Neumann boundary condition

Suppose $\nabla\phi = 0$ on boundaries, use cosine transform

$$\phi_{jl} = \frac{2}{J}\frac{2}{L}\sum_{m=0}^{J}{}''\sum_{n=0}^{L}{}''\tilde{\phi}_{mn}\cos\frac{\pi jm}{J}\cos\frac{\pi ln}{L}$$

$$\tilde{\phi}_{mn} = \sum_{j=0}^{J}{}''\sum_{l=0}^{L}{}''\phi_{jl}\cos\frac{\pi jm}{J}\cos\frac{\pi ln}{L}$$

$$\rho_{jl} = \frac{2}{J}\frac{2}{L}\sum_{m=0}^{J}{}''\sum_{n=0}^{L}{}''\tilde{\rho}_{mn}\cos\frac{\pi jm}{J}\cos\frac{\pi ln}{L}$$

$$\tilde{\rho}_{mn} = \sum_{j=0}^{J}{}''\sum_{l=0}^{L}{}''\rho_{jl}\cos\frac{\pi jm}{J}\cos\frac{\pi ln}{L}$$

$\sum''$: $\begin{matrix} j,m = 0 \text{ or } J \\ l,n = 0 \text{ or } L \end{matrix}$ $\Rightarrow$ multiplied by 1/2. c.f. $\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(mx)\cos(nx)dx = \begin{cases} 0 & m \neq n \\ \frac{1}{2} & m = n \neq 0 \\ 1 & m = n = 0 \end{cases}$

Again, the boundary condition $\nabla\phi = 0$ only determines the solution up to an arbitrary constant. One is free to choose arbitrary values for $\tilde{\phi}_{00}$.

Proof of DCT:

I shall prove 1D. Generalization to 2D is left as exercise.

- Here we have ϕ_j with j=0,1,…J
- Even extension: ϕ_j with $j = -1, -2, \ldots, -J+1$ and $\phi_{-j} = \phi_j$
- Period = $2J$

$$\begin{aligned}
\tilde{\phi}'_k &= \sum_{j=-J+1}^{J}\phi_j \exp\left(i\frac{2\pi kj}{2J}\right) \\
&= \sum_{j=-J+1}^{-1}\phi_j \exp\left(i\frac{2\pi kj}{2J}\right) + \phi_0 + \sum_{j=1}^{J-1}\phi_j \exp\left(i\frac{2\pi kj}{2J}\right) + (-1)^k\phi_J \\
&= \phi_0 + (-1)^k\phi_J + \sum_{j=1}^{J-1}\phi_j\left[\exp\left(i\frac{2\pi kj}{2J}\right) + \exp\left(-i\frac{2\pi kj}{2J}\right)\right] \\
&= 2\left[\frac{1}{2}\phi_0 + \frac{(-1)^k}{2}\phi_J + \sum_{j=1}^{J-1}\phi_j\cos\left(\frac{\pi kj}{J}\right)\right] \\
&= 2\sum_{j=0}^{J}{}''\phi_J\cos\left(\frac{\pi kj}{J}\right)
\end{aligned}$$

Define $\tilde{\phi}_k = \frac{1}{2}\tilde{\phi}_k{}'$

$$\tilde{\phi}_k = \sum_{j=1}^{J-1}{}''\phi_j \cos\left(\frac{\pi kj}{J}\right)$$

It is readily observed that $\tilde{\phi}_k$ has period $2J$ and $\tilde{\phi}_{-k} = \tilde{\phi}_k$.

The inverse transform is

$$\begin{aligned}
\phi_j &= \frac{1}{2J}\sum_{k=-J+1}^{J}\tilde{\phi}'_k \exp\left(-i\frac{2\pi kj}{2J}\right) \\
&= \frac{1}{2J}2\sum_{k=-J+1}^{J}\tilde{\phi}_k \exp\left(-i\frac{2\pi kj}{2J}\right) \\
&= \frac{1}{J}\left[\sum_{k=-J+1}^{-1}\tilde{\phi}_k \exp\left(-i\frac{2\pi kj}{2J}\right) + \tilde{\phi}_0 + \sum_{k=1}^{J-1}\tilde{\phi}_k \exp\left(-i\frac{2\pi kj}{2J}\right) + (-1)^k\tilde{\phi}_J\right] \\
&= \frac{1}{J}\left[\tilde{\phi}_0 + (-1)^k\tilde{\phi}_J + \sum_{k=1}^{J-1}\tilde{\phi}_k \exp\left(i\frac{2\pi kj}{2J}\right) + \sum_{k=1}^{J-1}\tilde{\phi}_k \exp\left(-i\frac{2\pi kj}{2J}\right)\right] \\
&= \frac{2}{J}\left[\frac{1}{2}\tilde{\phi}_0 + \frac{(-1)^k}{2}\tilde{\phi}_J + \sum_{k=1}^{J-1}\tilde{\phi}_k \cos\left(\frac{\pi kj}{J}\right)\right] \\
&= \frac{2}{J}\sum_{k=1}^{J-1}{}''\tilde{\phi}_k \cos\left(\frac{\pi kj}{J}\right)
\end{aligned}$$

Procedure:

Step 1) Compute DCT of ρ

$$\tilde{\rho}_{mn} = \sum_{j=0}^{J}{}''\sum_{l=0}^{L}{}''\rho_{jl}\cos\frac{\pi jm}{J}\cos\frac{\pi ln}{L}$$

Step 2) Compute $\tilde{\phi}_{mn}$ from $\tilde{\rho}_{mn}$ for $(m,n) \neq (0,0)$

$$\tilde{\phi}_{mn} = \frac{\tilde{\rho}_{mn}\Delta^2}{2\left(\cos\frac{\pi m}{J} + \cos\frac{\pi n}{L} - 2\right)}$$

And set $\tilde{\phi}_{00} =$ Arbitrary number

Step 3) Compute ϕ_{ij} using inverse DCT

$$\phi_{jl} = \frac{2}{J}\frac{2}{L}\sum_{m=0}^{J}{}''\sum_{n=0}^{L}{}''\tilde{\phi}_{mn}\cos\frac{\pi jm}{J}\cos\frac{\pi ln}{L}$$

Note:

(1) In Step 2), the angles in cosine are halved because the periods are doubled.

(2) Must have $\tilde{\rho}_{00} = 0$ for solution to exist.

For inhomogeneous B.C.:

- Suppose $\nabla\phi = g(y)$ at $x = 0$
- B.C. $\frac{\phi_{1,l} - \phi_{-1,l}}{2\Delta} = g_l$
- Write the solution as

$$\phi = \phi^I + \phi^B$$

where $\nabla\phi^I = 0$ on all boundaries and $\phi^B = 0$ everywhere except just outside the boundaries

$$\nabla\phi = g(y) = \nabla\phi^I + \nabla\phi^B = \nabla\phi^B$$

$$\Rightarrow \phi^B_{-1,l} = -2g_l\Delta$$

Finite differencing:

$$\phi^I_{j+1,l} + \phi^I_{j-1,l} + \phi^I_{j,l+1} + \phi^I_{j,l-1} - 4\phi^I_{jl} = \begin{cases} 2g_l\Delta + \rho_{0,l}\Delta^2 & j = 0 \\ \rho_{j,l}\Delta^2 & \text{otherwise} \end{cases}$$