TD - Constrained optimization: solution

Exercise 14 (Gaussian Channel, Water filling)
The problem is

$$\max_{x\in\mathbb{R}^n} \sum_{i=1}^{n} \log(1+\alpha_i x_i) \quad \text{under constraints: } \forall i, x_i \geq 0, \quad \sum_{i=1}^{n} x_i \leq P. \tag{1}$$

1. *Write problem (1) as a minimization problem under constraint $g(x) \preceq 0$. Show that this is a convex problem (objective and constraints both convex).*

   ► The constraints can be written as $g(x) \leq 0$ by taking $g(x) = [-x_1, ..., -x_n, \sum_i x_i - P]$. For all j, g_j is affine so it is convex.

   For the objective, we just need to remark that $\max_x \sum_i \log(1+\alpha_i x_i) = -\min_x - \sum_i \log(1+\alpha_i x_i)$. The objective is the sum of the composition of an affine function with the convex function $(z \mapsto -\log(z))$.

2. *Show that the constraints are qualified. (hint: Slater).*

   ► x given by $x_i = P/(2n)$ is strictly feasible. Hence, by Slater's qualification condition, the constraints are qualified.

3. *Write the Lagrangian function.*

   ► $L(x,\phi,\nu) = -\sum_i log(1+\alpha_i x_i) + \langle -x, \phi\rangle + \nu(\sum_i x_i - P) - \iota_{\mathbb{R}_+^n}(\phi) - \iota_{\mathbb{R}_+}(\nu)$

4. *Using the KKT theorem, show that a primal optimal x^* exists and satisfies:*

   - $\exists K > 0$ *such that* $x_i = \max(0, K - 1/\alpha_i)$.
   - K *is given by*

   $$\sum_{i=1}^{n} \max(K - 1/\alpha_i, 0) = P$$

   ► There exists a solution x because the objective is continuous and the set of constraints is compact. There exist Lagrange multipliers because Slater's qualification condition holds.

   Let (x, ϕ, ν) be a saddle point to the Lagrangian. Then it must satisfy the KKT conditions

$$\forall i: -\frac{\alpha_i}{1+\alpha_i x_i} - \phi_i + \nu = 0 \tag{2}$$

$$\phi \geq 0 \qquad x \geq 0 \qquad \forall i, x_i\phi_i = 0 \tag{3}$$

$$\nu \geq 0 \qquad \sum_i x_i - P \leq 0 \qquad \nu(\sum_i x_i - P) = 0 \tag{4}$$

Using (2), we deduce that $x_i = 1/(\nu - \phi_i) - 1/\alpha_i$ Using (3), we get that if $\phi_i > 0$, then $x_i = 0$, which implies that $0 \geq 1/\nu - 1/\alpha_i$ and if $\phi_i = 0$, then $x_i = 1/\nu - 1/\alpha_i$. In both cases $x_i = \max(0, 1/\nu - 1/\alpha_i)$

As $x_1 = \max(0, 1/\nu - 1/\alpha_1) \leq P$, $1/\nu - 1/\alpha_1 \leq P$ and thus $\nu \geq 1/(P + 1/\alpha_1) > 0$. This implies that $\sum_i x_i = P$ by (3). We conclude by taking $K = 1/\nu$.

5. *Justify the expression* water filling.

   ▶ An algorithm to compute x could be as follows:

   - start with a level $K = 0$.
   - increase K until $\sum_i \max(0, K - 1/\alpha_i) = P$

   An illustration of the process of the algorithm is similar to filling connected boxes with water, each box having lower level equal to $1/\alpha_i$. The question is then: how many boxes will have water in them at the end of the process?

Exercise 16 (Total-Variation-regularized least squares regression)
Let $x \in \mathbb{R}^n$ be a vector. We consider the following problem

$$\min_{x \in \mathbb{R}^n} \frac{1}{2}\|Ax - b\|_2^2 + \alpha \sum_{i=1}^{n-1} |x_{i+1} - x_i| \, ,$$

where A is a $m \times n$ matrix, b is a vector of $\mathbb{R}^m$. The second term is called the total-variation (TV) regularization term.

1. *Can you guess what type of solution is promoted by the TV regularization?*

   ▶ By analogy to the Lasso, solutions with $x_{i+1} = x_i$ will be promoted.

2. *Show that the problem writes as*

$$\min_{x \in \mathbb{R}^n} \frac{1}{2}\|Ax - b\|_2^2 + \alpha \|Dx\|_1 \, , \tag{5}$$

   where matrix D should be explicited.

   ▶ D has sizes $n - 1 \times n$.

$$D = \begin{bmatrix} -1 & 1 & 0 & \dots & 0 \\ 0 & -1 & 1 & & 0 \\ & & \ddots & \ddots & \\ 0 & \dots & 0 & -1 & 1 \end{bmatrix}$$

3. *Show that the problem (5) is convex.*

   ▶ $x \mapsto \frac{1}{2}\|Ax - b\|_2^2$ is the composition of a convex function with an affine function. It is thus convex. $x \mapsto \alpha\|Dx\|_1$ is also the composition of a convex function with an affine function. Hence, the sum is convex.

4. *By considering an auxiliary variable z and the constraint $z = Dx$, write an equivalent problem with an objective that can be written as $f_1(Ax) + f_2(z)$.*

   ▶
$$\min_{x\in\mathbb{R}^n} \frac{1}{2}\|Ax - b\|_2^2 + \alpha\|Dx\|_1 = \min_{x,z:z=Dx} \frac{1}{2}\|Ax - b\|_2^2 + \alpha\|z\|_1$$

   We take $f_1(w) = \frac{1}{2}\|w - b\|^2$ and $f_2(z) = \alpha\|z\|_1$.

5. *Write the Lagrangian of this new problem.*

   ▶ $L(x, z, \phi) = f_1(Ax) + f_2(z) + \langle Dx - z, \phi\rangle$

6. *Write the ADMM for this problem.*

   ▶
$$\begin{aligned}
x_{k+1} &\in \arg\min_x f_1(Ax) + \langle Dx, \phi_k\rangle + \frac{\gamma}{2}\|Dx - z_k\|_2^2 \\
z_{k+1} &= \arg\min_z f_2(z) - \langle z, \phi_k\rangle + \frac{\gamma}{2}\|Dx_{k+1} - z\|_2^2 \\
\phi_{k+1} &= \phi_k + \gamma(Dx_{k+1} - z_{k+1})
\end{aligned}$$

   The update for variable x can be computed by the resolution of a linear system.

   The update for z can be shown to involve the soft thresholding operator.