MSDM 5059

1. (a) $f'(x) = 2\frac{\sin x}{x} \cdot \left(\frac{x\cos x - \sin x}{x^2}\right)$ Let $f'(x)=0$

$= \frac{2\sin x(x\cos x - \sin x)}{x^3}$ Then $\sin x(x\cos x - \sin x) = 0$

~~$f''(x) = 2\cos x(x\cos x - \sin x) + 2\sin x(\cos x - x\sin x - \cos x)$~~

Let $g(x) = \frac{\sin x}{x}$ $f'(x) = 2g(x)g'(x)$ $f''(x) = 2g(x)g''(x) + 2g'(x)^2$

$g''(x) = \frac{x^2(-\sin x) - 2x(x\cos x - \sin x)}{x^4}$

Hence when $x = n\pi$, $g(x) = 0$. $f'(x) = 0$, $f''(x) > 0$

Thus $x = n\pi, n = \pm 1, \pm 2, \cdots$, are local minima

(b) Let $f'(x) = 0$. $x\cos x - \sin x = 0 \Rightarrow x = \tan x$ $x \neq 0$

At this time $g''(x) = -\frac{x^3 \sin x}{x^4} = -\frac{\sin x}{x} = -\cos x$

$f''(x) = -2 \cdot \frac{\sin x}{x} \cos x = -2\cos^2 x < 0$

if $\cos x = 0$, $\sin x = 0$, so $\cos x$ can't be 0

Hence $f''(x) < 0$ $x \neq 0$.

(c) ~~$x_{n+1} = \tan x_n$~~ ~~$x_0 = 1.3\pi$~~ I have tried $x_{n+1} = \tan x_n$, $x_0 = 1.3\pi$,

~~I've tried~~ and after 10 iterations, $\tan x$ growed quite fast,

and the iteration diverged. $|g'(x)| = |\sec^2 x| \geq 1$ always holds.

So I changed the scheme to ~~$x_{n+1} = (1-\lambda)x_n + \lambda \tan x_n$ and let $\lambda = 0.01$~~

~~At this time $x_{10} = 4.3232$~~ $x_{n+1} = \arctan x_n + \pi$

Then $x_{10} = 4.493409$

2. Let $\begin{cases} f_x = x+\cos y = 0 \\ f_y = -x\sin y = 0 \end{cases} \Rightarrow \begin{matrix} (0, \frac{\pi}{2}+k\pi),\ k\in\mathbb{Z} \\ ((-1)^{k+1}, k\pi),\ k\in\mathbb{Z} \end{matrix}$

$f_{xx} = 1$

$f_{yy} = -x\cos y$ $\quad H = \begin{bmatrix} 1 & -\sin y \\ -\sin y & -x\cos y \end{bmatrix}$ $D = \det(H) = -x\cos y - \sin^2 y$

$f_{xy} = -\sin y$

when $(0, \frac{\pi}{2}+k\pi)$ $D = -1$, saddle point

when $((-1)^{k+1}, k\pi)$

$D = (-1)^k \cos(k\pi) = 1$ ~~if k is a odd number, D = 1,~~ else $D = 1$. and $f_{xx} = 1 > 0$. H is positive definite

Hence ~~when k is a even number~~

$((-1)^{k+1}, k\pi)$ are local maximum

and $(0, \frac{\pi}{2}+k\pi)$ are neither a maximum nor a minimum

3. (a) $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha \\ \alpha \end{pmatrix}$ Thus $H(\alpha,\alpha) = 5e^{-\frac{1}{2}[(\alpha-1)^2+(\alpha-1)^2]} + 3e^{-(\alpha+1)^2-(\alpha+1)^2} + \sin\alpha\cos\alpha$

$= 5e^{-(\alpha-1)^2} + 3e^{-2(\alpha+1)^2} + \sin\alpha\cos\alpha$

$F(\alpha) = -H(\alpha,\alpha) = -5e^{-(\alpha-1)^2} - 3e^{-2(\alpha+1)^2} - \frac{1}{2}\sin 2\alpha$

(b) $F'(\alpha) = -5e^{-(\alpha-1)^2}\cdot(-2(\alpha-1)) - 3e^{-2(\alpha+1)^2}\cdot(-4(\alpha+1)) - \cos 2\alpha$

$= 10(\alpha-1)e^{-(\alpha-1)^2} + 12(\alpha+1)e^{-2(\alpha+1)^2} - \cos 2\alpha$

$F''(\alpha) = 10e^{-(\alpha-1)^2} + 10(\alpha-1)e^{-(\alpha-1)^2}(-2(\alpha-1)) + 12e^{-2(\alpha+1)^2} + 12(\alpha+1)e^{-2(\alpha+1)^2}(-4(\alpha+1)) + 2\sin 2\alpha$

(c) $F'(1) = 24e^{-8} - \cos 2$ $= 0.4241908$ $\quad F''(1) = 10 + 12e^{-8} + 24e^{-8}\cdot(-8) + 2\sin 2$

$F'(1) = 0.4249198$ $\quad$ $F''(1) = 11.75821$

$$\alpha_1 = \alpha_0 - \frac{F'(\alpha)}{F''(\alpha)} = 1 - \frac{0.4249198}{11.75821} = 0.963923$$

$F(\alpha_1) = -5.46330213$

(d) the minimizer is 0.9639456

The minimum value is -5.4633021

3 steps

(e) $\alpha_2 = 2-(2-0)\times 0.618 = 0.7639320$

$\alpha_3 = 0+(2-0)\times 0.618 = 1.2360680$

(f). minimizer 0.9639455

minimum value -5.4633021

31 steps