, c_6 > 0$, *a constant* $c_5 \in (0,1)$, *a constant* $Q_0 \in (0,1)$, *a integer* $T \geq 1$, *matrices* $\widetilde{\mathbf{W}}^{(0)}, \mathbf{W}_* \in \mathcal{W} \subset \mathbb{R}^{d\times m}$, *if it holds for a series of positive constants* $\{\varepsilon_t\}$ *that* $\sum_{t=1}^{T}Q_0^{-t}\sqrt{\varepsilon_t} = O(\sqrt{c_6 c_5^{-T}})$, *setting* $T = \Theta(\log_{1/c_5}(K^2 Lm/c_6))$, *we have*

$$\mathcal{E} = \frac{Q_0^T}{\sqrt{m}}\left(\|\widetilde{\mathbf{W}}^{(0)} - \mathbf{W}_*\|_F + 2\sum_{t=1}^{T}Q_0^{-t}\sqrt{\frac{2\varepsilon_t}{L}}\right)$$
$$= O\left(K\left[\frac{c_6}{K^2 Lm}\right]^{\log_{c_5}Q_0}\right).$$

Proof. Using Lemma 15, it suffices that

$$\mathcal{E} = O\left(\frac{Q_0^T}{\sqrt{m}}\left[K\sqrt{m} + \frac{1}{\sqrt{L}}\sum_{t=1}^{T}Q_0^{-t}\sqrt{\varepsilon_t}\right]\right)$$
$$= O\left(Q_0^T\left[K + \frac{1}{\sqrt{mL}}\sum_{t=1}^{T}Q_0^{-t}\sqrt{\varepsilon_t}\right]\right)$$
$$= O\left(Q_0^T\left[K + \frac{1}{\sqrt{mL}}\sqrt{c_6 c_5^{-T}}\right]\right).$$

Then setting $T = \Theta(\log_{1/c_5}(K^2 Lm/c_6))$, we complete the proof. □

Lemma 22. *For constants $L, \mu, c_6 > 0$, a constant $c_5 \in (0, 1)$, a constant $Q_0 \in (0, 1)$, a integer $T \geq 1$, matrices $\widetilde{\mathbf{W}}^{(0)}, \mathbf{W}_* \in \mathcal{W} \subset \mathbb{R}^{d \times m}$, if it holds for a series of positive constants $\{\varepsilon_t\}$ that $\sum_{t=1}^{T} \sqrt{\varepsilon_t Q_0^{-t}} = O(\sqrt{c_6 c_5^{-T}})$, setting $T = \Theta((K^2 Lm/c_4)^{1/c_3})$, we have*

$$\mathcal{E} = \frac{(Q_0)^T}{m} \left(K\sqrt{Lm} + 2\sqrt{\frac{L}{\mu}} \sum_{t=1}^{T} \sqrt{\varepsilon_t (Q_0)^{-t}} \right.$$
$$\left. + \sqrt{\left(\sum_{t=1}^{T} \varepsilon_t (Q_0)^{-t} \right)^2} \right)$$
$$= O\left(K^2 L \left[\frac{c_6}{K^2 \mu m} \right]^{\log_{c_5} Q_0} \right).$$

Proof. First, because $\varepsilon_t > 0$ for $t \in [T]$, we have

$$\sqrt{\sum_{t=1}^{T} \varepsilon_t Q_0^{-t}} \leq \sum_{t=1}^{T} \sqrt{\varepsilon_t Q_0^{-t}}.$$

Then it suffices that

$$\mathcal{E} = O\left(\frac{Q_0^T}{m} \left[K\sqrt{Lm} + \sqrt{\frac{L}{\mu}} \sum_{t=1}^{T} \sqrt{\varepsilon_t Q_0^{-t}} \right]^2 \right)$$
$$= O\left(Q_0^T \left[K\sqrt{L} + \sqrt{\frac{L}{m\mu}} \sum_{t=1}^{T} \sqrt{\varepsilon_t Q_0^{-t}} \right]^2 \right)$$
$$= O\left(Q_0^T \left[K\sqrt{L} + \sqrt{\frac{L}{m\mu}} \sqrt{c_6 c_5^{-T}} \right]^2 \right).$$

Then setting $T = \Theta(\log_{1/c_5}(K^2 \mu m/c_6))$, we complete the proof. $\square$

APPENDIX J
PROOF OF RESULTS IN THE MAIN TEXT

A. Proof of Lemma 1

Proof. Under the setting of single-task learning, each task is learned independently, and thus, we have for $i = 1, \ldots, m$, for any set $\mathcal{S}$,

$$\mathbb{P}(\hat{\mathbf{w}}_{[-i]} \in \mathcal{S} \mid \mathcal{B} = (\mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}_i, \mathcal{D}_i))$$
$$= \mathbb{P}(\hat{\mathbf{w}}_{[-i]} \in \mathcal{S} \mid \mathcal{B} = (\mathbf{w}_{[-i]}, \mathcal{D}_{[-i]})).$$

As such, we have for $i = 1, \ldots, m$,

$$\frac{\mathbb{P}(\hat{\mathbf{w}}_{[-i]} \in \mathcal{S} \mid \mathcal{B} = (\mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}_i, \mathcal{D}_i))}{\mathbb{P}(\hat{\mathbf{w}}_{[-i]} \in \mathcal{S} \mid \mathcal{B} = (\mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}'_i, \mathcal{D}'_i))}$$
$$= \frac{\mathbb{P}(\hat{\mathbf{w}}_{[-i]} \in \mathcal{S} \mid \mathcal{B} = (\mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}))}{\mathbb{P}(\hat{\mathbf{w}}_{[-i]} \in \mathcal{S} \mid \mathcal{B} = (\mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}))} = 1 \leq e^0.$$

$\square$

B. Proof of Proposition 1

Proof. First, for Algorithm 2, denoting $\mathbf{\Sigma}_0 = \widetilde{\mathbf{\Sigma}}^{(t)}$, the j-th diagonal element of $\mathbf{S}_{\eta\lambda}$ is

$$\max\left(0, 1 - \frac{\eta\lambda}{\sqrt{\sigma_j(\mathbf{\Sigma}_0 + \mathbf{E})}} \right)$$
$$\geq \max\left(0, 1 - \frac{\eta\lambda}{\sqrt{\sigma_j(\mathbf{\Sigma}_0) + \sigma_d(\mathbf{E})}} \right),$$

where $\sigma_d(\mathbf{E})$ is the d-th largest singular value, i.e., the smallest singular value, of $\mathbf{E}$. As such, when $\sigma_d(\mathbf{E}) = C\lambda^2$ for sufficiently large $C > 0$, $\max\left(0, 1 - \frac{\eta\lambda}{\sqrt{\sigma_j(\mathbf{\Sigma}_0) + \sigma_d(\mathbf{E})}} \right) \to 1$.

Then $\hat{\mathbf{w}}_i^{(t-1)} = \mathbf{U}\mathbf{S}_{\eta\lambda}\mathbf{U}^{\mathrm{T}}\mathbf{w}_i^{(t-1)} = \mathbf{U}\mathbf{U}^{\mathrm{T}}\mathbf{w}_i^{(t-1)} = \mathbf{w}_i^{(t-1)}$. Therefore, all the procedures can be decoupled to independently run for each task, thus Algorithm 2 degrades to an STL algorithm with no random perturbation.

Similarly, for Algorithm 3, for all $j \in [m]$, the j-th diagonal element of $\mathbf{S}_{\eta\lambda}$ is

$$\max\left(0, 1 - \frac{\eta\lambda}{\sqrt{|\mathbf{\Sigma}_{jj,0} + \mathbf{E}_{jj}|}} \right)$$
$$= \max\left(0, 1 - \frac{\eta\lambda}{\sqrt{\mathbf{\Sigma}_{jj,0} + \mathbf{E}_{jj}}} \right),$$

where the equality is because $\mathbf{\Sigma}_0$ is semi-positive definite and $\mathbf{E}$ is positive definite.

As such, when $\min_j \mathbf{E}_{jj} = C\lambda^2$ for sufficiently large $C > 0$, $\min_j \left[\max\left(0, 1 - \frac{\eta\lambda}{\sqrt{\mathbf{\Sigma}_{jj,0} + \mathbf{E}_{jj}}} \right) \right] \to 1$.

Then $\hat{\mathbf{w}}_i^{(t-1)} = \mathbf{S}_{\eta\lambda}\mathbf{w}_i^{(t-1)} = \mathbf{w}_i^{(t-1)}$. Therefore, all the procedures can be decoupled to independently run for each task, thus Algorithm 3 degrades to an STL algorithm with no random perturbation.

$\square$

C. Proof of Theorem 1

Proof. For simplicity, we omit the symbol $\mathcal{B}$ used to denote the input in the conditional events in some equations.

First, we show that for all $t \in [T]$, the mapping $\mathbf{W}^{(t-1)} \to \mathbf{\Sigma}^{(t)}$ is an $(\epsilon_t, 0)$-differentially private algorithm.

Case 1. For $\mathbf{\Sigma}^{(t)} = \widetilde{\mathbf{\Sigma}}^{(t)} + \mathbf{E} = \mathbf{W}^{(t-1)}(\mathbf{W}^{(t-1)})^{\mathrm{T}} + \mathbf{E}$, we follow the proof of Theorem 4 of Jiang et al. [7].

For all $i \in [m]$, consider two adjacent parameter matrices $\mathbf{W}^{(t-1)}$ and $(\mathbf{W}')^{(t-1)}$ that differ only in the i-th column such that $\mathbf{W}^{(t-1)} = [\mathbf{w}_1^{(t-1)} \cdots \mathbf{w}_i^{(t-1)} \cdots \mathbf{w}_m^{(t-1)}]$ and $(\mathbf{W}')^{(t-1)} = [\mathbf{w}_1^{(t-1)} \cdots (\mathbf{w}'_i)^{(t-1)} \cdots \mathbf{w}_m^{(t-1)}]$. Now, let

$$\widetilde{\mathbf{\Sigma}}^{(t)} = \mathbf{W}^{(t-1)}(\mathbf{W}^{(t-1)})^{\mathrm{T}} = \sum_{j=1}^{m} \mathbf{w}_j^{(t-1)}(\mathbf{w}_j^{(t-1)})^{\mathrm{T}}$$
$$(\widetilde{\mathbf{\Sigma}}')^{(t)} = (\mathbf{W}')^{(t-1)}((\mathbf{W}')^{(t-1)})^{\mathrm{T}}$$
$$= \sum_{j \in [m], j \neq i} \mathbf{w}_j^{(t-1)}(\mathbf{w}_j^{(t-1)})^{\mathrm{T}} + (\mathbf{w}'_i)^{(t-1)}((\mathbf{w}'_i)^{(t-1)})^{\mathrm{T}}$$
$$\Delta = \widetilde{\mathbf{\Sigma}}^{(t)} - (\widetilde{\mathbf{\Sigma}}')^{(t)}$$
$$= \mathbf{w}_i^{(t-1)}(\mathbf{w}_i^{(t-1)})^{\mathrm{T}} - (\mathbf{w}'_i)^{(t-1)}((\mathbf{w}'_i)^{(t-1)})^{\mathrm{T}}.$$

Then, we have for the conditional densities

$$\frac{p(\mathbf{\Sigma}^{(t)} \mid \mathbf{W}^{(t-1)})}{p(\mathbf{\Sigma}^{(t)} \mid (\mathbf{W}')^{(t-1)})} = \frac{p(\mathbf{\Sigma}^{(t)} = \mathbf{W}^{(t-1)}(\mathbf{W}^{(t-1)})^{\mathrm{T}} + \mathbf{E}_1)}{p(\mathbf{\Sigma}^{(t)} = (\mathbf{W}')^{(t-1)}((\mathbf{W}')^{(t-1)})^{\mathrm{T}} + \mathbf{E}_2)}.$$

Because $\mathbf{E}_1, \mathbf{E}_2 \sim W_D(\frac{\max_i s_i^{(t-1)}}{2\epsilon_t} \mathbf{I}_D, D+1)$, letting $\mathbf{V} =$

$\frac{\max_j s_j^{(t-1)}}{2\epsilon_t}\mathbf{I}_D$, $\alpha = \frac{\max_j s_j^{(t-1)}}{2\epsilon_t}$,

$$\frac{p(\mathbf{\Sigma}^{(t)} = \mathbf{W}^{(t-1)}(\mathbf{W}^{(t-1)})^{\mathrm{T}} + \mathbf{E}_1)}{p(\mathbf{\Sigma}^{(t)} = (\mathbf{W}')^{(t-1)}((\mathbf{W}')^{(t-1)})^{\mathrm{T}} + \mathbf{E}_2)}$$

$$= \frac{\exp[-\mathrm{tr}(\mathbf{V}^{-1}(\mathbf{\Sigma}^{(t)} - \mathbf{W}^{(t-1)}(\mathbf{W}^{(t-1)})^{\mathrm{T}}))/2]}{\exp[-\mathrm{tr}(\mathbf{V}^{-1}(\mathbf{\Sigma}^{(t)} - (\mathbf{W}')^{(t-1)}((\mathbf{W}')^{(t-1)})^{\mathrm{T}}))/2]}$$

$$= \exp[\mathrm{tr}(\mathbf{V}^{-1}(\mathbf{\Sigma}^{(t)} - (\mathbf{W}')^{(t-1)}((\mathbf{W}')^{(t-1)})^{\mathrm{T}}))/2$$
$$- \mathrm{tr}(\mathbf{V}^{-1}(\mathbf{\Sigma}^{(t)} - \mathbf{W}^{(t-1)}(\mathbf{W}^{(t-1)})^{\mathrm{T}}))/2]$$

$$= \exp[\mathrm{tr}(\mathbf{V}^{-1}\Delta)/2]$$

$$= \exp[\mathrm{tr}(\mathbf{w}_i^{(t-1)}(\mathbf{w}_i^{(t-1)})^{\mathrm{T}} - (\mathbf{w}_i')^{(t-1)}((\mathbf{w}_i')^{(t-1)})^{\mathrm{T}})/(2\alpha)]$$

$$= \exp[(\mathrm{tr}(\mathbf{w}_i^{(t-1)}(\mathbf{w}_i^{(t-1)})^{\mathrm{T}}) - \mathrm{tr}((\mathbf{w}_i')^{(t-1)}((\mathbf{w}_i')^{(t-1)})^{\mathrm{T}}))/(2\alpha)]$$

$$= \exp[(\mathrm{tr}((\mathbf{w}_i^{(t-1)})^{\mathrm{T}}\mathbf{w}_i^{(t-1)}) - \mathrm{tr}(((\mathbf{w}_i')^{(t-1)})^{\mathrm{T}}(\mathbf{w}_i')^{(t-1)}))/(2\alpha)]$$

$$= \exp[(\|\mathbf{w}_i^{(t-1)}\|_2^2 - \|(\mathbf{w}_i')^{(t-1)}\|_2^2)/(2\alpha)]$$

$$\leq \exp[s_i^{(t-1)}/(2\alpha)] = \exp\left[2\epsilon_t \frac{s_i^{(t-1)}}{2\max_j s_j^{(t-1)}}\right] \leq \exp(\epsilon_t).$$

As such, we have

$$\frac{p(\mathbf{\Sigma}^{(t)} \mid \mathbf{W}^{(t-1)})}{p(\mathbf{\Sigma}^{(t)} \mid (\mathbf{W}')^{(t-1)})} \leq \exp(\epsilon_t).$$

Case 2. Consider $\mathbf{\Sigma}^{(t)} = \widetilde{\mathbf{\Sigma}}^{(t)} + \mathbf{E} = (\mathbf{W}^{(t-1)})^{\mathrm{T}}\mathbf{W}^{(t-1)} + \mathbf{E}$.

For all $i \in [m]$, consider two adjacent parameter matrices $\mathbf{W}^{(t-1)}$ and $(\mathbf{W}')^{(t-1)}$ that differ only in the i-th column such that $\mathbf{W}^{(t-1)} = [\mathbf{w}_1^{(t-1)} \cdots \mathbf{w}_i^{(t-1)} \cdots \mathbf{w}_m^{(t-1)}]$ and $(\mathbf{W}')^{(t-1)} = [\mathbf{w}_1^{(t-1)} \cdots (\mathbf{w}_i')^{(t-1)} \cdots \mathbf{w}_m^{(t-1)}]$. Let

$$\widetilde{\mathbf{\Sigma}}^{(t)} = (\mathbf{W}^{(t-1)})^{\mathrm{T}}\mathbf{W}^{(t-1)}$$
$$(\widetilde{\mathbf{\Sigma}}')^{(t)} = ((\mathbf{W}')^{(t-1)})^{\mathrm{T}}\mathbf{W}'^{(t-1)}$$
$$\Delta = \widetilde{\mathbf{\Sigma}}^{(t)} - (\widetilde{\mathbf{\Sigma}}')^{(t)},$$

where the i-th diagonal element of Δ is $\|\mathbf{w}_i^{(t-1)}\|_2^2 - \|(\mathbf{w}_i')^{(t-1)}\|_2^2$ and the other diagonal elements of Δ are zeros.

Then, we have

$$\frac{p(\mathbf{\Sigma}^{(t)} \mid \mathbf{W}^{(t-1)})}{p(\mathbf{\Sigma}^{(t)} \mid (\mathbf{W}')^{(t-1)})} = \frac{p(\mathbf{\Sigma}^{(t)} = (\mathbf{W}^{(t-1)})^{\mathrm{T}}\mathbf{W}^{(t-1)} + \mathbf{E}_1)}{p(\mathbf{\Sigma}^{(t)} = ((\mathbf{W}')^{(t-1)})^{\mathrm{T}}\mathbf{W}'^{(t-1)} + \mathbf{E}_2)}.$$

Because $\mathbf{E}_1, \mathbf{E}_2 \sim W_m(\mathrm{diag}(\mathbf{s}^{(t-1)}/2\epsilon_t), m+1)$, letting $\mathbf{V} = \mathrm{diag}(\mathbf{s}^{(t-1)}/2\epsilon_t)$,

$$\frac{p(\mathbf{\Sigma}^{(t)} = (\mathbf{W}^{(t-1)})^{\mathrm{T}}\mathbf{W}^{(t-1)} + \mathbf{E}_1)}{p(\mathbf{\Sigma}^{(t)} = ((\mathbf{W}')^{(t-1)})^{\mathrm{T}}\mathbf{W}'^{(t-1)} + \mathbf{E}_2)}$$

$$= \frac{\exp[-\mathrm{tr}(\mathbf{V}^{-1}(\mathbf{\Sigma}^{(t)} - (\mathbf{W}^{(t-1)})^{\mathrm{T}}\mathbf{W}^{(t-1)}))/2]}{\exp[-\mathrm{tr}(\mathbf{V}^{-1}(\mathbf{\Sigma}^{(t)} - ((\mathbf{W}')^{(t-1)})^{\mathrm{T}}\mathbf{W}'^{(t-1)}))/2]}$$

$$= \exp[\mathrm{tr}(\mathbf{V}^{-1}(\mathbf{\Sigma}^{(t)} - ((\mathbf{W}')^{(t-1)})^{\mathrm{T}}\mathbf{W}'^{(t-1)}))/2$$
$$- \mathrm{tr}(\mathbf{V}^{-1}(\mathbf{\Sigma}^{(t)} - (\mathbf{W}^{(t-1)})^{\mathrm{T}}\mathbf{W}^{(t-1)}))/2]$$

$$= \exp[\mathrm{tr}(\mathbf{V}^{-1}\Delta)/2]$$

$$= \exp[(\|\mathbf{w}_i^{(t-1)}\|_2^2 - \|(\mathbf{w}_i')^{(t-1)}\|_2^2)/(2v_{ii})]$$

$$\leq \exp[s_i^{(t-1)}/(2v_{ii})] = \exp\left[2\epsilon_t \frac{s_i^{(t-1)}}{2s_i^{(t-1)}}\right] \leq \exp(\epsilon_t).$$

As such, we also have

$$\frac{p(\mathbf{\Sigma}^{(t)} \mid \mathbf{W}^{(t-1)})}{p(\mathbf{\Sigma}^{(t)} \mid (\mathbf{W}')^{(t-1)})} \leq \exp(\epsilon_t).$$

Next, given $t \in [T]$, $\mathbf{\Sigma}^{(1:t-1)}$ (when $t = 1$, $\mathbf{\Sigma}^{(1:t-1)} = \emptyset$) and the mapping $f : \mathbf{\Sigma}^{(1:t)} \rightarrow \mathbf{M}^{(t)}$, which does not touch any unperturbed sensitive information, using the *Post-Processing immunity* Lemma (Lemma 4) for the mapping $f' : \mathbf{\Sigma}^{(1:t)} \rightarrow (\mathbf{M}^{(t)}, \widetilde{\mathbf{\Sigma}}^{(t)})$, the

algorithm $(\mathbf{W}^{(t-1)}, \mathbf{\Sigma}^{(1:t-1)}) \rightarrow (\mathbf{M}^{(t)}, \mathbf{\Sigma}^{(t)})$ is still an $(\epsilon_t, 0)$-differentially private algorithm.

Then, because $\hat{\mathbf{w}}_i^{(t)} = \mathcal{A}_{\mathrm{st},i}(\mathbf{M}^{(t)}, \mathbf{w}_i^{(0:t-1)}, \mathbf{X}_i, \mathbf{y}_i)$ is an STL algorithm for the i-th task, for $i = 1, \ldots, m$, the mapping $(\mathbf{M}^{(t)}, \mathbf{w}_{[-i]}^{(0:t-1)}, \mathbf{X}_{[-i]}, \mathbf{y}_{[-i]}) \rightarrow (\hat{\mathbf{w}}_{[-i]}^{(t)})$ thus does not touch any unperturbed sensitive information for the i-th task. As such, applying the *Post-Processing immunity* Lemma again for the mapping $f'' : (\mathbf{M}^{(t)}, \mathbf{w}_{[-i]}^{(0:t-1)}, \mathbf{X}_{[-i]}, \mathbf{y}_{[-i]}, \mathbf{\Sigma}^{(1:t-1)}) \rightarrow (\hat{\mathbf{w}}_{[-i]}^{(t)}, \mathbf{M}^{(t)}, \mathbf{\Sigma}^{(t)})$, for the algorithm $(\mathbf{W}^{(t-1)}, \mathbf{\Sigma}^{(1:t-1)}, \mathbf{w}_{[-i]}^{(0:t-2)}, \mathbf{X}_{[-i]}, \mathbf{y}_{[-i]}) \rightarrow (\hat{\mathbf{w}}_{[-i]}^{(t)}, \mathbf{M}^{(t)}, \mathbf{\Sigma}^{(t)})$ (when $t = 1$, $\mathbf{w}_{[-i]}^{(0:t-2)} = \emptyset$), denoting $\vartheta_{t,i} = (\hat{\mathbf{w}}_{[-i]}^{(t)}, \mathbf{M}^{(t)}, \mathbf{\Sigma}^{(t)}) \in \mathcal{C}_{t,i}$, we have for any set $\mathcal{S}_{t,i} \subseteq \mathcal{C}_{t,i}$

$$\mathbb{P}(\vartheta_{t,i} \in \mathcal{S}_{t,i} \mid \mathbf{W}^{(t-1)}, \mathbf{\Sigma}^{(1:t-1)}, \mathbf{w}_{[-i]}^{(0:t-2)}, \mathcal{D}^m)$$
$$\leq e^{\epsilon_t}\mathbb{P}(\vartheta_{t,i} \in \mathcal{S}_{t,i} \mid (\mathbf{W}')^{(t-1)}, \mathbf{\Sigma}^{(1:t-1)}, \mathbf{w}_{[-i]}^{(0:t-2)}, (\mathcal{D}')^m),$$

where $\mathbf{W}^{(t-1)}$ and $(\mathbf{W}')^{(t-1)}$ differ only in the i-th column and $\mathcal{D}^m$ and $(\mathcal{D}')^m$ differ only in the i-th task.

Now, again, for $t = 1, \ldots, T$, we take the t-th dataset $\widetilde{\mathcal{D}}_t = \{(\mathbf{w}_1^{(t-1)}, \mathcal{D}_1), \ldots, (\mathbf{w}_m^{(t-1)}, \mathcal{D}_m)\}$. Given that $\mathbf{W}^{(t)} = \widehat{\mathbf{W}}^{(t)}$ for all $t \in [T]$, we have for any set $\mathcal{S}_{t,i} \subseteq \mathcal{C}_{t,i}$ that

$$\mathbb{P}(\vartheta_{t,i} \in \mathcal{S}_{t,i} \mid \widetilde{\mathcal{D}}_t, \boldsymbol{\vartheta}_{1:t-1})$$
$$\leq e^{\epsilon_t}\mathbb{P}(\vartheta_{t,i} \in \mathcal{S}_{t,i} \mid \widetilde{\mathcal{D}}_t', \boldsymbol{\vartheta}_{1:t-1}),$$

where $\widetilde{\mathcal{D}}_t$ and $\widetilde{\mathcal{D}}_t'$ are two adjacent datasets that differ in a single entry, the i-th data instance $(\mathbf{w}_i^{(t-1)}, \mathcal{D}_i = (\mathbf{X}_i, \mathbf{y}_i))$, and

$$\boldsymbol{\vartheta}_{1:t-1} = \begin{cases} \emptyset, & t = 1 \\ (\vartheta_{1,1}, \ldots, \vartheta_{1,m}) \ldots, (\vartheta_{t-1,1}, \ldots, \vartheta_{t-1,m}), & t \geq 2. \end{cases}$$

This renders the algorithm in the t-th iteration an $(\epsilon_t, 0)$-differentially private algorithm.

Now, again by the *Adaptive composition* Lemma (Lemma 7), for all $i \in [m]$ and for any set $\mathcal{S}' \subseteq \bigotimes_{t=1}^{T} \mathcal{C}_{t_i}$, we have

$$\mathbb{P}((\vartheta_{1,i}, \cdots, \vartheta_{T,i}) \in \mathcal{S}' \mid \bigcap_{t=1}^{T}(\mathcal{B}_t = (\widetilde{\mathcal{D}}_t, \boldsymbol{\vartheta}_{1:t-1})))$$

$$\leq e^{\tilde{\epsilon}}\mathbb{P}((\vartheta_{1,i}, \cdots, \vartheta_{T,i}) \in \mathcal{S}' \mid \bigcap_{t=1}^{T}(\mathcal{B}_t = (\widetilde{\mathcal{D}}_t', \boldsymbol{\vartheta}_{1:t-1})))$$
$$+ \delta,$$

where for all $t \in [T]$, $\mathcal{B}_t$ denotes the input for the t-th iteration.

Finally, taking $\theta_t = (\vartheta_{t,1}, \ldots, \vartheta_{t,m})$ for all $t \in [T]$, we have for any set $\mathcal{S} \subseteq \mathbb{R}^{d \times (m-1) \times T}$

$$\mathbb{P}(\hat{\mathbf{w}}_{[-i]}^{(1:T)} \in \mathcal{S} \mid \bigcap_{t=1}^{T} \mathcal{B}_t = (\mathbf{W}^{(t-1)}, \mathcal{D}^m, \boldsymbol{\theta}_{1:t-1}))$$

$$\leq e^{\epsilon}\mathbb{P}(\hat{\mathbf{w}}_{[-i]}^{(1:T)} \in \mathcal{S} \mid \bigcap_{t=1}^{T} \mathcal{B}_t = ((\mathbf{W}')^{(t-1)}, (\mathcal{D}')^m, \boldsymbol{\theta}_{1:t-1}))$$
$$+ \delta,$$

$\square$

D. Proof of Corollary 1

Proof. For simplicity, we omit the symbol $\mathcal{B}$ used to denote the input in the conditional events in some equations.

Using Theorem 1, we only need to show that Algorithm 2 complies with our MP-MTL framework in Algorithm 1.

Consider the t-th iteration for all $t \in [T]$. Because of the norm clipping, for $i = 1, \ldots, m$, the ℓ_2 norm of any parameter vector equals one. Then, we have for $i = 1, \ldots, m$ that

$$s_i^{(t-1)} = \max_{(\tilde{\mathbf{w}}_i')^{(t-1)}} |\|\tilde{\mathbf{w}}_i^{(t-1)}\|_2^2 - \|(\tilde{\mathbf{w}}_i')^{(t-1)}\|_2^2|$$

$$\leq \max_{(\tilde{\mathbf{w}}_i')^{(t-1)}} |\|\tilde{\mathbf{w}}_i^{(t-1)}\|_2^2| + |\|(\tilde{\mathbf{w}}_i')^{(t-1)}\|_2^2| = 2K.$$

Because the norm clipping is a deterministic STL algorithm and because the mapping $\widetilde{\mathbf{W}}^{(t-1)} \to \mathbf{\Sigma}^{(t)}$ is an $(\epsilon_t, 0)$ - differentially algorithm, we have for any set $\mathcal{S} \subseteq \mathbb{R}^{d \times d}$ that

$$
\begin{aligned}
&\mathbb{P}(\mathbf{\Sigma}^{(t)} \in \mathcal{S} \mid \mathbf{w}_{[-i]}^{(t-1)}, \mathbf{w}_i^{(t-1)}) \\
&= \mathbb{P}(\mathbf{\Sigma}^{(t)} \in \mathcal{S} \mid \tilde{\mathbf{w}}_{[-i]}^{(t-1)}, \tilde{\mathbf{w}}_i^{(t-1)}) \\
&\leq e^{\epsilon_t} \mathbb{P}(\mathbf{\Sigma}^{(t)} \in \mathcal{S} \mid \tilde{\mathbf{w}}_{[-i]}^{(t-1)}, (\tilde{\mathbf{w}}_i')^{(t-1)}) \\
&= e^{\epsilon_t} \mathbb{P}(\mathbf{\Sigma}^{(t)} \in \mathcal{S} \mid \mathbf{w}_{[-i]}^{(t-1)}, (\mathbf{w}_i')^{(t-1)}),
\end{aligned}
$$

which renders the mapping $\mathbf{W}^{(t-1)} \to \mathbf{\Sigma}^{(t)}$ as an $(\epsilon_t, 0)$ - differentially algorithm as well.

Let $\mathbf{M}^{(t)} = \mathbf{U}\mathbf{S}_{\eta\lambda}\mathbf{U}^{\mathrm{T}}$. As such, the 7-th step to the 9-th step can be treated as the process of first performing a mapping $f : \mathbf{\Sigma}^{(1:t)} \to \mathbf{M}^{(t)}$ and then applying an STL algorithm:

$$\hat{\mathbf{w}}_i^{(t)} = \mathbf{U}\mathbf{S}_{\eta\lambda}\mathbf{U}^{\mathrm{T}}\tilde{\mathbf{w}}_i^{(t-1)}, \text{ for all } i \in [m]. \tag{51}$$

Now, because (51), the 10-th step and the 11-th step are all STL algorithms, they can be treated as a complete STL algorithm performing the mapping: $(\mathbf{M}^{(t)}, \mathbf{w}_i^{(0:t-1)}, \mathbf{X}_i, \mathbf{y}_i) \to (\hat{\mathbf{w}}_i^{(t)}, \mathbf{w}_i^{(t)})$.

As such, in all the iterations, Algorithm 2 complies with Algorithm 1. Thus, the result of Theorem 1 can be applied to Algorithm 2.

Similarly, using Theorem 1, we only need to show that Algorithm 3 complies with our MP-MTL framework in Algorithm 1.

The proof for the sensitivity is the same.

Now, let $\mathbf{M}^{(t)} = \mathbf{S}_{\eta\lambda}$. As such, the 7-th step can be treated as a mapping $f : \mathbf{\Sigma}^{(1:t)} \to \mathbf{M}^{(t)}$.

Then, because the 8-th step, the 9-th step and the 10-th step are all STL algorithms, they can be treated as a complete STL algorithm performing the mapping: $(\mathbf{M}^{(t)}, \mathbf{w}_i^{(0:t-1)}, \mathbf{X}_i, \mathbf{y}_i) \to (\hat{\mathbf{w}}_i^{(t)}, \mathbf{w}_i^{(t)})$.

Therefore, in all the iterations, Algorithm 3 complies with Algorithm 1, and thus, the result of Theorem 1 can be applied to Algorithm 3. □

E. Proof of Lemma 2

Proof. We invoke the results of Schmidt et al. [11] to bound the empirical optimization error.

In the t-th step, a standard proximal operator (see Ji and Ye [6]) optimizes the following problem:

$$\min_{\mathbf{W}} \frac{1}{2\eta} \|\mathbf{W} - \mathbf{C}\|_F^2 + \lambda \|\mathbf{W}\|_*,$$

where $\mathbf{C} = \widetilde{\mathbf{W}}^{(t-1)}$. By Theorem 3.1 of Ji and Ye [6], denote the solution of the problem by $\widehat{\mathbf{W}}_0^{(t)} = \mathbf{U}_0\mathbf{S}_{\eta\lambda,0}\mathbf{U}_0^{\mathrm{T}}\mathbf{C}$, where $\mathbf{U}_0\mathbf{\Lambda}_0\mathbf{U}_0^{\mathrm{T}} = \mathbf{C}\mathbf{C}^{\mathrm{T}}$ is the SVD decomposition of $\mathbf{C}\mathbf{C}^{\mathrm{T}}$. $\mathbf{S}_{\eta\lambda,0}$ is also a diagonal matrix and $\mathbf{S}_{\eta\lambda,ii,0} = \max\{0, 1 - \eta\lambda/\sqrt{\mathbf{\Lambda}_{ii,0}}\}$ for $i = 1, \ldots, \min\{d, m\}$.

By Algorithm 2, $\widehat{\mathbf{W}}^{(t)} = \mathbf{U}\mathbf{S}_{\eta\lambda}\mathbf{U}^{\mathrm{T}}\mathbf{C}$.

Then we analyse the bound of $\frac{1}{2\eta}\|\widehat{\mathbf{W}}^{(t)} - \mathbf{C}\|_F^2 + \lambda\|\widehat{\mathbf{W}}^{(t)}\|_* - \{\frac{1}{2\eta}\|\widehat{\mathbf{W}}_0^{(t)} - \mathbf{C}\|_F^2 + \lambda\|\widehat{\mathbf{W}}_0^{(t)}\|_*\}$.

First, we have

$$
\begin{aligned}
&\|\widehat{\mathbf{W}}^{(t)} - \mathbf{C}\|_F^2 - \|\widehat{\mathbf{W}}_0^{(t)} - \mathbf{C}\|_F^2 \\
&= \mathrm{tr}((\widehat{\mathbf{W}}^{(t)} - \mathbf{C})^{\mathrm{T}}(\widehat{\mathbf{W}}^{(t)} - \mathbf{C})) - \mathrm{tr}((\widehat{\mathbf{W}}_0^{(t)} - \mathbf{C})^{\mathrm{T}}(\widehat{\mathbf{W}}_0^{(t)} - \mathbf{C})) \\
&= \mathrm{tr}((\widehat{\mathbf{W}}^{(t)})^{\mathrm{T}}\widehat{\mathbf{W}}^{(t)}) - \mathrm{tr}((\widehat{\mathbf{W}}_0^{(t)})^{\mathrm{T}}\widehat{\mathbf{W}}_0^{(t)}) - 2\mathrm{tr}((\widehat{\mathbf{W}}^{(t)} - \widehat{\mathbf{W}}_0^{(t)})^{\mathrm{T}}\mathbf{C}) \\
&= \mathrm{tr}((\widehat{\mathbf{W}}^{(t)} - \widehat{\mathbf{W}}_0^{(t)})^{\mathrm{T}}(\widehat{\mathbf{W}}^{(t)} + \widehat{\mathbf{W}}_0^{(t)})) - 2\mathrm{tr}((\widehat{\mathbf{W}}^{(t)} - \widehat{\mathbf{W}}_0^{(t)})^{\mathrm{T}}\mathbf{C}) \\
&= \mathrm{tr}((\widehat{\mathbf{W}}^{(t)} - \mathbf{C})^{\mathrm{T}}(\widehat{\mathbf{W}}^{(t)} - \widehat{\mathbf{W}}_0^{(t)})) \\
&\quad + \mathrm{tr}((\widehat{\mathbf{W}}_0^{(t)} - \mathbf{C})^{\mathrm{T}}(\widehat{\mathbf{W}}^{(t)} - \widehat{\mathbf{W}}_0^{(t)})) \\
&\leq \sigma_1(\widehat{\mathbf{W}}^{(t)} - \mathbf{C})\|\widehat{\mathbf{W}}^{(t)} - \widehat{\mathbf{W}}_0^{(t)}\|_* \\
&\quad + \sigma_1(\widehat{\mathbf{W}}_0^{(t)} - \mathbf{C})\|\widehat{\mathbf{W}}^{(t)} - \widehat{\mathbf{W}}_0^{(t)}\|_*,
\end{aligned}
$$

$$\tag{52}$$

where $\sigma_1(\cdot)$ denotes the largest singular value of the enclosed matrix.

Denote $\mathbf{T} = \mathbf{U}\mathbf{S}_{\eta\lambda}\mathbf{U}^{\mathrm{T}}, \mathbf{T}_0 = \mathbf{U}_0\mathbf{S}_{\eta\lambda,0}\mathbf{U}_0^{\mathrm{T}}$. Since $\mathbf{U}$ is decomposed from a symmetric matrix, we have

$$
\begin{aligned}
\sigma_1(\widehat{\mathbf{W}}^{(t)} - \mathbf{C}) &= \sigma_1(\mathbf{T}\mathbf{C} - \mathbf{C}) \leq \sigma_1(\mathbf{C})\sigma_1(\mathbf{T} - \mathbf{I}) \\
&= \sigma_1(\mathbf{C})\sigma_1(\mathbf{U}\mathbf{S}_{\eta\lambda}\mathbf{U}^{\mathrm{T}} - \mathbf{U}\mathbf{U}^{\mathrm{T}}) \\
&= \sigma_1(\mathbf{C})\sigma_1(\mathbf{U}(\mathbf{S}_{\eta\lambda} - \mathbf{I})\mathbf{U}^{\mathrm{T}}).
\end{aligned}
$$

Since $\mathbf{S}_{\eta\lambda} - \mathbf{I}$ is a diagonal matrix, whose i-th diagonal element is $\max\{0, 1 - \eta\lambda/\sqrt{\mathbf{\Lambda}_{ii}}\} - 1 \in [-1, 0)$, so $\sigma_1(\mathbf{U}(\mathbf{S}_{\eta\lambda} - \mathbf{I})\mathbf{U}^{\mathrm{T}}) \leq 1$ and

$$\sigma_1(\widehat{\mathbf{W}}^{(t)} - \mathbf{C}) \leq \sigma_1(\mathbf{C}). \tag{53}$$

Similarly,

$$\sigma_1(\widehat{\mathbf{W}}_0^{(t)} - \mathbf{C}) \leq \sigma_1(\mathbf{C}). \tag{54}$$

On the other hand,

$$
\begin{aligned}
&\|\widehat{\mathbf{W}}^{(t)} - \widehat{\mathbf{W}}_0^{(t)}\|_* = \|\mathbf{T}\mathbf{C} - \mathbf{T}_0\mathbf{C}\|_* \\
&= \left\| \sum_{j=1}^d \sigma_j(\mathbf{T})\mathbf{u}_j\mathbf{u}_j^{\mathrm{T}}\mathbf{C} - \sum_{j=1}^d \sigma_j(\mathbf{T}_0)\mathbf{u}_{j,0}\mathbf{u}_{j,0}^{\mathrm{T}}\mathbf{C} \right\|_* \\
&= \left\| \sum_{j=1}^d (\sigma_j(\mathbf{T}_0) + \sigma_j(\mathbf{T}) - \sigma_j(\mathbf{T}_0))\mathbf{u}_j\mathbf{u}_j^{\mathrm{T}}\mathbf{C} \right. \\
&\qquad \left. - \sum_{j=1}^d \sigma_j(\mathbf{T}_0)\mathbf{u}_{j,0}\mathbf{u}_{j,0}^{\mathrm{T}}\mathbf{C} \right\|_* \\
&= \left\| \sum_{j=1}^d \sigma_j(\mathbf{T}_0)(\mathbf{u}_j\mathbf{u}_j^{\mathrm{T}} - \mathbf{u}_{j,0}\mathbf{u}_{j,0}^{\mathrm{T}})\mathbf{C} \right. \\
&\qquad \left. + \sum_{j=1}^d (\sigma_j(\mathbf{T}) - \sigma_j(\mathbf{T}_0))\mathbf{u}_j\mathbf{u}_j^{\mathrm{T}}\mathbf{C} \right\|_* \\
&\leq \left\| \sum_{j=1}^d \sigma_j(\mathbf{T}_0)(\mathbf{u}_j\mathbf{u}_j^{\mathrm{T}} - \mathbf{u}_{j,0}\mathbf{u}_{j,0}^{\mathrm{T}})\mathbf{C} \right\|_* \\
&\qquad + \left\| \sum_{j=1}^d (\sigma_j(\mathbf{T}) - \sigma_j(\mathbf{T}_0))\mathbf{u}_j\mathbf{u}_j^{\mathrm{T}}\mathbf{C} \right\|_*,
\end{aligned}
$$

$$\tag{55}$$

where $\mathbf{u}_j$ and $\mathbf{u}_{j,0}$ are the j-th column of $\mathbf{U}$ and $\mathbf{U}_0$, respectively.

Let $r_c = \mathrm{rank}(\mathbf{C}) \leq \min\{d, m\}$ be the rank of $\mathbf{C}$. Then we have

$$
\begin{aligned}
&\left\| \sum_{j=1}^d (\sigma_j(\mathbf{T}) - \sigma_j(\mathbf{T}_0))\mathbf{u}_j\mathbf{u}_j^{\mathrm{T}}\mathbf{C} \right\|_* \\
&\leq \sum_{j=1}^{r_c} |\sigma_j(\mathbf{T}) - \sigma_j(\mathbf{T}_0)|\sigma_j(\mathbf{C}) \leq \sigma_1(\mathbf{C}) \sum_{j=1}^{r_c} |\sigma_j(\mathbf{T}) - \sigma_j(\mathbf{T}_0)|.
\end{aligned}
$$

$$\tag{56}$$

Denote $\mathbf{\Sigma}_0 = \widetilde{\mathbf{\Sigma}}^{(t)} = \mathbf{C}\mathbf{C}^{\mathrm{T}}$. Then we have for $j \in [r_c]$,

$$
\begin{aligned}
&|\sigma_j(\mathbf{T}) - \sigma_j(\mathbf{T}_0)| \\
&= \left| \max\left(0, 1 - \frac{\eta\lambda}{\sqrt{\sigma_j(\mathbf{\Sigma}_0 + \mathbf{E})}}\right) - \max\left(0, 1 - \frac{\eta\lambda}{\sqrt{\sigma_j(\mathbf{\Sigma}_0)}}\right) \right| \\
&\leq \left| \max\left(0, 1 - \frac{\eta\lambda}{\sqrt{\sigma_j(\mathbf{\Sigma}_0) + \sigma_1(\mathbf{E})}}\right) - \max\left(0, 1 - \frac{\eta\lambda}{\sqrt{\sigma_j(\mathbf{\Sigma}_0)}}\right) \right|.
\end{aligned}
$$

Case 1: $\eta\lambda > \sqrt{\sigma_j(\mathbf{\Sigma}_0)}$. Then

$$
\begin{aligned}
&|\sigma_j(\mathbf{T}) - \sigma_j(\mathbf{T}_0)| \\
&= \max\left(0, 1 - \frac{\eta\lambda}{\sqrt{\sigma_j(\mathbf{\Sigma}_0) + \sigma_1(\mathbf{E})}}\right) \leq 1 - \frac{\eta\lambda}{\sqrt{\eta^2\lambda^2 + \sigma_1(\mathbf{E})}} \\
&\leq 1 - \frac{\eta\lambda}{\eta\lambda + \sqrt{\sigma_1(\mathbf{E})}} = \frac{\sqrt{\sigma_1(\mathbf{E})}}{\eta\lambda + \sqrt{\sigma_1(\mathbf{E})}} \leq \frac{\sqrt{\sigma_1(\mathbf{E})}}{\eta\lambda}
\end{aligned}
$$

Case 2: $\eta\lambda \le \sqrt{\sigma_j(\boldsymbol{\Sigma}_0)}$. Then

$$
\begin{aligned}
&|\sigma_j(\mathbf{T}) - \sigma_j(\mathbf{T}_0)| \\
=&1 - \frac{\eta\lambda}{\sqrt{\sigma_j(\boldsymbol{\Sigma}_0) + \sigma_1(\mathbf{E})}} - 1 + \frac{\eta\lambda}{\sqrt{\sigma_j(\boldsymbol{\Sigma}_0)}} \\
=&\eta\lambda \cdot \frac{\sqrt{\sigma_j(\boldsymbol{\Sigma}_0) + \sigma_1(\mathbf{E})} - \sqrt{\sigma_j(\boldsymbol{\Sigma}_0)}}{\sqrt{\sigma_j^2(\boldsymbol{\Sigma}_0) + \sigma_j(\boldsymbol{\Sigma}_0)\sigma_1(\mathbf{E})}} \\
=&\frac{\eta\lambda\sigma_1(\mathbf{E})}{[\sqrt{\sigma_j(\boldsymbol{\Sigma}_0) + \sigma_1(\mathbf{E})} + \sqrt{\sigma_j(\boldsymbol{\Sigma}_0)}]\sqrt{\sigma_j^2(\boldsymbol{\Sigma}_0) + \sigma_j(\boldsymbol{\Sigma}_0)\sigma_1(\mathbf{E})}} \\
\le&\frac{\eta\lambda\sigma_1(\mathbf{E})}{[\sqrt{\eta^2\lambda^2 + 0} + \sqrt{\eta^2\lambda^2}]\sqrt{\eta^4\lambda^4 + 0}} = \frac{\sigma_1(\mathbf{E})}{2\eta^2\lambda^2}.
\end{aligned}
$$

Suppose that there exists an index $k \le d$ such that

$$
\sigma_k^2(\mathbf{C}) = \sigma_k(\boldsymbol{\Sigma}_0) > \eta^2\lambda^2, \sigma_{k+1}^2(\mathbf{C}) = \sigma_{k+1}(\boldsymbol{\Sigma}_0) \le \eta^2\lambda^2,
$$

then $\sigma_j(\mathbf{T}_0) > 0$ for $j \le k$, $k \le r_c$, and

$$
\begin{aligned}
&\sum_{j=1}^{r_c} |\sigma_j(\mathbf{T}) - \sigma_j(\mathbf{T}_0)| \\
\le&k\frac{\sigma_1(\mathbf{E})}{2\eta^2\lambda^2} + (r_c - k)I(r_c > k)\frac{\sqrt{\sigma_1(\mathbf{E})}}{\eta\lambda}.
\end{aligned} \tag{57}
$$

For another part of (55),

$$
\begin{aligned}
&\left\|\sum_{j=1}^{d} \sigma_j(\mathbf{T}_0)(\mathbf{u}_j\mathbf{u}_j^{\mathrm{T}} - \mathbf{u}_{j,0}\mathbf{u}_{j,0}^{\mathrm{T}})\mathbf{C}\right\|_* \\
=&\left\|\sum_{j=1}^{k} \sigma_j(\mathbf{T}_0)(\mathbf{u}_j\mathbf{u}_j^{\mathrm{T}} - \mathbf{u}_{j,0}\mathbf{u}_{j,0}^{\mathrm{T}})\mathbf{C}\right\|_* \\
\le&\sigma_1(\mathbf{C})\left\|\sum_{j=1}^{k} \sigma_j(\mathbf{T}_0)(\mathbf{u}_j\mathbf{u}_j^{\mathrm{T}} - \mathbf{u}_{j,0}\mathbf{u}_{j,0}^{\mathrm{T}})\right\|_*.
\end{aligned} \tag{58}
$$

Denote $\mathbf{U}_j = \sum_{j'=1}^{j} \mathbf{u}_{j'}\mathbf{u}_{j'}$, $\mathbf{U}_{j,0} = \sum_{j'=1}^{j} \mathbf{u}_{j',0}\mathbf{u}_{j',0}$ for $j \in [d]$. Let $\mathbf{U}_0 = \mathbf{U}_{0,0} = \mathbf{0}$. Then $\mathbf{u}_j\mathbf{u}_j = \mathbf{U}_j - \mathbf{U}_{j-1}$, $\mathbf{u}_{j,0}\mathbf{u}_{j,0} = \mathbf{U}_{j,0} - \mathbf{U}_{j-1,0}$ for $j \in [d]$.

Then we have

$$
\begin{aligned}
&\left\|\sum_{j=1}^{k} \sigma_j(\mathbf{T}_0)(\mathbf{u}_j\mathbf{u}_j^{\mathrm{T}} - \mathbf{u}_{j,0}\mathbf{u}_{j,0}^{\mathrm{T}})\right\|_* \\
=&\left\|\sum_{j=1}^{k} \sigma_j(\mathbf{T}_0)(\mathbf{U}_j - \mathbf{U}_{j,0} - (\mathbf{U}_{j-1} - \mathbf{U}_{j-1,0}))\right\|_* \\
=&\left\|\sum_{j=1}^{k-1} (\sigma_j(\mathbf{T}_0) - \sigma_{j+1}(\mathbf{T}_0))(\mathbf{U}_j - \mathbf{U}_{j,0})\right. \\
&\left. + \sigma_k(\mathbf{T}_0)(\mathbf{U}_k - \mathbf{U}_{k,0})\right\|_* \\
\le&\sum_{j=1}^{k-1} (\sigma_j(\mathbf{T}_0) - \sigma_{j+1}(\mathbf{T}_0))\|\mathbf{U}_j - \mathbf{U}_{j,0}\|_* \\
&+ \sigma_k(\mathbf{T}_0)\|\mathbf{U}_k - \mathbf{U}_{k,0}\|_*.
\end{aligned}
$$

We assume $2\sigma_1(\mathbf{E}) \le \sigma_j(\boldsymbol{\Sigma}_0) - \sigma_{j+1}(\boldsymbol{\Sigma}_0)$ for all $j \in [k]$, and apply the Theorem 6 of Jiang et al. [7]. Then for $j \in [k]$,

$$
\begin{aligned}
\|\mathbf{U}_j - \mathbf{U}_{j,0}\|_* &\le \min\{2j, k\}\|\mathbf{U}_j - \mathbf{U}_{j,0}\|_2 \\
&\le \min\{2j, k\}\frac{2\sigma_1(\mathbf{E})}{\sigma_j(\boldsymbol{\Sigma}_0) - \sigma_{j+1}(\boldsymbol{\Sigma}_0)}.
\end{aligned}
$$

Since $j \in [k-1]$,

$$
\begin{aligned}
\sigma_j(\mathbf{T}_0) - \sigma_{j+1}(\mathbf{T}_0) =&1 - \frac{\eta\lambda}{\sqrt{\sigma_j(\boldsymbol{\Sigma}_0)}} - \left(1 - \frac{\eta\lambda}{\sqrt{\sigma_{j+1}(\boldsymbol{\Sigma}_0)}}\right) \\
=&\eta\lambda\frac{\sqrt{\sigma_j(\boldsymbol{\Sigma}_0)} - \sqrt{\sigma_{j+1}(\boldsymbol{\Sigma}_0)}}{\sqrt{\sigma_j(\boldsymbol{\Sigma}_0)\sigma_{j+1}(\boldsymbol{\Sigma}_0)}},
\end{aligned}
$$

and

$$
\sigma_k(\mathbf{T}_0) =1 - \frac{\eta\lambda}{\sqrt{\sigma_k(\boldsymbol{\Sigma}_0)}} \le 1 - \frac{\sqrt{\sigma_{k+1}(\boldsymbol{\Sigma}_0)}}{\sqrt{\sigma_k(\boldsymbol{\Sigma}_0)}},
$$

therefore,

$$
\begin{aligned}
&\left\|\sum_{j=1}^{k} \sigma_j(\mathbf{T}_0)(\mathbf{u}_j\mathbf{u}_j^{\mathrm{T}} - \mathbf{u}_{j,0}\mathbf{u}_{j,0}^{\mathrm{T}})\right\|_* \\
\le&\sum_{j=1}^{k-1} \frac{2\eta\lambda\min\{2j, k\}\sigma_1(\mathbf{E})}{(\sqrt{\sigma_j(\boldsymbol{\Sigma}_0)} + \sqrt{\sigma_{j+1}(\boldsymbol{\Sigma}_0)})\sqrt{\sigma_j(\boldsymbol{\Sigma}_0)\sigma_{j+1}(\boldsymbol{\Sigma}_0)}} \\
&+ \frac{2\eta\lambda k\sigma_1(\mathbf{E})}{(\sqrt{\sigma_k(\boldsymbol{\Sigma}_0)} + \sqrt{\sigma_{k+1}(\boldsymbol{\Sigma}_0)})\sqrt{\sigma_k(\boldsymbol{\Sigma}_0)}} \\
\le&\sum_{j=1}^{k-1} \frac{2\eta\lambda\min\{2j, k\}\sigma_1(\mathbf{E})}{(\eta\lambda + \eta\lambda)\sqrt{\eta^2\lambda^2\eta^2\lambda^2}} + \frac{2\eta\lambda k\sigma_1(\mathbf{E})}{(\eta\lambda + 0)\eta\lambda} \\
\le&\left(\frac{k(k-1)}{\eta^2\lambda^2} + \frac{2k}{\eta\lambda}\right)\sigma_1(\mathbf{E}).
\end{aligned} \tag{59}
$$

Combining (52), (53), (54), (55), (56), (57), (58) and (59), it follows that

$$
\begin{aligned}
&\|\widehat{\mathbf{W}}^{(t)} - \mathbf{C}\|_F^2 - \|\widehat{\mathbf{W}}_0^{(t)} - \mathbf{C}\|_F^2 \\
\le&2\left[k\frac{\sigma_1(\mathbf{E})}{2\eta^2\lambda^2} + (r_c - k)I(r_c > k)\frac{\sqrt{\sigma_1(\mathbf{E})}}{\eta\lambda}\right. \\
&\left. + \left(\frac{k(k-1)}{\eta^2\lambda^2} + \frac{2k}{\eta\lambda}\right)\sigma_1(\mathbf{E})\right]\sigma_1^2(\mathbf{C}).
\end{aligned} \tag{60}
$$

On the other hand,

$$
\|\widehat{\mathbf{W}}^{(t)}\|_* - \|\widehat{\mathbf{W}}_0^{(t)}\|_* \le \|\widehat{\mathbf{W}}^{(t)} - \widehat{\mathbf{W}}_0^{(t)}\|_*.
$$

As such, we have

$$
\begin{aligned}
&\frac{1}{2\eta}\|\widehat{\mathbf{W}}^{(t)} - \mathbf{C}\|_F^2 + \lambda\|\widehat{\mathbf{W}}^{(t)}\|_* \\
&- \left\{\min_{\mathbf{W}} \frac{1}{2\eta}\|\mathbf{W} - \mathbf{C}\|_F^2 + \lambda\|\mathbf{W}\|_*\right\} \\
=&\frac{1}{2\eta}(\|\widehat{\mathbf{W}}^{(t)} - \mathbf{C}\|_F^2 - \|\widehat{\mathbf{W}}_0^{(t)} - \mathbf{C}\|_F^2) \\
&+ \lambda(\|\widehat{\mathbf{W}}^{(t)}\|_* - \|\widehat{\mathbf{W}}_0^{(t)}\|_*) \\
\le&\left(\frac{\sigma_1^2(\mathbf{C})}{\eta} + \lambda\sigma_1(\mathbf{C})\right)\left[k\frac{\sigma_1(\mathbf{E})}{2\eta^2\lambda^2}\right. \\
&\left. + (r_c - k)I(r_c > k)\frac{\sqrt{\sigma_1(\mathbf{E})}}{\eta\lambda} + \left(\frac{k(k-1)}{\eta^2\lambda^2} + \frac{2k}{\eta\lambda}\right)\sigma_1(\mathbf{E})\right] \\
=&\frac{1}{\eta}\left(\frac{\sigma_1^2(\mathbf{C})}{\eta\lambda} + \sigma_1(\mathbf{C})\right)\left[k\frac{\sigma_1(\mathbf{E})}{2\eta\lambda}\right. \\
&\left. + \max(0, r_c - k)\sqrt{\sigma_1(\mathbf{E})} + \left(\frac{k(k-1)}{\eta\lambda} + 2k\right)\sigma_1(\mathbf{E})\right].
\end{aligned} \tag{61}
$$

$\square$

F. Proof of Lemma 3

Proof. In the t-th step, a standard proximal operator (see Liu et al. [9]) optimizes the following problem:

$$\min_{\mathbf{W}} \frac{1}{2\eta} \|\mathbf{W} - \mathbf{C}\|_F^2 + \lambda \|\mathbf{W}\|_{2,1},$$

where $\mathbf{C} = \widetilde{\mathbf{W}}_i^{(t-1)}$. By Theorem 5 of Liu et al. [9], denote the solution of the problem by $\widehat{\mathbf{W}}_0^{(t)} = \mathbf{S}_{\eta\lambda,0}\mathbf{C}$, $\mathbf{\Lambda}_0$ is a diagonal matrix containing the diagonal elements of $\mathbf{C}\mathbf{C}^{\mathrm{T}}$, $\mathbf{S}_0$ is a diagonal matrix and suffices $\mathbf{S}_{ii,0} = \sqrt{\mathbf{\Lambda}_{ii,0}}$ for $i = 1, \dots, \min\{d, m\}$. $\mathbf{S}_{\eta\lambda,0}$ is also a diagonal matrix and $\mathbf{S}_{\eta\lambda,ii,0} = \max\{0, 1 - \eta\lambda/\mathbf{S}_{ii,0}\}$ for $i = 1, \dots, \min\{d, m\}$.

By Algorithm 2, $\widehat{\mathbf{W}}^{(t)} = \mathbf{U}\mathbf{S}_{\eta\lambda}\mathbf{U}^{\mathrm{T}}\mathbf{C}$.

Then we analyse the bound of $\frac{1}{2\eta}\|\widehat{\mathbf{W}}^{(t)} - \mathbf{C}\|_F^2 + \lambda\|\widehat{\mathbf{W}}^{(t)}\|_{2,1} - \{\frac{1}{2\eta}\|\widehat{\mathbf{W}}_0^{(t)} - \mathbf{C}\|_F^2 + \lambda\|\widehat{\mathbf{W}}_0^{(t)}\|_{2,1}\}$.

First, similarly as in (52), we have

$$
\begin{aligned}
&\|\widehat{\mathbf{W}}^{(t)} - \mathbf{C}\|_F^2 - \|\widehat{\mathbf{W}}_0^{(t)} - \mathbf{C}\|_F^2 \\
=&\mathrm{tr}((\widehat{\mathbf{W}}^{(t)} - \widehat{\mathbf{W}}_0^{(t)})(\widehat{\mathbf{W}}^{(t)} - \mathbf{C})^{\mathrm{T}}) \\
&+ \mathrm{tr}((\widehat{\mathbf{W}}^{(t)} - \widehat{\mathbf{W}}_0^{(t)})(\widehat{\mathbf{W}}_0^{(t)} - \mathbf{C})^{\mathrm{T}}) \\
=&\sum_{j=1}^{d} (\widehat{\mathbf{W}}^{(t)} - \widehat{\mathbf{W}}_0^{(t)})^j ((\widehat{\mathbf{W}}^{(t)} - \mathbf{C})^j)^{\mathrm{T}} \\
&+ \sum_{j=1}^{d} (\widehat{\mathbf{W}}^{(t)} - \widehat{\mathbf{W}}_0^{(t)})^j ((\widehat{\mathbf{W}}_0^{(t)} - \mathbf{C})^j)^{\mathrm{T}} \\
\leq&\|\widehat{\mathbf{W}}^{(t)} - \mathbf{C}\|_{2,1}\|\widehat{\mathbf{W}}^{(t)} - \widehat{\mathbf{W}}_0^{(t)}\|_{2,1} \\
&+ \|\widehat{\mathbf{W}}_0^{(t)} - \mathbf{C}\|_{2,1}\|\widehat{\mathbf{W}}^{(t)} - \widehat{\mathbf{W}}_0^{(t)}\|_{2,1},
\end{aligned}
\tag{62}
$$

where $(\cdot)^j$ denotes the j-th row vector of the enclosed matrix.

Denote $\mathbf{T} = \mathbf{S}_{\eta\lambda}, \mathbf{T}_0 = \mathbf{S}_{\eta\lambda,0}$. Denote the indices of non-zero rows of $\mathbf{C}$ by $\mathcal{I}_c = \{j : \mathbf{C}^j \neq \mathbf{0}\}$ and let $r_{c,s} = |\mathcal{I}_c| \leq d$.

We have

$$
\begin{aligned}
\|\widehat{\mathbf{W}}^{(t)} - \mathbf{C}\|_{2,1} &= \|(\mathbf{T} - \mathbf{I})\mathbf{C}\|_{2,1} \\
&= \sum_{j=1}^{d} \sqrt{\sum_{i=1}^{m} |(\mathbf{T} - \mathbf{I})^j \mathbf{C}_i|^2} = \sum_{j \in \mathcal{I}_c} \sqrt{\sum_{i=1}^{m} |(\mathbf{T} - \mathbf{I})_{jj} \mathbf{C}_{ij}|^2} \\
&= \sum_{j \in \mathcal{I}_c} \sqrt{\sum_{i=1}^{m} |(\mathbf{T} - \mathbf{I})_{jj}|^2 |\mathbf{C}_{ij}|^2} = \sum_{j \in \mathcal{I}_c} |(\mathbf{T} - \mathbf{I})_{jj}| \|\mathbf{C}^j\|_2.
\end{aligned}
$$

Since $\mathbf{S}_{\eta\lambda} - \mathbf{I}$ is a diagonal matrix, whose i-th diagonal element is $\max\{0, 1 - \eta\lambda/\mathbf{S}_{ii}\} - 1 \in [-1, 0)$, so

$$\|\widehat{\mathbf{W}}^{(t)} - \mathbf{C}\|_{2,1} \leq \sum_{j \in \mathcal{I}_c} \|\mathbf{C}^j\|_2 \leq r_{c,s} \max_{j \in [d]} \|\mathbf{C}^j\|_2. \tag{63}$$

Similarly,

$$\|\widehat{\mathbf{W}}_0^{(t)} - \mathbf{C}\|_{2,1} \leq \sum_{j \in \mathcal{I}_c} \|\mathbf{C}^j\|_2 \leq r_{c,s} \max_{j \in [d]} \|\mathbf{C}^j\|_2. \tag{64}$$

On the other hand,

$$
\begin{aligned}
\|\widehat{\mathbf{W}}^{(t)} - \widehat{\mathbf{W}}_0^{(t)}\|_{2,1} &= \|\mathbf{S}_{\eta\lambda}\mathbf{C} - \mathbf{S}_{\eta\lambda,0}\mathbf{C}\|_{2,1} \\
&= \sum_{j \in \mathcal{I}_c} |\mathbf{S}_{\eta\lambda,jj} - \mathbf{S}_{\eta\lambda,jj,0}| \|\mathbf{C}^j\|_2 \\
&\leq \max_{j' \in [d]} \|\mathbf{C}^{j'}\|_2 \sum_{j \in \mathcal{I}_c} |\mathbf{S}_{\eta\lambda,jj} - \mathbf{S}_{\eta\lambda,jj,0}|.
\end{aligned}
\tag{65}
$$

Denote $\mathbf{\Sigma}_0 = \widetilde{\mathbf{\Sigma}}^{(t)} = \mathbf{C}\mathbf{C}^{\mathrm{T}}$. Then we have for $j \in \mathcal{I}_c$,

$$
|\mathbf{S}_{\eta\lambda,jj} - \mathbf{S}_{\eta\lambda,jj,0}|
= \left| \max\left(0, 1 - \frac{\eta\lambda}{\sqrt{|\mathbf{\Sigma}_{jj,0} + \mathbf{E}_{jj}|}}\right) - \max\left(0, 1 - \frac{\eta\lambda}{\sqrt{|\mathbf{\Sigma}_{jj,0}|}}\right) \right|.
$$

Case 1: $\eta\lambda > \sqrt{\mathbf{\Sigma}_{jj,0}}$. Then

$$
\begin{aligned}
&|\mathbf{S}_{\eta\lambda,jj} - \mathbf{S}_{\eta\lambda,jj,0}| \\
=&\max\left(0, 1 - \frac{\eta\lambda}{\sqrt{|\mathbf{\Sigma}_{jj,0} + \mathbf{E}_{jj}|}}\right) \leq 1 - \frac{\eta\lambda}{\sqrt{\eta^2\lambda^2 + |\mathbf{E}_{jj}|}} \\
\leq&1 - \frac{\eta\lambda}{\eta\lambda + \sqrt{|\mathbf{E}_{jj}|}} = \frac{\sqrt{|\mathbf{E}_{jj}|}}{\eta\lambda + \sqrt{|\mathbf{E}_{jj}|}} \leq \frac{\sqrt{|\mathbf{E}_{jj}|}}{\eta\lambda}.
\end{aligned}
$$

Case 2: $\eta\lambda \leq \sqrt{\mathbf{\Sigma}_{jj,0}}$. Then

$$
\begin{aligned}
&|\mathbf{S}_{\eta\lambda,jj} - \mathbf{S}_{\eta\lambda,jj,0}| \\
\leq&1 - \frac{\eta\lambda}{\sqrt{|\mathbf{\Sigma}_{jj,0}| + |\mathbf{E}_{jj}|}} - 1 + \frac{\eta\lambda}{\sqrt{|\mathbf{\Sigma}_{jj,0}|}} \\
=&\eta\lambda \cdot \frac{\sqrt{|\mathbf{\Sigma}_{jj,0}| + |\mathbf{E}_{jj}|} - \sqrt{|\mathbf{\Sigma}_{jj,0}|}}{\sqrt{|\mathbf{\Sigma}_{jj,0}|(|\mathbf{\Sigma}_{jj,0}| + |\mathbf{E}_{jj}|)}} \\
=&\frac{\eta\lambda|\mathbf{E}_{jj}|}{[\sqrt{|\mathbf{\Sigma}_{jj,0}| + |\mathbf{E}_{jj}|} + \sqrt{|\mathbf{\Sigma}_{jj,0}|}]\sqrt{|\mathbf{\Sigma}_{jj,0}|^2 + |\mathbf{E}_{jj}||\mathbf{\Sigma}_{jj,0}|}} \\
\leq&\frac{\eta\lambda|\mathbf{E}_{jj}|}{[\sqrt{\eta^2\lambda^2 + 0} + \sqrt{\eta^2\lambda^2}]\sqrt{\eta^4\lambda^4 + 0}} = \frac{|\mathbf{E}_{jj}|}{2\eta^2\lambda^2}.
\end{aligned}
$$

Suppose that there exists an integer $k \leq d$ such that

$$\sum_{j=1}^{d} I(\sqrt{\mathbf{\Sigma}_{jj,0}} \geq \eta\lambda) = k$$

then $k \leq r_{c,s}$ and

$$
\begin{aligned}
&\sum_{j \in \mathcal{I}_c} |\mathbf{S}_{\eta\lambda,jj} - \mathbf{S}_{\eta\lambda,jj,0}| \\
\leq&\frac{k}{2\eta^2\lambda^2} \max_{j:\eta^2\lambda^2 \leq \mathbf{\Sigma}_{jj,0}} \mathbf{E}_{jj} \\
&+ \frac{(r_{c,s} - k)I(r_{c,s} > k)}{\eta\lambda} \max_{j:\eta^2\lambda^2 > \mathbf{\Sigma}_{jj,0}} \sqrt{\mathbf{E}_{jj}}.
\end{aligned}
\tag{66}
$$

Combining (62), (63), (64), (65) and (66), it follows that

$$
\begin{aligned}
&\|\widehat{\mathbf{W}}^{(t)} - \mathbf{C}\|_F^2 - \|\widehat{\mathbf{W}}_0^{(t)} - \mathbf{C}\|_F^2 \\
\leq&2r_{c,s}\left(\max_{j \in [d]} \|\mathbf{C}^j\|_2\right)^2 \left(\frac{k}{2\eta^2\lambda^2} \max_{j:\eta^2\lambda^2 \leq \mathbf{\Sigma}_{jj,0}} |\mathbf{E}_{jj}| \right. \\
&\left. + \frac{(r_{c,s} - k)I(r_{c,s} > k)}{\eta\lambda} \max_{j:\eta^2\lambda^2 > \mathbf{\Sigma}_{jj,0}} \sqrt{|\mathbf{E}_{jj}|}\right).
\end{aligned}
\tag{67}
$$

On the other hand,

$$\|\widehat{\mathbf{W}}^{(t)}\|_{2,1} - \|\widehat{\mathbf{W}}_0^{(t)}\|_{2,1} \leq \|\widehat{\mathbf{W}}^{(t)} - \widehat{\mathbf{W}}_0^{(t)}\|_{2,1}.$$

As such, we have

$$
\begin{aligned}
&\frac{1}{2\eta}\|\widehat{\mathbf{W}}^{(t)} - \mathbf{C}\|_F^2 + \lambda\|\widehat{\mathbf{W}}^{(t)}\|_{2,1} \\
&\quad - \left\{\min_{\mathbf{W}} \frac{1}{2\eta}\|\mathbf{W} - \mathbf{C}\|_F^2 + \lambda\|\mathbf{W}\|_{2,1}\right\} \\
&= \frac{1}{2\eta}(\|\widehat{\mathbf{W}}^{(t)} - \mathbf{C}\|_F^2 - \|\widehat{\mathbf{W}}_0^{(t)} - \mathbf{C}\|_F^2) \\
&\quad + \lambda(\|\widehat{\mathbf{W}}^{(t)}\|_{2,1} - \|\widehat{\mathbf{W}}_0^{(t)}\|_{2,1}) \\
&\leq \left[\frac{r_{c,s}}{\eta}\left(\max_{j\in[d]}\|\mathbf{C}^j\|_2\right)^2 + \lambda\left(\max_{j\in[d]}\|\mathbf{C}^j\|_2\right)\right] \\
&\quad \cdot \left[\frac{k}{2\eta^2\lambda^2} \max_{j:\eta^2\lambda^2\leq\mathbf{\Sigma}_{jj,0}}|\mathbf{E}_{jj}|\right. \\
&\quad \left. + \frac{(r_{c,s}-k)I(r_{c,s}>k)}{\eta\lambda} \max_{j:\eta^2\lambda^2>\mathbf{\Sigma}_{jj,0}}\sqrt{|\mathbf{E}_{jj}|}\right] \\
&= \frac{1}{\eta}\left[\frac{r_{c,s}}{\eta\lambda}\left(\max_{j\in[d]}\|\mathbf{C}^j\|_2\right)^2 + \left(\max_{j\in[d]}\|\mathbf{C}^j\|_2\right)\right] \\
&\quad \cdot \left[\frac{k}{2\eta\lambda} \max_{j:\eta^2\lambda^2\leq\mathbf{\Sigma}_{jj,0}}|\mathbf{E}_{jj}|\right. \\
&\quad \left. + \max(0, r_{c,s}-k) \max_{j:\eta^2\lambda^2>\mathbf{\Sigma}_{jj,0}}\sqrt{|\mathbf{E}_{jj}|}\right].
\end{aligned}
\tag{68}
$$

$\square$

G. Proof of Theorem 2

Proof. First, consider the case with no acceleration. We first use Proposition 1 of Schmidt et al. [11] by regarding procedures from Step 5 to Step 9 as approximation for the proximal operator in (8). Note that the norm clipping only bounds the parameter space and does not affect the results of Schmidt et al. [11]. Then for ε_t defined in Lemma 13 for $t\in[T]$, we have

$$
\mathcal{E} = \frac{2L}{m(T+1)^2}\left(\|\widetilde{\mathbf{W}}^{(0)} - \mathbf{W}_*\|_F \right.
$$
$$
\left. + 2\sum_{t=1}^{T} t\sqrt{\frac{2\varepsilon_t}{L}} + \sqrt{2\sum_{t=1}^{T} t^2\frac{\varepsilon_t}{L}}\right)^2.
$$

Meanwhile, by Lemma 13, we have

$$
\varepsilon_t = O\left(\frac{\kappa}{\epsilon_t}\right),
$$

where $\kappa = \frac{K^2\sqrt{m}kd\log d}{\eta}$.

On the other hand, because

$$
\epsilon = \sum_{t=1}^{T} \frac{(e^{\epsilon_t}-1)\epsilon_t}{(e^{\epsilon_t}+1)} + \sqrt{\sum_{t=1}^{T} 2\epsilon_t^2\log\left(e + \frac{\sqrt{\sum_{t=1}^{T}\epsilon_t^2}}{\delta}\right)},
$$

then by Lemma 16, we have

$$
\sqrt{\sum_{t=1}^{T}\epsilon_t^2} \geq \frac{\sqrt{2}\epsilon}{2\sqrt{\log(e + \epsilon/\sqrt{2}\delta) + 2\epsilon}} = c_2.
$$

Then by Lemma 17, we have

$$
\sum_{t=1}^{T}\sqrt{\varepsilon_t} =
\begin{cases}
O\left(\sqrt{\frac{\kappa T^{\alpha+1/2}}{c_2(\alpha/2-1)^2\sqrt{2\alpha+1}}}\right), & \alpha > 2; \\
O\left(\sqrt{\frac{\kappa T^{5/2}}{c_2(\alpha/2-1)^2\sqrt{2\alpha+1}}}\right), & -1/2 < \alpha < 2; \\
O\left(\sqrt{\frac{\kappa T^{2-\alpha}}{c_2(\alpha/2-1)^2\sqrt{-2\alpha-1}}}\right), & \alpha < -1/2.
\end{cases}
$$

Because $\widetilde{\mathbf{W}}^{(0)}$ is the result of the norm clipping, we have $\widetilde{\mathbf{W}}^{(0)} \in \mathcal{W}$.

Finally, taking $c_3 = \phi(\alpha)$ defined in (13) and $c_4 = \frac{\kappa}{c_2(\alpha/2-1)^2\sqrt{|2\alpha+1|}}$, under the assumption that $\mathbf{W}_* \in \mathcal{W}$, using Lemma 19, we have the results for the case with no acceleration.

For the accelerated case, we use Proposition 2 of Schmidt et al. [11] to have

$$
\mathcal{E} = \frac{2L}{m(T+1)^2}\left(\|\widetilde{\mathbf{W}}^{(0)} - \mathbf{W}_*\|_F \right.
$$
$$
\left. + 2\sum_{t=1}^{T} t\sqrt{\frac{2\varepsilon_t}{L}} + \sqrt{2\sum_{t=1}^{T} t^2\frac{\varepsilon_t}{L}}\right)^2.
$$

Then one can prove similarly combining Lemma 13, Lemma 16, Lemma 17 and Lemma 20.

$\square$

H. Proof of Theorem 3

Proof. First, consider the case with no acceleration. We use Proposition 1 of Schmidt et al. [11] and prove similarly as in Appendix J-G, combining Lemma 14, Lemma 16, Lemma 17 and Lemma 19.

For the accelerated case, we use Proposition 2 of Schmidt et al. [11] and prove similarly as in Appendix J-G, combining Lemma 14, Lemma 16, Lemma 17 and Lemma 20.

$\square$

I. Proof of Theorem 4

Proof. First, consider the case with no acceleration. We use Proposition 3 of Schmidt et al. [11] to have

$$
\mathcal{E} = \frac{Q_0^T}{\sqrt{m}}\left(\|\widetilde{\mathbf{W}}^{(0)} - \mathbf{W}_*\|_F + 2\sum_{t=1}^{T} Q_0^{-t}\sqrt{\frac{2\varepsilon_t}{L}}\right).
$$

Then one can prove similarly as in Appendix J-G, combining Lemma 13, Lemma 16, Lemma 17 and Lemma 21.

For the accelerated case, we use Proposition 4 of Schmidt et al. [11] to have

$$
\mathcal{E} = \frac{(Q_0)^T}{m}\left(\sqrt{2(f(\widehat{\mathbf{W}}^{(0)}) - f(\mathbf{W}_*))} + 2\sqrt{\frac{L}{\mu}}\sum_{t=1}^{T}\sqrt{\varepsilon_t(Q_0)^{-t}} \right.
$$
$$
\left. + \sqrt{\sum_{t=1}^{T}\varepsilon_t(Q_0)^{-t}}\right)^2.
$$

Then one can prove similarly as in Appendix J-G, using the assumption that $f(\widetilde{\mathbf{W}}^{(0)}) - f(\mathbf{W}_*) = O(K^2Lm)$, combining Lemma 13, Lemma 16, Lemma 17 and Lemma 22.

$\square$

J. Proof of Theorem 5

Proof. First, consider the case with no acceleration. We use Proposition 3 of Schmidt et al. [11] and prove similarly as in Appendix J-I, combining Lemma 14, Lemma 16, Lemma 17 and Lemma 21.

For the accelerated case, we use Proposition 4 of Schmidt et al. [11] and prove similarly as in Appendix J-I, using the assumption that $f(\widetilde{\mathbf{W}}^{(0)}) - f(\mathbf{W}_*) = O(K^2Lm)$, combining Lemma 14, Lemma 16, Lemma 17 and Lemma 22.

$\square$

K. Proof of Theorem 6

Proof. Consider the bound in (12), whose logarithm is

$$
\phi(\alpha)\log\left(\frac{kd\log d\sqrt{\log(e + \epsilon/\sqrt{2}\delta) + 2\epsilon}}{\sqrt{m}\epsilon}\right)
$$
$$
- \phi(\alpha)\log((\alpha/2-1)^2\sqrt{|2\alpha+1|}) + \log(K^2L)
$$

By Assumption 1, the first term dominates. Then we should firstly maximize $\phi(\alpha)$, which results in that $\phi(\alpha) = 2/5$ and $-1/2 < \alpha < 2$. Then since $\phi(\alpha)$ is now fixed, we maximize $(\alpha/2-1)^2\sqrt{|2\alpha+1|}$, which results in $\alpha = 0$. Results under other settings can be proved similarly. □

L. Proof of Proposition 2

Proof. First, consider the method of Pathak et al. [10].

By Definition 7, an (ϵ, δ)-Iterative DP-MTL algorithm with $T = 1$ should suffice for any set $\mathcal{S} \subseteq \mathbb{R}^{d\times(m-1)}$ and all $i \in [m]$ that

$$\mathbb{P}(\hat{\mathbf{w}}_{[-i]}^{(1)} \in \mathcal{S} \mid \mathbf{W}^{(0)}, \mathcal{D}^m)$$
$$\leq e^{\epsilon}\mathbb{P}(\hat{\mathbf{w}}_{[-i]}^{(1)} \in \mathcal{S} \mid (\mathbf{W}')^{(0)}, (\mathcal{D}')^m) + \delta.$$

On the other hand, for the ϵ given in the method of Pathak et al. [10], using Theorem 4.1 of Pathak et al. [10], taking $D = \mathcal{D}^m$ and $D' = (\mathcal{D}')^m$, we have for any set $\mathcal{S} \subseteq \mathbb{R}^d$,

$$\mathbb{P}(\hat{\mathbf{w}}^s \in \mathcal{S} \mid \mathcal{D}^m) \leq e^{\epsilon}\mathbb{P}(\hat{\mathbf{w}}^s \in \mathcal{S} \mid (\mathcal{D}')^m),$$

where $\hat{\mathbf{w}}^s$ is defined in Section 3.2 of Pathak et al. [10].

Because the method of Pathak et al. [10] uses $\hat{\mathbf{w}}^s$ for all the tasks, then we have $\hat{\mathbf{w}}_i^{(1)} = \hat{\mathbf{w}}^s$ for all $i \in [m]$.

As such, denote $\mathbf{W}^{(0)}$ and $(\mathbf{W}')^{(0)}$ as the collections of models independently learned using $\mathcal{D}^m$ and $(\mathcal{D}')^m$, respectively. Then $\mathcal{D}^m$ and $(\mathcal{D}')^m$ contain all the information of $\mathbf{W}^{(0)}$ and $(\mathbf{W}')^{(0)}$, respectively. As such, we have for any set $\mathcal{S} \subseteq \mathbb{R}^{d\times(m-1)}$, all $i \in [m]$ and $\delta = 0$ that

$$\mathbb{P}(\hat{\mathbf{w}}_{[-i]}^{(1)} \in \mathcal{S} \mid \mathbf{W}^{(0)}, \mathcal{D}^m)$$
$$\leq e^{\epsilon}\mathbb{P}(\hat{\mathbf{w}}_{[-i]}^{(1)} \in \mathcal{S} \mid (\mathbf{W}')^{(0)}, (\mathcal{D}')^m) + \delta,$$

which shows that the method of Pathak et al. [10] is an (ϵ, δ)-Iterative DP-MTL algorithm with $T = 1$ and $\delta = 0$.

Then we consider the method of Gupta et al. [5]. Assume a constant $\delta \geq 0$ and the number of iteration T is given.

Taking $T_0 = m, t = i$ for $i \in [m]$, $\beta_t = \hat{\mathbf{w}}_i$, $\mathscr{D} = \mathcal{D}^m$, for the ϵ given in the method of Gupta et al. [5], using Theorem 1 of Gupta et al. [5], for $t \in [T]$, we have *in the t-th each iteration*, for any set $\mathcal{S} \subseteq \mathbb{R}^{d\times m}$ and all $i \in [m]$,

$$\mathbb{P}(\hat{\mathbf{W}}^{(t)} \in \mathcal{S} \mid \mathcal{D}^m) \leq e^{\epsilon}\mathbb{P}(\hat{\mathbf{W}}^{(t)} \in \mathcal{S} \mid (\mathcal{D}')^m),$$

which suggests that for any set $\mathcal{S} \subseteq \mathbb{R}^{d\times(m-1)}$ and all $i \in [m]$,

$$\mathbb{P}(\hat{\mathbf{w}}_{[-i]}^{(t)} \in \mathcal{S} \mid \mathcal{D}^m) \leq e^{\epsilon}\mathbb{P}(\hat{\mathbf{w}}_{[-i]}^{(t)} \in \mathcal{S} \mid (\mathcal{D}')^m).$$

Then for all $i \in [m]$ and for all $t \in [T]$, take the t-th output $\theta_{t,i} = \hat{\mathbf{w}}_{[-i]}^{(t)}$ and $\delta_t = 0$.

Therefore by the *Adaptive composition* Lemma (Lemma 7), for all $i \in [m]$ and for any set $\mathcal{S} \subset \mathbb{R}^{d\times(m-1)\times T}$,

$$\mathbb{P}((\theta_{1,i}, \ldots, \theta_{T,i}) \in \mathcal{S} \mid \bigcap_{t=1}^{T}(\mathcal{B}_t = (\mathcal{D}^m, \boldsymbol{\theta}_{1:t-1})))$$
$$\leq e^{\tilde{\epsilon}}\mathbb{P}((\theta_{1,i}, \cdots, \theta_{T,i}) \in \mathcal{S} \mid \bigcap_{t=1}^{T}(\mathcal{B}_t = ((\mathcal{D}')^m, \boldsymbol{\theta}_{1:t-1})))$$
$$+ \delta,$$

where for all $t \in [T]$, $\mathcal{B}_t$ denotes the input for the t-th iteration,

$$\boldsymbol{\theta}_{1:t-1} = \begin{cases} \emptyset, & t = 1 \\ (\theta_{1,1}, \ldots, \theta_{1,m}) \ldots, (\theta_{t-1,1}, \ldots, \theta_{t-1,m}), & t \geq 2, \end{cases}$$

and $\tilde{\epsilon}$ is defined as follows.

$$\tilde{\epsilon} = \min\left\{\sum_{t=1}^{T}\epsilon, \sum_{t=1}^{T}\frac{(e^{\epsilon}-1)\epsilon}{(e^{\epsilon}+1)} + \sqrt{\sum_{t=1}^{T}2\epsilon^2\log\left(\frac{1}{\delta}\right)}, \right.$$
$$\left. \sum_{t=1}^{T}\frac{(e^{\epsilon}-1)\epsilon}{(e^{\epsilon}+1)} + \sqrt{\sum_{t=1}^{T}2\epsilon^2\log\left(e + \frac{\sqrt{\sum_{t=1}^{T}\epsilon^2}}{\delta}\right)}\right\}.$$

As such, in each t-th iteration, denote $\mathbf{W}^{(t-1)}$ and $(\mathbf{W}')^{(t-1)}$ as the collections of models independently learned using $(\mathcal{D}^m, \boldsymbol{\theta}_{1:t-1})$ and $(\mathcal{D}')^m, \boldsymbol{\theta}_{1:t-1})$, respectively. Then $(\mathcal{D}^m, \boldsymbol{\theta}_{1:t-1})$ and $(\mathcal{D}')^m, \boldsymbol{\theta}_{1:t-1})$ contain all the information of $\mathbf{W}^{(t-1)}$ and $(\mathbf{W}')^{(t-1)}$, respectively.

Therefore, we have for any set $\mathcal{S} \subset \mathbb{R}^{d\times(m-1)\times T}$,

$$\mathbb{P}(\mathbf{w}_{[-i]}^{(1:T)} \in \mathcal{S} \mid \bigcap_{t=1}^{T}(\mathcal{B}_t = (\mathbf{W}^{(t-1)}, \mathcal{D}^m, \boldsymbol{\theta}_{1:t-1})))$$
$$\leq e^{\tilde{\epsilon}}\mathbb{P}(\mathbf{w}_{[-i]}^{(1:T)} \in \mathcal{S} \mid \bigcap_{t=1}^{T}(\mathcal{B}_t = ((\mathbf{W}')^{(t-1)}, (\mathcal{D}')^m, \boldsymbol{\theta}_{1:t-1})))$$
$$+ \delta,$$

which shows that by Definition 7, the method of Gupta et al. [5] is an $(\tilde{\epsilon}, \delta)$-Iterative DP-MTL algorithm. □

M. Proof of Proposition 3

Proof. Given an (ϵ, δ) - Iterative DP-MTL algorithm $\mathcal{A}(\mathcal{B})$, by Definition 7, we have for any set $\mathcal{S} \subseteq \mathbb{R}^{d\times(m-1)\times T}$ that

$$\mathbb{P}(\hat{\mathbf{w}}_{[-i]}^{(1:T)} \in \mathcal{S} \mid \bigcap_{t=1}^{T}\mathcal{B}_t = (\mathbf{W}^{(t-1)}, \mathcal{D}^m, \boldsymbol{\theta}_{1:t-1}))$$
$$\leq \exp(\epsilon)\mathbb{P}(\hat{\mathbf{w}}_{[-i]}^{(1:T)} \in \mathcal{S} \mid \bigcap_{t=1}^{T}\mathcal{B}_t = ((\mathbf{W}')^{(t-1)}, (\mathcal{D}')^m, \boldsymbol{\theta}_{1:t-1}))$$
$$+ \delta.$$

Furthermore, following the proof of the *Group privacy* Lemma (Lemma 5), shown by Vadhan [12], for protecting the entire dataset, n data instances, of the i-th task, we construct a series of datasets, $\mathcal{D}_{(0)}^m, \mathcal{D}_{(1)}^m, \ldots, \mathcal{D}_{(n)}^m$, and let $\mathcal{D}_{(0)}^m = \mathcal{D}^m, \mathcal{D}_{(n)}^m = (\mathcal{D}')^m$ such that for $j = 0, \ldots, N-1$, $\mathcal{D}_{(j)}^m$ and $\mathcal{D}_{(j+1)}^m$ are two neighboring datasets that differ in one data instance. Let a series of model matrices, $\mathbf{W}_{(0)}, \ldots, \mathbf{W}_{(n)}$, where $\mathbf{W}_{(0)} = \mathbf{W}, \mathbf{W}_{(n)} = \mathbf{W}'$, be the input model matrices in those settings. Let a series of output objects $\boldsymbol{\theta}_{1:t-1}^{(0)}, \ldots, \boldsymbol{\theta}_{1:t-1}^{(n)}$, where $\boldsymbol{\theta}_{1:t-1}^{(0)} = \boldsymbol{\theta}_{1:t-1}, \boldsymbol{\theta}_{1:t-1}^{(n)} = \mathbf{W}'$, be the output objects in those settings.

Then, we have

$$\mathbb{P}(\hat{\mathbf{w}}_{[-i]}^{(1:T)} \in \mathcal{S} \mid \bigcap_{t=1}^{T}\mathcal{B}_t = (\mathbf{W}_{(0)}^{(t-1)}, \mathcal{D}_{(0)}^m, \boldsymbol{\theta}_{1:t-1}^{(0)}))$$
$$\leq \exp(\epsilon)\mathbb{P}(\hat{\mathbf{w}}_{[-i]}^{(1:T)} \in \mathcal{S} \mid \bigcap_{t=1}^{T}\mathcal{B}_t = (\mathbf{W}_{(1)}^{(t-1)}, \mathcal{D}_{(1)}^m, \boldsymbol{\theta}_{1:t-1}^{(1)}))$$
$$+ \delta$$
$$\vdots$$
$$\leq \exp(n\epsilon)\mathbb{P}(\hat{\mathbf{w}}_{[-i]}^{(1:T)} \in \mathcal{S} \mid \bigcap_{t=1}^{T}\mathcal{B}_t = (\mathbf{W}_{(n)}^{(t-1)}, \mathcal{D}_{(n)}^m, \boldsymbol{\theta}_{1:t-1}^{(n)}))$$
$$+ (1 + \exp(\epsilon) + \ldots + \exp((n-1)\epsilon))\delta$$
$$\leq \exp(n\epsilon)\mathbb{P}(\hat{\mathbf{w}}_{[-i]}^{(1:T)} \in \mathcal{S} \mid \bigcap_{t=1}^{T}\mathcal{B}_t = (\mathbf{W}_{(n)}^{(t-1)}, \mathcal{D}_{(n)}^m, \boldsymbol{\theta}_{1:t-1}^{(n)}))$$
$$+ n\exp(n\epsilon)\delta,$$

which renders $\mathcal{A}$ as an $(n\epsilon, n\exp(n\epsilon)\delta)$ - Iterative MP-MTL algorithm. $\square$

APPENDIX K
PROOF OF THE RESULTS IN APPENDIX C AND APPENDIX D

A. Proof of Corollary 2

Proof. For simplicity, we omit the symbol $\mathcal{B}$ to denote the input in the conditional events in some equations.

Use Corollary 1 and Theorem 1. Given $t \in [T]$, the algorithm $(\mathbf{P}^{(t-1)}, \mathbf{\Sigma}^{(1:t-1)}) \to (\mathbf{M}^{(t)}, \mathbf{\Sigma}^{(t)})$ is an $(\epsilon_t, 0)$-differentially private algorithm, where $\mathbf{M}^{(t)} = \mathbf{U}\mathbf{S}_{\eta\lambda}\mathbf{U}^{\mathrm{T}}$.

Now, for all $i \in [m]$, applying the *Post-Processing immunity* Lemma (Lemma 4) for the mapping $f : (\mathbf{M}^{(t)}, \mathbf{p}_{[-i]}^{(t-1)}) \to \hat{\mathbf{p}}_{[-i]}^{(t-1)}$, which does not touch any unperturbed sensitive information of the i-th task, we have for any set $\mathcal{S} \subseteq \mathbb{R}^{d \times (m-1)}$ that

$$\mathbb{P}(\hat{\mathbf{p}}_{[-i]}^{(t-1)} \in \mathcal{S} \mid \mathbf{P}^{(t-1)}, \mathbf{\Sigma}^{(1:t-1)})$$
$$\leq e^{\epsilon_t}\mathbb{P}(\hat{\mathbf{p}}_{[-i]}^{(t-1)} \in \mathcal{S} \mid (\mathbf{P}')^{(t-1)}, \mathbf{\Sigma}^{(1:t-1)}),$$

where $\mathbf{P}^{(t-1)}$ and $(\mathbf{P}')^{(t-1)}$ differ only in the i-th column.

Then, because in the t-th iteration the mapping $\mathbf{Q}^{(t-1)} \to \widehat{\mathbf{Q}}^{(t-1)}$ is a deterministic STL algorithm, we have for any set $\mathcal{S} \subseteq \mathbb{R}^{d \times (m-1)}$ that

$$\mathbb{P}(\hat{\mathbf{q}}_{[-i]}^{(t-1)} \in \mathcal{S} \mid \mathbf{Q}^{(t-1)})$$
$$=\mathbb{P}(\hat{\mathbf{q}}_{[-i]}^{(t-1)} \in \mathcal{S} \mid \mathbf{q}_{[-i]}^{(t-1)}, \mathbf{q}_i^{(t-1)})$$
$$=\mathbb{P}(\hat{\mathbf{q}}_{[-i]}^{(t-1)} \in \mathcal{S} \mid \mathbf{q}_{[-i]}^{(t-1)}, (\mathbf{q}_i')^{(t-1)})$$
$$=e^0\mathbb{P}(\hat{\mathbf{q}}_{[-i]}^{(t-1)} \in \mathcal{S} \mid (\mathbf{Q}')^{(t-1)}) + 0,$$

where $\mathbf{Q}^{(t-1)}$ and $(\mathbf{Q}')^{(t-1)}$ differ only in the i-th column.

Then applying *Combination* Lemma (Lemma 6), we have for any set $\mathcal{S} \subseteq \mathbb{R}^{d \times (m-1)} \times \mathbb{R}^{d \times (m-1)}$

$$\mathbb{P}((\hat{\mathbf{p}}_{[-i]}^{(t-1)}, \hat{\mathbf{q}}_{[-i]}^{(t-1)}) \in \mathcal{S} \mid \mathbf{P}^{(t-1)}, \mathbf{\Sigma}^{(1:t-1)}, \mathbf{Q}^{(t-1)})$$
$$\leq e^{\epsilon_t}\mathbb{P}((\hat{\mathbf{p}}_{[-i]}^{(t-1)}, \hat{\mathbf{q}}_{[-i]}^{(t-1)}) \in \mathcal{S} \mid (\mathbf{P}')^{(t-1)}, \mathbf{\Sigma}^{(1:t-1)}, (\mathbf{Q}')^{(t-1)}),$$

Because the mapping $(\widehat{\mathbf{P}}^{(t-1)}, \widehat{\mathbf{Q}}^{(t-1)}, \mathcal{D}^m) \to (\widehat{\mathbf{P}}^{(t)}, \widehat{\mathbf{Q}}^{(t)})$ is a deterministic STL algorithm, applying Lemma 8, we further have for any set $\mathcal{S} \subseteq \mathbb{R}^{d \times (m-1)} \times \mathbb{R}^{d \times (m-1)}$ that

$$\mathbb{P}((\hat{\mathbf{p}}_{[-i]}^{(t)}, \hat{\mathbf{q}}_{[-i]}^{(t)}) \in \mathcal{S} \mid \mathbf{P}^{(t-1)}, \mathbf{\Sigma}^{(1:t-1)}, \mathbf{Q}^{(t-1)}, \mathcal{D}^m)$$
$$\leq e^{\epsilon_t}\mathbb{P}((\hat{\mathbf{p}}_{[-i]}^{(t)}, \hat{\mathbf{q}}_{[-i]}^{(t)}) \in \mathcal{S} \mid (\mathbf{P}')^{(t-1)}, \mathbf{\Sigma}^{(1:t-1)}, (\mathbf{Q}')^{(t-1)}, (\mathcal{D}')^m),$$

where $(\mathcal{D}')^m$ differs from $\mathcal{D}^m$ in the entire dataset of the i-th task.

Now, using Theorem 1, for $t = 1, \ldots, T$, we again take the t-th dataset $\widetilde{\mathcal{D}}_t = \{(\mathbf{p}_1^{(t-1)}, \mathbf{q}_1^{(t-1)}, \mathcal{D}_1), \ldots, (\mathbf{p}_m^{(t-1)}, \mathbf{q}_m^{(t-1)}\mathcal{D}_m)\}$ and denote $\vartheta_{t,i} = (\hat{\mathbf{q}}_{[-i]}^{(t)}, \hat{\mathbf{q}}_{[-i]}^{(t)}, \mathbf{M}^{(t)}, \mathbf{\Sigma}^{(t)}) \in \mathcal{C}_{t,i}$. Given the fact that $\mathbf{P}^{(t)} = \widehat{\mathbf{P}}^{(t)}$ and $\mathbf{Q}^{(t)} = \widehat{\mathbf{Q}}^{(t)}$ for all $t \in [T]$, we have for any set $\mathcal{S}_{t,i} \subseteq \mathcal{C}_{t,i}$ that

$$\mathbb{P}(\vartheta_{t,i} \in \mathcal{S}_{t,i} \mid \widetilde{\mathcal{D}}_t, \boldsymbol{\vartheta}_{1:t-1})$$
$$\leq e^{\epsilon_t}\mathbb{P}(\vartheta_{t,i} \in \mathcal{S}_{t,i} \mid \widetilde{\mathcal{D}}_t', \boldsymbol{\vartheta}_{1:t-1}),$$

where $\widetilde{\mathcal{D}}_t$ and $\widetilde{\mathcal{D}}_t'$ are two adjacent datasets that differ in a single entry, the i-th "data instance" $(\mathbf{p}_i^{(t-1)}, \mathbf{q}_i^{(t-1)}, \mathcal{D}_i = (\mathbf{X}_i, \mathbf{y}_i))$, and

$$\boldsymbol{\vartheta}_{1:t-1} = \begin{cases} \emptyset, & t = 1 \\ (\vartheta_{1,1}, \ldots, \vartheta_{1,m}) \ldots, (\vartheta_{t-1,1}, \ldots, \vartheta_{t-1,m}), & t \geq 2. \end{cases}$$

This renders the algorithm in the t-th iteration as an $(\epsilon_t, 0)$-differentially private algorithm.

Then, again by the *Adaptive composition* Lemma (Lemma 7), for all $i \in [m]$ and for any set $\mathcal{S}' \subseteq \bigotimes_{t=1}^{T} \mathcal{C}_{t_i}$, we have

$$\mathbb{P}((\vartheta_{1,i}, \cdots, \vartheta_{T,i}) \in \mathcal{S}' \mid \bigcap_{t=1}^{T}(\mathcal{B}_t = (\widetilde{\mathcal{D}}_t, \boldsymbol{\vartheta}_{1:t-1})))$$

$$\leq e^{\tilde{\epsilon}}\mathbb{P}((\vartheta_{1,i}, \cdots, \vartheta_{T,i}) \in \mathcal{S}' \mid \bigcap_{t=1}^{T}(\mathcal{B}_t = (\widetilde{\mathcal{D}}_t', \boldsymbol{\vartheta}_{1:t-1})))$$
$$+ \delta,$$

where for all $t \in [T]$, $\mathcal{B}_t$ denotes the input for the t-th iteration.

Finally, for all $t \in [T]$, taking $\theta_t = (\vartheta_{t,1}, \ldots, \vartheta_{t,m})$ and given the fact that $\widehat{\mathbf{W}}^{(t)} = \widehat{\mathbf{P}}^{(t)} + \widehat{\mathbf{Q}}^{(t)}$, we have for any set $\mathcal{S} \subseteq \mathbb{R}^{d \times (m-1) \times T}$ that

$$\mathbb{P}(\hat{\mathbf{w}}_{[-i]}^{(1:T)} \in \mathcal{S} \mid \bigcap_{t=1}^{T} \mathcal{B}_t = (\mathbf{W}^{(t-1)}, \mathcal{D}^m, \boldsymbol{\theta}_{1:t-1}))$$

$$\leq e^{\epsilon}\mathbb{P}(\hat{\mathbf{w}}_{[-i]}^{(1:T)} \in \mathcal{S} \mid \bigcap_{t=1}^{T} \mathcal{B}_t = ((\mathbf{W}')^{(t-1)}, (\mathcal{D}')^m, \boldsymbol{\theta}_{1:t-1}))$$
$$+ \delta,$$

where $(\mathbf{W}')^{(t-1)}$ are associated with the setting in which the i-th task has been replaced. $\square$

B. Proof of Proposition 4

Proof. For simplicity, we omit the symbol $\mathcal{B}$ used to denote the input in the conditional events in some equations.

First, the procedure from the 4-th step to the 5-th step is a standard output perturbation of Chaudhuri et al. [2]; thus, we have for all $i \in [m]$, for all neighboring datasets $\mathcal{D}^m$ and $(\mathcal{D}')^m$ that differ in a single data instance of the i-th task, and for any set $\mathcal{S} \in \mathbb{R}^d$,

$$\mathbb{P}(\tilde{\mathbf{w}}_i^{(t-1)} \in \mathcal{S} \mid \tilde{\mathbf{w}}_i^{(0:t-2)}, \mathcal{D}^m, \mathbf{M}^{(t-1)})$$
$$\leq \exp(\epsilon_{\mathrm{dp},t})\mathbb{P}(\tilde{\mathbf{w}}_i^{(t-1)} \in \mathcal{S} \mid \tilde{\mathbf{w}}_i^{(0:t-2)}, (\mathcal{D}')^m, \mathbf{M}^{(t-1)}),$$

where $\tilde{\mathbf{w}}_i^{(0:t-2)} = \emptyset$ when $t = 1$.

Then, because the mapping $(\widetilde{\mathbf{W}}^{(t-1)}, \mathbf{\Sigma}^{(1:t-1)}) \to \theta_t = (\mathbf{\Sigma}^{(t)}, \mathbf{M}^{(t)}, \widetilde{\mathbf{W}}^{(t-1)}) \in \mathcal{C}_t$ does not touch any unperturbed sensitive information of $(\mathbf{X}_i, \mathbf{y}_i, \mathbf{w}_i^{(0:t-1)})$, the *Post-Processing immunity* Lemma (Lemma 4) can be applied such that we have for any set $\mathcal{S}' \subseteq \mathcal{C}_t$ that

$$\mathbb{P}(\theta_t \in \mathcal{S}' \mid \widetilde{\mathbf{W}}^{(0:t-2)}, \mathcal{D}^m, \mathbf{M}^{(t-1)})$$
$$\leq \exp(\epsilon_{\mathrm{dp},t})\mathbb{P}(\theta_t \in \mathcal{S}' \mid \widetilde{\mathbf{W}}^{(0:t-2)}, (\mathcal{D}')^m, \mathbf{M}^{(t-1)}),$$

which means that

$$\mathbb{P}(\theta_t \in \mathcal{S}' \mid \mathcal{D}^m, \boldsymbol{\theta}_{1:t-1})$$
$$\leq \exp(\epsilon_{\mathrm{dp},t})\mathbb{P}(\theta_t \in \mathcal{S}' \mid (\mathcal{D}')^m, \boldsymbol{\theta}_{1:t-1}),$$

where

$$\boldsymbol{\theta}_{1:t-1} = \begin{cases} \emptyset, & t = 1 \\ \theta_1, \theta_2, \cdots, \theta_{t-1}, & t \geq 2. \end{cases}$$

Then, by the *Adaptive composition* Lemma (Lemma 7), we have for any set $\mathcal{S}'' \subseteq \bigotimes_{t=1}^{T} \mathcal{C}_t$ that

$$\mathbb{P}(\theta_{1:T} \in \mathcal{S}'' \mid \bigcap_{t=1}^{T}(\mathcal{B}_t = (\mathcal{D}^m, \boldsymbol{\theta}_{1:t-1})))$$

$$\leq \exp(\epsilon_{\mathrm{dp}})\mathbb{P}(\theta_{1:T} \in \mathcal{S}'' \mid \bigcap_{t=1}^{T}(\mathcal{B}_t = ((\mathcal{D}')^m, \boldsymbol{\theta}_{1:t-1})))$$
$$+ \delta_{\mathrm{dp}}.$$

Because the mapping $(\theta_t, \mathcal{D}_{[-i]}, \tilde{\mathbf{w}}_{[-i]}^{(0:t-2)}, \mathbf{W}^{(t-1)}) \to \hat{\mathbf{w}}_{[-i]}^{(t)}$ does not touch any unperturbed sensitive information of $(\mathbf{X}_i, \mathbf{y}_i, \mathbf{w}_i^{(0:t-1)})$ for all $t \in [T]$ ($\mathbf{W}^{(t-1)}$ is actually not used in the mapping), the *Post-Processing immunity* Lemma (Lemma 4) can be applied such that we have for any set $\mathcal{S}_0 \subseteq \mathbb{R}^{d \times (m-1) \times T}$ that

$$\mathbb{P}(\hat{\mathbf{w}}_{[-i]}^{(1:T)} \in \mathcal{S}_0 \mid \bigcap_{t=1}^{T} (\mathcal{B}_t = (\mathcal{D}^m, \boldsymbol{\theta}_{1:t-1}, \mathbf{W}^{(t-1)})))$$

$$\leq e^{\epsilon_{\mathrm{dp},t}} \mathbb{P}(\hat{\mathbf{w}}_{[-i]}^{(1:T)} \in \mathcal{S}_0 \mid \bigcap_{t=1}^{T} (\mathcal{B}_t = ((\mathcal{D}')^m, \boldsymbol{\theta}_{1:t-1}, (\mathbf{W}')^{(t-1)})))$$

$$+ \delta_{\mathrm{dp}},$$

where $(\mathbf{W}')^{(t-1)}$ is associated with the setting in which a single data instance of the i-th task has been replaced.

Therefore, Algorithm 5 is an $(\epsilon_{\mathrm{dp}}, \delta_{\mathrm{dp}})$ - Iterative DP-MTL algorithm.

Next, for the conditional density of $\boldsymbol{\Sigma}^{(t)}$ given $\mathbf{W}^{(t-1)}$, we have

$$p(\boldsymbol{\Sigma}^{(t)} \mid \mathbf{W}^{(t-1)})$$

$$= \int_{\widetilde{\mathbf{W}}^{(t-1)}} p(\boldsymbol{\Sigma}^{(t)} \mid \mathbf{W}^{(t-1)}, \widetilde{\mathbf{W}}^{(t-1)})$$
$$p(\widetilde{\mathbf{W}}^{(t-1)} \mid \mathbf{W}^{(t-1)}) d\widetilde{\mathbf{W}}^{(t-1)}$$

$$= \int_{\widetilde{\mathbf{W}}^{(t-1)}} p(\boldsymbol{\Sigma}^{(t)} \mid \widetilde{\mathbf{W}}^{(t-1)}) p(\widetilde{\mathbf{W}}^{(t-1)} \mid \mathbf{W}^{(t-1)}) d\widetilde{\mathbf{W}}^{(t-1)}$$

$$= \int_{\widetilde{\mathbf{W}}^{(t-1)}} p(\boldsymbol{\Sigma}^{(t)} \mid \widetilde{\mathbf{W}}^{(t-1)}) \prod_{i=1}^{m} p(\tilde{\mathbf{w}}_i^{(t-1)} \mid \mathbf{w}_i^{(t-1)}) d\widetilde{\mathbf{W}}^{(t-1)}.$$

Because, for all $i \in [m]$ and some constant $c = \frac{\tilde{s}_i^{(t-1)}}{\epsilon_{\mathrm{dp},t}}$, we have

$$p(\tilde{\mathbf{w}}_i^{(t-1)} \mid \mathbf{w}_i^{(t-1)}) \propto \exp\left(-c \|\tilde{\mathbf{w}}_i^{(t-1)} - \mathbf{w}_i^{(t-1)}\|_2\right),$$

given $(\mathbf{W}')^{(t-1)}$ such that for some $i \in [m]$, $(\mathbf{w}_i')^{(t-1)} \neq \mathbf{w}_i^{(t-1)}$, letting $(\tilde{\mathbf{w}}_i')^{(t-1)} = \tilde{\mathbf{w}}_i^{(t-1)} - \mathbf{w}_i^{(t-1)} + (\mathbf{w}_i')^{(t-1)}$, we have

$$\|(\tilde{\mathbf{w}}_i')^{(t-1)} - (\mathbf{w}_i')^{(t-1)}\|_2 = \|\tilde{\mathbf{w}}_i^{(t-1)} - \mathbf{w}_i^{(t-1)}\|_2$$
$$\Rightarrow p((\tilde{\mathbf{w}}_i')^{(t-1)} \mid \mathbf{w}_i^{(t-1)}) = p(\tilde{\mathbf{w}}_i^{(t-1)} \mid \mathbf{w}_i^{(t-1)}),$$

and $d(\tilde{\mathbf{w}}_i')^{(t-1)} = d\tilde{\mathbf{w}}_i^{(t-1)}$.

Furthermore, based on the proof of Theorem 1 in Section J-C, we know that for neighboring matrices $\widetilde{\mathbf{W}}^{(t-1)}$ and $(\widetilde{\mathbf{W}}')^{(t-1)}$ that differ in the i-th column, we have

$$p(\boldsymbol{\Sigma}^{(t)} \mid \widetilde{\mathbf{W}}^{(t-1)}) \leq \exp(\epsilon_{\mathrm{mp},t}) p(\boldsymbol{\Sigma}^{(t)} \mid (\widetilde{\mathbf{W}}')^{(t-1)}).$$

Therefore, for all $i \in [m]$, given $(\mathbf{W}')^{(t-1)}$ such that $(\mathbf{w}_i')^{(t-1)} \neq \mathbf{w}_i^{(t-1)}$, under the choice for $(\tilde{\mathbf{w}}_i')^{(t-1)}$, we have

$$p(\boldsymbol{\Sigma}^{(t)} \mid \mathbf{W}^{(t-1)})$$

$$= \int_{\widetilde{\mathbf{W}}^{(t-1)}} p(\boldsymbol{\Sigma}^{(t)} \mid \widetilde{\mathbf{W}}^{(t-1)}) \prod_{j=1}^{m} p(\tilde{\mathbf{w}}_j^{(t-1)} \mid \mathbf{w}_j^{(t-1)}) d\widetilde{\mathbf{W}}^{(t-1)}$$

$$\leq \int_{(\widetilde{\mathbf{W}}')^{(t-1)}} e^{\epsilon_{\mathrm{mp},t}} p(\boldsymbol{\Sigma}^{(t)} \mid (\widetilde{\mathbf{W}}')^{(t-1)}) p((\tilde{\mathbf{w}}_i')^{(t-1)} \mid (\mathbf{w}_i')^{(t-1)})$$
$$\prod_{j \in [m], j \neq i} p(\tilde{\mathbf{w}}_j^{(t-1)} \mid \mathbf{w}_j^{(t-1)}) d(\widetilde{\mathbf{W}}')^{(t-1)}.$$

$$= \int_{(\widetilde{\mathbf{W}}')^{(t-1)}} \exp(\epsilon_{\mathrm{mp},t}) p(\boldsymbol{\Sigma}^{(t)} \mid (\widetilde{\mathbf{W}}')^{(t-1)})$$
$$p((\widetilde{\mathbf{W}}')^{(t-1)} \mid (\mathbf{W}')^{(t-1)}) d(\widetilde{\mathbf{W}}')^{(t-1)}$$
$$= \exp(\epsilon_{\mathrm{mp},t}) p(\boldsymbol{\Sigma}^{(t)} \mid (\mathbf{W}')^{(t-1)}),$$

which renders the mapping $\mathbf{W}^{(t-1)} \to \boldsymbol{\Sigma}^{(t)}$ as an $(\exp(\epsilon_{\mathrm{mp},t}), 0)$ - differentially private algorithm.

Then, according to the proof of Theorem 1 in Section J-C, Algorithm 5 is an $(\epsilon_{\mathrm{mp}}, \delta_{\mathrm{mp}})$ - Iterative MP-MTL algorithm. $\square$

APPENDIX L
PROOF OF RESULTS IN APPENDIX E-A

A. Proof of Theorem 7

Proof. First, consider the case with no acceleration. We first use Proposition 1 of Schmidt et al. [11] by regarding procedures from Step 5 to Step 9 as approximation for the proximal operator in (8). Note that the norm clipping only bounds the parameter space and does not affect the results of Schmidt et al. [11]. Then for ε_t defined in Lemma 13 for $t \in [T]$, we have

$$\mathcal{E} = \frac{2L}{m(T+1)^2} \left(\|\widetilde{\mathbf{W}}^{(0)} - \mathbf{W}_*\|_F \right.$$
$$\left. + 2\sum_{t=1}^{T} t \sqrt{\frac{2\varepsilon_t}{L}} + \sqrt{2\sum_{t=1}^{T} t^2 \frac{\varepsilon_t}{L}} \right)^2.$$

Meanwhile, by Lemma 13, we have

$$\varepsilon_t = O\left(\frac{\kappa}{\epsilon_t}\right),$$

where $\kappa = \frac{K^2 \sqrt{m} kd \log d}{\eta}.$

On the other hand, let

$$c_1 = \epsilon = \sum_{t=1}^{T} \epsilon_t,$$

then by Lemma 17, we have

$$\sum_{t=1}^{T} \sqrt{\varepsilon_t} = \begin{cases} O\left(\sqrt{\frac{\kappa T^{\alpha+1}}{c_1 (\alpha/2-1)^2 (\alpha+1)}}\right), & \alpha > 2; \\ O\left(\sqrt{\frac{\kappa T^3}{c_1 (\alpha/2-1)^2 (\alpha+1)}}\right), & -1 < \alpha < 2; \\ O\left(\sqrt{\frac{\kappa T^{2-\alpha}}{c_1 (\alpha/2-1)^2 (-\alpha-1)}}\right), & \alpha < -1, \end{cases}$$

Because $\widetilde{\mathbf{W}}^{(0)}$ is the result of the norm clipping, we have $\widetilde{\mathbf{W}}^{(0)} \in \mathcal{W}$.

Finally, taking $c_3 = \phi(\alpha)$ defined in (13) and $c_4 = \frac{\kappa}{c_2 (\alpha/2-1)^2 |\alpha+1|}$, under the assumption that $\mathbf{W}_* \in \mathcal{W}$, using Lemma 19, we have the results for the case with no acceleration.

For the accelerated case, we use Proposition 2 of Schmidt et al. [11] to have

$$\mathcal{E} = \frac{2L}{m(T+1)^2} \left(\|\widetilde{\mathbf{W}}^{(0)} - \mathbf{W}_*\|_F \right.$$
$$\left. + 2\sum_{t=1}^{T} t \sqrt{\frac{2\varepsilon_t}{L}} + \sqrt{2\sum_{t=1}^{T} t^2 \frac{\varepsilon_t}{L}} \right)^2.$$

Then one can prove similarly combining Lemma 13, Lemma 16, Lemma 17 and Lemma 20. $\square$

B. Proof of Theorem 8

Proof. First, consider the case with no acceleration. We use Proposition 1 of Schmidt et al. [11] and prove similarly as in Appendix L-A, combining Lemma 14, Lemma 16, Lemma 17 and Lemma 19.

For the accelerated case, we use Proposition 2 of Schmidt et al. [11] and prove similarly as in Appendix L-A, combining Lemma 14, Lemma 16, Lemma 17 and Lemma 20. $\square$

C. Proof of Theorem 9

Proof. First, consider the case with no acceleration. We use Proposition 3 of Schmidt et al. [11] to have

$$\mathcal{E} = \frac{Q_0^T}{\sqrt{m}}\left(\|\widetilde{\mathbf{W}}^{(0)} - \mathbf{W}_*\|_F + 2\sum_{t=1}^{T} Q_0^{-t}\sqrt{\frac{2\varepsilon_t}{L}}\right).$$

Then one can prove similarly as in Appendix L-A, combining Lemma 13, Lemma 16, Lemma 17 and Lemma 21.

For the accelerated case, we use Proposition 4 of Schmidt et al. [11] to have

$$\mathcal{E} = \frac{(Q_0)^T}{m}\left(\sqrt{2(f(\widehat{\mathbf{W}}^{(0)}) - f(\mathbf{W}_*))} + 2\sqrt{\frac{L}{\mu}}\sum_{t=1}^{T}\sqrt{\varepsilon_t(Q_0)^{-t}}\right.$$
$$\left. + \sqrt{\left(\sum_{t=1}^{T}\varepsilon_t(Q_0)^{-t}\right)^2}\right).$$

Then one can prove similarly as in Appendix L-A, using the assumption that $f(\widetilde{\mathbf{W}}^{(0)}) - f(\mathbf{W}_*) = O(K^2 Lm)$, combining Lemma 13, Lemma 16, Lemma 17 and Lemma 22. □

D. Proof of Theorem 10

Proof. First, consider the case with no acceleration. We use Proposition 3 of Schmidt et al. [11] and prove similarly as in Appendix J-I, combining Lemma 14, Lemma 16, Lemma 17 and Lemma 21.

For the accelerated case, we use Proposition 4 of Schmidt et al. [11] and prove similarly as in Appendix J-I, using the assumption that $f(\widetilde{\mathbf{W}}^{(0)}) - f(\mathbf{W}_*) = O(K^2 Lm)$, combining Lemma 14, Lemma 16, Lemma 17 and Lemma 22. □

E. Proof of Theorem 11

Proof. Consider the bound in (25). First, by Assumption 1, $\mathcal{E}$ is minimized by maximizing $\phi(\alpha)$ and $(\alpha/2 - 1)^2|\alpha + 1|$, which are maximized simultaneously when $\alpha = 0$. Results under other settings can be proved similarly. □

APPENDIX M
PROOF OF RESULTS IN APPENDIX E-B

Proof. Results in this settings are the corollaries of Theorem 2, Theorem 3, Theorem 4, Theorem 5 and Theorem 6, respectively, replacing the term $\sqrt{\log(e + \epsilon/\delta)}$ with the term $\sqrt{\log(1/\delta)}$ by Lemma 16. □

APPENDIX N
PROOF OF RESULTS IN APPENDIX H

A. Proof of Lemma 8

Proof. For simplicity, we omit the symbol $\mathcal{B}$ in the conditional events.

Because $\widetilde{\mathbf{W}} = \mathcal{A}_{mp}(\mathbf{W}, \mathbf{X}^m, \mathbf{y}^m)$ is an (ϵ, δ)-Non-iterative MP-MTL algorithm, by Definition 5, we have for $i \in [m]$ and for any set $\mathcal{S}' \subseteq \mathbb{R}^{d \times (m-1)}$,

$$\mathbb{P}(\tilde{\mathbf{w}}_{[-i]} \in \mathcal{S}' \mid \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}_i, \mathcal{D}_i) \\ \leq e^\epsilon \mathbb{P}(\tilde{\mathbf{w}}_{[-i]} \in \mathcal{S}_1 \mid \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}'_i, \mathcal{D}'_i) + \delta. \quad (69)$$

In the following, we follow the proof of Theorem B.1 of Dwork et al. [3].

For any $C_1 \subseteq \mathbb{R}^{d \times (m-1)}$, define

$$\mu(C_1) = (\mathbb{P}(\tilde{\mathbf{w}}_{[-i]} \in C_1 \mid \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}_i, \mathcal{D}_i) \\ - e^\epsilon \mathbb{P}(\tilde{\mathbf{w}}_{[-i]} \in C_1 \mid \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}'_i, \mathcal{D}'_i))_+$$

and then, μ is a measure on C_1 and $\mu(C_1) \leq \delta$ by (69). As a result, we have for all $s_1 \in C_1$,

$$\mathbb{P}(\tilde{\mathbf{w}}_{[-i]} \in ds_1 \mid \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}_i, \mathcal{D}_i) \\ \leq e^\epsilon \mathbb{P}(\tilde{\mathbf{w}}_{[-i]} \in ds_1 \mid \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}'_i, \mathcal{D}'_i) + \mu(ds_1).$$

As such, for any set $\mathcal{S} \subseteq \mathbb{R}^{d \times (m-1)} \times \mathbb{R}^{d \times (m-1)}$ and $\mathcal{S}_1$, which denotes the projection of $\mathcal{S}$ onto C_1:

$$\mathbb{P}((\hat{\mathbf{w}}_{[-i]}, \tilde{\mathbf{w}}_{[-i]}) \in \mathcal{S} \mid \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}_i, \mathcal{D}_i)$$
$$\leq \int_{\mathcal{S}_1} \mathbb{P}((\hat{\mathbf{w}}_{[-i]}, s_1) \in \mathcal{S} \mid \tilde{\mathbf{w}}_{[-i]}, \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}_i, \mathcal{D}_i)$$
$$\mathbb{P}(\tilde{\mathbf{w}}_{[-i]} \in ds_1 \mid \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}_i, \mathcal{D}_i)$$
$$\leq \int_{\mathcal{S}_1} \mathbb{P}((\hat{\mathbf{w}}_{[-i]}, s_1) \in \mathcal{S} \mid \tilde{\mathbf{w}}_{[-i]}, \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}_i, \mathcal{D}_i)$$
$$\left[e^\epsilon \mathbb{P}(\tilde{\mathbf{w}}_{[-i]} \in ds_1 \mid \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}'_i, \mathcal{D}'_i) + \mu(ds_1)\right]$$
$$= \int_{\mathcal{S}_1} \mathbb{P}((\hat{\mathbf{w}}_{[-i]}, s_1) \in \mathcal{S} \mid \tilde{\mathbf{w}}_{[-i]}, \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}_i, \mathcal{D}_i)$$
$$e^\epsilon \mathbb{P}(\tilde{\mathbf{w}}_{[-i]} \in ds_1 \mid \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}'_i, \mathcal{D}'_i) + \mu(S_1)$$
$$= \int_{\mathcal{S}_1} \mathbb{P}((\hat{\mathbf{w}}_{[-i]}, s_1) \in \mathcal{S} \mid \tilde{\mathbf{w}}_{[-i]}, \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}'_i, \mathcal{D}'_i)$$
$$e^\epsilon \mathbb{P}(\tilde{\mathbf{w}}_{[-i]} \in ds_1 \mid \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}'_i, \mathcal{D}'_i) + \mu(S_1)$$
$$\leq e^\epsilon \mathbb{P}((\hat{\mathbf{w}}_{[-i]}, \tilde{\mathbf{w}}_{[-i]}) \in \mathcal{S} \mid \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}'_i, \mathcal{D}'_i) + \delta.$$

The second equality uses the independence of the learning process for $\hat{\mathbf{w}}_{[-i]}$ given $(\tilde{\mathbf{w}}_{[-i]}, \mathcal{D}_{[-i]})$.

The procedure is similar to proving that the algorithm $\mathcal{A}_{st+mp}$ that first uses a *deterministic* STL algorithm $\mathcal{A}_{st}$ before applying $\mathcal{A}_{mp}$ is also an (ϵ, δ) - Non-iterative MP-MTL algorithm.

For a *deterministic* STL algorithm $\widetilde{\mathbf{W}} = \mathcal{A}_{st}(\mathbf{W}, \mathbf{X}^m, \mathbf{y}^m)$, for $i \in [m]$, by the independence of the learning process for $\tilde{\mathbf{w}}_{[-i]}$ given $(\mathbf{w}_{[-i]}, \mathcal{D}_{[-i]})$, we have

$$p(\tilde{\mathbf{w}}_{[-i]} \mid \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}_i, \mathcal{D}_i) \\ = p(\tilde{\mathbf{w}}_{[-i]} \mid \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}'_i, \mathcal{D}'_i). \quad (70)$$

Because the STL algorithm is *deterministic*, when the input is given, it is reasonable to assume that the output is given. As such, we also have for $i \in [m]$,

$$p(\cdot \mid \mathbf{w}_i, \mathcal{D}_i) = p(\cdot \mid \tilde{\mathbf{w}}_i, \mathcal{D}_i) \\ p(\cdot \mid \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}) = p(\cdot \mid \tilde{\mathbf{w}}_{[-i]}, \mathcal{D}_{[-i]}) \quad (71)$$

Then, for an (ϵ, δ)-Non-iterative MP-MTL algorithm $\widehat{\mathbf{W}} = \mathcal{A}_{mp}(\widetilde{\mathbf{W}}, \mathbf{X}^m, \mathbf{y}^m)$, by Definition 5, we have for $i \in [m]$ and for any set $\mathcal{S}' \subseteq \mathbb{R}^{d \times (m-1)}$

$$\mathbb{P}(\hat{\mathbf{w}}_{[-i]} \in \mathcal{S}' \mid \tilde{\mathbf{w}}_{[-i]}, \mathcal{D}_{[-i]}, \tilde{\mathbf{w}}_i, \mathcal{D}_i) \\ \leq e^\epsilon \mathbb{P}(\hat{\mathbf{w}}_{[-i]} \in \mathcal{S}' \mid \tilde{\mathbf{w}}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}'_i, \mathcal{D}'_i) + \delta, \quad (72)$$

where $\tilde{\mathbf{w}}'_i$ can be replaced with $\mathbf{w}'_i$ because Definition 5 allows replacing $\tilde{\mathbf{w}}_i$ with any different model.

As such,

$$
\mathbb{P}((\hat{\mathbf{w}}_{[-i]}, \tilde{\mathbf{w}}_{[-i]}) \in \mathcal{S} \mid \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}_i, \mathcal{D}_i)
$$

$$
\leq \int_{\mathcal{S}_1} \mathbb{P}((\hat{\mathbf{w}}_{[-i]}, s_1) \in \mathcal{S} \mid \tilde{\mathbf{w}}_{[-i]}, \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}_i, \mathcal{D}_i)
$$
$$
\mathbb{P}(\tilde{\mathbf{w}}_{[-i]} \in ds_1 \mid \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}_i, \mathcal{D}_i)
$$

$$
= \int_{\mathcal{S}_1} \mathbb{P}((\hat{\mathbf{w}}_{[-i]}, s_1) \in \mathcal{S} \mid \tilde{\mathbf{w}}_{[-i]}, \mathcal{D}_{[-i]}, \tilde{\mathbf{w}}_i, \mathcal{D}_i)
$$
$$
\mathbb{P}(\tilde{\mathbf{w}}_{[-i]} \in ds_1 \mid \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}_i, \mathcal{D}_i)
$$

$$
\leq \int_{\mathcal{S}_1} \left[\left(e^{\epsilon} \mathbb{P}((\hat{\mathbf{w}}_{[-i]}, s_1) \in \mathcal{S} \mid \tilde{\mathbf{w}}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}_i', \mathcal{D}_i') \right. \right.
$$
$$
\left. \left. + \delta \right) \wedge 1 \right] \mathbb{P}(\tilde{\mathbf{w}}_{[-i]} \in ds_1 \mid \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}_i, \mathcal{D}_i)
$$

$$
\leq \int_{\mathcal{S}_1} \left[e^{\epsilon} \mathbb{P}((\hat{\mathbf{w}}_{[-i]}, s_1) \in \mathcal{S} \mid \tilde{\mathbf{w}}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}_i', \mathcal{D}_i') \right.
$$
$$
\left. \wedge 1 + \delta \right] \mathbb{P}(\tilde{\mathbf{w}}_{[-i]} \in ds_1 \mid \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}_i, \mathcal{D}_i)
$$

$$
\leq \int_{\mathcal{S}_1} \left[e^{\epsilon} \mathbb{P}((\hat{\mathbf{w}}_{[-i]}, s_1) \in \mathcal{S} \mid \tilde{\mathbf{w}}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}_i', \mathcal{D}_i') \right.
$$
$$
\left. \wedge 1 \right] \mathbb{P}(\tilde{\mathbf{w}}_{[-i]} \in ds_1 \mid \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}_i, \mathcal{D}_i) + \delta
$$

$$
= \int_{\mathcal{S}_1} \left[e^{\epsilon} \mathbb{P}((\hat{\mathbf{w}}_{[-i]}, s_1) \in \mathcal{S} \mid \tilde{\mathbf{w}}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}_i', \mathcal{D}_i') \right.
$$
$$
\left. \wedge 1 \right] \mathbb{P}(\tilde{\mathbf{w}}_{[-i]} \in ds_1 \mid \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}_i', \mathcal{D}_i') + \delta
$$

$$
= e^{\epsilon} \mathbb{P}((\hat{\mathbf{w}}_{[-i]}, \tilde{\mathbf{w}}_{[-i]}) \in \mathcal{S} \mid \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}_i', \mathcal{D}_i') + \delta.
$$

The second inequality uses (72), the first equality uses (71), and the second equality uses (70). □

B. Proof of Lemma 9

Proof. For simplicity, we omit the symbol $\mathcal{B}$ to denote the input in the conditional events in some equations.

First, because for all $t \in [T]$, $\widehat{\mathbf{W}}^{(t)} = \mathcal{A}_{\mathrm{mp}}(\mathbf{W}^{(t-1)}, \mathbf{X}^m, \mathbf{y}^m)$ is an (ϵ_t, δ_t)-Non-iterative MP-MTL algorithm and because for all $i \in [m]$ $\mathbf{w}_i^{(t)} = \mathcal{A}_{\mathrm{st},i}(\hat{\mathbf{w}}_i^{(t)}, \mathbf{X}_i, \mathbf{y}_i)$ is a deterministic STL algorithm for the i-th task, then by the proof of Lemma 8, we have that the mapping $(\mathbf{X}^m, \mathbf{y}^m, \mathbf{W}^{(t-1)}) \to (\widehat{\mathbf{W}}^{(t)}, \mathbf{W}^{(t)})$ is an (ϵ_t, δ_t)-Non-iterative MP-MTL algorithm for all $t \in [T]$. In other words, for all $i \in [m]$, we have for any set $\mathcal{S} \subseteq \mathbb{R}^{d \times (m-1)} \times \mathbb{R}^{d \times (m-1)}$ that

$$
\mathbb{P}((\hat{\mathbf{w}}_{[-i]}^{(t)}, \mathbf{w}_{[-i]}^{(t)}) \in \mathcal{S} \mid \mathbf{w}_{[-i]}^{(t-1)}, \mathcal{D}_{[-i]}, \mathbf{w}_i^{(t-1)}, \mathcal{D}_i)
$$
$$
\leq e^{\epsilon_t} \mathbb{P}((\hat{\mathbf{w}}_{[-i]}^{(t)}, \mathbf{w}_{[-i]}^{(t)}) \in \mathcal{S} \mid \mathbf{w}_{[-i]}^{(t-1)}, \mathcal{D}_{[-i]}, (\mathbf{w}_i')^{(t-1)}, \mathcal{D}_i')
$$
$$
+ \delta_t. \tag{73}
$$

Then, for $t = 1, \ldots, T$, take the t-th dataset $\widetilde{\mathcal{D}}_t = \{(\mathbf{w}_1^{(t-1)}, \mathcal{D}_1 = (\mathbf{X}_1, \mathbf{y}_1)), \ldots, (\mathbf{w}_m^{(t-1)}, \mathcal{D}_m = (\mathbf{X}_m, \mathbf{y}_m))\}$, i.e., treat $(\mathbf{w}_i^{(t-1)}, \mathcal{D}_i = (\mathbf{X}_i, \mathbf{y}_i))$ as the i-th data instance of the dataset $\widetilde{\mathcal{D}}_t$ for all $i \in [m]$. For all $i \in [m]$ and for all $t \in [T]$, take the t-th output $\theta_{t,i} = (\hat{\mathbf{w}}_{[-i]}^{(t)}, \mathbf{w}_{[-i]}^{(t)})$. By (73), we have for all $t \in [T]$, for all $i \in [m]$, and for any set $\mathcal{S}_t \subseteq \mathbb{R}^{d \times (m-1)} \times \mathbb{R}^{d \times (m-1)}$ that

$$
\mathbb{P}(\theta_{t,i} \in \mathcal{S}_t \mid \widetilde{\mathcal{D}}_t) \leq e^{\epsilon_t} \mathbb{P}(\theta_{t,i} \in \mathcal{S}_t \mid \widetilde{\mathcal{D}}_t') + \delta_t,
$$

where $\widetilde{\mathcal{D}}_t$ and $\widetilde{\mathcal{D}}_t'$ are two adjacent datasets that differ in a single entry, the i-th data instance $(\mathbf{w}_i^{(t-1)}, \mathcal{D}_i = (\mathbf{X}_i, \mathbf{y}_i))$, which renders the algorithm in the t-th iteration an (ϵ_t, δ_t)-differentially private algorithm. As such, by the *Adaptive composition* Lemma (Lemma

7), for all $i \in [m]$ and for any set $\mathcal{S} \subseteq \bigotimes_{t=1}^{T} \mathcal{C}_t$, where $\mathcal{C}_t = \mathbb{R}^{d \times (m-1)} \times \mathbb{R}^{d \times (m-1)}$, we have

$$
\mathbb{P}((\theta_{1,i}, \cdots, \theta_{T,i}) \in \mathcal{S} \mid \bigcap_{t=1}^{T} (\mathcal{B}_t = (\widetilde{\mathcal{D}}_t, \boldsymbol{\theta}_{1:t-1})))
$$

$$
\leq e^{\epsilon} \mathbb{P}((\theta_{1,i}, \cdots, \theta_{T,i}) \in \mathcal{S} \mid \bigcap_{t=1}^{T} (\mathcal{B}_t = (\widetilde{\mathcal{D}}_t', \boldsymbol{\theta}_{1:t-1})))
$$

$$
+ 1 - (1 - \delta) \prod_{t=1}^{T} (1 - \delta_t),
$$

where for all $t \in [T]$, $\mathcal{B}_t$ denotes the input for the t-th iteration,

$$
\boldsymbol{\theta}_{1:t-1} = \begin{cases} \emptyset, & t = 1 \\ (\theta_{1,1}, \ldots, \theta_{1,m}) \ldots, (\theta_{t-1,1}, \ldots, \theta_{t-1,m}), & t \geq 2, \end{cases}
$$

and ϵ is defined in Lemma 7.

Then, we have for any set $\mathcal{S}' \subseteq \mathbb{R}^{d \times (m-1) \times T}$,

$$
\mathbb{P}(\hat{\mathbf{w}}_{[-i]}^{(1:T)} \in \mathcal{S}' \mid \bigcap_{t=1}^{T} \mathcal{B}_t = (\mathbf{W}^{(t-1)}, \mathcal{D}^m, \boldsymbol{\theta}_{1:t-1}))
$$

$$
\leq e^{\epsilon} \mathbb{P}(\hat{\mathbf{w}}_{[-i]}^{(1:T)} \in \mathcal{S}' \mid \bigcap_{t=1}^{T} \mathcal{B}_t = ((\mathbf{W}')^{(t-1)}, (\mathcal{D}')^m, \boldsymbol{\theta}_{1:t-1}))
$$

$$
+ 1 - (1 - \delta) \prod_{t=1}^{T} (1 - \delta_t).
$$

□

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