Exercise 2.5.1: solution

Let us denote

$$\operatorname{prox}_g(y) = \arg\min_{x\in\mathcal{X}} g(x) + \frac{1}{2}\|y-x\|^2$$

the proximal operator of g at y.
Fix $\gamma > 0$. Show that the fixed points of the nonlinear equation

$$x = \operatorname{prox}_{\gamma g}(x - \gamma\nabla f(x))$$

are the minimizers of the function $F = f + g$.
▶ Let us denote $h_y : x \mapsto \gamma g(x) + \frac{1}{2}\|y-x\|^2$ and $p = \operatorname{prox}_{\gamma g}(y)$. By Fermat's rule and Proposition 2.4.1,

$$0 \in \partial h_y(p) = \gamma\partial g(p) + \{p - y\}. \tag{1}$$

Let x^* be a fixed point of $x = \operatorname{prox}_{\gamma g}(x - \gamma\nabla f(x))$. We apply (1) with $y = x^* - \gamma\nabla f(x^*)$ and we get

$$0 \in \gamma\partial g(x^*) + \{x^* - (x^* - \gamma\nabla f(x^*))\} = \gamma\partial g(x^\star) + \gamma\{\nabla f(x^\star)\} = \gamma\partial(f+g)(x^*)$$

So, as $\gamma > 0$, we deduce that $0 \in \partial(f+g)(x^*)$, which is equivalent to $x^* \in \arg\min(f+g)$.

Conversely, let $x^* \in \arg\min(f+g)$. We have $0 \in \gamma\partial g(x^*) + \{x^* - (x^* - \gamma\nabla f(x^*))\}$, which implies that $x^* = \operatorname{prox}_{\gamma g}(x^* - \gamma\nabla f(x^*))$.