MSDM5004 Homework2 Part I

1. ~~Jacobi method~~

$$x_1 = \frac{1+x_2-x_3}{3}$$

$$x_2 = \frac{-3x_1-2x_3}{6}$$

$$x_3 = \frac{-3x_1-3x_2+4}{7}$$

Jacobi method:

first iteration:

$$x_1^{(1)} = \frac{1+0-0}{3} = \frac{1}{3}$$

$$x_2^{(1)} = \frac{0-0}{6} = 0$$

$$x_3^{(1)} = \frac{0-0+4}{7} = \frac{4}{7}$$

$$\vec{x}^{(1)} = \left(\frac{1}{3}, 0, \frac{4}{7}\right)$$

second iteration.

$$x_1^{(2)} = \frac{1+0-\frac{4}{7}}{3} = \frac{1}{7}$$

$$x_2^{(2)} = \frac{-3\cdot\frac{1}{3}-2\cdot\frac{4}{7}}{6} = -\frac{5}{14}$$

$$x_3^{(2)} = \frac{-3\cdot\frac{1}{3}-3\cdot 0+4}{7} = \frac{3}{7}$$

$$\vec{x}^{(2)} = \left(\frac{1}{7}, -\frac{5}{14}, \frac{3}{7}\right)$$

Gauss-Seidel method:

first ~~step~~ iteration:

$$x_1^{(1)} = \frac{1+0-0}{3} = \frac{1}{3}$$

$$x_2^{(1)} = \frac{-3\times\frac{1}{3}-2\cdot 0}{6} = -\frac{1}{6}$$

$$x_3^{(1)} = \frac{-3\cdot\frac{1}{3}-3\cdot\left(-\frac{1}{6}\right)+4}{7} = \frac{1}{2}$$

$$\vec{x}^{(1)} = \left(\frac{1}{3}, -\frac{1}{6}, \frac{1}{2}\right)$$

second iteration:

$$x_1^{(2)} = \frac{1-\frac{1}{6}-\frac{1}{2}}{3} = \frac{1}{9}$$

$$x_2^{(2)} = \frac{-3\cdot\frac{1}{9}-2\cdot\frac{1}{2}}{6} = -\frac{2}{9}$$

$$x_3^{(2)} = \frac{-3\cdot\frac{1}{9}-3\cdot\left(-\frac{2}{9}\right)+4}{7}$$

$$= \frac{13}{21}$$

$$\vec{x}^{(2)} = \left(\frac{1}{9}, -\frac{2}{9}, \frac{13}{21}\right)$$

Homework 2 Part II

5. Proof: Consider Taylor expansion

$$u(t+\Delta t, x) = u(t,x) + \Delta t\, u_t(t,x) + \frac{\Delta t^2}{2} u_{tt}(t,x) + \frac{\Delta t^3}{6} u_{ttt}(t,x) + O(\Delta t^4)$$

$$u(t-\Delta t, x) = u(t,x) - \Delta t\, u_t(t,x) + \frac{\Delta t^2}{2} u_{tt}(t,x) - \frac{\Delta t^3}{6} u_{ttt}(t,x) + O(\Delta t^4)$$

$$\frac{u(t+\Delta t, x) - u(t-\Delta t, x)}{2\Delta t} = u_t(t,x) + \frac{\Delta t^2}{6} u_{ttt}(t,x) + O(\Delta t^4)$$

Thus, the central time difference is ~~So the error is second order in time~~ second order accurate.

$$u(t, x+\Delta x) = u(t,x) + \Delta x\, u_x(t,x) + \frac{\Delta x^2}{2} u_{xx}(t,x) + \frac{\Delta x^3}{6} u_{xxx}(t,x) + \frac{\Delta x^4}{24} u_{xxxx}(t,x)$$

$$u(t, x-\Delta x) = u(t,x) - \Delta x\, u_x(t,x) + \frac{\Delta x^2}{2} u_{xx}(t,x) - \frac{\Delta x^3}{6} u_{xxx}(t,x) + \frac{\Delta x^4}{24} u_{xxxx}(t,x) + O(\Delta x^5)$$

$$+ O(\Delta x^5)$$

$$\frac{u(t, x+\Delta x) - 2u(t,x) + u(t, x-\Delta x)}{(\Delta x)^2} = u_{xx}(t,x) + \frac{\Delta x^2}{12} u_{xxxx}(t,x) + O(\Delta x^4)$$

Hence the central space difference is second order accurate.

Replacing $U_j^n$ with the exact solution $u(t_n, x_j)$ and using the above expansions, the local truncation error is $T = O(\Delta t^2) + O(\Delta x^2)$

Therefore, the scheme is second order accurate in both time and space.