Assortative Matching with Private Information

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Abstract

We study matching between heterogeneous agents when their types are private information. Competing platforms post terms of trade; Agents with private information chose where to search for each other and form matches. We reprise results from one-sided adverse selection and from full-information matching. Supermodularity in private value of matches leads to separation of types and sufficiency of downward incentive constraints. In the limit with vanishing role of platforms, we derive the set of conditions that guarantees positive assortative matching.

1 Introduction

Matching complementarities generate positively assortative matching, while substitutability generates negatively assortative matching (Koopmans and Beckmann, 1957; Becker, 1973). This conclusion holds in an environment where individual characteristics are common knowledge, so in particular everyone knows whom they are matching with when they match. This paper explores the robustness of sorting patterns to this informational assumption. We ask who matches with whom when individuals are privately informed about their own characteristics, and those characteristics affect the value an individual gets from matching and the value a partner gets from matching with them.

In order to have sorting in an environment where characteristics are completely hidden, there must be a mechanism for getting individuals with different types to separate themselves. The mechanism we propose is their choice of competing markets, where a market is defined as a combination of access fees, matching frequencies, and composition of the population. Individuals may self-select into different markets because they have different preferences over this triad. We stress that markets do not have any special ability to screen individuals, but instead rely on this self-selection.

Importantly, we assume that there are three inputs into matching in a market. First, there are two sides to each market. These two sides are intrinsically the same, but the market may charge a different fee for accessing each side, and as a result the composition of individuals on each side of the market may be different. And second, there is another type of actor, competitive platforms, which facilitate matching through their costly postings to markets. Free entry pins down the number of postings, and postings are only made in the most profitable markets, shutting the other ones down. When a market matches a platform with a random, independently selected, and privately-informed individual from each side of the market, the individuals pay the stated fee and then collect any benefit from matching.

This structure is rich enough to allow for both positive and negative sorting between the privately-informed individuals. Positive sorting arises, for example, if each market only attracts one type of person, the same type on both sides of the market. Negative sorting arises if each market attracts one type of person on one side of the market and very different type on the other side of the market. Platform postings play an important role in this analysis because they permit us to have independent variation in the ease of matching on both sides of the market. We find that this is important when we have negative sorting.

In this environment, we find that the sorting patterns that arise are broadly in line with those predicted in the common knowledge environment. In particular, when there are matching complementarities, we find that different types of individuals sort into different markets, where they match only with their own type. Thus our model delivers positively assortative matching. Under some conditions, active markets are unaffected by the presence of other types of individuals, and self-selection ensures that the desired type of individual comes to each market. Under other conditions, the access fees and matching frequencies in an active markets is distorted so as to ensure that the market is attractive only to the types that it is supposed to attract. That is, incentive constraints may bind. Still, matching is positively assortative.

When there are matching substitutabilities, more possibilities arise, even in the simplest case with only two types of individuals. With some parameter values, there is negatively assortative matching while with others there is no sorting at all. For example, equilibrium may have only one active market, with the two types of individuals sorting between the two sides of the market. This means that high types match only with low types and vice versa, negatively assortative matching. In this case, high types may pay a negative fee (i.e. receive a transfer), while low types are willing to pay a positive fee to match with high types.

Alternatively, a negatively sorted market may coexist with another market that only attracts one type of individual, soaking up the excess supply of individuals from the relatively abundant side of the market. Again, this case is broadly in line with results from the prior literature.

When incentive issues are severe, however, another type of market may arise when there are matching substitutabilities. In this market, the fee is the same on the two sides, as is the mix of people who show up. The mix of people on both sides of the market is such that both types of people are willing to go to both sides of the market. As a result, matching is random within this market, so there is no sorting, even though we are in the case with substitutes. This type of market may also coexist with a negatively sorted market.

Before turning to a more detailed description of our theory, we first offer an illustrative model of learning that may be useful for grounding ideas. There are two types of people, knowledgable and ignorant. People interact in pairs, getting a utility benefit from those interactions. In particular, everyone gets the same utility from interacting with an ignorant person, and they get more utility from interacting with a knowledgable person. Complementarity (substitutability) corresponds to the case where the utility benefit enjoyed by a knowledgable person interacting with another knowledgable person is higher (lower) than the benefit an ignorant person gets from interacting with a knowledgable person. We can imagine why either might be the case. For example, an ignorant person may be unable to fully appreciate the nuance of the knowledgable person's argument (substitutes), or an ignorant person simply has more to learn (complements).

To model interactions in this framework, we use an extension to the competitive search equilibrium framework, in particular building on Guerrieri, Shimer and Wright (2010) and Guerrieri and Shimer (2014). One of our innovations is to introduce matching between pairs of privately-informed individuals, facilitated by a platform's postings, so we have a matching function with three arguments. Otherwise, our notion of competitive search equilibrium draws heavily on that earlier research, although as we discuss, some interesting new issues arise in this framework.

A market is defined by a pair of transfers from successfully matched agents on each side to the platform to the platform. In competitive search, privately-informed people direct their search to a particular side of a particular market, and platforms make postings in the most profitable market. A constant returns to scale matching function with these three inputs then delivers the number of matches between two agents and a platform. Through variation in the ratios of the three arguments, we obtain independent variation in the ease of matching for the privately informed individuals on both sides of the market. In equilibrium, markets are distinguished by what types of people go to each side, how hard it is to match on each side, and the fee paid by each side, with the first two objects endogenous. People choose to go to the market that delivers the highest expected utility, which disciplines the relationship between these objects.

In a competitive search equilibrium, platforms and privately-informed individuals have rational beliefs about the ease of matching on each side of each market and about the composition of each side of each market. They assume that their own behavior does not affect those outcomes. They then go to the market, and the side of the market, that delivers them the highest expected utility. In turn, the beliefs about who is on each side of a market and how hard it is to match on that side must be consistent with individuals behaving rationally in their choice of markets.

One technical issue that arises in our framework is that there may be multiple rational beliefs about who goes to each market. For example, with complementarities, knowledgable people may be more willing to pay a particular fee if they believe other knowledgable people will go to the market, while they will shy away from the same market if they believe ignorant people will come. As a result, there may be a rational belief that the market will have only ignorant people. Of course, this makes the market less attractive to everyone than a rational belief that the market will have only knowledgable people. We resolve this multiplicity by generally focusing on the outcome in a particular market that is optimal from the perspective of the platform. That is, if the platform wants knowledgable people to come to the market, and it is rational for knowledgable people to come to the market when they expect to meet knowledgable people there, the platform can coordinate beliefs so as to achieve this outcome. Still, we find that a restriction to platform-optimal market outcomes is inconsistent with existence of a competitive search equilibrium for some parameter values in the submodular case.

Prior research has explored sorting in competitive search equilibrium with observable types. Shi (2001) characterizes efficient sorting patterns and shows how they can be decentralized through a competitive search equilibrium. Heterogeneous firms post wages and skill requirements, and workers apply for the job yielding the highest expected utility. Eeckhout and Kircher (2010) shows that negatively assortative matching can arise when there are matching complementarities, and prove that complementarity and n-root-complementarity together are sufficient to ensure positively assortative matching.

Our main innovation relative to these papers is the introduction of private information, which means (in the language of Shi (2001)) that firms cannot post skill requirements, but instead must accept the mix of workers who apply for the job. Additionally, we assume that both sides of the market care about their partner's type. In Shi (2001), workers only care about wages, not the type of firm they work for. In Eeckhout and Kircher (2010), sellers only care about the price they get, not who buys the object for sale.

There are also papers exploring sorting with private information. Damiano and Li (2007) and Hoppe, Moldovanu and Ozdenoren (2011) characterize the screening of privately-informed types in a matching environment by a platform with monopolist power. Our paper makes two innovations relative to this literature. First, we allow for endogenous matching rates that allows for screening not only through transfers but also contact rates. We show equilibrium screening could happen through either margins, showing the non-triviality of considering search friction; Secondly, we consider the problem in a competitive equilibrium, where platforms also face competitions from other platforms. This leads to different predictions regarding separation of types into different markets.

In Section 2, we develop a general framework for analyzing sorting in competitive search equilibrium when types are private information. We briefly discuss the conditions that lead to separation of types and assortative matching when types are observable, in section 4. With private types, one would need stronger conditions to guarantee separation and positive assortative matching. We discuss these results respectively in section 3 and section 3.4.

2 Model

2.1 Platforms and Agents

Time is continuous and lasts forever. There are two sets of individuals, platforms and agents. Agents are privately-informed about their type $i \in \{1, \ldots, I\} \equiv \mathbb{I}$. Let $\bar{\omega}_i > 0$ denote the exogenous measure of type i agents in the economy, with $\sum_{i=1}^{I} \bar{\omega}_i = 1$. There is a large number of homogeneous platforms which can post to a market by paying a flow cost c. Platforms can make as many postings as they want, potentially to different markets. Agents and platforms are risk-neutral, infinitely-lived, and discount the future at rate r.

2.2 Matching Function

Platforms intermediate matches between two agents. We assume each market has two sides, labeled by side a and side b. If a particular market has a measure N^a of agents on side a, N^b of agents on side b, and a measure N^p of postings by platforms, then the flow of matches in the market is $(N^a N^b)^{\frac{1-\gamma}{2}}(N^p)^{\gamma}$, where $0 < \gamma < 1$. It follows that a platform posting in this market matches with two agents according to a Poisson process with arrival rate $(N^a N^b)^{\frac{1-\gamma}{2}}(N^p)^{\gamma}/N^p = (n^a n^b)^{\frac{1-\gamma}{2}}$ where $n^s \equiv N^s/N^p$ is the number of agents going to side s per posting. Similarly, an agent going to side s finds a match according to a Poisson process with arrival rate $\lambda^s \equiv (N^a N^b)^{\frac{1-\gamma}{2}}(N^p)^{\gamma}/N^s = (n^s)^{-\frac{1+\gamma}{2}}(n^{-s})^{\frac{1-\gamma}{2}}$. Here we use the notation $-s = b$ if $s = a$ and $-s = a$ if $s = b$. Given any pair of contact rate (λ^a, λ^b), we can invert the matching function to solve for (n^a, n^b) as:

$$n^s = (\lambda^s)^{-\frac{1+\gamma}{2\gamma}} (\lambda^{-s})^{-\frac{1-\gamma}{2\gamma}}. \tag{1}$$

We can then recover the platform matching probability as $(n^a n^b)^{\frac{1-\gamma}{2}} = (\lambda^a \lambda^b)^{-\frac{1-\gamma}{2\gamma}}$. It is convenient in what follows to treat λ rather than n as a primitive feature of a market.

2.3 Markets and Payoffs

A market is a vector $m = \left(\lambda^s, \tau^s, \left(\omega_i^s\right)_{i=1}^I\right)_{s=a,b}$ satisfying $\lambda^s > 0$, $\omega_i^s \geq 0$, and $\sum_i^I \omega_i^s = 1$. We interpret λ^s to be the contact rate for agents on side s; τ^s to be the fee paid by a agent who matches on side s of the market; and ω_i^s to be the share of type i agents on side s of the market. Let $\mathbb{M}$ denote the set of markets, i.e. the set of vectors satisfying these conditions. A platform posting in market $m \in \mathbb{M}$ matches two agents at rate $(\lambda^a \lambda^b)^{-\frac{1-\gamma}{2\gamma}}$. When this happens, the probability it is an (i, j) match is $\omega_i^a \omega_j^b$, which delivers fees of $\tau^a + \tau^b$. Thus the gross flow profit from the posting is

$$rV(m) \equiv (\lambda^a \lambda^b)^{-\frac{1-\gamma}{2\gamma}} \left(\tau^a + \tau^b\right). \tag{2}$$

From this, we must subtract the flow cost of postings rc to obtain the flow value of the posting.

Similarly, if a type i agent goes to side $s \in \{a, b\}$ of market $m \in \mathbb{M}$, he matches at rate λ^s, in which event he pays a fee τ^s and is matched to a type j agent on the $-s$ side of the market with probability ω_j^{-s}, earning payoff $u_{i,j}$. Thus the value to i of this action is

$$rU_i^s(m) \equiv \lambda^s \left(\sum_{j=1}^I \omega_j^{-s} u_{i,j} - \tau^s \right). \tag{3}$$

Throughout the paper, we assume the payoff from matching is positive and strictly increasing in the partner's type. Thus a higher type in our environment means a type that is strictly preferred by everyone else.

Assumption 1 (Monotonicity) *For every i and $j > j'$, $u_{i,j} > u_{i,j'} > 0$.*

2.4 Partial Equilibrium

Let $N^p(m)$ denote the measure of postings in market m. A market is active if $N^p(m) > 0$ and otherwise inactive. Let $M \subset \mathbb{M}$ denote the (nonempty) set of markets that are active in equilibrium. We can now define a partial equilibrium with observable types:

Definition 1 *A **partial equilibrium with sorted matching technology** is a nonempty set of active markets $M \subset \mathbb{M}$ and nonnegative numbers $\bar{V}, \bar{U}_1, \ldots, \bar{U}_I$ such that*

1. *$\forall m \in M$, (a) $\bar{V} = V(m)$; (b) $\bar{U}_i \geq U_i^s(m)$, $\forall i \in \mathbb{I}, s = a, b$, with equality if $\omega_i^s(m) > 0$;*
2. *$\nexists m \in \mathbb{M}$, (a) $V(m) > \bar{V}$; (b) $\bar{U}_i \geq U_i^s(m)$, $\forall i \in \mathbb{I}, s = a, b$, with equality if $\omega_i^s(m) > 0$.*

First, $\bar{V}$ is the profit that a posting can earn in any active market. Second, $\bar{U}_i$ is the maximal utility that a type i agent can earn in any active market if he truthfully reports his type, and he earns this utility in any active market that expects to attract his type, $\omega_i^s(m) > 0$. Finally, we require that there is no other market that yields profits in excess of $\bar{V}$ while delivering utility no more than $\bar{U}_i$ on side s to a type i agent who truthfully reports his type, and exactly this level of utility if type i agents are supposed to be on side s.

2.5 Free Entry and Market Clearing

Definition 2 *A* ***competitive search equilibrium with sorted matching technology*** *is a partial equilibrium with sorted matching technology and measures $N^p(m)$ for all $m \in M$ such that*

1. *free entry: $\bar{V} = c$*
2. *market clearing: $\bar{\omega}_i = \sum_{m \in M} N^p(m) \sum_{s \in \{a,b\}} \omega_i^s n^s$, where n^s satisfies equation (1).*

2.6 Separation and Assortative Matching

Two characteristics of the competitive search equilibrium are of particular interest to this paper. First, do the active markets in an equilibrium involve different pairs of types matching together, or is there only one pair of matching types possible? To answer this question, we define a notion of separating markets. Second, are high-type agents more likely to match with high-type or low-type agents? To answer this question, we define *positive assortative matching* (PAM) and *negative assortative matching* (NAM). We analyze these two characteristics in the model with both one-door matching function and two-door matching function.

Definition 3 *A market $m \in \mathbb{M}$ is separating if $\omega_i^s \in \{0, 1\}$, $\forall\, i = 1, ..., I$ and $s = a, b$. A competitive search equilibrium (or a partial equilibrium) is separating if M contains only separating markets.*

Our definition of sorting is based the set of matches made within active markets. More specifically, we define $\epsilon(M) = \{(i, j)|\omega_i^a \omega_j^b > 0$ for some $m \in M\}$ for M that is separating.

Definition 4 *A set of separating markets M has positive (negative) assortative matching if $\forall$ $(i_1, j_1), (i_2, j_2) \in \epsilon(M)$, $i_1 > i_2$ implies $j_1 \geq (\leq) j_2$. A competitive search equilibrium (or partial equilibrium) has PAM (NAM) if the equilibrium set of markets has PAM (NAM).*

2.7 Sorting with Conventional Matching Function

The key novelty of the environment in this paper is the assumption of a matching function that with two sides of a match which can have different contact rates. This assumption allows for the technological possibility of negative sorting. Before proceeding to the characterization of equilibrium, we consider an alternative matching technology that involves only agents and platform postings. Platforms intermediate matches between two agents. Instead of the matching function in section 2.2, suppose the flow of matches in the market is $(N^a)^{1-\gamma}(N^p)^{\gamma}$ where $0 < \gamma < 1$, where N^a is the measure of agents, and N^p is the measure of postings by platforms. It follows that a platform posting in this market matches with two agents according to a Poisson process with arrival rate $(N^a)^{1-\gamma}(N^p)^{\gamma-1} = n^{1-\gamma}$ where $n \equiv N^a/N^p$ is the number of agents per platform. Similarly, an agent finds a match according to a Poisson process with arrival rate $\lambda \equiv (N^a)^{1-\gamma}(N^p)^{\gamma}/N^a = n^{-\gamma}$. This matching technology is a direct extension of the standard bilateral matching technology into our environment. The core result of this section highlights a physical restriction due to random matching technology that rules out negative sorting among agents, which motivates us to adapt our environment to a richer setting where negative sorting is possible.

The core result of this section is the impossibility of positive sorting with random matching technology. To show this, we define a set of active market as M, and the measure of platform postings as N^p, where $N^p(m)$ is the measure of postings in market $m \in M$. We define the covariance of matching types as:

$$\mathbf{COV}(M, N^p) = \sum_{m \in M} \sum_{i=1}^{I} \sum_{j=1}^{J} N^p(m) \lambda^{-\frac{\gamma}{1-\gamma}} \omega_i \omega_j \left(i - \mathbf{E}(i)\right) \left(j - \mathbf{E}(j)\right), \tag{4}$$

where

$$\mathbf{E}(i) = \sum_{m \in M} \sum_{i=1}^{I} N^p(m) \lambda^{-\frac{\gamma}{1-\gamma}} \omega_i i,$$

$$\mathbf{E}(j) = \sum_{m \in M} \sum_{j=1}^{I} N^p(m) \lambda^{-\frac{\gamma}{1-\gamma}} \omega_j j.$$

The covariance between agents' types and their matching partners' types must be non-negative with random matching technology, as stated in the following lemma:

Lemma 1 *Under random matching technology,* $\mathbf{COV}(M, N^p) \geq 0$*.*

Proof. Due to random matching within markets, $\mathbf{E}(i) = \mathbf{E}(j)$. Thus we can write the covariance as

$$\mathbf{COV}(M, N^p) = \sum_{m \in M} N^p(m) \lambda^{-\frac{\gamma}{1-\gamma}} \left[\sum_{i=1}^{I} \omega_i \left(i - \mathbf{E}(i) \right) \right]^2 \geq 0.$$

■

Lemma 1 highlights a physical restriction on sorting pattern due to random matching within markets. It is impossible to have negative assortative matching because for each match made between a high-type agent and a low-type agent, there must be another match between each types within themselves. The core assumption that leads to this result is that the contact rates for agents within each market equal. To allow for the possibility of negative assortative matching, we introduce the notion of a *sorted matching technology* within markets. One could characterize a positively sorted equilibrium with random technology, and the equilibrium outcomes resemble the ones we will derive with sorted matching technology. We give the details in section **??**.

3 PAM with Private Information

3.1 General Results

We start by showing that the outcomes of a competitive search equilibrium can be characterized by a set of optimization problem, even with private information. Consider the following problem given $\bar{\boldsymbol{U}}$:

$$\bar{V} = \max_{\tau^a, \tau^b, \lambda^a > 0, \lambda^b > 0, \omega^s \in \Delta^I} \left(\lambda^a \lambda^b \right)^{-\frac{1-\gamma}{2\gamma}} (\tau^a + \tau^b), \tag{P}$$

s.t.
for $s = a, b$

$$\bar{U}_i \geq \lambda^s \left(\sum_{i=1}^{I} \omega_j^{-s} u_{i,j} - \tau^s \right),$$

with equality if $\omega_i^s > 0$.

We start by stating the equivalence of a partial equilibrium and a solution to problem (P).

Lemma 2 *Suppose* $\{M, \bar{V}, \bar{\boldsymbol{U}}\}$ *is a partial equilibrium with private information. For any* $m \in M$*, denote* $\tau^s = \tau_i^s$*.* $\{\bar{V}, (\lambda^s, \tau^s, (\omega_i^s)_{i=1}^I)_{s=a,b}\}$ *must solve (P); Conversely, if* $\{\bar{V}, (\lambda^{s*}, \tau^{s*}, (\omega_i^{s*})_{i=1}^I)_{s=a,b}\}$ *solve problem (P) given* $\bar{\boldsymbol{U}}$*, then there is a partial equilibrium with* $\{M, \bar{V}, \bar{\boldsymbol{U}}\}$ *with* $m \in M$ *such that* $\tau_i^s = \tau^{s*}$ *for* i*,* $\lambda^s = \lambda^{s*}$*, and* $\omega_i^s = \omega_i^{s*}$*.*

Proof. First, suppose $m \in M$ in a partial equilibrium and m is not a solution to problem (P). Then there exists $(\tau^{s'}, \lambda^{s'}, (\omega_i^{s'})_{i\in\mathbb{I}})_{s=a,b}$ that satisfies the constraints in problem (P) and:

$$(\lambda^{a'}\lambda^{b'})^{-\frac{1-\gamma}{2\gamma}}(\tau^{a'}+\tau^{b'}) > (\lambda^a\lambda^b)^{-\frac{1-\gamma}{2\gamma}}(\tau^a+\tau^b).$$

We can thus create a market with higher $V(m')$ by setting the contact rate to be λ^s, transfers to be τ^s, and type distribution to be $(\omega_i^s)_{i\in\mathbb{I}}$. This violates the point 2 for the definition of a partial equilibrium with private information.

Conversely, suppose $(\lambda^{S*}, \tau^{s*}, (\omega_i^{s*}))$ solve problem (P), yet we cannot construct a market as in the lemma for any partial equilibrium. Denote the constructed market m^*. Then in these partial equilibrium, either $\bar{V} > V(m^*)$, which violates m^* being a solution, or $\bar{V} < V(m^*)$, which violates the definition of a partial equilibrium. ■

3.2 Implications of Supermodularity

With private information, contact rates and transfers take an additional role of screening agents. Naturally, we would need additional assumptions to guarantee separation. It turns out the supermodularity in $u_{i,j}$ is sufficient for separation.

Definition 5 *$u_{i,j}$ is supermodular if for $i > i'$ and $j > j'$, $u_{i,j} + u_{i',j'} > u_{i,j'} + u_{i',j}$.*

When $u_{i,j}$ is supermodular in (i, j), we can further simply the problem by considering the following problem:

$$V_{i,j} \equiv \max_{\lambda,\tau} (\lambda^a\lambda^b)^{-\frac{1-\gamma}{2\gamma}}(\tau^a+\tau^b) \tag{PS-1}$$

s.t.

$$\begin{aligned}
\bar{U}_i &= \lambda^a(u_{i,j} - \tau^a),\\
\bar{U}_j &= \lambda^b(u_{j,i} - \tau^b),\\
\bar{U}_k &\geq \lambda^a(u_{k,j} - \tau^a), \forall k < i,\\
\bar{U}_k &\geq \lambda^b(u_{k,i} - \tau^b), \forall k < j,
\end{aligned}$$

and

$$\bar{V} = \max_{i,j} V_{i,j} \tag{PS-2}$$

In relaxed problem (PS-1), there are two simplifications. First, we only consider separating markets. Secondly, we only take into account the incentive constraints with types below the focal participants, the *downward incentive constraints*. The following lemma states that it is without loss of generality to only consider the relaxed problem (PS-1) and (PS-2), when $u_{i,j}$ is supermodular.

Lemma 3 *Suppose $u_{i,j}$ is supermodular. If $(\bar{V}, (\lambda^s, \tau^s, (\omega_i^s)_{i=1\ldots,i})_{s=a,b}$ solve problem (P), then $\omega_i^a \omega_j^b = 1$ for some (i,j), and (λ^s, τ^s) is solution to (PS-1) given (i,j) and (i,j) is solution to (PS-2); If $(\lambda^{s*}, \tau^{s*})$ solve (PS-1) given (i,j) and $(\bar{V}, (i,j))$ solve (PS-2), then $(\bar{V}, (\lambda^s, \tau^s, (\omega_i^s)_{i=1\ldots,i})_{s=a,b}$ solve problem (P) such that $\lambda^s = \lambda^{s*}$, $\tau^s = \tau^{s*}$, and $\omega_i^a \omega_j^b = 1$.*

Proof. The proof is by contradiction. Specifically, for each market with non-separation and binding upward incentive constraints, we look for a profitable deviation for platforms. In this deviation, we seek to single out the highest type whose incentive constraint holds with equality (,which could be a participating type of non-participating type.)

To start, we define:

$$\bar{i}_0 = \max \left\{ i | \bar{U}_i = \lambda^s (\sum_j \omega_j^{-s} u_{i,j} - \tau^s), \ for \ s = a, b \right\}.$$

We define a set of types:

$$\mathcal{I}^D(i) = \left\{ i' | \lambda^s \left(u_{i',i} - u_{i,i} \right) + \bar{U}_i - \bar{U}_{i'} \geq 0, i' > i \right\},$$

and

$$\iota(i) \equiv \begin{cases} \max \mathcal{I}^D(i) & \mathcal{I}^D(i) \neq \emptyset \\ i & \mathcal{I}^D(i) = \emptyset \end{cases}.$$

Starting from $\bar{i}_0$, we define a sequence $\bar{i}_{t+1} = \iota\left(\bar{i}_t\right)$. This is an weakly increasing sequence, and it must converge to a fixed point $i_* = \iota\left(i_*\right)$. Take this i_*, we propose the following deviation $m-1$, denoted as $\left(\lambda_1^s, \omega_i^{s'}, \tau_1^s\right)$.The deviation is constructed such that:

1. $\lambda_1^s = \lambda^s, for s = a, b$;
2. $\omega_i^{s'} = \begin{cases} 1 & \text{if } i = i_* \\ 0 & \text{otherwise} \end{cases}$;
3. $\bar{U}_{i_*} = \lambda_1^s \left(u_{i_*, i_*} - \tau_1^s \right).$

$m1$ is profitable for the platform because:

$$\begin{aligned}
0 &\leq \lambda_0^s \left(u_{i_*,\bar{i}_0} - u_{\bar{i}_0,\bar{i}_0}\right) + \bar{U}_{\bar{i}_0} - \bar{U}_{i_*} \\
&\leq \lambda_0^s \left(u_{i_* i_*} - u_{\bar{i}_0,\bar{i}_0}\right) + \bar{U}_{\bar{i}_0} - \bar{U}_{i_*} \\
&< \lambda_0^s \left(u_{i_* i_*} - \sum_j \omega_j u_{\bar{i}_0,j}\right) + \bar{U}_{\bar{i}_0} - \bar{U}_{i_*} \\
&= \lambda^s \left(\tau_1^s - \tau_0^s\right),
\end{aligned}$$

where the first inequality comes from the construction of i_*, the second inequality comes from the monotonicity of $u_{i,j}$ in j, the last inequality comes from the ICs in m and $m1$. This implies that $\tau_1^s > \tau_0^s$. The contact rates stay the same while the fees per match increase. Thus the deviation must be strictly profitable.

By the definition of i_*, the incentive constraint of any $i > i_*$ must hold with strict inequality. For any $i < i_*$, the ICs are also satisfied: $\bar{U}_i - \lambda_0^s \left(u_{i,i_*} - \tau_1^s\right) = \bar{U}_i - \lambda_0^s u_{i,i_*} + \lambda_0^s \tau_1^s = \bar{U}_i - \bar{U}_{i_*} - \lambda_0^s u_{i,i_*} + \lambda_0^s u_{i_*,i_*} > \lambda_0^s \left(u_{i,i} - u_{i_*,i} - u_{i,i_*} + u_{i_*,i_*}\right) > 0$. ■

Corollary 1 *If $u_{i,j}$ is supermodular, any partial equilibrium must be separating.*

Corollary 2 *If $u_{i,j}$ is supermodular, upward incentive constraints must be slack for any market in a partial equilibrium.*

Example: Non-separating Equilibrium without Supermodularity. Most of the results in this paper is discussed under supermodularity and separation of markets. Before departure, we briefly discuss a case with non-separating competitive search equilibrium when we do not assume supermodularity. We consider a case where $I = 2$ and $\alpha = 0.6$. The utility is set such that $u_{0,0} = 1$, $u_{0,1} = 1.5$, $u_{1,0} = 1$, and $u_{1,1} = 1.2$. It is easy to verify that $u_{0,0} + u_{1,1} < u_{0,1} + u_{1,0}$. Thus $u_{i,j}$ does not satisfy supermodularity. With this set of parameters, a non-separating market always dominates the separating markets, when $\frac{\bar{U}_1}{\bar{U}_2} \in (1.4618, 1.5)$. For these ratio of $\frac{\bar{U}_1}{\bar{U}_2}$, the partial equilibrium is non-separating. We can find the corresponding measure of platform postings to clear the market when $\bar{\omega}_1 \in (0.88973, 1.0)$. So the competitive search equilibrium is non-separating.

3.3 Characterization of PAM Equilibrium

This section provides a characterization of competitive search equilibrium with PAM. In a competitive search equilibrium with PAM, each type must be matched with its own type. Thus all markets in a competitive search equilibrium must be in the form of (i, i). We start with a program that solves market (i, i) problem recursively.

Definition 6

$$
\begin{aligned}
c = & \max_{\lambda^s>0,\tau^s,s=a,b} (\lambda^a\lambda^b)^{\frac{1-\gamma}{2\gamma}}(\tau^a+\tau^b) \\
s.t.\ & \bar{U}_i = \lambda^s(u_{i,i}-\tau^s)\ \forall s=a,b, \\
& \bar{U}_j \geq \lambda^s(u_{j,i}-\tau^s)\ \forall s=a,b,j<i,
\end{aligned} \tag{5}
$$

and $c = \max_{\lambda^s>0,\tau^s,s=a,b}(\lambda^a\lambda^b)^{\frac{1-\gamma}{2\gamma}}(\tau^a+\tau^b)$ *s.t* $\bar{U}_i = \lambda^s(u_{i,i}-\tau^s)\ \forall s=a,b$.

In the recursive program in definition 6, we start from a $(1,1)$ market that is unconstrained. This gives a value $\bar{U}_1$. Recursively, we solve the problem of market (i,i) given the market utilities $\bar{U}_1,...,\bar{U}_{i-1}$, for $i>1$. For each problem (i,i), it has a concave objective function defined on a compact set. Thus if the feasible set is non-empty, there is always a unique solution. It is likely the feasible set becomes empty for some $i<I$. That is: there is no feasible set of contact rates transfers that jointly satisfies the incentive constraints of all downward incentive constraints. This is a case noted as countervailing incentives in the literature (Lewis and Sappington, 1989). When this happens, there is no competitive search equilibrium with PAM. The following section focuses on the case where $u_{i,j}$ is also monotonic in i, where the countervailing incentives do not show up. We could also simply the characterization by focusing only on local incentive constraints.

3.4 PAM with Monotonicity

We consider two cases: (1) one where $u_{i,j}$ is increasing in i, so higher types get higher payoffs from every match and (2) one where $u_{i,j}$ is decreasing in i, so higher types get lower payoffs from every match. For each case, we characterize the competitive search equilibrium imposing PAM and focus on only local incentive constraints, and verify that satisfying local incentive constraints implies satisfying all downward incentive constraints.

Definition 7

$$
\begin{aligned}
\bar{U}_i = & \max_{\lambda>0,\tau} \lambda(u_{i,i}-\tau) \\
s.t.\ & \bar{U}_{i-1} \geq g_{i-1,i}(\lambda),
\end{aligned} \tag{P$_i$}
$$

where

$$
g_{i,j}(\lambda) \equiv u_{i,j}\lambda - \frac{c}{2}\lambda^{\frac{1}{1-2\alpha}}, \tag{6}
$$

and $\bar{U}_1 = \max_{\lambda\geq 0} g_{1,1}(\lambda)$.

Lemma 4 *Suppose $(\bar{V}, M, N^p)$ is a competitive search equilibrium with PAM and only local incentive constraints. Denote the contact rate for the market (i,i) as λ_i. $\{\lambda_i\}_{i=1}^{I}$ must solve the following sequence of problem:*
$(i \geq 2)$

$$\bar{U}_i = \max_{\lambda \geq 0, \tau} g_{i,i}(\lambda) \tag{P$_i$}$$
$$s.t.\ \bar{U}_{i-1} \geq g_{i-1,i}(\lambda),$$

where

$$g_{i,j}(\lambda) \equiv u_{i,j}\lambda - \frac{c}{2}\lambda^{\frac{1}{1-2\alpha}}, \tag{7}$$

and $\bar{U}_1 = \max_{\lambda \geq 0} g_{1,1}(\lambda)$.

Proof. In a competitive search equilibrium $V_{i,i} = c$. Imposing positive assortative matching, we look for markets that only involve the same type. With on only local incentive constraint, the problem (PS-1) can be written as:

$$c = \max_{\tau^a, \tau^b, \lambda^a, \lambda^b} \left(\lambda^a \lambda^b\right)^{-\frac{1-\gamma}{2\gamma}} (\tau^a + \tau^b)$$
$$\text{s.t. } \bar{U}_i = \lambda^a (u_{i,i} - \tau^a),$$
$$\bar{U}_i = \lambda^b (u_{i,i} - \tau^b),$$
$$\bar{U}_{i-1} \geq \lambda^a (u_{i-1,j} - \tau^a),$$
$$\bar{U}_{i-1} \geq \lambda^b (u_{i-1,i} - \tau^b).$$

This is a strictly convex problem, the solution $(\lambda, \tau, \bar{U}_i)$ must also solve the following dual problem:

$$\bar{U}_i = \max_{\lambda \geq 0, \tau} \lambda (u_{i,i} - \tau)$$
$$\text{s.t. } \bar{U}_{i-1} \geq \lambda (u_{i-1,i} - \tau)$$
$$c = \lambda^{-\frac{2\alpha}{1-2\alpha}} 2\tau.$$

We can eliminate transfers from this problem to write it as

$$\bar{U}_i = \max_{\lambda \geq 0, \tau} g_{i,i}(\lambda) \tag{P$_i$}$$
$$\text{s.t. } \bar{U}_j \geq g_{j,i}(\lambda) \text{ for all } j,$$

where

$$g_{i,j}(\lambda) \equiv u_{i,j}\lambda - \frac{c}{2}\lambda^{\frac{1}{1-2\alpha}}, \tag{8}$$

defined on the positive reals. By construction, $\bar{U}_1$ does not have any local downward constraints. Thus it must solve:

$$\bar{U}_i = \max_{\lambda \geq 0, \tau} g_{i,i}(\lambda). \tag{9}$$

Recursively, if we have solved problem i, $\bar{U}_i$ is known. Using the definition of $g_{i,j}(\lambda)$, we can rewrite the constrained optimization problem for $i+1$ in the form proposed by the lemma. ■

A immediate question to ask when a solution that satisfies local incentive constraints also satisfies all global constraints.

Lemma 5 *A sequence $\{\lambda_i\}_{i \in \mathbb{I}}$ satisfies global incentive constraints if:*

1. *$\{\lambda_i\}_{i \in \mathbb{I}}$ satisfies local incentive constraints;*
2. *$u_{i,j}$ is supermodular;*
3. *$(\lambda_i - \lambda_{i'})(u_{i',i'} - u_{i'-1,i'}) > 0$ for $i > i'$.*

Proof. If $\{\lambda_i\}_{i=1,\ldots,I}$ satisfies local incentive constraints. Take a fixed i:

$$\bar{U}_{i-1} - \bar{U}_i \geq \lambda_i (u_{i-1,i} - u_{i,i})$$

We prove this statement by induction. Suppose we have already established that for $i' < i$ the IC holds, where $i' \geq 2$:

$$\bar{U}_{i'} - \bar{U}_i \geq \lambda_i (u_{i',i} - u_{i,i}).$$

Because the local IC for i' holds:

$$\bar{U}_{i'-1} - \bar{U}_{i'} \geq \lambda_{i'} (u_{i'-1,i'} - u_{i',i'}).$$

Adding these two inequalities above:

$$\begin{aligned}
\bar{U}_{i'-1} - \bar{U}_i \geq & \lambda_{i'}(u_{i'-1,i'} - u_{i',i'}) + \lambda_i(u_{i',i} - u_{i,i}) \\
> & \lambda_i(u_{i'-1,i'} - u_{i',i'}) + \lambda_i(u_{i',i} - u_{i,i}) \\
= & \lambda_i(u_{i'-1,i'} - u_{i',i'} + u_{i',i} - u_{i,i}) \\
= & \lambda_i(u_{i'-1,i'} - u_{i',i'} + u_{i',i} - u_{i'-1,i} + u_{i'-1,i} - u_{i,i}) \\
> & \lambda_i(u_{i-1',i} - u_{i,i})
\end{aligned}$$

The second inequality comes from condition 3 of the lemma. The third inequality comes from $u_{j',j'+1} - u_{j'+1,j'+1} > u_{j',j} - u_{j'+1,j}$ due to supermodularity in $u_{i,j}$. ■

Lemma 4 and Lemma 5 together provide a tractable way to analyze a PAM competitive search equilibrium. We first solve the problem in Lemma 4. Under the case of $u_{i,j}$ increasing in i, the conditions in Lemma 5 is to check if λ_i is decreasing in i; Under the case of $u_{i,j}$ decreasing in i, the conditions in Lemma 5 is to check if λ_i is increasing in i. Next we show, these two conditions are indeed satisfied.

Increasing Case. First we start with the case $u_{i,j}$ is increasing in i. In this case, we need to verify that λ_i is increasing in i for global incentive constraints.

Lemma 6 *If $u_{i,j}$ is increasing in i, λ_i is increasing in a competitive search equilibrium with PAM. The solution to Lemma 4 also satisfies all global incentive constraints.*

Proof. By construction, we have $\bar{U}_1 = g(\lambda^*_{1,1})$, and the (1,1) market has $\lambda_1 = \lambda^*_{1,1}$.

We now proceed by induction. Suppose we have solved problems (L_2)–(L_{j-1}) for some $j > 1$. Assume that for all $1 < i < j$, the maximizer of problem (L_i) is $\lambda_i > \lambda_{i-1}$, $\lambda_i \geq \lambda^*_{i,i}$, and that λ_i satisfies all the downward incentive constraints in problem (D_i). Note that this is trivially true when $j = 2$. We look at problem (L_j) and prove $\lambda_j > \lambda_{j-1}$, $\lambda_j \geq \lambda^*_{j,j}$, and λ_j satisfies all the downward incentive constraints in problem (D_j).

We break the analysis into two cases.

1. Suppose $\lambda_j = \lambda^*_{j,j}$, with $\bar{U}_{j-1} \geq g_{j-1,j}(\lambda^*_{j,j})$. In this case, the solution is unconstrained. We claim that $\lambda_j > \lambda_{j-1}$. By equation (**??**) and monotonicity of u, this is true if $\lambda_{j-1} = \lambda^*_{j-1,j-1}$. Alternatively, we may have $\lambda_{j-1} > \lambda^*_{j-1,j-1}$. If $\lambda_{j-1} \geq \lambda^*_{j,j} > \lambda^*_{j-1,j-1}$, the fact that $g_{j-1,j-1}(\lambda)$ is decreasing for $\lambda > \lambda^*_{j-1,j-1}$ implies $\bar{U}_{j-1} = g_{j-1,j-1}(\lambda_{j-1}) \leq g_{j-1,j-1}(\lambda^*_{j,j})$. And then g is increasing in each of its subindices because u is increasing, so $g_{j-1,j-1}(\lambda^*_{j,j}) < g_{j-1,j}(\lambda^*_{j,j})$. This implies $\bar{U}_{j-1} < g_{j-1,j}(\lambda^*_{j,j})$, and so $\lambda = \lambda^*_{j,j}$ does not satisfy the constraint in problem (L_j), a contradiction. This proves that if $\lambda_{j-1} > \lambda^*_{j-1,j-1}$ and $\lambda_j = \lambda^*_{j,j}$, then $\lambda_j > \lambda_{j-1}$.

2. Alternatively the constraint is binding, so $\bar{U}_{j-1} = g_{j-1,j}(\lambda_j)$. We know $\bar{U}_{j-1} = g_{j-1,j-1}(\lambda_{j-1}) < g_{j-1,j}(\lambda_{j-1})$, where the inequality uses monotonicity of u. This means that there are two solutions to the binding constraint, one bigger than λ_{j-1} and one smaller: $a < \lambda_{j-1} < b$ with $\bar{U}_{j-1} = g_{j-1,j}(a) = g_{j-1,j}(b)$. Use the definition of g in equation (8), together with the definitions of a and b:

$$\bar{U}_{j-1} = u_{j-1,j}a - \frac{c}{2}a^{\frac{1}{1-2\alpha}} = u_{j-1,j}b - \frac{c}{2}b^{\frac{1}{1-2\alpha}}$$

Since $a < b$ and $u_{j-1,j} < u_{j,j}$, $(u_{j,j} - u_{j-1,j})a < (u_{j,j} - u_{j-1,j})b$. Add this to the previous equation to get

$$g_{j,j}(a) = u_{j,j}a - \frac{c}{2}a^{\frac{1}{1-2\alpha}} < u_{j,j}b - \frac{c}{2}b^{\frac{1}{1-2\alpha}} = g_{j,j}(b),$$

This proves $g_{j,j}(a) < g_{j,j}(b)$, hence $\lambda_j = b > \lambda_{j-1}$, the larger solution. Finally, in this case $\lambda_j > \lambda^*_{j,j}$, because otherwise the unconstrained solution would have satisfied the incentive constraint.

This proves that in the solution to problem (L_j) satisfies $\lambda_j > \lambda_{j-1}$ and $\lambda_j \geq \lambda^*_{j,j}$. ■

Decreasing Case. We now move to the case $u_{i,j}$ is decreasing in i. In this case, we need to verify that λ_i is decreasing in i for global incentive constraints.

Lemma 7 *If $u_{i,j}$ is decreasing in i, λ_i is decreasing in a competitive search equilibrium with PAM. The solution to Lemma 4 also satisfies all global incentive constraints.*

Proof.

Case a: $u_{j,j} < u_{j-1,j-1}$. First suppose $\lambda_j = \lambda^*_{j,j}$ and $\bar{U}_j \geq g_{j-1,j}(\lambda^*_{j,j})$. If $\lambda^*_{j-1,j-1} = \lambda_{j-1}$, then the unconstrained problem directly implies $\lambda_j < \lambda_{j-1}$. Otherwise suppose $\lambda_{j-1} < \lambda^*_{j-1,j-1}$. We want to show:

$$\lambda^*_{j-1,j-1} > \lambda_{j-1} > \lambda^*_{j,j}.$$

Suppose, to the contrary, $\lambda^*_{j-1,j-1} > \lambda^*_{j,j} > \lambda_{j-1}$. The following inequality holds:

$$\bar{U}_{j-1} = g_{j-1,j-1}(\lambda_{j-1}) \leq g_{j-1,j-1}(\lambda^*_{j,j}) < g_{j-1,j}(\lambda^*_{j,j}),$$

where the first inequality comes from $g_{j-1,j-1}(\lambda)$ is increasing for $\lambda < \lambda^*_{j-1,j-1}$; The second inequality comes from $u_{j-1,j-1} < u_{j-1,j}$. Thus the IC in market j must be violated. A contradiction. Thus it must be $\lambda^*_{j,j} < \lambda_{j-1}$.

Secondly suppose the market j is constrained. Thus $\bar{U}_{j-1} = g_{j-1,j}(\lambda^*_j)$. There are two solutions to this equation. Denote them as $a < \lambda_{j-1} < b$. Because $a < b$ and $u_{j,j} < u_{j-1,j}$:

$$a(u_{j,j} - u_{j-1,j}) > b(u_{j,j} - u_{j-1,j})$$

Adding this inequality to the equation $g_{j-1,j}(a) = g_{j-1,j}(b)$:

$$g_{j,j}(a) = u_{j,j}a - \frac{c}{2}a^{\frac{1}{1-2\alpha}} > u_{j,j}b - \frac{c}{2}b^{\frac{1}{1-2\alpha}} = g_{j,j}(b)$$

Thus $\lambda_j = a < \lambda_{j-1}$. In addition, it must be $\lambda_j < \lambda^*_{j,j}$ because otherwise:

$$\begin{aligned}\bar{U}_{j-1} &= u_{j-1,j}\lambda_j - \frac{c}{2}\lambda_j^{\frac{1}{1-2\alpha}} \\ &\geq u_{j-1,j}\lambda_{j-1} - \frac{c}{2}\lambda_{j-1}\end{aligned}$$

and $\lambda^*_{j,j}$ would be feasible

Case b: $u_{j,j} \geq u_{j-1,j-1}$**.** We start by showing that the IC in AMO j must binds. Suppose, to the contrary $\lambda_j = \lambda^*_{j,j}$. The following inequality holds:

$$\bar{U}_{j-1} \leq g_{j-1.j-1}(\lambda^*_{j-1,j-1}) < g_{j,j}(\lambda^*_{j-1,j-1}) \leq g_{j,j}(\lambda^*_{j,j}) < g_{j-1,j}(\lambda^*_{j,j}),$$

where the first inequality uses the fact $g_{j-1,j-1}(\lambda)$ is maximized at $\lambda = \lambda^*_{j-1,j-1}$; The second inequality uses the fact $u_{j-1,j-1} < u_{j,j}$; The third inequality uses the fact $g_{j,j}(\lambda)$ is maximized at $\lambda = \lambda^*_{j,j}$; The last inequality uses the fact $u_{j-1,j} > u_{j,j}$. This violates the IC in market j. Thus it must be $\lambda_j < \lambda^*_{j,j}$ and $\bar{U}_{j-1} = g_{j-1,j}(\lambda_j)$.

There are two solutions to $\bar{U}_{j-1} = g_{j-1,j}(\lambda)$. Denote them as $a < \lambda_{j-1} < b$. Because $a < b$ and $u_{j,j} < u_{j-1,j}$:

$$a(u_{j,j} - u_{j-1,j}) > b(u_{j,j} - u_{j-1,j}).$$

Adding the inequality to the previous equation:

$$g_{j,j}(a) = u_{j,j}a - \frac{c}{2}a^{\frac{1}{1-2\alpha}} > u_{j,j}b - \frac{c}{2}b^{\frac{1}{1-2\alpha}} = g_{j,j}(b).$$

Thus $\lambda_j = a < \lambda_{j-1}$. Similar logic to the earlier case implies $\lambda_j < \lambda^*_{j,j}$. ■

3.5 Conditions for PAM as $\gamma \to 0$

If there ia PAM, we thus have $\bar{U}_1 = g_{1,1}(\lambda^*_{1,1})$. We then proceed as follows. Assume we know $\bar{U}_1, \ldots, \bar{U}_{i-1}$. We check if $\bar{U}_{i-1} \geq g_{i-1,i}(\lambda^*_{i,i})$. If it is, $\lambda_i = \lambda^*_{i,i}$, while otherwise λ_i is implicitly defined as the larger solution to $\bar{U}_{i-1} \geq g_{i-1,i}(\lambda_i)$. In either case, we then obtain $\bar{U}_i = g_{i,i}(\lambda_i)$. It still remains to check whether there is PAM. To do this, we need to solve the following problem when i matches with j:

$$\max_{\lambda_a,\lambda_b}(\lambda_a\lambda_b)^{-\frac{1-\gamma}{2\gamma}}\left(u_{i,j}+u_{j,i}-\frac{\bar{U}_i}{\lambda_a}-\frac{\bar{U}_j}{\lambda_b}\right)$$
$$\text{s.t. } \bar{U}_i-\bar{U}_{i'}\le\lambda_a(u_{i,j}-u_{i',j}) \text{ for all } i'<i$$
$$\bar{U}_j-\bar{U}_{j'}\le\lambda_b(u_{j,i}-u_{j',i}) \text{ for all } j'<j$$

There is a PAM CSE if and only if the solution to this is smaller than c for all (i,j). An open question is whether there is an efficient way to check this condition. For example, do only local constraints bind? Can we focus on a particular subset of (i,j) pairs? Giving answer to this question is complicated without restricting the parameters further. In a limiting case of $\alpha\to\frac{1}{2}$, we are able to derive a sufficient condition that guarantees PAM. This limit corresponds to the case studied in the search and matching literature. That is: the role of platforms in the matching process vanishes, and the matching function only takes the number of agents on two sides of the market as inputs.

Proposition 1 *Assume $u_{i,j}$ is supermodular and $u_{i,j}-u_{i-1,j}$ has log-supermodularity. As $\alpha\to\frac{1}{2}$, there exists a competitive search equilibrium that has positive assortative matching. In any such competitive search equilibrium, all local incentive constraints bind, and for the market with type (i,i), the contact rate $\lambda_i=1$, and the fee on both sides are $\tau_{i+1}=\tau_i+u_{i,i+1}-u_{i,i}$, with $\lambda_1=0$.*

Proof. Note that $\lambda_1=(2(1-2\alpha)u_{1,1}/c)^{\frac{1-2\alpha}{2\alpha}}$ and λ_I satisfies $g_{I,I}(\lambda_I)\ge 0$, which implies $\lambda_I\le(2u_{I,I}/c)^{\frac{1-2\alpha}{2\alpha}}$. In the limit as $\alpha\to 1/2$, this implies $\lambda_1\to 1$ and $\lambda_I\to 1$. Since λ is increasing at any $\alpha\in(0,1/2)$, it must be constant and equal to 1 in this limit.

Reexpressing things in terms of transfers, we have that when $\lambda_i=1$ for all i, the value of an agent is $\bar{U}_i=(u_{i,i}-\tau_i)$, while the local downward incentive constraint gives us $\bar{U}_i=u_{i,i+1}-\tau_{i+1}$. Combining these, we get $\tau_{i+1}-\tau_i=u_{i,i+1}-u_{i,i}$. Moreover, the transfers in the (1,1) match are $(1-2\alpha)u_{1,1}\to 0$. Thus we have an ordinary difference equation for transfers, increasing in the type of the match, when $\alpha\to 1/2$.

Can we find conditions for PAM in this case? Note that we have $\bar{U}_i-\bar{U}_{i-1}=u_{i,i}-u_{i-1,i}$, so the local downward incentive constraints reduce to

$$\lambda_a\ge\frac{\bar{U}_i-\bar{U}_{i-1}}{u_{i,j}-u_{i-1,j}}=\frac{u_{i,i}-u_{i-1,i}}{u_{i,j}-u_{i-1,j}},$$
$$\lambda_b\ge\frac{\bar{U}_j-\bar{U}_{j-1}}{u_{j,i}-u_{j-1,i}}=\frac{u_{j,j}-u_{j-1,j}}{u_{j,i}-u_{j-1,i}}$$

This implies $\lambda_a\lambda_b>1$ for all $i\neq j$ if $u_{i,j}-u_{i-1,j}$ is log-supermodular, i.e. log-supermodularity in first differences, as in Pratt (1964), Shimer-Smith (2000), and

Bonneton-Sandmann. ■

Proposition 2 *Assume $u_{i,j}$ is supermodular and $u_{i,j}+u_{j,i} \leq u_{i,i}\frac{u_{i-1,j}-u_{i,j}}{u_{i-1,i}-u_{i,i}}+u_{j,j}\frac{u_{j-1,i}-u_{j,i}}{u_{j-1,j}-u_{j,j}}$. As $\alpha \to \frac{1}{2}$, there exists a competitive search equilibrium that has positive assortative matching. In any such competitive search equilibrium, all local incentive constraints bind, and for the market with type (i,i), the contact rate $\lambda_{i+1} = \lambda_i \frac{u_{i,i}}{u_{i,i+1}}$, with $\lambda_1 = 1$, and the fees on both sides are $\tau_i = 0$.*

Proof. We have shown that λ_i must be decreasing in this case. The free-entry condition for platforms requires $c = 2\lambda_i^{-\frac{1-\gamma}{2\gamma}}\tau_i$. Thus, τ_i is also decreasing. As $\alpha \to \frac{1}{2}$, $\tau_1 = (1-2\alpha)u_{1,1} \to 0$. Meanwhile $\tau_i \geq 0$. In this limit, it must be $\tau_i = 0$. For the platforms to break even, it has to be $\lambda_i^{-\frac{1-\gamma}{2\gamma}} \to \infty$, and $\lambda_i \in (0,1)$.

Next we show the incentive constraints must bind for any (i,i) with $i > 1$. Suppose for $i > 1$ the incentive constraint is slack, $\lambda_i = \lambda^*_{i,i}$. The incentive constraint requires:

$$\frac{\lambda_{i-1}(u_{i-1,i-1} - \tau_{i-1})}{\lambda_i(u_{i-1,i} - \tau_i)} \geq 1.$$

We have shown that $\lambda_{i-1} \leq \lambda^*_{i-1,i-1}$. The inequality above thus implies

$$\frac{\lambda^*_{i-1,i-1}(u_{i-1,i-1} - \tau_{i-1})}{\lambda^*_{i,i}(u_{i-1,i} - \tau_i)} \geq 1$$

Plugging in the expression for $\lambda^*_{i,i}$ we have:

$$\frac{u_{i-1,i-1}^{\frac{1-2\alpha}{2\alpha}}(u_{i-1,i-1} - \tau_{i-1})}{u_{i,i}^{\frac{1-2\alpha}{2\alpha}}(u_{i-1,i} - \tau_i)} \geq 1$$

Taking $\alpha \to \frac{1}{2}$ and $\tau_i \to 0$:

$$\frac{u_{i-1,i-1}}{u_{i-1,i}} \geq 1,$$

which is contradicting to the assumption $u_{i,j}$ is strictly increasing in j. Thus the local incentive constraints must bind for any $i > 1$ when $\alpha \to \frac{1}{2}$. From the incentive constraints: $\lambda_{i-1}u_{i-1,i-1} = \lambda_i u_{i-1,i}$. Take the limit of $\lambda^*_{1,1}$ we get $\lambda_1 = 1$.

Lastly, we check under the conditions proposed there is not profitable deviation to wards to markets that involve different types. Suppose there is a market that involves (i,j). On side a, the incentive constraints require:

$$\bar{U}_{i-1} \geq \lambda_{i-1}u_{i-1,i-1} = \lambda^a(u_{i-1,j} - \tau^a)$$

$$\bar{U}_i = \lambda_{i-1}u_{i-1,i} = \lambda^a(u_{i,j} - \tau^a)$$

Together they require:

$$\tau^a \le \frac{u_{i-1,j}u_{i,i} - u_{i,j}u_{i-1,i}}{u_{i-1,i} - u_{i,i}} = u_{i,j} - u_{i,i}\frac{u_{i-1,j} - u_{i,j}}{u_{i-1,i} - u_{i,i}}.$$

Similarly, the incentive constraints on side b requires:

$$\tau^b \le u_{j,i} - u_{j,j}\frac{u_{j-1,i} - u_{j,i}}{u_{j-1,j} - u_{j,j}}.$$

Under the condition $u_{i,j}+u_{j,i} \le u_{i,i}\frac{u_{i-1,j}-u_{i,j}}{u_{i-1,i}-u_{i,i}}+u_{j,j}\frac{u_{j-1,i}-u_{j,i}}{u_{j-1,j}-u_{j,j}}$, $\tau^a+\tau^b \le 0$. Any market that involves (i,j) leads to non-positive profit. ■

4 Comparison with Observable Types

First consider the competitive search equilibrium with observable types. This environment is similar to Shi (2001) and Eeckhout and Kircher (2010). The recast of their result in our definition will be a benchmark for our characterization of competitive search equilibrium with private information.

We start by showing that the outcomes of a competitive search equilibrium can be characterized by a set of optimization problem. Consider the following problem given $\bar{\boldsymbol{U}} \equiv U_1, \dots, U_I$:

$$\bar{V} = \max_{m\in\mathbb{M}} (\lambda^a\lambda^b)^{-\frac{1-\gamma}{2\gamma}} \sum_{s=a,b}\sum_{i=1}^{I} \tau_i^s\omega_i^s, \tag{O}$$

$$\text{s.t. } \bar{U}_i \ge \lambda^s\left(\sum_{j=1}^{I}\omega_j^{-s}u_{i,j} - \tau_i^s\right) \text{ with equality if } \omega_i^s > 0,$$

$$\text{for } s = a, b \text{ and } i \in \{1,\dots,I\}$$

We say that $(M,\bar{V})$ solves optimization problem (O) if $\bar{V}$ is the maximum value of the optimization problem O and any $m \in M$ achieves this value given $\bar{\boldsymbol{U}}$.

Lemma 8 *Suppose $\{M,\bar{V},\bar{\boldsymbol{U}}\}$ is a partial equilibrium with observable types. Then $(M,\bar{V})$ must solve problem* (O) *given $\bar{\boldsymbol{U}}$. Conversely, if $(M,\bar{V})$ solves optimization problem* (O) *given $\bar{\boldsymbol{U}}$, then $\{M,\bar{V},\bar{\boldsymbol{U}}\}$ is a partial equilibrium with observable types.*

Proof. First, suppose $\{M,\bar{V},\bar{\boldsymbol{U}}\}$ is a partial equilibrium, yet $(M,\bar{V})$ is not a solution to (O). Thus there is $m' \in \mathbb{M}$ such that the constraint of (O) is satisfied and $V(m') > \bar{V}$. This contradicts point 2 of the definition for a partial equilibrium.

Conversely, suppose $(M,\bar{V})$ is the solution to (O) given $\bar{\boldsymbol{U}}$ yet $(M,\bar{V},\bar{\boldsymbol{U}})$ is not a partial equilibrium. If $\exists m \in M$ such that $V(m) \ne \bar{V}$. This violates that M is the solution to (O) because either m is not a maximizer or $\bar{V}$ is not the optimal value; If there is $m \in \mathbb{M}$ that violates condition 2 of the definition for a partial equilibrium, this m satisfies constraints of O, and $V(m) > \bar{V}$, which contradicts $(M, \bar{V})$ solving (O). ■

We first restrict attention to separating competitive search equilibrium, then show that under monotonicity of $f_{i,j}$ in j, any competitive search equilibrium must be separating. With separation, the problem O can be analyzed in two steps: (1) conditional on a fixed pair (i,j), find the $(\boldsymbol{\lambda}, \boldsymbol{\tau})$ that maximizes the platform's value. (2) choose (i,j) that delivers the highest value to platforms. We define this step-one problem as

$$V_{i,j} = \max_{\boldsymbol{\lambda}>0, \boldsymbol{\tau}>0} \left(\lambda^a \lambda^b\right)^{-\frac{1-\gamma}{2\gamma}} \left(\tau^a + \tau^b\right), \tag{OS-1}$$

s.t.

$$\bar{U}_i = \lambda^a \left(u_{i,j} - \tau^a\right),$$
$$\bar{U}_j = \lambda^b \left(u_{j,i} - \tau^b\right).$$

Lemma 9 *Given (i,j), a unique solution to (OS-1) exists:*

$$\lambda^a_{i,j} = \frac{\bar{U}_i}{\frac{1-\gamma}{2} f_{i,j}},\ \lambda^b_{i,j} = \frac{\bar{U}_j}{\frac{1-\gamma}{2} f_{i,j}},\ \tau^a_{i,j} = u_{i,j} - \frac{1-\gamma}{2} f_{i,j},\ \tau^b_{i,j} = u_{j,i} - \frac{1-\gamma}{2} f_{i,j},$$

$$V_{i,j} = \gamma \left(\frac{1-\gamma}{2}\right)^{\frac{1-\gamma}{\gamma}} \left(\bar{U}_i \bar{U}_j\right)^{-\frac{1-\gamma}{2\gamma}} f_{i,j}^{\frac{1}{\gamma}}.$$

Proof. By definition:

$$V_{i,j} = \max_{\lambda^a, \lambda^b} v(\lambda^a, \lambda^b) = \max_{\lambda^a, \lambda^b} (\lambda^a \lambda^b)^{-\frac{1-\gamma}{2\gamma}} \left(f_{i,j} - \frac{\bar{U}_i}{\lambda^a} - \frac{\bar{U}_j}{\lambda^b}\right).$$

The partial derivatives of $V_{1,1}$ are:

$$\frac{\partial \log v}{\partial \lambda^a}(\lambda^a, \lambda^b) = -\frac{1-\gamma}{2\gamma}\frac{1}{\lambda^a} + \frac{\bar{U}_i}{\lambda^{a2}} \frac{1}{f_{i,j} - \frac{\bar{U}_i}{\lambda^a} - \frac{\bar{U}_j}{\lambda^b}}.$$

For $\lambda^a, \lambda^b > 0$ and $f_{i,j} - \frac{\bar{U}_i}{\lambda^a} - \frac{\bar{U}_j}{\lambda^b} > 0$, this implies that

$$\frac{\partial \log v}{\partial \lambda^a}(\lambda^a, \lambda^b) \gtreqless 0 \Leftrightarrow (1 - \frac{1-\gamma}{2})\frac{\bar{U}_i}{\lambda^a} + \frac{1-\gamma}{2}\frac{\bar{U}_j}{\lambda^b} \gtreqless \alpha f_{i,j}$$
$$\frac{\partial \log v}{\partial \lambda^b}(\lambda^a, \lambda^b) \gtreqless 0 \Leftrightarrow (1 - \frac{1-\gamma}{2})\frac{\bar{U}_j}{\lambda^b} + \frac{1-\gamma}{2}\frac{\bar{U}_i}{\lambda^a} \gtreqless \alpha f_{i,j}$$

Thus when λ^a and λ^b are both close to 0, raising either of them increases v. When both are sufficiently large, reducing either of them increases v. If $\frac{\bar{U}_i}{\lambda^a} > \frac{\bar{U}_j}{\lambda^b}$, $0 < \gamma < 1$ implies $(1 - \frac{1-\gamma}{2})\frac{\bar{U}_i}{\lambda^a} + \frac{1-\gamma}{2}\frac{\bar{U}_j}{\lambda^b} > (1 - \frac{1-\gamma}{2})\frac{\bar{U}_j}{\lambda^b} + \frac{1-\gamma}{2}\frac{\bar{U}_i}{\lambda^a}$, and so we can have $\frac{\partial v}{\partial \lambda^a}(\lambda^a, \lambda^b) >$

$0 > \frac{\partial v}{\partial \lambda^b}(\lambda^a, \lambda^b)$, i.e. we increase v by increasing λ^a and decreasing λ^b. We can reach the opposite conclusion when $\frac{\bar{U}_i}{\lambda^a} < \frac{\bar{U}_j}{\lambda^b}$. This implies that v is single-peaked in $\boldsymbol{\lambda}$ such that $\lambda^a, \lambda^b > 0$ and $f_{i,j} - \frac{\bar{U}_i}{\lambda^a} - \frac{\bar{U}_j}{\lambda^b} > 0$. That peak comes at $\lambda^a = \frac{\bar{U}_i}{\frac{1-\gamma}{2} f_{i,j}}$ and $\lambda^b = \frac{\bar{U}_j}{\frac{1-\gamma}{2} f_{i,j}}$.

Plugging the optimal solution back to the incentive constraints of participants and objective function, we reach the results in lemma. ■

It turns on a competitive search equilibrium must be separating, if the joint value $f_{i,j} \neq f_{i,j'}$ for a fixed i and $j \neq j'$. This is the assumption made in models from literature such as Eeckhout and Kircher (2010). This result comes from that a non-separating market can be viewed as a collection of separating markets, with an additional restriction that all of these separating markets must share the identical contact rates. When $f_{i,j}$ varies in the matching partners' types (dimension j), this identical contact rate constraint is binding and strictly decreases payoffs for the platform.

Lemma 10 *If $f_{i,j} \neq f_{i,j'}$ for $j \neq j'$, any competitive search equilibrium is separating.*

Proof. We first replace (τ_i^a, τ_j^b) in the objective function by the constraints in problem (O): The original problem is now an unconstrained problem in terms of (τ^a, τ^b) and (ω_i^a, ω_j^b):

$$\max_{\boldsymbol{\lambda}>0, \boldsymbol{\omega}^s \in \Delta^I} (\lambda^a \lambda^b)^{-\frac{1-\gamma}{2\gamma}} \sum_{i=1}^{I} \sum_{j}^{I} \omega_i^a \omega_j^b \left(f_{i,j} - \frac{\bar{U}_i}{\lambda^a} - \frac{\bar{U}_j}{\lambda^b} \right).$$

Given any $\bar{\boldsymbol{U}}$, we have can write the value of objective function as:

$$\sum_{i=1}^{I} \sum_{j}^{I} \omega_i^a \omega_j^b (\lambda^a \lambda^b)^{-\frac{1-\gamma}{2\gamma}} \left(f_{i,j} - \frac{\bar{U}_i}{\lambda^a} - \frac{\bar{U}_j}{\lambda^b} \right) \leq \sum_{i=1}^{I} \sum_{j}^{I} \omega_i^a \omega_j^b V_{i,j}^*,$$

where the inequality comes from the fact $V_{i,j}^*$ is a solution to problem OS-1 and any positive (λ^a, λ^b) is feasible for OS-1. The inequality holds with equality only if $\lambda^s = \lambda_{i,j}^s$ for any (i, j) such that $\omega_i^s \omega_j^{-s} > 0$.

Suppose there is a CSE that is non-separating while $f_{i,j}$ is strictly increasing in j. WLOG, assume there is $m \in M$ such that $\omega_j^b > 0$, $\omega_{j'}^b > 0$, and $j \neq j'$. From lemma 9, for some i such that $\omega_i^a > 0$:

$$\lambda_{i,j}^a = \frac{\bar{U}_i}{\frac{1-\gamma}{2} f_{i,j}} \neq \frac{\bar{U}_i}{\frac{1-\gamma}{2} f_{i,j'}} = \lambda_{i,j'}^a.$$

This means λ^a cannot equal to both contact rates in separating markets. This means:

$$c = \sum_{i=1}^{I} \sum_{j}^{I} \omega_i^a \omega_j^b (\lambda^a \lambda^b)^{-\frac{1-\gamma}{2\gamma}} \left(f_{i,j} - \frac{\bar{U}_i}{\lambda^a} - \frac{\bar{U}_j}{\lambda^b} \right) < \sum_{i=1}^{I} \sum_{j}^{I} \omega_i^a \omega_j^b V_{i,j}^*.$$

Thus one of the separating markets such that $\omega_i^a \omega_j^b > 0$ must yield strictly higher payoff for the platform. This contradicts to the second condition of the definition for a partial equilibrium (,and thus competitive search equilibrium). ■

Lemma 10 implies that we should only look for separating competitive search equilibrium. From here on we use the enumeration defined in section 2.6, and index markets by the pair that shows up on the two sides. Given the results in Lemma 9, the step-two problem is to find the set of (i, j) that maximize platforms' value:

$$\bar{V} = \max_{i,j} V_{i,j}. \tag{OS-2}$$

Naturally, the partial-equilibrium set of markets M is the set of maximizers to problem (OS-2).

Proposition 3 *There exists a unique PAM (NAM) competitive search equilibrium if $f_{i,j}$ is log-supermodulary (log-submodular).*

Proof. From (OS-2), $M = \arg\max_{i,j} V_{i,j}$. This is a maximization problem on a finite set. Thus a solution (and correspondingly a partial equilibrium) must exist given $\bar{\boldsymbol{U}}$. We now show (1) M must have stated sorting pattern and (2) we can construct a unique competitive search equilibrium. From lemma 9:

$$V_{i,j} = \gamma \left(\frac{1-\gamma}{2}\right)^{\frac{1-\gamma}{\gamma}} \left(\bar{U}_i \bar{U}_j\right)^{-\frac{1-\gamma}{2\gamma}} f_{i,j}^{\frac{1}{\gamma}}.$$

If $f_{i,j}$ is log-supermodular. We prove PAM by contradiction. More precisely, we want to show that if $(i, j) \in M$, then $i = j$. Suppose otherwise. The formula of $V_{i,j}$ implies

$$\frac{V_{i,j} V_{j,j}}{V_{i,j}^2} = \left(\frac{f_{i,i} f_{j,j}}{f_{i,j}^2}\right)^{\frac{1}{\gamma}} > 1.$$

Thus $\max\{V_{i,i}, V_{j,j}\} > V_{i,j}$. It is profitable for the platforms to deviate from the focal active market to either a market with (i, i) or a market (j, j). A contradiction to the optimality. So any separating partial equilibrium has markets with identical types on both sides, which satisfies the definition of positive assortative matching.

We then show there is a unique separating competitive search equilibrium. In a competitive search equilibrium, $\bar{V} = V_{i,i} = c$:

$$c = V_{i,j} = \gamma \left(\frac{1-\gamma}{2}\right)^{\frac{1-\gamma}{\gamma}} \left(\bar{U}_i \bar{U}_j\right)^{-\frac{1-\gamma}{2\gamma}} f_{i,j}^{\frac{1}{\gamma}}.$$

Solving this equation gives a unique value of $\bar{U}_i$. With this value of $\bar{U}_i$, there is a unique number of participants per posting $n^s_{i,i}$ from Lemma 9. Lastly, from the market to clear for type i, we compute $N^P(m_{i,i}) = \frac{\bar{\omega}_i}{2n_{i,i}}$. This construction leads to unique separating competitive search equilibrium with positive assortative matching.

If $f_{i,j}$ is log-submodular. Suppose there are two active markets for (i_1, j_1) and (i_2, j_2) and $i_1 > i_2$, $j_1 > j_2$.

$$\frac{V_{i_1,j_2}V_{i_2,j_1}}{V_{i_1,j_1}V_{i_2,j_2}} = \left(\frac{f_{i_1,j_2}f_{i_2,j_1}}{f_{i_1,j_1}f_{i_2,j_2}}\right)^{\frac{1}{\gamma}} > 1.$$

This contradicts to optimality of (i_1, j_1) or (i_2, j_2).

The negative assortative matching in M and the clear marketing implies that all active markets have the pairs in the form of $(1, I)$, $(2, I-1)$, ..., $(I/2, I/2+1)$ if I is even, and in the form of $(1, I)$, $(2, I-1)$, ..., $((I-1)/2, (I-1)/2)$ if I is odd. Take each of these pair (i, j), we clear the market in the following order. First solve $\bar{U}_i$ and $\bar{U}_j$ as solution to:

$$c = \gamma\left(\frac{1-\gamma}{2}\right)^{\frac{1-\gamma}{\gamma}} \left(\bar{U}_i\bar{U}_j\right)^{-\frac{1-\gamma}{2\gamma}} f_{i,j}^{\frac{1}{\gamma}},$$

$$\frac{\bar{\omega}_i}{\bar{\omega}_j} = \frac{\bar{U}_i}{\bar{U}_j}.$$

This is a system of two equations with unique solution $(\bar{U}_i, \bar{U}_j)$. We can solve from lemma 9 the correspondingly $n^s_{i,j}$. The measure of postings is given by $N^P_{i,j} = N^P(i,j) = \frac{\bar{\omega}_i}{n^a_{i,j}}$. ■

5 Concluding Remarks

How does private information regarding payoff-relevant types affect equilibrium matching pattern? We provide a competitive search model to answer this question. Supermodularity in private value from matches guarantees separation of types into different active markets, and a new set of conditions is derived to guarantee positive assortative matching.

There are two open questions. First, although a tractable characterization of a PAM equilibrium is provided, how would one solve for a NAM equilibrium? Second, is the equilibrium allocation in the environment with private information constrained efficient? We hope to address these questions in the future iterations of the current project.

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