Supplementary Material for "Online Operation of Renewable Energy and Battery Integrated Electric Vehicle Parking Lots with an Improved Priority Rule"

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***Abstract*—This document contains the supplementary material for our paper titled "Online Operation of Renewable Energy and Battery Integrated Electric Vehicle Parking Lots with an Improved Priority Rule", accepted for publication in IEEE Transactions on Transportation Electrification. Due to space constraints in the main manuscript, this supplement includes the detailed proof of the primary theoretical result presented in the paper.**

I. Preliminaries

For completeness, we restate the problem setup from the main paper, including all relevant definitions and notations required for the theoretical analysis in this supplement.

A. Problem Setup

We consider a parking lot containing N chargers operating over discrete timeslots $t \in \mathcal{T} = \{1, 2, \cdots, T\}$. In each timeslot t, a set of EVs, denoted by $\mathcal{N}_t$, arrive at the parking lot. Each EV $i \in \mathcal{N}_t$ has a known charging demand E_i and a departure time (charging deadline) d_i. Every EV is assigned to one charger, and all chargers share the same maximum charging rate p_c^{max}. The normalized total charging demand of EV i is defined as:

$$e_{i,a_i} = \frac{E_i}{p_c^{max}\Delta t}, \tag{1}$$

where a_i denotes EV i's arrival time, and Δt is the duration of one timeslot. For any subsequent timeslot t', we use $e_{i,t'}$ to represent the normalized remaining charging demand of EV i.

Let $\mathcal{I}_t$ represent all EVs present at the parking lot during timeslot t, including those just arriving. Each EV $i \in \mathcal{I}_t$ is charged at power $p_{i,t} \in [0, p_c^{max}]$, with normalized charging rate defined as:

$$v_{i,t} = \frac{p_{i,t}}{p_c^{max}}, \ \forall t. \tag{2}$$

EV states evolve according to:

$$e_{i,t+1} = e_{i,t} - v_{i,t}, \ \forall t, \ \forall i \in \mathcal{I}_t, \tag{3}$$

subject to the constraints:

$$0 \le v_{i,t} \le 1, \ \forall t, \ \forall i \in \mathcal{I}_t, \tag{4}$$

$$e_{i,d_i} = 0, \ \forall i. \tag{5}$$

The energy storage has state-of-charge u_t at timeslot t, and its charging power is denoted by p_t (positive for charging, negative for discharging). The storage dynamics are modeled by:

$$u_{t+1} = u_t + \mathbf{1}_{p_t \ge 0}\eta_c p_t \Delta_t + \mathbf{1}_{p_t < 0}\frac{p_t \Delta_t}{\eta_d}, \ \forall t, \tag{6}$$

where $\mathbf{1}_{p_t \ge 0}/\mathbf{1}_{p_t \le 0}$ are indicator functions, and η_c/η_d are the charging/discharging efficiency. The storage level and power are bounded by:

$$0 \le u_t \le u^{\max}, \ \forall t, \tag{7}$$

$$p_s^{\min} \le p_t \le p_s^{\max}, \ \forall t. \tag{8}$$

Denote the renewable energy generation at timeslot t by $g_t \in \mathbb{R}_+$. Then, the net grid power demand l_t is given by:

$$l_t = p_c^{max} \sum_{i \in \mathcal{I}_t} v_{i,t} + p_t - g_t, \ \forall t. \tag{9}$$

The electricity cost in each timeslot is:

$$c_t = \lambda_t [l_t]_+, \ \forall t, \tag{10}$$

where $[x]_+ = \max(x, 0)$, and λ_t is the electricity price.

The parking lot operation problem is formulated as a Markov decision process (MDP) [1] as follows:

- State: State of charger i is represented by $x_{i,t} = (e_{i,t}, d_i)$, and the joint charger state is denoted by $\boldsymbol{x}_t$. The endogenous state is denoted by $\boldsymbol{s}_t = (\boldsymbol{x}_t, u_t)$, while the exogenous state containing uncertain renewable generation and electricity price is denoted as $\boldsymbol{w_t} = (g_t, \lambda_t)$.
- Action: Action $\boldsymbol{a}_t$ includes normalized charging rates for all chargers (denoted by $\boldsymbol{v}_t$) and the storage charing decision p_t, i.e. $\boldsymbol{a}_t = (\boldsymbol{v}_t, p_t)$.
- Cost: The cost function is defined as:

$$C_t(\boldsymbol{s}_t, \boldsymbol{w}_t, \boldsymbol{a}_t) = \begin{cases} c_t & t = 1 : T-1 \\ c_T - \beta u_{T+1} & t = T \end{cases}$$

The endogenous state transitions function $\boldsymbol{s}_{t+1} = f_t(\boldsymbol{s}_t, \boldsymbol{a}_t)$ is described by equations (3) and (6), with feasible actions $\mathcal{A}_t$

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constrained by (4)(5)(7)(8). Exogenous states evolve according to an unknown stochastic process.

We denote by $\pi = (\pi_1, \cdots, \pi_T)$ the charging policy mapping states $(\boldsymbol{s}_t, \boldsymbol{w}_t)$ to actions $\boldsymbol{a}_t = \pi_t(\boldsymbol{s}_t, \boldsymbol{w}_t)$. The objective is to determine an optimal policy minimizing the expected cumulative cost:

$$\min_{\pi} \mathbb{E}_{\pi} \sum_{\tau=1}^{T} [C_\tau(\boldsymbol{s}_\tau, \boldsymbol{w}_\tau, \pi_\tau(\boldsymbol{s}_\tau, \boldsymbol{w}_\tau))]. \tag{11}$$

B. Definitions and Main Result

The following definitions specify the proposed priority rule-compliant policy.

Definition 1. *For two chargers i and j with state (e_i, d_i) and (e_j, d_j), we define the charging order $\prec$ as follows:*

1) *if $d_i = d_j$, then $(e_i, d_i) \prec (e_j, d_j) \iff e_i < e_j$,*
2) *if $d_i < d_j$, then $(e_i, d_i) \prec (e_j, d_j) \iff \lceil e_i \rceil \leq \lceil e_j \rceil - (d_j - d_i)$,*
3) *if $d_i > d_j$, then $(e_i, d_i) \prec (e_j, d_j) \iff \lceil e_i \rceil \leq \lceil e_j \rceil$,*

and $(e_i, d_i) \nprec (e_j, d_j)$ means $(e_i, d_i) \prec (e_j, d_j)$ does not hold.

Definition 2. *A charging policy π is said to be priority rule-compliant, if for any two chargers i and j, in any state $(\boldsymbol{s}_t, \boldsymbol{w}_t)$ where $t < \min(d_i, d_j)$, the following conditions hold. Let $v_{i,t}$ and $v_{j,t}$ denote the charging decisions for i and j under policy π. If there exists $\epsilon > 0$ such that*

$$(e_{i,t+1} + \epsilon, d_i) \prec (e_{j,t+1}, d_j), \tag{12}$$

where $e_{i,t+1} = e_{i,t} - v_{i,t}$ and $e_{j,t+1} = e_{j,t} - v_{j,t}$, then at least one of the following conditions must be satisfied:

1) *$v_{i,t} = 0$,*
2) *$v_{j,t} = \min(1, e_{j,t})$.*

Based on the above definitions, our main theoretical result is stated as follows:

Theorem 1. *There exists an optimal charging policy π^* which is priority rule-compliant.*

II. Proofs

We now conduct the proof for Theorem 1. The core idea of the proof is that we can modify an optimal policy to comply with the priority rule, while preserving optimality throughout the adjustment process. In problem (11), the optimal value is influenced by the total charging power V_t and dispatched power p_t of energy storage (ES), so we write the optimal policy as $\pi^* = (\pi^*_{EV}, \pi^*_{ES})$. For simplicity of notation, in the following proof, we focus on modifying π^*_{EV} to compliy with the priority rule and omit π^*_{ES}. However, if π^*_{ES} is considered, the proof process remains essentially the same. Also, we slightly change the MDP defined in Section I-A: We relax the constraint (5), and therefore the charging demand of EVs are not necessarily satisfied. However, for each EV departing with unsatisfied charging demand e, we will add a penalty term γe in the cost of the departing timeslot. We set the γ to be greater than the maximum possible electricity price to ensure that optimal policy will satisfy all charging demands.

We first introduce a few notations. The state value function and the state-action value function are defined as:

$$V_t^{\pi}(\boldsymbol{x}_t, \boldsymbol{w_t}) = \mathbb{E}_{\pi} \sum_{\tau=t}^{T} [C_\tau(\boldsymbol{x}_\tau, \boldsymbol{w}_\tau, \pi_\tau(\boldsymbol{x}_\tau, \boldsymbol{w}_\tau))], \tag{13}$$

$$V_t(\boldsymbol{x}_t, \boldsymbol{w_t}) = \mathbb{E}_{\pi^*} \sum_{\tau=t}^{T} [C_\tau(\boldsymbol{x}_\tau, \boldsymbol{w}_\tau, \pi^*_\tau(\boldsymbol{x}_\tau, \boldsymbol{w}_\tau))], \tag{14}$$

$$\begin{aligned} Q_t(\boldsymbol{x}_t, \boldsymbol{w_t}, \boldsymbol{v}_t) = C_t(\boldsymbol{x}_t, \boldsymbol{w_t}, \boldsymbol{v}_t) + \\ \mathbb{E}_{\pi^*} \sum_{\tau=t+1}^{T} [C_\tau(\boldsymbol{x}_\tau, \boldsymbol{w}_\tau, \pi^*_\tau(\boldsymbol{x}_\tau, \boldsymbol{w}_\tau))], \end{aligned} \tag{15}$$

where π^* is an optimal policy. Let $\mathcal{D}_t$ denote the set of chargers for which the EVs they charge will leave the parking lot at the beginning of timeslot t. Let $\mathcal{G}_{d,t}$ denote the set of chargers with charging deadline d at timeslot t, i.e.,

$$\mathcal{G}_{d,t} = \{i \in \mathcal{I}_t | d_i = d\}. \tag{16}$$

Lemma 1. *Consider two charger states $\boldsymbol{x}_t = \{(e_{i,t}, d_i)\}_{i=1}^N$ and $\hat{\boldsymbol{x}}_t = \{(\hat{e}_{i,t}, d_i)\}_{i=1}^N$, which are identical except for the remaining charging demand at charger i, where*

$$\hat{e}_{i,t} = e_{i,t} + \epsilon, \tag{17}$$

where $\epsilon > 0$. Then, the following holds:

$$0 \leq V_t(\hat{\boldsymbol{x}}_t, \boldsymbol{w_t}) - V_t(\boldsymbol{x}_t, \boldsymbol{w_t}) \leq \gamma\epsilon. \tag{18}$$

Proof. Since $0 \leq V_t(\hat{\boldsymbol{x}}_t, \boldsymbol{w_t}) - V_t(\boldsymbol{x}_t, \boldsymbol{w_t})$ is obvious, we only need to prove $V_t(\hat{\boldsymbol{x}}_t, \boldsymbol{w_t}) - V_t(\boldsymbol{x}_t, \boldsymbol{w_t}) \leq \gamma\epsilon$.

Let π^* denote the optimal policy, and starting from state $\hat{\boldsymbol{x}}_t$, we construct another policy $\hat{\pi}$ that mimics the charging behavior of π^* starting from state $\boldsymbol{x}_t$:

$$\hat{\pi}(\hat{\boldsymbol{x}}_{t+\tau}, \boldsymbol{w}_{t+\tau}) = \pi^*(\boldsymbol{x}_{t+\tau}, \boldsymbol{w}_{t+\tau}),\ \forall \tau = 1, ..., T-t. \tag{19}$$

We can see that the difference of $V_t^{\hat{\pi}}(\hat{\boldsymbol{x}}_t, \boldsymbol{w_t})$ and $V_t(\boldsymbol{x}_t, \boldsymbol{w_t})$ merely comes from the penalty term in timeslot d_i:

$$V_t^{\hat{\pi}}(\hat{\boldsymbol{x}}_t, \boldsymbol{w_t}) - V_t(\boldsymbol{x}_t, \boldsymbol{w_t}) = \gamma(e_{i,d_i} + \epsilon) - \gamma(e_{i,d_i}) = \gamma\epsilon. \tag{20}$$

Then we conclude that

$$V_t(\hat{\boldsymbol{x}}_t, \boldsymbol{w_t}) \leq V_t^{\hat{\pi}}(\hat{\boldsymbol{x}}_t, \boldsymbol{w_t}) = V_t(\boldsymbol{x}_t, \boldsymbol{w_t}) + \gamma\epsilon. \tag{21}$$

□

Lemma 2. *Consider any charger state $\boldsymbol{x}_t = \{(e_{i,t}, d_i)\}_{i=1}^N$ where two chargers i and j satisfy $(e_{i,t}, d_i) \prec (e_{j,t}, d_j)$. Let $m^* = \inf\{m | (e_{i,t} + m, d_i) \nprec (e_{j,t} - m, d_j)\}$. $\forall m \in [0, m^*]$, if we construct another charger state $\tilde{\boldsymbol{x}}_t = \{(\tilde{e}_{i,t}, d_i)\}_{i=1}^N$ that is identical with $\boldsymbol{x}_t$, except for chargers i and j where the following conditions hold:*

$$\tilde{e}_{i,t} = e_{i,t} + m, \tag{22}$$

$$\tilde{e}_{j,t} = e_{j,t} - m, \tag{23}$$

then we have

$$V_t(\tilde{\boldsymbol{x}}_t, \boldsymbol{w_t}) \leq V_t(\boldsymbol{x}_t, \boldsymbol{w_t}). \tag{24}$$

Proof. Since Lemma 1 implies that the value function $V_t(\boldsymbol{x}_t, \boldsymbol{w_t})$ is continuous with respect to the remaining demand of any charger, it is enough to prove (24) for $m \in [0, m^*)$. Note that in this case we have

$$(\tilde{e}_{i,t}, d_i) \prec (\tilde{e}_{j,t}, d_j). \tag{25}$$

Then we proceed with proof using backward induction with the following two steps:

Step 1. When $t = \min(d_i, d_j)$, we discuss the following cases:

Case 1. If $d_i < d_j$, and $t = d_i$, then by (25) and Definition 1 we have

$$\lceil \tilde{e}_{i,t} \rceil \leq \lceil \tilde{e}_{j,t} \rceil - (d_j - d_i). \tag{26}$$

If $m = 0$, then $\tilde{\boldsymbol{x}}_t = \boldsymbol{x}$, and (24) holds trivially. If $m > 0$, then $\tilde{e}_{i,t} > 0$, and by (26) we have

$$\tilde{e}_{j,t} > d_j - d_i, \tag{27}$$

which means charger j is destined to fail to fulfill the charging demand at timeslot d_j. Let π^* denote the optimal policy, and starting from state $\tilde{\boldsymbol{x}}_t$, we construct another policy $\tilde{\pi}$ that mimics the charging behavior of π^* starting from state $\boldsymbol{x}_t$:

$$\tilde{\pi}(\tilde{\boldsymbol{x}}_{t+\tau}, \boldsymbol{w}_{t+\tau}) = \pi^*(\boldsymbol{x}_{t+\tau}, \boldsymbol{w}_{t+\tau}),\ \forall \tau = 1, ..., T - t. \tag{28}$$

The feasibility of policy $\tilde{\pi}$ is guaranteed by (27), as there is no chance that $\tilde{v}_{j,t+\tau} = v_{j,t+\tau} > \tilde{e}_{j,t+\tau}$. We can see that the difference of $V_t^{\tilde{\pi}}(\hat{\boldsymbol{x}}_t, \boldsymbol{w_t})$ and $V_t(\boldsymbol{x}_t, \boldsymbol{w_t})$ merely comes from the penalty term in timeslot d_i and d_j:

$$\begin{aligned}
&V_t^{\tilde{\pi}}(\tilde{\boldsymbol{x}}_t, \boldsymbol{w_t}) - V_t(\boldsymbol{x}_t, \boldsymbol{w_t}) \\
=&[\gamma \tilde{e}_{i,d_i} + \gamma \tilde{e}_{j,d_j}] - [\gamma e_{i,d_i} + \gamma e_{j,d_j}] \\
=&[\gamma (e_{i,d_i} + \epsilon) + \gamma (e_{j,d_j} - \epsilon)] - [\gamma e_{i,d_i} + \gamma e_{j,d_j}] \\
=&0.
\end{aligned} \tag{29}$$

Then we conclude that

$$V_t(\tilde{\boldsymbol{x}}_t, \boldsymbol{w_t}) \leq V_t^{\tilde{\pi}}(\tilde{\boldsymbol{x}}_t, \boldsymbol{w_t}) = V_t(\boldsymbol{x}_t, \boldsymbol{w_t}). \tag{30}$$

Case 2. If $d_j < d_i$ and $t = d_j$, let $\boldsymbol{v}_t$ denote the optimal charging decisions at state $(\boldsymbol{x}_t, \boldsymbol{w}_t)$, then we have

$$\begin{aligned}
V_t(\boldsymbol{x}_t, \boldsymbol{w_t}) =& \gamma e_{j,t} + \gamma \sum_{k \in \mathcal{D}_t \setminus \{j\}} e_{k,t} + \lambda_t [\boldsymbol{1}^T \boldsymbol{v}_t p_c^{\max} - g_t]_+ \\
&+ \mathop{\mathbb{E}}_{\boldsymbol{w_{t+1}}} [V_{t+1}(\boldsymbol{x}_{t+1}, \boldsymbol{w}_{t+1})],
\end{aligned} \tag{31}$$

and

$$\begin{aligned}
V_t(\tilde{\boldsymbol{x}}_t, \boldsymbol{w_t}) \leq& \gamma \tilde{e}_{j,t} + \gamma \sum_{k \in \mathcal{D}_t \setminus \{j\}} e_{k,t} + \lambda_t [\boldsymbol{1}^T \boldsymbol{v}_t p_c^{\max} - g_t]_+ \\
&+ \mathop{\mathbb{E}}_{\boldsymbol{w_{t+1}}} [V_{t+1}(\tilde{\boldsymbol{x}}_{t+1}, \boldsymbol{w}_{t+1})],
\end{aligned} \tag{32}$$

where $\tilde{\boldsymbol{x}}_{t+1}$ is identical to $\boldsymbol{x}_{t+1}$ except for the charger i, where

$$\tilde{e}_{i,t+1} = e_{i,t+1} + m. \tag{33}$$

According to Lemma 1, we have

$$\mathop{\mathbb{E}}_{\boldsymbol{w_{t+1}}} [V_{t+1}(\tilde{\boldsymbol{x}}_{t+1}, \boldsymbol{w}_{t+1})] \leq \mathop{\mathbb{E}}_{\boldsymbol{w_{t+1}}} [V_{t+1}(\boldsymbol{x}_{t+1}, \boldsymbol{w}_{t+1})] + \gamma m. \tag{34}$$

By (23)(31)(32)(34) we have

$$V_t(\tilde{\boldsymbol{x}}_t, \boldsymbol{w_t}) \leq V_t(\boldsymbol{x}_t, \boldsymbol{w_t}). \tag{35}$$

Case 3. If $t = d_j = d_i$, the only difference between $\boldsymbol{x}_t$ and $\tilde{\boldsymbol{x}}_t$ is the difference remaining demands of the EVs that are going to depart from chargers i and j. As $\tilde{e}_{i,t} + \tilde{e}_{j,t} = e_{i,t} + e_{j,t}$, we have

$$\gamma \sum_{k \in \mathcal{D}_t} e_{k,t} = \gamma \sum_{k \in \mathcal{D}_t} \tilde{e}_{k,t}. \tag{36}$$

Then we conclude that

$$V_t(\tilde{\boldsymbol{x}}_t, \boldsymbol{w_t}) = V_t(\boldsymbol{x}_t, \boldsymbol{w_t}). \tag{37}$$

Step 2. When $t < \min(d_i, d_j)$, suppose (24) holds for timeslot $t + 1$. Let $\boldsymbol{v}_t$ denote the optimal charging decisions at state $(\boldsymbol{x}_t, \boldsymbol{w}_t)$, and $\boldsymbol{x}_{t+1}$ the next state after the charging action $\boldsymbol{v}_t$. Specifically, we have

$$e_{i,t+1} = e_{i,t} - v_{i,t}, \tag{38}$$
$$e_{j,t+1} = e_{j,t} - v_{j,t}. \tag{39}$$

For the convenience of the following discussion, we first list some properties that the mentioned variables satisfy:

$$e_{i,t} + e_{j,t} = \tilde{e}_{i,t} + \tilde{e}_{j,t}, \tag{40}$$
$$0 \leq v_{i,t} \leq \min(e_{i,t}, 1), \tag{41}$$
$$0 \leq v_{j,t} \leq \min(e_{j,t}, 1), \tag{42}$$
$$\tilde{e}_{i,t} \geq e_{i,t}, \tag{43}$$
$$\tilde{e}_{j,t} \leq e_{j,t}. \tag{44}$$

We will complete the proof by discussing the following two cases.

Case 1. If $\tilde{e}_{j,t} \geq e_{j,t+1}$, for state $\tilde{\boldsymbol{x}}_t$ (and $\boldsymbol{w}_t$), we will construct the charging decision $\tilde{\boldsymbol{v}}_t$ that is identical to $\boldsymbol{v}_t$ except for charger i and j, where:

$$\tilde{v}_{i,t} = \min(\tilde{e}_{i,t}, v_{i,t} + v_{j,t} - (\tilde{e}_{j,t} - e_{j,t+1}), 1), \tag{45}$$
$$\tilde{v}_{j,t} = v_{i,t} + v_{j,t} - \tilde{v}_{i,t}. \tag{46}$$

First, we check the feasibility of $\tilde{v}_{i,t}$ and $\tilde{v}_{j,t}$ defined in (45) and (46). Note that by (39) and (44) we have

$$\tilde{e}_{j,t} - e_{j,t+1} \leq v_{j,t}. \tag{47}$$

Combining (45) and (47) we have

$$0 \leq \tilde{v}_{i,t} \leq \min(\tilde{e}_{i,t}, 1). \tag{48}$$

The non-negative charging constraint for $\tilde{v}_{j,t}$ holds trivially by (45) and (46):

$$\tilde{v}_{j,t} \geq 0. \tag{49}$$

As we will demonstrate, the maximum charging constraint for $\tilde{v}_{j,t}$ also holds, and the proof will be given later in the discussion:

$$\tilde{v}_{j,t} \leq \min(\tilde{e}_{j,t}, 1). \tag{50}$$

We now proceed to verify the correctness of (24), where

$$\begin{aligned}
V_t(\boldsymbol{x}_t, \boldsymbol{w_t}) = \gamma \sum_{k \in \mathcal{D}_t} e_{k,t} + \lambda_t [\boldsymbol{1}^T \boldsymbol{v}_t p_c^{\max} - g_t]_+ \\
+ \mathop{\mathbb{E}}_{\boldsymbol{w_{t+1}}} [V_{t+1}(\boldsymbol{x}_{t+1}, \boldsymbol{w}_{t+1})],
\end{aligned} \tag{51}$$

and

$$V_t(\tilde{\boldsymbol{x}}_t, \boldsymbol{w_t}) \le \gamma \sum_{k\in\mathcal{D}_t} e_{k,t} + \lambda_t[\mathbf{1}^T\tilde{\boldsymbol{v}}_t p_c^{\max} - g_t]_+ + \mathop{\mathbb{E}}_{\boldsymbol{w_{t+1}}}[V_{t+1}(\tilde{\boldsymbol{x}}_{t+1}, \boldsymbol{w}_{t+1})], \tag{52}$$

where $\tilde{\boldsymbol{x}}_{t+1}$ is identical to $\boldsymbol{x}_{t+1}$ except for the chargers i and j which satisfies

$$\tilde{e}_{i,t+1} = \tilde{e}_{i,t} - \tilde{v}_{i,t}, \tag{53}$$

$$\tilde{e}_{j,t+1} = \tilde{e}_{j,t} - \tilde{v}_{j,t}. \tag{54}$$

By (40)(46)(53)(54) we have

$$\tilde{e}_{i,t+1} + \tilde{e}_{j,t+1} = e_{i,t+1} + e_{j,t+1}. \tag{55}$$

By (46) we have $\mathbf{1}^T\boldsymbol{v}_t = \mathbf{1}^T\tilde{\boldsymbol{v}}_t$, and therefore to prove (24) we only need to prove the following property:

$$\mathop{\mathbb{E}}_{\boldsymbol{w_{t+1}}}[V_{t+1}(\tilde{\boldsymbol{x}}_{t+1}, \boldsymbol{w}_{t+1})] \le \mathop{\mathbb{E}}_{\boldsymbol{w_{t+1}}}[V_{t+1}(\boldsymbol{x}_{t+1}, \boldsymbol{w}_{t+1})]. \tag{56}$$

According to the value of $\tilde{v}_{i,t}$, Case 1 can be further divided into 3 sub-cases. In each sub-case, we will check the correctness of (50) and (56).

Case 1.1. If $\tilde{v}_{i,t} = \tilde{e}_{i,t}$, then we have

$$\tilde{e}_{i,t} \le v_{i,t} + v_{j,t} - (\tilde{e}_{j,t} - e_{j,t+1}). \tag{57}$$

By (39)(40)(57) we have

$$v_{i,t} \ge e_{i,t}. \tag{58}$$

Combining with (41) we have

$$v_{i,t} = e_{i,t}. \tag{59}$$

Then we have

$$\begin{aligned} \tilde{v}_{j,t} &= v_{i,t} + v_{j,t} - \tilde{v}_{i,t} \\ &= e_{i,t} + v_{j,t} - \tilde{e}_{i,t} \\ &\le v_{j,t} \\ &\le 1, \end{aligned} \tag{60}$$

and

$$\begin{aligned} \tilde{e}_{j,t} - \tilde{v}_{j,t} &= (e_{i,t} + e_{j,t} - \tilde{e}_{i,t}) - (v_{i,t} + v_{j,t} - \tilde{v}_{i,t}) \\ &= (e_{i,t} - v_{i,t}) + (e_{j,t} - v_{j,t}) \\ &\ge 0. \end{aligned} \tag{61}$$

From (60) and (61) we can see (50) holds.

Since $\tilde{v}_{i,t} = \tilde{e}_{i,t}$ we have

$$\tilde{e}_{i,t+1} = 0 = e_{i,t+1}. \tag{62}$$

Then from (55) we have

$$\tilde{e}_{j,t+1} = e_{j,t+1}. \tag{63}$$

Therefore (56) holds as $\tilde{\boldsymbol{x}}_{t+1} = \boldsymbol{x}_{t+1}$.

Case 1.2. If $\tilde{v}_{i,t} = v_{i,t} + v_{j,t} - (\tilde{e}_{j,t} - e_{j,t+1})$, then combining (46) we have

$$\tilde{v}_{j,t} = \tilde{e}_{j,t} - e_{j,t+1}, \tag{64}$$

Combining (64) with (39)(42)(44) we can see (50) holds. Alao, by (64) and (54) we have

$$\tilde{e}_{j,t+1} = e_{j,t+1}. \tag{65}$$

Then from (55) we have

$$\tilde{e}_{i,t+1} = e_{i,t+1}. \tag{66}$$

Therefore (56) holds as $\tilde{\boldsymbol{x}}_{t+1} = \boldsymbol{x}_{t+1}$.

Case 1.3. If $\tilde{v}_{i,t} = 1$, then $\tilde{e}_{i,t} \ge 1$. We can assume $\tilde{e}_{i,t} > 1$ as the case for $\tilde{e}_{i,t} = 1$ can be covered by Case 1.1. Then from (25) and Definition 1 we can see $\tilde{e}_{j,t} > 1$. On the other hand, combining $\tilde{e}_{i,t} > 1$ with (41)(42)(46) we have

$$\tilde{v}_{j,t} \le 1 < \tilde{e}_{j,t}, \tag{67}$$

and therefore (50) holds.

Recall that $(\tilde{e}_{i,t}, d_i) \prec (\tilde{e}_{j,t}, d_j)$ from (25). According to the Definition 1, the fact $\tilde{v}_{i,t} = 1$, and the relationship (53)(54), we can see that

$$(\tilde{e}_{i,t+1}, d_i) \prec (\tilde{e}_{j,t+1}, d_j). \tag{68}$$

On the other hand, combining (45)(46)(54) we have

$$\tilde{e}_{j,t+1} \le e_{j,t+1}. \tag{69}$$

Then from (55) we can see that there exists $\hat{m} \in [0, \hat{m}^*]$ such that

$$\tilde{e}_{i,t+1} = e_{i,t+1} + \hat{m}, \tag{70}$$

$$\tilde{e}_{j,t+1} = e_{j,t+1} - \hat{m}, \tag{71}$$

where $\hat{m}^* = \inf\{m | (e_{i,t+1} + m, d_i) \not\prec (e_{j,t+1} - m, d_j)\}$. Then according to the property (24) for timeslot $t+1$ we can see (56) holds.

Case 2. If $\tilde{e}_{j,t} < e_{j,t+1}$, for state $(\tilde{\boldsymbol{x}}_t, \boldsymbol{w}_t)$, we will construct the charging decision $\tilde{\boldsymbol{v}}_t$ that is identical to $\boldsymbol{v}_t$ except for charger i and j which satisfies:

$$\tilde{v}_{i,t} = \min(v_{i,t} + v_{j,t}, 1), \tag{72}$$

$$\tilde{v}_{j,t} = v_{i,t} + v_{j,t} - \tilde{v}_{i,t}. \tag{73}$$

For the feasibility of $\tilde{v}_{i,t}$ and $\tilde{v}_{j,t}$, it is obvious that

$$0 \le \tilde{v}_{i,t} \le 1, \tag{74}$$

$$0 \le \tilde{v}_{j,t}. \tag{75}$$

We prove $\tilde{v}_{i,t} \le \tilde{e}_{i,t}$ by showing that $\tilde{e}_{i,t} \ge v_{i,t} + v_{j,t}$. Otherwise,

$$\begin{aligned} \tilde{e}_{i,t} + \tilde{e}_{j,t} &< v_{i,t} + v_{j,t} + e_{j,t+1} \\ &\le e_{i,t} + e_{j,t}, \end{aligned} \tag{76}$$

which contradicts with (40). Then similar to Case 1, we only need to prove (50) and (56) in 2 sub-cases:

Case 2.1. If $\tilde{v}_{i,t} = v_{i,t} + v_{j,t}$, then combining (73) we have

$$\tilde{v}_{j,t} = 0, \tag{77}$$

and then (50) holds trivially. Recall that $(\tilde{e}_{i,t}, d_i) \prec (\tilde{e}_{j,t}, d_j)$ from (25). According to the Definition 1, the fact $\tilde{v}_{j,t} = 0$, and the relationship (53)(54), we can see that

$$(\tilde{e}_{i,t+1}, d_i) \prec (\tilde{e}_{j,t+1}, d_j). \tag{78}$$

On the other hand, Combining $\tilde{e}_{j,t} < e_{j,t+1}$ with (54) we have

$$\tilde{e}_{j,t+1} < e_{j,t+1}. \tag{79}$$

Then from (55) we can see that there exists $\hat{m} \in [0, \hat{m}^*]$ such that

$$\tilde{e}_{i,t+1} = e_{i,t+1} + \hat{m}, \tag{80}$$
$$\tilde{e}_{j,t+1} = e_{j,t+1} - \hat{m}, \tag{81}$$

where $\hat{m}^* = \inf\{m|(e_{i,t+1}+m, d_i) \nprec (e_{j,t+1}-m, d_j)\}$. Then according to the property (24) for timeslot $t+1$ we can see (56) holds.

Case 2.2. If $\tilde{v}_{i,t} = 1$, then $\tilde{e}_{i,t} \geq 1$. If $\tilde{e}_{i,t} = 1$, then

$$1 = \tilde{v}_{i,t} \leq v_{i,t} + v_{j,t} \leq \tilde{e}_{i,t} = 1, \tag{82}$$

and therefore

$$v_{i,t} + v_{j,t} = 1, \tag{83}$$

and this case can be covered by Case 2.1. Then we consider the case $\tilde{e}_{i,t} > 1$. From (25) and Definition 1 we can see $\tilde{e}_{j,t} > 1$. On the other hand, combining $\tilde{e}_{i,t} > 1$ with (41)(42)(73) we have

$$\tilde{v}_{j,t} \leq 1 < \tilde{e}_{j,t}, \tag{84}$$

and therefore (50) holds.

Recall that $(\tilde{e}_{i,t}, d_i) \prec (\tilde{e}_{j,t}, d_j)$ from (25). According to the Definition 1, the fact $\tilde{v}_{i,t} = 1$, and the relationship (53)(54), we can see that

$$(\tilde{e}_{i,t+1}, d_i) \prec (\tilde{e}_{j,t+1}, d_j). \tag{85}$$

On the other hand, we have

$$\tilde{e}_{j,t+1} \leq \tilde{e}_{j,t} < e_{j,t+1}. \tag{86}$$

Then from (55) we can see that there exists $\hat{m} \in [0, \hat{m}^*]$ such that

$$\tilde{e}_{i,t+1} = e_{i,t+1} + \hat{m}, \tag{87}$$
$$\tilde{e}_{j,t+1} = e_{j,t+1} - \hat{m}, \tag{88}$$

where $\hat{m}^* = \inf\{m|(e_{i,t+1}+m, d_i) \nprec (e_{j,t+1}-m, d_j)\}$. Then according to the property (24) for timeslot $t+1$ we can see (56) holds.

□

Next, we introduce the concept of differentiated threshold charging (DTC) policy, which has been discussed in [2] and is very important for our following proofs. In a DTC-type policy, at each timeslot t, there is a (normalized) charging threshold $\mu_{d,t} \geq 0$ for each deadline d, and the normalized charging rate for charger i with deadline d is determined by

$$v_{i,t} = \min(\max(e_{i,t} - \mu_{d,t}, 0), 1), \ \forall t, \ \forall i \in \mathcal{G}_{d,t}. \tag{89}$$

It is shown in [2] that there exists an optimal DTC-type charging policy. Therefore, in the following proofs, we focus on the DTC-type policies. First, we will define the concept of charging order-compliance for thresholds in a DTC-type policy.

Definition 3. *In a DTC-type charging policy, for two charging deadlines $d_1 < d_2$, we say that charging thresholds $\mu_{d_1,t}$ and $\mu_{d_2,t}$ are charging order-compliant, if the following two conditions hold:*

1) $\forall \epsilon > 0$, $(\mu_{d_1,t} + \epsilon, d_1) \nprec (\mu_{d_2,t}, d_2)$,
2) $\forall \epsilon > 0$, $(\mu_{d_2,t} + \epsilon, d_2) \nprec (\mu_{d_1,t}, d_1)$.

Furthermore, let $d_1 < d_2 < \cdots < d_m$ be m different charging deadlines, then we say $\mu_{d_1,t}, \mu_{d_2,t}, \cdots, \mu_{d_m,t}$ are charging order-compliant, if $\mu_{d_k,t}$ and $\mu_{d_{k+1},t}$ are charging order-compliant for all $1 \leq k \leq m-1$.

Lemma 3. *In a DTC-type charging policy, let $d_1 < d_2$ be two charging deadlines and $\mu_{d_1,t}, \mu_{d_2,t}$ be corresponding charging thresholds. Then the following conditions are equivalent:*

1) *$\mu_{d_1,t}, \mu_{d_2,t}$ are charging order-compliant,*
2) $\lceil \mu_{d_1,t} \rceil \leq \mu_{d_2,t} \leq \lfloor \mu_{d_1,t} \rfloor + (d_2 - d_1)$,
3) $\lceil \mu_{d_2,t} \rceil - (d_2 - d_1) \leq \mu_{d_1,t} \leq \lfloor \mu_{d_2,t} \rfloor$.

Proof. The equivalence can be established by

$$\forall \epsilon > 0, (\mu_{d_1,t} + \epsilon, d_1) \nprec (\mu_{d_2,t}, d_2) \tag{90}$$
$$\iff \forall \epsilon > 0, \lceil \mu_{d_1,t} + \epsilon \rceil > \lceil \mu_{d_2,t} \rceil - (d_2 - d_1) \tag{91}$$
$$\iff \mu_{d_1,t} \geq \lceil \mu_{d_2,t} \rceil - (d_2 - d_1) \tag{92}$$
$$\iff \lfloor \mu_{d_1,t} \rfloor \geq \lceil \mu_{d_2,t} \rceil - (d_2 - d_1) \tag{93}$$
$$\iff \mu_{d_2,t} \leq \lfloor \mu_{d_1,t} \rfloor + (d_2 - d_1), \tag{94}$$

and

$$\forall \epsilon > 0, (\mu_{d_2,t} + \epsilon, d_2) \nprec (\mu_{d_1,t}, d_1) \tag{95}$$
$$\iff \forall \epsilon > 0, \lceil \mu_{d_2,t} + \epsilon \rceil > \lceil \mu_{d_1,t} \rceil \tag{96}$$
$$\iff \mu_{d_2,t} \geq \lceil \mu_{d_1,t} \rceil \tag{97}$$
$$\iff \lfloor \mu_{d_2,t} \rfloor \geq \lceil \mu_{d_1,t} \rceil \tag{98}$$
$$\iff \mu_{d_1,t} \leq \lfloor \mu_{d_2,t} \rfloor. \tag{99}$$

□

Lemma 4. *In a DTC-type charging policy, let $d_1 < d_2 < d_3$ be three different charging deadlines, if $\mu_{d_1,t}$ and $\mu_{d_2,t}$ are charging order-compliant, and $\mu_{d_2,t}$ and $\mu_{d_3,t}$ are charging order-compliant, then $\mu_{d_1,t}$ and $\mu_{d_3,t}$ are charging order-compliant.*

Proof. Because $\mu_{d_1,t}$ and $\mu_{d_2,t}$ are charging order-compliant, and $\mu_{d_2,t}$ and $\mu_{d_3,t}$ are charging order-compliant, according to Lemma 3 we have

$$\lceil \mu_{d_1,t} \rceil \leq \mu_{d_2,t} \leq \lfloor \mu_{d_1,t} \rfloor + (d_2 - d_1), \tag{100}$$
$$\lceil \mu_{d_2,t} \rceil \leq \mu_{d_3,t} \leq \lfloor \mu_{d_2,t} \rfloor + (d_3 - d_2), \tag{101}$$

and therefore

$$\lceil \mu_{d_1,t} \rceil \leq \mu_{d_3,t} \leq \lfloor \mu_{d_1,t} \rfloor + (d_3 - d_1). \tag{102}$$

then according to Lemma 3, $\mu_{d_1,t}$ and $\mu_{d_3,t}$ are charging order-compliant. □

Next, we will show that we can modify charging thresholds to be charging order-compliant without increasing the state-action value function. We first discuss a simple case, where two charging thresholds are adjusted.

Lemma 5. *For any charging threshold $\boldsymbol{\mu}_t$, and any deadline d, $d+\tau$ satisfying $t < d < d+\tau \leq T$, there is another charging threshold $\tilde{\boldsymbol{\mu}}_t$ that is identical to $\boldsymbol{\mu}_t$ except for $\tilde{\mu}_{d,t}$ and $\tilde{\mu}_{d+\tau,t}$ which are charging order-compliant, and $Q_t(\boldsymbol{x}_t, \boldsymbol{w}_t, \tilde{\boldsymbol{\mu}}_t) \leq Q_t(\boldsymbol{x}_t, \boldsymbol{w}_t, \boldsymbol{\mu}_t)$.*

Proof. We only need to consider the following cases where $\mu_{d,t}$ and $\mu_{d+\tau,t}$ are not charging order-compliant:

Case 1: $\exists\epsilon > 0$ such that $(\mu_{d,t}+\epsilon, d) \prec (\mu_{d+\tau,t}, d+\tau)$. From Lemma 3, we have the following properties:

$$\mu_{d,t} < \lceil \mu_{d+\tau,t} \rceil - \tau, \tag{103}$$
$$\mu_{d+\tau,t} > \lfloor \mu_{d,t} \rfloor + \tau. \tag{104}$$

Let $A = \lceil \mu_{d+\tau,t} \rceil - \lfloor \mu_{d,t} \rfloor - \tau$, we have $A \geq 1$ by (104). Then we define two sequences $\{\hat{\mu}_{d,t}^{(i)}\}_{i=0}^{A}$ and $\{\hat{\mu}_{d+\tau,t}^{(i)}\}_{i=0}^{A}$ as follows:

$$\hat{\mu}_{d,t}^{(i)} = \begin{cases} \mu_{d,t}, & i = 0, \\ \lfloor \mu_{d,t} \rfloor + i, & i = 1, 2, ..., A. \end{cases} \tag{105}$$

$$\hat{\mu}_{d+\tau,t}^{(i)} = \begin{cases} \lfloor \mu_{d,t} \rfloor + i + \tau, & i = 0, 1, ..., A-1, \\ \mu_{d+\tau,t}, & i = A. \end{cases} \tag{106}$$

We define the (normalized) total charging power of EVs with deadline d and $d+\tau$ with charging threshold $\bar{\mu}_d$ and $\bar{\mu}_{d+\tau}$ as:

$$C_t^{d,d+\tau}(\bar{\mu}_d, \bar{\mu}_{d+\tau}) = \sum_{i \in \mathcal{G}_{d,t}} \min(\max(e_{i,t} - \bar{\mu}_d, 0), 1) + \sum_{i \in \mathcal{G}_{d+\tau,t}} \min(\max(e_{i,t} - \bar{\mu}_{d+\tau}, 0), 1). \tag{107}$$

Specifically, we define

$$c = C_t^{d,d+\tau}(\mu_{d,t}, \mu_{d+\tau,t}), \tag{108}$$
$$\hat{c}^{(i)} = C_t^{d,d+\tau}(\hat{\mu}_{d,t}^{(i)}, \hat{\mu}_{d+\tau,t}^{(i)}), \qquad i = 0, 1, ..., A. \tag{109}$$

Then we can verify that $\hat{c}^{(0)} \geq \hat{c}^{(1)} \geq \cdots \geq \hat{c}^{(A)}$ and $\hat{c}^{(0)} \geq c \geq \hat{c}^{(A)}$. Then we let

$$k = \min\{0 \leq k' \leq A-1 | \hat{c}^{(k')} \geq c \text{ and } \hat{c}^{(k'+1)} \leq c\}. \tag{110}$$

If $C_t^{d,d+\tau}(\hat{\mu}_{d,t}^{(k+1)}, \hat{\mu}_{d+\tau,t}^{(k)}) \leq c$, then we define

$$\tilde{\mu}_{d,t} = \min\{\hat{\mu}_{d,t}^{(k)} \leq \bar{\mu}_d \leq \hat{\mu}_{d,t}^{(k+1)} | C_t^{d,d+\tau}(\bar{\mu}_d, \hat{\mu}_{d+\tau,t}^{(k)}) = c\}, \tag{111}$$

$$\tilde{\mu}_{d+\tau,t} = \hat{\mu}_{d+\tau,t}^{(k)}. \tag{112}$$

Otherwise, we define

$$\tilde{\mu}_{d,t} = \hat{\mu}_{d,t}^{(k+1)}, \tag{113}$$

$$\tilde{\mu}_{d+\tau,t} = \min\{\hat{\mu}_{d+\tau,t}^{(k)} \leq \bar{\mu}_{d+\tau} \leq \hat{\mu}_{d+\tau,t}^{(k+1)} | C_t^{d,d+\tau}(\hat{\mu}_{d,t}^{(k+1)}, \bar{\mu}_{d+\tau}) = c\}. \tag{114}$$

Note that the total charging power remains the same after changing $\boldsymbol{\mu}_t$ to $\tilde{\boldsymbol{\mu}}_t$. Now we will show that $\tilde{\mu}_{d,t}$ and $\tilde{\mu}_{d+\tau,t}$ are charging order-compliant, which is equivalent to

$$\lceil \tilde{\mu}_{d,t} \rceil \leq \tilde{\mu}_{d+\tau,t} \leq \lfloor \tilde{\mu}_{d,t} \rfloor + \tau. \tag{115}$$

If $C_t^{d,d+\tau}(\hat{\mu}_{d,t}^{(k+1)}, \hat{\mu}_{d+\tau,t}^{(k)}) \leq c$, assume $1 \leq k \leq A-1$ (the case for $k = 0$ can be proved likewise), according to (105)(106)(111)(112) we have

$$\lfloor \mu_{d,t} \rfloor + k \leq \tilde{\mu}_{d,t} \leq \lfloor \mu_{d,t} \rfloor + k + 1, \tag{116}$$
$$\tilde{\mu}_{d+\tau,t} = \lfloor \mu_{d,t} \rfloor + k + \tau. \tag{117}$$

From (116)(117) we know (115) holds.

If $C_t^{d,d+\tau}(\hat{\mu}_{d,t}^{(k+1)}, \hat{\mu}_{d+\tau,t}^{(k)}) > c$, assume $0 \leq k \leq A-2$ (the case for $k = A-1$ can be proved likewise), according to (105)(106)(113)(114) we have

$$\tilde{\mu}_{d,t} = \lfloor \mu_{d,t} \rfloor + k + 1, \tag{118}$$
$$\lfloor \mu_{d,t} \rfloor + k + \tau \leq \tilde{\mu}_{d+\tau,t} \leq \lfloor \mu_{d,t} \rfloor + k + 1 + \tau. \tag{119}$$

From (118)(119) we know (115) holds.

Next we will prove that $Q_t(\boldsymbol{x}_t, \boldsymbol{w}_t, \tilde{\boldsymbol{\mu}}_t) \leq Q_t(\boldsymbol{x}_t, \boldsymbol{w}_t, \boldsymbol{\mu}_t)$. Note that by (116)(117)(118)(119) we have

$$(\tilde{\mu}_{d,t} - \epsilon, d) \prec (\tilde{\mu}_{d+\tau,t} + \epsilon, d+\tau), \ \forall \epsilon > 0. \tag{120}$$

Also, it is obvious that

$$\tilde{\mu}_{d,t} \geq \mu_{d,t}, \tag{121}$$
$$\tilde{\mu}_{d+\tau,t} \leq \mu_{d+\tau,t}. \tag{122}$$

We denote all chargers in $\mathcal{G}_{d,t}$ as $k_1, k_2, ..., k_p$ and all chargers in $\mathcal{G}_{d+\tau,t}$ as $l_1, l_2, ..., l_q$. Then the sequence $\mathcal{E} = \{e_{k_1,t+1}, ..., e_{k_p,t+1}, e_{l_1,t+1}, ..., e_{l_q,t+1}\}$ can be calculated by

$$e_{k_n,t+1} = \begin{cases} e_{k_n,t}, & e_{k_n,t} \leq \mu_{d,t}, \\ \mu_{d,t}, & \mu_{d,t} < e_{k_n,t} \leq \mu_{d,t} + 1, \\ e_{k_n,t} - 1, & e_{k_n,t} > \mu_{d,t} + 1. \end{cases} \tag{123}$$

$$e_{l_n,t+1} = \begin{cases} e_{l_n,t}, & e_{l_n,t} \leq \mu_{d+\tau,t}, \\ \mu_{d+\tau,t}, & \mu_{d+\tau,t} < e_{l_n,t} \leq \mu_{d+\tau,t} + 1, \\ e_{l_n,t} - 1, & e_{l_n,t} > \mu_{d+\tau,t} + 1. \end{cases} \tag{124}$$

Similarly, we can calculate the the sequence $\tilde{\mathcal{E}} = \{\tilde{e}_{k_1,t+1}, ..., \tilde{e}_{k_p,t+1}, \tilde{e}_{l_1,t+1}, ..., \tilde{e}_{l_q,t+1}\}$ by

$$\tilde{e}_{k_n,t+1} = \begin{cases} e_{k_n,t}, & e_{k_n,t} \leq \tilde{\mu}_{d,t}, \\ \tilde{\mu}_{d,t}, & \tilde{\mu}_{d,t} < e_{k_n,t} \leq \tilde{\mu}_{d,t} + 1, \\ e_{k_n,t} - 1, & e_{k_n,t} > \tilde{\mu}_{d,t} + 1. \end{cases} \tag{125}$$

$$\tilde{e}_{l_n,t+1} = \begin{cases} e_{l_n,t}, & e_{l_n,t} \leq \tilde{\mu}_{d+\tau,t}, \\ \tilde{\mu}_{d+\tau,t}, & \tilde{\mu}_{d+\tau,t} < e_{l_n,t} \leq \tilde{\mu}_{d+\tau,t} + 1, \\ e_{l_n,t} - 1, & e_{l_n,t} > \tilde{\mu}_{d+\tau,t} + 1. \end{cases} \tag{126}$$

Combining with (121)(122) we know

$$\tilde{e}_{k_n,t+1} \geq e_{k_n,t+1}, \ \forall k_n, \tag{127}$$
$$\tilde{e}_{l_n,t+1} \leq e_{l_n,t+1}, \ \forall l_n. \tag{128}$$

If $\hat{\mathcal{E}} = \mathcal{E}$, then the changing of $\boldsymbol{\mu}_t$ to $\tilde{\boldsymbol{\mu}}_t$ does not alter the actual changing power of any charger, then naturally $Q_t(\boldsymbol{x}_t, \boldsymbol{w}_t, \tilde{\boldsymbol{\mu}}_t) \leq Q_t(\boldsymbol{x}_t, \boldsymbol{w}_t, \boldsymbol{\mu}_t)$.

Otherwise, from (111)(114)(121)(122) we know both set $\{k_n | \tilde{e}_{k_n,t+1} > e_{k_n,t+1}\}$ and $\{l_n | \tilde{e}_{l_n,t+1} < e_{l_n,t+1}\}$ are not empty. Then we recursively define multiple sequences $\hat{\mathcal{E}}^{(0)}, \hat{\mathcal{E}}^{(1)}, \hat{\mathcal{E}}^{(2)}, ...$ as follows:

1) We define $\hat{\mathcal{E}}^{(0)} = \mathcal{E}$,
2) If $\hat{\mathcal{E}}^{(i)} = \{\hat{e}_{k_1,t+1}^{(i)}, ..., \hat{e}_{k_p,t+1}^{(i)}, \hat{e}_{l_1,t+1}^{(i)}, ..., \hat{e}_{l_q,t+1}^{(i)}\}$ is not identical to $\tilde{\mathcal{E}}$, then we select any $k^{(i)}$ and $l^{(i)}$ that satisfy:

$$k^{(i)} \in \{k_n | \hat{e}_{k_n,t+1}^{(i)} < \tilde{e}_{k_n,t+1}\}, \tag{129}$$
$$l^{(i)} \in \{l_n | \hat{e}_{l_n,t+1}^{(i)} > \tilde{e}_{l_n,t+1}\}. \tag{130}$$

Let $m = \min(\tilde{e}_{k^{(i)},t+1} - \hat{e}^{(i)}_{k^{(i)},t+1}, \hat{e}^{(i)}_{l^{(i)},t+1} - \tilde{e}_{l^{(i)},t+1})$, and we construct $\hat{\mathcal{E}}^{(i+1)}$ which is identical to $\hat{\mathcal{E}}^{(i)}$ except for $k^{(i)}$ and $l^{(i)}$:

$$\hat{e}^{(i+1)}_{k^{(i)},t+1} = \hat{e}^{(i)}_{k^{(i)},t+1} + m, \tag{131}$$
$$\hat{e}^{(i+1)}_{l^{(i)},t+1} = \hat{e}^{(i)}_{l^{(i)},t+1} - m. \tag{132}$$

From the construction process of $\hat{\mathcal{E}}^{(i)}$, we have

$$\tilde{e}_{k^{(i)},t+1} > e_{k^{(i)},t+1}, \tag{133}$$
$$\tilde{e}_{l^{(i)},t+1} < e_{l^{(i)},t+1}. \tag{134}$$

Combining (133) with (123)(125) we can infer that

$$\tilde{e}_{k^{(i)},t+1} \leq \tilde{\mu}_{d,t}, \tag{135}$$

and therefore

$$\hat{e}^{(i+1)}_{k^{(i)},t+1} \leq \tilde{\mu}_{d,t}. \tag{136}$$

Similarly, we have

$$\hat{e}^{(i+1)}_{l^{(i)},t+1} \geq \tilde{\mu}_{d+\tau,t}. \tag{137}$$

Let $\hat{\boldsymbol{v}}^{(i)}_t$ and $\hat{\boldsymbol{v}}^{(i+1)}_t$ denote the corresponding charging action of $\hat{\mathcal{E}}^{(i)}$ and $\hat{\mathcal{E}}^{(i+1)}$, then $\hat{\boldsymbol{v}}^{(i)}$ and $\hat{\boldsymbol{v}}^{(i+1)}$ are identical except for $k^{(i)}$ and $l^{(i)}$, with the following property:

$$\mathbf{1}^T \hat{\boldsymbol{v}}^{(i)}_t = \mathbf{1}^T \hat{\boldsymbol{v}}^{(i+1)}_t. \tag{138}$$

On the other hand, combining (131)(132)(136)(137) with (120) we have

$$(\hat{e}^{(i)}_{k^{(i)},t+1} + m - \epsilon, d) \prec (\hat{e}^{(i)}_{l^{(i)},t+1} - m + \epsilon, d+\tau),\ \forall \epsilon > 0, \tag{139}$$

and therefore

$$m \leq \inf\{m' | (\hat{e}^{(i)}_{k^{(i)},t+1} + m', d) \not\prec (\hat{e}^{(i)}_{l^{(i)},t+1} - m', d+\tau)\}. \tag{140}$$

Then according to Lemma 2 we know

$$\mathop{\mathbb{E}}_{\boldsymbol{w}_{t+1}}[V_{t+1}(\hat{\boldsymbol{x}}^{(i+1)}_{t+1}, \boldsymbol{w}_{t+1})] \leq \mathop{\mathbb{E}}_{\boldsymbol{w}_{t+1}}[V_{t+1}(\hat{\boldsymbol{x}}^{(i)}_{t+1}, \boldsymbol{w}_{t+1})]. \tag{141}$$

By (138)(141) we have

$$Q_t(\boldsymbol{x}_t, \boldsymbol{w}_t, \hat{\boldsymbol{v}}^{(i+1)}_t) \leq Q_t(\boldsymbol{x}_t, \boldsymbol{w}_t, \hat{\boldsymbol{v}}^{(i)}_t). \tag{142}$$

Note that $\hat{\mathcal{E}}^{(i+1)}$ and $\tilde{\mathcal{E}}$ share at least one more identical element than $\hat{\mathcal{E}}^{(i)}$ and $\tilde{\mathcal{E}}$. Then we know there is some $\tilde{i}$ such that $\hat{\mathcal{E}}^{(\tilde{i})} = \tilde{\mathcal{E}}$, which means $\hat{\boldsymbol{v}}^{(\tilde{i})}$ is the charging action corresponding to $\tilde{\boldsymbol{\mu}}_t$. Repeatedly applying Lemma 2 to the recursive definition of $\hat{\mathcal{E}}^{(i+1)}$ we know $Q_t(\boldsymbol{x}_t, \boldsymbol{w}_t, \tilde{\boldsymbol{\mu}}_t) \leq Q_t(\boldsymbol{x}_t, \boldsymbol{w}_t, \boldsymbol{\mu}_t)$.

Case 2: $\exists \epsilon > 0$ such that $(\mu_{d+\tau,t} + \epsilon, d+\tau) \prec (\mu_{d,t}, d)$. From Lemma 3, we have the following properties:

$$\mu_{d,t} > \lfloor \mu_{d+\tau,t}, t \rfloor, \tag{143}$$
$$\mu_{d+\tau,t} < \lceil \mu_{d,t} \rceil. \tag{144}$$

Let $A = \lceil \mu_{d,t} \rceil - \lfloor \mu_{d+\tau,t} \rfloor$, we have $A \geq 1$ by (144). Then we define two sequences $\{\hat{\mu}^{(i)}_{d,t}\}^A_{i=0}$ and $\{\hat{\mu}^{(i)}_{d+\tau,t}\}^A_{i=0}$ as follows:

$$\hat{\mu}^{(i)}_{d,t} = \begin{cases} \lfloor \mu_{d+\tau,t} \rfloor + i, & i = 0, 1, ..., A-1, \\ \mu_{d,t}, & i = A. \end{cases} \tag{145}$$

$$\hat{\mu}^{(i)}_{d+\tau,t} = \begin{cases} \mu_{d+\tau,t}, & i = 0, \\ \lfloor \mu_{d+\tau,t} \rfloor + i, & i = 1, ..., A. \end{cases} \tag{146}$$

We can also verify that $\hat{c}^{(0)} \geq \hat{c}^{(1)} \geq \cdots \geq \hat{c}^{(A)}$ and $\hat{c}^{(0)} \geq c \geq \hat{c}^{(A)}$. Then we let

$$k = \min\{0 \leq k' \leq A-1 | \hat{c}^{(k')} \geq c \text{ and } \hat{c}^{(k'+1)} \leq c\}. \tag{147}$$

If $C^{d,d+\tau}_t(\hat{\mu}^{(k)}_{d,t}, \hat{\mu}^{(k+1)}_{d+\tau,t}) \leq c$, then we define

$$\tilde{\mu}_{d,t} = \hat{\mu}^{(k)}_{d,t}, \tag{148}$$

$$\tilde{\mu}_{d+\tau,t} = \min\{\hat{\mu}^{(k)}_{d+\tau,t} \leq \mu_{d+\tau} \leq \hat{\mu}^{(k+1)}_{d+\tau,t} | C^{d,d+\tau}_t(\hat{\mu}^{(k)}_{d,t}, \mu_{d+\tau}) = c\}. \tag{149}$$

Otherwise, we define

$$\tilde{\mu}_{d,t} = \min\{\hat{\mu}^{(k)}_{d,t} \leq \mu_d \leq \hat{\mu}^{(k+1)}_{d,t} | C^{d,d+\tau}_t(\mu_d, \hat{\mu}^{(k+1)}_{d+\tau,t}) = c\}, \tag{150}$$

$$\tilde{\mu}_{d+\tau,t} = \hat{\mu}^{(k+1)}_{d+\tau,t}. \tag{151}$$

The rest of the proof is similar to Case 1 and is therefore omitted. □

With Lemma 5, we denote the mapping from $\mu_{d_1,t}, \mu_{d_2,t}$ to $\tilde{\mu}_{d_1,t}, \tilde{\mu}_{d_2,t}$ as $\mathcal{F}^{d_1,d_2}_{2,t}$:

$$(\tilde{\mu}_{d_1,t}, \tilde{\mu}_{d_2,t}) = \mathcal{F}^{d_1,d_2}_{2,t}(\mu_{d_1,t}, \mu_{d_2,t}). \tag{152}$$

Combining (105)(105)(116)(117)(118)(119)(145)(146) in proof of Lemma 5 with Lemma 3, we can verify the following corollaries:

Corollary 1. *Suppose $\mu_{d_1,t}, \mu_{d_2,t}$ are not charging order-compliant. Let*

$$(\tilde{\mu}_{d_1,t}, \tilde{\mu}_{d_2,t}) = \mathcal{F}^{d_1,d_2}_{2,t}(\mu_{d_1,t}, \mu_{d_2,t}). \tag{153}$$

If $\tilde{\mu}_{d_1,t} > \mu_{d_1,t}$, then $\tilde{\mu}_{d_2,t} \leq \mu_{d_2,t}$.

Corollary 2. *Suppose $\mu_{d_1,t}, \mu_{d_2,t}$ are not charging order-compliant. Let*

$$(\tilde{\mu}_{d_1,t}, \tilde{\mu}_{d_2,t}) = \mathcal{F}^{d_1,d_2}_{2,t}(\mu_{d_1,t}, \mu_{d_2,t}). \tag{154}$$

Then $\exists j \in \{1,2\}$ such that $\tilde{\mu}_{d_j,t} \neq \mu_{d_j,t}$ and also $\tilde{\mu}_{d_j,t}$ is an integer.

Corollary 3. *Suppose $\mu_{d_1,t}, \mu_{d_2,t}$ are charging order-compliant, if $\hat{\mu}_{d_2,t} > \mu_{d_2,t}$ is another charging threshold for deadline d_2 with $\mu_{d_1,t}, \hat{\mu}_{d_2,t}$ not charging order-compliant. Let*

$$(\tilde{\mu}_{d_1,t}, \tilde{\mu}_{d_2,t}) = \mathcal{F}^{d_1,d_2}_{2,t}(\mu_{d_1,t}, \hat{\mu}_{d_2,t}), \tag{155}$$

then

$$\tilde{\mu}_{d_1,t} \geq \mu_{d_1,t}, \quad (156)$$
$$\mu_{d_2,t} \leq \tilde{\mu}_{d_2,t} \leq \hat{\mu}_{d_2,t}, \quad (157)$$
$$\tilde{\mu}_{d_1,t} \leq \lfloor \tilde{\mu}_{d_2,t} \rfloor - (d_2 - d_1) + 1, \quad (158)$$
$$\tilde{\mu}_{d_2,t} \geq \lceil \tilde{\mu}_{d_1,t} \rceil + (d_2 - d_1) - 1, \quad (159)$$
$$\tilde{\mu}_{d_1,t} \leq \lceil \hat{\mu}_{d_2,t} \rceil - (d_2 - d_1), \quad (160)$$
$$\tilde{\mu}_{d_2,t} \geq \lfloor \mu_{d_1,t} \rfloor + (d_2 - d_1). \quad (161)$$

Corollary 4. *Suppose $\mu_{d_1,t}, \mu_{d_2,t}$ are charging order-compliant, if $\hat{\mu}_{d_1,t} < \mu_{d_1,t}$ is another charging threshold for deadline d_1 with $\hat{\mu}_{d_1,t}, \mu_{d_2,t}$ not charging order-compliant. Let*

$$(\tilde{\mu}_{d_1,t}, \tilde{\mu}_{d_2,t}) = \mathcal{F}_{2,t}^{d_1,d_2}(\hat{\mu}_{d_1,t}, \mu_{d_2,t}), \quad (162)$$

then

$$\tilde{\mu}_{d_2,t} \leq \mu_{d_2,t}, \quad (163)$$
$$\hat{\mu}_{d_1,t} \leq \tilde{\mu}_{d_1,t} \leq \mu_{d_1,t}, \quad (164)$$
$$\tilde{\mu}_{d_1,t} \leq \lfloor \tilde{\mu}_{d_2,t} \rfloor - (d_2 - d_1) + 1, \quad (165)$$
$$\tilde{\mu}_{d_2,t} \geq \lceil \tilde{\mu}_{d_1,t} \rceil + (d_2 - d_1) - 1, \quad (166)$$
$$\tilde{\mu}_{d_1,t} \leq \lceil \mu_{d_2,t} \rceil - (d_2 - d_1), \quad (167)$$
$$\tilde{\mu}_{d_2,t} \geq \lfloor \hat{\mu}_{d_1,t} \rfloor + (d_2 - d_1). \quad (168)$$

Lemma 6. *For any charging threshold $\boldsymbol{\mu}_t$, and any m deadlines $d_1, \cdots, d_{m-1}, d_m$, satisfying $t < d_1 < \cdots < d_{m-1} < d_m \leq T$, there is another charging threshold $\tilde{\boldsymbol{\mu}}_t$ that is identical to $\boldsymbol{\mu}_t$ except for $\tilde{\mu}_{d_1,t}, \cdots, \tilde{\mu}_{d_{m-1},t}, \tilde{\mu}_{d_m,t}$ which are charging order-compliant, and $Q(\boldsymbol{x}_t, \boldsymbol{w}_t, \tilde{\boldsymbol{\mu}}_t) \leq Q(\boldsymbol{x}_t, \boldsymbol{w}_t, \boldsymbol{\mu}_t)$.*

Proof. We prove this lemma by induction. Notice that the case for $m = 2$ have been proved in Lemma 5.

Suppose the lemma holds for any m deadlines $d_1, \cdots, d_{m-1}, d_m$. Then we denote the mapping from $(\mu_{d_1,t}, \cdots, \mu_{d_{m-1},t}, \mu_{d_m,t})$ to $(\tilde{\mu}_{d_1,t}, \cdots, \tilde{\mu}_{d_{m-1},t}, \tilde{\mu}_{d_m,t})$ as $\mathcal{F}_{m,t}^{d_1,\cdots,d_m}$:

$$(\tilde{\mu}_{d_1,t}, \cdots, \tilde{\mu}_{d_m,t}) = \mathcal{F}_{m,t}^{d_1,\cdots,d_m}(\mu_{d_1,t}, \cdots, \mu_{d_m,t}). \quad (169)$$

Now we will prove the case for $m + 1$ charging thresholds $\mu_{d_1,t}, \cdots, \mu_{d_{m-1},t}, \mu_{d_m,t}, \mu_{d_{m+1},t}$. We first assume that $\mu_{d_1,t}, \cdots, \mu_{d_{m-1},t}, \mu_{d_m,t}$ are charging order-compliant. Otherwise, we replace it with $\mathcal{F}_{m,t}^{d_1,\cdots,d_m}(\mu_{d_1,t}, \cdots, \mu_{d_m,t})$ without increasing $Q(\boldsymbol{x}_t, \boldsymbol{w}_t, \boldsymbol{\mu}_t)$.

We need two additional auxiliary properties to complete the inductive proof, which are described as follows.

Property 1: Suppose $\mu_{d_1,t}, \cdots, \mu_{d_{m-1},t}, \mu_{d_m,t}$ are charging order-compliant, if $\hat{\mu}_{d_m,t} > \mu_{d_m,t}$ is another charging threshold for deadline d_m with $\mu_{d_{m-1},t}, \hat{\mu}_{d_m,t}$ not charging order-compliant. Let

$$(\tilde{\mu}_{d_1,t}, \cdots, \tilde{\mu}_{d_m,t}) = \mathcal{F}_{m,t}^{d_1,\cdots,d_m}(\mu_{d_1,t}, \cdots, \mu_{d_{m-1},t}, \hat{\mu}_{d_m,t}), \quad (170)$$

then

$$\tilde{\mu}_{d_j,t} \geq \mu_{d_j,t}, \ j = 1, \cdots, m-1, \quad (171)$$
$$\mu_{d_m,t} \leq \tilde{\mu}_{d_m,t} \leq \hat{\mu}_{d_m,t}, \quad (172)$$
$$\tilde{\mu}_{d_{m-1},t} \leq \lfloor \tilde{\mu}_{d_m,t} \rfloor - (d_m - d_{m-1}) + 1, \quad (173)$$
$$\tilde{\mu}_{d_m,t} \geq \lceil \tilde{\mu}_{d_{m-1},t} \rceil + (d_m - d_{m-1}) - 1, \quad (174)$$
$$\tilde{\mu}_{d_{m-1},t} \leq \lceil \hat{\mu}_{d_m,t} \rceil - (d_m - d_{m-1}), \quad (175)$$
$$\tilde{\mu}_{d_m,t} \geq \lfloor \mu_{d_{m-1},t} \rfloor + (d_m - d_{m-1}). \quad (176)$$

Property 2: Suppose $\mu_{d_1,t}, \cdots, \mu_{d_m,t}$ are not charging order-compliant. Let

$$(\tilde{\mu}_{d_1,t}, \cdots, \tilde{\mu}_{d_m,t}) = \mathcal{F}_{m,t}^{d_1,\cdots,d_m}(\mu_{d_1,t}, \cdots, \mu_{d_m,t}). \quad (177)$$

Then $\exists j \in \{1, \cdots, m\}$ such that $\tilde{\mu}_{d_j,t} \neq \mu_{d_j,t}$ and also $\tilde{\mu}_{d_j,t}$ is an integer.

From the Corollary 3 and Corollary 2 we know the two properties above hold for $m = 2$. Now we suppose these two properties hold for any deadline $d_1, \cdots, d_{m-1}, d_m$, and we will also prove these two properties for the case of $m + 1$ charging thresholds. We will perform our proof in multiple steps.

Step 1: if $\mu_{d_m,t}, \mu_{d_{m+1},t}$ are charging order-compliant, then $\mu_{d_1,t}, \cdots, \mu_{d_{m-1},t}, \mu_{d_m,t}, \mu_{d_{m+1},t}$ have been charging order-compliant, and the desired properties hold naturally. Otherwise, we define $\tilde{\boldsymbol{\mu}}_t^{(1)}$ as follows[1]:

$$(\tilde{\mu}_{d_1,t}^{(1)}, \cdots, \tilde{\mu}_{d_{m-1},t}^{(1)}) = (\mu_{d_1,t}, \cdots, \mu_{d_{m-1},t}), \quad (178)$$
$$(\tilde{\mu}_{d_m,t}^{(1)}, \tilde{\mu}_{d_{m+1},t}^{(1)}) = \mathcal{F}_{2,t}^{d_m,d_{m+1}}(\mu_{d_m,t}, \mu_{d_{m+1},t}). \quad (179)$$

According to the induction hypothesis for 2 charging thresholds, we have

$$Q(\boldsymbol{x}_t, \boldsymbol{w}_t, \tilde{\boldsymbol{\mu}}_t^{(1)}) \leq Q(\boldsymbol{x}_t, \boldsymbol{w}_t, \boldsymbol{\mu}_t). \quad (180)$$

If $\tilde{\mu}_{d_{m-1},t}^{(1)}, \tilde{\mu}_{d_m,t}^{(1)}$ are charging order-compliant, then $\tilde{\mu}_{d_1,t}^{(1)}, \cdots, \tilde{\mu}_{d_{m-1},t}^{(1)}, \tilde{\mu}_{d_m,t}^{(1)}, \tilde{\mu}_{d_{m+1},t}^{(1)}$ are charging order-compliant, and $\tilde{\boldsymbol{\mu}}_t = \tilde{\boldsymbol{\mu}}_t^{(1)}$ satisfy the desired properties. Otherwise, since $\tilde{\mu}_{d_{m-1},t}^{(1)}, \mu_{d_m,t}$ are charging order-compliant, we have

$$\tilde{\mu}_{d_m,t}^{(1)} \neq \mu_{d_m,t}. \quad (181)$$

In the following discussion, we will assume

$$\tilde{\mu}_{d_m,t}^{(1)} > \mu_{d_m,t}. \quad (182)$$

In fact, the case for $\tilde{\mu}_{d_m,t}^{(1)} < \mu_{d_m,t}$ can be proved likewise. Then according to Corollary 1, we have

$$\tilde{\mu}_{d_{m+1},t}^{(1)} \leq \mu_{d_{m+1},t}. \quad (183)$$

Then we define $\bar{\boldsymbol{\mu}}_t^{(1)}$ as follows:

$$(\bar{\mu}_{d_1,t}^{(1)}, \cdots, \bar{\mu}_{d_m,t}^{(1)}) = \mathcal{F}_{m,t}^{d_1,\cdots,d_m}(\tilde{\mu}_{d_1,t}^{(1)}, \cdots, \tilde{\mu}_{d_m,t}^{(1)}), \quad (184)$$
$$\bar{\mu}_{d_{m+1},t}^{(1)} = \tilde{\mu}_{d_{m+1},t}^{(1)}. \quad (185)$$

According to the induction hypothesis for m charging thresholds, we have

$$Q(\boldsymbol{x}_t, \boldsymbol{w}_t, \bar{\boldsymbol{\mu}}_t^{(1)}) \leq Q(\boldsymbol{x}_t, \boldsymbol{w}_t, \tilde{\boldsymbol{\mu}}_t^{(1)}) \leq Q(\boldsymbol{x}_t, \boldsymbol{w}_t, \boldsymbol{\mu}_t). \quad (186)$$

[1]In the following proof, we will only change the charging thresholds for deadlines $d_1, \cdots, d_m, d_{m+1}$ and keep other charging thresholds unchanged.

According to (182) and Property 1, we have

$$\bar{\mu}^{(1)}_{d_j,t} \geq \tilde{\mu}^{(1)}_{d_j,t} = \mu_{d_j,t},\ j = 1,\cdots,m-1, \tag{187}$$
$$\mu_{d_m,t} \leq \bar{\mu}^{(1)}_{d_m,t} \leq \tilde{\mu}^{(1)}_{d_m,t}, \tag{188}$$
$$\bar{\mu}^{(1)}_{d_{m-1},t} \leq \left\lfloor \bar{\mu}^{(1)}_{d_m,t} \right\rfloor - (d_m - d_{m-1}) + 1, \tag{189}$$
$$\bar{\mu}^{(1)}_{d_m,t} \geq \left\lceil \bar{\mu}^{(1)}_{d_{m-1},t} \right\rceil + (d_m - d_{m-1}) - 1, \tag{190}$$
$$\tilde{\mu}^{(1)}_{d_{m-1},t} \leq \left\lceil \tilde{\mu}^{(1)}_{d_m,t} \right\rceil - (d_m - d_{m-1}), \tag{191}$$
$$\tilde{\mu}^{(1)}_{d_m,t} \geq \left\lfloor \tilde{\mu}^{(1)}_{d_{m-1},t} \right\rfloor + (d_m - d_{m-1}). \tag{192}$$

If $\bar{\mu}^{(1)}_{d_m,t}, \bar{\mu}^{(1)}_{d_{m+1},t}$ are charging order-compliant, then $\bar{\mu}^{(1)}_{d_1,t},\cdots,\bar{\mu}^{(1)}_{d_{m-1},t},\bar{\mu}^{(1)}_{d_m,t},\bar{\mu}^{(1)}_{d_{m+1},t}$ are charging order-compliant, and $\tilde{\boldsymbol{\mu}}_t = \bar{\boldsymbol{\mu}}^{(1)}_t$ satisfy the desired properties. Otherwise, since $\tilde{\mu}^{(1)}_{d_m,t}, \tilde{\mu}^{(1)}_{d_{m+1},t}$ are charging order-compliant, we have

$$\bar{\mu}^{(1)}_{d_m,t} < \tilde{\mu}^{(1)}_{d_m,t}. \tag{193}$$

Then we define $\tilde{\boldsymbol{\mu}}^{(2)}_t$ as follows:

$$(\tilde{\mu}^{(2)}_{d_1,t},\cdots,\tilde{\mu}^{(2)}_{d_{m-1},t}) = (\bar{\mu}^{(1)}_{d_1,t},\cdots,\bar{\mu}^{(1)}_{d_{m-1},t}), \tag{194}$$
$$(\tilde{\mu}^{(2)}_{d_m,t},\tilde{\mu}^{(2)}_{d_{m+1},t}) = \mathcal{F}^{d_m,d_{m+1}}_{2,t}(\bar{\mu}^{(1)}_{d_m,t},\bar{\mu}^{(1)}_{d_{m+1},t}). \tag{195}$$

According to the induction hypothesis for 2 charging thresholds, we have

$$Q(\boldsymbol{x}_t,\boldsymbol{w}_t,\tilde{\boldsymbol{\mu}}^{(2)}_t) \leq Q(\boldsymbol{x}_t,\boldsymbol{w}_t,\bar{\boldsymbol{\mu}}^{(1)}_t) \leq Q(\boldsymbol{x}_t,\boldsymbol{w}_t,\boldsymbol{\mu}_t). \tag{196}$$

According to (193) and Corollary 4, we have

$$\tilde{\mu}^{(2)}_{d_{m+1},t} \leq \bar{\mu}^{(1)}_{d_{m+1},t}, \tag{197}$$
$$\bar{\mu}^{(1)}_{d_m,t} \leq \tilde{\mu}^{(2)}_{d_m,t} \leq \tilde{\mu}^{(1)}_{d_m,t}, \tag{198}$$
$$\tilde{\mu}^{(2)}_{d_m,t} \leq \left\lfloor \tilde{\mu}^{(2)}_{d_{m+1},t} \right\rfloor - (d_{m+1} - d_m) + 1, \tag{199}$$
$$\tilde{\mu}^{(2)}_{d_{m+1},t} \geq \left\lceil \tilde{\mu}^{(2)}_{d_m,t} \right\rceil + (d_{m+1} - d_m) - 1, \tag{200}$$
$$\tilde{\mu}^{(2)}_{d_m,t} \leq \left\lceil \bar{\mu}^{(1)}_{d_{m+1},t} \right\rceil - (d_{m+1} - d_m), \tag{201}$$
$$\tilde{\mu}^{(2)}_{d_{m+1},t} \geq \left\lfloor \bar{\mu}^{(1)}_{d_m,t} \right\rfloor + (d_{m+1} - d_m). \tag{202}$$

Step 2: Now we repeat the process in Step 1 for $i = 2,3,\cdots$ to define $\bar{\boldsymbol{\mu}}^{(i)}_t$ and $\tilde{\boldsymbol{\mu}}^{(i+1)}_t$:

$$(\bar{\mu}^{(i)}_{d_1,t},\cdots,\bar{\mu}^{(i)}_{d_m,t}) = \mathcal{F}^{d_1,\cdots,d_m}_{m,t}(\tilde{\mu}^{(i)}_{d_1,t},\cdots,\tilde{\mu}^{(i)}_{d_m,t}), \tag{203}$$
$$\bar{\mu}^{(i)}_{d_{m+1},t} = \tilde{\mu}^{(i)}_{d_{m+1},t}, \tag{204}$$
$$(\tilde{\mu}^{(i+1)}_{d_1,t},\cdots,\tilde{\mu}^{(i+1)}_{d_{m-1},t}) = (\bar{\mu}^{(i)}_{d_1,t},\cdots,\bar{\mu}^{(i)}_{d_{m-1},t}), \tag{205}$$
$$(\tilde{\mu}^{(i+1)}_{d_m,t},\tilde{\mu}^{(i+1)}_{d_{m+1},t}) = \mathcal{F}^{d_m,d_{m+1}}_{2,t}(\bar{\mu}^{(i)}_{d_m,t},\bar{\mu}^{(i)}_{d_{m+1},t}). \tag{206}$$

According to the induction hypothesis for 2 charging thresholds, we have

$$Q(\boldsymbol{x}_t,\boldsymbol{w}_t,\bar{\boldsymbol{\mu}}^{(i)}_t) \leq Q(\boldsymbol{x}_t,\boldsymbol{w}_t,\boldsymbol{\mu}_t), \tag{207}$$
$$Q(\boldsymbol{x}_t,\boldsymbol{w}_t,\tilde{\boldsymbol{\mu}}^{(i+1)}_t) \leq Q(\boldsymbol{x}_t,\boldsymbol{w}_t,\boldsymbol{\mu}_t). \tag{208}$$

Besides, we have the following properties:

$$\bar{\mu}^{(i)}_{d_j,t} \geq \tilde{\mu}^{(i)}_{d_j,t},\ j = 1,\cdots,m-1, \tag{209}$$
$$\bar{\mu}^{(i-1)}_{d_m,t} \leq \bar{\mu}^{(i)}_{d_m,t} \leq \tilde{\mu}^{(i)}_{d_m,t}, \tag{210}$$
$$\tilde{\mu}^{(i+1)}_{d_{m+1},t} \leq \bar{\mu}^{(i)}_{d_{m+1},t}, \tag{211}$$
$$\bar{\mu}^{(i)}_{d_m,t} \leq \tilde{\mu}^{(i+1)}_{d_m,t} \leq \tilde{\mu}^{(i)}_{d_m,t}, \tag{212}$$
$$\bar{\mu}^{(i)}_{d_m,t} \geq \left\lfloor \tilde{\mu}^{(i)}_{d_{m-1},t} \right\rfloor + (d_m - d_{m-1}), \tag{213}$$
$$\tilde{\mu}^{(i+1)}_{d_m,t} \leq \left\lceil \bar{\mu}^{(i)}_{d_{m+1},t} \right\rceil - (d_{m+1} - d_m). \tag{214}$$

If for some $i \in \{2,3,\cdots\}$, $\bar{\boldsymbol{\mu}}^{(i)}_t$ or $\tilde{\boldsymbol{\mu}}^{(i+1)}_t$ is order-compatible, then we can finish our proof. Otherwise, we have the following infinite sequences:

$$\mu_{d_j,t} = \tilde{\mu}^{(1)}_{d_j,t} \leq \bar{\mu}^{(1)}_{d_j,t} = \tilde{\mu}^{(2)}_{d_j,t} \leq \bar{\mu}^{(2)}_{d_j,t} = \cdots,\ 1 \leq j \leq m-1, \tag{215}$$
$$\mu_{d_m,t} \leq \bar{\mu}^{(1)}_{d_m,t} \leq \bar{\mu}^{(2)}_{d_m,t} \leq \cdots, \tag{216}$$
$$\tilde{\mu}^{(1)}_{d_m,t} \geq \tilde{\mu}^{(2)}_{d_m,t} \geq \cdots, \tag{217}$$
$$\mu_{d_{m+1},t} \geq \tilde{\mu}^{(1)}_{d_{m+1},t} = \bar{\mu}^{(1)}_{d_{m+1},t} \geq \tilde{\mu}^{(2)}_{d_{m+1},t} = \bar{\mu}^{(2)}_{d_{m+1},t} \geq \cdots. \tag{218}$$

We can verify that all the infinite sequences are bounded by 0 and $\mu_{d_{m+1},t}$. Since monotonic sequence converges if it is bounded [3], we have

$$\lim_{i\to\infty} \tilde{\mu}^{(i)}_{d_j,t} = \lim_{i\to\infty} \bar{\mu}^{(i)}_{d_j,t} = \mu^{(\infty)}_{d_j,t}, 1 \leq j \leq m-1, \tag{219}$$
$$\lim_{i\to\infty} \bar{\mu}^{(i)}_{d_m,t} = \bar{\mu}^{(\infty)}_{d_m,t}, \tag{220}$$
$$\lim_{i\to\infty} \tilde{\mu}^{(i)}_{d_m,t} = \tilde{\mu}^{(\infty)}_{d_m,t}, \tag{221}$$
$$\lim_{i\to\infty} \tilde{\mu}^{(i)}_{d_{m+1},t} = \lim_{i\to\infty} \bar{\mu}^{(i)}_{d_{m+1},t} = \mu^{(\infty)}_{d_{m+1},t}. \tag{222}$$
$$\tag{223}$$

And according to (210), we have

$$\bar{\mu}^{(\infty)}_{d_m,t} \leq \tilde{\mu}^{(\infty)}_{d_m,t}. \tag{224}$$

Step 3: In this step, we will show that $\bar{\mu}^{(\infty)}_{d_m,t} < \tilde{\mu}^{(\infty)}_{d_m,t}$, and both $\bar{\mu}^{(\infty)}_{d_m,t}$ and $\tilde{\mu}^{(\infty)}_{d_m,t}$ are integers.

Let $k'_{d_{m+1}} = \min\{i | \left|\tilde{\mu}^{(i+1)}_{d_{m+1},t} - \mu^{(\infty)}_{d_{m+1},t}\right| < 1\}$, then there is at most one integer $k''_{d_{m+1}} > k'_{d_{m+1}}$ such that $\tilde{\mu}^{(k''_{d_{m+1}}+1)}_{d_{m+1},t}$ is an integer and different from $\bar{\mu}^{(k''_{d_{m+1}})}_{d_{m+1},t}$. Otherwise, if two numbers $k'''_{d_{m+1}} > k''_{d_{m+1}} > k'_{d_{m+1}}$ satisfy

1) $\tilde{\mu}^{(k''_{d_{m+1}}+1)}_{d_{m+1},t}$ is an integer and different from $\bar{\mu}^{(k''_{d_{m+1}})}_{d_{m+1},t}$,
2) $\tilde{\mu}^{(k'''_{d_{m+1}}+1)}_{d_{m+1},t}$ is an integer and different from $\bar{\mu}^{(k'''_{d_{m+1}})}_{d_{m+1},t}$,

then according to the monotonicity described in (218), we have

$$\left|\tilde{\mu}^{(k'''_{d_{m+1}}+1)}_{d_{m+1},t} - \tilde{\mu}^{(k'_{d_{m+1}}+1)}_{d_{m+1},t}\right| > 1, \tag{225}$$

which contradicts with the definition of $k'_{d_{m+1}}$.

Then we let $k_{d_{m+1}} = k''_{d_{m+1}}$(or $k_{d_{m+1}} = k'_{d_{m+1}}$ if $k''_{d_{m+1}}$ does not exist). Then $\forall i > k_{d_{m+1}}$, the situation where $\tilde{\mu}^{(i+1)}_{d_{m+1},t}$ is an integer different from $\bar{\mu}^{(i)}_{d_{m+1},t}$ will not happen. Therefore, according to Corollary 2, $\forall i > k_{d_{m+1}}$, $\tilde{\mu}^{(i+1)}_{d_m,t}$ is an integer different from $\bar{\mu}^{(i)}_{d_m,t}$.

Similarly, we can find k_{d_j} for each $j \in \{1, \cdots, m-1\}$ such that $\forall i > k_{d_j}$, the situation where $\bar{\mu}^{(i)}_{d_j,t}$ is an integer different from $\tilde{\mu}^{(i)}_{d_j,t}$ will not happen. Let $k' = \max\{k_{d_1}, \cdots, k_{d_{m-1}}\}$, according to Property 2, $\forall i > k'$, $\bar{\mu}^{(i)}_{d_m,t}$ is an integer different from $\tilde{\mu}^{(i)}_{d_m,t}$.

Therefore, $\forall i > \max(k_{d_{m+1}}, k')$, $\bar{\mu}^{(i)}_{d_m,t}$ is an integer different from $\tilde{\mu}^{(i)}_{d_m,t}$, and $\tilde{\mu}^{(i+1)}_{d_m,t}$ is an integer different from $\bar{\mu}^{(i)}_{d_m,t}$. Considering the monotonicity described in (216) and (217), there is some integer K such that $\forall i \geq K$, $\tilde{\mu}^{(i)}_{d_m,t} = \tilde{\mu}^{(\infty)}_{d_m,t}$, and $\bar{\mu}^{(i)}_{d_m,t} = \bar{\mu}^{(\infty)}_{d_m,t}$. This shows that $\tilde{\mu}^{(\infty)}_{d_m,t}$ and $\bar{\mu}^{(\infty)}_{d_m,t}$ are two different integers. Combining (224), we have

$$\bar{\mu}^{(\infty)}_{d_m,t} < \tilde{\mu}^{(\infty)}_{d_m,t}. \tag{226}$$

Step 4: In this step, we will prove that changing the charging threshold of deadline d_m from $\bar{\mu}^{(\infty)}_{d_m,t}$ to $\tilde{\mu}^{(\infty)}_{d_m,t}$ will not alter actual charging power of any chargers. In other word, $\forall l \in \mathcal{G}_{d_m,t}$, we have $e_{l,t} \leq \bar{\mu}^{(\infty)}_{d_m,t}$ or $e_{l,t} \geq \tilde{\mu}^{(\infty)}_{d_m,t} + 1$.

We will prove this by contradiction. First we define the (normalized) total charging rate that is needed to charge chargers with deadline d to charging threshold μ_d as:

$$C^d_t(\mu_d) = \sum_{i \in \mathcal{G}_{d,t}} \min(\max(e_{i,t} - \mu_d, 0), 1). \tag{227}$$

With this definition, if the claim does not hold, then $\exists B > 0$, $\forall i \in \mathbb{N}$, such that

$$\left| C^{d_m}_t(\tilde{\mu}^{(i+1)}_{d_m}) - C^{d_m}_t(\bar{\mu}^{(i)}_{d_m}) \right| \geq B. \tag{228}$$

Then by (111)(114) we have

$$\left| C^{d_{m+1}}_t(\tilde{\mu}^{(i+1)}_{d_{m+1}}) - C^{d_{m+1}}_t(\bar{\mu}^{(i)}_{d_{m+1}}) \right| \geq B. \tag{229}$$

However, by (222) we have

$$\begin{aligned} &\left| C^{d_{m+1}}_t(\tilde{\mu}^{(i+1)}_{d_{m+1}}) - C^{d_{m+1}}_t(\bar{\mu}^{(i)}_{d_{m+1}}) \right| \\ \leq& \left| \mathcal{G}_{d_{m+1},t} \right| \left| \tilde{\mu}^{(i+1)}_{d_{m+1}} - \bar{\mu}^{(i)}_{d_{m+1}} \right| \\ \to& 0, \ i \to \infty, \end{aligned} \tag{230}$$

which contradicts with (229).

Step 5: Recall that in Step 3, we prove that there is some integer K such that

1) $\bar{\mu}^{(K)}_{d_m,t} = \bar{\mu}^{(\infty)}_{d_m,t} < \tilde{\mu}^{(\infty)}_{d_m,t} = \tilde{\mu}^{(K)}_{d_m,t}$,
2) $\bar{\mu}^{(K)}_{d_1,t}, \cdots, \bar{\mu}^{(K)}_{d_{m-1},t}, \bar{\mu}^{(K)}_{d_m,t}$ are charging order-compliant,
3) $\tilde{\mu}^{(K)}_{d_m,t}, \tilde{\mu}^{(K)}_{d_{m+1},t}$ are charging order-compliant.

In the rest of Step 5, we will prove that $\bar{\mu}^{(K)}_{d_{m-1},t}$ and $\tilde{\mu}^{(K)}_{d_{m+1},t}$ are not charging order-compliant.

We will prove this by contradiction. If $\bar{\mu}^{(K)}_{d_{m-1},t}$ and $\tilde{\mu}^{(K)}_{d_{m+1},t}$ are charging order-compliant, by Lemma 3 we have

$$\tilde{\mu}^{(K)}_{d_{m+1},t} \leq \left\lfloor \bar{\mu}^{(K)}_{d_{m-1},t} \right\rfloor + d_{m+1} - d_{m-1}, \tag{231}$$

and therefore

$$\left\lceil \tilde{\mu}^{(K)}_{d_{m+1},t} \right\rceil - \left\lfloor \bar{\mu}^{(K)}_{d_{m-1},t} \right\rfloor \leq d_{m+1} - d_{m-1}. \tag{232}$$

On the other hand, by (205)(213) we have

$$\bar{\mu}^{(K+1)}_{d_m,t} - \left\lfloor \bar{\mu}^{(K)}_{d_{m-1},t} \right\rfloor \geq d_m - d_{m-1}. \tag{233}$$

By (204)(214) we have

$$\left\lceil \tilde{\mu}^{(K)}_{d_{m+1},t} \right\rceil - \tilde{\mu}^{(K+1)}_{d_m,t} \geq d_{m+1} - d_m. \tag{234}$$

Since $\bar{\mu}^{(K+1)}_{d_m,t}$ and $\tilde{\mu}^{(K+1)}_{d_m,t}$ are two different integers, we have

$$\tilde{\mu}^{(K+1)}_{d_m,t} - \bar{\mu}^{(K+1)}_{d_m,t} \geq 1. \tag{235}$$

Combining (233)(234)(235) we have

$$\left\lceil \tilde{\mu}^{(K)}_{d_{m+1},t} \right\rceil - \left\lfloor \bar{\mu}^{(K)}_{d_{m-1},t} \right\rfloor \geq d_{m+1} - d_{m-1} + 1, \tag{236}$$

which contradicts with (232).

Step 6: Now we define $\tilde{\mu}_{d_1,t}, \cdots, \tilde{\mu}_{d_{m-1},t}, \tilde{\mu}_{d_{m+1},t}$ as follows:

$$\begin{aligned} &(\tilde{\mu}_{d_1,t}, \cdots, \tilde{\mu}_{d_{m-1},t}, \tilde{\mu}_{d_{m+1},t}) \\ =& \mathcal{F}^{d_1,\cdots,d_{m-1},d_{m+1}}_{m,t}(\bar{\mu}^{(K)}_{d_1,t}, \cdots, \bar{\mu}^{(K)}_{d_{m-1},t}, \tilde{\mu}^{(K)}_{d_{m+1},t}). \end{aligned} \tag{237}$$

Note that we have not defined the value of $\tilde{\mu}_{d_m,t}$. Instead, we will first find the range of $\tilde{\mu}_{d_m,t}$ such that $\tilde{\mu}_{d_1,t}, \cdots, \tilde{\mu}_{d_{m-1},t}, \tilde{\mu}_{d_m,t}, \tilde{\mu}_{d_{m+1},t}$ is order compliant. Since $\tilde{\mu}_{d_1,t}, \cdots, \tilde{\mu}_{d_{m-1},t}, \tilde{\mu}_{d_{m+1},t}$ have been charging order-compliant, all we need is that $\tilde{\mu}_{d_{m-1},t}, \tilde{\mu}_{d_m,t}, \tilde{\mu}_{d_{m+1},t}$ are charging order-compliant. According to Lemma 3, this is equivalent to

$$\begin{aligned} \tilde{\mu}_{d_m,t} \in& \left[\left\lceil \tilde{\mu}_{d_{m-1},t} \right\rceil, \left\lfloor \tilde{\mu}_{d_{m-1},t} \right\rfloor + (d_m - d_{m-1}) \right] \\ &\cap \left[\left\lceil \tilde{\mu}_{d_{m+1},t} \right\rceil - (d_{m+1} - d_m), \left\lfloor \tilde{\mu}_{d_{m+1},t} \right\rfloor \right]. \end{aligned} \tag{238}$$

In the rest of step 6, we will prove that (238) is equivalent to the following condition:

$$\begin{aligned} &\tilde{\mu}_{d_m,t} \\ \in& \left[\left\lceil \tilde{\mu}_{d_{m+1},t} \right\rceil - (d_{m+1} - d_m), \left\lfloor \tilde{\mu}_{d_{m-1},t} \right\rfloor + (d_m - d_{m-1}) \right]. \end{aligned} \tag{239}$$

First, by (173) we have

$$\begin{aligned} \tilde{\mu}_{d_{m-1},t} &\leq \left\lfloor \tilde{\mu}_{d_{m+1},t} \right\rfloor - (d_{m+1} - d_{m-1}) + 1 \\ &\leq \left\lceil \tilde{\mu}_{d_{m+1},t} \right\rceil - d_{m+1} + (d_{m-1} + 1) \\ &\leq \left\lceil \tilde{\mu}_{d_{m+1},t} \right\rceil - (d_{m+1} - d_m), \end{aligned} \tag{240}$$

and therefore

$$\left\lceil \tilde{\mu}_{d_{m-1},t} \right\rceil \leq \left\lceil \tilde{\mu}_{d_{m+1},t} \right\rceil - (d_{m+1} - d_m). \tag{241}$$

Second, by Lemma 3 we have

$$\tilde{\mu}_{d_{m+1},t} \leq \left\lfloor \tilde{\mu}_{d_{m-1},t} \right\rfloor + (d_{m+1} - d_{m-1}), \tag{242}$$

and therefore

$$\left\lceil \tilde{\mu}_{d_{m+1},t} \right\rceil - (d_{m+1} - d_m) \leq \left\lfloor \tilde{\mu}_{d_{m-1},t} \right\rfloor + (d_m - d_{m-1}). \tag{243}$$

Third, by (174) we have

$$\begin{aligned} \tilde{\mu}_{d_{m+1},t} &\geq \left\lceil \tilde{\mu}_{d_{m-1},t} \right\rceil + (d_{m+1} - d_{m-1}) - 1 \\ &\geq \left\lfloor \tilde{\mu}_{d_{m-1},t} \right\rfloor + (d_{m+1} - 1) - d_{m-1} \\ &\geq \left\lfloor \tilde{\mu}_{d_{m-1},t} \right\rfloor + (d_m - d_{m-1}), \end{aligned} \tag{244}$$

and therefore

$$\left\lfloor \tilde{\mu}_{d_{m-1},t} \right\rfloor + (d_m - d_{m-1}) \leq \left\lfloor \tilde{\mu}_{d_{m+1},t} \right\rfloor. \tag{245}$$

From (241)(243)(245) we know (238) and (239) are equivalent.

Step 7: In this step, with the following definition:

$$\mathcal{S}_1 = \left[\lceil\tilde{\mu}_{d_{m+1},t}\rceil - (d_{m+1} - d_m), \lfloor\tilde{\mu}_{d_{m-1},t}\rfloor + d_m - d_{m-1}\right], \tag{246}$$

$$\mathcal{S} = \mathcal{S}_1 \cap [\bar{\mu}^{(\infty)}_{d_m,t}, \tilde{\mu}^{(\infty)}_{d_m,t}], \tag{247}$$

we will show that $\mathcal{S}$ is not empty.

We prove this by contradiction. If $\mathcal{S}$ is empty, and one of (248) and (249) must hold:

$$\tilde{\mu}^{(\infty)}_{d_m,t} < \lceil\tilde{\mu}_{d_{m+1},t}\rceil - (d_{m+1} - d_m), \tag{248}$$
$$\bar{\mu}^{(\infty)}_{d_m,t} > \lfloor\tilde{\mu}_{d_{m-1},t}\rfloor + d_m - d_{m-1}. \tag{249}$$

However, since $\tilde{\mu}^{(\infty)}_{d_m,t}(= \tilde{\mu}^{(K)}_{d_m,t}), \tilde{\mu}^{(K)}_{d_{m+1},t}$ are charging order-compliant, by Lemma 3 we have

$$\tilde{\mu}^{(\infty)}_{d_m,t} \geq \left\lceil\tilde{\mu}^{(K)}_{d_{m+1},t}\right\rceil - (d_{m+1} - d_m). \tag{250}$$

On the other hand, since $\bar{\mu}^{(K)}_{d_{m-1},t}, \bar{\mu}^{(K)}_{d_m,t}$ are charging order-compliant, if we let $\mathring{\mu}_{d_{m+1},t} = \bar{\mu}^{(K)}_{d_m,t}$ (for charging deadline d_{m+1}), from Lemma 3 we know $\bar{\mu}^{(K)}_{d_{m-1},t}$ and $\mathring{\mu}_{d_{m+1},t}$ will be charging order-compliant. Since by Step 5 we know $\bar{\mu}^{(K)}_{d_{m-1},t}, \tilde{\mu}^{(K)}_{d_{m+1},t}$ are not charging order-compliant and $\mathring{\mu}_{d_{m+1},t} < \tilde{\mu}^{(K)}_{d_m,t} \leq \tilde{\mu}^{(K)}_{d_{m+1},t}$, by (172) in Property 1 we have

$$\tilde{\mu}_{d_{m+1},t} \leq \tilde{\mu}^{(K)}_{d_{m+1},t}, \tag{251}$$

and therefore

$$\tilde{\mu}^{(\infty)}_{d_m,t} \geq \lceil\tilde{\mu}_{d_{m+1},t}\rceil - (d_{m+1} - d_m), \tag{252}$$

which contradicts with (248).

Similarly, we have

$$\begin{aligned}\bar{\mu}^{(\infty)}_{d_m,t} &\leq \left\lfloor\bar{\mu}^{(K)}_{d_{m-1},t}\right\rfloor + (d_m - d_{m-1}) \\ &\leq \lfloor\tilde{\mu}_{d_{m-1},t}\rfloor + (d_m - d_{m-1}),\end{aligned} \tag{253}$$

which contradicts with (249). And we can conclude that $\mathcal{S}$ is not empty.

Step 8: Now we can choose $\tilde{\mu}_{d_m,t} \in \mathcal{S}$, then $\tilde{\mu}_{d_{m-1},t}, \tilde{\mu}_{d_m,t}, \tilde{\mu}_{d_{m+1},t}$ are charging order-compliant. In the proof process above, we can see that change charging thresholds $\mu_{d_1,t}, \cdots, \mu_{d_{m-1},t}, \mu_{d_m,t}, \mu_{d_{m+1},t}$ to $\bar{\mu}^{(K)}_{d_1,t}, \cdots, \bar{\mu}^{(K)}_{d_{m-1},t}, \bar{\mu}^{(K)}_{d_m,t}, \tilde{\mu}^{(K)}_{d_{m+1},t}$, and further to $\tilde{\mu}_{d_1,t}, \cdots, \tilde{\mu}_{d_{m-1},t}, \bar{\mu}^{(K)}_{d_m,t}, \tilde{\mu}_{d_{m+1},t}$ will do no harm to the charging performance. From the results of Step 4, we know that changing $\bar{\mu}^{(K)}_{d_m,t}$ to $\tilde{\mu}_{d_m,t} \in [\bar{\mu}^{(\infty)}_{d_m,t}, \tilde{\mu}^{(\infty)}_{d_m,t}]$ will not alter charging behavior of any chargers. As the conclusion, $\tilde{\mu}_{d_1,t}, \cdots, \tilde{\mu}_{d_m,t}, \tilde{\mu}_{d_{m+1},t}$ are charging order-compliant, and $Q(\boldsymbol{x}_t, \boldsymbol{w}_t, \tilde{\boldsymbol{\mu}}_t) \leq Q(\boldsymbol{x}_t, \boldsymbol{w}_t, \boldsymbol{\mu}_t)$.

The only thing we need to do next is to prove Property 1 and Property 2 for the case of $m+1$ charging thresholds.

Step 9: We now prove Property 1 for $m+1$ charging thresholds. For the consistency of notations, we suppose $\mu_{d_1,t}, \cdots, \mu_{d_{m-1},t}, \mu_{d_m,t}, \mu'_{d_{m+1},t}$ are charging order-compliant, $\mu_{d_{m+1},t} > \mu'_{d_{m+1},t}$ is another charging threshold for deadline d_{m+1} with $\mu_{d_m,t}, \mu_{d_{m+1},t}$ not charging order-compliant. Let

$$(\tilde{\mu}^{(1)}_{d_m,t}, \tilde{\mu}^{(1)}_{d_{m+1},t}) = \mathcal{F}^{d_m,d_{m+1}}_{2,t}(\mu_{d_m,t}, \mu_{d_{m+1},t}), \tag{254}$$

then by Corollary 3 we have

$$\tilde{\mu}^{(1)}_{d_m,t} \geq \mu_{d_m,t}. \tag{255}$$

If $\tilde{\mu}^{(1)}_{d_m,t} = \mu_{d_m,t}$, then $\tilde{\mu}^{(1)}_{d_1,t}, \cdots, \tilde{\mu}^{(1)}_{d_{m-1},t}, \tilde{\mu}^{(1)}_{d_m,t}, \tilde{\mu}^{(1)}_{d_{m+1},t}$ are charging order-compliant, and Property 1 can be established by Corollary 3. Otherwise, we have

$$\tilde{\mu}^{(1)}_{d_m,t} > \mu_{d_m,t}, \tag{256}$$

which satisfies the assumption (182) in Step 1. Therefore, all the properties established in steps above can be used in current step.

Note that the final $\tilde{\boldsymbol{\mu}}_t$ can be generated in Step1/Step2, where $\tilde{\mu}^{(i)}_{d_1,t}, \cdots, \tilde{\mu}^{(i)}_{d_m,t}, \tilde{\mu}^{(i)}_{d_{m+1},t}$ or $\bar{\mu}^{(i)}_{d_1,t}, \cdots, \bar{\mu}^{(i)}_{d_m,t}, \bar{\mu}^{(i)}_{d_{m+1},t}$ is order-compatible for some finite i, or in Step6 and Step8 otherwise. We now prove the more complex case where $\tilde{\boldsymbol{\mu}}_t$ is generated in Step6 and Step8, and the case where $\tilde{\boldsymbol{\mu}}_t$ is generated in Step1/Step2 can be proved likewise.

Now we establish some more properties that $\tilde{\mu}_{d_1,t}, \cdots, \tilde{\mu}_{d_{m-1},t}, \tilde{\mu}_{d_{m+1},t}$ in Step 6 should satisfy. Note that if we define $\hat{\mu}^{(K)}_{d_{m+1},t} = \left\lfloor\bar{\mu}^{(K)}_{d_{m-1},t}\right\rfloor + d_{m+1} - d_{m-1}$, then $\bar{\mu}^{(K)}_{d_{m-1},t}, \hat{\mu}^{(K)}_{d_{m+1},t}$ are charging order-compliant. Also by (236) we have

$$\tilde{\mu}^{(K)}_{d_{m+1},t} > \hat{\mu}^{(K)}_{d_{m+1},t}. \tag{257}$$

Then according to Property 1 for m thresholds, we have

$$\tilde{\mu}_{d_j,t} \geq \bar{\mu}^{(K)}_{d_j,t},\ j = 1, \cdots, m-1, \tag{258}$$
$$\hat{\mu}^{(K)}_{d_{m+1},t} \leq \tilde{\mu}_{d_{m+1},t} \leq \tilde{\mu}^{(K)}_{d_{m+1},t}, \tag{259}$$
$$\tilde{\mu}_{d_{m+1},t} \geq \left\lfloor\bar{\mu}^{(K)}_{d_{m-1},t}\right\rfloor + (d_{m+1} - d_{m-1}), \tag{260}$$
$$\tilde{\mu}_{d_{m-1},t} \leq \lfloor\tilde{\mu}_{d_{m+1},t}\rfloor - (d_{m+1} - d_{m-1}) + 1. \tag{261}$$

Then by (215)(258) we know

$$\tilde{\mu}_{d_j,t} \geq \mu_{d_j,t},\ j = 1, \cdots, m-1. \tag{262}$$

Also, by (216) and $\tilde{\mu}_{d_m,t} \in \mathcal{S}$ we have

$$\tilde{\mu}_{d_m,t} \geq \mu_{d_m,t}. \tag{263}$$

Since $\mu_{d_{m-1},t}, \mu_{d_m,t}, \mu'_{d_{m+1},t}$ is order compliant, by Lemma 3 we have

$$\mu'_{d_{m+1},t} \leq \lfloor\mu_{d_{m-1},t}\rfloor + (d_{m+1} - d_{m-1}), \tag{264}$$
$$\mu_{d_m,t} \leq \lfloor\mu_{d_{m-1},t}\rfloor + (d_m - d_{m-1}). \tag{265}$$

Then by (215)(260)(264) we have

$$\tilde{\mu}_{d_{m+1},t} \geq \mu'_{d_{m+1},t}. \tag{266}$$

By (218)(259) we have

$$\tilde{\mu}_{d_{m+1},t} \leq \mu_{d_{m+1},t}. \tag{267}$$

By (215)(260)(265) we have

$$\tilde{\mu}_{d_{m+1},t} \geq \lfloor\mu_{d_m,t}\rfloor + (d_{m+1} - d_m). \tag{268}$$

Since $\mu_{d_{m+1},t} > \mu'_{d_{m+1},t}$, according to (179) and Corollary 3 we have

$$\tilde{\mu}^{(1)}_{d_m,t} \leq \lceil \mu_{d_{m+1},t} \rceil - (d_{m+1} - d_m). \tag{269}$$

Then by (217)(269) and $\tilde{\mu}_{d_m,t} \in \mathcal{S}$ we have

$$\tilde{\mu}_{d_m,t} \leq \lceil \mu_{d_{m+1},t} \rceil - (d_{m+1} - d_m). \tag{270}$$

Since $\tilde{\mu}_{d_m,t} \in \mathcal{S}$, we have

$$\tilde{\mu}_{d_m,t} \leq \lfloor \tilde{\mu}_{d_{m-1},t} \rfloor + d_m - d_{m-1}. \tag{271}$$

Combining (261)(271) we have

$$\tilde{\mu}_{d_m,t} \leq \lfloor \tilde{\mu}_{d_{m+1},t} \rfloor - (d_{m+1} - d_m) + 1, \tag{272}$$

and equivalently

$$\tilde{\mu}_{d_{m+1},t} \geq \lceil \tilde{\mu}_{d_m,t} \rceil + (d_{m+1} - d_m) - 1. \tag{273}$$

Now we have finished the proof for Property 1 for $m+1$ thresholds.

Step 10: We now prove Property 2 for $m+1$ thresholds. According to how the final modified charging thresholds $\tilde{\boldsymbol{\mu}}_t$ are generated, we can discuss in following 3 cases:

1) $\tilde{\boldsymbol{\mu}}_t$ is generated in Step 1 or Step 2, and $\tilde{\boldsymbol{\mu}}_t = \tilde{\boldsymbol{\mu}}^{(i)}_t$ for some $i \in \{1, 2, \cdots\}$. In this case, if $\tilde{\mu}^{(i)}_{d_m,t}$ is an integer different from $\bar{\mu}^{(i-1)}_{d_m,t}$, then by (212)(216) we know $\tilde{\mu}^{(i)}_{d_m,t}$ is an integer different from $\mu_{d_m,t}$. If $\tilde{\mu}^{(i)}_{d_{m+1},t}$ is an integer different from $\bar{\mu}^{(i-1)}_{d_{m+1},t}$, then by (211)(218) we know $\tilde{\mu}^{(i)}_{d_{m+1},t}$ is an integer different from $\mu_{d_{m+1},t}$.
2) $\tilde{\boldsymbol{\mu}}_t$ is generated in Step 1 or Step 2, and $\tilde{\boldsymbol{\mu}}_t = \bar{\boldsymbol{\mu}}^{(i)}_t$ for some $i \in \{1, 2, \cdots\}$. In this case, if for some $j \in \{1, \cdots, m-1\}$, $\bar{\mu}^{(i)}_{d_j,t}$ is an integer different from $\tilde{\mu}^{(i)}_{d_j,t}$. Then by (215) we know $\bar{\mu}^{(i)}_{d_j,t}$ is an integer different from $\mu_{d_j,t}$.
Otherwise, according to Property 2 for m thresholds, $\bar{\mu}^{(i)}_{d_m,t}$ is an integer different from $\tilde{\mu}^{(i)}_{d_m,t}$. By Corollary 3 we know that

$$\tilde{\mu}^{(i)}_{d_m,t} \leq \left\lfloor \tilde{\mu}^{(i)}_{d_{m+1},t} \right\rfloor - (d_{m+1} - d_m) + 1, \tag{274}$$

and since $\bar{\mu}^{(i)}_{d_m,t}$ is an integer different from $\tilde{\mu}^{(i)}_{d_m,t}$, with $\bar{\mu}^{(i)}_{d_{m+1},t} = \tilde{\mu}^{(i)}_{d_{m+1},t}$ we have

$$\bar{\mu}^{(i)}_{d_m,t} \leq \left\lfloor \bar{\mu}^{(i)}_{d_{m+1},t} \right\rfloor - (d_{m+1} - d_m). \tag{275}$$

However, since $\bar{\mu}^{(i)}_{d_m,t}, \bar{\mu}^{(i)}_{d_{m+1},t}$ are charging order-compliant, and by Lemma 3 we know

$$\bar{\mu}^{(i)}_{d_m,t} \geq \left\lceil \bar{\mu}^{(i)}_{d_{m+1},t} \right\rceil - (d_{m+1} - d_m). \tag{276}$$

From (275)(276) we know $\bar{\mu}^{(i)}_{d_{m+1},t}$ is an integer. Since $\mu_{d_m,t}, \mu_{d_{m+1},t}$ is not order compliant, we conclude that either $\bar{\mu}^{(i)}_{d_m,t}$ is an integer different from $\mu_{d_m,t}$, or $\bar{\mu}^{(i)}_{d_{m+1},t}$ is an integer different from $\mu_{d_{m+1},t}$.
3) $\tilde{\boldsymbol{\mu}}_t$ is generated in Step 6 and Step 8. Then applying Property 2 to (237) we know there is some $j \in \{1 \cdots, m-1, m+1\}$ such that $\tilde{\mu}_{d_j,t}$ is an integer different from $\bar{\mu}^{(K)}_{d_j,t}$. Then according to (215) or (218) we know $\tilde{\mu}_{d_j,t}$ is an integer different from $\mu_{d_j,t}$.

□

Now we can establish the proof for Theorem 1.

Proof. It is shown in [2] that there exists an optimal DTC-type charging policy, and we denote it as π. Then according to Lemma 6, we can convert π into an DTC-type optimal charging policy π^*, such that for any state $(\boldsymbol{x}_t, \boldsymbol{w}_t)$, let $\boldsymbol{\mu}_t = \pi^*_t(\boldsymbol{x}_t, \boldsymbol{w}_t)$, then $\mu_{t+1,t}, \cdots, \mu_{T,t}$ are charging order-compliant.

We will prove that policy π^* is priority rule-compliant. We prove it by contradiction. Suppose π^* is not priority rule-compliant, then there is a timeslot t and two chargers i and j with $t < \min(d_i, d_j)$, whose charging decisions given by π^* are $v_{i,t}$ and $v_{j,t}$ satisfying

1) $\exists \epsilon > 0$ such that $(e_{i,t+1} + \epsilon, d_i) \prec (e_{j,t+1}, d_j)$,
2) $v_{i,t} > 0$,
3) $v_{j,t} < \min(1, e_{j,t})$.

Then by 2) we know $\mu_{d_i,t} \leq e_{i,t+1}$, by 3) we know $\mu_{d_j,t} \geq e_{j,t+1}$, and combining with 1) we know $\exists \epsilon > 0$ such that

$$(\mu_{d_i,t} + \epsilon, d_i) \prec (\mu_{d_j,t}, d_j), \tag{277}$$

which contradicts with the fact that $\mu_{d_i,t}$ and $\mu_{d_j,t}$ are charging order-compliant. This suggests that π^* is priority rule-compliant.

□

References

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