Assortative Matching with Private Information

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Abstract

We study matching between heterogeneous agents when their types are private information. Competing platforms post terms of trade. Agents with private information choose where to search and form matches. Positively assortative matching arises when each market attracts only one type of agent. We characterize an equilibrium with positively assortative matching when one exists, and provide sufficient conditions to ensure that matching is positively assortative in a limit with a vanishing role for platforms. When more desirable partners have a higher willingness-to-pay for matches, they pay high fees to platforms to avoid less desirable types. When more desirable partners have a lower willingness-to-pay, they match at a low rate to keep out less desirable types.

1 Introduction

Matching complementarities generate positively assortative matching, while substitutability generates negatively assortative matching (Koopmans and Beckmann, 1957; Becker, 1973). This conclusion holds in an environment where individual characteristics are common knowledge, so in particular everyone knows whom they are matching with when they match. This paper examines who matches with whom when individuals are privately informed about their own characteristics, and those characteristics affect the value an individual gets from matching and the value a partner gets from matching with them.

In order to have sorting in an environment where characteristics are completely hidden, there must be a mechanism for getting individuals with different types to separate themselves. The mechanism we explore is their choice of competing markets, where a market is defined as a combination of an access fee and a composition of other individuals in the market. Individuals self-select into different markets because they have different preferences over access fees and composition. We stress that markets do not have any special ability to screen individuals, but instead rely on this self-selection.

Importantly, we assume that there are three inputs into matching in a market. First, there are two sides to each market. These two sides are intrinsically the same, but the market may charge a different fee for accessing each side. As a result, the composition of individuals on each side of the market may be different. And second, there is another kind of actor, homogeneous competitive platforms, which facilitate matching through their costly postings to markets. Free entry pins down the number of postings, and postings are only made in the most profitable markets. When a market matches a posting with a random, independently selected, and privately-informed individual from each side of the market, the individuals pay the stated fees and then collect any benefit from matching.

This structure is rich enough to allow for both positive and negative sorting between the privately-informed individuals. Positive sorting arises if each market only attracts one type of privately-informed individual, the same type on both sides of the market. Negative sorting arises if each market attracts one type of individual on one side of the market and very different type on the other side of the market. Platform postings play an important role in this analysis because they permit us to have independent variation in the ease of matching on both sides of the market. For example, it may be easier to match on either side of one market than on either side of another market. We find that this possibility is critical in markets with positive sorting.

A couple of examples may be useful for illustrating our environment. The first is a model of partnership, as in the labor or marriage market. Individuals are privately informed about their own characteristics, something that cannot be observed by platforms (e.g. job search or dating apps) or potential partners until after a partnership is formed. A critical object is the payoff that an individual with hidden characteristic i gets from matching with someone with hidden characteristic j, before paying any fee to the platform. We call this the payoff function $u_{i,j}$ and assume it captures everything that happens after i and j match and possibly learn each others' type. In the partnership model, it may be natural to assume that u is increasing in each of its arguments and supermodular, so the cross-partial derivative is positive. The assumption that u is increasing in its first argument means higher types have a higher willingness-to-pay for any partner. The assumption that it is increasing in its second argument means that everyone agrees that higher types are more desirable partners. And supermodularity implies that higher types have a higher willingness-to-pay for partners.

In this example, we find that if there is positively assortative matching, higher types match at a higher rate, pay higher fees, and get higher utility from matching. In a limit where the role of platforms in the matching function disappears, separation occurs exclusively through differences in fees paid to the platform, with everyone matching at the same rate. In that limit, we also provide a sufficient condition for positively assortative matching.

Our second example is one of disease transmission, such as of HIV. Individuals are privately informed about their probability of being infected, again something that cannot be observed by platforms or potential partners, and prefer to match with healthier partners. More precisely, if the type is the probability of being healthy, then the value of a match is increasing in the partner's health status but decreasing in own health status, since healthier people face a higher risk of becoming sick in any match. Finally, the payoff function is still supermodular, so a healthier person gains more from a given improvement in partner health than does a sick person.

In this case, we find that if there is positively assortative matching, higher (healthier) types match at a lower rate, pay lower fees, and get a lower utility from matching, the opposite of the results in the partnership model. In a limit where the role of platforms in the matching function disappears, separation occurs exclusively through different matching rates, with all fees converging to zero. That is, healthy people isolate themselves from the market by reducing the number of partners, rather than trying to screen out undesirable partners through fees. Again, we provide a sufficient condition for positively assortative matching in this environment.

Our model is an extension of the competitive search equilibrium framework to an environment with two-sided private information. We build on Guerrieri, Shimer and Wright (2010) and Guerrieri and Shimer (2014), who analyzed competitive search equilibrium with one-sided private information. A key novelty in the present environment is that privately informed individuals care about who they match with and their partner also has private information. Nevertheless, our notion of competitive search equilibrium draws heavily on that earlier research.

A market is defined by a pair of transfers from successfully matched agents on each side to the platform, as a well as a recommendation about which types of agents should show up on each side of the market. Privately-informed agents direct their search to a particular side of a particular market, and platforms make postings in the most profitable market. A constant returns to scale matching function with these three inputs then delivers the number of matches between two agents and a platform. Through variation in the ratios of the three arguments of the matching function, we obtain independent variation in the ease of matching for the privately informed individuals on both sides of the market. In equilibrium, the ease of matching on each side of a market is determined endogenously and platforms' recommendation about what types of individuals show up on each side of the market must be fulfilled in any active market, a restriction that we call promise-keeping.

In a competitive search equilibrium, platforms and privately-informed individuals have rational beliefs about the ease of matching on each side of each market. They assume that their own behavior does not affect this. They then go to the market, and the side of the market, that delivers the highest expected utility. In turn, the recommendation about who is on each side of a market and the belief about how hard it is to match on that side are consistent with individuals behaving rationally in their choice of markets.

As is common in the competitive search literature, we prove that a competitive search equilibrium can be characterized through the solution to an optimization problem. But differently than the existing literature, we find that without further assumptions, the equilibrium may have pooling, meaning that more than one type of individual comes to one side of an active market. We obtain separation under two assumptions. First, we assume that all types have a common ranking over partners, so the payoff function $u_{i,j}$ is increasing in j for all i. Second, we assume that higher types gain more from an increase in their partner's type, so the payoff function is supermodular in i and j. These assumptions also imply that incentive constraints bind downwards, so equilibrium is constrained only by the need to keep lower types out of the market.

Prior research has explored sorting in competitive search equilibrium with observable types. Shi (2001) characterizes efficient sorting patterns and shows how they can be decentralized through a competitive search equilibrium. Heterogeneous firms post wages and skill requirements, and workers apply for the job yielding the highest expected utility. Eeckhout and Kircher (2010) shows that negatively assortative matching can arise when there are matching complementarities, and prove that complementarity and n-root-complementarity together are sufficient to ensure positively assortative matching.

Our main innovation relative to these papers is the introduction of private information, which means (in the language of Shi (2001)) that firms cannot post skill requirements, but instead must accept the mix of workers who apply for the job. Additionally, we assume that both sides of the market care about their partner's type. In Shi (2001), workers only care about wages, not the type of firm they work for. In Eeckhout and Kircher (2010), sellers only care about the price they get, not who buys the object for sale.

There are also papers exploring sorting with private information. Hoppe, Moldovanu and Sela (2009) characterize assortative matching with private information. They assume that agents send costly signals and matching is assortative in signals. Thus there is positively assortative matching if higher types send higher signals. A single-crossing property ensures that this happens in equilibrium. Damiano and Li (2007) and Hoppe, Moldovanu and Ozdenoren (2011) characterize the screening of privately-informed types in a matching environment by a monopoly platform. We make three innovations relative to this literature. First, we allow for endogenous matching rates, so screening works both through platform fees and through contact rates. We show equilibrium screening can happen through either margin. Second, we consider a competitive search equilibrium, where platforms face competition from other platforms. This leads to different predictions regarding the separation of types into different markets. Lastly, we allow for more general payoff functions, showing that the characterization of equilibrium—whether separation works through platform fees or contact rates—depends critically on whether more desirable types have a higher or lower willingness-to-pay.

In Section 2, we develop a general framework for analyzing sorting in competitive search equilibrium when types are private information. Section 3 analyzes equilibrium, including finding sufficient conditions for positively assortative matching. For comparison with the literature, we briefly discuss the conditions that lead assortative matching when types are observable in Section 4.

2 Model

2.1 Platforms and Agents

Time is continuous and lasts forever. There are two sets of individuals, platforms and agents. Agents are privately-informed about their type $i \in \{1, \ldots, I\} \equiv \mathbb{I}$. Let $\bar{\omega}_i > 0$ denote the exogenous measure of type i agents in the economy, with $\sum_{i=1}^{I} \bar{\omega}_i = 1$. There is a large number of homogeneous platforms which can post to a market by paying a flow cost c. Platforms can make as many postings as they want, potentially to different markets. Agents and platforms are risk-neutral, infinitely-lived, and discount the future at rate r.

2.2 Matching Function

Platforms intermediate matches between two agents. There are three parties to a match, a platform's posting, an agent on side a, and an agent on side b, with a Cobb-Douglas, constant returns to scale matching technology. Let n^a denote the ratio of agents on side a of the market to platform postings and n^b denote the ratio of agents on side b of the market to platform postings. Then postings match an agent on side a with an agent on side b at a rate $(n^a n^b)^{\frac{1-\gamma}{2}}$, where $0 < \gamma < 1$ measures the importance of the platform. The matching rate for an agent on side $s = a, b$ is equal to the matching rate of postings divided by n^s, the ratio of agents on side s to postings. That is, an agent going to side s finds a match according to a Poisson process with arrival rate $\lambda^s \equiv (n^s)^{-\frac{1+\gamma}{2}}(n^{-s})^{\frac{1-\gamma}{2}}$. Throughout we use the notation $-s = b$ if $s = a$ and $-s = a$ if $s = b$[1].

2.3 Markets and Payoffs

A market is a vector $m = (\phi^s, \omega^s)_{s=a,b}$ satisfying $\omega^s \equiv (\omega_1^s, \ldots, \omega_I^s) \in \Delta^I$, the standard probability simplex. We interpret ϕ^s to be the fee paid by an agent on side s of the market when he matches; and ω_i^s to be the share of type i agents on side s of the market. Let $\mathbb{M}$ denote the set of markets, i.e. the set of vectors satisfying these conditions. The number of agents on the side s of market m is given by an equilibrium mapping $N : \{a, b\} \times \mathbb{M} \to \mathbb{R}_+$, which is determined by the optimal search decision of agents detailed later.

A platform posting in market $m \in \mathbb{M}$ matches two agents at rate $\left[N^a(m)N^b(m)\right]^{\frac{1-\gamma}{2}}$. When this happens, the platform earns fees of $\phi^a + \phi^b$. Thus the gross flow profit from the posting is

$$rV(m) \equiv \left[N^a(m)N^b(m)\right]^{\frac{1-\gamma}{2}} \left(\phi^a + \phi^b\right). \tag{1}$$

From this, we must subtract the flow cost of postings rc to obtain the flow value of the posting.

Similarly, if a type i agent goes to side $s \in \{a, b\}$ of market $m \in \mathbb{M}$, he matches at rate $\frac{1}{N^s(m)}\left[N^a(m)N^b(m)\right]^{\frac{1-\gamma}{2}}$, in which event he pays a fee ϕ^s and is matched to a type j agent on the $-s$ side of the market with probability ω_j^{-s}, earning an exogenous

[1] This matching process can be derived from a Cobb-Douglas matching function that takes the number of agents and the number of postings as arguments: $(\bar{n}^a\bar{n}^b)^{1-\gamma}(n^p)^\gamma$, where $\bar{n}^s$ is the total number of agents in a particular market and n^p is the number of postings in the same market.

payoff $u_{i,j}$. Thus the value to i of this action is

$$rU_i^s(m) \equiv \frac{1}{N^s(m)} \left[N^a(m)N^b(m)\right]^{\frac{1-\gamma}{2}} \left(\sum_{j=1}^{I} \omega_j^{-s} u_{i,j} - \phi^s\right). \tag{2}$$

Note that if $N^{-s}(m) > 0$ and we consider the limit as $N^s(m)$ converges to zero, $U_i^s(m)$ grows (shrinks) without bound if $\sum_{j=1}^{I} \omega_j^{-s} u_{i,j} - \phi^s$ is positive (negative). We impose that more generally:

$$U_i^s(m) \equiv \begin{cases} \infty \\ 0 & \text{if } \sum_{j=1}^{I} \omega_j^{-s} u_{i,j} \gtreqless \phi^s \text{ and } N^s(m) = 0. \\ -\infty \end{cases} \tag{3}$$

If there are zero agents per posting on side s of market m, an agent expects to match instantaneously, with unbounded payoff.

Throughout the paper, we assume the payoff from matching $u_{i,j}$ is positive and strictly increasing in the partner's type. The real content of this assumption is that everyone has a common ranking over partners.[2] Given this, we normalize the order such that everyone strictly prefers to match with a higher type.

Assumption 1 (Common Ranking) *For every i and $j > j'$, $u_{i,j} > u_{i,j'} > 0$.*

2.4 Equilibrium

We start by defining a partial equilibrium.

Definition 1 *A **partial equilibrium** $\{N, M^p, M, \bar{\boldsymbol{U}}\}$ is a mapping $N : \{a, b\} \times \mathbb{M} \to \mathbb{R}_+$, two nonempty sets $M \subseteq M^p \subseteq \mathbb{M}$, and strictly positive numbers $\bar{\boldsymbol{U}} \equiv \bar{U}_1, \ldots, \bar{U}_I$ such that:*

1. *(Optimal Search) $\forall m \in \mathbb{M}$, $s \in \{a, b\}$, and $i \in \mathbb{I}$, $\bar{U}_i \geq U_i^s(m)$; and if $N^s(m) > 0$, $\bar{U}_i = U_i^s(m)$ for some $i \in \mathbb{I}$;*
2. *(Promise Keeping) $M^p = \{m \in \mathbb{M} | \omega_i^s > 0 \Rightarrow \bar{U}_i = U_i^s(m)\}$;*
3. *(Profit Maximization) $M = \arg\max_{m \in M^p} V(m)$.*

There are three pieces to this definition. First, in any market, platforms should expect that agents will come to side s until one type of agent attains their equilibrium value and all other types attain weakly less than their equilibrium value. The only exception is that no one will come to side s of a market if matching there delivers a non-positive

[2]The assumption that $u_{i,j}$ is positive ensures that everyone goes to some market in competitive search equilibrium, simplifying our exposition.

payoff, $U_i^s(m) \le 0$, as will be the case if $\sum_{j=1}^I \omega_j^{-s} u_{i,j} \le \phi^s$ for all i; see equation (3). We illustrate this with an example. Suppose there are two types and consider a market m with $\phi^a = \phi^b = \phi$ and $\omega_2^a = \omega_2^b = 1$, so the market is supposed to only attract type 2 agents. The symmetry of the market implies $N^a(m) = N^b(m) = N(m)$, and optimal search implies

$$N(m) = \max\left\{ \left(\frac{u_{1,2} - \phi}{r\bar{U}_1}\right)^{\frac{1}{\gamma}}, \left(\frac{u_{2,2} - \phi}{r\bar{U}_2}\right)^{\frac{1}{\gamma}} \right\},$$

assuming at least one of these ratios is positive; otherwise $N(m) = 0$. The expected agent-posting ratio is determined by the type willing to take the higher ratio $N(m)$, which in turn depends on which type has the higher gain from trade, $u_{i,2} - \phi$, relative to the equilibrium value. Note that in making this calculation, agents believe the market's commitment to deliver a mixture ω^s, but we turn to the implications of that restriction next.

The second part of the definition of equilibrium restricts attention to markets $m \in M^p$ where the platform can attract the promised types. That is, any type with $\omega_i^s > 0$ must attain their equilibrium value when going to that market. Continuing the previous example, $m \in M^p$ if and only if

$$\frac{u_{2,2} - \phi}{r\bar{U}_2} \ge \frac{u_{1,2} - \phi}{r\bar{U}_1} \text{ and } u_{2,2} > \phi,$$

so $\bar{U}_2 = U_2^s(m)$. Otherwise a platform cannot deliver the promised type to the market. For a given ω, this condition generally holds only for some values of ϕ.

Finally, the third part of the definition of equilibrium defines M to be subset of promise-keeping markets which are profit maximizing. In equilibrium, only these markets can be active.

Armed with the definition of a partial equilibrium, we can define a *competitive search equilibrium* as follows:

Definition 2 *A* ***competitive search equilibrium*** *is a partial equilibrium* $\{N, M^p, M, \bar{\boldsymbol{U}}\}$ *and a measure* μ *on the set of active market* M *such that*
1. *(free entry)* $c = V(m)$ *for all* $m \in M$*;*
2. *(market clearing)* $\bar{\omega}_i = \int_M \left[\sum_{s=a,b} \omega_i^s N^s(m)\right] d\mu(m)$.

In addition to the requirement of a partial equilibrium, a competitive search equilibrium requires that the maximal gross profit from a promise-keeping posting is exactly the flow cost of maintaining a posting. It also imposes that the total measure of agents of type i participating in active markets equals their exogenous supply.

The notion of (partial and competitive search) equilibrium builds on the competitive search literature, going back to Moen (1997). It is worth noting how private information changes the definition of equilibrium compared to a model with full information (Shi, 2001; Eeckhout and Kircher, 2010). With private information, the fee ϕ^s cannot be type-dependent and the type distribution ω_i^s must be consistent with agents' self selection, as captured in part 2 of the definition of partial equilibrium. In contrast, when types are observable, platforms may set a type-specific fee schedule to keep unwanted types from visiting a particular market. We could then modify parts 1 and 2 in Definition 1 to impose only that $\bar{U}_i = U_i^s(m)$ if $\omega_i^s > 0$, with no promise-keeping constraint and no restriction on $U_i^s(m)$ for types with $\omega_i^s(m) = 0$.

The definition of equilibrium is also a bit different than in Guerrieri, Shimer and Wright (2010). The main difference is that in this earlier paper, the mixture of agents in a market was not part of the description of a market, but instead was determined in equilibrium, analogous with how $N^s(m)$ is determined in part 1 of Definition 1 here. As a result, in that earlier environment there is no analog to the promise-keeping constraint in part 2 of our definition of equilibrium. We include ω^s in the definition of a market here because (unlike in the earlier work) transfers do not uniquely define a market. To see why, suppose there are two types and a payoff function u and a transfer $\phi < \min_{i,j} u_{i,j}$ such that

$$(u_{1,1} - \phi)(u_{2,2} - \phi) \geq (u_{2,1} - \phi)(u_{1,2} - \phi).$$

Then for some $(\bar{U}_1, \bar{U}_2)$,

$$\frac{u_{1,1} - \phi}{r\bar{U}_1} \geq \frac{u_{2,1} - \phi}{r\bar{U}_2} > 0 \text{ and } \frac{u_{2,2} - \phi}{r\bar{U}_2} \geq \frac{u_{1,2} - \phi}{r\bar{U}_1} > 0,$$

Then the first inequality implies the market $(\phi, (\omega_i^s)_{i=1}^I)_{s=a,b}$ with $\omega_1^a = \omega_1^b = 1$ satisfies the promise-keeping constraint, while second inequality implies that $(\phi, (\omega_i^s)_{i=1}^I)_{s=a,b}$ with $\omega_2^a = \omega_2^b = 1$ satisfies the constraint. When both markets satisfy the promise-keeping constraint, the specification of ω^s acts as a coordination device, allowing the platform to attain the desired mix of agents.

2.5 Separation and Assortative Matching

Two characteristics of the competitive search equilibrium are of particular interest to this paper. First, do the active markets in an equilibrium involve different pairs of types matching together, or is there only one pair of matching types possible? To answer this question, we define a notion of separating markets. Second, are high-type agents more likely to match with high-type or low-type agents? To answer this question, we define *positive assortative matching* (PAM) and *negative assortative matching* (NAM).

Definition 3 *A market $m \in \mathbb{M}$ is separating if $\omega_i^s \in \{0, 1\}$, $\forall\, i = 1, ..., I$ and $s = a, b$.*

A (partial or competitive search) equilibrium is separating if M contains only separating markets.

Our definition of sorting is based the set of matches made within active markets. Here we define positively assortative matching (PAM) and negatively assortative matching (NAM) for separating markets:

Definition 4 *Consider any set of separating markets M. For any market $m_k \in M$ and any side $s_k \in \{a, b\}$, fix (i_k, j_k) such that $\omega_{i_k}^{s_k}\omega_{j_k}^{-s_k} = 1$ in market m_k. If for any markets m_1 and m_2 and types (i_1, j_1) and (i_2, j_2) defined in this way, $i_1 > i_2$ implies $j_1 \geq (\leq) j_2$, the markets M have PAM (NAM). A (partial or competitive search) equilibrium has PAM (NAM) if the equilibrium set of markets M has PAM (NAM).*

Note that if there is a market $m \in M$ with $\omega_i^a\omega_j^b = 1$ for some $i \neq j$, the markets M do not have PAM. That is, a set of markets has PAM if and only if all markets are homogeneous, with the same type on both sides of the market. On the other hand, a nontrivial set markets M may have NAM even if some are homogeneous. For example, assume M consists of two markets, one with $\omega_1^a = \omega_1^b = 1$ and one with $\omega_1^a\omega_2^b = 1$. These markets have NAM.

2.6 Sorting with Conventional Matching Function

The key novelty of the environment in this paper is the assumption of a matching function that with two sides of a match which can have different contact rates. This assumption allows for the technological possibility of negative sorting. Before proceeding to the characterization of equilibrium, we consider an alternative "random" matching technology that involves only agents and platform postings. Platforms intermediate matches between two agents. Instead of the matching function in section 2.2, suppose the flow of matches in the market is $(N^a)^{1-\gamma}(N^p)^{\gamma}$ where $0 < \gamma < 1$, where N^a is the measure of agents, and N^p is the measure of postings by platforms. At this rate, there is a match between two randomly selected agents in the market, so if there are a shares ω_i of type i agents in a particular market, the share of (i, j) matches is $\omega_i\omega_j$. It follows that a platform posting in this market matches with two agents according to a Poisson process with arrival rate $(N^a)^{\gamma}(N^p)^{\gamma-1} = n^{\gamma}$ where $n \equiv N^a/N^p$ is the number of agents per platform. Similarly, an agent finds a match according to a Poisson process with arrival rate $\lambda \equiv (N^a)^{\gamma}(N^p)^{1-\gamma}/N^a = n^{-(1-\gamma)}$. This matching technology is a direct extension of the standard bilateral matching technology into our environment.

The core result of this section is the impossibility of positive sorting with this random matching technology. To show this, we define a set of active market as M, and the measure of platform postings as N^p, where $N^p(m)$ is the measure of postings in market $m \in M$. We define the covariance of matching types as:

$$\mathbf{COV}(M, N^p) = \frac{\sum_{m \in M} N^p(m) \lambda^{-\frac{1-\gamma}{\gamma}} \sum_{i=1}^{I} \sum_{j=1}^{J} \omega_i \omega_j \left(i - \mathbf{E}(i)\right) \left(j - \mathbf{E}(j)\right)}{\sum_{m \in M} N^p(m) \lambda^{-\frac{1-\gamma}{\gamma}}}, \tag{4}$$

where

$$\mathbf{E}(i) = \sum_{m \in M} N^p(m) \lambda^{-\frac{1-\gamma}{\gamma}} \sum_{i=1}^{I} \omega_i i \quad \text{and} \quad \mathbf{E}(j) = \sum_{m \in M} N^p(m) \lambda^{-\frac{1-\gamma}{\gamma}} \sum_{j=1}^{I} \omega_j j.$$

The covariance between agents' types and their matching partners' types must be non-negative with random matching technology, as stated in the following lemma:

Lemma 1 *Under a random matching technology,* $\mathbf{COV}(M, N^p) \geq 0$.

Proof. Due to random matching within markets, $\mathbf{E}(i) = \mathbf{E}(j)$. Thus we can write the covariance as

$$\mathbf{COV}(M, N^p) = \frac{\sum_{m \in M} N^p(m) \lambda^{-\frac{1-\gamma}{\gamma}} \left[\sum_{i=1}^{I} \omega_i \left(i - \mathbf{E}(i)\right)\right]^2}{\sum_{m \in M} N^p(m) \lambda^{-\frac{1-\gamma}{\gamma}}} \geq 0.$$

■

Lemma 1 highlights a physical restriction on sorting pattern due to random matching within markets. It is impossible to have negative assortative matching because for each match made between a high-type agent and a low-type agent, there must be another match between each types within themselves. The core assumption that leads to this result is that the contact rates for agents within each market equal. To allow for the possibility of negative assortative matching, we introduce our matching technology with two sides and a roll for platforms.

3 PAM with Private Information

3.1 Characterization of A Partial Equilibrium

We start by showing the equivalence between a partial equilibrium and an optimization problem. For given $\bar{\boldsymbol{U}}$,

$$\max_{\{\lambda^s>0,\phi^s,\omega^s\in\Delta^I\}_{s=a,b}} (\lambda^a\lambda^b)^{-\frac{1-\gamma}{2\gamma}}(\phi^a+\phi^b), \tag{5}$$

$$\text{s.t. } \bar{U}_i \geq \lambda^s\left(\sum_{j=1}^{I}\omega_j^{-s}u_{i,j}-\phi^s\right) \text{ for } s=a,b,$$

$$\omega_i^s\bar{U}_i = \omega_i^s\lambda^s\left(\sum_{j=1}^{I}\omega_j^{-s}u_{i,j}-\phi^s\right) \text{ for } s=a,b.$$

Our first result shows how to move back and forth between a partial equilibrium and a solution to problem (5). Intuitively, the objective is platforms' gross profit, where we invert the matching function $\lambda^s \equiv (n^s)^{-\frac{1+\gamma}{2}}(n^{-s})^{\frac{1-\gamma}{2}}$ to recover $(n^a n^b)^{\frac{1-\gamma}{2}} = (\lambda^a\lambda^b)^{-\frac{1-\gamma}{2\gamma}}$. The first constraint states agents' contact rate, the mix of agents on the other side of the market, and the fee must ensure that agents get utility no higher than $\bar{U}_i$. The second constraints states that they get exactly this utility when they are part of the mix in the market.

Lemma 2 *Suppose $\{N, M^p, M, \bar{\mathbf{U}}\}$ is a partial equilibrium. For any $m=(\phi^s,\omega^s)_{s=a,b}\in M$, denote $\lambda^s=(N^s(m))^{-\frac{1+\gamma}{2}}(N^{-s}(m))^{\frac{1-\gamma}{2}}$. Then $(\lambda^s,\phi^s,\omega^s)_{s=a,b}$ must solve problem (5) given $\bar{\mathbf{U}}$.*

Suppose $(\lambda^s,\phi^s,\omega^s)_{s=a,b}$ solves problem (5) given $\bar{\boldsymbol{U}}$. Then there is a partial equilibrium $\{N, M^p, M, \bar{\boldsymbol{U}}\}$ with $m=(\phi^s,\omega^s)_{s=a,b}\in M$ and $N^s(m)=(\lambda^s)^{-\frac{1+\gamma}{2\gamma}}(\lambda^{-s})^{-\frac{1-\gamma}{2\gamma}}$ for $s=a,b$.

Proof. First, take a partial equilibrium $\{N, M^p, M, \bar{\mathbf{U}}\}$. Take any $m=(\phi^s,\omega^s)_{s=a,b}\in M^p$ and denote $\lambda^s=(N^s(m))^{-\frac{1+\gamma}{2}}(N^{-s}(m))^{\frac{1-\gamma}{2}}$. Then the first two parts of the definition of a partial equilibrium imply $(\lambda^s,\phi^s,\omega^s)_{s=a,b}$ satisfies the constraints in problem (5). Conversely, if $m\in M^P$ and λ^s takes a different value, or if $m\notin M^P$ and λ^s takes an arbitrary values, $(\lambda^s,\phi^s,\omega^s)_{s=a,b}$ does not satisfy the constraints in problem (5). Moreover, the third part of the definition of partial equilibrium implies that if $m\in M\subseteq M^p$, then $(\lambda^s,\phi^s,\omega^s)_{s=a,b}$ attains a weakly higher value for the objective in problem (5) than any other tuple satisfying the constraints.

Next, take a tuple $(\lambda^s,\phi^s,\omega^s)_{s=a,b}$ that satisfies the constraints in problem (5). Let $m=(\phi^s,\omega^s)_{s=a,b}$ and $N^s(m)=(\lambda^s)^{-\frac{1+\gamma}{2\gamma}}(\lambda^{-s})^{-\frac{1-\gamma}{2\gamma}}$ for $s=a,b$. Then the first constraint in problem (5) implies $\bar{U}_i\geq U_i^s(m)$, while the second constraint implies

$\bar{U}_i = U_i^s(m)$ if $\omega_i^s > 0$. This implies $m \in M^p$ in the definition of partial equilibrium. Again, any tuple $(\lambda^s, \phi^s, \omega^s)_{s=a,b}$ that does not satisfy the constraints in problem (5) has $m = (\phi^s, \omega^s)_{s=a,b} \notin M^p$, though we can recover $N^s(m)$ using the first part of the definition of equilibrium. Finally, among the tuples that satisfies the constraints in problem (5), the ones that maximize the objective have $(\phi^s, \omega^s)_{s=a,b} \in M$. Thus we find the partial equilibrium $\{N, M^p, M, \bar{\boldsymbol{U}}\}$. ■

Problem (5) is a mathematical program with equilibrium constraints. Such problems are difficult to work with because of an inherent nonconvexity: Increasing ω_i^s from 0 to a small positive number changes an inequality constraint into an equality constraint. To make progress, we impose restrictions on the payoff function u which simplify the problem considerably. We turn to those next.

3.2 Implications of Supermodularity

In this section, we introduce the assumption that the payoff function u is supermodular:

Assumption 2 (Supermodularity) *For every $i > i'$ and $j > j'$, $u_{i,j} + u_{i',j'} > u_{i,j'} + u_{i',j}$.*

This allows us to simplify problem (5) as follows:

$$\max_{\{\lambda^s > 0, \phi^s, k^s\}_{s=a,b}} (\lambda^a \lambda^b)^{-\frac{1-\gamma}{2\gamma}} (\phi^a + \phi^b) \tag{6}$$
$$\text{s.t. } \bar{U}_{k^s} = \lambda^s (u_{k^s, k^{-s}} - \phi^s) \text{ for } s = a, b,$$
$$\bar{U}_i \geq \lambda^s (u_{i, k^{-s}} - \phi^s) \text{ for } i < k^s \text{ and } s = a, b.$$

Problem (6) is more restricted than problem (5) because we only look at separating markets, where $\omega_{k^a}^a \omega_{k^b}^b = 1$ for some types k^a and k^b. On the other hand, problem (6) is more relaxed than problem (5) because we only impose the incentive constraints for types below k^a and k^b, the *downward incentive constraints*. We find that when u is monotone in its second argument and supermodular, a solution to problem (6) solves the original problem (5):

Lemma 3 *Assume Common Ranking and Supermodularity. If $(\lambda^s, \phi^s, \omega^s)_{s=a,b}$ solves problem* (5) *then there exists an (k^a, k^b) such that $\omega_{k^a}^a \omega_{k^b}^b = 1$ and $(\lambda^s, \phi^s, k^s)_{s=a,b}$ solves problem* (6)*. Conversely, if $(\lambda^s, \phi^s, k^s)_{s=a,b}$ solves problem* (6)*, then $(\lambda^s, \phi^s, \omega^s)_{s=a,b}$ with $\omega_{k^a}^a \omega_{k^b}^b = 1$ solves problem* (5)*.*

Proof. To find a contradiction, suppose there is a solution to problem (5) with $\omega_i^a > 0$ and a binding incentive constraint on side a for some type $i' > i$, $\bar{U}_{i'} =$

$\lambda^a \left(\sum_{j=1}^I \omega_j^b u_{i',j} - \phi^a\right)$. Let k_1^a be the largest such type and $\phi_1^a = \phi^a$. We look for a feasible and more profitable solution to problem (5).

Next, let k_1^b be a maximizer of $u_{j',k_1^a} - \bar{U}_{j'}/\lambda^b$. Common Ranking implies $u_{j,k_1^a} > \sum_{i'=1}^I \omega_{i'}^a u_{j,i'}$, since the construction of k_1^a implies that $i' \leq k_1^a$ if $\omega_{i'}^a > 0$ and by assumption there is an $i < k_1^a$ with $\omega_i^a > 0$. Next, supermodularity, the fact that j maximizes $\sum_{i'=1}^I \omega_{i'}^a u_{j',i'} - \bar{U}_{j'}/\lambda^b$, and the fact that $i' \leq k_1^a$ if $\omega_{i'}^a > 0$ imply $k_1^b \geq j$. Finally, let ϕ_1^b by the maximized value of $u_{j',k_1^a} - \bar{U}_{j'}/\lambda^b$. Again, monotonicity implies $\phi_1^b > \phi^b$

We then proceed iteratively. Having found k_{n-1}^a, ϕ_{n-1}^a, k_{n-1}^b, and ϕ_{n-1}^b, let k_n^a be the largest i' that maximizes $u_{i',k_{n-1}^b} - \bar{U}_{i'}/\lambda^a$ and let ϕ_n^a be the maximized value. Common Ranking and Supermodularity imply that $k_n^a \geq k_{n-1}^a$ and $\phi_n^a \geq \phi_{n-1}^a$, with the latter strict if $k_{n-1}^b > k_{n-2}^b$.

Next, let k_n^b be the largest j' that maximizes $u_{j',k_n^a} - \bar{U}_{j'}/\lambda^b$ and let ϕ_n^b be the maximized value. Common Ranking and Supermodularity implies that $k_n^b \geq k_{n-1}^b$ and $\phi_n^b \geq \phi_{n-1}^b$, with the latter strict if $k_n^a > k_{n-1}^a$. We then increment n by 1 and iterate.

The sequences (k_n^a, k_n^b) are nondecreasing on the finite set $1, \ldots, I$ and so converge to (k^{a*}, k^{b*}). This means the maximized values converge as well, to (ϕ^{a*}, ϕ^{b*}), with $\phi^{a*} \geq \phi^a$ and $\phi^{b*} > \phi^b$.

Now consider the policy $(\lambda^s, \phi^{s*}, \omega^{s*})_{s=a,b}$ where $\omega_{k^{a*}}^{a*} \omega_{k^{b*}}^{b*} = 1$. By construction this satisfies the constraints in problem (5) and it attains a higher value than the original policy because $\phi^{a*} + \phi^{b*} > \phi^a + \phi^b$ and $\lambda^a \lambda^b$ is unchanged. This implies any solution to problem (5) has $\omega_i^a = 1$ for some i and $\bar{U}_{i'} > \lambda^a \left(\sum_{j=1}^I \omega_j^b u_{i',j} - \phi^a\right)$ for all $i' > i$. The proof that any solution to problem (5) has $\omega_j^b = 1$ for some j and $\bar{U}_{j'} > \lambda^b \left(\sum_{i=1}^I \omega_i^a u_{j',i} - \phi^b\right)$ for all $j' > j$ is analogous. Thus any solution to problem (5) satisfies all the constraints in problem (6) and attains the same value.

We can similarly take any solution to problem (6) and follow a similar logic. Suppose the solution has $\bar{U}_{i'} < \lambda^s(u_{i',k^{-s}} - \phi^s)$ for some s and $i' > k^s$. We again iteratively find higher types (k^{a*}, k^{b*}) who each pay higher fees to match with each other at the same rates (λ^a, λ^b), thus yielding higher profits, a contradiction. This means that any solution to problem (5) satisfies all the constraints in problem (6) and attains the same value. Thus a solution to either problem yields the solution to the other problem. ■

This problem yields two immediate corollaries. First, since any solution to problem (5) has a degenerate mixture ω, Lemmas 2 and 3 imply any equilibrium is separating:

Corollary 1 *Assume Common Ranking and Supermodularity. Any (partial or competitive search) equilibrium is separating.*

An example illustrates that supermodularity is critical to this result. Assume $I = 2$

with $u_{1,1} = 1$, $u_{1,2} = 1.5$, $u_{2,1} = 1$, and $u_{2,2} = 1.2$, satisfying monotonicity but not supermodularity. Fix $\gamma = 0.6$. We find that when $\bar{U}_1/\bar{U}_2 \in (1.4618, 1.5)$, the unique partial equilibrium pools both types on both sides of a single market. This is also the competitive search equilibrium when $\bar{\omega}_2 \in (0.88973, 1.0)$, so most individuals are high types.

Second, in any partial equilibrium $\{N, M^p, M, \bar{\boldsymbol{U}}\}$ and any active market $m \in M$, only downward incentive constraints can bind:

Corollary 2 *Assume Common Ranking and Supermodularity. In any partial equilibrium $\{N, M^p, M, \bar{\boldsymbol{U}}\}$ and for any market $m \in M$, $\omega_i^s > 0$ implies $\bar{U}_{i'} > U_{i'}^s(m)$ for all $i' > i$.*

As we will see, this result is particularly useful when a competitive search equilibrium has PAM, since it allows us to find the equilibrium value of $\bar{U}_i$ recursively from the lowest type on up.

This result rules out the possibility of countervailing incentives (Lewis and Sappington, 1989) in active markets. We do not think this result is trivial. To understand why, note that if the equilibrium value of a high type is small and we ignore their incentive constraint in a market attracting low types, the high type may prefer to go to that market. As a result, either the low market shuts down or it is distorted by the need to keep out high types. To understand why this does not contradict the claim, note that in this case, a market attracting the high type will be more profitable than one attracting only the low type. That is, the low market cannot be active. It follows that countervailing incentives are relevant, but only in inactive markets.

Add a generic example with upward incentive constraints binding but only one type in the market.

3.3 Characterization of PAM Equilibrium

In a competitive search equilibrium with PAM, every market $m \in M$ has an i such that $\omega_i^a \omega_i^b = 1$, with one active market for each type i. The measure of markets μ then ensures that the market clearing condition holds. Under Common Ranking and Supermodularity, Lemmas 2 and 3 imply that each such market must solve a version of problem (6) with the restriction that $k^a = k^b = i$, and that the maximized value must be c. That is, if there is a competitive search equilibrium with PAM, then it must be the case that for all i,

$$c = \max_{\{\lambda^s>0,\phi^s\}_{s=a,b}} (\lambda^a\lambda^b)^{-\frac{1-\gamma}{2\gamma}}(\phi^a+\phi^b) \tag{7}$$
$$\text{s.t. } \bar{U}_i = \lambda^s(u_{i,i}-\phi^s) \text{ for } s=a,b,$$
$$\bar{U}_j \geq \lambda^s(u_{j,i}-\phi^s) \text{ for } j<i \text{ and } s=a,b.$$

Equation (7) is a recursive problem. The problem for $i=1$ depends only on $\bar{U}_1$, and so we can adjust $\bar{U}_1$ to find the level such that the maximized value of the problem is exactly equal to c. Having found $\bar{U}_1,\ldots,\bar{U}_{i-1}$, we turn to the problem for i, adjusting $\bar{U}_i$ until we find a level such that the maximized value of the problem is $\bar{U}_i$, assuming this is feasible.

Unfortunately, without making further assumptions, there may be no solution to equation (7) when $i\geq 3$. To understand why, note that for each type i, the problem in equation (7) has a concave objective with a compact constraint set. Thus if the feasible set is non-empty, there is always a unique solution for any value of $\bar{U}_i$. But the constraint set may be empty because it is impossible to simultaneously exclude all the lower types from the (i,i) market. Some types may be kept out with a high contact rate and high transfer, others with a low contact rate and low transfer, and no incentive feasible combination delivers payoff c to the platform. To see how this works, it helps to eliminate transfers from equation (7):

$$c = \max_{\{\lambda^s>0,\phi^s\}_{s=a,b}} (\lambda^a\lambda^b)^{-\frac{1-\gamma}{2\gamma}}\left(2u_{i,i}-\frac{\bar{U}_i}{\lambda^a}-\frac{\bar{U}_i}{\lambda^b}\right) \tag{8}$$
$$\text{s.t. } \bar{U}_j-\bar{U}_i \geq \lambda^s(u_{j,i}-u_{i,i}) \text{ for } j<i \text{ and } s=a,b.$$

If $u_{1,3}<u_{3,3}<u_{2,3}$, the constraint for $i=3$ and $j=1$ implies $\lambda^s\geq\frac{\bar{U}_3-\bar{U}_1}{u_{3,3}-u_{1,3}}$, while the constraint for $i=3$ and $j=2$ implies $\lambda^s\leq\frac{\bar{U}_2-\bar{U}_3}{u_{2,3}-u_{3,3}}$, which are mutually exclusive when $\bar{U}_3$ is large. On the other hand, the objective function is decreasing in $\bar{U}_3$, so there may be no value of $\bar{U}_3$ such that the solution to equation (7) is a small positive number. When this happens, there is no competitive search equilibrium with PAM.

The key to this example is that the willingness-to-pay (WTP) for matching with a type 3 agent, $u_{i,3}$, is not a monotonic function of the own-type i. We avoid this problem by assuming a Monotonic WTP, meaning the payoff function has either Increasing WTP or Decreasing WTP:

Assumption 3 (Increasing WTP) *For every $i>i'$ and j, $u_{i,j}\geq u_{i',j}$.*

Assumption 4 (Decreasing WTP) *For every $i>i'$ and j, $u_{i,j}\leq u_{i',j}$.*

We further characterize equilibrium under these restrictions in the remainder of this section. We continue to maintain the assumptions of Common Ranking and Supermodularity throughout. Note that under the assumption of Supermodularity, the assumption of increasing WTP implies that for any $i > 1$, the inequality is strict; the assumption of decreasing WTP implies that for any $i < I$, the inequality is strict.

Under Common Ranking, Supermodularity, and Monotonic WTP, we prove that if there is a competitive search equilibrium with PAM, the equilibrium value $\bar{U}_i$ is uniquely determined as the solution to the following problem. First, define

$$g_{i,j}(\lambda) \equiv u_{i,j}\lambda - \tfrac{1}{2}c\lambda^{\frac{1}{\gamma}}. \tag{9}$$

Since $\gamma \in (0,1)$, this is a strictly concave function of λ. Its maximum is unique and positive since $u_{i,j}$ and c are positive. We let λ_i^* denote the unique maximizer of $g_{i,i}(\lambda)$:

$$\lambda_i^* \equiv \left(\frac{2\gamma u_{i,i}}{c}\right)^{\frac{\gamma}{1-\gamma}} \tag{10}$$

In an environment without private information, this turns out to be the equilibrium contact rate for type i agents when there is PAM; see Section 4.

With private information, we prove that the same function describes the equilibrium contact rate with PAM. Let $\bar{U}_1 = \max_{\lambda>0} g_{1,1}(\lambda)$ and for $i > 1$,

$$\bar{U}_i = \max_{\lambda>0} g_{i,i}(\lambda) \text{ s.t. } \bar{U}_{i-1} \geq g_{i-1,i}(\lambda). \tag{11}$$

This recursively and uniquely defines the whole vector $\bar{\boldsymbol{U}}$ in a competitive search equilibrium with PAM. Let λ_i denote the (we prove) unique maximizer in problem (11). Finally, let $\phi_i \equiv u_{i,i} - \bar{U}_i/\lambda_i = \frac{1}{2}c\lambda_i^{\frac{1-\gamma}{\gamma}}$ denote the transfer that delivers the equilibrium value $\bar{U}_i$ when the contact rate is λ_i. We then establish the following characterization:

Proposition 1 *Assume Common Ranking, Supermodularity, and Monotonic WTP. If there is a competitive search equilibrium with PAM, $\bar{U}_i$ solves problem* (11)*. In the market with $\omega_i^s = 1$ for $s = a, b$, the contacts rates are $\lambda^a = \lambda^b = \lambda_i$ and the fees are $\phi^a = \phi^b = \phi_i$. With Increasing WTP, $\bar{U}_i$, λ_i, and ϕ_i are increasing in i and $\lambda_i \geq \lambda_i^*$. With Decreasing WTP, $\bar{U}_i$, λ_i, and ϕ_i are decreasing in i and $\lambda_i \leq \lambda_i^*$.*

It is natural that markets with higher contact rates have higher fees; this is necessary for platforms to break even. The rest of the results are more meaningful. First, there is a simple problem which characterizes the equilibrium value of each type of agent. The problem depends on the utility of the next highest type of agent, a local incentive constraint, but not directly on other types.

Second, the direction of Monotonic WTP dictates which agents have higher utility, contact rates, and fees. With Increasing WTP, then each of these are increasing, as is the case without private information (see Section 4). Conversely with Decreasing WTP, each of these objects decreasing. As we show in the proof, this reflects the need to exclude lower types from the market in as an inexpensive a manner as possible. In contrast, with observable types, whether the contact rate and fee is increasing or decreasing depends on whether the equilibrium payoff $u_{i,i}$ is increasing or decreasing; again see Section 4.

Third, private information always distorts contact rates and transfers in the direction of the WTP. With Increasing WTP, equilibrium contact rates are at least as high with private information as with observable types. The free entry condition then implies that fees are weakly higher with private information. With Decreasing WTP, this is reversed. Contact rates and fees are weakly lower with private information than with observable types.

We put the proof of Proposition 1 in Appendix A. The proof proceeds in several steps to prove the equivalence between problems (8) and (11). First, we prove that in any active market with $\omega_i^a = \omega_i^b = 1$ for some i, contact rates and fees are the same on both sides of the market. We then consider a relaxed problem, where in the active market with $\omega_i^a = \omega_i^b = 1$ only worries about the local incentive constraint, keeping out type $i-1$. We prove that with Monotone WTP, problem (11) is the equivalent to this relaxed problem and use that to establish monotonicity of $\bar{U}_i$, λ_i, and ϕ_i, as well as the relationship between λ_i and λ_i^*. Finally, we verify that the solution to the relaxed problem solves the original problem with all the downward incentive constraints.

We note one thing that is missing from Proposition 1: it characterizes a competitive search equilibrium conditional on their being PAM, but does not tell us whether there is PAM. We address this in two ways. First, once we recover a candidate PAM equilibrium and associated equilibrium values $\bar{U}_i$, we can check the profitability of every other separating market by solving the following optimization problem:

$$
\begin{aligned}
\max_{\lambda^a,\lambda^b} (\lambda^a\lambda^b)^{-\frac{1-\gamma}{2\gamma}} &\left(u_{i,j} + u_{j,i} - \frac{\bar{U}_i}{\lambda^a} - \frac{\bar{U}_j}{\lambda^b} \right) \qquad (12) \\
\text{s.t. } \bar{U}_i - \bar{U}_{i'} &\le \lambda^a (u_{i,j} - u_{i',j}) \text{ for all } i' < i \\
\bar{U}_j - \bar{U}_{j'} &\le \lambda^b (u_{j,i} - u_{j',i}) \text{ for all } j' < j
\end{aligned}
$$

If the value of this program is no higher than c for all pairs (i, j), we have found a competitive search equilibrium with PAM. This is easy to check on a computer.

Alternatively, in the next section, we derive a sufficient condition for PAM in a limiting case of the model, when the platform share in the matching technology is small, $\gamma \to 0$.

3.4 Conditions for PAM in the Limit without Platforms

This section studies the limit of equilibria when $\gamma \to 0$, so platforms do not play a role in the matching process. This is a natural limit, analogous to a case often studied in the search and matching literature (albeit with observable types), where the matching function only takes the number of agents on the two sides of the market as inputs. We study separately two cases, first with an Increasing WTP and then with a Decreasing WTP. The characterization in the two cases is notably different, an issue we return to at the end of this section.

Proposition 2 *Assume Common Ranking, Supermodularity, and Increasing WTP. Also assume* $\log(u_{i,j} - u_{i-1,j})$ *is supermodular and* $u_{i,1} = u_{1,1}$ *for all* i*. For all* $\gamma < \bar{\gamma}$*, there exists a unique competitive search equilibrium with PAM. In it, local incentive constraints bind in all active markets. In the limit as* $\gamma \to 0$*, in the market attracting type* i *agents, the contact rate is* $\lambda_i = 1$*; the fee solves* $\phi_{i+1} - \phi_i = u_{i,i+1} - u_{i,i} > 0$ *with* $\phi_1 = 0$*; equilibrium utility solves* $\bar{U}_{i+1} - \bar{U}_i = u_{i+1,i+1} - u_{i,i+1} > 0$ *with* $\bar{U}_1 = u_{1,1}$*; and the reciprocal of the agent-platform ratio satisfies* $N_{i+1}^{-1} - N_i^{-1} = (u_{i,i+1} - u_{i,i})/c$ *with* $N_1^{-1} = 0$*.*

We illustrate this proposition with a simple example. Let $g(i, j)$ be a CES production function:

$$g(i,j) = \left(\frac{1}{2} i^{\frac{\rho-1}{\rho}} + \frac{1}{2} j^{\frac{\rho-1}{\rho}}\right)^{\frac{\rho}{\rho-1}} + \kappa,$$

where $\kappa > 0$ ensures $g(0, 0) > 0$ and $\rho > 0$ is the elasticity of substitution. Let $u_{i,j} = g((i-1)/I, (j-1)/I)$ for all i and j. We think of this as a model of partnership in the labor or marriage market, where the utility that i gets from matching with j is the same as the utility j gets from matching with i. This function satisfies Common Ranking, Supermodularity, and Increasing WTP. Additionally it satisfies $\log(u_{i,j} - u_{i-1,j})$ supermodular when $\rho \in (0, 1)$, so types are less substitutable than Cobb-Douglas. Finally, it satisfies $u_{i,1} = u_{1,1} = \kappa$ under the same restriction on the elasticity of substitution. The proposition then establishes that there is a PAM competitive search equilibrium with this payoff function and characterizes it completely.

Proof of Proposition 2. We break the proof into three steps. We first prove that local incentive constraints bind when γ is sufficiently small. This part of the proof does not use Increasing WTP and so also applies in the proof of Proposition 3. We then characterizing a PAM competitive search equilibrium in the limit as $\gamma \to 0$, assuming one exists. Finally, we find sufficient conditions for a PAM competitive search equilibrium when γ is small.

Binding Local Incentive Constraints Assume that there exists a competitive search equilibrium with PAM. We claim that for a sufficiently small but positive value of γ, all local incentive constraints bind. From problem (11), we have $\bar{U}_{i-1} \leq g_{i-1,i-1}(\lambda^*_{i-1})$, since λ^*_{i-1} maximizes $g_{i-1,i-1}$. Additionally, $g_{i-1,i-1}(\lambda^*_{i-1}) < g_{i-1,i}(\lambda^*_i)$ if and only if

$$u_{i-1,i-1}\left(\frac{2\gamma u_{i-1,i-1}}{c}\right)^{\frac{\gamma}{1-\gamma}} - \tfrac{1}{2}c\left(\frac{2\gamma u_{i-1,i-1}}{c}\right)^{\frac{1}{1-\gamma}} < u_{i-1,i}\left(\frac{2\gamma u_{i,i}}{c}\right)^{\frac{\gamma}{1-\gamma}} - \tfrac{1}{2}c\left(\frac{2\gamma u_{i,i}}{c}\right)^{\frac{1}{1-\gamma}}.$$

Common ranking implies $u_{i-1,i-1} < u_{i-1,i}$, so this condition holds for all $\gamma < \bar{\gamma}_i$. It follows that for all $\gamma < \bar{\gamma}_i$,, $\bar{U}_{i-1} \leq g_{i-1,i}(\lambda^*_i)$, so λ^*_i is not in the constraint set of problem (11) for type i, i.e. the local incentive constraint binds, so $\bar{U}_{i-1} = g_{i-1,i}(\lambda_i)$. Finally, $\gamma < \min_{i=2,\ldots,I} \bar{\gamma}_i$ ensures that local incentive constraints bind for all i.

Characterization Still assume there exists a competitive search equilibrium with PAM. With Common Ranking, Supermodularity, and Increasing WTP, Proposition 1 describes the equilibrium allocation. In particular, $\bar{U}_i$ and λ_i solve problem (11), with $\phi_i = u_{i,i} - \bar{U}_i/\lambda_i = \frac{1}{2}c\lambda_i^{\frac{1-\gamma}{\gamma}}$.

Equation (10) implies $\lim_{\gamma\to 0}\lambda^*_i = 1$ for all i. In particular, this implies $\lambda_1 \to 1$ and

$$\bar{U}_1 \to \lim_{\gamma\to 0} g_{1,1}(\lambda^*_1) = \lim_{\gamma\to 0}\left(u_{1,1}\left(\frac{2\gamma u_{1,1}}{c}\right)^{\frac{\gamma}{1-\gamma}} - \tfrac{1}{2}c\left(\frac{2\gamma u_{1,1}}{c}\right)^{\frac{1}{1-\gamma}}\right) = u_{1,1}.$$

Since we know $\bar{U}_i$ is increasing, it follows that $\bar{U}_i$ is strictly positive for all i, which implies $g_{i,i}(\lambda_i) > 0$ or $\lambda_i < \left(\frac{2u_{i,i}}{c}\right)^{\frac{\gamma}{1-\gamma}}$. This upper bound converges to 1 as $\gamma \to 0$. Since $\lambda_1 \to 1$ and λ_i is increasing for any positive γ, it follows that $\lambda_i \to 1$ for all i.

Turn next to fees. In a $(1,1)$ market, problem (7) implies fees are $\phi_i = \gamma u_{1,1}$, which converges to 0 as $\gamma \to 0$. For other matches, the objective in problem (7) gives $\bar{U}_i = u_{i,i} - \phi_i$ and the binding local incentive constraint gives $\bar{U}_i = u_{i,i+1} - \phi_{i+1}$. Combining these, we gives $\phi_{i+1} - \phi_i = u_{i,i+1} - u_{i,i}$. Under Common Ranking it follows that fees are strictly increasing in i in this limit.

Next, $\bar{U}_i = u_{i,i} - \phi_i$ pins down equilibrium utility. In particular, $\bar{U}_1 = u_{1,1}$ and $\bar{U}_{i+1} - \bar{U}_i = u_{i+1,i+1} - u_{i,i} - \phi_{i+1} + \phi_i = u_{i+1,i+1} - u_{i,i+1}$. Under Increasing WTP, equilibrium utility is strictly increasing in i.

In the last piece of the characterization, we look at the agent-principal ratio $N^a_i = N^b_i = N_i$ in the market with $\omega^a_i = \omega^b_i = 1$. With $\lambda^a = \lambda^b$, recall from the matching function that $\lambda = N^{-\gamma}$. Thus from equation (10) we have

$$N_1^{-1} = \left(\frac{2\gamma u_{1,1}}{c}\right)^{\frac{1}{1-\gamma}} \to 0.$$

For higher types, the objective function in problem (7) gives $\frac{1}{2}c = N_i^{1-\gamma}\phi_i$. In the limit as $\gamma \to 0$, this implies $N_i = c/\phi_i$, so $N_{i+1}^{-1} - N_i^{-1} = (u_{i,i+1} - u_{i,i})/c > 0$.

Sufficient Conditions We look for sufficient conditions for a PAM CSE for small γ. We conjecture that there is a PAM CSE and look at a market which attracts type $i \geq 2$ agents on side a and type $j \geq 2$ agents on side b. If all such markets are unprofitable, then there is a PAM CSE.

In the conjectured market, the local incentive constraints give

$$\bar{U}_{i-1} - \bar{U}_i \geq \lambda^a(u_{i-1,j} - u_{i,j}) \text{ and } \bar{U}_{j-1} - \bar{U}_j \geq \lambda^b(u_{j-1,i} - u_{j,i}),$$

as in problem (8). Using Increasing WTP, we can rewrite these as

$$\lambda^a \geq \frac{\bar{U}_i - \bar{U}_{i-1}}{u_{i,j} - u_{i-1,j}} \text{ and } \lambda^b \geq \frac{\bar{U}_j - \bar{U}_{j-1}}{u_{j,i} - u_{j-1,i}}.$$

We have proved that for small γ, if there is a PAM competitive search equilibrium, $\bar{U}_i - \bar{U}_{i-1} \to u_{i,i} - u_{i-1,i}$, so these reduce to

$$\lambda^a \geq \frac{u_{i,i} - u_{i-1,i}}{u_{i,j} - u_{i-1,j}} \text{ and } \lambda^b \geq \frac{u_{j,j} - u_{j-1,j}}{u_{j,i} - u_{j-1,i}}.$$

Multiply these together to get

$$\lambda^a\lambda^b \geq \frac{(u_{i,i} - u_{i-1,i})(u_{j,j} - u_{j-1,j})}{(u_{i,j} - u_{i-1,j})(u_{j,i} - u_{j-1,i})}.$$

Supermodularity of $\log(u_{i,j} - u_{i-1,j})$ implies that the right hand side is strictly bigger than 1 if $i \neq j$, proving $\lambda^a\lambda^b > 1$ in any market with different types on each side. Finally, platforms' matching rate in this market is $(\lambda^a\lambda^b)^{-\frac{1-\gamma}{\gamma}} \to 0$ when $\gamma \to 0$, while the transfers are necessarily finite, e.g. bounded above by the gains from trade $u_{i,j} + u_{j,i}$. This means that a market that attracts different types on each side yields 0 gross profits in the limit as $\gamma \to 0$, and so does not cover the fixed costs c. It follows that there exists a competitive search equilibrium with PAM under these conditions when γ is sufficiently small.

Finally we look at a market which attracts type 1 on side a and type $j \geq 2$ on side b. From problem (6), the incentive constraint on side b requires $\bar{U}_{j'} - \bar{U}_j \geq \lambda^b(u_{j',1} - u_{j,1})$ for all $j' < j$. By assumption, $u_{j',1} = u_{j,1} = u_{1,1}$, so the right hand side is zero. On the other hand, we have proved that $\bar{U}$ is increasing, so the left hand side is negative. This implies that a $(1, j)$ market cannot satisfy the incentive constraint and so is infeasible. ■

The novel condition here is log-supermodularity in first differences. This condition first appeared in Pratt (1964), who showed that it is equivalent to an assumption that lower types are more risk-averse over partners. It reappears in the search literature, e.g. in Shimer and Smith (2000) and Bonneton and Sandmann (2019), who use this condition to establish assortative matching in an environment with random search and observable types. In a random search model, individuals are making a decision to accept a sure thing (a match they have found) versus searching for an uncertain partner (a potential future match), and so it seems natural that relative risk-aversion will affect matching decisions. It is less clear to us why this assumption plays an important role in ensuring PAM in this environment.

Proposition 3 *Assume Common Ranking, Supermodularity, and Decreasing WTP. Also assume*

$$u_{i,j} + u_{j,i} < u_{i,i}\frac{u_{i-1,j} - u_{i,j}}{u_{i-1,i} - u_{i,i}} + u_{j,j}\frac{u_{j-1,i} - u_{j,i}}{u_{j-1,j} - u_{j,j}} \tag{13}$$

for all $i \in 2, \ldots, I$ *and* $j \in 2, \ldots, I$*; and*

$$u_{1,i} + u_{i,1} < 2u_{1,1}\sqrt{\prod_{j=2}^{i}\frac{u_{j,j}}{u_{j-1,j}}} \tag{14}$$

for all $i \in 2, \ldots, I$*. For all* $\gamma < \bar{\gamma}$*, there exists a competitive search equilibrium with PAM. In it, local incentive constraints bind in all active markets. In the market attracting type* i *agents, the contact rate solves* $\lambda_{i+1}/\lambda_i = u_{i,i}/u_{i,i+1} \in (0, 1)$*, with* $\lambda_1 = 1$ *and the fee is* $\phi_i = 0$*; equilibrium utility solves* $\bar{U}_{i+1}/\bar{U}_i = u_{i+1,i+1}/u_{i,i+1} \in (0, 1)$*, with* $\bar{U}_1 = u_{1,1}$*; and the agent-platform ratio is* $N_i = \infty$*.*

Again, we illustrate this proposition with a simple example. Let $g(i, j)$ satisfy

$$g(i, j) = 1 - \kappa i(1 - j)$$

for $(i, j) \in [0, 1]^2$ and $0 < \kappa \leq 1$. Let $u_{i,j} = g((i-1)/I, (j-1)/I)$ for all i and j. We think of this as a model of disease transmission. i is the probabliity that an individual is healthy. If he interacts with anyone, he gets a utility benefit of 1, but if he interacts with a sick person and was previously healthy, he gets the disease, costing utility κ. Thus the cost of interacting with someone who is healthy with probability j is $\kappa i(1 - j)$. Again, this function satisfies Common Ranking, Supermodularity, and Decreasing WTP, as well as conditions (13) and (14). The proposition then establishes that there is a PAM competitive search equilibrium with this payoff function and characterizes it completely.

Proof of Proposition 3. We follow the structure of the proof of Proposition 3. The proof that in a competitive search equilibrium with PAM, local incentive constraints all bind for sufficiently small positive γ is unchanged, and so we omit it.

Characterization. Assume that there exists a competitive search equilibrium with PAM. With Common Ranking, Supermodularity, and Decreasing WTP, the same logic as in the proof of Proposition 2 implies that: $\lambda_1 \to 1$, $\phi_1 \to 0$, $\bar{U}_1 \to u_{1,1}$, and $N_1 = \infty$.

With decreasing WTP, Proposition 1 proves that ϕ_i is decreasing in i. Platforms' free-entry condition implies $\phi_i > 0$ when γ is positive. It follows that when $\gamma \to 0$, $\phi_i \to 0$ for all i.

Next, the objective in Problem 7 implies $\bar{U}_i \to \lambda_i u_{i,i}$, and the binding local incentive constraint implies $\bar{U}_i \to \lambda_{i+1} u_{i,i+1}$, which gives $\lambda_{i+1}/\lambda_i = \frac{u_{i,i}}{u_{i,i+1}}$ in the limit as $\gamma \to 0$. With Common Ranking, this $\lambda_{i+1} < \lambda_i$. The same equations give $\bar{U}_{i+1}/\bar{U}_i = \frac{u_{i+1,i+1}}{u_{i,i+1}}$. With Decreasing WTP, this implies $\bar{U}_{i+1} < \bar{U}_i$.

Lastly, we look at the agent-platform ratio. With $\lambda_i = N_i^{-\gamma} < 1$, it follows that in the $N_i \to \infty$ as $\gamma \to 0$.

Sufficient Conditions We look for sufficient conditions for a PAM CSE for small γ. We conjecture that there is a PAM CSE and look at a market which attracts type i agents on side a and type j agents on side b. If all such markets are unprofitable, then there is a PAM CSE. To start, we assume that both $i \geq 2$ and $j \geq 2$.

In the conjectured market, the ratio of the local incentive constraints to the objective function gives

$$\frac{\bar{U}_{i-1}}{\bar{U}_i} \geq \frac{u_{i-1,j} - \phi^a}{u_{i,j} - \phi^a} \text{ and } \frac{\bar{U}_{j-1}}{\bar{U}_j} \geq \frac{u_{j-1,i} - \phi^b}{u_{j,i} - \phi^b}.$$

We have proved that for small γ, if there is a PAM competitive search equilibrium, $\bar{U}_{i-1}/\bar{U}_i = u_{i-1,i}/u_{i,i}$. Substitute this into the preceding equations and simplify using the facts that $\phi^a < u_{i,j}$ and $\phi^b < u_{j,i}$ (since $\bar{U}_i$ and $\bar{U}_j$ are positive) and $u_{i-1,i} > u_{i,i}$ and $u_{j-1,j} > u_{j,j}$ (since there is Decreasing WTP):

$$\phi^a \leq u_{i,j} - u_{i,i}\frac{u_{i-1,j} - u_{i,j}}{u_{i-1,i} - u_{i,i}} \text{ and } \phi^b \leq u_{j,i} - u_{j,j}\frac{u_{j-1,i} - u_{j,i}}{u_{j-1,j} - u_{j,j}}.$$

Under condition (13), $\phi^a + \phi^b < 0$, so any (i, j) market yields negative profits.

Now look at a market with $i \geq 2$ and $j = 1$. In this case, the sum of the fees in the objective function (12) is

$$u_{1,i} + u_{i,1} - \frac{\bar{U}_1}{\lambda^a} - \frac{\bar{U}_i}{\lambda^b}.$$

We claim that under condition (14), this is negative whenever $\lambda^a \lambda^b \leq 1$, so platforms' meeting rate is strictly positive in the limit as $\gamma \to 0$. To prove this, we maximize the sum of fees subject to the constraint $\lambda^a \lambda^b \leq 1$. This gives an upper bound on fees, $u_{1,i} + u_{i,1} - 2\sqrt{\bar{U}_1 \bar{U}_i}$. Substituting for equilibruim utility given the values we have characterized, we obtain that the upper bound on fees is strictly negative if condition (14) is satisfied. ■

There are two ways markets can induce agents to self-select, through fees and contact rates. We find that with Increasing WTP, fees are critical in the limit without platforms. All active markets have the same contact rate, but the more desirable agents, who have a higher willingness-to-pay for matches, exclude lower types by paying higher fees. Contact rates are not useful for excluding lower types because higher types value meetings more than lower types do.

In contrast, with a Decreasing WTP, contact rates play the critical role. Now fees are zero in all markets, but the more desirable agents, who have a lower willingness-to-pay for matches, exclude lower types by suppressing their contact rate. Now fees are not useful for excluding lower types because higher types are unwilling to pay high fees for matching.

4 Model with Observable Types

First consider the competitive search equilibrium with observable types. This environment is similar to Shi (2001) and Eeckhout and Kircher (2010). The recast of their result in our definition will be a benchmark for our characterization of competitive search equilibrium with private information. In the observable-type model, the platforms can directly control the type distributions by charging type-specific fees.

A market is a vector $m = (\boldsymbol{\phi}^s, \omega^s)_{s=a,b}$ satisfying $\boldsymbol{\phi}^s \equiv (\phi_1^s, \ldots, \phi_I^s) \in \mathbb{R}^I$ and $\omega^s \equiv (\omega_1^s, \ldots, \omega_I^s) \in \Delta^I$, the standard probability simplex. We interpret ϕ_i^s to be the fee paid by a type i agent on side s of the market when he matches; and ω_i^s to be the share of type i agents on side s of the market. Let $\mathbb{M}_0$ denote the set of markets, i.e. the set of vectors satisfying these conditions. With this definition of markets, the value of platforms and agents are accordingly

$$r\tilde{V}(m) \equiv \left[N^a(m)N^b(m)\right]^{\frac{1-\gamma}{2}} \left(\sum_{i=1}^{I} \phi_i^a \omega_i^a + \sum_{j=1}^{I} \phi_j^b \omega_j^b \right), \tag{15}$$

and

$$r\tilde{U}_i^s(m) \equiv \frac{1}{N^s(m)} \left[N^a(m)N^b(m)\right]^{\frac{1-\gamma}{2}} \left(\sum_{j=1}^{I} \omega_j^{-s} u_{i,j} - \phi_i^s\right), \tag{16}$$

with

$$\tilde{U}_i^s(m) \equiv \begin{cases} \infty \\ 0 & \text{if } \sum_{j=1}^{I} \omega_j^{-s} u_{i,j} \gtreqless \phi_i^s \text{ and } N^s(m) = 0. \\ -\infty \end{cases} \tag{17}$$

The logic closely follows equations (1)–(3), extended to allow for type-specific fees.

Definition 5 *A* ***partial equilibrium with observable types*** $\{N, M^p, M, \bar{\boldsymbol{U}}\}$ *is a mapping* $N : \{a, b\} \times \mathbb{M}_0 \to \mathbb{R}_+$*, two nonempty sets* $M \subseteq M^p \subseteq \mathbb{M}_0$*, and strictly positive numbers* $\bar{\boldsymbol{U}} \equiv \bar{U}_1, \ldots, \bar{U}_I$ *such that:*

1. *(Optimal Search)* $\forall m \in \mathbb{M}_0$, $s \in \{a, b\}$, *and* $i \in \mathbb{I}$, $\bar{U}_i \geq \tilde{U}_i^s(m)$*; and if* $N^s(m) > 0$, $\bar{U}_i = \tilde{U}_i^s(m)$ *for some* $i \in \mathbb{I}$*;*
2. *(Promise Keeping)* $M^p = \{m \in \mathbb{M}_0 | \omega_i^s > 0 \Rightarrow \bar{U}_i = \tilde{U}_i^s(m)\}$*;*
3. *(Profit Maximization)* $M = \arg\max_{m \in M^p} \tilde{V}(m)$.

With these definition of partial equilibrium, the definition of a competitive search equilibrium with observable types is:

Definition 6 *A* ***competitive search equilibrium with observable types*** *is a partial equilibrium* $\{N, M^p, M, \bar{\boldsymbol{U}}\}$ *and a measure* μ *on the set of active market* M *such that*

1. *(free entry)* $c = V(m)$ *for all* $m \in M$*;*
2. *(market clearing)* $\bar{\omega}_i = \int_M \left[\sum_{s=a,b} \omega_i^s N^s(m)\right] d\mu(m)$.

We start by showing that the outcomes of a competitive search equilibrium can be characterized by a set of optimization problem. Consider the following problem given $\bar{\boldsymbol{U}} \equiv U_1, \ldots, U_I$:

$$\bar{V} = \max_{m \in \mathbb{M}} (\lambda^a \lambda^b)^{-\frac{1-\gamma}{2\gamma}} \sum_{s=a,b} \sum_{i=1}^{I} \phi_i^s \omega_i^s, \tag{18}$$

$$\text{s.t. } \bar{U}_i \geq \lambda^s \left(\sum_{j=1}^{I} \omega_j^{-s} u_{i,j} - \phi_i^s\right) \text{ with equality if } \omega_i^s > 0,$$

$$\text{for } s = a, b \text{ and } i \in \{1, \ldots, I\}$$

We say that $(M, \bar{V})$ solves optimization problem (18) if $\bar{V}$ is the maximum value of the optimization problem (18) and any $m \in M$ achieves this value given $\bar{\boldsymbol{U}}$.

Lemma 4 *Suppose $\{M, \bar{V}, \bar{\boldsymbol{U}}\}$ is a partial equilibrium with observable types. Then $(M, \bar{V})$ must solve problem* (18) *given $\bar{\boldsymbol{U}}$. Conversely, if $(M, \bar{V})$ solves optimization problem* (18) *given $\bar{\boldsymbol{U}}$, then $\{M, \bar{V}, \bar{\boldsymbol{U}}\}$ is a partial equilibrium with observable types.*

Proof. First, suppose $\{M, \bar{V}, \bar{\boldsymbol{U}}\}$ is a partial equilibrium, yet $(M, \bar{V})$ is not a solution to (18). Thus there is $m' \in \mathbb{M}$ such that the constraint of (18) is satisfied and $V(m') > \bar{V}$. This contradicts point 2 of the definition for a partial equilibrium.

Conversely, suppose $(M, \bar{V})$ is the solution to (18) given $\bar{\boldsymbol{U}}$ yet $(M, \bar{V}, \bar{\boldsymbol{U}})$ is not a partial equilibrium. If $\exists m \in M$ such that $V(m) \neq \bar{V}$. This violates that M is the solution to (18) because either m is not a maximizer or $\bar{V}$ is not the optimal value; If there is $m \in \mathbb{M}$ that violates condition 2 of the definition for a partial equilibrium, this m satisfies constraints of (18), and $V(m) > \bar{V}$, which contradicts $(M, \bar{V})$ solving (18). ■

We first restrict attention to separating competitive search equilibrium, then show that under monotonicity of $f_{i,j}$ in j, any competitive search equilibrium must be separating. With separation, the problem (18) can be analyzed in two steps: (1) conditional on a fixed pair (i, j), find the $(\boldsymbol{\lambda}, \boldsymbol{\phi})$ that maximizes the platform's value. (2) choose (i, j) that delivers the highest value to platforms. We define this step-one problem as

$$V_{i,j} = \max_{\boldsymbol{\lambda}>0, \boldsymbol{\phi}>0} \left(\lambda^a \lambda^b\right)^{-\frac{1-\gamma}{2\gamma}} \left(\phi^a + \phi^b\right), \tag{19}$$

s.t.

$$\bar{U}_i = \lambda^a \left(u_{i,j} - \phi^a\right),$$
$$\bar{U}_j = \lambda^b \left(u_{j,i} - \phi^b\right).$$

With observable types, the relevant value for the solution of problem 19 is the joint surplus in a (i, j) match, which we define as $f_{ij} \equiv u_{i,j} + u_{j,i}$.

Lemma 5 *Given (i, j), a unique solution to* (19) *exists:*

$$\lambda^a_{i,j} = \frac{\bar{U}_i}{\frac{1-\gamma}{2} f_{i,j}}, \ \lambda^b_{i,j} = \frac{\bar{U}_j}{\frac{1-\gamma}{2} f_{i,j}}, \ \phi^a_{i,j} = u_{i,j} - \frac{1-\gamma}{2} f_{i,j}, \ \phi^b_{i,j} = u_{j,i} - \frac{1-\gamma}{2} f_{i,j},$$

$$V_{i,j} = \gamma \left(\frac{1-\gamma}{2}\right)^{\frac{1-\gamma}{\gamma}} \left(\bar{U}_i \bar{U}_j\right)^{-\frac{1-\gamma}{2\gamma}} f_{i,j}^{\frac{1}{\gamma}}.$$

Proof. By definition:

$$V_{i,j} = \max_{\lambda^a, \lambda^b} v(\lambda^a, \lambda^b) = \max_{\lambda^a, \lambda^b} (\lambda^a \lambda^b)^{-\frac{1-\gamma}{2\gamma}} \left(f_{i,j} - \frac{\bar{U}_i}{\lambda^a} - \frac{\bar{U}_j}{\lambda^b}\right).$$

The partial derivatives of $V_{1,1}$ are:

$$\frac{\partial \log v}{\partial \lambda^a}(\lambda^a, \lambda^b) = -\frac{1-\gamma}{2\gamma}\frac{1}{\lambda^a} + \frac{\bar{U}_i}{\lambda^{a2}}\frac{1}{f_{i,j} - \frac{\bar{U}_i}{\lambda^a} - \frac{\bar{U}_j}{\lambda^b}}.$$

For $\lambda^a, \lambda^b > 0$ and $f_{i,j} - \frac{\bar{U}_i}{\lambda^a} - \frac{\bar{U}_j}{\lambda^b} > 0$, this implies that

$$\frac{\partial \log v}{\partial \lambda^a}(\lambda^a, \lambda^b) \gtreqless 0 \Leftrightarrow (1 - \frac{1-\gamma}{2})\frac{\bar{U}_i}{\lambda^a} + \frac{1-\gamma}{2}\frac{\bar{U}_j}{\lambda^b} \gtreqless \frac{1-\gamma}{2} f_{i,j}$$
$$\frac{\partial \log v}{\partial \lambda^b}(\lambda^a, \lambda^b) \gtreqless 0 \Leftrightarrow (1 - \frac{1-\gamma}{2})\frac{\bar{U}_j}{\lambda^b} + \frac{1-\gamma}{2}\frac{\bar{U}_i}{\lambda^a} \gtreqless \frac{1-\gamma}{2} f_{i,j}$$

Thus when λ^a and λ^b are both close to 0, raising either of them increases v. When both are sufficiently large, reducing either of them increases v. If $\frac{\bar{U}_i}{\lambda^a} > \frac{\bar{U}_j}{\lambda^b}$, $0 < \gamma < 1$ implies $(1 - \frac{1-\gamma}{2})\frac{\bar{U}_i}{\lambda^a} + \frac{1-\gamma}{2}\frac{\bar{U}_j}{\lambda^b} > (1 - \frac{1-\gamma}{2})\frac{\bar{U}_j}{\lambda^b} + \frac{1-\gamma}{2}\frac{\bar{U}_i}{\lambda^a}$, and so we can have $\frac{\partial v}{\partial \lambda^a}(\lambda^a, \lambda^b) > 0 > \frac{\partial v}{\partial \lambda^b}(\lambda^a, \lambda^b)$, i.e. we increase v by increasing λ^a and decreasing λ^b. We can reach the opposite conclusion when $\frac{\bar{U}_i}{\lambda^a} < \frac{\bar{U}_j}{\lambda^b}$. This implies that v is single-peaked in $\boldsymbol{\lambda}$ such that $\lambda^a, \lambda^b > 0$ and $f_{i,j} - \frac{\bar{U}_i}{\lambda^a} - \frac{\bar{U}_j}{\lambda^b} > 0$. That peak comes at $\lambda^a = \frac{\bar{U}_i}{\frac{1-\gamma}{2} f_{i,j}}$ and $\lambda^b = \frac{\bar{U}_j}{\frac{1-\gamma}{2} f_{i,j}}$.

Plugging the optimal solution back to the incentive constraints of participants and objective function, we reach the results in lemma. ■

It turns on a competitive search equilibrium must be separating, if the joint value $f_{i,j} \neq f_{i,j'}$ for a fixed i and $j \neq j'$. This is the assumption made in models from literature such as Eeckhout and Kircher (2010). This result comes from that a non-separating market can be viewed as a collection of separating markets, with an additional restriction that all of these separating markets must share the identical contact rates. When $f_{i,j}$ varies in the matching partners' types (dimension j), this identical contact rate constraint is binding and strictly decreases payoffs for the platform.

Lemma 6 *If $f_{i,j} \neq f_{i,j'}$ for $j \neq j'$, any competitive search equilibrium with observable types is separating.*

Proof. We first replace (ϕ_i^a, ϕ_j^b) in the objective function by the constraints in problem (18): The original problem is now an unconstrained problem in terms of (ϕ^a, ϕ^b) and (ω_i^a, ω_j^b):

$$\max_{\boldsymbol{\lambda} > 0, \boldsymbol{\omega}^s \in \Delta^I} (\lambda^a \lambda^b)^{-\frac{1-\gamma}{2\gamma}} \sum_{i=1}^{I} \sum_{j}^{I} \omega_i^a \omega_j^b \left(f_{i,j} - \frac{\bar{U}_i}{\lambda^a} - \frac{\bar{U}_j}{\lambda^b} \right).$$

Given any $\bar{\boldsymbol{U}}$, we have can write the value of objective function as:

$$\sum_{i=1}^{I} \sum_{j}^{I} \omega_i^a \omega_j^b (\lambda^a \lambda^b)^{-\frac{1-\gamma}{2\gamma}} \left(f_{i,j} - \frac{\bar{U}_i}{\lambda^a} - \frac{\bar{U}_j}{\lambda^b} \right) \leq \sum_{i=1}^{I} \sum_{j}^{I} \omega_i^a \omega_j^b V_{i,j}^*,$$

where the inequality comes from the fact $V^*_{i,j}$ is a solution to problem (19) and any positive (λ^a, λ^b) is feasible for (19). The inequality holds with equality only if $\lambda^s = \lambda^s_{i,j}$ for any (i,j) such that $\omega_i^s \omega_j^{-s} > 0$.

Suppose there is a CSE that is non-separating while $f_{i,j}$ is strictly increasing in j. WLOG, assume there is $m \in M$ such that $\omega_j^b > 0$, $\omega_{j'}^b > 0$, and $j \neq j'$. From lemma 5, for some i such that $\omega_i^a > 0$:

$$\lambda^a_{i,j} = \frac{\bar{U}_i}{\frac{1-\gamma}{2} f_{i,j}} \neq \frac{\bar{U}_i}{\frac{1-\gamma}{2} f_{i,j'}} = \lambda^a_{i,j'}.$$

This means λ^a cannot equal to both contact rates in separating markets. This means:

$$c = \sum_{i=1}^{I} \sum_{j}^{I} \omega_i^a \omega_j^b (\lambda^a \lambda^b)^{-\frac{1-\gamma}{2\gamma}} \left(f_{i,j} - \frac{\bar{U}_i}{\lambda^a} - \frac{\bar{U}_j}{\lambda^b} \right) < \sum_{i=1}^{I} \sum_{j}^{I} \omega_i^a \omega_j^b V^*_{i,j}.$$

Thus one of the separating markets such that $\omega_i^a \omega_j^b > 0$ must yield strictly higher payoff for the platform. This contradicts to the second condition of the definition for a partial equilibrium (,and thus competitive search equilibrium). ∎

Lemma 6 implies that we should only look for separating competitive search equilibrium. From here on we use the enumeration defined in section 2.5, and index markets by the pair that shows up on the two sides. Given the results in Lemma 5, the step-two problem is to find the set of (i,j) that maximize platforms' value:

$$\bar{V} = \max_{i,j} V_{i,j}. \tag{20}$$

Naturally, the partial-equilibrium set of markets M is the set of maximizers to problem (20).

Proposition 4 *Assume $f_{i,j} \neq f_{i,j'}$ for $j \neq j'$ and $\log f_{i,j}$ is supermodular (submodular). There exists a unique PAM (NAM) competitive search equilibrium with observable types.*

Proof. From (20), $M = \arg\max_{i,j} V_{i,j}$. This is a maximization problem on a finite set. Thus a solution (and correspondingly a partial equilibrium) must exist given $\bar{\boldsymbol{U}}$. We now show (1) M must have stated sorting pattern and (2) we can construct a unique competitive search equilibrium. From lemma 5:

$$V_{i,j} = \gamma \left(\frac{1-\gamma}{2} \right)^{\frac{1-\gamma}{\gamma}} \left(\bar{U}_i \bar{U}_j \right)^{-\frac{1-\gamma}{2\gamma}} f_{i,j}^{\frac{1}{\gamma}}.$$

If $f_{i,j}$ is log-supermodular. We prove PAM by contradiction. More precisely, we want to show that if $(i,j) \in M$, then $i = j$. Suppose otherwise. The formula of $V_{i,j}$ implies

$$\frac{V_{i,j}V_{j,j}}{V_{i,j}^2} = \left(\frac{f_{i,i}f_{j,j}}{f_{i,j}^2}\right)^{\frac{1}{\gamma}} > 1.$$

Thus $\max\{V_{i,i}, V_{j,j}\} > V_{i,j}$. It is profitable for the platforms to deviate from the focal active market to either a market with (i,i) or a market (j,j). A contradiction to the optimality. So any separating partial equilibrium has markets with identical types on both sides, which satisfies the definition of positive assortative matching.

We then show there is a unique separating competitive search equilibrium. In a competitive search equilibrium, $\bar{V} = V_{i,i} = c$:

$$c = V_{i,j} = \gamma\left(\frac{1-\gamma}{2}\right)^{\frac{1-\gamma}{\gamma}} \left(\bar{U}_i\bar{U}_j\right)^{-\frac{1-\gamma}{2\gamma}} f_{i,j}^{\frac{1}{\gamma}}.$$

Solving this equation gives a unique value of $\bar{U}_i$. With this value of $\bar{U}_i$, there is a unique number of participants per posting $n_{i,i}^s$ from Lemma 5. Lastly, from the market to clear for type i, we compute $N^P(m_{i,i}) = \frac{\bar{\omega}_i}{2n_{i,i}}$. This construction leads to unique separating competitive search equilibrium with positive assortative matching.

If $f_{i,j}$ is log-submodular. Suppose there are two active markets for (i_1, j_1) and (i_2, j_2) and $i_1 > i_2$, $j_1 > j_2$.

$$\frac{V_{i_1,j_2}V_{i_2,j_1}}{V_{i_1,j_1}V_{i_2,j_2}} = \left(\frac{f_{i_1,j_2}f_{i_2,j_1}}{f_{i_1,j_1}f_{i_2,j_2}}\right)^{\frac{1}{\gamma}} > 1.$$

This contradicts to optimality of (i_1, j_1) or (i_2, j_2).

The negative assortative matching in M and the clear marketing implies that all active markets have the pairs in the form of $(1, I)$, $(2, I-1)$, ..., $(I/2, I/2+1)$ if I is even, and in the form of $(1, I)$, $(2, I-1)$, ..., $((I-1)/2, (I-1)/2)$ if I is odd. Take each of these pair (i,j), we clear the market in the following order. First solve $\bar{U}_i$ and $\bar{U}_j$ as solution to:

$$c = \gamma\left(\frac{1-\gamma}{2}\right)^{\frac{1-\gamma}{\gamma}} \left(\bar{U}_i\bar{U}_j\right)^{-\frac{1-\gamma}{2\gamma}} f_{i,j}^{\frac{1}{\gamma}},$$

$$\frac{\bar{\omega}_i}{\bar{\omega}_j} = \frac{\bar{U}_i}{\bar{U}_j}.$$

This is a system of two equations with unique solution $(\bar{U}_i, \bar{U}_j)$. We can solve from lemma 5 the correspondingly $n_{i,j}^s$. The measure of postings is given by $N_{i,j}^P = N^P(i,j) =$

$\frac{\bar{\omega}_i}{n^a_{i,j}}$. ■

5 Concluding Remarks

How do individuals sort in the presence of private information? We use a competitive search model to answer that question. We prove that under Common Ranking and Supermodularity, each side of every active market only attracts one type of agent. We then characterize positively assortative matching under those two assumptions and Monotonic Willingness-to-Pay. We prove that with Increasing WTP, higher types have higher utility, match at a faster rate, and pay higher fees. With Decreasing WTP, higher types have lower utility, match at a lower rate, and pay lower fees. Finally, we provide sufficient conditions for positively assortative matching when the role of platforms vanishes. With Increasing WTP, only fees differ across active markets, and the matching rate is driven to 1. With Decreasing WTP, only matching rates differ across markets, and fees are driven to zero.

We leave two important questions for future work. First, what is a Pareto efficient allocation with private information. We imagine a planner with the ability to choose which markets are open. This allows for cross-subsidization between markets and also allows the planner to shut down markets that would otherwise disrupt desirable outcomes. Under what conditions is the competitive search equilibrium Pareto efficient?

Second, how can we characterize an equilibrium with negatively assortative matching (NAM)? This is tricky because of the market clearing condition, Part 2 of Definition 2. With positively assortative matching, we just need to ensure that a market matching type i agents to type i agents is profitable, and then determine the number of them to clear the market for type i agents. With negatively assortative matching, there may need markets matching multiple types of agents to a single type j in order to clear the market for type j. We conjecture that this problem will be easier to solve in a version of the model with a continuum of types.

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Appendix

A Proof of Proposition 1.

The proof proceeds in steps.

A.1 Active PAM Markets are Symmetric

Take a partial equilibrium $\{N, M^p, M, \bar{\boldsymbol{U}}\}$. In any market $m \in M$ with $w_i^s = 1$, $s = a, b$, we prove $\lambda^a = \lambda^b = \lambda_i$ and $\phi^a = \phi^b = \phi_i$. Suppose not, so there is an active market m with $w_i^s = 1$, $s = a, b$ and $\lambda^a > \lambda^b$. These contact must solve problem (8).

Now consider an alternative allocation (λ', λ') with $\lambda' = \sqrt{\lambda^a \lambda^b}$. By construction $\lambda^a > \lambda' > \lambda^b$. The allocation (λ', λ') satisfies the constraints in Problem (8) because for $j < i$ such that $u_{j,i} - u_{i,i} \geq 0$,

$$\bar{U}_j - \bar{U}_i \geq \lambda^a (u_{j,i} - u_{i,i}) \geq \lambda' (u_{j,i} - u_{i,i}),$$

and for $j < i$ such that $u_{j,i} - u_{i,i} < 0$,

$$\bar{U}_j - \bar{U}_i \geq \lambda^b (u_{j,i} - u_{i,i}) > \lambda' (u_{j,i} - u_{i,i}).$$

(λ', λ') also delivers strictly higher payoff to the platform because:

$$(\lambda' \lambda')^{-\frac{1-\gamma}{2\gamma}} \left(2u_{i,i} - \frac{\bar{U}_i}{\lambda'} - \frac{\bar{U}_i}{\lambda'} \right) > (\lambda^a \lambda^b)^{-\frac{1-\gamma}{2\gamma}} \left(2u_{i,i} - \frac{\bar{U}_i}{\lambda^a} - \frac{\bar{U}_i}{\lambda^b} \right),$$

The inequality uses the fact that $\lambda' = \sqrt{\lambda^a \lambda^b}$ to get that the first terms are equivalent. It uses the inequality of arithmetic and geometric means, $\frac{\lambda^a + \lambda^b}{2} > \sqrt{\lambda^a \lambda^b}$, and $\bar{U}_i > 0$ to get the inequality in the second term. This proves that any market $m \in M$ with $w_i^s = 1$, $s = a, b$ has $\lambda^a = \lambda^b = \lambda_i$.

Next, the constraint in problem (6) implies that $\phi^s = u_{i,i} - \frac{\bar{U}_i}{\lambda^s} \equiv \phi_i$ since $\lambda^a = \lambda^b$.

We thus simplify Problem (8) to focus on symmetric allocations:

$$\tfrac{1}{2} c = \max_{\lambda} u_{i,i} \lambda^{1-\frac{1}{\gamma}} - \bar{U}_i \lambda^{-\frac{1}{\gamma}} \tag{21}$$

$$\text{s.t. } \bar{U}_j - \bar{U}_i \geq \lambda (u_{j,i} - u_{i,i}) \text{ for } j < i.$$

A.2 Existence of a Solution to Problem (21)

Define

$$v(\bar{U}) \equiv \max_{\lambda>0} h(\lambda;\bar{U}) \equiv u_{i,i}\lambda^{1-\frac{1}{\gamma}} - \bar{U}\lambda^{-\frac{1}{\gamma}}$$
$$\text{s.t. } \bar{U}_j - \bar{U} \geq \lambda(u_{j,i} - u_{i,i}) \text{ for } j < i.$$

The problem is to find $\bar{U}_i$ such that $\frac{1}{2}c = v(\bar{U}_i)$. First, assume Increasing WTP, so for all $j < i$, $u_{j,i} < u_{i,i}$. This means the constraint in this problem is $\lambda \geq \frac{\bar{U}_j - \bar{U}}{u_{j,i} - u_{i,i}}$. Gathering all of these constraints, we can find the largest lower bound: $\lambda \geq \bar{\lambda}(\bar{U}) \equiv \max_{j<i} \frac{\bar{U}_j - \bar{U}}{u_{j,i} - u_{i,i}}$, with $\bar{\lambda}(\bar{U}) = -\infty$ if $i = 1$.

Next, note that for fixed $\bar{U}$, the objective function $h(\lambda;\bar{U})$ is a single-peaked function of λ, negative if $\lambda < \bar{U}/u_{i,i}$ and otherwise positive, attaining its maximum at $\lambda^*(\bar{U}) = \frac{\bar{U}}{(1-\gamma)u_{i,i}}$. It follows that the constrained optimum is $\lambda(\bar{U}) \equiv \max\{\lambda^*(\bar{U}), \bar{\lambda}(\bar{U})\}$.

It is easy to verify $\lambda(\bar{U})$ is continuous in $\bar{U}$ and $\lim_{\bar{U}\to 0} \lambda(\bar{U}) = 0$ and $\lim_{\bar{U}\to\infty} \lambda(\bar{U}) = \infty$. Because $h(\lambda;\bar{U})$ is continuous in both arguments, $v(\bar{U}) = h(\lambda(\bar{U}))$ is continuous in $\bar{U}$ as well. Moreover, $\lim_{\bar{U}\to 0} v(\bar{U}) = \infty$ and $\lim_{\bar{U}\to\infty} v(\bar{U}) = 0$. The intemediate value theorem implies that there is least one $\bar{U}_i$ such that $v(\bar{U}_i) = c$.

The same logic applies with Decreasing WTP.

A.3 Relaxed Problem and Dual Problem

We next consider a relaxed problem with only the local incentive constraint, and later verify the solution indeed satisfies all global constraints:

$$\frac{1}{2}c = \max_{\lambda} u_{i,i}\lambda^{1-\frac{1}{\gamma}} - \bar{U}_i\lambda^{-\frac{1}{\gamma}} \tag{22}$$
$$\text{s.t. } \bar{U}_{i-1} - \bar{U}_i \geq \lambda(u_{i-1,i} - u_{i,i}).$$

If $i = 1$, we drop the constraint so this is simply the same problem as (21). We prove that if λ and $\bar{U}_i$ solve problem (22), then λ must also solve the dual problem (11), delivering value $\bar{U}_i$, and vice versa.

First, we prove that any solution to the primal problem also solves the dual problem. To find a contradiction, suppose $(\lambda, \bar{U}_i)$ solves problem (22) but does not solve the dual problem (11). Note from the definition of g in equation (9) and the objective function in (22) that $\bar{U}_i = g_{i,i}(\lambda)$. That means there must be a $\lambda' \neq \lambda$ in the constraint set of problem (11), $\bar{U}_{i-1} \geq g_{i-1,i}(\lambda')$, which attains a higher value of the objective function, $g_{i,i}(\lambda') > \bar{U}_i$.

Now observe that λ' must be in the constraint set in problem (22) because:

$$\lambda'(u_{i-1,i} - u_{i,i}) = g_{i-1,i}(\lambda') - g_{i,i}(\lambda') \leq \bar{U}_{i-1} - g_{i,i}(\lambda') < \bar{U}_{i-1} - \bar{U}_i,$$

where the first equality uses the definition of $g_{i,j}(\lambda)$, the second inequality uses the constraint of the dual problem, and the last inequality uses the assumption $g_{i,i}(\lambda') > \bar{U}_i$. Additionally, inverting $g_{i,i}(\lambda') > \bar{U}_i$ implies

$$u_{i,i}(\lambda')^{1-\frac{1}{\gamma}} - \bar{U}_i(\lambda')^{-\frac{1}{\gamma}} > \tfrac{1}{2}c.$$

This contradicts $(\lambda, \bar{U})$ solving the primal problem.

We have now proved that the primal problem has a solution (Appendix A.2) and that any such solution also solves the dual problem. We also know that the dual problem uniquely determines $\bar{U}_i$, although we have not (yet) proved that it uniquely determines λ. This means that any $\bar{U}_i$ obtained from the dual problem must also be the unique $\bar{U}_i$ in any solution to the primal problem.

A.4 Increasing WTP

We now analyze separately the case of an Increasing and Decreasing WTP, starting with the former. In both cases, we prove that the maximizer to problem (11), λ_i, is unique establish monotonicity of $\bar{U}_i$ and λ_i, along with the relationship between λ_i and λ_i^*. Finally, we can recover the transfer from the free entry condition, which under symmetry reduces to

$$\phi_i = \tfrac{1}{2}c\lambda_i^{\frac{1-\gamma}{\gamma}}.$$

Thus monotonicity of λ_i immediately implies monotonicity of ϕ_i.

Now assume Increasing WTP. By construction, we have $\bar{U}_1 = g(\lambda_{1,1}^*)$, and the (1,1) market has $\lambda_1 = \lambda_1^*$.

We then proceed by induction. Assume that for $j = 1, \ldots, i-1$, we have proved that $\lambda_j > \lambda_{j-1}$ and $\lambda_j \geq \lambda_j^*$. We look at problem (11) for type i and prove $\lambda_i > \lambda_{i-1}$ and $\lambda_i \geq \lambda_i^*$. We break the analysis into two cases.

1. Assume $\bar{U}_{i-1} \geq g_{i-1,i}(\lambda_i^*)$. In this case, the solution is unconstrained, $\lambda_i = \lambda_i^*$. We claim that $\lambda_i > \lambda_{i-1}$. By equation (10) and monotonicity of u, this is true if $\lambda_{i-1} = \lambda_{i-1}^*$. Alternatively, we may have $\lambda_{i-1} > \lambda_{i-1}^*$. If $\lambda_{i-1} \geq \lambda_i^* > \lambda_{i-1}^*$, $g_{i-1,i-1}(\lambda)$ decreasing for $\lambda > \lambda_{i-1}^*$ implies $\bar{U}_{i-1} = g_{i-1,i-1}(\lambda_{i-1}) \leq g_{i-1,i-1}(\lambda_i^*)$. And then g is increasing in its second subindex because u is increasing in its second subindex (Common Ranking), so $g_{i-1,i-1}(\lambda_i^*) < g_{i-1,i}(\lambda_i^*)$. This implies $\bar{U}_{i-1} < g_{i-1,i}(\lambda_i^*)$, and so $\lambda = \lambda_i^*$ does not satisfy the constraint in problem (11), a contradiction. This proves that if $\lambda_{i-1} > \lambda_{i-1}^*$ and $\lambda_i = \lambda_i^*$, then $\lambda_i > \lambda_{i-1}$.

2. If $\bar{U}_{i-1} < g_{i-1,i}(\lambda_i^*)$, the constraint is binding, so $\bar{U}_{i-1} = g_{i-1,i}(\lambda_i)$. We know $\bar{U}_{i-1} = g_{i-1,i-1}(\lambda_{i-1}) < g_{i-1,i}(\lambda_{i-1})$, where the inequality again uses Common Ranking. This means that there are two solutions to $\bar{U}_{i-1} = g_{i-1,i}(\lambda_i)$, one bigger than λ_{i-1} and one smaller: $\ell_1 < \lambda_{i-1} < \ell_2$ with $\bar{U}_{i-1} = g_{i-1,i}(\ell_1) = g_{i-1,i}(\ell_2)$. Use the definition of g in equation (9), together with the definitions of ℓ_1 and ℓ_2:

$$\bar{U}_{i-1} = u_{i-1,i}\ell_1 - \tfrac{1}{2}c\ell_1^{\frac{1}{\gamma}} = u_{i-1,i}\ell_2 - \tfrac{1}{2}c\ell_2^{\frac{1}{\gamma}}$$

Since $\ell_1 < \ell_2$ and $u_{i-1,i} < u_{i,i}$ (Increasing WTP), $(u_{i,i}-u_{i-1,i})\ell_1 < (u_{i,i}-u_{i-1,i})\ell_2$. Add this to the previous equation to get

$$g_{i,i}(\ell_1) = u_{i,i}\ell_1 - \frac{c}{2}\ell_1^{\frac{1}{\gamma}} < u_{i,i}\ell_2 - \frac{c}{2}\ell_2^{\frac{1}{\gamma}} = g_{i,i}(\ell_2),$$

This proves $g_{i,i}(\ell_1) < g_{i,i}(\ell_2)$, hence $\lambda_i = \ell_2 > \lambda_{i-1}$, the larger solution. Finally, in this case $\lambda_i > \lambda_i^*$, because otherwise the unconstrained solution would have satisfied the incentive constraint.

This proves that in the solution to problem (11) is unique and has $\lambda_i > \lambda_{i-1}$ and $\lambda_i \geq \lambda_i^*$. As discussed before, it implies $\phi_i > \phi_{i-1}$ as well.

To prove $\bar{U}_i > \bar{U}_{i-1}$, we again look at two cases.

1. If $\lambda_i = \lambda_i^*$, then we have $\bar{U}_i = g_{i,i}(\lambda_i^*) > g_{i-1,i-1}(\lambda_{i-1}^*) \geq \bar{U}_{i-1}$, where the first inequality uses the fact that g is increasing in u and $u_{i,i} > u_{i-1,i-1}$ under Common Ranking and Increasing WTP.
2. If $\lambda_i > \lambda_i^*$, the local incentive constriant binds, so we have $\bar{U}_{i-1} = g_{i-1,i}(\lambda_i) < g_{i,i}(\lambda_i) = \bar{U}_i$, where again the inequality uses the fact that g is increasing in u and $u_{i-1,i} < u_{i,i}$ under Increasing WTP.

A.5 Decreasing WTP

Now assume Increasing WTP. By construction, we have $\bar{U}_1 = g(\lambda_{1,1}^*)$, and the (1,1) market has $\lambda_1 = \lambda_1^*$. We again proceed by induction. Assume that for $j = 1, \ldots, i-1$, we have proved that $\lambda_j < \lambda_{j-1}$ and $\lambda_j \leq \lambda_j^*$. We look at problem (11) for type i and prove $\lambda_i < \lambda_{i-1}$ and $\lambda_i \leq \lambda_i^*$. We break the analysis into two cases.

Case A: $u_{i,i} < u_{i-1,i-1}$. There are two subcases:

1. Assume $\bar{U}_i \geq g_{i-1,i}(\lambda_i^*)$. In this case the solution is unconstrained, $\lambda_i = \lambda_i^*$. If $\lambda_{i-1}^* = \lambda_{i-1}$, then the unconstrained problem directly implies $\lambda_i < \lambda_{i-1}$. Otherwise suppose $\lambda_{i-1} < \lambda_{i-1}^*$. If $\lambda_{i-1}^* > \lambda_i^* > \lambda_{i-1}$, we have the following inequality holds:

$$\bar{U}_{i-1} = g_{i-1,i-1}(\lambda_{i-1}) \leq g_{i-1,i-1}(\lambda_i^*) < g_{i-1,i}(\lambda_i^*),$$

where the first inequality comes from $g_{i-1,i-1}(\lambda)$ increasing for $\lambda < \lambda_{i-1}^*$ and the second inequality comes from Common Ranking. Thus the IC in market i must be violated. This is a contradiction, proving that $\lambda_{i-1} > \lambda_i^* = \lambda_i$.

2. If $\bar{U}_{i-1} < g_{i-1,i}(\lambda_i^*)$, the constraint is binding, so $\bar{U}_{i-1} = g_{i-1,i}(\lambda_i)$. We know $\bar{U}_{i-1} = g_{i-1,i-1}(\lambda_{i-1}) < g_{i-1,i}(\lambda_{i-1})$, where the inequality again uses Common Ranking. This means that there are two solutions to $\bar{U}_{i-1} = g_{i-1,i}(\lambda_i)$, one bigger than λ_{i-1} and one smaller: $\ell_1 < \lambda_{i-1} < \ell_2$ with $\bar{U}_{i-1} = g_{i-1,i}(\ell_1) = g_{i-1,i}(\ell_2)$. Use the definition of g in equation (9), together with the definitions of ℓ_1 and ℓ_2:

$$\bar{U}_{i-1} = u_{i-1,i}\ell_1 - \tfrac{1}{2}c\ell_1^{\frac{1}{\gamma}} = u_{i-1,i}\ell_2 - \tfrac{1}{2}c\ell_2^{\frac{1}{\gamma}}$$

Because $\ell_1 < \ell_2$ and $u_{i,i} < u_{i-1,i}$ (Decreasing WTP), $\ell_1(u_{i,i} - u_{i-1,i}) > \ell_2(u_{i,i} - u_{i-1,i})$. Adding this inequality to the previous equation $g_{i-1,i}(\ell_1) = g_{i-1,i}(\ell_2)$:

$$g_{i,i}(\ell_1) = u_{i,i}\ell_1 - \frac{c}{2}\ell_1^{\frac{1}{\gamma}} > u_{i,i}\ell_2 - \frac{c}{2}\ell_2^{\frac{1}{\gamma}} = g_{i,i}(\ell_2)$$

Thus $\lambda_i = \ell_1 < \lambda_{i-1}$. In addition, it must be $\lambda_i < \lambda_i^*$ because the unconstrained solution would have satisfied the incentive constraint.

Case B: $u_{i,i} \geq u_{i-1,i-1}$**.** We start by showing that $\bar{U}_i < g_{i-1,i}(\lambda_i^*)$. If not, we have $\lambda_i = \lambda_i^*$ and the following inequality holds:

$$\bar{U}_{i-1} \leq g_{i-1.i-1}(\lambda_{i-1}^*) < g_{i,i}(\lambda_{i-1}^*) \leq g_{i,i}(\lambda_i^*) < g_{i-1,i}(\lambda_i^*),$$

where the first inequality uses the fact $g_{i-1,i-1}(\lambda)$ is maximized at $\lambda = \lambda_{i-1}^*$; the second inequality uses the assumption in Case B that $u_{i-1,i-1} < u_{i,i}$; the third inequality uses the fact $g_{i,i}(\lambda)$ is maximized at $\lambda = \lambda_i^*$; and the last inequality uses $u_{i-1,i} > u_{i,i}$, as is the case with Decreasing WTP. This violates constraint in problem (11). This proves that there is a binding constraint, $\bar{U}_{i-1} = g_{i-1,i}(\lambda_i)$.

As before, there are two solutions to $\bar{U}_{i-1} = g_{i-1,i}(\lambda)$. Denote them as $\ell_1 < \lambda_{i-1} < \ell_2$ with $\bar{U}_{i-1} = g_{i-1,i}(\ell_1) = g_{i-1,i}(\ell_2)$, so

$$\bar{U}_{i-1} = u_{i-1,i}\ell_1 - \tfrac{1}{2}c\ell_1^{\frac{1}{\gamma}} = u_{i-1,i}\ell_2 - \tfrac{1}{2}c\ell_2^{\frac{1}{\gamma}}$$

Because $\ell_1 < \ell_2$ and $u_{i,i} < u_{i-1,i}$ (Decreasing WTP), $\ell_1(u_{i,i} - u_{i-1,i}) > \ell_2(u_{i,i} - u_{i-1,i})$. Adding the inequality to the previous equation:

$$g_{i,i}(\ell_1) = u_{i,i}\ell_1 - \frac{c}{2}\ell_1^{\frac{1}{\gamma}} > u_{i,i}\ell_2 - \frac{c}{2}\ell_2^{\frac{1}{\gamma}} = g_{i,i}(\ell_2).$$

Thus $\lambda_i = \ell_1 < \lambda_{i-1}$. A similar logic to the earlier case implies $\lambda_i < \lambda_i^*$.

This proves that in the solution to problem (11) is unique and has $\lambda_i < \lambda_{i-1}$ and $\lambda_i \leq \lambda_i^*$. As discussed before, it implies $\phi_i < \phi_{i-1}$ as well.

To prove $\bar{U}_i < \bar{U}_{i-1}$, note that the local incentive constraint implies $\bar{U}_{i-1} \geq g_{i-1,i}(\lambda_i)$. Since g is increasing in u and $u_{i-1,i} > u_{i,i}$ under Decreasing WTP, $g_{i-1,i}(\lambda_i) > g_{i,i}(\lambda_i) = \bar{U}_i$. Combining inequalities gives the desired result.

A.6 Local IC implies Global IC

We now prove that under Monotone WTP, any solution to problem (22) satisfies all the downward incentive constraints and so solves problem (21). The result holds trivially for types 1 and 2. We proceed by induction. Fix $i \geq 3$. Suppose that we have already shown that for $i-1$ all incentive constraints are satisfied. That is, for $j < i-1$,

$$\bar{U}_j - \bar{U}_{i-1} \geq \lambda_{i-1}(u_{j,i-1} - u_{i-1,i-1}). \tag{23}$$

Increasing WTP implies $u_{j,i-1} < u_{i-1,i-1}$ and we proved in Appendix A.4 that $\lambda_i > \lambda_{i-1}$. This implies $\lambda_{i-1}(u_{j,i-1} - u_{i-1,i-1}) > \lambda_i(u_{j,i-1} - u_{i-,i-1})$. Likewise, Decreasing WTP implies $u_{j,i-1} > u_{i-1,i-1}$ and we proved in Appendix A.4 that $\lambda_i < \lambda_{i-1}$, which gives the same inequality. In either case, combining this with inequality (23) gives

$$\bar{U}_j - \bar{U}_{i-1} > \lambda_i(u_{j,i-1} - u_{i-1,i-1}). \tag{24}$$

Additionally, Supermodularity implies $u_{j,i-1} - u_{i-1,i-1} > u_{j,i} - u_{i-1,i}$. Combining this with inequality (24) gives

$$\bar{U}_j - \bar{U}_{i-1} > \lambda_i(u_{j,i} - u_{i-1,i}). \tag{25}$$

Finally, the local IC constraint for i is $\bar{U}_{i-1} - \bar{U}_i \geq \lambda_i(u_{i-1,i} - u_{i,i})$. Adding this to inequality (25) gives

$$\bar{U}_j - \bar{U}_i > \lambda_i(u_{j,i} - u_{i,i}). \tag{26}$$

This is the global incentive constraint in problem (21), establishing that any solution to the relaxed problem (22) also solves the original problem (21).

A.7 Summary

At this point, we have characterized the unique solution to problem (11) and proved that it solves the relaxed primal problem (22). We then showwed that any solution to the relaxed primal problem (21) also solves the more restricted problem (21). The proof that active markets are symmetric implies that the solution to problem (21) also solves the asymmetric problem (8) and hence, with ϕ defined as above, problem (7).

This in turn is a special case of problem (6) for a market with PAM, which in turn is a special case of problem (5) for separating markets. Lemma 2 then tells us that if there is a PAM competitive search equilibrium, the unique solution to problem (11) characterizes it.