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Exam of MDI 210

Duration: 3 h.
Authorized documents: two double-sided sheets (four pages), A4 size;
dictionary authorized for students whose mother tongue is not French.
Calculators and computers are forbidden,
as well as any object allowing to communicate with the outside.
The exam consists of four independent exercises.
An unjustified result may be considered as false. A result obtained by a method which is not the one indicated in the statement may be considered as false.
Detail the calculations, even when not explicitly requested.

Exercise 1

Let A be the following matrix: $A = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix}$.

1. Give the LU decomposition of A; detail the calculations.
2. Deduce from the LU decomposition the solution of the linear system $AX = b$ where $b = (b_1, b_2, b_3, b_4)^t$ is a given vector of $\mathbf{R}^4$ (specify the components of X using those of b).
3. If A is invertible, deduce from the above the expression of A^{-1}; otherwise, justify the fact that A is not invertible.

Exercise 2

Let A be the following matrix: $A = \begin{pmatrix} 16 & -11 & -6\sqrt{2} \\ -11 & 16 & 6\sqrt{2} \\ -6\sqrt{2} & 6\sqrt{2} & 34 \end{pmatrix}$.

1. By applying Jacobi's method, compute the eigenvalues and a base of orthonormal eigenvectors of A.

Indication. We recall that the first squares of integers are the following numbers: 0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400, 441, 484, 529, 576, 625, 676, 729, 784, 841, 900...

2. Let b be a given vector of $\mathbf{R}^3$. We consider the linear system $AX = b$. Give the value of the conditioning of this system for the norm $\| \|_2$.

3. Let b be a given vector of $\mathbf{R}^3$. We focus on the global minimum on $\mathbf{R}^3$ of the quadratic form Q defined by $Q(X) = \frac{1}{2} X^t AX + bX$. For which values of b does the form Q reach a value which is a global minimum on $\mathbf{R}^3$? For these values, can there be several vectors X reaching this global minimum or is such a vector necessarily unique? For which vectors b are there local minima which are not the global minimum?

Exercise 3

We consider the following linear programming problem (P):

$$(P) \qquad \begin{array}{l} \text{Maximize } z(x_1, x_2) = 2x_1 + 3x_2 \\ \text{with the constraints } \begin{cases} 2x_1 + x_2 \le 14 \\ x_1 + x_2 \le 8 \\ x_1 + 3x_2 \le 18 \\ x_1 \ge 0, x_2 \ge 0 \end{cases} \end{array}$$

1. Solve the problem (P) by applying the simplex algorithm (one will meet fractions whose denominator contains only one digit); $\left(x_1^*, x_2^*\right)$ will denote the optimal solution of (P).
2. Specify the dual problem of (P) and give its optimal solution (detail how to obtain the optimal value of the dual variables).
3. We assume in this question that the function z is in fact:

$$z(x_1, x_2) = ax_1 + 3x_2$$

where a is a real parameter. Indicate for which values of a the solution $\left(x_1^*, x_2^*\right)$ remains optimal (one will not rely on a graphical reasoning).

Exercise 4

Let α be a real parameter. We consider the following problem (P_α):

$$\begin{array}{l} \text{Minimize } f_\alpha(x, y) = x^2 + y^2 + xy + \alpha x \\ \text{with the constraints } \begin{cases} x + y \ge 1 \\ x \ge 0 \end{cases} \end{array}$$

We will admit that, for every α, any local minimum of (P_α) is global.

1. Indicate, according to α, the points of the realizable domain where the constraints are qualified.
2. By applying the condition of Karush, Kuhn and Tucker, determine in terms of α the coordinates of the point where f_α reaches its minimum on the considered domain.
3. We now consider the problem (P_1) obtained for $\alpha = 1$. Find back the result of the previous question by applying the steepest feasible descent method (at each iteration, the feasible direction which is the direction of steepest descent is adopted) with optimal steps from the point (1, 0). A drawing will be made that clearly represents the situation; such a drawing can be used to justify the adopted directions or, at the end, to stop the method.