Model-Protected Multi-Task Learning

Supplementary Material

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APPENDIX B
WISHART DISTRIBUTION

Definition 8 (Gupta and Nagar [4]). *A $d \times d$ random symmetric positive definite matrix $\mathbf{E}$ is said to have a Wishart distribution $\mathbf{E} \sim W_d(\nu, \mathbf{V})$ if its probability density function is*

$$p(\mathbf{E}) = \frac{|\mathbf{E}|^{(\nu-d-1)/2}\exp(-\mathrm{tr}(\mathbf{V}^{-1}\mathbf{E})/2)}{2^{\frac{\nu d}{2}}|\mathbf{V}|^{1/2}\Gamma_d(\nu/2)},$$

where $\nu > d-1$ and $\mathbf{V}$ is a $d \times d$ positive definite matrix.

APPENDIX C
MODEL-DECOMPOSED MP-MTL METHODS

In this section, we consider the extension of our MP-MTL framework for MTL methods using the decomposed parameter/model matrix. Specifically, we focus on the following problem, where the trace norm is used for knowledge sharing across tasks and the $\|\cdot\|_1$ norm (sum of the ℓ_1 norm for vectors) is used for entry-wise outlier detection, as described in Algorithm 4.

$$\begin{aligned}&\min_{\mathbf{W}} \sum\nolimits_{i=1}^{m} \mathcal{L}_i(\mathbf{X}_i\mathbf{w}_i, \mathbf{y}_i) + \lambda_1\|\mathbf{P}\|_* + \lambda_2\|\mathbf{Q}\|_1 \\ &s.t.\ \mathbf{W} = \mathbf{P} + \mathbf{Q},\end{aligned} \tag{24}$$

where $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{d\times m}$.

We note that in Algorithm 4, the role of $\mathbf{P}$ is the same as the role of $\mathbf{W}$ in Algorithm 2, and the additional procedures introduced to update $\mathbf{Q}$ are still STL algorithms. As such, we have the result in Corollary 2.

Corollary 2. *Algorithm 4 is an (ϵ, δ) - iterative MP-MTL algorithm.*

Remark 3. *Based on Algorithm 4, this will result in a similar procedure and identical theoretical results with respect to privacy by replacing the trace norm with the $\ell_{2,1}$ norm to force group sparsity in $\mathbf{P}$ or by replacing the $\|\cdot\|_1$ norm with the $\ell_{1,2}$ norm (sum of the ℓ_2 norm of column vectors) or $\|\cdot\|_F^2$ (square of the Frobenius norm).*

APPENDIX D
MP-MTL FRAMEWORK WITH SECURE MULTI-PARTY COMPUTATION

Pathak et al. [10] considered the demand for secure multi-party computation (SMC): protecting data instances from leaking to the curator and leaking between tasks during joint learning. However, by Proposition 3, the method of Pathak et al. [10] may introduce excess noise to protect both the data instances and the models simultaneously. To avoid unnecessary noise, we consider a divide-and-conquer strategy to ensure privacy for a single data instance and the model separately. Specifically, in each iteration of the Iterative MP-MTL algorithms, we perform private sharing after introducing the perturbation to the parameter matrix to protect a single data instance, as described in Algorithm 5, where a noise vector is added in Step 5 to the model vector based on sensitivity of replacing a single data instance.

Algorithm 4 Model-Protected Low-Rank and SParse (MP-LR-SP) Estimator

Input: Datasets $(\mathbf{X}^m, \mathbf{y}^m) = \{(\mathbf{X}_1, \mathbf{y}_1), \ldots, (\mathbf{X}_m, \mathbf{y}_m)\}$, where $\forall i \in [m]$, $\mathbf{X}_i \in \mathbb{R}^{n_i\times d}$ and $\mathbf{y}_i \in \mathbb{R}^{n_i \times 1}$. Privacy loss $\epsilon, \delta \geq 0$. Number of iterations T. Step size η. Regularization parameter $\lambda_1, \lambda_2 > 0$. Norm clipping parameter $K > 0$. Acceleration parameters $\{\beta_t\}$. Initial models of tasks $\mathbf{W}^{(0)}$.

Output: $\widehat{\mathbf{W}}^{(1:T)}$.

For $t = 1, \ldots, T$, set ϵ_t such that $\tilde{\epsilon} \leq \epsilon$, where $\tilde{\epsilon}$ is defined in (7).
Let $\mathbf{P}^{(0)} = \mathbf{Q}^{(0)} = \widehat{\mathbf{Q}}^{(0)} = \mathbf{W}^{(0)}$.
for $t = 1 : T$ **do**
  Norm clipping: $\tilde{\mathbf{p}}_i^{(t-1)} = \mathbf{p}_i^{(t-1)}/\max(1, \frac{\|\mathbf{p}_i^{(t-1)}\|_2}{K})$ for all $i \in [m]$. Let $\widehat{\mathbf{P}}^{(0)} = \widetilde{\mathbf{P}}^{(0)}$.
  Compute sensitivity: $s_i^{(t-1)} = 2$ for all $i \in [m]$.
  $\widetilde{\mathbf{\Sigma}}^{(t)} = \widetilde{\mathbf{P}}^{(t-1)}(\widetilde{\mathbf{P}}^{(t-1)})^{\mathrm{T}}$.
  $\mathbf{\Sigma}^{(t)} = \widetilde{\mathbf{\Sigma}}^{(t-1)} + \mathbf{E}$, where $\mathbf{E} \sim W_d(d+1, \frac{\max_i s_i^{(t-1)}}{2\epsilon_t}\mathbf{I}_d)$ is a sample of the Wishart distribution.
  Perform SVD decomposition: $\mathbf{U}\mathbf{\Lambda}\mathbf{U}^{\mathrm{T}} = \mathbf{\Sigma}^{(t)}$.
  Let $\mathbf{S}_{\eta\lambda_1}$ be a diagonal matrix and let $\mathbf{S}_{\eta\lambda_1, ii} = \max\{0, 1 - \eta\lambda_1/\sqrt{\mathbf{\Lambda}_{ii}}\}$ for $i = 1, \ldots, \min\{d, m\}$.
  Let $\hat{\mathbf{p}}_i^{(t)} = \mathbf{U}\mathbf{S}_{\eta\lambda_1}\mathbf{U}^{\mathrm{T}}\mathbf{p}_i^{(t-1)}$ for all $i \in [m]$.
  Let $\hat{\mathbf{q}}_i^{(t)} = \mathrm{sign}(\mathbf{q}_i^{(t-1)}) \circ \max\{0, |\mathbf{q}_i^{(t-1)}| - \eta\lambda_2\}$ for all $i \in [m]$, where $\circ$ denotes the entry-wise product.
  Let $\widehat{\mathbf{W}}^{(t)} = \widehat{\mathbf{P}}^{(t)} + \widehat{\mathbf{Q}}^{(t)}$.
  Let $\boldsymbol{z}_{i,p}^{(t)} = \hat{\mathbf{p}}_i^{(t)} + \beta_t(\hat{\mathbf{p}}_i^{(t)} - \hat{\mathbf{p}}_i^{(t-1)})$ for all $i \in [m]$.
  Let $\boldsymbol{z}_{i,q}^{(t)} = \hat{\mathbf{q}}_i^{(t)} + \beta_t(\hat{\mathbf{q}}_i^{(t)} - \hat{\mathbf{q}}_i^{(t-1)})$ for all $i \in [m]$.
  $\mathbf{p}_i^{(t)} = \boldsymbol{z}_{i,p}^{(t)} - \eta\frac{\partial\mathcal{L}_i(\mathbf{X}_i(\boldsymbol{z}_{i,p}^{(t)} + \boldsymbol{z}_{i,q}^{(t)}), \mathbf{y}_i)}{\partial\hat{\mathbf{p}}_i^{(t)}}$ for all $i \in [m]$.
  $\mathbf{q}_i^{(t)} = \boldsymbol{z}_{i,q}^{(t)} - \eta\frac{\partial\mathcal{L}_i(\mathbf{X}_i(\boldsymbol{z}_{i,p}^{(t)} + \boldsymbol{z}_{i,q}^{(t)}), \mathbf{y}_i)}{\partial\hat{\mathbf{q}}_i^{(t)}}$ for all $i \in [m]$.
end for

The results in Proposition 4 show that we can simultaneously protect a single data instance and the model using such a divide-and-conquer strategy. Because it is not necessary to protect all the data instances in each task using data-protected algorithms, the perturbation for data-instance protection can be reduced.

Proposition 4. *Use Lemma 4 and Theorem 1. Algorithm 5 is an $(\epsilon_{mp}, \delta_{mp})$ - iterative MP-MTL algorithm and an $(\epsilon_{dp}, \delta_{dp})$ - iterative DP-MTL algorithm.*

Algorithm 5 MP-MTL framework with Secure Multi-party Computation (SMC)

Input: Datasets $(\mathbf{X}^m, \mathbf{y}^m) = \{(\mathbf{X}_1, \mathbf{y}_1), \ldots, (\mathbf{X}_m, \mathbf{y}_m)\}$, where $\forall i \in [m]$, $\mathbf{X}_i \in \mathbb{R}^{n_i \times d}$ and $\mathbf{y}_i \in \mathbb{R}^{n_i \times 1}$. Privacy loss for model protection $\epsilon_{\text{mp}}, \delta_{\text{mp}} \geq 0$. Privacy loss for single data instance protection $\epsilon_{\text{dp}} \geq 0$. Number of iterations T. Shared information matrices $\mathbf{M}^{(0)}$. Initial models of tasks $\mathbf{W}^{(0)}$.

Output: $\widehat{\mathbf{W}}^{(1:T)}$.

1: For $t = 1, \ldots, T$, set $\epsilon_{\text{mp},t}$ such that $\tilde{\epsilon}_{\text{mp}} \leq \epsilon_{\text{mp}}$, where $\tilde{\epsilon}_{\text{mp}}$ is defined in (7), taking $\epsilon_t = \epsilon_{\text{mp},t}, \epsilon = \epsilon_{\text{mp}}, \delta = \delta_{\text{mp}}$.

2: For $t = 1, \ldots, T$, set $\epsilon_{\text{dp},t}$ such that $\tilde{\epsilon}_{\text{dp}} \leq \epsilon_{\text{dp}}$, where $\tilde{\epsilon}_{\text{dp}}$ is defined in (7), taking $\epsilon_t = \epsilon_{\text{dp},t}, \epsilon = \epsilon_{\text{dp}}, \delta = \delta_{\text{dp}}$.

3: **for** $t = 1 : T$ **do**

4: Compute the sensitivity vector $\tilde{\mathbf{s}}^{(t-1)} = [\tilde{s}_1^{(t-1)}, \ldots, \tilde{s}_m^{(t-1)}]^{\mathrm{T}}$, which is defined for all $i \in [m]$,

$$\tilde{s}_i^{(t-1)} = \max_{(\mathbf{w}_i')^{(t-1)}} \|\mathbf{w}_i^{(t-1)} - (\mathbf{w}_i')^{(t-1)}\|_2,$$

where $(\mathbf{w}_i')^{(t-1)}$ is assumed to be generated using $\mathcal{D}_i'$, which differ with $\mathcal{D}_i$ in a single data instance.

5: $\tilde{\mathbf{w}}_i^{(t-1)} = \mathbf{w}_i^{(t-1)} + \mathbf{b}_i$, where $\mathbf{b}_i$ is a sample with the density function of

$$p(\mathbf{b}_i) \propto \exp\left(-\frac{\tilde{s}_i^{(t-1)}}{\epsilon_{\text{dp},t}}\|\mathbf{b}_i\|_2\right),$$

for all $i \in [m]$.

6: Compute the sensitivity vector $\mathbf{s}^{(t-1)} = [s_1^{(t-1)}, \ldots, s_m^{(t-1)}]^{\mathrm{T}}$, which is defined for all $i \in [m]$,

$$s_i^{(t-1)} = \max_{(\mathbf{w}_i')^{(t-1)}} |\|\tilde{\mathbf{w}}_i^{(t-1)}\|_2^2 - \|(\tilde{\mathbf{w}}_i')^{(t-1)}\|_2^2|,$$

where $(\tilde{\mathbf{w}}_i')^{(t-1)}$ is assumed to be generated arbitrarily.

7: $\widetilde{\mathbf{\Sigma}}^{(t)} = \widetilde{\mathbf{W}}^{(t-1)}(\widetilde{\mathbf{W}}^{(t-1)})^{\mathrm{T}}$ (or $\widetilde{\mathbf{\Sigma}}^{(t)} = (\widetilde{\mathbf{W}}^{(t-1)})^{\mathrm{T}}\widetilde{\mathbf{W}}^{(t-1)}$).

8: $\mathbf{\Sigma}^{(t)} = \widetilde{\mathbf{\Sigma}}^{(t)} + \mathbf{E}$, where $\mathbf{E} \sim W_d(d+1, \frac{\max_i s_i^{(t-1)}}{2\epsilon_{\text{mp},t}}\mathbf{I}_d)$ (or $\mathbf{E} \sim W_m(m+1, \text{diag}(\mathbf{s}^{(t-1)}/2\epsilon_{\text{mp},t}))$) is a sample of the Wishart distribution.

9: Perform an arbitrary mapping $f : \mathbf{\Sigma}^{(1:t)} \to \mathbf{M}^{(t)}$.

10: $\hat{\mathbf{w}}_i^{(t)} = \mathcal{A}_{\text{st},i}(\mathbf{M}^{(t)}, \tilde{\mathbf{w}}_i^{(0:t-1)}, \mathbf{X}_i, \mathbf{y}_i)$ for all $i \in [m]$, where $\mathbf{w}_i^{(0:t-1)}$ are for the initialization.

11: Set the input for the next iteration: $\mathbf{W}^{(t)} = \widehat{\mathbf{W}}^{(t)}$.

12: **end for**

APPENDIX E
RESULTS OF UTILITY ANALYSES UNDER OTHER TWO SETTINGS

Here we consider the other two settings of $\{\epsilon_t\}$.

A. Setting No.1

In this setting, we have

$$\epsilon = \sum_{t=1}^{T} \epsilon_t.$$

Theorem 7 (Low rank - Convexity - Setting No.1). *Consider Algorithm 2. For an index $k \leq q$ that suffices the definition in Lemma 2 for all $t \in [T]$, $\eta = 1/L$, $\lambda = \Theta(LK\sqrt{m})$, assume $\epsilon_t \leq 4Kk^2 d(\log d)/q^2$ for $t \in [T]$.*

***No acceleration**: If we set $\beta_t = 0$ for $t \in [m]$, then setting*

$$T = \Theta\left(\left[\frac{(\alpha/2-1)^2|\alpha+1|\sqrt{m}\epsilon}{kd\log d}\right]^{\phi(\alpha)}\right)$$

for $\mathcal{E} = f(\frac{1}{T}\sum_{t=1}^{T}\widehat{\mathbf{W}}^{(t)}) - f(\mathbf{W}_*)$, we have with high probability,

$$\mathcal{E} = O\left(K^2 L\left[\frac{kd\log d}{(\alpha/2-1)^2|\alpha+1|\sqrt{m}\epsilon}\right]^{\phi(\alpha)}\right), \tag{25}$$

where

$$\phi(\alpha) = \begin{cases} 1/(\alpha+1), & \alpha > 2; \\ 1/3, & -1 < \alpha < 2; \\ 1/(2-\alpha), & \alpha < -1. \end{cases} \tag{26}$$

***Use acceleration**: If we set $\beta_t = (t-1)/(t+2)$ for $t \in [m]$, then setting*

$$T = \Theta\left(\left[\frac{(\alpha/2-2)^2|\alpha+1|\sqrt{m}\epsilon}{kd\log d}\right]^{\phi(\alpha)/2}\right)$$

for $\mathcal{E} = f(\widehat{\mathbf{W}}^{(T)}) - f(\mathbf{W}_*)$, we have with high probability,

$$\mathcal{E} = O\left(K^2 L\left[\frac{kd\log d}{(\alpha/2-2)^2|\alpha+1|\sqrt{m}\epsilon}\right]^{\phi(\alpha)}\right), \tag{27}$$

where

$$\phi(\alpha) = \begin{cases} 2/(\alpha+1), & \alpha > 4; \\ 2/5, & -1 < \alpha < 4; \\ 2/(4-\alpha), & \alpha < -1. \end{cases} \tag{28}$$

Theorem 8 (Group sparse - Convexity - Setting No.1). *Consider Algorithm 3. For an index $k \leq d$ that suffices the definition in Lemma 3 for all $t \in [T]$, $\eta = 1/L$, $\lambda = \Theta(LKd\sqrt{m})$, assume $\epsilon_t \leq k^2\log(d)/4Kd(d-k)^2 m$ for $t \in [T]$.*

***No acceleration**: If we set $\beta_t = 0$ for $t \in [m]$, then setting*

$$T = \Theta\left(\left[\frac{(\alpha/2-1)^2|\alpha+1|Km\epsilon}{k\log d}\right]^{\phi(\alpha)}\right).$$

for $\mathcal{E} = f(\frac{1}{T}\sum_{t=1}^{T}\widehat{\mathbf{W}}^{(t)}) - f(\mathbf{W}_*)$, we have with high probability,

$$\mathcal{E} = O\left(K^2 L\left[\frac{k\log d}{(\alpha/2-1)^2|\alpha+1|Km\epsilon}\right]^{\phi(\alpha)}\right), \tag{29}$$

where $\phi(\alpha)$ is defined in (26).

***Use acceleration**: If we set $\beta_t = (t-1)/(t+2)$ for $t \in [m]$, then setting*

$$T = \Theta\left(\left[\frac{(\alpha/2-2)^2|\alpha+1|Km\epsilon}{k\log d}\right]^{\phi(\alpha)/2}\right).$$

for $\mathcal{E} = f(\widehat{\mathbf{W}}^{(T)}) - f(\mathbf{W}_*)$, we have with high probability,

$$\mathcal{E} = O\left(K^2 L\left[\frac{k\log d}{(\alpha/2-2)^2|\alpha+1|Km\epsilon}\right]^{\phi(\alpha)}\right), \tag{30}$$

where $\phi(\alpha)$ is defined in (28).

Now we further assume that $mf(\mathbf{W})$ is μ-strongly convex and has L-Lipschitz-continuous gradient, where $\mu < L$. We set $\epsilon_t = \Theta(Q^{-t})$ for $Q > 0$ and $t \in [T]$ for this case.

Theorem 9 (Low rank - Strong convexity - Setting No.1). *Consider Algorithm 2. For an index $k \leq q$ that suffices the definition in Lemma 2 for all $t \in [T]$, $\eta = 1/L$, $\lambda = \Theta(LK\sqrt{m})$, assume $\epsilon_t \leq 4Kk^2 d(\log d)/q^2$ for $t \in [T]$.*

***No acceleration**: If we set $\beta_t = 0$ for $t \in [m]$, then denoting $Q_0 = 1 - \mu/L$ and setting*

$$T = \Theta\left(\log_{1/\psi(Q,Q_0^2)}\left[\frac{(Q_0/\sqrt{Q}-1)^2|1-Q|\sqrt{m}\epsilon}{kd\log d}\right]\right)$$

for $\mathcal{E} = \frac{1}{\sqrt{m}}\|\widehat{\mathbf{W}}^{(T)} - \mathbf{W}_*\|_F$, we have with high probability,

$$\mathcal{E} = O\left(K\left[\frac{kd\log d}{(Q_0/\sqrt{Q}-1)^2|1-Q|\sqrt{m}\epsilon}\right]^{\log_{\psi(Q,Q_0^2)} Q_0}\right), \tag{31}$$

where $\psi(\cdot,\cdot)$ is defined in (19).
Use acceleration*: If we set $\beta_t = (1-\sqrt{\mu/L})/(1+\sqrt{\mu/L})$ for $t \in [m]$, then denoting $Q_0' = 1-\sqrt{\mu/L}$ and setting*

$$T = \Theta\bigg(\log_{1/\psi(Q,Q_0')}\bigg[\frac{(\sqrt{Q_0'}/\sqrt{Q}-1)^2|1-Q|\sqrt{m}\epsilon}{kd\log d}\bigg]\bigg)$$

for $\mathcal{E} = f(\widehat{\mathbf{W}}^{(T)}) - f(\mathbf{W}_*)$, we have with high probability,

$$\mathcal{E} = O\bigg(K\bigg[\frac{kd\log d}{(\sqrt{Q_0'}/\sqrt{Q}-1)^2|1-Q|\sqrt{m}\epsilon}\bigg]^{\log_{\psi(Q,Q_0')}Q_0'}\bigg), \quad (32)$$

where $\psi(\cdot,\cdot)$ is defined in (19).

Theorem 10 (Group sparse - Strong convexity - Setting No.1)**.** *Consider Algorithm 3. For an index $k \le d$that suffices the definition in Lemma 3 for all $t \in [T]$, $\eta = 1/L$, $\lambda = \Theta(LKd\sqrt{m})$, assume $\epsilon_t \le k^2\log(d)/4Kd(d-k)^2m$ for $t \in [T]$.*
No acceleration*: If we set $\beta_t = 0$ for $t \in [m]$, then denoting $Q_0 = 1-\mu/L$ and setting*

$$T = \Theta\bigg(\log_{1/\psi(Q,Q_0^2)}\bigg[\frac{(Q_0/\sqrt{Q}-1)^2|1-Q|Km\epsilon}{k\log d}\bigg]\bigg)$$

for $\mathcal{E} = \frac{1}{\sqrt{m}}\|\widehat{\mathbf{W}}^{(T)} - \mathbf{W}_*\|_F$, we have with high probability,

$$\mathcal{E} = O\bigg(K\bigg[\frac{k\log d}{(Q_0/\sqrt{Q}-1)^2|1-Q|Km\epsilon}\bigg]^{\log_{\psi(Q,Q_0^2)}Q_0}\bigg), \quad (33)$$

where $\psi(\cdot,\cdot)$ is defined in (19).
Use acceleration*: If we set $\beta_t = (1-\sqrt{\mu/L})/(1+\sqrt{\mu/L})$ for $t \in [m]$, then denoting $Q_0' = 1-\sqrt{\mu/L}$ and setting*

$$T = \Theta\bigg(\log_{1/\psi(Q,Q_0')}\bigg[\frac{(\sqrt{Q_0'}/\sqrt{Q}-1)^2|1-Q|Km\epsilon}{k\log d}\bigg]\bigg)$$

for $\mathcal{E} = f(\widehat{\mathbf{W}}^{(T)}) - f(\mathbf{W}_*)$, we have with high probability,

$$\mathcal{E} = O\bigg(K\bigg[\frac{k\log d}{(\sqrt{Q_0'}/\sqrt{Q}-1)^2|1-Q|Km\epsilon}\bigg]^{\log_{\psi(Q,Q_0')}Q_0'}\bigg), \quad (34)$$

where $\psi(\cdot,\cdot)$ is defined in (19).

Then we optimize the utility bounds with respect to the respective budget allocation strategies.

Theorem 11 (Budget allocation - Setting No.1)**.** *Consider Algorithm 2 and Algorithm 3.*

For convex *f, use Theorem 7 and Theorem 8.*

(1) No acceleration*: Both the bounds in (25) and (29) achieve their respective minimums w.r.t. α at $\alpha = 0$. Meanwhile, $\phi(\alpha) = 1/3$.*

(2) Accelerated*: Both the bounds in (27) and (30) achieve their respective minimums w.r.t. α at $\alpha = 2/3$. Meanwhile, $\phi(\alpha) = 2/5$.*

For strongly convex *f, use Theorem 9 and Theorem 10.*

(1) No acceleration*: Both the bounds in (31) and (33) achieve their respective minimums w.r.t. Q at $Q = Q_0^{2/3}$. Meanwhile, $\log_{\psi(Q,Q_0^2)} Q_0 = 1/2$.*

(2) Accelerated*: Both the bounds in (32) and (34) achieve their respective minimums w.r.t. Q at $Q = (Q_0')^{1/3}$. Meanwhile, $\log_{\psi(Q,Q_0')} Q_0' = 1$.*

B. Setting No.2

In this setting, we have

$$\epsilon = \sum_{t=1}^{T}\frac{(e^{\epsilon_t}-1)\epsilon_t}{(e^{\epsilon_t}+1)} + \sqrt{\sum_{t=1}^{T}2\epsilon_t^2\log\Big(\frac{1}{\delta}\Big)}.$$

Theorem 12 (Low rank - Convexity - Setting No.2)**.** *Consider Algorithm 2. For an index $k \le q$ that suffices the definition in Lemma 2 for all $t \in [T]$, $\eta = 1/L$, $\lambda = \Theta(LK\sqrt{m})$, assume $\epsilon_t \le 4Kk^2d(\log d)/q^2$ for $t \in [T]$.*
No acceleration*: If we set $\beta_t = 0$ for $t \in [m]$, then setting*

$$T = \Theta\bigg(\bigg[\frac{(\alpha/2-1)^2\sqrt{|2\alpha+1|}\sqrt{m}\epsilon}{kd\log d\sqrt{\log(1/\delta)+2\epsilon}}\bigg]^{\phi(\alpha)}\bigg)$$

for $\mathcal{E} = f(\frac{1}{T}\sum_{t=1}^{T}\widehat{\mathbf{W}}^{(t)}) - f(\mathbf{W}_*)$, we have with high probability,

$$\mathcal{E} = O\bigg(K^2L\bigg[\frac{kd\log d\sqrt{\log(1/\delta)+2\epsilon}}{(\alpha/2-1)^2\sqrt{|2\alpha+1|}\sqrt{m}\epsilon}\bigg]^{\phi(\alpha)}\bigg), \quad (35)$$

where

$$\phi(\alpha) = \begin{cases} 2/(2\alpha+1), & \alpha > 2; \\ 2/5, & -1/2 < \alpha < 2; \\ 1/(2-\alpha), & \alpha < -1/2. \end{cases} \quad (36)$$

Use acceleration*: If we set $\beta_t = (t-1)/(t+2)$ for $t \in [m]$, then setting*

$$T = \Theta\bigg(\bigg[\frac{(\alpha/2-2)^2\sqrt{|2\alpha+1|}\sqrt{m}\epsilon}{kd\log d\sqrt{\log(1/\delta)+2\epsilon}}\bigg]^{\phi(\alpha)/2}\bigg)$$

for $\mathcal{E} = f(\widehat{\mathbf{W}}^{(T)}) - f(\mathbf{W}_*)$, we have with high probability,

$$\mathcal{E} = O\bigg(K^2L\bigg[\frac{kd\log d\sqrt{\log(1/\delta)+2\epsilon}}{(\alpha/2-2)^2\sqrt{|2\alpha+1|}\sqrt{m}\epsilon}\bigg]^{\phi(\alpha)}\bigg), \quad (37)$$

where

$$\phi(\alpha) = \begin{cases} 4/(2\alpha+1), & \alpha > 4; \\ 4/9, & -1/2 < \alpha < 4; \\ 2/(4-\alpha), & \alpha < -1/2. \end{cases} \quad (38)$$

Theorem 13 (Group sparse - Convexity - Setting No.2)**.** *Consider Algorithm 3. For an index $k \le d$ that suffices the definition in Lemma 3 for all $t \in [T]$, $\eta = 1/L$, $\lambda = \Theta(LKd\sqrt{m})$, assume $\epsilon_t \le k^2\log(d)/4Kd(d-k)^2m$ for $t \in [T]$.*
No acceleration*: If we set $\beta_t = 0$ for $t \in [m]$, then setting*

$$T = \Theta\bigg(\bigg[\frac{(\alpha/2-1)^2\sqrt{|2\alpha+1|}Km\epsilon}{k\log d\sqrt{\log(1/\delta)+2\epsilon}}\bigg]^{\phi(\alpha)}\bigg).$$

for $\mathcal{E} = f(\frac{1}{T}\sum_{t=1}^{T}\widehat{\mathbf{W}}^{(t)}) - f(\mathbf{W}_*)$, we have with high probability,

$$\mathcal{E} = O\bigg(K^2L\bigg[\frac{k\log d\sqrt{\log(1/\delta)+2\epsilon}}{(\alpha/2-1)^2\sqrt{|2\alpha+1|}Km\epsilon}\bigg]^{\phi(\alpha)}\bigg), \quad (39)$$

where $\phi(\alpha)$ is defined in (36).
Use acceleration*: If we set $\beta_t = (t-1)/(t+2)$ for $t \in [m]$, then setting*

$$T = \Theta\bigg(\bigg[\frac{(\alpha/2-2)^2\sqrt{|2\alpha+1|}Km\epsilon}{k\log d\sqrt{\log(1/\delta)+2\epsilon}}\bigg]^{\phi(\alpha)/2}\bigg).$$

for $\mathcal{E} = f(\widehat{\mathbf{W}}^{(T)}) - f(\mathbf{W}_*)$, we have with high probability,

$$\mathcal{E} = O\bigg(K^2L\bigg[\frac{k\log d\sqrt{\log(1/\delta)+2\epsilon}}{(\alpha/2-2)^2\sqrt{|2\alpha+1|}Km\epsilon}\bigg]^{\phi(\alpha)}\bigg), \quad (40)$$

where $\phi(\alpha)$ is defined in (38).

Now we further assume that $mf(\mathbf{W})$ is μ-strongly convex and has L-Lipschitz-continuous gradient, where $\mu < L$. We set $\epsilon_t = \Theta(Q^{-t})$ for $Q > 0$ and $t \in [T]$ for this case.

Theorem 14 (Low rank - Strong convexity - Setting No.2)**.** *Consider Algorithm 2. For an index $k \le q$ that suffices the definition in Lemma 2 for all $t \in [T]$, $\eta = 1/L$, $\lambda = \Theta(LK\sqrt{m})$, assume $\epsilon_t \le 4Kk^2d(\log d)/q^2$ for $t \in [T]$.*

***No acceleration**: If we set $\beta_t = 0$ for $t \in [m]$, then denoting $Q_0 = 1 - \mu/L$ and setting*

$$T = \Theta\left(\log_{1/\psi(Q,Q_0^2)}\left[\frac{(Q_0/\sqrt{Q}-1)^2\sqrt{|1-Q^2|}\sqrt{m}\epsilon}{kd\log d\sqrt{\log(1/\delta)+2\epsilon}}\right]\right)$$

for $\mathcal{E} = \frac{1}{\sqrt{m}}\|\widehat{\mathbf{W}}^{(T)} - \mathbf{W}_*\|_F$, we have with high probability,

$$\mathcal{E} = O\left(K\left[\frac{kd\log d\sqrt{\log(1/\delta)+2\epsilon}}{(Q_0/\sqrt{Q}-1)^2\sqrt{|1-Q^2|}\sqrt{m}\epsilon}\right]^{\log_{\psi(Q,Q_0^2)} Q_0}\right), \tag{41}$$

where $\psi(\cdot,\cdot)$ is defined in (19).
***Use acceleration**: If we set $\beta_t = (1-\sqrt{\mu/L})/(1+\sqrt{\mu/L})$ for $t \in [m]$, then denoting $Q_0' = 1 - \sqrt{\mu/L}$ and setting*

$$T = \Theta\left(\log_{1/\psi(Q,Q_0')}\left[\frac{(\sqrt{Q_0'}/\sqrt{Q}-1)^2\sqrt{|1-Q^2|}\sqrt{m}\epsilon}{kd\log d\sqrt{\log(1/\delta)+2\epsilon}}\right]\right)$$

for $\mathcal{E} = f(\widehat{\mathbf{W}}^{(T)}) - f(\mathbf{W}_*)$, we have with high probability,

$$\mathcal{E} = O\left(K\left[\frac{kd\log d\sqrt{\log(1/\delta)+2\epsilon}}{(\sqrt{Q_0'}/\sqrt{Q}-1)^2\sqrt{|1-Q^2|}\sqrt{m}\epsilon}\right]^{\log_{\psi(Q,Q_0')} Q_0'}\right), \tag{42}$$

where $\psi(\cdot,\cdot)$ is defined in (19).

Theorem 15 (Group sparse - Strong convexity - Setting No.2)**.** *Consider Algorithm 3. For an index $k \le d$that suffices the definition in Lemma 3 for all $t \in [T]$, $\eta = 1/L$, $\lambda = \Theta(LKd\sqrt{m})$, assume $\epsilon_t \le k^2\log(d)/4Kd(d-k)^2 m$ for $t \in [T]$.*
***No acceleration**: If we set $\beta_t = 0$ for $t \in [m]$, then denoting $Q_0 = 1 - \mu/L$ and setting*

$$T = \Theta\left(\log_{1/\psi(Q,Q_0^2)}\left[\frac{(Q_0/\sqrt{Q}-1)^2\sqrt{|1-Q^2|}Km\epsilon}{k\log d\sqrt{\log(1/\delta)+2\epsilon}}\right]\right)$$

for $\mathcal{E} = \frac{1}{\sqrt{m}}\|\widehat{\mathbf{W}}^{(T)} - \mathbf{W}_*\|_F$, we have with high probability,

$$\mathcal{E} = O\left(K\left[\frac{k\log d\sqrt{\log(1/\delta)+2\epsilon}}{(Q_0/\sqrt{Q}-1)^2\sqrt{|1-Q^2|}Km\epsilon}\right]^{\log_{\psi(Q,Q_0^2)} Q_0}\right), \tag{43}$$

where $\psi(\cdot,\cdot)$ is defined in (19).
***Use acceleration**: If we set $\beta_t = (1-\sqrt{\mu/L})/(1+\sqrt{\mu/L})$ for $t \in [m]$, then denoting $Q_0' = 1 - \sqrt{\mu/L}$ and setting*

$$T = \Theta\left(\log_{1/\psi(Q,Q_0')}\left[\frac{(\sqrt{Q_0'}/\sqrt{Q}-1)^2\sqrt{|1-Q^2|}Km\epsilon}{k\log d\sqrt{\log(1/\delta)+2\epsilon}}\right]\right)$$

for $\mathcal{E} = f(\widehat{\mathbf{W}}^{(T)}) - f(\mathbf{W}_*)$, we have with high probability,

$$\mathcal{E} = O\left(K\left[\frac{k\log d\sqrt{\log(1/\delta)+2\epsilon}}{(\sqrt{Q_0'}/\sqrt{Q}-1)^2\sqrt{|1-Q^2|}Km\epsilon}\right]^{\log_{\psi(Q,Q_0')} Q_0'}\right), \tag{44}$$

where $\psi(\cdot,\cdot)$ is defined in (19).

Then we optimize the utility bounds with respect to the respective budget allocation strategies.

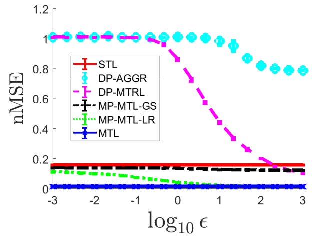
Theorem 16 (Budget allocation - Setting No.2)**.** *Consider Algorithm 2 and Algorithm 3.*

For convex f*, use Theorem 12 and Theorem 13.*

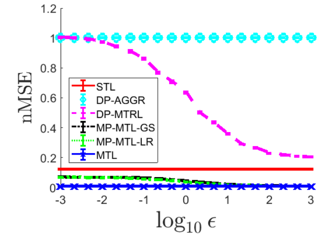
(1) No acceleration*: Both the bounds in (35) and (39) achieve their respective minimums w.r.t. α at $\alpha = 0$. Meanwhile, $\phi(\alpha) = 2/5$.*

(2) Accelerated*: Both the bounds in (37) and (40) achieve their respective minimums w.r.t. α at $\alpha = 2/5$. Meanwhile, $\phi(\alpha) = 4/9$.*

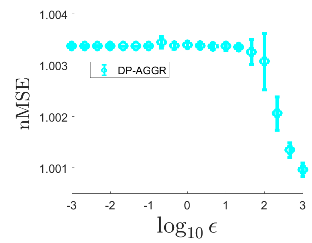
For strongly convex f*, use Theorem 14 and Theorem 15.*



(a) Low-rank pattern



(b) Group-sparse pattern



(c) Group-sparse pattern

Figure 1. Privacy-accuracy tradeoff on synthetic datasets. For (a), the data that associated with the low-rank model matrix were used; for (b) and (c), the data that associated with the group-sparse model matrix were used. In (c), the plot shows the same performances of DP-AGGR as in (b) with a finer vertical axis. MP-MTL-LR denotes Algorithm 2, MP-MTL-GS denotes Algorithm 3, and STL denotes the ℓ_2-norm-penalized method. In both panels, STL and MTL denote non-private methods.

(1) No acceleration*: Both the bounds in (41) and (43) achieve their respective minimums w.r.t. Q at $Q = Q_0^{2/5}$. Meanwhile, $\log_{\psi(Q,Q_0^2)} Q_0 = 1/2$.*

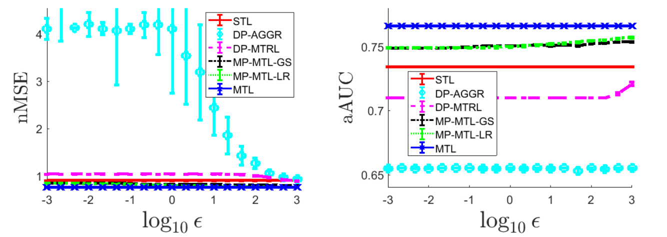
(2) Accelerated*: Both the bounds in (42) and (44) achieve their respective minimums w.r.t. Q at $Q = (Q_0')^{1/5}$. Meanwhile, $\log_{\psi(Q,Q_0')} Q_0' = 1$.*

Appendix F
Detailed Privacy-Accuracy Tradeoff for Baseline Methods on Synthetic Datasets

In Fig. 1, the detailed performances of both of DP-MTRL and DP-AGGR are shown. Note that we have tuned the regularization parameters for both DP-MTRL and DP-AGGR for acceptable accuracies. Our Algorithm 2 outperforms DP-MTRL and DP-AGGR. In Fig. 1 (a), DP-MTRL outperforms the STL method and our Algorithm 3 when ϵ is large, because it suits the true model matrix, in which the relatedness among tasks is modeled by a graph. However, the true model matrix is not group-sparse, hence our Algorithm 3 underperforms comparing with Algorithm 2 and DP-MTRL when ϵ is large. By contrast, in Fig. 1 (b), the true model matrix is group-sparse and is not suitable for DP-MTRL, hence DP-MTRL underperforms comparing with the STL method even when ϵ is large. Fig. 1 (c) is used to show that the accuracy of DP-AGGR grows with ϵ under the same setting as in Fig. 1 (b). As we discussed, DP-AGGR only performs model-averaging, which is not suitable for the true model matrices in both settings of Fig. 1 (a) and (b), hence the accuracies of DP-AGGR are much worse than those of the respective STL methods.

Appendix G
Detailed Privacy-Accuracy Tradeoff for DP-AGGR on Real-World Datasets

In Fig. 2, the detailed performances of DP-AGGR are shown. Because the dimension is large and the number of tasks is not sufficient, the accuracy of DP-AGGR barely grows with ϵ; other



(a) School Data (b) Patient Survey Data

Figure 2. Privacy-accuracy tradeoff on real-world datasets. In both panels, MTL denotes the method with the best performance among the four non-private MTL methods proposed by Ji and Ye [6], Liu et al. [9], Zhang and Yeung [13] and DP-AGGR without perturbations; MP-MTL-LR denotes Algorithm 2, whereas MP-MTL-GS denotes Algorithm 3; STL denotes the method with the better performance between the ℓ_1- and ℓ_2-regularized methods.

private-preserving methods, such as DP-MTRL and our algorithms, grow slowly with ϵ as well.

Appendix H
Varying Training-Data Percentage

Since the MTL behavior may change when the training-data percentage (the size of the training data divided by the size of the entire dataset) changes, we evaluated the methods on both real-world datasets at different training-data percentages. Here, we present the results mostly for our low-rank algorithm (denoted by MP-MTL-LR) because it always outperforms our group-sparse algorithm (MP-MTL-GS) in the above experiments. The results corresponding to School Data are shown in Fig. 3; the results corresponding to Patient Survey Data are shown in Fig. 4. From those plots, we observe that on both real-world datasets, our MP-MTL method behaves similarly at different training-data percentages and outperforms DP-MTRL and DP-AGGR, especially when ϵ is small.

Appendix I
Lemmas of Differential Privacy

- Post-Processing immunity. This property helps us safely use the output of a differentially private algorithm without additional information leaking, as long as we do not touch the dataset $\mathcal{D}$ again.

  Lemma 4 (Post-Processing immunity. Proposition 2.1 in Dwork et al. [3]). *Let algorithm $\mathcal{A}_1(\mathcal{B}_1) : \mathcal{D} \to \theta_1 \in \mathcal{C}_1$ be an (ϵ, δ) - differential privacy algorithm, and let $f : \mathcal{C}_1 \to \mathcal{C}_2$ be an arbitrary mapping. Then, algorithm $\mathcal{A}_2(\mathcal{B}_2) : \mathcal{D} \to \theta_2 \in \mathcal{C}_2$ is still (ϵ, δ) - differentially private, i.e., for any set $\mathcal{S} \subseteq \mathcal{C}_2$,*

$$\mathbb{P}(\theta_2 \in \mathcal{S} \mid \mathcal{B}_2 = \mathcal{D}) \leq e^{\epsilon}\mathbb{P}(\theta_2 \in \mathcal{S} \mid \mathcal{B}_2 = \mathcal{D}') + \delta.$$

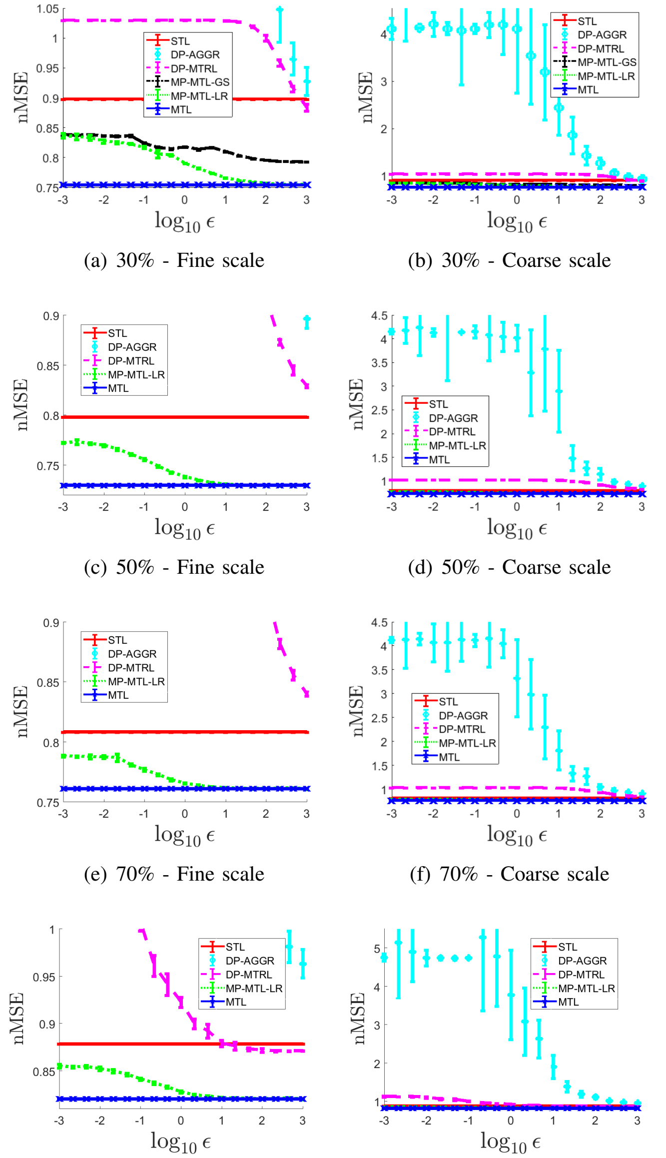
- Group privacy. This property guarantees the graceful increment of the privacy budget when more output variables need differentially private protection.

  Lemma 5 (Group privacy. Lemma 2.2 in Vadhan [12]). *Let algorithm $\mathcal{A}(\mathcal{B}) : \mathcal{D} \to \theta \in \mathcal{C}$ be an (ϵ, δ) - differential privacy algorithm. Then, considering two neighboring datasets $\mathcal{D}$ and $\mathcal{D}'$ that differ in k entries, the algorithm satisfies for any set $\mathcal{S} \subseteq \mathcal{C}$*

$$\mathbb{P}(\theta \in \mathcal{S} \mid \mathcal{B} = \mathcal{D}) \leq e^{k\epsilon}\mathbb{P}(\theta \in \mathcal{S} \mid \mathcal{B} = \mathcal{D}') + ke^{k\epsilon}\delta.$$

- Combination. This property guarantees the linear incrementing of the privacy budget when the dataset $\mathcal{D}$ is repeatedly used.

  Lemma 6 (Combination. Theorem 3.16 in Dwork et al. [3]). *Let algorithm $\mathcal{A}_i : \mathcal{D} \to \theta_i \in \mathcal{C}_i$ be an (ϵ_i, δ_i) - differential privacy algorithm for all $i \in [k]$. Then, for $\mathcal{A}_{[k]} : \mathcal{D} \to$*



(a) 30% - Fine scale (b) 30% - Coarse scale

(c) 50% - Fine scale (d) 50% - Coarse scale

(e) 70% - Fine scale (f) 70% - Coarse scale

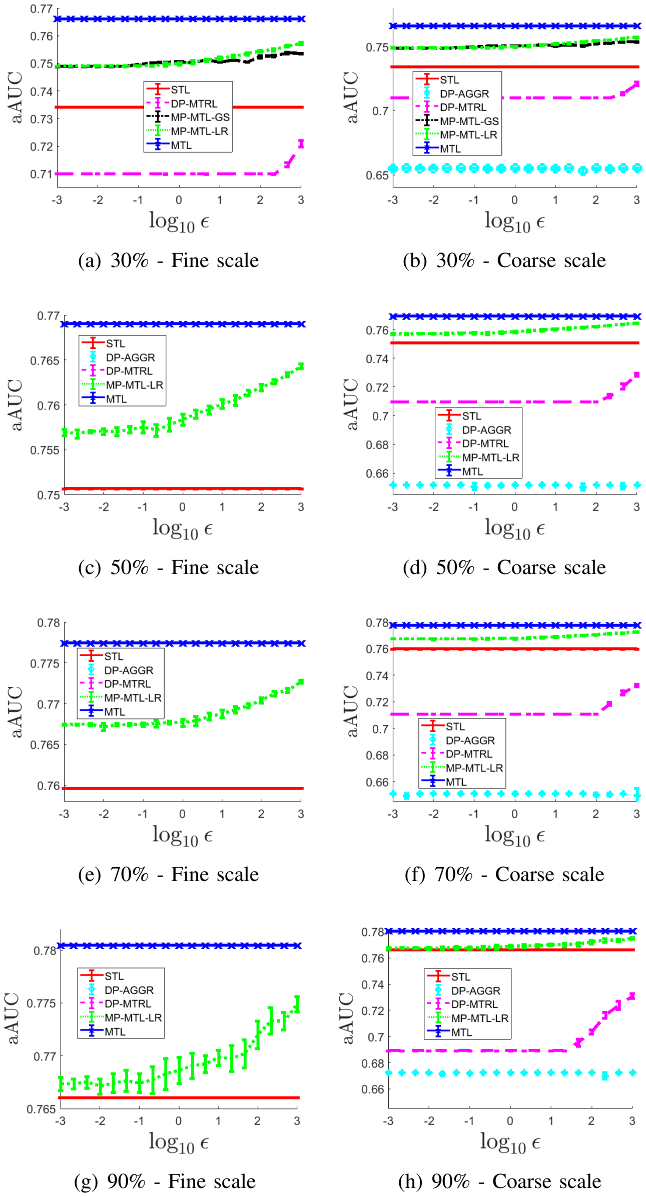
(g) 90% - Fine scale (h) 90% - Coarse scale

Figure 3. Privacy-accuracy tradeoff on School Data. (a) and (b) correspond to a training-data percentage of 30%, (c) and (d) correspond to a training-data percentage of 50%, (e) and (f) correspond to a training-data percentage of 70%, (g) and (h) correspond to a training-data percentage of 90%. (a), (c), (e) and (g) use fine scales of vertical axes to focus on the performances of our algorithms; (b), (d), (f) and (h) use coarse scales of vertical axes to focus on the baseline algorithms. In all the panels, MTL denotes the method with the best performance among the four non-private MTL methods proposed by Ji and Ye [6], Liu et al. [9], Zhang and Yeung [13] and DP-AGGR without perturbations; MP-MTL-LR denotes Algorithm 2, whereas MP-MTL-GS denotes Algorithm 3; STL denotes the method with the better performance between the ℓ_1- and ℓ_2-regularized methods.

$(\theta_1, \theta_2, \cdots, \theta_k) \in \bigotimes_{j=1}^{k} \mathcal{C}_j$ is a $(\sum_i \epsilon_i, \sum_i \delta_i)$ - differentially private algorithm.

- Adaptive composition. This property guarantees privacy when an iterative algorithm is adopted on *different* datasets that may nevertheless contain information relating to the same individual.

  Lemma 7 (Adaptive composition. Directly taken Theorem 3.5 in Kairouz et al. [8]). *Let algorithm $\mathcal{A}_1(\mathcal{B}_1) : \mathcal{D}_1 \to \theta_1$ be an (ϵ_1, δ_1) - differential privacy algorithm, and for $t = 2, \ldots, T$, let $\mathcal{A}_t(\mathcal{B}_t) : (\mathcal{D}_t, \theta_1, \theta_2, \cdots, \theta_{t-1}) \to \theta_t \in \mathcal{C}_t$ be (ϵ_t, δ_t) -*



(a) 30% - Fine scale (b) 30% - Coarse scale
(c) 50% - Fine scale (d) 50% - Coarse scale
(e) 70% - Fine scale (f) 70% - Coarse scale
(g) 90% - Fine scale (h) 90% - Coarse scale

Figure 4. Privacy-accuracy tradeoff on Patient Survey Data. (a) and (b) correspond to a training-data percentage of 30%, (c) and (d) correspond to a training-data percentage of 50%, (e) and (f) correspond to a training-data percentage of 70%, (g) and (h) correspond to a training-data percentage of 90%. (a), (c), (e) and (g) use fine scales of vertical axes to focus on the performances of our algorithms; (b), (d), (f) and (h) use coarse scales of vertical axes to focus on the baseline algorithms. In all the panels, MTL denotes the method with the best performance among the four non-private MTL methods proposed by Ji and Ye [6], Liu et al. [9], Zhang and Yeung [13] and DP-AGGR without perturbations; MP-MTL-LR denotes Algorithm 2, whereas MP-MTL-GS denotes Algorithm 3; STL denotes the method with the better performance between the ℓ_1- and ℓ_2-regularized methods.

differentially private for all given $(\theta_1, \theta_2, \cdots, \theta_{t-1}) \in \bigotimes_{t'=1}^{t-1} \mathcal{C}_{t'}$. Then, for all neighboring datasets $\mathcal{D}_t$ and $\mathcal{D}'_t$ that differ in a single entry relating to the same individual and for any set $\mathcal{S} \subseteq \bigotimes_{t=1}^{T} \mathcal{C}_t$,

$$
\begin{aligned}
&\mathbb{P}((\theta_1, \cdots, \theta_T) \in \mathcal{S} \mid \bigcap_{t=1}^{T} (\mathcal{B}_t = (\mathcal{D}_t, \boldsymbol{\theta}_{1:t-1}))) \\
&\leq e^{\epsilon} \mathbb{P}((\theta_1, \cdots, \theta_T) \in \mathcal{S} \mid \bigcap_{t=1}^{T} (\mathcal{B}_t = (\mathcal{D}'_t, \boldsymbol{\theta}_{1:t-1}))) \\
&+ 1 - (1-\delta) \prod_{t=1}^{T} (1-\delta_t),
\end{aligned}
\tag{45}
$$

where

$$
\boldsymbol{\theta}_{1:t-1} = \begin{cases} \emptyset, & t = 1 \\ \theta_1, \theta_2, \cdots, \theta_{t-1}, & t \geq 2, \end{cases}
$$

and

$$
\begin{aligned}
\epsilon = \min \Bigg\{ &\sum_{t=1}^{T} \epsilon_t, \sum_{t=1}^{T} \frac{(e^{\epsilon_t}-1)\epsilon_t}{(e^{\epsilon_t}+1)} + \sqrt{\sum_{t=1}^{T} 2\epsilon_t^2 \log\left(\frac{1}{\delta}\right)}, \\
&\sum_{t=1}^{T} \frac{(e^{\epsilon_t}-1)\epsilon_t}{(e^{\epsilon_t}+1)} + \sqrt{\sum_{t=1}^{T} 2\epsilon_t^2 \log\left(e + \frac{\sqrt{\sum_{t=1}^{T} \epsilon_t^2}}{\delta}\right)} \Bigg\}.
\end{aligned}
$$

Appendix J
Lemmas for Privacy Guarantees

The following lemma shows that STL algorithms do not increase the privacy budget when they are concatenated with an MTL algorithm.

Lemma 8. *For an (ϵ, δ) - non-iterative MP-MTL algorithm $\mathcal{A}_{mp} : (\mathbf{W} \in \mathbb{R}^{d\times m}, \mathcal{D}^m) \to \widetilde{\mathbf{W}} \in \mathbb{R}^{d\times m}$ and any STL algorithm $\mathcal{A}_{st} : (\widetilde{\mathbf{W}}, \mathcal{D}^m) \to \widehat{\mathbf{W}} \in \mathbb{R}^{d\times m}$, an algorithm $\mathcal{A}_{mp+st} : (\mathbf{W} \in \mathbb{R}^{d\times m}, \mathcal{D}^m) \to \widehat{\mathbf{W}} \in \mathbb{R}^{d\times m}$ that first uses $\mathcal{A}_{mp}$ before applying $\mathcal{A}_{st}$ is still an (ϵ, δ) - non-iterative MP-MTL algorithm. Moreover, an algorithm $\mathcal{A}_{st+mp}$ that first uses a* deterministic *STL algorithm before applying an (ϵ, δ) - non-iterative MP-MTL algorithm is also an (ϵ, δ) - non-iterative MP-MTL algorithm.*

The following result shows that adopting a series of Non-iterative MP-MTL algorithms defined in Definition 5 iteratively, we can develop an Iterative MP-MTL algorithm, as described in Algorithm 6.

Algorithm 6 Iterative MP-MTL build by Non-iterative MP-MTL

Input: Datasets $(\mathbf{X}^m, \mathbf{y}^m) = \{(\mathbf{X}_1, \mathbf{y}_1), \ldots, (\mathbf{X}_m, \mathbf{y}_m)\}$, where $\forall i \in [m]$, $\mathbf{X}_i \in \mathbb{R}^{n_i \times d}$ and $\mathbf{y}_i \in \mathbb{R}^{n_i \times 1}$. Number of iterations T. Privacy loss $\{\epsilon_t, \delta_t\}_{t=1,\ldots,T}, \delta \geq 0$. Initial models of tasks $\mathbf{W}^{(0)}$.
Output: $\widehat{\mathbf{W}}^{(1:T)}$.
1: **for** $t = 1 : T$ **do**
2: $\widehat{\mathbf{W}}^{(t)} = \mathcal{A}_{\mathrm{mp}}(\mathbf{W}^{(t-1)}, \mathbf{X}^m, \mathbf{y}^m)$, where $\mathcal{A}_{\mathrm{mp}}$ denotes an (ϵ_t, δ_t)-Non-iterative MP-MTL algorithm.
3: $\mathbf{w}_i^{(t)} = \mathcal{A}_{\mathrm{st},i}(\hat{\mathbf{w}}_i^{(t)}, \mathbf{X}_i, \mathbf{y}_i)$, for $i = 1, \ldots, m$, where $\mathcal{A}_{\mathrm{st},i}$ denotes a deterministic STL algorithm for the i-th task.
4: **end for**

Lemma 9. *Use Lemmas 7 and 8. Algorithm 6 is an $(\epsilon, 1 - (1-\delta)\prod_{t=1}^{T}(1-\delta_t))$ - iterative MP-MTL algorithm, where ϵ is defined in Lemma 7.*

APPENDIX K
LEMMAS FOR UTILITY ANALYSIS

Lemma 10. *For a integer $T \geq 1$, a constant $\alpha \in \mathbb{R}$, by EulerMaclaurin formula [1], we have*

$$\sum_{t=1}^{T} t^{\alpha} = \begin{cases} O(T^{\alpha+1}/(\alpha+1)), & \alpha > -1; \\ O(1/(-\alpha-1)), & \alpha < -1. \end{cases}$$

Proof. This is the direct result of EulerMaclaurin formula [1]. □

Lemma 11. *For a integer $T \geq 1$ and a constant $Q > 0$, we have*

$$\sum_{t=1}^{T} Q^{-t} \leq \begin{cases} \frac{1}{Q-1}, & Q > 1; \\ \frac{Q^{-T}}{1-Q}, & Q < 1. \end{cases}$$

Proof. Because $\sum_{t=1}^{T} Q^{-t} = Q^{-1}\frac{1-Q^{-T}}{1-Q^{-1}}$, we complete the proof. □

Lemma 12. *For a constant $c_1, c_2 > 0$, a constant $\epsilon_0 > 0$, a integer $T \geq 1$, a mapping $s : t \in [T] \to s(t) > 0$, a mapping $S_1 : T \to S_1(T) > 0$ and a mapping $S_2 : T \to S_2(T) > 0$, then if $\sum_{t=1}^{T} \epsilon_0 s(t) \geq c_1$ and $\sum_{t=1}^{T} s(t) \leq S_1(T)$, we have*

$$1/\epsilon_0 \leq S_1(T)/c_1.$$

On the other hand, if $\sqrt{\sum_{t=1}^{T} \epsilon_0^2 s^2(t)} \geq c_2$ and $\sum_{t=1}^{T} s^2(t) \leq S_2(T)$, we have

$$1/\epsilon_0 \leq \sqrt{S_2(T)}/c_2.$$

Proof. If $\sum_{t=1}^{T} \epsilon_0 s(t) \geq c_1$, $1/\epsilon_0 \leq \sum_{t=1}^{T} s(t)/c_1 \leq S_1(T)/c_1$.

On the other hand, if $\sqrt{\sum_{t=1}^{T} \epsilon_0^2 s^2(t)} \geq c_2$, $1/\epsilon_0 \leq \sqrt{\sum_{t=1}^{T} s^2(t)}/c_2 \leq \sqrt{S_2(T)}/c_2$. □

Lemma 13. *Consider Algorithm 2. For an index $k \leq q$ that suffices the definition in Lemma 2 for all $t \in [T]$, $\eta = 1/L$, $\lambda = \Theta(LK\sqrt{m})$, set $\epsilon_t \leq 4Kk^2 d(\log d)/q^2$ for $t \in [T]$. Assume in each iteration, $\mathbf{E}$ is the defined Wishart random matrix. We have with probability at least $1 - d^{-c}$ for some constant $c > 1$ that*

$$\begin{aligned} \varepsilon_t =& \frac{1}{2\eta}\|\widehat{\mathbf{W}}^{(t)} - \mathbf{C}\|_F^2 + \lambda\|\widehat{\mathbf{W}}^{(t)}\|_* \\ &- \left\{\min_{\mathbf{W}} \frac{1}{2\eta}\|\mathbf{W} - \mathbf{C}\|_F^2 + \lambda\|\mathbf{W}\|_*\right\} \\ =& O\left(\frac{K^2\sqrt{m}kd\log d}{\eta\epsilon_t}\right). \end{aligned} \tag{46}$$

Proof. First, using Lemma 1 of Jiang et al. [7], we have in the t-th step, with probability at least $1 - d^{-c}$ for some constant $c > 1$,

$$\sigma_1(\mathbf{E}) = O\left(d(\log d)\sigma_1\left(\frac{\max_i s_i^{(t-1)}}{2\epsilon_t}\mathbf{I}_d\right)\right) = O(d(\log d)K/\epsilon_t).$$

We also have $\sigma_1(\mathbf{C}) \leq \|\mathbf{C}\|_F \leq \sqrt{m}\max_i \|\mathbf{C}_i\|_2 \leq K\sqrt{m}$, where $\mathbf{C}_i$ is the i-th column of $\mathbf{C}$.

As such, by Lemma 2, in the t-th iteration, for $\epsilon_t \leq 4Kk^2 d(\log d)/q^2$, where $q = \min\{d, m\}$, we have

$$\begin{aligned} \varepsilon_t =& \frac{1}{2\eta}\|\widehat{\mathbf{W}}^{(t)} - \mathbf{C}\|_F^2 + \lambda\|\widehat{\mathbf{W}}^{(t)}\|_* \\ &- \left\{\min_{\mathbf{W}} \frac{1}{2\eta}\|\mathbf{W} - \mathbf{C}\|_F^2 + \lambda\|\mathbf{W}\|_*\right\} \\ \leq& \frac{1}{\eta}\left(\frac{\sigma_1^2(\mathbf{C})}{\eta\lambda} + \sigma_1(\mathbf{C})\right)\left[k\frac{\sigma_1(\mathbf{E})}{2\eta\lambda}\right. \\ &\left. + (r_c - k)I(r_c > k)\sqrt{\sigma_1(\mathbf{E})} + \left(\frac{k(k-1)}{\eta\lambda} + 2k\right)\sigma_1(\mathbf{E})\right] \\ \leq& \frac{1}{\eta}\left(\frac{K^2 m}{\eta\lambda} + K\sqrt{m}\right)\left[k\frac{\sigma_1(\mathbf{E})}{2\eta\lambda}\right. \\ &\left. + q\sqrt{\sigma_1(\mathbf{E})} + \left(\frac{k(k-1)}{\eta\lambda} + 2k\right)\sigma_1(\mathbf{E})\right] \\ =& O\left(\frac{1}{\eta}\left(\frac{K^2 m}{\eta\lambda} + K\sqrt{m}\right)\left(\frac{k^2}{\eta\lambda} + 2k\right)\frac{d(\log d)K}{\epsilon_t}\right), \end{aligned}$$

where in the second inequality, the terms with $\sigma_1(\mathbf{E})$ nominate due to the condition on ϵ_t.

Further assuming $\eta = 1/L$ and $\lambda = \Theta(LK\sqrt{m})$, we complete the proof. □

Lemma 14. *Consider Algorithm 3. For an index $k \leq d$ that suffices the definition in Lemma 3 for all $t \in [T]$, $\eta = 1/L$, $\lambda = \Theta(LKd\sqrt{m})$, set $\epsilon_t \leq k^2\log(d)/4Kd(d-k)^2 m$ for $t \in [T]$. Assume in each iteration, $\mathbf{E}$ is the defined Wishart random matrix. We have with probability at least $1 - d^{-c}$ for some constant $c > 1$ that*

$$\begin{aligned} \varepsilon_t =& \frac{1}{2\eta}\|\widehat{\mathbf{W}}^{(t)} - \mathbf{C}\|_F^2 + \lambda\|\widehat{\mathbf{W}}^{(t)}\|_{2,1} \\ &- \left\{\min_{\mathbf{W}} \frac{1}{2\eta}\|\mathbf{W} - \mathbf{C}\|_F^2 + \lambda\|\mathbf{W}\|_{2,1}\right\} \\ =& O\left(\frac{Kk\log d}{\eta\epsilon_t}\right). \end{aligned} \tag{47}$$

Proof. Similarly as in proof for Lemma 13, by Lemma 3, in the t-th iteration, we have

$$\begin{aligned} \varepsilon_t =& \frac{1}{2\eta}\|\widehat{\mathbf{W}}^{(t)} - \mathbf{C}\|_F^2 + \lambda\|\widehat{\mathbf{W}}^{(t)}\|_{2,1} \\ &- \left\{\min_{\mathbf{W}} \frac{1}{2\eta}\|\mathbf{W} - \mathbf{C}\|_F^2 + \lambda\|\mathbf{W}\|_{2,1}\right\} \\ \leq& \frac{1}{\eta}\left[\frac{r_{c,s}}{\eta\lambda}\left(\max_{j\in[d]}\|\mathbf{C}^j\|_2\right)^2 + \left(\max_{j\in[d]}\|\mathbf{C}^j\|_2\right)\right] \\ &\cdot\left[\frac{k}{2\eta\lambda}\max_{j:\eta^2\lambda^2 \leq \mathbf{\Sigma}_{jj,0}}|\mathbf{E}_{jj}|\right. \\ &\left. + (r_{c,s} - k)I(r_{c,s} > k)\max_{j:\eta^2\lambda^2 > \mathbf{\Sigma}_{jj,0}}\sqrt{|\mathbf{E}_{jj}|}\right] \\ \leq& \frac{1}{\eta}\left[\frac{r_{c,s}}{\eta\lambda}\|\mathbf{C}\|_F^2 + \|\mathbf{C}\|_F\right] \\ &\cdot\left[\frac{k}{2\eta\lambda}\sigma_1(\mathbf{E}) + (r_{c,s} - k)I(r_{c,s} > k)\sqrt{\sigma_1(\mathbf{E})}\right] \\ \leq& \frac{1}{\eta}\left(\frac{dK^2 m}{\eta\lambda} + K\sqrt{m}\right)\left[k\frac{\sigma_1(\mathbf{E})}{2\eta\lambda} + (d-k)\sqrt{\sigma_1(\mathbf{E})}\right]. \end{aligned}$$

Further setting $\eta = 1/L$ and $\lambda = \Theta(LKd\sqrt{m})$, assuming $\epsilon_t \leq k^2\log(d)/4Kd(d-k)^2 m$, we have

$$\begin{aligned} \varepsilon_t =& O\left(\frac{1}{\eta}\left(\frac{dK^2 m}{\eta\lambda} + K\sqrt{m}\right)\frac{k}{\eta\lambda}\frac{d(\log d)K}{\epsilon_t}\right) \\ =& O\left(\frac{Kk\log d}{\eta\epsilon_t}\right). \end{aligned} \tag{48}$$

□

Lemma 15. *For matrices $\mathbf{W}_1, \mathbf{W}_2 \in \mathcal{W} \subset \mathbb{R}^{d\times m}$, we have*

$$\|\mathbf{W}_1 - \mathbf{W}_2\|_F = O(K\sqrt{m}).$$

Proof. Because $\mathbf{W}_1, \mathbf{W}_2 \in \mathcal{W}$, $\max_{i\in[m]} \|\mathbf{w}_{i,1}\|_2 \le K$. Therefore,

$$\|\mathbf{W}_1 - \mathbf{W}_2\|_F \le 2\|\mathbf{W}_1\|_F \le 2\sqrt{m}\max_{i\in[m]} \|\mathbf{w}_{i,1}\|_2 \le 2K\sqrt{m}.$$

□

Lemma 16. *For constants $\epsilon, \delta \ge 0$, a integer $T \ge 1$, a series constants $\epsilon_t > 0$ for $t \in [T]$, then if*

$$\epsilon = \sum_{t=1}^T \frac{(e^{\epsilon_t}-1)\epsilon_t}{(e^{\epsilon_t}+1)} + \sqrt{\sum_{t=1}^T 2\epsilon_t^2 \log\left(\frac{1}{\delta}\right)},$$

we have

$$\sqrt{\sum_{t=1}^T \epsilon_t^2} \ge \frac{\sqrt{2}\epsilon}{2\sqrt{\log(1/\delta)+2\epsilon}}.$$

On the other hand, if

$$\epsilon = \sum_{t=1}^T \frac{(e^{\epsilon_t}-1)\epsilon_t}{(e^{\epsilon_t}+1)} + \sqrt{\sum_{t=1}^T 2\epsilon_t^2 \log\left(e + \frac{\sqrt{\sum_{t=1}^T \epsilon_t^2}}{\delta}\right)},$$

we have

$$\sqrt{\sum_{t=1}^T \epsilon_t^2} \ge \max\left\{\sqrt{\frac{\epsilon}{1+\sqrt{2}/(e\delta)}}, \frac{\sqrt{2}\epsilon}{2\sqrt{\log(e+\epsilon/\sqrt{2}\delta)+2\epsilon}}\right\}.$$

Proof. If $\epsilon = \sum_{t=1}^T \frac{(e^{\epsilon_t}-1)\epsilon_t}{(e^{\epsilon_t}+1)} + \sqrt{\sum_{t=1}^T 2\epsilon_t^2 \log\left(\frac{1}{\delta}\right)}$,

Because $(e^x-1)/(e^x+1) \le x$ for $x \ge 0$, then

$$\epsilon \le \sum_{t=1}^T \epsilon_t^2 + \sqrt{\sum_{t=1}^T 2\epsilon_t^2 \log\left(\frac{1}{\delta}\right)}.$$

Solving the inequality with respect to $\sqrt{\sum_{t=1}^T \epsilon_t^2}$, we get

$$\sqrt{\sum_{t=1}^T \epsilon_t^2} \ge \frac{\sqrt{2}\epsilon}{\sqrt{\log(1/\delta)+2\epsilon}+\sqrt{\log(1/\delta)}} \ge \frac{\sqrt{2}\epsilon}{2\sqrt{\log(1/\delta)+2\epsilon}}.$$

If we have

$$\epsilon = \sum_{t=1}^T \frac{(e^{\epsilon_t}-1)\epsilon_t}{(e^{\epsilon_t}+1)} + \sqrt{\sum_{t=1}^T 2\epsilon_t^2 \log\left(e + \frac{\sqrt{\sum_{t=1}^T \epsilon_t^2}}{\delta}\right)}. \quad (49)$$

Because $(e^x-1)/(e^x+1) \le x$ for $x \ge 0$, then

$$\begin{aligned}\epsilon &\le \sum_{t=1}^T \epsilon_t^2 + \sqrt{\sum_{t=1}^T 2\epsilon_t^2 \log\left(e + \frac{\sqrt{\sum_{t=1}^T \epsilon_t^2}}{\delta}\right)} \\ &\le \sum_{t=1}^T \epsilon_t^2 + \frac{\sqrt{2}\sum_{t=1}^T \epsilon_t^2}{e\delta},\end{aligned}$$

where the second inequality is because $\log(e+x) \le x/e+1$ for $x \ge 0$.

As such,

$$\sqrt{\sum_{t=1}^T \epsilon_t^2} \ge \sqrt{\frac{\epsilon}{1+\sqrt{2}/(e\delta)}}.$$

On the other hand, by (49), it also holds that $\sqrt{2}\sqrt{\sum_{t=1}^T \epsilon_t^2} \le \epsilon$. Then we have

$$\begin{aligned}\epsilon &\le \sum_{t=1}^T \epsilon_t^2 + \sqrt{\sum_{t=1}^T 2\epsilon_t^2 \log\left(e + \frac{\sqrt{\sum_{t=1}^T \epsilon_t^2}}{\delta}\right)} \\ &\le \sum_{t=1}^T \epsilon_t^2 + \sqrt{\sum_{t=1}^T 2\epsilon_t^2 \log\left(e + \frac{\epsilon}{\sqrt{2}\delta}\right)}.\end{aligned}$$

Solving the inequality with respect to $\sqrt{\sum_{t=1}^T \epsilon_t^2}$, we also get

$$\begin{aligned}\sqrt{\sum_{t=1}^T \epsilon_t^2} &\ge \frac{\sqrt{2}\epsilon}{\sqrt{\log(e+\epsilon/\sqrt{2}\delta)+2\epsilon}+\sqrt{\log(e+\epsilon/\sqrt{2}\delta)}} \\ &\ge \frac{\sqrt{2}\epsilon}{2\sqrt{\log(e+\epsilon/\sqrt{2}\delta)+2\epsilon}}.\end{aligned}$$

□

Lemma 17. *For constants $\kappa, \epsilon_0 > 0$, $c_1, c_2 > 0$, $\alpha \in \mathbb{R}$, a integer $T \ge 1$, assuming $\epsilon_t = \epsilon_0 t^\alpha$, $\varepsilon_t = O(\kappa/\epsilon_t)$ for $t \in [T]$, if $\sum_{t=1}^T \epsilon_t \ge c_1$, we have*

$$\sum_{t=1}^T \sqrt{\varepsilon_t} = \begin{cases} O\left(\sqrt{\frac{\kappa T^{\alpha+1}}{c_1(\alpha/2-1)^2(\alpha+1)}}\right), & \alpha > 2; \\ O\left(\sqrt{\frac{\kappa T^3}{c_1(\alpha/2-1)^2(\alpha+1)}}\right), & -1 < \alpha < 2; \\ O\left(\sqrt{\frac{\kappa T^{2-\alpha}}{c_1(\alpha/2-1)^2(-\alpha-1)}}\right), & \alpha < -1, \end{cases}$$

and

$$\sum_{t=1}^T t\sqrt{\varepsilon_t} = \begin{cases} O\left(\sqrt{\frac{\kappa T^{\alpha+1}}{c_1(\alpha/2-2)^2(\alpha+1)}}\right), & \alpha > 4; \\ O\left(\sqrt{\frac{\kappa T^5}{c_1(\alpha/2-2)^2(\alpha+1)}}\right), & -1 < \alpha < 4; \\ O\left(\sqrt{\frac{\kappa T^{4-\alpha}}{c_1(\alpha/2-2)^2(-\alpha-1)}}\right), & \alpha < -1. \end{cases}$$

If $\sqrt{\sum_{t=1}^T \epsilon_t^2} \ge c_2$, we have

$$\sum_{t=1}^T \sqrt{\varepsilon_t} = \begin{cases} O\left(\sqrt{\frac{\kappa T^{\alpha+1/2}}{c_2(\alpha/2-1)^2\sqrt{2\alpha+1}}}\right), & \alpha > 2; \\ O\left(\sqrt{\frac{\kappa T^{5/2}}{c_2(\alpha/2-1)^2\sqrt{2\alpha+1}}}\right), & -1/2 < \alpha < 2; \\ O\left(\sqrt{\frac{\kappa T^{2-\alpha}}{c_2(\alpha/2-1)^2\sqrt{-2\alpha-1}}}\right), & \alpha < -1/2, \end{cases}$$

and

$$\sum_{t=1}^T t\sqrt{\varepsilon_t} = \begin{cases} O\left(\sqrt{\frac{\kappa T^{\alpha+1/2}}{c_2(\alpha/2-2)^2\sqrt{2\alpha+1}}}\right), & \alpha > 4; \\ O\left(\sqrt{\frac{\kappa T^{9/2}}{c_2(\alpha/2-2)^2\sqrt{2\alpha+1}}}\right), & -1/2 < \alpha < 4; \\ O\left(\sqrt{\frac{\kappa T^{4-\alpha}}{c_2(\alpha/2-2)^2\sqrt{-2\alpha-1}}}\right), & \alpha < -1/2. \end{cases}$$

Proof. If $\sum_{t=1}^T \epsilon_t = \sum_{t=1}^T \epsilon_0 t^\alpha \geq c_1$. We have

$$\sum_{t=1}^T \sqrt{\varepsilon_t} = O\left(\sum_{t=1}^T \sqrt{\kappa/\epsilon_t}\right) = O\left(\sum_{t=1}^T \sqrt{\frac{\kappa}{\epsilon_0 t^\alpha}}\right) = O\left(\sum_{t=1}^T t^{-\alpha/2}\sqrt{\frac{\kappa}{\epsilon_0}}\right).$$

Using Lemma 12, we have

$$\sum_{t=1}^T \sqrt{\varepsilon_t} = O\left(\sum_{t=1}^T t^{-\alpha/2}\sqrt{\frac{\kappa}{c_1}\sum_{t=1}^T t^\alpha}\right).$$

Then using Lemma 10, if $\alpha > 2$, i.e., $-\alpha/2 < -1$, we have

$$\sum_{t=1}^T \sqrt{\varepsilon_t} = O\left(\frac{1}{-(-\alpha/2)-1}\sqrt{\frac{\kappa}{c_1}\frac{T^{\alpha+1}}{\alpha+1}}\right) = O\left(\sqrt{\frac{\kappa}{c_1}\frac{T^{\alpha+1}}{(\alpha/2-1)^2(\alpha+1)}}\right).$$

Results under other conditions can be proved similarly. $\square$

Lemma 18. *For constants $\kappa, \epsilon_0 > 0$, $c_1, c_2 > 0$, $Q_0 \in (0,1)$, $Q > 0$, a integer $T \geq 1$, assuming $\epsilon_t = \epsilon_0 Q^{-t}$, $\varepsilon_t = O(\kappa/\epsilon_t)$ for $t \in [T]$, if $\sum_{t=1}^T \epsilon_t \geq c_1$, we have*

$$\sum_{t=1}^T Q_0^{-t}\sqrt{\varepsilon_t} = \begin{cases} O\left(\sqrt{\frac{\kappa Q^{-T}}{c_1(Q_0/\sqrt{Q}-1)^2(1-Q)}}\right), & 0 < Q < Q_0^2; \\ O\left(\sqrt{\frac{\kappa (Q_0^2)^{-T}}{c_1(Q_0/\sqrt{Q}-1)^2(1-Q)}}\right), & Q_0^2 < Q < 1; \\ O\left(\sqrt{\frac{\kappa (Q_0^2/Q)^{-T}}{c_1(Q_0/\sqrt{Q}-1)^2(Q-1)}}\right), & Q > 1, \end{cases}$$

and

$$\sum_{t=1}^T \sqrt{\varepsilon_t Q_0^{-t}} = \begin{cases} O\left(\sqrt{\frac{\kappa Q^{-T}}{c_1(\sqrt{Q_0}/\sqrt{Q}-1)^2(1-Q)}}\right), & 0 < Q < Q_0; \\ O\left(\sqrt{\frac{\kappa Q_0^{-T}}{c_1(\sqrt{Q_0}/\sqrt{Q}-1)^2(1-Q)}}\right), & Q_0 < Q < 1; \\ O\left(\sqrt{\frac{\kappa (Q_0/Q)^{-T}}{c_1(\sqrt{Q_0}/\sqrt{Q}-1)^2(Q-1)}}\right), & Q > 1. \end{cases}$$

If $\sqrt{\sum_{t=1}^T \epsilon_t^2} \geq c_2$, we have

$$\sum_{t=1}^T Q_0^{-t}\sqrt{\varepsilon_t} = \begin{cases} O\left(\sqrt{\frac{\kappa Q^{-T}}{c_2(Q_0/\sqrt{Q}-1)^2\sqrt{1-Q^2}}}\right), & 0 < Q < Q_0^2; \\ O\left(\sqrt{\frac{\kappa (Q_0^2)^{-T}}{c_2(Q_0/\sqrt{Q}-1)^2\sqrt{1-Q^2}}}\right), & Q_0^2 < Q < 1; \\ O\left(\sqrt{\frac{\kappa (Q_0^2/Q)^{-T}}{c_2(Q_0/\sqrt{Q}-1)^2\sqrt{Q^2-1}}}\right), & Q > 1, \end{cases}$$

and

$$\sum_{t=1}^T \sqrt{\varepsilon_t Q_0^{-t}} = \begin{cases} O\left(\sqrt{\frac{\kappa Q^{-T}}{c_2(\sqrt{Q_0}/\sqrt{Q}-1)^2\sqrt{1-Q^2}}}\right), & 0 < Q < Q_0; \\ O\left(\sqrt{\frac{\kappa Q_0^{-T}}{c_2(\sqrt{Q_0}/\sqrt{Q}-1)^2\sqrt{1-Q^2}}}\right), & Q_0 < Q < 1; \\ O\left(\sqrt{\frac{\kappa (Q_0/Q)^{-T}}{c_2(\sqrt{Q_0}/\sqrt{Q}-1)^2\sqrt{Q^2-1}}}\right), & Q > 1. \end{cases}$$

Proof. If $\sum_{t=1}^T \epsilon_t = \sum_{t=1}^T \epsilon_0 Q^{-t} \geq c_1$. We have

$$\sum_{t=1}^T Q_0^{-t}\sqrt{\varepsilon_t} = O\left(\sum_{t=1}^T Q_0^{-t}\sqrt{\kappa/\epsilon_t}\right) = O\left(\sum_{t=1}^T Q_0^{-t}\sqrt{\frac{\kappa}{\epsilon_0 Q^{-t}}}\right) = O\left(\sum_{t=1}^T (Q_0/\sqrt{Q})^{-t}\sqrt{\frac{\kappa}{\epsilon_0}}\right).$$

Using Lemma 12, we have

$$\sum_{t=1}^T \sqrt{\varepsilon_t} = O\left(\sum_{t=1}^T (Q_0/\sqrt{Q})^{-t}\sqrt{\frac{\kappa}{c_1}\sum_{t=1}^T Q^{-t}}\right).$$

Then using Lemma 11, if $Q < Q_0^2 < 1$, i.e., $Q_0/\sqrt{Q} > 1$, we have

$$\sum_{t=1}^T \sqrt{\varepsilon_t} = O\left(\frac{1}{Q_0/\sqrt{Q}-1}\sqrt{\frac{\kappa}{c_1}\frac{Q^{-T}}{1-Q}}\right) = O\left(\sqrt{\frac{\kappa Q^{-T}}{c_1(Q_0/\sqrt{Q}-1)^2(1-Q)}}\right).$$

Results under other conditions can be proved similarly. $\square$

Lemma 19. *For constants $L, c_3, c_4 > 0$, a integer $T \geq 1$, matrices $\widetilde{\mathbf{W}}^{(0)}, \mathbf{W}_* \in \mathcal{W} \subset \mathbb{R}^{d\times m}$, if it holds for a series of positive constants $\{\varepsilon_t\}$ that $\sum_{t=1}^T \sqrt{\varepsilon_t} = O(\sqrt{c_4 T^{c_3}})$, setting $T = \Theta((K^2 Lm/c_4)^{1/c_3})$, we have*

$$\mathcal{E} = \frac{L}{2mT}\left(\|\widetilde{\mathbf{W}}^{(0)} - \mathbf{W}_*\|_F + 2\sum_{t=1}^T \sqrt{\frac{2\varepsilon_t}{L}} + \sqrt{2\sum_{t=1}^T \frac{\varepsilon_t}{L}}\right)^2 = O\left(K^2 L\left[\frac{c_4}{K^2 Lm}\right]^{1/c_3}\right).$$

Proof. First, because $\varepsilon_t > 0$ for $t \in [T]$, we have

$$\sqrt{\sum_{t=1}^T \varepsilon_t} \leq \sum_{t=1}^T \sqrt{\varepsilon_t}.$$

Then combining Lemma 15, it suffices that

$$\mathcal{E} = O\left(\frac{L}{mT}\left[K\sqrt{m} + \frac{1}{\sqrt{L}}\sum_{t=1}^T \sqrt{\varepsilon_t}\right]^2\right) = O\left(\left[K\sqrt{\frac{L}{T}} + \frac{1}{\sqrt{mT}}\sum_{t=1}^T \sqrt{\varepsilon_t}\right]^2\right) = O\left(\left[K\sqrt{\frac{L}{T}} + \frac{1}{\sqrt{mT}}\sqrt{c_4 T^{c_3}}\right]^2\right).$$

Then setting $T = \Theta((K^2 Lm/c_4)^{1/c_3})$, we complete the proof. $\square$

Lemma 20. *For constants $L, c_3, c_4 > 0$, a integer $T \geq 1$, matrices $\widetilde{\mathbf{W}}^{(0)}, \mathbf{W}_* \in \mathcal{W} \subset \mathbb{R}^{d\times m}$, if it holds for a series of positive constants $\{\varepsilon_t\}$ that $\sum_{t=1}^T \sqrt{\varepsilon_t} = O(\sqrt{c_4 T^{c_3}})$, setting $T = \Theta((K^2 Lm/c_4)^{1/c_3})$, we have*

$$\mathcal{E} = \frac{2L}{m(T+1)^2}\left(\|\widetilde{\mathbf{W}}^{(0)} - \mathbf{W}_*\|_F + 2\sum_{t=1}^T t\sqrt{\frac{2\varepsilon_t}{L}} + \sqrt{2\sum_{t=1}^T t^2\frac{\varepsilon_t}{L}}\right)^2 = O\left(K^2 L\left[\frac{c_4}{K^2 Lm}\right]^{2/c_3}\right).$$

Proof. First, because $\varepsilon_t > 0$ for $t \in [T]$, we have

$$\sqrt{\sum_{t=1}^T t^2\varepsilon_t} \leq \sum_{t=1}^T \sqrt{t^2\varepsilon_t} = \sum_{t=1}^T t\sqrt{\varepsilon_t}.$$

Then combining Lemma 15, it suffices that

$$\begin{aligned}\mathcal{E} &= O\bigg(\frac{L}{mT^2}\bigg[K\sqrt{m}+\frac{1}{\sqrt{L}}\sum_{t=1}^{T}t\sqrt{\varepsilon_t}\bigg]^2\bigg)\\ &= O\bigg(\bigg[K\frac{\sqrt{L}}{T}+\frac{1}{\sqrt{m}T}\sum_{t=1}^{T}t\sqrt{\varepsilon_t}\bigg]^2\bigg)\\ &= O\bigg(\bigg[K\frac{\sqrt{L}}{T}+\frac{1}{\sqrt{m}T}\sqrt{c_4T^{c_3}}\bigg]^2\bigg).\end{aligned}$$

Then setting $T=\Theta((K^2Lm/c_4)^{1/c_3})$, we complete the proof. □

Lemma 21. *For constants $L, c_6 > 0$, a constant $c_5 \in (0,1)$, a constant $Q_0 \in (0,1)$, a integer $T \geq 1$, matrices $\widetilde{\mathbf{W}}^{(0)}, \mathbf{W}_* \in \mathcal{W} \subset \mathbb{R}^{d\times m}$, if it holds for a series of positive constants $\{\varepsilon_t\}$ that $\sum_{t=1}^{T}Q_0^{-t}\sqrt{\varepsilon_t} = O(\sqrt{c_6{c_5}^{-T}})$, setting $T=\Theta(\log_{1/c_5}(K^2Lm/c_6))$, we have*

$$\begin{aligned}\mathcal{E} &= \frac{Q_0^T}{\sqrt{m}}\bigg(\|\widetilde{\mathbf{W}}^{(0)}-\mathbf{W}_*\|_F+2\sum_{t=1}^{T}Q_0^{-t}\sqrt{\frac{2\varepsilon_t}{L}}\bigg)\\ &= O\bigg(K\bigg[\frac{c_6}{K^2Lm}\bigg]^{\log_{c_5}Q_0}\bigg).\end{aligned}$$

Proof. Using Lemma 15, it suffices that

$$\begin{aligned}\mathcal{E} &= O\bigg(\frac{Q_0^T}{\sqrt{m}}\bigg[K\sqrt{m}+\frac{1}{\sqrt{L}}\sum_{t=1}^{T}Q_0^{-t}\sqrt{\varepsilon_t}\bigg]\bigg)\\ &= O\bigg(Q_0^T\bigg[K+\frac{1}{\sqrt{mL}}\sum_{t=1}^{T}Q_0^{-t}\sqrt{\varepsilon_t}\bigg]\bigg)\\ &= O\bigg(Q_0^T\bigg[K+\frac{1}{\sqrt{mL}}\sqrt{c_6{c_5}^{-T}}\bigg]\bigg).\end{aligned}$$

Then setting $T=\Theta(\log_{1/c_5}(K^2Lm/c_6))$, we complete the proof. □

Lemma 22. *For constants $L, \mu, c_6 > 0$, a constant $c_5 \in (0,1)$, a constant $Q_0 \in (0,1)$, a integer $T \geq 1$, matrices $\widetilde{\mathbf{W}}^{(0)}, \mathbf{W}_* \in \mathcal{W} \subset \mathbb{R}^{d\times m}$, if it holds for a series of positive constants $\{\varepsilon_t\}$ that $\sum_{t=1}^{T}\sqrt{\varepsilon_t Q_0^{-t}} = O(\sqrt{c_6{c_5}^{-T}})$, setting $T=\Theta((K^2Lm/c_4)^{1/c_3})$, we have*

$$\begin{aligned}\mathcal{E} =&\frac{(Q_0)^T}{m}\bigg(K\sqrt{Lm}+2\sqrt{\frac{L}{\mu}}\sum_{t=1}^{T}\sqrt{\varepsilon_t(Q_0)^{-t}}\\ &+\sqrt{\sum_{t=1}^{T}\varepsilon_t(Q_0)^{-t}}\bigg)^2\\ =&O\bigg(K^2L\bigg[\frac{c_6}{K^2\mu m}\bigg]^{\log_{c_5}Q_0}\bigg).\end{aligned}$$

Proof. First, because $\varepsilon_t > 0$ for $t\in[T]$, we have

$$\sqrt{\sum_{t=1}^{T}\varepsilon_t Q_0^{-t}} \leq \sum_{t=1}^{T}\sqrt{\varepsilon_t Q_0^{-t}}.$$

Then it suffices that

$$\begin{aligned}\mathcal{E} &= O\bigg(\frac{Q_0^T}{m}\bigg[K\sqrt{Lm}+\sqrt{\frac{L}{\mu}}\sum_{t=1}^{T}\sqrt{\varepsilon_t Q_0^{-t}}\bigg]^2\bigg)\\ &= O\bigg(Q_0^T\bigg[K\sqrt{L}+\sqrt{\frac{L}{m\mu}}\sum_{t=1}^{T}\sqrt{\varepsilon_t Q_0^{-t}}\bigg]^2\bigg)\\ &= O\bigg(Q_0^T\bigg[K\sqrt{L}+\sqrt{\frac{L}{m\mu}}\sqrt{c_6{c_5}^{-T}}\bigg]^2\bigg).\end{aligned}$$

Then setting $T=\Theta(\log_{1/c_5}(K^2\mu m/c_6))$, we complete the proof. □

Appendix L
Proof of Results in the Main Text

A. Proof of Lemma 1

Proof. Under the setting of single-task learning, each task is learned independently, and thus, we have for $i=1,\ldots,m$, for any set $\mathcal{S}$,

$$\begin{aligned}&\mathbb{P}(\hat{\mathbf{w}}_{[-i]}\in\mathcal{S}\mid\mathcal{B}=(\mathbf{w}_{[-i]},\mathcal{D}_{[-i]},\mathbf{w}_i,\mathcal{D}_i))\\ &\quad=\mathbb{P}(\hat{\mathbf{w}}_{[-i]}\in\mathcal{S}\mid\mathcal{B}=(\mathbf{w}_{[-i]},\mathcal{D}_{[-i]})).\end{aligned}$$

As such, we have for $i=1,\ldots,m$,

$$\begin{aligned}&\frac{\mathbb{P}(\hat{\mathbf{w}}_{[-i]}\in\mathcal{S}\mid\mathcal{B}=(\mathbf{w}_{[-i]},\mathcal{D}_{[-i]},\mathbf{w}_i,\mathcal{D}_i))}{\mathbb{P}(\hat{\mathbf{w}}_{[-i]}\in\mathcal{S}\mid\mathcal{B}=(\mathbf{w}_{[-i]},\mathcal{D}_{[-i]},\mathbf{w}_i',\mathcal{D}_i'))}\\ =&\frac{\mathbb{P}(\hat{\mathbf{w}}_{[-i]}\in\mathcal{S}\mid\mathcal{B}=(\mathbf{w}_{[-i]},\mathcal{D}_{[-i]}))}{\mathbb{P}(\hat{\mathbf{w}}_{[-i]}\in\mathcal{S}\mid\mathcal{B}=(\mathbf{w}_{[-i]},\mathcal{D}_{[-i]}))}=1\leq e^0.\end{aligned}$$

□

B. Proof of Proposition 1

Proof. First, for Algorithm 2, denoting $\mathbf{\Sigma}_0=\widetilde{\mathbf{\Sigma}}^{(t)}$, the j-th diagonal element of $\mathbf{S}_{\eta\lambda}$ is

$$\begin{aligned}&\max\bigg(0,1-\frac{\eta\lambda}{\sqrt{\sigma_j(\mathbf{\Sigma}_0+\mathbf{E})}}\bigg)\\ \geq&\max\bigg(0,1-\frac{\eta\lambda}{\sqrt{\sigma_j(\mathbf{\Sigma}_0)+\sigma_d(\mathbf{E})}}\bigg),\end{aligned}$$

where $\sigma_d(\mathbf{E})$ is the d-th largest singular value, i.e., the smallest singular value, of $\mathbf{E}$. As such, when $\sigma_d(\mathbf{E})=C\lambda^2$ for sufficiently large $C>0$, $\max\Big(0,1-\frac{\eta\lambda}{\sqrt{\sigma_j(\mathbf{\Sigma}_0)+\sigma_d(\mathbf{E})}}\Big)\to1$.

Then $\hat{\mathbf{w}}_i^{(t-1)}=\mathbf{U}\mathbf{S}_{\eta\lambda}\mathbf{U}^{\mathrm{T}}\mathbf{w}_i^{(t-1)}=\mathbf{U}\mathbf{U}^{\mathrm{T}}\mathbf{w}_i^{(t-1)}=\mathbf{w}_i^{(t-1)}$. Therefore, all the procedures can be decoupled to independently run for each task, thus Algorithm 2 degrades to an STL algorithm with no random perturbation.

Similarly, for Algorithm 3, for all $j\in[m]$, the j-th diagonal element of $\mathbf{S}_{\eta\lambda}$ is

$$\begin{aligned}&\max\bigg(0,1-\frac{\eta\lambda}{\sqrt{|\mathbf{\Sigma}_{jj,0}+\mathbf{E}_{jj}|}}\bigg)\\ =&\max\bigg(0,1-\frac{\eta\lambda}{\sqrt{\mathbf{\Sigma}_{jj,0}+\mathbf{E}_{jj}}}\bigg),\end{aligned}$$

where the equality is because $\mathbf{\Sigma}_0$ is semi-positive definite and $\mathbf{E}$ is positive definite.

As such, when $\min_j\mathbf{E}_{jj}=C\lambda^2$ for sufficiently large $C>0$, $\min_j\Big[\max\Big(0,1-\frac{\eta\lambda}{\sqrt{\mathbf{\Sigma}_{jj,0}+\mathbf{E}_{jj}}}\Big)\Big]\to1$.

Then $\hat{\mathbf{w}}_i^{(t-1)}=\mathbf{S}_{\eta\lambda}\mathbf{w}_i^{(t-1)}=\mathbf{w}_i^{(t-1)}$. Therefore, all the procedures can be decoupled to independently run for each task, thus Algorithm 3 degrades to an STL algorithm with no random perturbation.

□

C. Proof of Theorem 1

Proof. For simplicity, we omit the symbol $\mathcal{B}$ used to denote the input in the conditional events in some equations.

First, we show that for all $t\in[T]$, the mapping $\mathbf{W}^{(t-1)}\to\mathbf{\Sigma}^{(t)}$ is an $(\epsilon_t,0)$-differentially private algorithm.

Case 1. For $\mathbf{\Sigma}^{(t)}=\widetilde{\mathbf{\Sigma}}^{(t)}+\mathbf{E}=\mathbf{W}^{(t-1)}(\mathbf{W}^{(t-1)})^{\mathrm{T}}+\mathbf{E}$, we follow the proof of Theorem 4 of Jiang et al. [7].

For all $i\in[m]$, consider two adjacent parameter matrices $\mathbf{W}^{(t-1)}$ and $(\mathbf{W}')^{(t-1)}$ that differ only in the i-th column such that $\mathbf{W}^{(t-1)} = [\mathbf{w}_1^{(t-1)} \cdots \mathbf{w}_i^{(t-1)} \cdots \mathbf{w}_m^{(t-1)}]$ and $(\mathbf{W}')^{(t-1)} = [\mathbf{w}_1^{(t-1)} \cdots (\mathbf{w}_i')^{(t-1)} \cdots \mathbf{w}_m^{(t-1)}]$. Now, let

$$\begin{aligned}\widetilde{\mathbf{\Sigma}}^{(t)} &= \mathbf{W}^{(t-1)}(\mathbf{W}^{(t-1)})^{\mathrm{T}} = \sum_{j=1}^{m} \mathbf{w}_j^{(t-1)}(\mathbf{w}_j^{(t-1)})^{\mathrm{T}} \\ (\widetilde{\mathbf{\Sigma}}')^{(t)} &= (\mathbf{W}')^{(t-1)}((\mathbf{W}')^{(t-1)})^{\mathrm{T}} \\ &= \sum_{j\in[m],j\neq i} \mathbf{w}_j^{(t-1)}(\mathbf{w}_j^{(t-1)})^{\mathrm{T}} + (\mathbf{w}_i')^{(t-1)}((\mathbf{w}_i')^{(t-1)})^{\mathrm{T}} \\ \Delta &= \widetilde{\mathbf{\Sigma}}^{(t)} - (\widetilde{\mathbf{\Sigma}}')^{(t)} \\ &= \mathbf{w}_i^{(t-1)}(\mathbf{w}_i^{(t-1)})^{\mathrm{T}} - (\mathbf{w}_i')^{(t-1)}((\mathbf{w}_i')^{(t-1)})^{\mathrm{T}}.\end{aligned}$$

Then, we have for the conditional densities

$$\frac{p(\mathbf{\Sigma}^{(t)} \mid \mathbf{W}^{(t-1)})}{p(\mathbf{\Sigma}^{(t)} \mid (\mathbf{W}')^{(t-1)})} = \frac{p(\mathbf{\Sigma}^{(t)} = \mathbf{W}^{(t-1)}(\mathbf{W}^{(t-1)})^{\mathrm{T}} + \mathbf{E}_1)}{p(\mathbf{\Sigma}^{(t)} = (\mathbf{W}')^{(t-1)}((\mathbf{W}')^{(t-1)})^{\mathrm{T}} + \mathbf{E}_2)}.$$

Because $\mathbf{E}_1, \mathbf{E}_2 \sim W_D(\frac{\max_i s_i^{(t-1)}}{2\epsilon_t}\mathbf{I}_D, D+1)$, letting $\mathbf{V} = \frac{\max_j s_j^{(t-1)}}{2\epsilon_t}\mathbf{I}_D$, $\alpha = \frac{\max_j s_j^{(t-1)}}{2\epsilon_t}$,

$$\begin{aligned}&\frac{p(\mathbf{\Sigma}^{(t)} = \mathbf{W}^{(t-1)}(\mathbf{W}^{(t-1)})^{\mathrm{T}} + \mathbf{E}_1)}{p(\mathbf{\Sigma}^{(t)} = (\mathbf{W}')^{(t-1)}((\mathbf{W}')^{(t-1)})^{\mathrm{T}} + \mathbf{E}_2)} \\ &= \frac{\exp[-\mathrm{tr}(\mathbf{V}^{-1}(\mathbf{\Sigma}^{(t)} - \mathbf{W}^{(t-1)}(\mathbf{W}^{(t-1)})^{\mathrm{T}}))/2]}{\exp[-\mathrm{tr}(\mathbf{V}^{-1}(\mathbf{\Sigma}^{(t)} - (\mathbf{W}')^{(t-1)}((\mathbf{W}')^{(t-1)})^{\mathrm{T}}))/2]} \\ &= \exp[\mathrm{tr}(\mathbf{V}^{-1}(\mathbf{\Sigma}^{(t)} - (\mathbf{W}')^{(t-1)}((\mathbf{W}')^{(t-1)})^{\mathrm{T}}))/2 \\ &- \mathrm{tr}(\mathbf{V}^{-1}(\mathbf{\Sigma}^{(t)} - \mathbf{W}^{(t-1)}(\mathbf{W}^{(t-1)})^{\mathrm{T}}))/2] \\ &= \exp[\mathrm{tr}(\mathbf{V}^{-1}\Delta)/2] \\ &= \exp[\mathrm{tr}(\mathbf{w}_i^{(t-1)}(\mathbf{w}_i^{(t-1)})^{\mathrm{T}} - (\mathbf{w}_i')^{(t-1)}((\mathbf{w}_i')^{(t-1)})^{\mathrm{T}})/(2\alpha)] \\ &= \exp[(\mathrm{tr}(\mathbf{w}_i^{(t-1)}(\mathbf{w}_i^{(t-1)})^{\mathrm{T}}) - \mathrm{tr}((\mathbf{w}_i')^{(t-1)}((\mathbf{w}_i')^{(t-1)})^{\mathrm{T}}))/(2\alpha)] \\ &= \exp[(\mathrm{tr}((\mathbf{w}_i^{(t-1)})^{\mathrm{T}}\mathbf{w}_i^{(t-1)}) - \mathrm{tr}(((\mathbf{w}_i')^{(t-1)})^{\mathrm{T}}(\mathbf{w}_i')^{(t-1)}))/(2\alpha)] \\ &= \exp[(\|\mathbf{w}_i^{(t-1)}\|_2^2 - \|(\mathbf{w}_i')^{(t-1)}\|_2^2)/(2\alpha)] \\ &\leq \exp[s_i^{(t-1)}/(2\alpha)] = \exp\left[2\epsilon_t \frac{s_i^{(t-1)}}{2\max_j s_j^{(t-1)}}\right] \leq \exp(\epsilon_t).\end{aligned}$$

As such, we have

$$\frac{p(\mathbf{\Sigma}^{(t)} \mid \mathbf{W}^{(t-1)})}{p(\mathbf{\Sigma}^{(t)} \mid (\mathbf{W}')^{(t-1)})} \leq \exp(\epsilon_t).$$

Case 2. Consider $\mathbf{\Sigma}^{(t)} = \widetilde{\mathbf{\Sigma}}^{(t)} + \mathbf{E} = (\mathbf{W}^{(t-1)})^{\mathrm{T}}\mathbf{W}^{(t-1)} + \mathbf{E}$.

For all $i \in [m]$, consider two adjacent parameter matrices $\mathbf{W}^{(t-1)}$ and $(\mathbf{W}')^{(t-1)}$ that differ only in the i-th column such that $\mathbf{W}^{(t-1)} = [\mathbf{w}_1^{(t-1)} \cdots \mathbf{w}_i^{(t-1)} \cdots \mathbf{w}_m^{(t-1)}]$ and $(\mathbf{W}')^{(t-1)} = [\mathbf{w}_1^{(t-1)} \cdots (\mathbf{w}_i')^{(t-1)} \cdots \mathbf{w}_m^{(t-1)}]$. Let

$$\begin{aligned}\widetilde{\mathbf{\Sigma}}^{(t)} &= (\mathbf{W}^{(t-1)})^{\mathrm{T}}\mathbf{W}^{(t-1)} \\ (\widetilde{\mathbf{\Sigma}}')^{(t)} &= ((\mathbf{W}')^{(t-1)})^{\mathrm{T}}(\mathbf{W}')^{(t-1)} \\ \Delta &= \widetilde{\mathbf{\Sigma}}^{(t)} - (\widetilde{\mathbf{\Sigma}}')^{(t)},\end{aligned}$$

where the i-th diagonal element of Δ is $\|\mathbf{w}_i^{(t-1)}\|_2^2 - \|(\mathbf{w}_i')^{(t-1)}\|_2^2$ and the other diagonal elements of Δ are zeros.

Then, we have

$$\frac{p(\mathbf{\Sigma}^{(t)} \mid \mathbf{W}^{(t-1)})}{p(\mathbf{\Sigma}^{(t)} \mid (\mathbf{W}')^{(t-1)})} = \frac{p(\mathbf{\Sigma}^{(t)} = (\mathbf{W}^{(t-1)})^{\mathrm{T}}\mathbf{W}^{(t-1)} + \mathbf{E}_1)}{p(\mathbf{\Sigma}^{(t)} = ((\mathbf{W}')^{(t-1)})^{\mathrm{T}}(\mathbf{W}')^{(t-1)} + \mathbf{E}_2)}.$$

Because $\mathbf{E}_1, \mathbf{E}_2 \sim W_m(\mathrm{diag}(\mathbf{s}^{(t-1)}/2\epsilon_t), m+1)$, letting $\mathbf{V} = \mathrm{diag}(\mathbf{s}^{(t-1)}/2\epsilon_t)$,

$$\begin{aligned}&\frac{p(\mathbf{\Sigma}^{(t)} = (\mathbf{W}^{(t-1)})^{\mathrm{T}}\mathbf{W}^{(t-1)} + \mathbf{E}_1)}{p(\mathbf{\Sigma}^{(t)} = ((\mathbf{W}')^{(t-1)})^{\mathrm{T}}(\mathbf{W}')^{(t-1)} + \mathbf{E}_2)} \\ &= \frac{\exp[-\mathrm{tr}(\mathbf{V}^{-1}(\mathbf{\Sigma}^{(t)} - (\mathbf{W}^{(t-1)})^{\mathrm{T}}\mathbf{W}^{(t-1)}))/2]}{\exp[-\mathrm{tr}(\mathbf{V}^{-1}(\mathbf{\Sigma}^{(t)} - ((\mathbf{W}')^{(t-1)})^{\mathrm{T}}(\mathbf{W}')^{(t-1)}))/2]} \\ &= \exp[\mathrm{tr}(\mathbf{V}^{-1}(\mathbf{\Sigma}^{(t)} - ((\mathbf{W}')^{(t-1)})^{\mathrm{T}}(\mathbf{W}')^{(t-1)}))/2 \\ &\quad - \mathrm{tr}(\mathbf{V}^{-1}(\mathbf{\Sigma}^{(t)} - (\mathbf{W}^{(t-1)})^{\mathrm{T}}\mathbf{W}^{(t-1)}))/2] \\ &= \exp[\mathrm{tr}(\mathbf{V}^{-1}\Delta)/2] \\ &= \exp[(\|\mathbf{w}_i^{(t-1)}\|_2^2 - \|(\mathbf{w}_i')^{(t-1)}\|_2^2)/(2v_{ii})] \\ &\quad \leq \exp[s_i^{(t-1)}/(2v_{ii})] = \exp\left[2\epsilon_t \frac{s_i^{(t-1)}}{2s_i^{(t-1)}}\right] \leq \exp(\epsilon_t).\end{aligned}$$

As such, we also have

$$\frac{p(\mathbf{\Sigma}^{(t)} \mid \mathbf{W}^{(t-1)})}{p(\mathbf{\Sigma}^{(t)} \mid (\mathbf{W}')^{(t-1)})} \leq \exp(\epsilon_t).$$

Next, given $t \in [T]$, $\mathbf{\Sigma}^{(1:t-1)}$ (when $t=1$, $\mathbf{\Sigma}^{(1:t-1)} = \emptyset$) and the mapping $f : \mathbf{\Sigma}^{(1:t)} \to \mathbf{M}^{(t)}$, which does not touch any unperturbed sensitive information, using the *Post-Processing immunity* Lemma (Lemma 4) for the mapping $f' : \mathbf{\Sigma}^{(1:t)} \to (\mathbf{M}^{(t)}, \mathbf{\Sigma}^{(t)})$, the algorithm $(\mathbf{W}^{(t-1)}, \mathbf{\Sigma}^{(1:t-1)}) \to (\mathbf{M}^{(t)}, \mathbf{\Sigma}^{(t)})$ is still an $(\epsilon_t, 0)$-differentially private algorithm.

Then, because $\hat{\mathbf{w}}_i^{(t)} = \mathcal{A}_{\mathrm{st},i}(\mathbf{M}^{(t)}, \mathbf{w}_i^{(0:t-1)}, \mathbf{X}_i, \mathbf{y}_i)$ is an STL algorithm for the i-th task, for $i = 1, \ldots, m$, the mapping $(\mathbf{M}^{(t)}, \mathbf{w}_{[-i]}^{(0:t-1)}, \mathbf{X}_{[-i]}, \mathbf{y}_{[-i]}) \to (\hat{\mathbf{w}}_{[-i]}^{(t)})$ thus does not touch any unperturbed sensitive information for the i-th task. As such, applying the *Post-Processing immunity* Lemma again for the mapping $f'' : (\mathbf{M}^{(t)}, \mathbf{w}_{[-i]}^{(0:t-1)}, \mathbf{X}_{[-i]}, \mathbf{y}_{[-i]}, \mathbf{\Sigma}^{(1:t-1)}) \to (\hat{\mathbf{w}}_{[-i]}^{(t)}, \mathbf{M}^{(t)}, \mathbf{\Sigma}^{(t)})$, for the algorithm $(\mathbf{W}^{(t-1)}, \mathbf{\Sigma}^{(1:t-1)}, \mathbf{w}_{[-i]}^{(0:t-2)}, \mathbf{X}_{[-i]}, \mathbf{y}_{[-i]}) \to (\hat{\mathbf{w}}_{[-i]}^{(t)}, \mathbf{M}^{(t)}, \mathbf{\Sigma}^{(t)})$ (when $t=1$, $\mathbf{w}_{[-i]}^{(0:t-2)} = \emptyset$), denoting $\vartheta_{t,i} = (\hat{\mathbf{w}}_{[-i]}^{(t)}, \mathbf{M}^{(t)}, \mathbf{\Sigma}^{(t)}) \in \mathcal{C}_{t,i}$, we have for any set $\mathcal{S}_{t,i} \subseteq \mathcal{C}_{t,i}$

$$\begin{aligned}&\mathbb{P}(\vartheta_{t,i} \in \mathcal{S}_{t,i} \mid \mathbf{W}^{(t-1)}, \mathbf{\Sigma}^{(1:t-1)}, \mathbf{w}_{[-i]}^{(0:t-2)}, \mathcal{D}^m) \\ &\leq e^{\epsilon_t}\mathbb{P}(\vartheta_{t,i} \in \mathcal{S}_{t,i} \mid (\mathbf{W}')^{(t-1)}, \mathbf{\Sigma}^{(1:t-1)}, \mathbf{w}_{[-i]}^{(0:t-2)}, (\mathcal{D}')^m),\end{aligned}$$

where $\mathbf{W}^{(t-1)}$ and $(\mathbf{W}')^{(t-1)}$ differ only in the i-th column and $\mathcal{D}^m$ and $(\mathcal{D}')^m$ differ only in the i-th task.

Now, again, for $t = 1, \ldots, T$, we take the t-th dataset $\widetilde{\mathcal{D}}_t = \{(\mathbf{w}_1^{(t-1)}, \mathcal{D}_1), \ldots, (\mathbf{w}_m^{(t-1)}, \mathcal{D}_m)\}$. Given that $\mathbf{W}^{(t)} = \widehat{\mathbf{W}}^{(t)}$ for all $t \in [T]$, we have for any set $\mathcal{S}_{t,i} \subseteq \mathcal{C}_{t,i}$ that

$$\begin{aligned}&\mathbb{P}(\vartheta_{t,i} \in \mathcal{S}_{t,i} \mid \widetilde{\mathcal{D}}_t, \boldsymbol{\vartheta}_{1:t-1}) \\ &\leq e^{\epsilon_t}\mathbb{P}(\vartheta_{t,i} \in \mathcal{S}_{t,i} \mid \widetilde{\mathcal{D}}_t', \boldsymbol{\vartheta}_{1:t-1}),\end{aligned}$$

where $\widetilde{\mathcal{D}}_t$ and $\widetilde{\mathcal{D}}_t'$ are two adjacent datasets that differ in a single entry, the i-th data instance $(\mathbf{w}_i^{(t-1)}, \mathcal{D}_i = (\mathbf{X}_i, \mathbf{y}_i))$, and

$$\boldsymbol{\vartheta}_{1:t-1} = \begin{cases} \emptyset, & t = 1 \\ (\vartheta_{1,1}, \ldots, \vartheta_{1,m}) \ldots, (\vartheta_{t-1,1}, \ldots, \vartheta_{t-1,m}), & t \geq 2. \end{cases}$$

This renders the algorithm in the t-th iteration an $(\epsilon_t, 0)$-differentially private algorithm.

Now, again by the *Adaptive composition* Lemma (Lemma 7), for all $i \in [m]$ and for any set $\mathcal{S}' \subseteq \bigotimes_{t=1}^{T} \mathcal{C}_{t_i}$, we have

$$\begin{aligned}&\mathbb{P}((\vartheta_{1,i}, \cdots, \vartheta_{T,i}) \in \mathcal{S}' \mid \bigcap_{t=1}^{T}(\mathcal{B}_t = (\widetilde{\mathcal{D}}_t, \boldsymbol{\vartheta}_{1:t-1}))) \\ &\leq e^{\tilde{\epsilon}}\mathbb{P}((\vartheta_{1,i}, \cdots, \vartheta_{T,i}) \in \mathcal{S}' \mid \bigcap_{t=1}^{T}(\mathcal{B}_t = (\widetilde{\mathcal{D}}_t', \boldsymbol{\vartheta}_{1:t-1}))) \\ &\quad + \delta,\end{aligned}$$

where for all $t \in [T]$, $\mathcal{B}_t$ denotes the input for the t-th iteration.

Finally, taking $\theta_t = (\vartheta_{t,1}, \dots, \vartheta_{t,m})$ for all $t \in [T]$, we have for any set $\mathcal{S} \subseteq \mathbb{R}^{d\times(m-1)\times T}$

$$\begin{aligned}&\mathbb{P}(\hat{\mathbf{w}}_{[-i]}^{(1:T)} \in \mathcal{S} \mid \bigcap_{t=1}^{T}\mathcal{B}_t = (\mathbf{W}^{(t-1)}, \mathcal{D}^m, \boldsymbol{\theta}_{1:t-1}))\\ \le& e^{\epsilon}\mathbb{P}(\hat{\mathbf{w}}_{[-i]}^{(1:T)} \in \mathcal{S} \mid \bigcap_{t=1}^{T}\mathcal{B}_t = ((\mathbf{W}')^{(t-1)}, (\mathcal{D}')^m, \boldsymbol{\theta}_{1:t-1}))\\ &+\delta,\end{aligned}$$

□

D. Proof of Corollary 1

Proof. For simplicity, we omit the symbol $\mathcal{B}$ used to denote the input in the conditional events in some equations.

Using Theorem 1, we only need to show that Algorithm 2 complies with our MP-MTL framework in Algorithm 1.

Consider the t-th iteration for all $t \in [T]$. Because of the norm clipping, for $i = 1, \dots, m$, the ℓ_2 norm of any parameter vector equals one. Then, we have for $i = 1, \dots, m$ that

$$\begin{aligned}s_i^{(t-1)} =& \max_{(\tilde{\mathbf{w}}_i')^{(t-1)}} |\|\tilde{\mathbf{w}}_i^{(t-1)}\|_2^2 - \|(\tilde{\mathbf{w}}_i')^{(t-1)}\|_2^2|\\ \le& \max_{(\tilde{\mathbf{w}}_i')^{(t-1)}} |\|\tilde{\mathbf{w}}_i^{(t-1)}\|_2^2| + |\|(\tilde{\mathbf{w}}_i')^{(t-1)}\|_2^2| = 2K.\end{aligned}$$

Because the norm clipping is a deterministic STL algorithm and because the mapping $\widetilde{\mathbf{W}}^{(t-1)} \to \boldsymbol{\Sigma}^{(t)}$ is an $(\epsilon_t, 0)$ - differentially algorithm, we have for any set $\mathcal{S} \subseteq \mathbb{R}^{d\times d}$ that

$$\begin{aligned}&\mathbb{P}(\boldsymbol{\Sigma}^{(t)} \in \mathcal{S} \mid \mathbf{w}_{[-i]}^{(t-1)}, \mathbf{w}_i^{(t-1)})\\ &=\mathbb{P}(\boldsymbol{\Sigma}^{(t)} \in \mathcal{S} \mid \tilde{\mathbf{w}}_{[-i]}^{(t-1)}, \tilde{\mathbf{w}}_i^{(t-1)})\\ &\le e^{\epsilon_t}\mathbb{P}(\boldsymbol{\Sigma}^{(t)} \in \mathcal{S} \mid \tilde{\mathbf{w}}_{[-i]}^{(t-1)}, (\tilde{\mathbf{w}}_i')^{(t-1)})\\ &=e^{\epsilon_t}\mathbb{P}(\boldsymbol{\Sigma}^{(t)} \in \mathcal{S} \mid \mathbf{w}_{[-i]}^{(t-1)}, (\mathbf{w}_i')^{(t-1)}),\end{aligned}$$

which renders the mapping $\mathbf{W}^{(t-1)} \to \boldsymbol{\Sigma}^{(t)}$ as an $(\epsilon_t, 0)$ - differentially algorithm as well.

Let $\mathbf{M}^{(t)} = \mathbf{U}\mathbf{S}_{\eta\lambda}\mathbf{U}^{\mathrm{T}}$. As such, the 7-th step to the 9-th step can be treated as the process of first performing a mapping $f : \boldsymbol{\Sigma}^{(1:t)} \to \mathbf{M}^{(t)}$ and then applying an STL algorithm:

$$\hat{\mathbf{w}}_i^{(t)} = \mathbf{U}\mathbf{S}_{\eta\lambda}\mathbf{U}^{\mathrm{T}}\tilde{\mathbf{w}}_i^{(t-1)}, \text{ for all } i \in [m]. \tag{50}$$

Now, because (50), the 10-th step and the 11-th step are all STL algorithms, they can be treated as a complete STL algorithm performing the mapping: $(\mathbf{M}^{(t)}, \mathbf{w}_i^{(0:t-1)}, \mathbf{X}_i, \mathbf{y}_i) \to (\hat{\mathbf{w}}_i^{(t)}, \mathbf{w}_i^{(t)})$.

As such, in all the iterations, Algorithm 2 complies with Algorithm 1. Thus, the result of Theorem 1 can be applied to Algorithm 2.

Similarly, using Theorem 1, we only need to show that Algorithm 3 complies with our MP-MTL framework in Algorithm 1.

The proof for the sensitivity is the same.

Now, let $\mathbf{M}^{(t)} = \mathbf{S}_{\eta\lambda}$. As such, the 7-th step can be treated as a mapping $f : \boldsymbol{\Sigma}^{(1:t)} \to \mathbf{M}^{(t)}$.

Then, because the 8-th step, the 9-th step and the 10-th step are all STL algorithms, they can be treated as a complete STL algorithm performing the mapping: $(\mathbf{M}^{(t)}, \mathbf{w}_i^{(0:t-1)}, \mathbf{X}_i, \mathbf{y}_i) \to (\hat{\mathbf{w}}_i^{(t)}, \mathbf{w}_i^{(t)})$.

Therefore, in all the iterations, Algorithm 3 complies with Algorithm 1, and thus, the result of Theorem 1 can be applied to Algorithm 3.

□

E. Proof of Lemma 2

Proof. We invoke the results of Schmidt et al. [11] to bound the empirical optimization error.

In the t-th step, a standard proximal operator (see Ji and Ye [6]) optimizes the following problem:

$$\min_{\mathbf{W}} \frac{1}{2\eta}\|\mathbf{W} - \mathbf{C}\|_F^2 + \lambda\|\mathbf{W}\|_*,$$

where $\mathbf{C} = \widetilde{\mathbf{W}}^{(t-1)}$. By Theorem 3.1 of Ji and Ye [6], denote the solution of the problem by $\widehat{\mathbf{W}}_0^{(t)} = \mathbf{U}_0\mathbf{S}_{\eta\lambda,0}\mathbf{U}_0^{\mathrm{T}}\mathbf{C}$, where $\mathbf{U}_0\boldsymbol{\Lambda}_0\mathbf{U}_0^{\mathrm{T}} = \mathbf{C}\mathbf{C}^{\mathrm{T}}$ is the SVD decomposition of $\mathbf{C}\mathbf{C}^{\mathrm{T}}$. $\mathbf{S}_{\eta\lambda,0}$ is also a diagonal matrix and $\mathbf{S}_{\eta\lambda,ii,0} = \max\{0, 1 - \eta\lambda/\sqrt{\boldsymbol{\Lambda}_{ii,0}}\}$ for $i = 1, \dots, \min\{d, m\}$.

By Algorithm 2, $\widehat{\mathbf{W}}^{(t)} = \mathbf{U}\mathbf{S}_{\eta\lambda}\mathbf{U}^{\mathrm{T}}\mathbf{C}$.

Then we analyse the bound of $\frac{1}{2\eta}\|\widehat{\mathbf{W}}^{(t)} - \mathbf{C}\|_F^2 + \lambda\|\widehat{\mathbf{W}}^{(t)}\|_* - \{\frac{1}{2\eta}\|\widehat{\mathbf{W}}_0^{(t)} - \mathbf{C}\|_F^2 + \lambda\|\widehat{\mathbf{W}}_0^{(t)}\|_*\}$.

First, we have

$$\begin{aligned}&\|\widehat{\mathbf{W}}^{(t)} - \mathbf{C}\|_F^2 - \|\widehat{\mathbf{W}}_0^{(t)} - \mathbf{C}\|_F^2\\ =&\mathrm{tr}((\widehat{\mathbf{W}}^{(t)} - \mathbf{C})^{\mathrm{T}}(\widehat{\mathbf{W}}^{(t)} - \mathbf{C})) - \mathrm{tr}((\widehat{\mathbf{W}}_0^{(t)} - \mathbf{C})^{\mathrm{T}}(\widehat{\mathbf{W}}_0^{(t)} - \mathbf{C}))\\ =&\mathrm{tr}((\widehat{\mathbf{W}}^{(t)})^{\mathrm{T}}\widehat{\mathbf{W}}^{(t)}) - \mathrm{tr}((\widehat{\mathbf{W}}_0^{(t)})^{\mathrm{T}}\widehat{\mathbf{W}}_0^{(t)}) - 2\mathrm{tr}((\widehat{\mathbf{W}}^{(t)} - \widehat{\mathbf{W}}_0^{(t)})^{\mathrm{T}}\mathbf{C})\\ =&\mathrm{tr}((\widehat{\mathbf{W}}^{(t)} - \widehat{\mathbf{W}}_0^{(t)})^{\mathrm{T}}(\widehat{\mathbf{W}}^{(t)} + \widehat{\mathbf{W}}_0^{(t)})) - 2\mathrm{tr}((\widehat{\mathbf{W}}^{(t)} - \widehat{\mathbf{W}}_0^{(t)})^{\mathrm{T}}\mathbf{C})\\ =&\mathrm{tr}((\widehat{\mathbf{W}}^{(t)} - \mathbf{C})^{\mathrm{T}}(\widehat{\mathbf{W}}^{(t)} - \widehat{\mathbf{W}}_0^{(t)}))\\ &+\mathrm{tr}((\widehat{\mathbf{W}}_0^{(t)} - \mathbf{C})^{\mathrm{T}}(\widehat{\mathbf{W}}^{(t)} - \widehat{\mathbf{W}}_0^{(t)}))\\ \le&\sigma_1(\widehat{\mathbf{W}}^{(t)} - \mathbf{C})\|\widehat{\mathbf{W}}^{(t)} - \widehat{\mathbf{W}}_0^{(t)}\|_*\\ &+\sigma_1(\widehat{\mathbf{W}}_0^{(t)} - \mathbf{C})\|\widehat{\mathbf{W}}^{(t)} - \widehat{\mathbf{W}}_0^{(t)}\|_*,\end{aligned} \tag{51}$$

where $\sigma_1(\cdot)$ denotes the largest singular value of the enclosed matrix.

Denote $\mathbf{T} = \mathbf{U}\mathbf{S}_{\eta\lambda}\mathbf{U}^{\mathrm{T}}, \mathbf{T}_0 = \mathbf{U}_0\mathbf{S}_{\eta\lambda,0}\mathbf{U}_0^{\mathrm{T}}$. Since $\mathbf{U}$ is decomposed from a symmetric matrix, we have

$$\begin{aligned}\sigma_1(\widehat{\mathbf{W}}^{(t)} - \mathbf{C}) &= \sigma_1(\mathbf{T}\mathbf{C} - \mathbf{C}) \le \sigma_1(\mathbf{C})\sigma_1(\mathbf{T} - \mathbf{I})\\ &= \sigma_1(\mathbf{C})\sigma_1(\mathbf{U}\mathbf{S}_{\eta\lambda}\mathbf{U}^{\mathrm{T}} - \mathbf{U}\mathbf{U}^{\mathrm{T}})\\ &= \sigma_1(\mathbf{C})\sigma_1(\mathbf{U}(\mathbf{S}_{\eta\lambda} - \mathbf{I})\mathbf{U}^{\mathrm{T}}).\end{aligned}$$

Since $\mathbf{S}_{\eta\lambda} - \mathbf{I}$ is a diagonal matrix, whose i-th diagonal element is $\max\{0, 1 - \eta\lambda/\sqrt{\boldsymbol{\Lambda}_{ii}}\} - 1 \in [-1, 0)$, so $\sigma_1(\mathbf{U}(\mathbf{S}_{\eta\lambda} - \mathbf{I})\mathbf{U}^{\mathrm{T}}) \le 1$ and

$$\sigma_1(\widehat{\mathbf{W}}^{(t)} - \mathbf{C}) \le \sigma_1(\mathbf{C}). \tag{52}$$

Similarly,

$$\sigma_1(\widehat{\mathbf{W}}_0^{(t)} - \mathbf{C}) \le \sigma_1(\mathbf{C}). \tag{53}$$

On the other hand,

$$\begin{aligned}
&\|\widehat{\mathbf{W}}^{(t)} - \widehat{\mathbf{W}}_0^{(t)}\|_* = \|\mathbf{T}\mathbf{C} - \mathbf{T}_0\mathbf{C}\|_* \\
=&\left\|\sum_{j=1}^{d} \sigma_j(\mathbf{T})\mathbf{u}_j\mathbf{u}_j^{\mathrm{T}}\mathbf{C} - \sum_{j=1}^{d} \sigma_j(\mathbf{T}_0)\mathbf{u}_{j,0}\mathbf{u}_{j,0}^{\mathrm{T}}\mathbf{C}\right\|_* \\
=&\left\|\sum_{j=1}^{d}(\sigma_j(\mathbf{T}_0) + \sigma_j(\mathbf{T}) - \sigma_j(\mathbf{T}_0))\mathbf{u}_j\mathbf{u}_j^{\mathrm{T}}\mathbf{C}\right. \\
&\left.- \sum_{j=1}^{d} \sigma_j(\mathbf{T}_0)\mathbf{u}_{j,0}\mathbf{u}_{j,0}^{\mathrm{T}}\mathbf{C}\right\|_* \\
=&\left\|\sum_{j=1}^{d} \sigma_j(\mathbf{T}_0)(\mathbf{u}_j\mathbf{u}_j^{\mathrm{T}} - \mathbf{u}_{j,0}\mathbf{u}_{j,0}^{\mathrm{T}})\mathbf{C}\right. \\
&\left.+ \sum_{j=1}^{d}(\sigma_j(\mathbf{T}) - \sigma_j(\mathbf{T}_0))\mathbf{u}_j\mathbf{u}_j^{\mathrm{T}}\mathbf{C}\right\|_* \\
\le&\left\|\sum_{j=1}^{d} \sigma_j(\mathbf{T}_0)(\mathbf{u}_j\mathbf{u}_j^{\mathrm{T}} - \mathbf{u}_{j,0}\mathbf{u}_{j,0}^{\mathrm{T}})\mathbf{C}\right\|_* \\
&+ \left\|\sum_{j=1}^{d}(\sigma_j(\mathbf{T}) - \sigma_j(\mathbf{T}_0))\mathbf{u}_j\mathbf{u}_j^{\mathrm{T}}\mathbf{C}\right\|_*,
\end{aligned} \tag{54}$$

where $\mathbf{u}_j$ and $\mathbf{u}_{j,0}$ are the j-th column of $\mathbf{U}$ and $\mathbf{U}_0$, respectively.

Let $r_c = \mathrm{rank}(\mathbf{C}) \le \min\{d, m\}$ be the rank of $\mathbf{C}$. Then we have

$$\begin{aligned}
&\left\|\sum_{j=1}^{d}(\sigma_j(\mathbf{T}) - \sigma_j(\mathbf{T}_0))\mathbf{u}_j\mathbf{u}_j^{\mathrm{T}}\mathbf{C}\right\|_* \\
\le&\sum_{j=1}^{r_c} |\sigma_j(\mathbf{T}) - \sigma_j(\mathbf{T}_0)|\sigma_j(\mathbf{C}) \le \sigma_1(\mathbf{C}) \sum_{j=1}^{r_c} |\sigma_j(\mathbf{T}) - \sigma_j(\mathbf{T}_0)|.
\end{aligned} \tag{55}$$

Denote $\mathbf{\Sigma}_0 = \widetilde{\mathbf{\Sigma}}^{(t)} = \mathbf{C}\mathbf{C}^{\mathrm{T}}$. Then we have for $j \in [r_c]$,

$$\begin{aligned}
&|\sigma_j(\mathbf{T}) - \sigma_j(\mathbf{T}_0)| \\
=&\left|\max\left(0, 1 - \frac{\eta\lambda}{\sqrt{\sigma_j(\mathbf{\Sigma}_0 + \mathbf{E})}}\right) - \max\left(0, 1 - \frac{\eta\lambda}{\sqrt{\sigma_j(\mathbf{\Sigma}_0)}}\right)\right| \\
\le&\left|\max\left(0, 1 - \frac{\eta\lambda}{\sqrt{\sigma_j(\mathbf{\Sigma}_0) + \sigma_1(\mathbf{E})}}\right) - \max\left(0, 1 - \frac{\eta\lambda}{\sqrt{\sigma_j(\mathbf{\Sigma}_0)}}\right)\right|.
\end{aligned}$$

Case 1: $\eta\lambda > \sqrt{\sigma_j(\mathbf{\Sigma}_0)}$. Then

$$\begin{aligned}
&|\sigma_j(\mathbf{T}) - \sigma_j(\mathbf{T}_0)| \\
&= \max\left(0, 1 - \frac{\eta\lambda}{\sqrt{\sigma_j(\mathbf{\Sigma}_0) + \sigma_1(\mathbf{E})}}\right) \le 1 - \frac{\eta\lambda}{\sqrt{\eta^2\lambda^2 + \sigma_1(\mathbf{E})}} \\
&\le 1 - \frac{\eta\lambda}{\eta\lambda + \sqrt{\sigma_1(\mathbf{E})}} = \frac{\sqrt{\sigma_1(\mathbf{E})}}{\eta\lambda + \sqrt{\sigma_1(\mathbf{E})}} \le \frac{\sqrt{\sigma_1(\mathbf{E})}}{\eta\lambda}
\end{aligned}$$

Case 2: $\eta\lambda \le \sqrt{\sigma_j(\mathbf{\Sigma}_0)}$. Then

$$\begin{aligned}
&|\sigma_j(\mathbf{T}) - \sigma_j(\mathbf{T}_0)| \\
=&1 - \frac{\eta\lambda}{\sqrt{\sigma_j(\mathbf{\Sigma}_0) + \sigma_1(\mathbf{E})}} - 1 + \frac{\eta\lambda}{\sqrt{\sigma_j(\mathbf{\Sigma}_0)}} \\
=&\eta\lambda \cdot \frac{\sqrt{\sigma_j(\mathbf{\Sigma}_0) + \sigma_1(\mathbf{E})} - \sqrt{\sigma_j(\mathbf{\Sigma}_0)}}{\sqrt{\sigma_j^2(\mathbf{\Sigma}_0) + \sigma_j(\mathbf{\Sigma}_0)\sigma_1(\mathbf{E})}} \\
=&\frac{\eta\lambda\sigma_1(\mathbf{E})}{[\sqrt{\sigma_j(\mathbf{\Sigma}_0) + \sigma_1(\mathbf{E})} + \sqrt{\sigma_j(\mathbf{\Sigma}_0)}]\sqrt{\sigma_j^2(\mathbf{\Sigma}_0) + \sigma_j(\mathbf{\Sigma}_0)\sigma_1(\mathbf{E})}} \\
\le&\frac{\eta\lambda\sigma_1(\mathbf{E})}{[\sqrt{\eta^2\lambda^2 + 0} + \sqrt{\eta^2\lambda^2}]\sqrt{\eta^4\lambda^4 + 0}} = \frac{\sigma_1(\mathbf{E})}{2\eta^2\lambda^2}.
\end{aligned}$$

Suppose that there exists an index $k \le d$ such that

$$\sigma_k^2(\mathbf{C}) = \sigma_k(\mathbf{\Sigma}_0) > \eta^2\lambda^2, \sigma_{k+1}^2(\mathbf{C}) = \sigma_{k+1}(\mathbf{\Sigma}_0) \le \eta^2\lambda^2,$$

then $\sigma_j(\mathbf{T}_0) > 0$ for $j \le k$, $k \le r_c$, and

$$\begin{aligned}
&\sum_{j=1}^{r_c} |\sigma_j(\mathbf{T}) - \sigma_j(\mathbf{T}_0)| \\
\le&k\frac{\sigma_1(\mathbf{E})}{2\eta^2\lambda^2} + (r_c - k)I(r_c > k)\frac{\sqrt{\sigma_1(\mathbf{E})}}{\eta\lambda}.
\end{aligned} \tag{56}$$

For another part of (54),

$$\begin{aligned}
&\left\|\sum_{j=1}^{d} \sigma_j(\mathbf{T}_0)(\mathbf{u}_j\mathbf{u}_j^{\mathrm{T}} - \mathbf{u}_{j,0}\mathbf{u}_{j,0}^{\mathrm{T}})\mathbf{C}\right\|_* \\
=&\left\|\sum_{j=1}^{k} \sigma_j(\mathbf{T}_0)(\mathbf{u}_j\mathbf{u}_j^{\mathrm{T}} - \mathbf{u}_{j,0}\mathbf{u}_{j,0}^{\mathrm{T}})\mathbf{C}\right\|_* \\
\le&\sigma_1(\mathbf{C})\left\|\sum_{j=1}^{k} \sigma_j(\mathbf{T}_0)(\mathbf{u}_j\mathbf{u}_j^{\mathrm{T}} - \mathbf{u}_{j,0}\mathbf{u}_{j,0}^{\mathrm{T}})\right\|_*.
\end{aligned} \tag{57}$$

Denote $\mathbf{U}_j = \sum_{j'=1}^{j} \mathbf{u}_{j'}\mathbf{u}_{j'}, \mathbf{U}_{j,0} = \sum_{j'=1}^{j} \mathbf{u}_{j',0}\mathbf{u}_{j',0}$ for $j \in [d]$. Let $\mathbf{U}_0 = \mathbf{U}_{0,0} = \mathbf{0}$. Then $\mathbf{u}_j\mathbf{u}_j = \mathbf{U}_j - \mathbf{U}_{j-1}, \mathbf{u}_{j,0}\mathbf{u}_{j,0} = \mathbf{U}_{j,0} - \mathbf{U}_{j-1,0}$ for $j \in [d]$.

Then we have

$$\begin{aligned}
&\left\|\sum_{j=1}^{k} \sigma_j(\mathbf{T}_0)(\mathbf{u}_j\mathbf{u}_j^{\mathrm{T}} - \mathbf{u}_{j,0}\mathbf{u}_{j,0}^{\mathrm{T}})\right\|_* \\
=&\left\|\sum_{j=1}^{k} \sigma_j(\mathbf{T}_0)(\mathbf{U}_j - \mathbf{U}_{j,0} - (\mathbf{U}_{j-1} - \mathbf{U}_{j-1,0}))\right\|_* \\
=&\left\|\sum_{j=1}^{k-1}(\sigma_j(\mathbf{T}_0) - \sigma_{j+1}(\mathbf{T}_0))(\mathbf{U}_j - \mathbf{U}_{j,0})\right. \\
&\left.+ \sigma_k(\mathbf{T}_0)(\mathbf{U}_k - \mathbf{U}_{k,0})\right\|_* \\
\le&\sum_{j=1}^{k-1}(\sigma_j(\mathbf{T}_0) - \sigma_{j+1}(\mathbf{T}_0))\|\mathbf{U}_j - \mathbf{U}_{j,0}\|_* \\
&+ \sigma_k(\mathbf{T}_0)\|\mathbf{U}_k - \mathbf{U}_{k,0}\|_*.
\end{aligned}$$

We assume $2\sigma_1(\mathbf{E}) \le \sigma_j(\mathbf{\Sigma}_0) - \sigma_{j+1}(\mathbf{\Sigma}_0)$ for all $j \in [k]$, and apply the Theorem 6 of Jiang et al. [7]. Then for $j \in [k]$,

$$\begin{aligned}
\|\mathbf{U}_j - \mathbf{U}_{j,0}\|_* \le& \min\{2j, k\}\|\mathbf{U}_j - \mathbf{U}_{j,0}\|_2 \\
\le& \min\{2j, k\}\frac{2\sigma_1(\mathbf{E})}{\sigma_j(\mathbf{\Sigma}_0) - \sigma_{j+1}(\mathbf{\Sigma}_0)}.
\end{aligned}$$

Since $j \in [k-1]$,

$$\begin{aligned}
\sigma_j(\mathbf{T}_0) - \sigma_{j+1}(\mathbf{T}_0) =&1 - \frac{\eta\lambda}{\sqrt{\sigma_j(\mathbf{\Sigma}_0)}} - \left(1 - \frac{\eta\lambda}{\sqrt{\sigma_{j+1}(\mathbf{\Sigma}_0)}}\right) \\
=&\eta\lambda\frac{\sqrt{\sigma_j(\mathbf{\Sigma}_0)} - \sqrt{\sigma_{j+1}(\mathbf{\Sigma}_0)}}{\sqrt{\sigma_j(\mathbf{\Sigma}_0)\sigma_{j+1}(\mathbf{\Sigma}_0)}},
\end{aligned}$$

and

$$\sigma_k(\mathbf{T}_0) =1 - \frac{\eta\lambda}{\sqrt{\sigma_k(\mathbf{\Sigma}_0)}} \le 1 - \frac{\sqrt{\sigma_{k+1}(\mathbf{\Sigma}_0)}}{\sqrt{\sigma_k(\mathbf{\Sigma}_0)}},$$

therefore,

$$
\begin{aligned}
&\left\|\sum_{j=1}^{k}\sigma_j(\mathbf{T}_0)(\mathbf{u}_j\mathbf{u}_j^{\mathrm{T}}-\mathbf{u}_{j,0}\mathbf{u}_{j,0}^{\mathrm{T}})\right\|_*\\
\le&\sum_{j=1}^{k-1}\frac{2\eta\lambda\min\{2j,k\}\sigma_1(\mathbf{E})}{(\sqrt{\sigma_j(\mathbf{\Sigma}_0)}+\sqrt{\sigma_{j+1}(\mathbf{\Sigma}_0)})\sqrt{\sigma_j(\mathbf{\Sigma}_0)\sigma_{j+1}(\mathbf{\Sigma}_0)}}\\
&+\frac{2\eta\lambda k\sigma_1(\mathbf{E})}{(\sqrt{\sigma_k(\mathbf{\Sigma}_0)}+\sqrt{\sigma_{k+1}(\mathbf{\Sigma}_0)})\sqrt{\sigma_k(\mathbf{\Sigma}_0)}} \qquad (58)\\
\le&\sum_{j=1}^{k-1}\frac{2\eta\lambda\min\{2j,k\}\sigma_1(\mathbf{E})}{(\eta\lambda+\eta\lambda)\sqrt{\eta^2\lambda^2\eta^2\lambda^2}}+\frac{2\eta\lambda k\sigma_1(\mathbf{E})}{(\eta\lambda+0)\eta\lambda}\\
\le&\left(\frac{k(k-1)}{\eta^2\lambda^2}+\frac{2k}{\eta\lambda}\right)\sigma_1(\mathbf{E}).
\end{aligned}
$$

Combining (51), (52), (53), (54), (55), (56), (57) and (58), it follows that

$$
\begin{aligned}
&\|\widehat{\mathbf{W}}^{(t)}-\mathbf{C}\|_F^2-\|\widehat{\mathbf{W}}_0^{(t)}-\mathbf{C}\|_F^2\\
\le&2\Bigg[k\frac{\sigma_1(\mathbf{E})}{2\eta^2\lambda^2}+(r_c-k)I(r_c>k)\frac{\sqrt{\sigma_1(\mathbf{E})}}{\eta\lambda} \qquad (59)\\
&+\left(\frac{k(k-1)}{\eta^2\lambda^2}+\frac{2k}{\eta\lambda}\right)\sigma_1(\mathbf{E})\Bigg]\sigma_1^2(\mathbf{C}).
\end{aligned}
$$

On the other hand,

$$\|\widehat{\mathbf{W}}^{(t)}\|_*-\|\widehat{\mathbf{W}}_0^{(t)}\|_*\le\|\widehat{\mathbf{W}}^{(t)}-\widehat{\mathbf{W}}_0^{(t)}\|_*.$$

As such, we have

$$
\begin{aligned}
&\frac{1}{2\eta}\|\widehat{\mathbf{W}}^{(t)}-\mathbf{C}\|_F^2+\lambda\|\widehat{\mathbf{W}}^{(t)}\|_*\\
&-\left\{\min_{\mathbf{W}}\frac{1}{2\eta}\|\mathbf{W}-\mathbf{C}\|_F^2+\lambda\|\mathbf{W}\|_*\right\}\\
=&\frac{1}{2\eta}(\|\widehat{\mathbf{W}}^{(t)}-\mathbf{C}\|_F^2-\|\widehat{\mathbf{W}}_0^{(t)}-\mathbf{C}\|_F^2)\\
&+\lambda(\|\widehat{\mathbf{W}}^{(t)}\|_*-\|\widehat{\mathbf{W}}_0^{(t)}\|_*)\\
\le&\left(\frac{\sigma_1^2(\mathbf{C})}{\eta}+\lambda\sigma_1(\mathbf{C})\right)\Bigg[k\frac{\sigma_1(\mathbf{E})}{2\eta^2\lambda^2}\\
&+(r_c-k)I(r_c>k)\frac{\sqrt{\sigma_1(\mathbf{E})}}{\eta\lambda}+\left(\frac{k(k-1)}{\eta^2\lambda^2}+\frac{2k}{\eta\lambda}\right)\sigma_1(\mathbf{E})\Bigg]\\
=&\frac{1}{\eta}\left(\frac{\sigma_1^2(\mathbf{C})}{\eta\lambda}+\sigma_1(\mathbf{C})\right)\Bigg[k\frac{\sigma_1(\mathbf{E})}{2\eta\lambda}\\
&+\max(0,r_c-k)\sqrt{\sigma_1(\mathbf{E})}+\left(\frac{k(k-1)}{\eta\lambda}+2k\right)\sigma_1(\mathbf{E})\Bigg].
\end{aligned} \qquad (60)
$$

□

F. Proof of Lemma 3

Proof. In the t-th step, a standard proximal operator (see Liu et al. [9]) optimizes the following problem:

$$\min_{\mathbf{W}}\frac{1}{2\eta}\|\mathbf{W}-\mathbf{C}\|_F^2+\lambda\|\mathbf{W}\|_{2,1},$$

where $\mathbf{C}=\widetilde{\mathbf{W}}_i^{(t-1)}$. By Theorem 5 of Liu et al. [9], denote the solution of the problem by $\widehat{\mathbf{W}}_0^{(t)}=\mathbf{S}_{\eta\lambda,0}\mathbf{C}$, $\mathbf{\Lambda}_0$ is a diagonal matrix containing the diagonal elements of $\mathbf{C}\mathbf{C}^{\mathrm{T}}$, $\mathbf{S}_0$ is a diagonal matrix and suffices $\mathbf{S}_{ii,0}=\sqrt{\mathbf{\Lambda}_{ii,0}}$ for $i=1,\ldots,\min\{d,m\}$. $\mathbf{S}_{\eta\lambda,0}$ is also a diagonal matrix and $\mathbf{S}_{\eta\lambda,ii,0}=\max\{0,1-\eta\lambda/\mathbf{S}_{ii,0}\}$ for $i=1,\ldots,\min\{d,m\}$.

By Algorithm 2, $\widehat{\mathbf{W}}^{(t)}=\mathbf{U}\mathbf{S}_{\eta\lambda}\mathbf{U}^{\mathrm{T}}\mathbf{C}$.

Then we analyse the bound of $\frac{1}{2\eta}\|\widehat{\mathbf{W}}^{(t)}-\mathbf{C}\|_F^2+\lambda\|\widehat{\mathbf{W}}^{(t)}\|_{2,1}$ $-\{\frac{1}{2\eta}\|\widehat{\mathbf{W}}_0^{(t)}-\mathbf{C}\|_F^2+\lambda\|\widehat{\mathbf{W}}_0^{(t)}\|_{2,1}\}$.

First, similarly as in (51), we have

$$
\begin{aligned}
&\|\widehat{\mathbf{W}}^{(t)}-\mathbf{C}\|_F^2-\|\widehat{\mathbf{W}}_0^{(t)}-\mathbf{C}\|_F^2\\
=&\mathrm{tr}((\widehat{\mathbf{W}}^{(t)}-\widehat{\mathbf{W}}_0^{(t)})(\widehat{\mathbf{W}}^{(t)}-\mathbf{C})^{\mathrm{T}})\\
&+\mathrm{tr}((\widehat{\mathbf{W}}^{(t)}-\widehat{\mathbf{W}}_0^{(t)})(\widehat{\mathbf{W}}_0^{(t)}-\mathbf{C})^{\mathrm{T}})\\
=&\sum_{j=1}^{d}(\widehat{\mathbf{W}}^{(t)}-\widehat{\mathbf{W}}_0^{(t)})^j((\widehat{\mathbf{W}}^{(t)}-\mathbf{C})^j)^{\mathrm{T}} \qquad (61)\\
&+\sum_{j=1}^{d}(\widehat{\mathbf{W}}^{(t)}-\widehat{\mathbf{W}}_0^{(t)})^j((\widehat{\mathbf{W}}_0^{(t)}-\mathbf{C})^j)^{\mathrm{T}}\\
\le&\|\widehat{\mathbf{W}}^{(t)}-\mathbf{C}\|_{2,1}\|\widehat{\mathbf{W}}^{(t)}-\widehat{\mathbf{W}}_0^{(t)}\|_{2,1}\\
&+\|\widehat{\mathbf{W}}_0^{(t)}-\mathbf{C}\|_{2,1}\|\widehat{\mathbf{W}}^{(t)}-\widehat{\mathbf{W}}_0^{(t)}\|_{2,1},
\end{aligned}
$$

where $(\cdot)^j$ denotes the j-th row vector of the enclosed matrix.

Denote $\mathbf{T}=\mathbf{S}_{\eta\lambda},\mathbf{T}_0=\mathbf{S}_{\eta\lambda,0}$. Denote the indices of non-zero rows of $\mathbf{C}$ by $\mathcal{I}_c=\{j:\mathbf{C}^j\neq\mathbf{0}\}$ and let $r_{c,s}=|\mathcal{I}_c|\le d$.

We have

$$
\begin{aligned}
&\|\widehat{\mathbf{W}}^{(t)}-\mathbf{C}\|_{2,1}=\|(\mathbf{T}-\mathbf{I})\mathbf{C}\|_{2,1}\\
&=\sum_{j=1}^{d}\sqrt{\sum_{i=1}^{m}|(\mathbf{T}-\mathbf{I})^j\mathbf{C}_i|^2}=\sum_{j\in\mathcal{I}_c}\sqrt{\sum_{i=1}^{m}|(\mathbf{T}-\mathbf{I})_{jj}\mathbf{C}_{ij}|^2}\\
&=\sum_{j\in\mathcal{I}_c}\sqrt{\sum_{i=1}^{m}|(\mathbf{T}-\mathbf{I})_{jj}|^2|\mathbf{C}_{ij}|^2}=\sum_{j\in\mathcal{I}_c}|(\mathbf{T}-\mathbf{I})_{jj}|\|\mathbf{C}^j\|_2.
\end{aligned}
$$

Since $\mathbf{S}_{\eta\lambda}-\mathbf{I}$ is a diagonal matrix, whose i-th diagonal element is $\max\{0,1-\eta\lambda/\mathbf{S}_{ii}\}-1\in[-1,0)$, so

$$\|\widehat{\mathbf{W}}^{(t)}-\mathbf{C}\|_{2,1}\le\sum_{j\in\mathcal{I}_c}\|\mathbf{C}^j\|_2\le r_{c,s}\max_{j\in[d]}\|\mathbf{C}^j\|_2. \qquad (62)$$

Similarly,

$$\|\widehat{\mathbf{W}}_0^{(t)}-\mathbf{C}\|_{2,1}\le\sum_{j\in\mathcal{I}_c}\|\mathbf{C}^j\|_2\le r_{c,s}\max_{j\in[d]}\|\mathbf{C}^j\|_2. \qquad (63)$$

On the other hand,

$$
\begin{aligned}
&\|\widehat{\mathbf{W}}^{(t)}-\widehat{\mathbf{W}}_0^{(t)}\|_{2,1}=\|\mathbf{S}_{\eta\lambda}\mathbf{C}-\mathbf{S}_{\eta\lambda,0}\mathbf{C}\|_{2,1}\\
=&\sum_{j\in\mathcal{I}_c}|\mathbf{S}_{\eta\lambda,jj}-\mathbf{S}_{\eta\lambda,jj,0}|\|\mathbf{C}^j\|_2 \qquad (64)\\
\le&\max_{j'\in[d]}\|\mathbf{C}^{j'}\|_2\sum_{j\in\mathcal{I}_c}|\mathbf{S}_{\eta\lambda,jj}-\mathbf{S}_{\eta\lambda,jj,0}|.
\end{aligned}
$$

Denote $\mathbf{\Sigma}_0=\widetilde{\mathbf{\Sigma}}^{(t)}=\mathbf{C}\mathbf{C}^{\mathrm{T}}$. Then we have for $j\in\mathcal{I}_c$,

$$
\begin{aligned}
&|\mathbf{S}_{\eta\lambda,jj}-\mathbf{S}_{\eta\lambda,jj,0}|\\
=&\left|\max\left(0,1-\frac{\eta\lambda}{\sqrt{|\mathbf{\Sigma}_{jj,0}+\mathbf{E}_{jj}|}}\right)-\max\left(0,1-\frac{\eta\lambda}{\sqrt{|\mathbf{\Sigma}_{jj,0}|}}\right)\right|.
\end{aligned}
$$

Case 1: $\eta\lambda>\sqrt{\mathbf{\Sigma}_{jj,0}}$. Then

$$
\begin{aligned}
&|\mathbf{S}_{\eta\lambda,jj}-\mathbf{S}_{\eta\lambda,jj,0}|\\
&=\max\left(0,1-\frac{\eta\lambda}{\sqrt{|\mathbf{\Sigma}_{jj,0}+\mathbf{E}_{jj}|}}\right)\le1-\frac{\eta\lambda}{\sqrt{\eta^2\lambda^2+|\mathbf{E}_{jj}|}}\\
&\le1-\frac{\eta\lambda}{\eta\lambda+\sqrt{|\mathbf{E}_{jj}|}}=\frac{\sqrt{|\mathbf{E}_{jj}|}}{\eta\lambda+\sqrt{|\mathbf{E}_{jj}|}}\le\frac{\sqrt{|\mathbf{E}_{jj}|}}{\eta\lambda}
\end{aligned}
$$

Case 2: $\eta\lambda \leq \sqrt{\mathbf{\Sigma}_{jj,0}}$. Then

$$\begin{aligned}&|\mathbf{S}_{\eta\lambda,jj} - \mathbf{S}_{\eta\lambda,jj,0}| \\ \leq&1 - \frac{\eta\lambda}{\sqrt{|\mathbf{\Sigma}_{jj,0}| + |\mathbf{E}_{jj}|}} - 1 + \frac{\eta\lambda}{\sqrt{|\mathbf{\Sigma}_{jj,0}|}} \\ =&\eta\lambda \cdot \frac{\sqrt{|\mathbf{\Sigma}_{jj,0}| + |\mathbf{E}_{jj}|} - \sqrt{|\mathbf{\Sigma}_{jj,0}|}}{\sqrt{|\mathbf{\Sigma}_{jj,0}|(|\mathbf{\Sigma}_{jj,0}| + |\mathbf{E}_{jj}|)}} \\ =&\frac{\eta\lambda|\mathbf{E}_{jj}|}{[\sqrt{|\mathbf{\Sigma}_{jj,0}| + |\mathbf{E}_{jj}|} + \sqrt{|\mathbf{\Sigma}_{jj,0}|}]\sqrt{|\mathbf{\Sigma}_{jj,0}|^2 + |\mathbf{E}_{jj}||\mathbf{\Sigma}_{jj,0}|}} \\ \leq&\frac{\eta\lambda|\mathbf{E}_{jj}|}{[\sqrt{\eta^2\lambda^2 + 0} + \sqrt{\eta^2\lambda^2}]\sqrt{\eta^4\lambda^4 + 0}} = \frac{|\mathbf{E}_{jj}|}{2\eta^2\lambda^2}.\end{aligned}$$

Suppose that there exists an integer $k \leq d$ such that

$$\sum_{j=1}^{d} I(\sqrt{\mathbf{\Sigma}_{jj,0}} \geq \eta\lambda) = k$$

then $k \leq r_{c,s}$ and

$$\begin{aligned}&\sum_{j\in\mathcal{I}_c} |\mathbf{S}_{\eta\lambda,jj} - \mathbf{S}_{\eta\lambda,jj,0}| \\ \leq&\frac{k}{2\eta^2\lambda^2} \max_{j:\eta^2\lambda^2\leq\mathbf{\Sigma}_{jj,0}} \mathbf{E}_{jj} \\ &+ \frac{(r_{c,s}-k)I(r_{c,s}>k)}{\eta\lambda} \max_{j:\eta^2\lambda^2>\mathbf{\Sigma}_{jj,0}} \sqrt{\mathbf{E}_{jj}}.\end{aligned} \tag{65}$$

Combining (61), (62), (63), (64) and (65), it follows that

$$\begin{aligned}&\|\widehat{\mathbf{W}}^{(t)} - \mathbf{C}\|_F^2 - \|\widehat{\mathbf{W}}_0^{(t)} - \mathbf{C}\|_F^2 \\ \leq&2r_{c,s}\left(\max_{j\in[d]}\|\mathbf{C}^j\|_2\right)^2\left(\frac{k}{2\eta^2\lambda^2}\max_{j:\eta^2\lambda^2\leq\mathbf{\Sigma}_{jj,0}}|\mathbf{E}_{jj}|\right. \\ &\left.+ \frac{(r_{c,s}-k)I(r_{c,s}>k)}{\eta\lambda}\max_{j:\eta^2\lambda^2>\mathbf{\Sigma}_{jj,0}}\sqrt{|\mathbf{E}_{jj}|}\right).\end{aligned} \tag{66}$$

On the other hand,

$$\|\widehat{\mathbf{W}}^{(t)}\|_{2,1} - \|\widehat{\mathbf{W}}_0^{(t)}\|_{2,1} \leq \|\widehat{\mathbf{W}}^{(t)} - \widehat{\mathbf{W}}_0^{(t)}\|_{2,1}.$$

As such, we have

$$\begin{aligned}&\frac{1}{2\eta}\|\widehat{\mathbf{W}}^{(t)} - \mathbf{C}\|_F^2 + \lambda\|\widehat{\mathbf{W}}^{(t)}\|_{2,1} \\ &- \left\{\min_{\mathbf{W}} \frac{1}{2\eta}\|\mathbf{W} - \mathbf{C}\|_F^2 + \lambda\|\mathbf{W}\|_{2,1}\right\} \\ =&\frac{1}{2\eta}(\|\widehat{\mathbf{W}}^{(t)} - \mathbf{C}\|_F^2 - \|\widehat{\mathbf{W}}_0^{(t)} - \mathbf{C}\|_F^2) \\ &+ \lambda(\|\widehat{\mathbf{W}}^{(t)}\|_{2,1} - \|\widehat{\mathbf{W}}_0^{(t)}\|_{2,1}) \\ \leq&\left[\frac{r_{c,s}}{\eta}\left(\max_{j\in[d]}\|\mathbf{C}^j\|_2\right)^2 + \lambda\left(\max_{j\in[d]}\|\mathbf{C}^j\|_2\right)\right] \\ &\cdot\left[\frac{k}{2\eta^2\lambda^2}\max_{j:\eta^2\lambda^2\leq\mathbf{\Sigma}_{jj,0}}|\mathbf{E}_{jj}|\right. \\ &\left.+ \frac{(r_{c,s}-k)I(r_{c,s}>k)}{\eta\lambda}\max_{j:\eta^2\lambda^2>\mathbf{\Sigma}_{jj,0}}\sqrt{|\mathbf{E}_{jj}|}\right] \\ =&\frac{1}{\eta}\left[\frac{r_{c,s}}{\eta\lambda}\left(\max_{j\in[d]}\|\mathbf{C}^j\|_2\right)^2 + \left(\max_{j\in[d]}\|\mathbf{C}^j\|_2\right)\right] \\ &\cdot\left[\frac{k}{2\eta\lambda}\max_{j:\eta^2\lambda^2\leq\mathbf{\Sigma}_{jj,0}}|\mathbf{E}_{jj}|\right. \\ &\left.+ \max(0, r_{c,s}-k)\max_{j:\eta^2\lambda^2>\mathbf{\Sigma}_{jj,0}}\sqrt{|\mathbf{E}_{jj}|}\right].\end{aligned} \tag{67}$$

□

G. Proof of Theorem 2

Proof. First, consider the case with no acceleration. We first use Proposition 1 of Schmidt et al. [11] by regarding procedures from Step 5 to Step 9 as approximation for the proximal operator in (8). Note that the norm clipping only bounds the parameter space and does not affect the results of Schmidt et al. [11]. Then for ε_t defined in Lemma 13 for $t \in [T]$, we have

$$\begin{aligned}\mathcal{E} =&\frac{2L}{m(T+1)^2}\Bigg(\|\widetilde{\mathbf{W}}^{(0)} - \mathbf{W}_*\|_F \\ &+ 2\sum_{t=1}^{T} t\sqrt{\frac{2\varepsilon_t}{L}} + \sqrt{2\sum_{t=1}^{T} t^2\frac{\varepsilon_t}{L}}\Bigg)^2.\end{aligned}$$

Meanwhile, by Lemma 13, we have

$$\varepsilon_t = O\left(\frac{\kappa}{\epsilon_t}\right),$$

where $\kappa = \frac{K^2\sqrt{m}kd\log d}{\eta}$

On the other hand, because

$$\epsilon = \sum_{t=1}^{T} \frac{(e^{\epsilon_t}-1)\epsilon_t}{(e^{\epsilon_t}+1)} + \sqrt{\sum_{t=1}^{T} 2\epsilon_t^2 \log\left(e + \frac{\sqrt{\sum_{t=1}^{T}\epsilon_t^2}}{\delta}\right)},$$

then by Lemma 16, we have

$$\sqrt{\sum_{t=1}^{T}\epsilon_t^2} \geq \frac{\sqrt{2}\epsilon}{2\sqrt{\log(e+\epsilon/\sqrt{2}\delta) + 2\epsilon}} = c_2.$$

Then by Lemma 17, we have

$$\sum_{t=1}^{T}\sqrt{\varepsilon_t} = \begin{cases} O\left(\sqrt{\frac{\kappa T^{\alpha+1/2}}{c_2(\alpha/2-1)^2\sqrt{2\alpha+1}}}\right), & \alpha > 2; \\ O\left(\sqrt{\frac{\kappa T^{5/2}}{c_2(\alpha/2-1)^2\sqrt{2\alpha+1}}}\right), & -1/2 < \alpha < 2; \\ O\left(\sqrt{\frac{\kappa T^{2-\alpha}}{c_2(\alpha/2-1)^2\sqrt{-2\alpha-1}}}\right), & \alpha < -1/2. \end{cases}$$

Because $\widetilde{\mathbf{W}}^{(0)}$ is the result of the norm clipping, we have $\widetilde{\mathbf{W}}^{(0)} \in \mathcal{W}$.

Finally, taking $c_3 = \phi(\alpha)$ defined in (13) and $c_4 = \frac{\kappa}{c_2(\alpha/2-1)^2\sqrt{|2\alpha+1|}}$, under the assumption that $\mathbf{W}_* \in \mathcal{W}$, using Lemma 19, we have the results for the case with no acceleration.

For the accelerated case, we use Proposition 2 of Schmidt et al. [11] to have

$$\begin{aligned}\mathcal{E} =&\frac{2L}{m(T+1)^2}\Bigg(\|\widetilde{\mathbf{W}}^{(0)} - \mathbf{W}_*\|_F \\ &+ 2\sum_{t=1}^{T} t\sqrt{\frac{2\varepsilon_t}{L}} + \sqrt{2\sum_{t=1}^{T} t^2\frac{\varepsilon_t}{L}}\Bigg)^2.\end{aligned}$$

Then one can prove similarly combining Lemma 13, Lemma 16, Lemma 17 and Lemma 20.

□

H. Proof of Theorem 3

Proof. First, consider the case with no acceleration. We use Proposition 1 of Schmidt et al. [11] and prove similarly as in Appendix L-G, combining Lemma 14, Lemma 16, Lemma 17 and Lemma 19.

For the accelerated case, we use Proposition 2 of Schmidt et al. [11] and prove similarly as in Appendix L-G, combining Lemma 14, Lemma 16, Lemma 17 and Lemma 20. □

I. Proof of Theorem 4

Proof. First, consider the case with no acceleration. We use Proposition 3 of Schmidt et al. [11] to have

$$\mathcal{E} = \frac{Q_0^T}{\sqrt{m}}\Big(\|\widetilde{\mathbf{W}}^{(0)} - \mathbf{W}_*\|_F + 2\sum_{t=1}^{T} Q_0^{-t}\sqrt{\frac{2\varepsilon_t}{L}}\Big).$$

Then one can prove similarly as in Appendix L-G, combining Lemma 13, Lemma 16, Lemma 17 and Lemma 21.

For the accelerated case, we use Proposition 4 of Schmidt et al. [11] to have

$$\mathcal{E} =\frac{(Q_0)^T}{m}\Bigg(\sqrt{2(f(\widehat{\mathbf{W}}^{(0)}) - f(\mathbf{W}_*))} + 2\sqrt{\frac{L}{\mu}}\sum_{t=1}^{T}\sqrt{\varepsilon_t (Q_0)^{-t}} + \sqrt{\sum_{t=1}^{T}\varepsilon_t (Q_0)^{-t}}\Bigg)^2.$$

Then one can prove similarly as in Appendix L-G, using the assumption that $f(\widetilde{\mathbf{W}}^{(0)}) - f(\mathbf{W}_*) = O(K^2 Lm)$, combining Lemma 13, Lemma 16, Lemma 17 and Lemma 22. □

J. Proof of Theorem 5

Proof. First, consider the case with no acceleration. We use Proposition 3 of Schmidt et al. [11] and prove similarly as in Appendix L-I, combining Lemma 14, Lemma 16, Lemma 17 and Lemma 21.

For the accelerated case, we use Proposition 4 of Schmidt et al. [11] and prove similarly as in Appendix L-I, using the assumption that $f(\widetilde{\mathbf{W}}^{(0)}) - f(\mathbf{W}_*) = O(K^2 Lm)$, combining Lemma 14, Lemma 16, Lemma 17 and Lemma 22. □

K. Proof of Theorem 6

Proof. Consider the bound in (12), whose logarithm is

$$\phi(\alpha)\log\Big(\frac{kd\log d\sqrt{\log(e+\epsilon/\sqrt{2}\delta)+2\epsilon}}{\sqrt{m}\epsilon}\Big) - \phi(\alpha)\log((\alpha/2-1)^2\sqrt{|2\alpha+1|}) + \log(K^2 L)$$

By Assumption 1, the first term dominates. Then we should firstly maximize $\phi(\alpha)$, which results in that $\phi(\alpha) = 2/5$ and $-1/2 < \alpha < 2$. Then since $\phi(\alpha)$ is now fixed, we maximize $(\alpha/2-1)^2\sqrt{|2\alpha+1|}$, which results in $\alpha = 0$. Results under other settings can be proved similarly. □

L. Proof of Proposition 2

Proof. First, consider the method of Pathak et al. [10].

By Definition 7, an (ϵ,δ)-Iterative DP-MTL algorithm with $T = 1$ should suffice for any set $\mathcal{S} \subseteq \mathbb{R}^{d\times(m-1)}$ and all $i \in [m]$ that

$$\begin{aligned}&\mathbb{P}(\hat{\mathbf{w}}_{[-i]}^{(1)} \in \mathcal{S} \mid \mathbf{W}^{(0)}, \mathcal{D}^m)\\ \le& e^{\epsilon}\mathbb{P}(\hat{\mathbf{w}}_{[-i]}^{(1)} \in \mathcal{S} \mid (\mathbf{W}')^{(0)}, (\mathcal{D}')^m) + \delta.\end{aligned}$$

On the other hand, for the ϵ given in the method of Pathak et al. [10], using Theorem 4.1 of Pathak et al. [10], taking $D = \mathcal{D}^m$ and $D' = (\mathcal{D}')^m$, we have for any set $\mathcal{S} \subseteq \mathbb{R}^d$,

$$\mathbb{P}(\hat{\mathbf{w}}^s \in \mathcal{S} \mid \mathcal{D}^m) \le e^{\epsilon}\mathbb{P}(\hat{\mathbf{w}}^s \in \mathcal{S} \mid (\mathcal{D}')^m),$$

where $\hat{\mathbf{w}}^s$ is defined in Section 3.2 of Pathak et al. [10].

Because the method of Pathak et al. [10] uses $\hat{\mathbf{w}}^s$ for all the tasks, then we have $\hat{\mathbf{w}}_i^{(1)} = \hat{\mathbf{w}}^s$ for all $i \in [m]$.

As such, denote $\mathbf{W}^{(0)}$ and $(\mathbf{W}')^{(0)}$ as the collections of models independently learned using $\mathcal{D}^m$ and $(\mathcal{D}')^m$, respectively. Then $\mathcal{D}^m$ and $(\mathcal{D}')^m$ contain all the information of $\mathbf{W}^{(0)}$ and $(\mathbf{W}')^{(0)}$, respectively. As such, we have for any set $\mathcal{S} \subseteq \mathbb{R}^{d\times(m-1)}$, all $i \in [m]$ and $\delta = 0$ that

$$\begin{aligned}&\mathbb{P}(\hat{\mathbf{w}}_{[-i]}^{(1)} \in \mathcal{S} \mid \mathbf{W}^{(0)}, \mathcal{D}^m)\\ \le& e^{\epsilon}\mathbb{P}(\hat{\mathbf{w}}_{[-i]}^{(1)} \in \mathcal{S} \mid (\mathbf{W}')^{(0)}, (\mathcal{D}')^m) + \delta,\end{aligned}$$

which shows that the method of Pathak et al. [10] is an (ϵ,δ)-Iterative DP-MTL algorithm with $T = 1$ and $\delta = 0$.

Then we consider the method of Gupta et al. [5]. Assume a constant $\delta \ge 0$ and the number of iteration T is given.

Taking $T_0 = m, t = i$ for $i \in [m]$, $\beta_t = \hat{\mathbf{w}}_i$, $\mathscr{D} = \mathcal{D}^m$, for the ϵ given in the method of Gupta et al. [5], using Theorem 1 of Gupta et al. [5], for $t \in [T]$, we have *in the t-th each iteration*, for any set $\mathcal{S} \subseteq \mathbb{R}^{d\times m}$ and all $i \in [m]$,

$$\mathbb{P}(\hat{\mathbf{W}}^{(t)} \in \mathcal{S} \mid \mathcal{D}^m) \le e^{\epsilon}\mathbb{P}(\hat{\mathbf{W}}^{(t)} \in \mathcal{S} \mid (\mathcal{D}')^m),$$

which suggests that for any set $\mathcal{S} \subseteq \mathbb{R}^{d\times(m-1)}$ and all $i \in [m]$,

$$\mathbb{P}(\hat{\mathbf{w}}_{[-i]}^{(t)} \in \mathcal{S} \mid \mathcal{D}^m) \le e^{\epsilon}\mathbb{P}(\hat{\mathbf{w}}_{[-i]}^{(t)} \in \mathcal{S} \mid (\mathcal{D}')^m).$$

Then for all $i \in [m]$ and for all $t \in [T]$, take the t-th output $\theta_{t,i} = \hat{\mathbf{w}}_{[-i]}^{(t)}$ and $\delta_t = 0$.

Therefore by the *Adaptive composition* Lemma (Lemma 7), for all $i \in [m]$ and for any set $\mathcal{S} \subset \mathbb{R}^{d\times(m-1)\times T}$,

$$\begin{aligned}&\mathbb{P}((\theta_{1,i},\ldots,\theta_{T,i}) \in \mathcal{S} \mid \bigcap_{t=1}^{T}(\mathcal{B}_t = (\mathcal{D}^m, \boldsymbol{\theta}_{1:t-1})))\\ \le& e^{\tilde{\epsilon}}\mathbb{P}((\theta_{1,i},\cdots,\theta_{T,i}) \in \mathcal{S} \mid \bigcap_{t=1}^{T}(\mathcal{B}_t = ((\mathcal{D}')^m, \boldsymbol{\theta}_{1:t-1})))\\ &+\delta,\end{aligned}$$

where for all $t \in [T]$, $\mathcal{B}_t$ denotes the input for the t-th iteration,

$$\boldsymbol{\theta}_{1:t-1} = \begin{cases}\emptyset, & t=1\\ (\theta_{1,1},\ldots,\theta_{1,m})\ldots,(\theta_{t-1,1},\ldots,\theta_{t-1,m}), & t\ge 2,\end{cases}$$

and $\tilde{\epsilon}$ is defined as follows.

$$\begin{aligned}\tilde{\epsilon} = \min\Bigg\{&\sum_{t=1}^{T}\epsilon, \sum_{t=1}^{T}\frac{(e^{\epsilon}-1)\epsilon}{(e^{\epsilon}+1)} + \sqrt{\sum_{t=1}^{T}2\epsilon^2\log\Big(\frac{1}{\delta}\Big)},\\ &\sum_{t=1}^{T}\frac{(e^{\epsilon}-1)\epsilon}{(e^{\epsilon}+1)} + \sqrt{\sum_{t=1}^{T}2\epsilon^2\log\Big(e+\frac{\sqrt{\sum_{t=1}^{T}\epsilon^2}}{\delta}\Big)}\Bigg\}.\end{aligned}$$

As such, in each t-th iteration, denote $\mathbf{W}^{(t-1)}$ and $(\mathbf{W}')^{(t-1)}$ as the collections of models independently learned using $(\mathcal{D}^m, \boldsymbol{\theta}_{1:t-1})$ and $((\mathcal{D}')^m, \boldsymbol{\theta}_{1:t-1})$, respectively. Then $(\mathcal{D}^m, \boldsymbol{\theta}_{1:t-1})$ and $(\mathcal{D}')^m, \boldsymbol{\theta}_{1:t-1})$ contain all the information of $\mathbf{W}^{(t-1)}$ and $(\mathbf{W}')^{(t-1)}$, respectively.

Therefore, we have for any set $\mathcal{S} \subset \mathbb{R}^{d\times(m-1)\times T}$,

$$\begin{aligned}&\mathbb{P}(\mathbf{w}_{[-i]}^{(1:T)} \in \mathcal{S} \mid \bigcap_{t=1}^{T}(\mathcal{B}_t = (\mathbf{W}^{(t-1)}, \mathcal{D}^m, \boldsymbol{\theta}_{1:t-1})))\\ \le& e^{\tilde{\epsilon}}\mathbb{P}(\mathbf{w}_{[-i]}^{(1:T)} \in \mathcal{S} \mid \bigcap_{t=1}^{T}(\mathcal{B}_t = ((\mathbf{W}')^{(t-1)}, (\mathcal{D}')^m, \boldsymbol{\theta}_{1:t-1})))\\ &+\delta,\end{aligned}$$

which shows that by Definition 7, the method of Gupta et al. [5] is an $(\tilde{\epsilon},\delta)$-Iterative DP-MTL algorithm. □

M. Proof of Proposition 3

Proof. Given an (ϵ, δ) - iterative DP-MTL algorithm $\mathcal{A}(\mathcal{B})$, by Definition 7, we have for any set $\mathcal{S} \subseteq \mathbb{R}^{d\times(m-1)\times T}$ that

$$
\begin{aligned}
&\mathbb{P}(\hat{\mathbf{w}}_{[-i]}^{(1:T)} \in \mathcal{S} \mid \bigcap_{t=1}^{T} \mathcal{B}_t = (\mathbf{W}^{(t-1)}, \mathcal{D}^m, \boldsymbol{\theta}_{1:t-1}))\\
&\leq \exp(\epsilon)\mathbb{P}(\hat{\mathbf{w}}_{[-i]}^{(1:T)} \in \mathcal{S} \mid \bigcap_{t=1}^{T} \mathcal{B}_t = ((\mathbf{W}')^{(t-1)}, (\mathcal{D}')^m, \boldsymbol{\theta}_{1:t-1}))\\
&\quad + \delta.
\end{aligned}
$$

Furthermore, following the proof of the *Group privacy* Lemma (Lemma 5), shown by Vadhan [12], for protecting the entire dataset, n data instances, of the i-th task, we construct a series of datasets, $\mathcal{D}_{(0)}^m, \mathcal{D}_{(1)}^m, \ldots, \mathcal{D}_{(n)}^m$, and let $\mathcal{D}_{(0)}^m = \mathcal{D}^m, \mathcal{D}_{(n)}^m = (\mathcal{D}')^m$ such that for $j = 0, \ldots, n-1$, $\mathcal{D}_{(j)}^m$ and $\mathcal{D}_{(j+1)}^m$ are two neighboring datasets that differ in one data instance. Let a series of model matrices, $\mathbf{W}_{(0)}, \ldots, \mathbf{W}_{(n)}$, where $\mathbf{W}_{(0)} = \mathbf{W}, \mathbf{W}_{(n)} = \mathbf{W}'$, be the input model matrices in those settings. Let a series of output objects $\boldsymbol{\theta}_{1:t-1}^{(0)}, \ldots, \boldsymbol{\theta}_{1:t-1}^{(n)}$, where $\boldsymbol{\theta}_{1:t-1}^{(0)} = \boldsymbol{\theta}_{1:t-1}, \boldsymbol{\theta}_{1:t-1}^{(n)} = \mathbf{W}'$, be the output objects in those settings.

Then, we have

$$
\begin{aligned}
&\mathbb{P}(\hat{\mathbf{w}}_{[-i]}^{(1:T)} \in \mathcal{S} \mid \bigcap_{t=1}^{T} \mathcal{B}_t = (\mathbf{W}_{(0)}^{(t-1)}, \mathcal{D}_{(0)}^m, \boldsymbol{\theta}_{1:t-1}^{(0)}))\\
&\leq \exp(\epsilon)\mathbb{P}(\hat{\mathbf{w}}_{[-i]}^{(1:T)} \in \mathcal{S} \mid \bigcap_{t=1}^{T} \mathcal{B}_t = (\mathbf{W}_{(1)}^{(t-1)}, \mathcal{D}_{(1)}^m, \boldsymbol{\theta}_{1:t-1}^{(1)}))\\
&\quad + \delta\\
&\quad \vdots\\
&\leq \exp(n\epsilon)\mathbb{P}(\hat{\mathbf{w}}_{[-i]}^{(1:T)} \in \mathcal{S} \mid \bigcap_{t=1}^{T} \mathcal{B}_t = (\mathbf{W}_{(n)}^{(t-1)}, \mathcal{D}_{(n)}^m, \boldsymbol{\theta}_{1:t-1}^{(n)}))\\
&\quad + (1 + \exp(\epsilon) + \ldots + \exp((n-1)\epsilon))\delta\\
&\leq \exp(n\epsilon)\mathbb{P}(\hat{\mathbf{w}}_{[-i]}^{(1:T)} \in \mathcal{S} \mid \bigcap_{t=1}^{T} \mathcal{B}_t = (\mathbf{W}_{(n)}^{(t-1)}, \mathcal{D}_{(n)}^m, \boldsymbol{\theta}_{1:t-1}^{(n)}))\\
&\quad + n\exp(n\epsilon)\delta,
\end{aligned}
$$

which renders $\mathcal{A}$ as an $(n\epsilon, n\exp(n\epsilon)\delta)$ - iterative MP-MTL algorithm. □

Appendix M
Proof of the Results in Appendix C and Appendix D

A. Proof of Corollary 2

Proof. For simplicity, we omit the symbol $\mathcal{B}$ to denote the input in the conditional events in some equations.

Use Corollary 1 and Theorem 1. Given $t \in [T]$, the algorithm $(\mathbf{P}^{(t-1)}, \boldsymbol{\Sigma}^{(1:t-1)}) \to (\mathbf{M}^{(t)}, \boldsymbol{\Sigma}^{(t)})$ is an $(\epsilon_t, 0)$-differentially private algorithm, where $\mathbf{M}^{(t)} = \mathbf{U}\mathbf{S}_{\eta\lambda}\mathbf{U}^{\mathrm{T}}$.

Now, for all $i \in [m]$, applying the *Post-Processing immunity* Lemma (Lemma 4) for the mapping $f : (\mathbf{M}^{(t)}, \mathbf{p}_{[-i]}^{(t-1)}) \to \hat{\mathbf{p}}_{[-i]}^{(t-1)}$, which does not touch any unperturbed sensitive information of the i-th task, we have for any set $\mathcal{S} \subseteq \mathbb{R}^{d\times(m-1)}$ that

$$
\begin{aligned}
&\mathbb{P}(\hat{\mathbf{p}}_{[-i]}^{(t-1)} \in \mathcal{S} \mid \mathbf{P}^{(t-1)}, \boldsymbol{\Sigma}^{(1:t-1)})\\
&\leq e^{\epsilon_t}\mathbb{P}(\hat{\mathbf{p}}_{[-i]}^{(t-1)} \in \mathcal{S} \mid (\mathbf{P}')^{(t-1)}, \boldsymbol{\Sigma}^{(1:t-1)}),
\end{aligned}
$$

where $\mathbf{P}^{(t-1)}$ and $(\mathbf{P}')^{(t-1)}$ differ only in the i-th column.

Then, because in the t-th iteration the mapping $\mathbf{Q}^{(t-1)} \to \widehat{\mathbf{Q}}^{(t-1)}$ is a deterministic STL algorithm, we have for any set $\mathcal{S} \subseteq \mathbb{R}^{d\times(m-1)}$ that

$$
\begin{aligned}
&\mathbb{P}(\hat{\mathbf{q}}_{[-i]}^{(t-1)} \in \mathcal{S} \mid \mathbf{Q}^{(t-1)})\\
&=\mathbb{P}(\hat{\mathbf{q}}_{[-i]}^{(t-1)} \in \mathcal{S} \mid \mathbf{q}_{[-i]}^{(t-1)}, \mathbf{q}_i^{(t-1)})\\
&=\mathbb{P}(\hat{\mathbf{q}}_{[-i]}^{(t-1)} \in \mathcal{S} \mid \mathbf{q}_{[-i]}^{(t-1)}, (\mathbf{q}_i')^{(t-1)})\\
&=e^0\mathbb{P}(\hat{\mathbf{q}}_{[-i]}^{(t-1)} \in \mathcal{S} \mid (\mathbf{Q}')^{(t-1)}) + 0,
\end{aligned}
$$

where $\mathbf{Q}^{(t-1)}$ and $(\mathbf{Q}')^{(t-1)}$ differ only in the i-th column.

Then applying *Combination* Lemma (Lemma 6), we have for any set $\mathcal{S} \subseteq \mathbb{R}^{d\times(m-1)} \times \mathbb{R}^{d\times(m-1)}$

$$
\begin{aligned}
&\mathbb{P}((\hat{\mathbf{p}}_{[-i]}^{(t-1)}, \hat{\mathbf{q}}_{[-i]}^{(t-1)}) \in \mathcal{S} \mid \mathbf{P}^{(t-1)}, \boldsymbol{\Sigma}^{(1:t-1)}, \mathbf{Q}^{(t-1)})\\
&\leq e^{\epsilon_t}\mathbb{P}((\hat{\mathbf{p}}_{[-i]}^{(t-1)}, \hat{\mathbf{q}}_{[-i]}^{(t-1)}) \in \mathcal{S} \mid (\mathbf{P}')^{(t-1)}, \boldsymbol{\Sigma}^{(1:t-1)}, (\mathbf{Q}')^{(t-1)}),
\end{aligned}
$$

Because the mapping$(\widehat{\mathbf{P}}^{(t-1)}, \widehat{\mathbf{Q}}^{(t-1)}, \mathcal{D}^m) \to (\widehat{\mathbf{P}}^{(t)}, \widehat{\mathbf{Q}}^{(t)})$ is a deterministic STL algorithm, applying Lemma 8, we further have for any set $\mathcal{S} \subseteq \mathbb{R}^{d\times(m-1)} \times \mathbb{R}^{d\times(m-1)}$ that

$$
\begin{aligned}
&\mathbb{P}((\hat{\mathbf{p}}_{[-i]}^{(t)}, \hat{\mathbf{q}}_{[-i]}^{(t)}) \in \mathcal{S} \mid \mathbf{P}^{(t-1)}, \boldsymbol{\Sigma}^{(1:t-1)}, \mathbf{Q}^{(t-1)}, \mathcal{D}^m)\\
&\leq e^{\epsilon_t}\mathbb{P}((\hat{\mathbf{p}}_{[-i]}^{(t)}, \hat{\mathbf{q}}_{[-i]}^{(t)}) \in \mathcal{S} \mid (\mathbf{P}')^{(t-1)}, \boldsymbol{\Sigma}^{(1:t-1)}, (\mathbf{Q}')^{(t-1)}, (\mathcal{D}')^m),
\end{aligned}
$$

where $(\mathcal{D}')^m$ differs from $\mathcal{D}^m$ in the entire dataset of the i-th task.

Now, using Theorem 1, for $t = 1, \ldots, T$, we again take the t-th dataset $\widetilde{\mathcal{D}}_t = \{(\mathbf{p}_1^{(t-1)}, \mathbf{q}_1^{(t-1)}, \mathcal{D}_1), \ldots, (\mathbf{p}_m^{(t-1)}, \mathbf{q}_m^{(t-1)}\mathcal{D}_m)\}$ and denote $\vartheta_{t,i} = (\hat{\mathbf{q}}_{[-i]}^{(t)}, \hat{\mathbf{q}}_{[-i]}^{(t)}, \mathbf{M}^{(t)}, \boldsymbol{\Sigma}^{(t)}) \in \mathcal{C}_{t,i}$. Given the fact that $\mathbf{P}^{(t)} = \widehat{\mathbf{P}}^{(t)}$ and $\mathbf{Q}^{(t)} = \widehat{\mathbf{Q}}^{(t)}$ for all $t \in [T]$, we have for any set $\mathcal{S}_{t,i} \subseteq \mathcal{C}_{t,i}$ that

$$
\begin{aligned}
&\mathbb{P}(\vartheta_{t,i} \in \mathcal{S}_{t,i} \mid \widetilde{\mathcal{D}}_t, \boldsymbol{\vartheta}_{1:t-1})\\
&\leq e^{\epsilon_t}\mathbb{P}(\vartheta_{t,i} \in \mathcal{S}_{t,i} \mid \widetilde{\mathcal{D}}_t', \boldsymbol{\vartheta}_{1:t-1}),
\end{aligned}
$$

where $\widetilde{\mathcal{D}}_t$ and $\widetilde{\mathcal{D}}_t'$ are two adjacent datasets that differ in a single entry, the i-th "data instance" $(\mathbf{p}_i^{(t-1)}, \mathbf{q}_i^{(t-1)}, \mathcal{D}_i = (\mathbf{X}_i, \mathbf{y}_i))$, and

$$
\boldsymbol{\vartheta}_{1:t-1} = \begin{cases} \emptyset, & t = 1\\ (\vartheta_{1,1}, \ldots, \vartheta_{1,m}) \ldots, (\vartheta_{t-1,1}, \ldots, \vartheta_{t-1,m}), & t \geq 2. \end{cases}
$$

This renders the algorithm in the t-th iteration as an $(\epsilon_t, 0)$-differentially private algorithm.

Then, again by the *Adaptive composition* Lemma (Lemma 7), for all $i \in [m]$ and for any set $\mathcal{S}' \subseteq \bigotimes_{t=1}^{T} \mathcal{C}_{t_i}$, we have

$$
\begin{aligned}
&\mathbb{P}((\vartheta_{1,i}, \cdots, \vartheta_{T,i}) \in \mathcal{S}' \mid \bigcap_{t=1}^{T} (\mathcal{B}_t = (\widetilde{\mathcal{D}}_t, \boldsymbol{\vartheta}_{1:t-1})))\\
&\leq e^{\tilde{\epsilon}}\mathbb{P}((\vartheta_{1,i}, \cdots, \vartheta_{T,i}) \in \mathcal{S}' \mid \bigcap_{t=1}^{T} (\mathcal{B}_t = (\widetilde{\mathcal{D}}_t', \boldsymbol{\vartheta}_{1:t-1})))\\
&\quad + \delta,
\end{aligned}
$$

where for all $t \in [T]$, $\mathcal{B}_t$ denotes the input for the t-th iteration.

Finally, for all $t \in [T]$, taking $\theta_t = (\vartheta_{t,1}, \ldots, \vartheta_{t,m})$ and given the fact that $\widehat{\mathbf{W}}^{(t)} = \widehat{\mathbf{P}}^{(t)} + \widehat{\mathbf{Q}}^{(t)}$, we have for any set $\mathcal{S} \subseteq \mathbb{R}^{d\times(m-1)\times T}$ that

$$
\begin{aligned}
&\mathbb{P}(\hat{\mathbf{w}}_{[-i]}^{(1:T)} \in \mathcal{S} \mid \bigcap_{t=1}^{T} \mathcal{B}_t = (\mathbf{W}^{(t-1)}, \mathcal{D}^m, \boldsymbol{\theta}_{1:t-1}))\\
&\leq e^{\epsilon}\mathbb{P}(\hat{\mathbf{w}}_{[-i]}^{(1:T)} \in \mathcal{S} \mid \bigcap_{t=1}^{T} \mathcal{B}_t = ((\mathbf{W}')^{(t-1)}, (\mathcal{D}')^m, \boldsymbol{\theta}_{1:t-1}))\\
&\quad + \delta,
\end{aligned}
$$

where $(\mathbf{W}')^{(t-1)}$ are associated with the setting in which the i-th task has been replaced. □

B. Proof of Proposition 4

Proof. For simplicity, we omit the symbol $\mathcal{B}$ used to denote the input in the conditional events in some equations.

First, the procedure from the 4-th step to the 5-th step is a standard output perturbation of Chaudhuri et al. [2]; thus, we have for all $i \in [m]$, for all neighboring datasets $\mathcal{D}^m$ and $(\mathcal{D}')^m$ that differ in a single data instance of the i-th task, and for any set $\mathcal{S} \in \mathbb{R}^d$,

$$\begin{aligned}&\mathbb{P}(\tilde{\mathbf{w}}_i^{(t-1)} \in \mathcal{S} \mid \tilde{\mathbf{w}}_i^{(0:t-2)}, \mathcal{D}^m, \mathbf{M}^{(t-1)})\\ &\leq \exp(\epsilon_{\text{dp},t})\mathbb{P}(\tilde{\mathbf{w}}_i^{(t-1)} \in \mathcal{S} \mid \tilde{\mathbf{w}}_i^{(0:t-2)}, (\mathcal{D}')^m, \mathbf{M}^{(t-1)}),\end{aligned}$$

where $\tilde{\mathbf{w}}_i^{(0:t-2)} = \emptyset$ when $t = 1$.

Then, because the mapping $(\widetilde{\mathbf{W}}^{(t-1)}, \mathbf{\Sigma}^{(1:t-1)}) \rightarrow \theta_t = (\mathbf{\Sigma}^{(t)}, \mathbf{M}^{(t)}, \widetilde{\mathbf{W}}^{(t-1)}) \in \mathcal{C}_t$ does not touch any unperturbed sensitive information of $(\mathbf{X}_i, \mathbf{y}_i, \mathbf{w}_i^{(0:t-1)})$, the *Post-Processing immunity* Lemma (Lemma 4) can be applied such that we have for any set $\mathcal{S}' \subseteq \mathcal{C}_t$ that

$$\begin{aligned}&\mathbb{P}(\theta_t \in \mathcal{S}' \mid \widetilde{\mathbf{W}}^{(0:t-2)}, \mathcal{D}^m, \mathbf{M}^{(t-1)})\\ &\leq \exp(\epsilon_{\text{dp},t})\mathbb{P}(\theta_t \in \mathcal{S}' \mid \widetilde{\mathbf{W}}^{(0:t-2)}, (\mathcal{D}')^m, \mathbf{M}^{(t-1)}),\end{aligned}$$

which means that

$$\begin{aligned}&\mathbb{P}(\theta_t \in \mathcal{S}' \mid \mathcal{D}^m, \boldsymbol{\theta}_{1:t-1})\\ &\leq \exp(\epsilon_{\text{dp},t})\mathbb{P}(\theta_t \in \mathcal{S}' \mid (\mathcal{D}')^m, \boldsymbol{\theta}_{1:t-1}),\end{aligned}$$

where

$$\boldsymbol{\theta}_{1:t-1} = \begin{cases} \emptyset, & t = 1\\ \theta_1, \theta_2, \cdots, \theta_{t-1}, & t \geq 2.\end{cases}$$

Then, by the *Adaptive composition* Lemma (Lemma 7), we have for any set $\mathcal{S}'' \subseteq \bigotimes_{t=1}^T \mathcal{C}_t$ that

$$\begin{aligned}&\mathbb{P}(\theta_{1:T} \in \mathcal{S}'' \mid \bigcap_{t=1}^T (\mathcal{B}_t = (\mathcal{D}^m, \boldsymbol{\theta}_{1:t-1})))\\ &\leq \exp(\epsilon_{\text{dp}})\mathbb{P}(\theta_{1:T} \in \mathcal{S}'' \mid \bigcap_{t=1}^T (\mathcal{B}_t = ((\mathcal{D}')^m, \boldsymbol{\theta}_{1:t-1})))\\ &\quad + \delta_{\text{dp}}.\end{aligned}$$

Because the mapping $(\theta_t, \mathcal{D}_{[-i]}, \tilde{\mathbf{w}}_{[-i]}^{(0:t-2)}, \mathbf{W}^{(t-1)}) \rightarrow \hat{\mathbf{w}}_{[-i]}^{(t)}$ does not touch any unperturbed sensitive information of $(\mathbf{X}_i, \mathbf{y}_i, \mathbf{w}_i^{(0:t-1)})$ for all $t \in [T]$ ($\mathbf{W}^{(t-1)}$ is actually not used in the mapping), the *Post-Processing immunity* Lemma (Lemma 4) can be applied such that we have for any set $\mathcal{S}_0 \subseteq \mathbb{R}^{d\times(m-1)\times T}$ that

$$\begin{aligned}&\mathbb{P}(\hat{\mathbf{w}}_{[-i]}^{(1:T)} \in \mathcal{S}_0 \mid \bigcap_{t=1}^T (\mathcal{B}_t = (\mathcal{D}^m, \boldsymbol{\theta}_{1:t-1}, \mathbf{W}^{(t-1)})))\\ &\leq e^{\epsilon_{\text{dp},t}}\mathbb{P}(\hat{\mathbf{w}}_{[-i]}^{(1:T)} \in \mathcal{S}_0 \mid \bigcap_{t=1}^T (\mathcal{B}_t = ((\mathcal{D}')^m, \boldsymbol{\theta}_{1:t-1}, (\mathbf{W}')^{(t-1)})))\\ &\quad + \delta_{\text{dp}},\end{aligned}$$

where $(\mathbf{W}')^{(t-1)}$ is associated with the setting in which a single data instance of the i-th task has been replaced.

Therefore, Algorithm 5 is an $(\epsilon_{\text{dp}}, \delta_{\text{dp}})$ - iterative DP-MTL algorithm.

Next, for the conditional density of $\mathbf{\Sigma}^{(t)}$ given $\mathbf{W}^{(t-1)}$, we have

$$\begin{aligned}&p(\mathbf{\Sigma}^{(t)} \mid \mathbf{W}^{(t-1)})\\ &= \int_{\widetilde{\mathbf{W}}^{(t-1)}} p(\mathbf{\Sigma}^{(t)} \mid \mathbf{W}^{(t-1)}, \widetilde{\mathbf{W}}^{(t-1)})\\ &\quad p(\widetilde{\mathbf{W}}^{(t-1)} \mid \mathbf{W}^{(t-1)}) d\widetilde{\mathbf{W}}^{(t-1)}\\ &= \int_{\widetilde{\mathbf{W}}^{(t-1)}} p(\mathbf{\Sigma}^{(t)} \mid \widetilde{\mathbf{W}}^{(t-1)}) p(\widetilde{\mathbf{W}}^{(t-1)} \mid \mathbf{W}^{(t-1)}) d\widetilde{\mathbf{W}}^{(t-1)}\\ &= \int_{\widetilde{\mathbf{W}}^{(t-1)}} p(\mathbf{\Sigma}^{(t)} \mid \widetilde{\mathbf{W}}^{(t-1)}) \prod_{i=1}^m p(\tilde{\mathbf{w}}_i^{(t-1)} \mid \mathbf{w}_i^{(t-1)}) d\widetilde{\mathbf{W}}^{(t-1)}.\end{aligned}$$

Because, for all $i \in [m]$ and some constant $c = \frac{\tilde{s}_i^{(t-1)}}{\epsilon_{\text{dp},t}}$, we have

$$p(\tilde{\mathbf{w}}_i^{(t-1)} \mid \mathbf{w}_i^{(t-1)}) \propto \exp\Big(-c\|\tilde{\mathbf{w}}_i^{(t-1)} - \mathbf{w}_i^{(t-1)}\|_2\Big),$$

given $(\mathbf{W}')^{(t-1)}$ such that for some $i \in [m]$, $(\mathbf{w}_i')^{(t-1)} \neq \mathbf{w}_i^{(t-1)}$, letting $(\tilde{\mathbf{w}}_i')^{(t-1)} = \tilde{\mathbf{w}}_i^{(t-1)} - \mathbf{w}_i^{(t-1)} + (\mathbf{w}_i')^{(t-1)}$, we have

$$\begin{aligned}&\|(\tilde{\mathbf{w}}_i')^{(t-1)} - (\mathbf{w}_i')^{(t-1)}\|_2 = \|\tilde{\mathbf{w}}_i^{(t-1)} - \mathbf{w}_i^{(t-1)}\|_2\\ &\Rightarrow p((\tilde{\mathbf{w}}_i')^{(t-1)} \mid \mathbf{w}_i^{(t-1)}) = p(\tilde{\mathbf{w}}_i^{(t-1)} \mid \mathbf{w}_i^{(t-1)}),\end{aligned}$$

and $d(\tilde{\mathbf{w}}_i')^{(t-1)} = d\tilde{\mathbf{w}}_i^{(t-1)}$.

Furthermore, based on the proof of Theorem 1 in Section L-C, we know that for neighboring matrices $\widetilde{\mathbf{W}}^{(t-1)}$ and $(\widetilde{\mathbf{W}}')^{(t-1)}$ that differ in the i-th column, we have

$$p(\mathbf{\Sigma}^{(t)} \mid \widetilde{\mathbf{W}}^{(t-1)}) \leq \exp(\epsilon_{\text{mp},t}) p(\mathbf{\Sigma}^{(t)} \mid (\widetilde{\mathbf{W}}')^{(t-1)}).$$

Therefore, for all $i \in [m]$, given $(\mathbf{W}')^{(t-1)}$ such that $(\mathbf{w}_i')^{(t-1)} \neq \mathbf{w}_i^{(t-1)}$, under the choice for $(\tilde{\mathbf{w}}_i')^{(t-1)}$, we have

$$\begin{aligned}&p(\mathbf{\Sigma}^{(t)} \mid \mathbf{W}^{(t-1)})\\ &= \int_{\widetilde{\mathbf{W}}^{(t-1)}} p(\mathbf{\Sigma}^{(t)} \mid \widetilde{\mathbf{W}}^{(t-1)}) \prod_{j=1}^m p(\tilde{\mathbf{w}}_j^{(t-1)} \mid \mathbf{w}_j^{(t-1)}) d\widetilde{\mathbf{W}}^{(t-1)}\\ &\leq \int_{(\widetilde{\mathbf{W}}')^{(t-1)}} e^{\epsilon_{\text{mp},t}} p(\mathbf{\Sigma}^{(t)} \mid (\widetilde{\mathbf{W}}')^{(t-1)}) p((\tilde{\mathbf{w}}_i')^{(t-1)} \mid (\mathbf{w}_i')^{(t-1)})\\ &\quad \prod_{j\in[m], j\neq i} p(\tilde{\mathbf{w}}_j^{(t-1)} \mid \mathbf{w}_j^{(t-1)}) d(\widetilde{\mathbf{W}}')^{(t-1)}.\\ &= \int_{(\widetilde{\mathbf{W}}')^{(t-1)}} \exp(\epsilon_{\text{mp},t}) p(\mathbf{\Sigma}^{(t)} \mid (\widetilde{\mathbf{W}}')^{(t-1)})\\ &\quad p((\widetilde{\mathbf{W}}')^{(t-1)} \mid (\mathbf{W}')^{(t-1)}) d(\widetilde{\mathbf{W}}')^{(t-1)}\\ &= \exp(\epsilon_{\text{mp},t}) p(\mathbf{\Sigma}^{(t)} \mid (\mathbf{W}')^{(t-1)}),\end{aligned}$$

which renders the mapping $\mathbf{W}^{(t-1)} \rightarrow \mathbf{\Sigma}^{(t)}$ as an $(\exp(\epsilon_{\text{mp},t}), 0)$ - differentially private algorithm.

Then, according to the proof of Theorem 1 in Section L-C, Algorithm 5 is an $(\epsilon_{\text{mp}}, \delta_{\text{mp}})$ - iterative MP-MTL algorithm. □

APPENDIX N
PROOF OF RESULTS IN APPENDIX E-A

A. Proof of Theorem 7

Proof. First, consider the case with no acceleration. We first use Proposition 1 of Schmidt et al. [11] by regarding procedures from Step 5 to Step 9 as approximation for the proximal operator in (8). Note that the norm clipping only bounds the parameter space and does not affect the results of Schmidt et al. [11]. Then for ε_t defined in Lemma 13 for $t \in [T]$, we have

$$\begin{aligned}\mathcal{E} =& \frac{2L}{m(T+1)^2}\Bigg(\|\widetilde{\mathbf{W}}^{(0)} - \mathbf{W}_*\|_F \\ &+ 2\sum_{t=1}^{T} t\sqrt{\frac{2\varepsilon_t}{L}} + \sqrt{2\sum_{t=1}^{T} t^2 \frac{\varepsilon_t}{L}}\Bigg)^2.\end{aligned}$$

Meanwhile, by Lemma 13, we have

$$\varepsilon_t = O\left(\frac{\kappa}{\epsilon_t}\right),$$

where $\kappa = \frac{K^2\sqrt{m}kd\log d}{\eta}$

On the other hand, let

$$c_1 = \epsilon = \sum_{t=1}^{T} \epsilon_t,$$

then by Lemma 17, we have

$$\sum_{t=1}^{T}\sqrt{\varepsilon_t} = \begin{cases} O\left(\sqrt{\frac{\kappa T^{\alpha+1}}{c_1(\alpha/2-1)^2(\alpha+1)}}\right), & \alpha > 2; \\ O\left(\sqrt{\frac{\kappa T^{3}}{c_1(\alpha/2-1)^2(\alpha+1)}}\right), & -1 < \alpha < 2; \\ O\left(\sqrt{\frac{\kappa T^{2-\alpha}}{c_1(\alpha/2-1)^2(-\alpha-1)}}\right), & \alpha < -1, \end{cases}$$

Because $\widetilde{\mathbf{W}}^{(0)}$ is the result of the norm clipping, we have $\widetilde{\mathbf{W}}^{(0)} \in \mathcal{W}$.

Finally, taking $c_3 = \phi(\alpha)$ defined in (13) and $c_4 = \frac{\kappa}{c_2(\alpha/2-1)^2|\alpha+1|}$, under the assumption that $\mathbf{W}_* \in \mathcal{W}$, using Lemma 19, we have the results for the case with no acceleration.

For the accelerated case, we use Proposition 2 of Schmidt et al. [11] to have

$$\begin{aligned}\mathcal{E} =& \frac{2L}{m(T+1)^2}\Bigg(\|\widetilde{\mathbf{W}}^{(0)} - \mathbf{W}_*\|_F \\ &+ 2\sum_{t=1}^{T} t\sqrt{\frac{2\varepsilon_t}{L}} + \sqrt{2\sum_{t=1}^{T} t^2 \frac{\varepsilon_t}{L}}\Bigg)^2.\end{aligned}$$

Then one can prove similarly combining Lemma 13, Lemma 16, Lemma 17 and Lemma 20. □

B. Proof of Theorem 8

Proof. First, consider the case with no acceleration. We use Proposition 1 of Schmidt et al. [11] and prove similarly as in Appendix N-A, combining Lemma 14, Lemma 16, Lemma 17 and Lemma 19.

For the accelerated case, we use Proposition 2 of Schmidt et al. [11] and prove similarly as in Appendix N-A, combining Lemma 14, Lemma 16, Lemma 17 and Lemma 20. □

C. Proof of Theorem 9

Proof. First, consider the case with no acceleration. We use Proposition 3 of Schmidt et al. [11] to have

$$\mathcal{E} = \frac{Q_0^T}{\sqrt{m}}\left(\|\widetilde{\mathbf{W}}^{(0)} - \mathbf{W}_*\|_F + 2\sum_{t=1}^{T} Q_0^{-t}\sqrt{\frac{2\varepsilon_t}{L}}\right).$$

Then one can prove similarly as in Appendix N-A, combining Lemma 13, Lemma 16, Lemma 17 and Lemma 21.

For the accelerated case, we use Proposition 4 of Schmidt et al. [11] to have

$$\begin{aligned}\mathcal{E} =& \frac{(Q_0)^T}{m}\Bigg(\sqrt{2(f(\widehat{\mathbf{W}}^{(0)}) - f(\mathbf{W}_*))} + 2\sqrt{\frac{L}{\mu}}\sum_{t=1}^{T}\sqrt{\varepsilon_t(Q_0)^{-t}} \\ &+ \sqrt{\sum_{t=1}^{T}\varepsilon_t(Q_0)^{-t}}\Bigg)^2.\end{aligned}$$

Then one can prove similarly as in Appendix N-A, using the assumption that $f(\widetilde{\mathbf{W}}^{(0)}) - f(\mathbf{W}_*) = O(K^2Lm)$, combining Lemma 13, Lemma 16, Lemma 17 and Lemma 22. □

D. Proof of Theorem 10

Proof. First, consider the case with no acceleration. We use Proposition 3 of Schmidt et al. [11] and prove similarly as in Appendix L-I, combining Lemma 14, Lemma 16, Lemma 17 and Lemma 21.

For the accelerated case, we use Proposition 4 of Schmidt et al. [11] and prove similarly as in Appendix L-I, using the assumption that $f(\widetilde{\mathbf{W}}^{(0)}) - f(\mathbf{W}_*) = O(K^2Lm)$, combining Lemma 14, Lemma 16, Lemma 17 and Lemma 22. □

E. Proof of Theorem 11

Proof. Consider the bound in (25). First, by Assumption 1, $\mathcal{E}$ is minimized by maximizing $\phi(\alpha)$ and $(\alpha/2-1)^2|\alpha+1|$, which are maximized simultaneously when $\alpha = 0$. Results under other settings can be proved similarly. □

Appendix O
Proof of Results in Appendix E-B

Proof. Results in this settings are the corollaries of Theorem 2, Theorem 3, Theorem 4, Theorem 5 and Theorem 6, respectively, replacing the term $\sqrt{\log(e+\epsilon/\delta)}$ with the term $\sqrt{\log(1/\delta)}$ by Lemma 16. □

Appendix P
Proof of Results in Appendix J

A. Proof of Lemma 8

Proof. For simplicity, we omit the symbol $\mathcal{B}$ in the conditional events.

Because $\widetilde{\mathbf{W}} = \mathcal{A}_{\mathrm{MP}}(\mathbf{W}, \mathbf{X}^m, \mathbf{y}^m)$ is an (ϵ,δ)-Non-iterative MP-MTL algorithm, by Definition 5, we have for $i \in [m]$ and for any set $\mathcal{S}' \subseteq \mathbb{R}^{d\times(m-1)}$,

$$\begin{aligned}&\mathbb{P}(\tilde{\mathbf{w}}_{[-i]} \in \mathcal{S}' \mid \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}_i, \mathcal{D}_i) \\ &\quad\le e^{\epsilon}\mathbb{P}(\tilde{\mathbf{w}}_{[-i]} \in \mathcal{S}_1 \mid \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}_i', \mathcal{D}_i') + \delta.\end{aligned} \tag{68}$$

In the following, we follow the proof of Theorem B.1 of Dwork et al. [3].

For any $C_1 \subseteq \mathbb{R}^{d\times(m-1)}$, define

$$\begin{aligned}\mu(C_1) =&(\mathbb{P}(\tilde{\mathbf{w}}_{[-i]} \in C_1 \mid \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}_i, \mathcal{D}_i) \\ &- e^{\epsilon}\mathbb{P}(\tilde{\mathbf{w}}_{[-i]} \in C_1 \mid \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}_i', \mathcal{D}_i'))_+\end{aligned}$$

and then, μ is a measure on C_1 and $\mu(C_1) \le \delta$ by (68). As a result, we have for all $s_1 \in C_1$,

$$\begin{aligned}&\mathbb{P}(\tilde{\mathbf{w}}_{[-i]} \in ds_1 \mid \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}_i, \mathcal{D}_i) \\ &\le e^{\epsilon}\mathbb{P}(\tilde{\mathbf{w}}_{[-i]} \in ds_1 \mid \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}_i', \mathcal{D}_i') + \mu(ds_1).\end{aligned}$$

As such, for any set $\mathcal{S} \subseteq \mathbb{R}^{d\times(m-1)} \times \mathbb{R}^{d\times(m-1)}$ and $\mathcal{S}_1$, which denotes the projection of $\mathcal{S}$ onto C_1:

$$\begin{aligned}
&\mathbb{P}((\hat{\mathbf{w}}_{[-i]}, \tilde{\mathbf{w}}_{[-i]}) \in \mathcal{S} \mid \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}_i, \mathcal{D}_i) \\
&\leq \int_{\mathcal{S}_1} \mathbb{P}((\hat{\mathbf{w}}_{[-i]}, s_1) \in \mathcal{S} \mid \tilde{\mathbf{w}}_{[-i]}, \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}_i, \mathcal{D}_i) \\
&\quad \mathbb{P}(\tilde{\mathbf{w}}_{[-i]} \in ds_1 \mid \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}_i, \mathcal{D}_i) \\
&\leq \int_{\mathcal{S}_1} \mathbb{P}((\hat{\mathbf{w}}_{[-i]}, s_1) \in \mathcal{S} \mid \tilde{\mathbf{w}}_{[-i]}, \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}_i, \mathcal{D}_i) \\
&\quad \left[e^{\epsilon}\mathbb{P}(\tilde{\mathbf{w}}_{[-i]} \in ds_1 \mid \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}'_i, \mathcal{D}'_i) + \mu(ds_1)\right] \\
&= \int_{\mathcal{S}_1} \mathbb{P}((\hat{\mathbf{w}}_{[-i]}, s_1) \in \mathcal{S} \mid \tilde{\mathbf{w}}_{[-i]}, \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}_i, \mathcal{D}_i) \\
&\quad e^{\epsilon}\mathbb{P}(\tilde{\mathbf{w}}_{[-i]} \in ds_1 \mid \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}'_i, \mathcal{D}'_i) + \mu(S_1) \\
&= \int_{\mathcal{S}_1} \mathbb{P}((\hat{\mathbf{w}}_{[-i]}, s_1) \in \mathcal{S} \mid \tilde{\mathbf{w}}_{[-i]}, \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}'_i, \mathcal{D}'_i) \\
&\quad e^{\epsilon}\mathbb{P}(\tilde{\mathbf{w}}_{[-i]} \in ds_1 \mid \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}'_i, \mathcal{D}'_i) + \mu(S_1) \\
&\leq e^{\epsilon}\mathbb{P}((\hat{\mathbf{w}}_{[-i]}, \tilde{\mathbf{w}}_{[-i]}) \in \mathcal{S} \mid \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}'_i, \mathcal{D}'_i) + \delta.
\end{aligned}$$

The second equality uses the independence of the learning process for $\hat{\mathbf{w}}_{[-i]}$ given $(\tilde{\mathbf{w}}_{[-i]}, \mathcal{D}_{[-i]})$.

The procedure is similar to proving that the algorithm $\mathcal{A}_{\text{st+mp}}$ that first uses a *deterministic* STL algorithm $\mathcal{A}_{\text{st}}$ before applying $\mathcal{A}_{\text{mp}}$ is also an (ϵ, δ) - non-iterative MP-MTL algorithm.

For a *deterministic* STL algorithm $\widetilde{\mathbf{W}} = \mathcal{A}_{\text{st}}(\mathbf{W}, \mathbf{X}^m, \mathbf{y}^m)$, for $i \in [m]$, by the independence of the learning process for $\tilde{\mathbf{w}}_{[-i]}$ given $(\mathbf{w}_{[-i]}, \mathcal{D}_{[-i]})$, we have

$$\begin{aligned}
&p(\tilde{\mathbf{w}}_{[-i]} \mid \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}_i, \mathcal{D}_i) \\
&\quad = p(\tilde{\mathbf{w}}_{[-i]} \mid \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}'_i, \mathcal{D}'_i).
\end{aligned} \tag{69}$$

Because the STL algorithm is *deterministic*, when the input is given, it is reasonable to assume that the output is given. As such, we also have for $i \in [m]$,

$$\begin{aligned}
&p(\cdot \mid \mathbf{w}_i, \mathcal{D}_i) = p(\cdot \mid \tilde{\mathbf{w}}_i, \mathcal{D}_i) \\
&p(\cdot \mid \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}) = p(\cdot \mid \tilde{\mathbf{w}}_{[-i]}, \mathcal{D}_{[-i]})
\end{aligned} \tag{70}$$

Then, for an (ϵ, δ)-Non-iterative MP-MTL algorithm $\widehat{\mathbf{W}} = \mathcal{A}_{\text{mp}}(\widetilde{\mathbf{W}}, \mathbf{X}^m, \mathbf{y}^m)$, by Definition 5, we have for $i \in [m]$ and for any set $\mathcal{S}' \subseteq \mathbb{R}^{d\times(m-1)}$

$$\begin{aligned}
&\mathbb{P}(\hat{\mathbf{w}}_{[-i]} \in \mathcal{S}' \mid \tilde{\mathbf{w}}_{[-i]}, \mathcal{D}_{[-i]}, \tilde{\mathbf{w}}_i, \mathcal{D}_i) \\
&\quad \leq e^{\epsilon}\mathbb{P}(\hat{\mathbf{w}}_{[-i]} \in \mathcal{S}' \mid \tilde{\mathbf{w}}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}'_i, \mathcal{D}'_i) + \delta,
\end{aligned} \tag{71}$$

where $\tilde{\mathbf{w}}'_i$ can be replaced with $\mathbf{w}'_i$ because Definition 5 allows replacing $\tilde{\mathbf{w}}_i$ with any different model.

As such,

$$\begin{aligned}
&\mathbb{P}((\hat{\mathbf{w}}_{[-i]}, \tilde{\mathbf{w}}_{[-i]}) \in \mathcal{S} \mid \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}_i, \mathcal{D}_i) \\
&\leq \int_{\mathcal{S}_1} \mathbb{P}((\hat{\mathbf{w}}_{[-i]}, s_1) \in \mathcal{S} \mid \tilde{\mathbf{w}}_{[-i]}, \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}_i, \mathcal{D}_i) \\
&\quad \mathbb{P}(\tilde{\mathbf{w}}_{[-i]} \in ds_1 \mid \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}_i, \mathcal{D}_i) \\
&= \int_{\mathcal{S}_1} \mathbb{P}((\hat{\mathbf{w}}_{[-i]}, s_1) \in \mathcal{S} \mid \tilde{\mathbf{w}}_{[-i]}, \mathcal{D}_{[-i]}, \tilde{\mathbf{w}}_i, \mathcal{D}_i) \\
&\quad \mathbb{P}(\tilde{\mathbf{w}}_{[-i]} \in ds_1 \mid \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}_i, \mathcal{D}_i) \\
&\leq \int_{\mathcal{S}_1} \left[\left(e^{\epsilon}\mathbb{P}((\hat{\mathbf{w}}_{[-i]}, s_1) \in \mathcal{S} \mid \tilde{\mathbf{w}}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}'_i, \mathcal{D}'_i) \right.\right. \\
&\quad \left.\left. + \delta\right) \wedge 1\right] \mathbb{P}(\tilde{\mathbf{w}}_{[-i]} \in ds_1 \mid \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}_i, \mathcal{D}_i) \\
&\leq \int_{\mathcal{S}_1} \left[e^{\epsilon}\mathbb{P}((\hat{\mathbf{w}}_{[-i]}, s_1) \in \mathcal{S} \mid \tilde{\mathbf{w}}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}'_i, \mathcal{D}'_i) \right. \\
&\quad \left. \wedge 1 + \delta\right] \mathbb{P}(\tilde{\mathbf{w}}_{[-i]} \in ds_1 \mid \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}_i, \mathcal{D}_i) \\
&\leq \int_{\mathcal{S}_1} \left[e^{\epsilon}\mathbb{P}((\hat{\mathbf{w}}_{[-i]}, s_1) \in \mathcal{S} \mid \tilde{\mathbf{w}}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}'_i, \mathcal{D}'_i) \right. \\
&\quad \left. \wedge 1\right] \mathbb{P}(\tilde{\mathbf{w}}_{[-i]} \in ds_1 \mid \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}_i, \mathcal{D}_i) + \delta \\
&= \int_{\mathcal{S}_1} \left[e^{\epsilon}\mathbb{P}((\hat{\mathbf{w}}_{[-i]}, s_1) \in \mathcal{S} \mid \tilde{\mathbf{w}}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}'_i, \mathcal{D}'_i) \right. \\
&\quad \left. \wedge 1\right] \mathbb{P}(\tilde{\mathbf{w}}_{[-i]} \in ds_1 \mid \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}'_i, \mathcal{D}'_i) + \delta \\
&= e^{\epsilon}\mathbb{P}((\hat{\mathbf{w}}_{[-i]}, \tilde{\mathbf{w}}_{[-i]}) \in \mathcal{S} \mid \mathbf{w}_{[-i]}, \mathcal{D}_{[-i]}, \mathbf{w}'_i, \mathcal{D}'_i) + \delta.
\end{aligned}$$

The second inequality uses (71), the first equality uses (70), and the second equality uses (69). ☐

B. Proof of Lemma 9

Proof. For simplicity, we omit the symbol $\mathcal{B}$ to denote the input in the conditional events in some equations.

First, because for all $t \in [T]$, $\widehat{\mathbf{W}}^{(t)} = \mathcal{A}_{\text{mp}}(\mathbf{W}^{(t-1)}, \mathbf{X}^m, \mathbf{y}^m)$ is an (ϵ_t, δ_t)-Non-iterative MP-MTL algorithm and because for all $i \in [m]$ $\mathbf{w}_i^{(t)} = \mathcal{A}_{\text{st},i}(\hat{\mathbf{w}}_i^{(t)}, \mathbf{X}_i, \mathbf{y}_i)$ is a deterministic STL algorithm for the i-th task, then by the proof of Lemma 8, we have that the mapping $(\mathbf{X}^m, \mathbf{y}^m, \mathbf{W}^{(t-1)}) \rightarrow (\widehat{\mathbf{W}}^{(t)}, \mathbf{W}^{(t)})$ is an (ϵ_t, δ_t)-Non-iterative MP-MTL algorithm for all $t \in [T]$. In other words, for all $i \in [m]$, we have for any set $\mathcal{S} \subseteq \mathbb{R}^{d\times(m-1)} \times \mathbb{R}^{d\times(m-1)}$ that

$$\begin{aligned}
&\mathbb{P}((\hat{\mathbf{w}}_{[-i]}^{(t)}, \mathbf{w}_{[-i]}^{(t)}) \in \mathcal{S} \mid \mathbf{w}_{[-i]}^{(t-1)}, \mathcal{D}_{[-i]}, \mathbf{w}_i^{(t-1)}, \mathcal{D}_i) \\
&\quad \leq e^{\epsilon_t}\mathbb{P}((\hat{\mathbf{w}}_{[-i]}^{(t)}, \mathbf{w}_{[-i]}^{(t)}) \in \mathcal{S} \mid \mathbf{w}_{[-i]}^{(t-1)}, \mathcal{D}_{[-i]}, (\mathbf{w}'_i)^{(t-1)}, \mathcal{D}'_i) \\
&\qquad + \delta_t.
\end{aligned} \tag{72}$$

Then, for $t = 1, \ldots, T$, take the t-th dataset $\widetilde{\mathcal{D}}_t = \{(\mathbf{w}_1^{(t-1)}, \mathcal{D}_1 = (\mathbf{X}_1, \mathbf{y}_1)), \ldots, (\mathbf{w}_m^{(t-1)}, \mathcal{D}_m = (\mathbf{X}_m, \mathbf{y}_m))\}$, i.e., treat $(\mathbf{w}_i^{(t-1)}, \mathcal{D}_i = (\mathbf{X}_i, \mathbf{y}_i))$ as the i-th data instance of the dataset $\widetilde{\mathcal{D}}_t$ for all $i \in [m]$. For all $i \in [m]$ and for all $t \in [T]$, take the t-th output $\theta_{t,i} = (\hat{\mathbf{w}}_{[-i]}^{(t)}, \mathbf{w}_{[-i]}^{(t)})$. By (72), we have for all $t \in [T]$, for all $i \in [m]$, and for any set $\mathcal{S}_t \subseteq \mathbb{R}^{d\times(m-1)} \times \mathbb{R}^{d\times(m-1)}$ that

$$\mathbb{P}(\theta_{t,i} \in \mathcal{S}_t \mid \widetilde{\mathcal{D}}_t) \leq e^{\epsilon_t}\mathbb{P}(\theta_{t,i} \in \mathcal{S}_t \mid \widetilde{\mathcal{D}}'_t) + \delta_t,$$

where $\widetilde{\mathcal{D}}_t$ and $\widetilde{\mathcal{D}}'_t$ are two adjacent datasets that differ in a single entry, the i-th data instance $(\mathbf{w}_i^{(t-1)}, \mathcal{D}_i = (\mathbf{X}_i, \mathbf{y}_i))$, which renders the algorithm in the t-th iteration an (ϵ_t, δ_t)-differentially private algorithm. As such, by the *Adaptive composition* Lemma (Lemma

7), for all $i \in [m]$ and for any set $\mathcal{S} \subseteq \bigotimes_{t=1}^{T} \mathcal{C}_t$, where $\mathcal{C}_t = \mathbb{R}^{d\times(m-1)} \times \mathbb{R}^{d\times(m-1)}$, we have

$$\begin{aligned}&\mathbb{P}((\theta_{1,i},\cdots,\theta_{T,i}) \in \mathcal{S} \mid \bigcap_{t=1}^{T}(\mathcal{B}_t = (\widetilde{\mathcal{D}}_t, \boldsymbol{\theta}_{1:t-1})))\\ \leq & e^{\epsilon}\mathbb{P}((\theta_{1,i},\cdots,\theta_{T,i}) \in \mathcal{S} \mid \bigcap_{t=1}^{T}(\mathcal{B}_t = (\widetilde{\mathcal{D}}'_t, \boldsymbol{\theta}_{1:t-1})))\\ &+ 1 - (1-\delta)\prod_{t=1}^{T}(1-\delta_t),\end{aligned}$$

where for all $t \in [T]$, $\mathcal{B}_t$ denotes the input for the t-th iteration,

$$\boldsymbol{\theta}_{1:t-1} = \begin{cases} \emptyset, & t=1\\ (\theta_{1,1},\ldots,\theta_{1,m})\ldots,(\theta_{t-1,1},\ldots,\theta_{t-1,m}), & t\geq 2,\end{cases}$$

and ϵ is defined in Lemma 7.

Then, we have for any set $\mathcal{S}' \subseteq \mathbb{R}^{d\times(m-1)\times T}$,

$$\begin{aligned}&\mathbb{P}(\hat{\mathbf{w}}_{[-i]}^{(1:T)} \in \mathcal{S}' \mid \bigcap_{t=1}^{T}\mathcal{B}_t = (\mathbf{W}^{(t-1)}, \mathcal{D}^m, \boldsymbol{\theta}_{1:t-1}))\\ \leq & e^{\epsilon}\mathbb{P}(\hat{\mathbf{w}}_{[-i]}^{(1:T)} \in \mathcal{S}' \mid \bigcap_{t=1}^{T}\mathcal{B}_t = ((\mathbf{W}')^{(t-1)}, (\mathcal{D}')^m, \boldsymbol{\theta}_{1:t-1}))\\ &+ 1 - (1-\delta)\prod_{t=1}^{T}(1-\delta_t).\end{aligned}$$

□

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