Supplementary Material: Multiobjective Test Problems with Degenerate Pareto Fronts†

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I. Sampling Reference Points on The True PFs

The PF of m-objective DPF1 with d essential objectives is:

$$
\begin{aligned}
f_1(\mathbf{x}) &= \gamma_1(\mathbf{x}) = \frac{1}{2}x_1x_2\ldots x_{d-1};\\
f_2(\mathbf{x}) &= \gamma_2(\mathbf{x}) = \frac{1}{2}x_1x_2\ldots(1-x_{d-1});\\
&\vdots\\
f_d(\mathbf{x}) &= \gamma_d(\mathbf{x}) = \frac{1}{2}(1-x_1);\\
f_{d+1}(\mathbf{x}) &= (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_d(\mathbf{x}))\mathbf{u}_1;\\
&\vdots\\
f_m(\mathbf{x}) &= (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_d(\mathbf{x}))\mathbf{u}_{m-d},
\end{aligned}
\tag{1}
$$

where $0 \leq x_i \leq 1$ is the decision variable and f_j is the objective value. We can see that the PF can be written as

$$
\begin{aligned}
&f_1 + f_2 + \cdots + f_d = 0.5;\\
&f_{d+1}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_d(\mathbf{x}))\mathbf{u}_1;\\
&\qquad\vdots\\
&f_m(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_d(\mathbf{x}))\mathbf{u}_{m-d},
\end{aligned}
\tag{2}
$$

with $0 \leq f_j \leq 0.5$. Since the PF of DPF1 is always lie on $f_1 + f_2 + \cdots + f_d = 0.5$ and $0 \leq f_j \leq 0.5$, the reference points for DPF1 can be obtained by sampling points on this unit simplex and extending them to the other objectives, i.e.,

1) sample a number of uniformly distributed points on the unit simplex $f_1 + f_2 + \cdots + f_d = 0.5$ with Deb and Jain's method [32].

2) calculate the objectives of the corresponding reference point via (2).

Similarly, the PF of DPF3 can be written as

$$
\begin{aligned}
&f_1(\mathbf{x}) = \gamma_1(\mathbf{x});\\
&\qquad\vdots\\
&f_{d-1}(\mathbf{x}) = \gamma_{d-1}(\mathbf{x});\\
&f_d(\mathbf{x}) = \min(\gamma_d(\mathbf{x}), \eta_1);\\
&f_{d+1}(\mathbf{x}) = \min(\max(\gamma_d(\mathbf{x}), \eta_1), \eta_2);\\
&\qquad\vdots\\
&f_{m-1}(\mathbf{x}) = \min(\max(\gamma_d(\mathbf{x}), \eta_{m-d-1}), \eta_{m-d});\\
&f_m(\mathbf{x}) = \max(\gamma_d(\mathbf{x}), \eta_{m-d}),\\
&\text{s.t.}\quad 0 \leq x_i \leq 1, \quad \text{for} \quad i = 1, 2, \ldots, n
\end{aligned}
\tag{3}
$$

with

$$(1-\gamma_1(\mathbf{x}))^2 + (1-\gamma_2(\mathbf{x}))^2 + \cdots + (1-\gamma_d(\mathbf{x}))^2 = 1, 0 < \gamma_j(\mathbf{x}) < 1. \tag{4}$$

Therefore, we sample the points from unit simplex and map these points to unit hypersphere, i.e., calculating the intersection of the line connecting each point and the origin on unit hypersphere. Then, we compute the objective values of the corresponding point via (3). In the same way, we can obtain the reference points for the PF of DPF4.

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The PF of DPF2 consists of a number of disconnected segments. From the definition of DPF2

$$
\begin{aligned}
& f_1(\mathbf{x}) = \gamma_1(\mathbf{x}) = x_1(1+g(\mathbf{x}^r)); \\
& \qquad \vdots \\
& f_{d-1}(\mathbf{x}) = \gamma_{d-1}(\mathbf{x}) = x_{d-1}(1+g(\mathbf{x}^r)); \\
& f_d(\mathbf{x}) = \gamma_d(\mathbf{x}) \\
& = (d - \sum_{i=1}^{d-1} \frac{f_i(\mathbf{x})}{1+g(\mathbf{x}^r)}(1+\sin(3\pi f_i(\mathbf{x}))))(1+g(\mathbf{x}^r)); \\
& f_{d+1}(\mathbf{x}) = \phi_1((f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_d(\mathbf{x}))\mathbf{u}_1); \\
& \qquad \vdots \\
& f_m(\mathbf{x}) = \phi_{m-d}((f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_d(\mathbf{x}))\mathbf{u}_{m-d}), \\
& \text{s.t.} \quad 0 \le x_i \le 1, \quad \text{for} \quad i = 1, 2, \dots, n,
\end{aligned}
\tag{5}
$$

we can see that the essential objective functions of DPF2 are same as the objective functions of DTLZ7. We follow the method in [32] to sample uniformly distributed points in the space spanned by the essential objectives. Then, we calculate the objective values of the reference points via (5).

From the definition of DPF5, we have that

$$
\beta_1(\mathbf{x})^2 + \beta_2(\mathbf{x})^2 + \cdots + \beta_{m-d+1}^2(\mathbf{x}) + f_{m-d+2}(\mathbf{x})^2 + \cdots + f_m(\mathbf{x})^2 = 1 \tag{6}
$$

and

$$
(m-d+1)\gamma_1(\mathbf{x})^2 + f_{m-d+2}(\mathbf{x})^2 + \cdots + f_m(\mathbf{x})^2 = 1 \tag{7}
$$

with

$$
\begin{aligned}
& f_1(\mathbf{x}) = \begin{cases} \beta_1(\mathbf{x}), & x_1 < \frac{1}{3}, \\ \gamma_1(\mathbf{x}), & \text{otherwise}; \end{cases} \\
& \qquad \vdots \\
& f_{m-d+1}(\mathbf{x}) = \begin{cases} \beta_{m-d+1}(\mathbf{x}), & x_1 < \frac{1}{3}, \\ \gamma_1(\mathbf{x}), & \text{otherwise}; \end{cases} \\
& f_{m-d+2}(\mathbf{x}) = \gamma_2(\mathbf{x}); \\
& \qquad \vdots \\
& f_m(\mathbf{x}) = \gamma_d(\mathbf{x}).
\end{aligned}
\tag{8}
$$

We sample the uniformly distributed points from unit simplex and map these points to unit hypersphere. Then, for the non-degenerated PF segment, we can obtain the objective values of the reference points directly. For the degenerated PF segment, we keep the objective values of $f_{m-d+2}(\mathbf{x}), \dots, f_m(\mathbf{x})$ unchanged, and calculate the objective values of $f_1(\mathbf{x}), \dots, f_{m-d+1}(\mathbf{x})$ as

$$
f_1(\mathbf{x}) = f_2(\mathbf{x}) = \cdots = f_{m-d+1}(\mathbf{x}) = \sqrt{\frac{1-(f_{m-d+2}(\mathbf{x})^2 + \cdots + f_m(\mathbf{x})^2}{m-d+1}}. \tag{9}
$$

II. Results of MOEAs on Proposed Problems with Different Metrics

The statistical results of the ten MOEAs with the inverted generational distance (IGD) [28, 29] and generational distance (GD) [30] metrics are reported in Table I and Table II, respectively. GD is a distance indicator to measure the distance from the solution sets and the reference points that sampled from the true PF of the problem. IGD measures the distance from the reference points to the individuals in the measured solution set.

From Table I, we have that: 1) For three-objective test instances, all of the tested algorithms perform well on most cases. The Pareto-dominance-based algorithms NSGA-II and SPEA2+SDE achieve slightly higher HV values than decomposition-based and indicator-based algorithms on DPF1-DPF4. The objective reduction-based algorithms δ-MOSS and NCIE perform as good as NSGA-II on DPF1 and DPF2, while they are inferior to NSGA-II on DPF3 and DPF4. This is due to that these two objective reduction-based algorithms select a subset of the original objective set as the criteria to optimise the problem, it performs well on the problems with explicitly redundant objectives, *e.g.*, DPF1 and DPF2, but might not work on the problems with implicitly redundant objectives, *e.g.*, DPF3 and DPF4. The decomposition-based methods, MOEA/D, RVEA and IDBEA perform not as well as Pareto dominance-based algorithms on these degenerate problems as only a small proportion of the weight vectors are close to the PF. 2) For six-objective test instances, SPEA2+SDE obtains the best HV values on DPF1 and DPF2. NSGA-II achieves the best results on DPF3 and DPF4, and IBEA outperforms the others on DPF5. In addition, the

TABLE I
THE STATISTICAL RESULTS (MEAN AND STANDARD DEVIATION) OF THE IGD VALUES ON THE PROPOSED TEST PROBLEMS. THE BEST RESULT REGARDING THE MEAN FOR EACH PROBLEM INSTANCE IS HIGHLIGHTED IN BOLDFACE.

# of Objs	Method	DPF1	DPF2	DPF3	DPF4	DPF5
m = 3, d = 2	NSGA-II	2.42E-03 (2.23E-04)	2.13E-02 (9.94E-04)	**5.18E-03** (2.70E-04)	**6.57E-03** (2.70E-04)	4.14E-02 (2.95E-03)
	SPEA2+SDE	**2.13E-03** (6.84E-05)	2.06E-02 (7.30E-04)	9.30E-03 (1.37E-03)	9.79E-02 (2.39E-02)	4.87E-02 (3.37E-03)
	NSGA-III	6.25E-03 (8.60E-04)	6.01E-02 (1.28E-02)	1.19E-02 (2.09E-03)	5.02E-02 (1.16E-02)	**4.05E-02** (1.80E-03)
	MOEA/D	4.48E-03 (9.85E-05)	1.86E+00 (7.05E-02)	3.30E-02 (1.43E-05)	1.83E-01 (2.09E-03)	4.60E-02 (1.12E-05)
	IDBEA	7.47E-03 (5.47E-03)	6.89E-02 (1.44E-02)	2.11E-02 (1.83E-03)	4.21E+01 (4.35E+01)	4.60E-02 (3.63E-05)
	RVEA	4.56E-02 (2.65E-02)	4.07E-01 (9.76E-02)	3.98E-02 (9.33E-05)	2.98E-01 (9.11E-02)	4.62E-02 (1.09E-05)
	IBEA	8.55E-02 (1.14E-02)	1.11E-01 (3.42E-01)	8.37E-03 (1.82E-03)	7.25E-01 (2.49E-02)	5.51E-02 (4.58E-03)
	δ-MOSS	2.40E-03 (2.11E-04)	2.21E-02 (5.55E-03)	5.27E-03 (1.90E-04)	3.49E-02 (7.48E-02)	4.06E-02 (2.54E-03)
	OSP	2.36E-02 (3.68E-02)	1.10E-01 (2.20E-01)	3.65E-02 (1.36E-02)	1.81E-01 (1.49E-01)	2.87E-01 (6.33E-02)
	NCIE	2.40E-03 (1.71E-04)	**2.04E-02** (7.21E-04)	8.21E-02 (1.22E-04)	2.11E-01 (9.08E-02)	9.60E-02 (1.72E-01)
m = 6, d = 3	NSGA-II	2.76E-02 (1.27E-03)	3.42E-01 (1.12E-01)	**6.05E-02** (2.04E-03)	**8.68E-02** (3.97E-03)	3.14E-01 (1.37E-02)
	SPEA2+SDE	**2.05E-02** (2.50E-04)	**2.73E-01** (1.67E-02)	6.35E-02 (3.49E-03)	1.95E-01 (1.62E-02)	2.51E-01 (1.07E-02)
	NSGA-III	3.54E-02 (3.03E-03)	3.55E-01 (3.76E-02)	8.02E-02 (5.25E-03)	1.08E-01 (1.09E-02)	2.62E-01 (1.12E-02)
	MOEA/D	4.49E-02 (1.89E-04)	6.66E+00 (1.68E+00)	1.74E-01 (1.14E-01)	3.29E-01 (4.37E-03)	2.66E-01 (1.57E-05)
	IDBEA	4.99E-02 (1.89E-02)	8.63E-01 (1.17E-01)	1.57E-01 (6.23E-03)	8.94E+01 (8.79E+01)	2.66E-01 (1.75E-04)
	RVEA	9.82E-02 (2.63E-02)	3.44E+00 (2.00E+00)	1.90E-01 (7.68E-02)	3.01E-01 (5.83E-02)	2.75E-01 (1.62E-04)
	IBEA	1.74E-01 (2.75E-02)	3.19E-01 (1.83E-02)	6.06E-02 (3.01E-03)	9.53E-01 (1.82E-02)	**2.43E-01** (4.24E-03)
	δ-MOSS	5.82E-02 (7.58E-02)	6.24E-01 (5.30E-01)	6.61E-02 (3.10E-03)	1.31E-01 (5.64E-02)	3.15E-01 (1.12E-02)
	OSP	3.45E-02 (2.49E-03)	1.15E+00 (8.09E-01)	1.76E-01 (6.25E-02)	2.29E-01 (6.82E-02)	7.33E-01 (4.05E-02)
	NCIE	2.74E-02 (1.61E-03)	3.26E+00 (2.35E+00)	1.88E-01 (1.37E-01)	4.13E-01 (2.10E-01)	7.76E-01 (2.14E-01)
m = 10, d = 5	NSGA-II	9.34E-02 (6.19E-03)	2.10E+00 (6.05E-02)	1.88E-01 (2.76E-03)	**2.54E-01** (5.40E-03)	2.38E+00 (3.77E-01)
	SPEA2+SDE	**5.42E-02** (2.13E-04)	2.05E+00 (7.55E-02)	1.81E-01 (3.47E-03)	3.74E-01 (1.19E-02)	**4.01E-01** (4.12E-03)
	NSGA-III	9.61E-02 (1.87E-02)	2.04E+00 (1.02E-01)	2.18E-01 (1.01E-02)	2.97E-01 (1.40E-02)	4.20E-01 (2.84E-03)
	MOEA/D	9.36E-02 (1.02E-03)	2.20E+01 (6.76E+00)	3.97E-01 (1.03E-01)	5.46E-01 (7.73E-03)	4.22E-01 (1.74E-04)
	IDBEA	8.11E-02 (3.55E-04)	9.68E+00 (2.95E+00)	4.71E-01 (3.76E-02)	3.16E+01 (2.63E+01)	4.60E-01 (1.41E-01)
	RVEA	1.68E-01 (3.00E-02)	2.01E+01 (5.86E+00)	3.83E-01 (2.22E-02)	4.75E-01 (2.78E-02)	4.22E-01 (5.80E-04)
	IBEA	2.09E-01 (2.67E-02)	2.03E+00 (3.89E-01)	**1.74E-01** (4.67E-03)	1.17E+00 (6.91E-03)	4.11E-01 (3.86E-03)
	δ-MOSS	9.08E-02 (7.58E-03)	2.12E+00 (7.69E-02)	1.94E-01 (4.53E-03)	3.50E-01 (1.44E-01)	2.48E+00 (8.67E-02)
	OSP	8.94E-02 (5.17E-03)	**1.87E+00** (1.55E-01)	2.17E-01 (1.90E-02)	2.96E-01 (2.02E-02)	6.64E-01 (3.82E-02)
	NCIE	8.56E-01 (9.41E-01)	8.74E+00 (3.67E+00)	3.61E-01 (1.48E-01)	8.05E-01 (1.66E-01)	1.01E+00 (1.33E-01)

objective reduction-based methods, OSP and NCIE, can obtain competitive performance compared with the best performance algorithms on DPF1, where the essential objectives are explicitly included in the objective set of the problem. However, they are much inferior to the best performance algorithms on DPF3 and DPF4, where the essential objectives are not explicitly included. The objective reduction methods have to extract the essential objectives instead of selecting them from the objective set of the problem in DPF3 and DPF4. The decomposition-based methods, MOEA/D, RVEA and IDBEA obtain a lower IGD values than the Pareto dominance-based algorithms on DPF2 and DPF4, where the problem objectives are nonlinearly dependent on the essential objectives. The decomposition-based algorithms have a smaller proportion of the weight vectors that are close to the PF in this six-dimensional objective space than in the three-dimensional objective space. 3) In terms of the ten-objective test instances, SPEA2+SDE achieves the best IGD values on DPF1 and DPF5, δ-MOSS on DPF2, IBEA on DPF3. NSGA-II achieve the best performance on DPF4, but it performs poorly on DPF5. In the objective reduction-based algorithms, OSP performs better than its integrated method on DPF1, DPF2 and DPF5, but it is inferior to its integrated method on DPF3 and DPF4. The improved version of decomposition-based algorithm IDBEA obtains a much better IGD values than MOEA/D on DPF1 and DPF2, but it inferiors MOEA/D on DPF3- DPF5.

From Table II, we can see that: 1) MOEA/D can obtain solutions that approached the true PF of the problem on all the tested instances, which illustrates the advantage of the decomposition-based algorithm. IBEA also performs good on these degenerate test instances in terms of convergence. NSGA-II is hard to converge to the PF on ten-objective instances, and the objective reduction-based algorithm OSP is potentially able to improve the convergence of NSGA-II. 2) The decomposition-based algorithms have a poor performance on six-objective and ten-objective degenerate instances. The Pareto dominance-based algorithms perform better than decomposition-based algorithms on DPF1-DPF4, which can also be seen from the visual results of the solution sets shown in the next section.

III. VISUALISATION RESULTS OF RUNS ASSOCIATED WITH MEDIAN HV VALUES

The results of the run with the median HV value (measured in the essential objective space) on DTLZ5(I, M) and on the proposed problems are shown in Fig. 1 and Fig. 2, respectively. The parallel coordinates plot of the solution sets obtained by five algorithms on 10-objective DPF5A in the run associated with the median HV value is shown in Fig. 3.

From the results in Fig. 1, we can see that δ-MOSS and NCIE obtain promising performance both in terms of the convergence and the diversity on DTLZ5(I, M). This is due to that the first $(M-I+1)$ objectives of DTLZ5(I, M) are linearly dependent

TABLE II
THE STATISTICAL RESULTS (MEAN AND STANDARD DEVIATION) OF THE GD VALUES ON THE PROPOSED TEST PROBLEMS. THE BEST RESULT REGARDING THE MEAN FOR EACH PROBLEM INSTANCE IS HIGHLIGHTED IN BOLDFACE.

Test instance	Method	DPF1	DPF2	DPF3	DPF4	DPF5
m = 3, d = 2	NSGA-II	6.36E-05 (4.51E-05)	1.18E-04 (5.95E-05)	8.11E-05 (1.46E-05)	4.39E-05 (4.05E-05)	1.01E-03 (1.89E-04)
	SPEA2+SDE	3.81E-02 (2.10E-01)	**3.65E-05** (1.02E-05)	2.40E-05 (7.93E-06)	**2.21E-05** (1.51E-05)	5.87E-04 (3.43E-05)
	NSGA-III	5.54E-02 (2.21E-01)	8.53E-04 (2.10E-03)	1.28E-04 (1.48E-04)	1.79E+01 (6.71E+01)	5.82E-04 (5.98E-05)
	MOEA/D	2.64E-03 (3.49E-04)	2.44E-03 (2.49E-04)	1.79E-02 (1.75E-06)	6.61E-03 (1.04E-04)	**4.98E-04** (9.07E-07)
	IDBEA	8.08E-03 (9.20E-03)	7.27E-02 (1.63E-01)	1.43E-02 (3.51E-03)	2.82E+03 (6.93E+03)	5.01E-04 (1.71E-05)
	RVEA	2.42E+00 (1.51E+00)	3.57E+00 (8.99E-01)	7.39E-02 (7.09E-04)	7.57E+04 (1.85E+04)	6.04E-04 (1.64E-06)
	IBEA	2.48E-03 (1.00E-02)	4.25E-05 (4.57E-05)	**1.02E-05** (2.59E-05)	1.77E-04 (6.63E-04)	1.15E-03 (2.14E-03)
	δ-MOSS	6.48E-05 (4.25E-05)	1.13E-04 (4.62E-05)	8.56E-05 (1.63E-05)	5.84E-05 (3.74E-05)	1.11E-03 (2.62E-04)
	OSP	8.26E-01 (8.31E-01)	2.21E-04 (1.45E-04)	1.45E-04 (5.35E-05)	9.38E+02 (6.91E+02)	2.03E-03 (2.68E-03)
	NCIE	**6.18E-05** (3.73E-05)	1.11E-04 (4.52E-05)	8.99E-05 (7.64E-05)	3.93E+02 (7.23E+02)	1.05E-03 (3.71E-04)
m = 6, d = 3	NSGA-II	3.41E-04 (1.83E-04)	1.56E-02 (6.88E-03)	1.21E-03 (1.43E-04)	6.97E-04 (1.50E-04)	3.51E-02 (4.53E-03)
	SPEA2+SDE	2.89E-03 (1.50E-02)	2.50E-03 (2.00E-04)	5.35E-04 (1.90E-05)	5.43E-04 (1.85E-05)	7.09E-03 (2.84E-04)
	NSGA-III	2.36E-01 (4.40E-01)	8.12E-03 (3.22E-03)	6.61E-04 (8.87E-05)	8.10E+01 (2.91E+02)	6.42E-03 (1.39E-04)
	MOEA/D	**1.89E-04** (5.83E-06)	**2.07E-03** (6.76E-04)	1.71E-03 (2.60E-05)	**5.25E-04** (1.92E-05)	6.18E-03 (1.40E-06)
	IDBEA	1.13E-02 (6.19E-02)	7.06E-01 (7.12E-01)	2.12E-03 (8.31E-04)	1.36E+03 (2.97E+03)	6.18E-03 (3.39E-05)
	RVEA	6.89E-01 (1.34E+00)	8.53E-02 (1.37E-01)	1.14E-01 (2.15E-02)	4.93E+04 (2.06E+04)	6.78E-03 (5.15E-06)
	IBEA	4.76E-03 (1.40E-02)	4.33E-03 (5.82E-04)	**5.05E-04** (2.61E-05)	1.16E-03 (3.62E-03)	6.90E-03 (3.40E-04)
	δ-MOSS	6.24E-02 (3.46E-01)	1.00E-02 (4.89E-03)	9.83E-04 (1.20E-04)	6.87E-04 (2.30E-04)	3.43E-02 (3.76E-03)
	OSP	2.44E-04 (9.68E-05)	6.18E-03 (5.03E-03)	1.09E-03 (4.68E-04)	2.13E+03 (3.72E+03)	**4.96E-03** (1.24E-03)
	NCIE	3.09E-04 (1.30E-04)	3.49E-03 (1.38E-03)	7.87E-04 (2.01E-04)	2.34E+02 (1.20E+03)	8.63E-03 (7.11E-03)
m = 10, d = 5	NSGA-II	8.03E-03 (1.05E-03)	2.41E-01 (2.61E-02)	7.06E-03 (4.61E-04)	1.08E-02 (1.03E-03)	1.77E-01 (1.23E-03)
	SPEA2+SDE	1.22E-03 (1.94E-05)	1.87E-02 (7.12E-04)	4.05E-03 (5.70E-05)	5.47E-03 (8.97E-05)	1.11E-02 (2.43E-04)
	NSGA-III	7.45E-01 (4.80E-01)	1.83E-01 (4.45E-02)	**3.51E-03** (1.42E-04)	5.51E+01 (1.37E+02)	7.87E-03 (1.05E-04)
	MOEA/D	1.24E-03 (1.32E-05)	**1.46E-02** (2.88E-03)	4.58E-03 (4.39E-04)	4.69E-03 (3.94E-05)	7.70E-03 (3.39E-06)
	IDBEA	**1.15E-03** (8.16E-06)	1.57E-02 (9.55E-04)	1.74E-02 (3.18E-03)	3.72E+03 (4.05E+03)	7.36E-03 (1.26E-03)
	RVEA	1.15E+00 (2.27E+00)	1.18E-01 (3.90E-02)	2.21E-02 (1.40E-02)	1.37E+04 (1.09E+04)	7.84E-03 (1.82E-05)
	IBEA	6.87E-03 (1.38E-02)	5.41E-02 (4.90E-02)	3.71E-03 (5.91E-05)	**1.50E-03** (5.52E-04)	9.35E-03 (2.94E-04)
	δ-MOSS	1.84E-02 (6.30E-02)	1.37E-01 (2.64E-02)	5.83E-03 (3.98E-04)	6.58E+00 (3.66E+01)	1.77E-01 (1.54E-03)
	OSP	3.92E-03 (1.51E-02)	4.32E-02 (8.49E-03)	3.61E-03 (5.67E-04)	2.57E+01 (8.73E+01)	**6.41E-03** (1.02E-03)
	NCIE	1.57E+00 (2.95E+00)	2.11E-02 (4.53E-03)	4.10E-03 (1.11E-03)	1.18E+02 (2.74E+02)	1.51E-02 (1.92E-02)

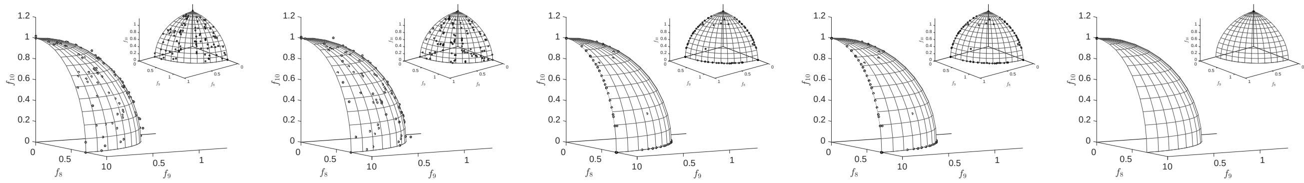
with each other, which makes it easy for the objective reduction algorithms to discover the essential objectives of the problem. However, other three algorithms, OSP, IBEA and IDBEA are still hard to obtain a diverse solution set.

From Fig. 2 and Fig. 3, we have the following observations:

1) The results obtained by the objective reduction-based algorithms, δ-MOSS, OSP and NCIE, on DPF1A and DPF2A are not as diverse as the results on DTLZ5(I, M) since the relation between the redundant objectives and the essential objectives is mutually linearly correlated in DTLZ5(I, M), linearly correlated (but not mutually linearly correlated) in DPF1A, and nonlinearly correlated in DPF2A.

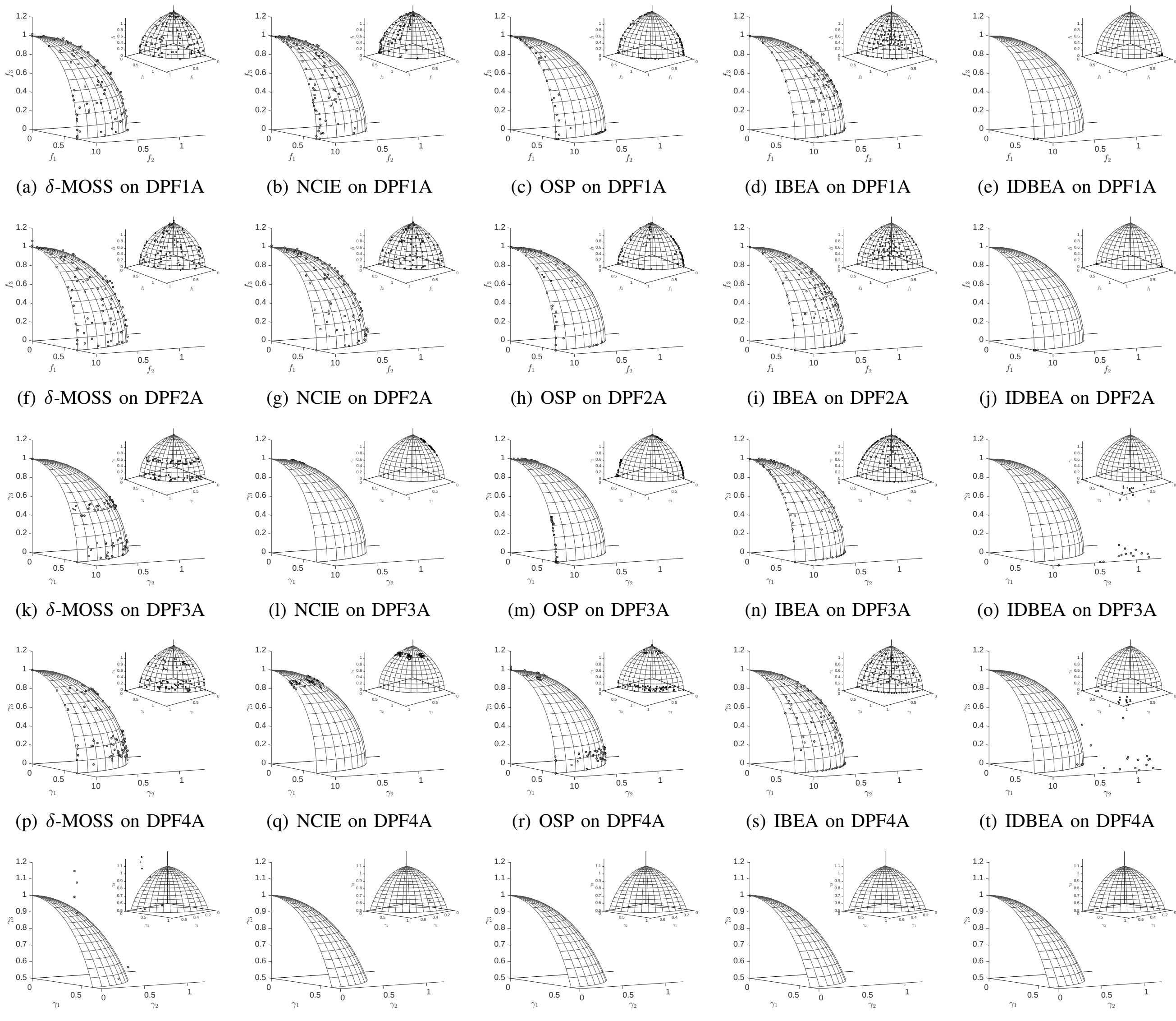
2) IDBEA fails to converge to the PF on DPF3A and DPF4A. The three objective reduction-based methods can converge to the PF, but they fail to maintain the diversity of the solution set. These two test problems are harder than DPF1A and DPF2A since their redundant objectives exist implicitly whereas the redundant objectives of DPF1A and DPF2A exist explicitly. IBEA can achieves promising results on DPF3A and DPF4A since the solution set with a better value in terms of the adopted indicator on the problem objectives would have the better value on the essential objectives as well.

3) None of these algorithms can obtain good results on the degenerate part of the PF on DPF5A. We also show the parallel coordinate plot of the whole solution sets of these five algorithms in Fig. 3, from which we can see that a large number of the solutions obtained by δ-MOSS do not converge to the PF. The objective value range of δ-MOSS's solutions is approximately from 0 to 3.5, which is far from that of the PF (from 0 to 1). OSP, IBEA and IDBEA do not have any individuals lying on the degenerate part of the PF as shown in Fig. 2(w)-(y). Even though NCIE can obtain two solutions that lie on the degenerate part of the PF on DPF5A, it fails to obtain diverse solutions in the non-degenerate part of the PF as a large part of the PF is not overlaid with the solutions of NCIE.



(a) δ-MOSS (b) NCIE (c) OSP (d) IBEA (e) IDBEA

Fig. 1. The solution set of the five objective reduction-based algorithms on DTLZ5(3, 10) in the run associated with its median HV value, where the grid mesh denotes the PF of the problem.



(a) δ-MOSS on DPF1A (b) NCIE on DPF1A (c) OSP on DPF1A (d) IBEA on DPF1A (e) IDBEA on DPF1A

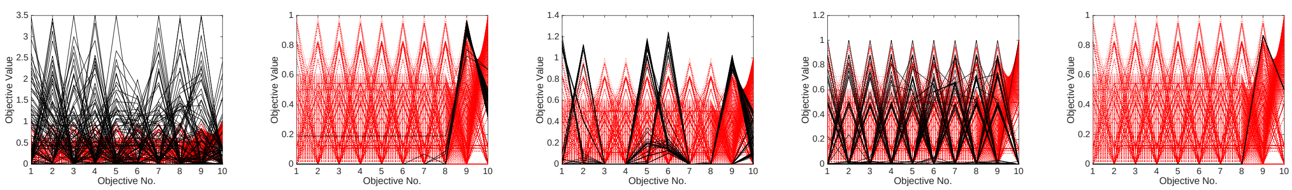
(f) δ-MOSS on DPF2A (g) NCIE on DPF2A (h) OSP on DPF2A (i) IBEA on DPF2A (j) IDBEA on DPF2A

(k) δ-MOSS on DPF3A (l) NCIE on DPF3A (m) OSP on DPF3A (n) IBEA on DPF3A (o) IDBEA on DPF3A

(p) δ-MOSS on DPF4A (q) NCIE on DPF4A (r) OSP on DPF4A (s) IBEA on DPF4A (t) IDBEA on DPF4A

(u) δ-MOSS on DPF5A (v) NCIE on DPF5A (w) OSP on DPF5A (x) IBEA on DPF5A (y) IDBEA on DPF5A

Fig. 2. The solution set of the five tested algorithms on DPF1A–PDF5A with $m = 10$, $d = 3$ in the run associated with its median HV value (measured in essential objective space), where the grid mesh denotes the PF of the problem in the essential objective space. From top to the bottom are the results on DPF1A to PDF4A, and the degenerate part of DPF5A. For the test instance of DPF5A, the scatter plot only shows the degenerate part of the PF and the solutions whose last objective values are not less than 0.5 (since the last objective value of the degenerate part of the PF is not less than 0.5.)



(a) δ-MOSS on DPF5A (b) NCIE on DPF5A (c) OSP on DPF5A (d) IBEA on DPF5A (e) IDBEA on DPF5A

Fig. 3. The solution set of the five algorithms on 10-objective PDF5A with $d = 3$ in the run associated with its median HV value, where the red dot lines represent the reference points sampled from the PF of the problem and the black lines denote the solutions.