工科数学分析

贺丹（东南大学）

3.5 多元复合函数的偏导数和全微分

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本节主要内容：

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- 复合函数的中间变量均为一元函数

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- 复合函数的中间变量均为一元函数
- 复合函数的中间变量为多元函数

3.5 多元复合函数的偏导数和全微分

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- 复合函数的中间变量均为一元函数
- 复合函数的中间变量为多元函数
- 一阶全微分形式不变性

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所以$\Delta z=\dfrac{\partial z}{\partial u}\Delta u+\dfrac{\partial z}{\partial v}\Delta v+o(\rho)$，其中$\rho=\sqrt{(\Delta u)^2+(\Delta v)^2}$.

于是 $\dfrac{\Delta z}{\Delta x} = \dfrac{\partial z}{\partial u}\dfrac{\Delta u}{\Delta x} + \dfrac{\partial z}{\partial v}\dfrac{\Delta v}{\Delta x} + \dfrac{o(\rho)}{\Delta x}.$

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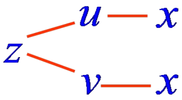

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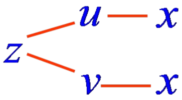

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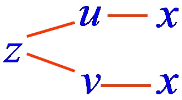

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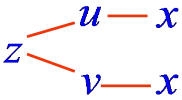

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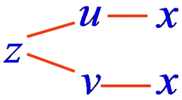

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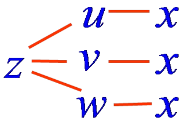

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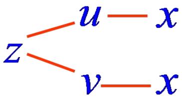

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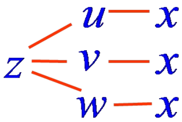

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设$z=f(u,v)$, 而$u=u(x,y)$, $v=v(x,y)$, 若$u=u(x,y)$, $v=v(x,y)$ 在点(x,y) 处偏导数都存在, 且$z=f(u,v)$ 在相应点(u,v) 可微, 则复合函数$z=f[u(x,y),v(x,y)]$ 在点(x,y) 处存在偏导数, 且

$$\begin{cases} \dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial x}+\dfrac{\partial z}{\partial v}\cdot\dfrac{\partial v}{\partial x} \\ \dfrac{\partial z}{\partial y}=\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial y}+\dfrac{\partial z}{\partial v}\cdot\dfrac{\partial v}{\partial y} \end{cases}$$

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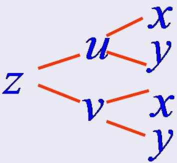

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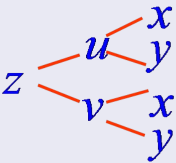

链式法则: 按线相乘, 分线相加.

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$$\mathrm{d}z=\left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}\right)\mathrm{d}x+\left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}\right)\mathrm{d}y.$$

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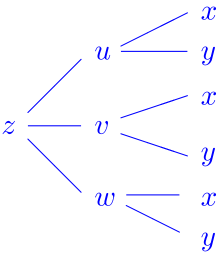

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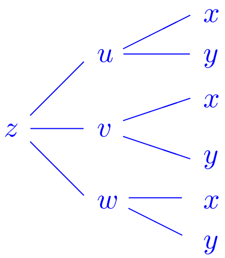

$$\begin{cases} \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial x} + \dfrac{\partial z}{\partial w}\dfrac{\partial w}{\partial x} \\ \dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial y} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial y} + \dfrac{\partial z}{\partial w}\dfrac{\partial w}{\partial y} \end{cases}$$

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注意: 在复合函数的求导过程中, 如果出现某一函数的中间变量是一元函数, 则涉及它的偏导数的记号应改为一元函数的导数记号.

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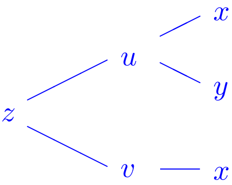

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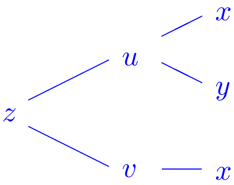

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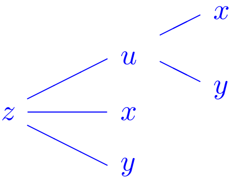

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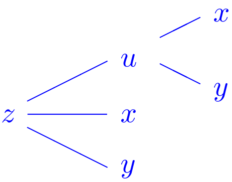

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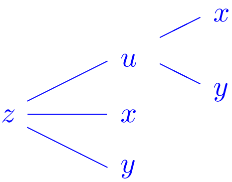

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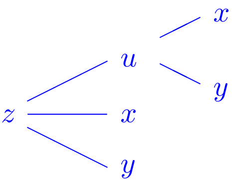

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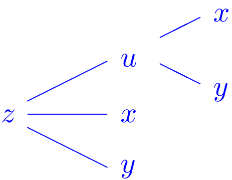

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$$\begin{tikzcd}
 & & x \\
 & u \arrow[ur, no head] \arrow[dr, no head] & \\
z \arrow[ur, no head] \arrow[r, no head] \arrow[dr, no head] & x & y \\
 & y &
\end{tikzcd}$$

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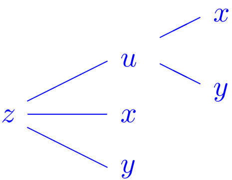

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其中f_i $(i=1,2,3)$ 表示函数f 对第i 个变量的偏导数.

例3. 设$z = xy + xF(u)$, 而$u = \dfrac{y}{x}$, $F(u)$ 为可导函数, 证明:

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z + xy.$$

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例4. 设$z = f(x-y, xy^2)$, f 有二阶连续偏导数,

求$\dfrac{\partial z}{\partial x}$, $\dfrac{\partial^2 z}{\partial x^2}$, $\dfrac{\partial^2 z}{\partial x \partial y}$.

例4. 设 $z = f(x-y, xy^2)$, f 有二阶连续偏导数,

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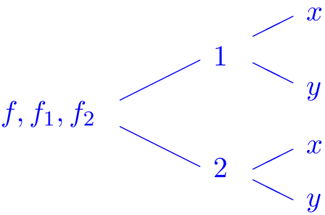

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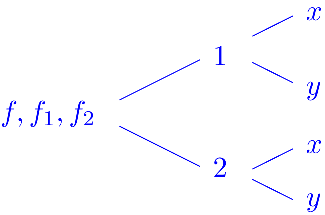

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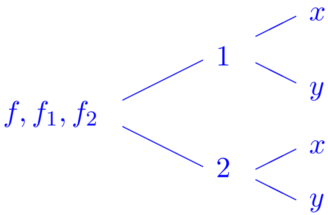

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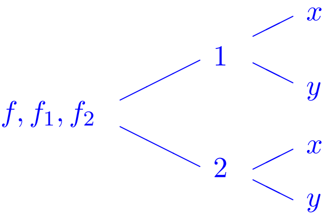

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$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(f_1 + y^2 f_2)$$

$$= f_{11} + y^2 f_{12} + y^2(f_{21} + y^2 f_{22}) = f_{11} + 2y^2 f_{12} + y^4 f_{22}.$$

例4. 设 $z=f(x-y,xy^2)$, f 有二阶连续偏导数,

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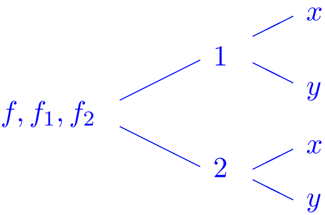

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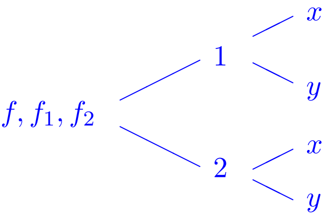

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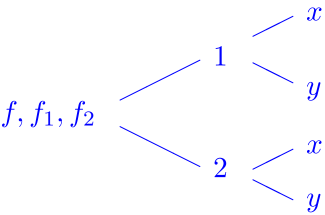

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例5. 设$z = x^3 f(xy, \dfrac{y}{x})$, f 有二阶连续偏导数, 求$\dfrac{\partial z}{\partial y}$, $\dfrac{\partial^2 z}{\partial y^2}$, $\dfrac{\partial^2 z}{\partial x \partial y}$.

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答: $\dfrac{\partial z}{\partial y}=x^4f_1+x^2f_2$, $\dfrac{\partial^2 z}{\partial y^2}=x^5f_{11}+2x^3f_{12}+xf_{22}$,

$$\frac{\partial^2 z}{\partial x\partial y}=4x^3f_1+2xf_2+x^4yf_{11}-yf_{22}.$$

例6. 设$u=u(\xi,\eta)$具有二阶连续偏导数, 其中$\xi=x+ay$,

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$u_{xx}+4u_{xy}+3u_{yy}=0$ 变换为$u_{\xi\eta}=0$.

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$$u_{yy}=a^2u_{\xi\xi}+2abu_{\xi\eta}+b^2u_{\eta\eta},$$

$$\therefore u_{xx}+4u_{xy}+3u_{yy}=(3a^2+4a+1)u_{\xi\xi}+(6ab+4a+4b+2)u_{\xi\eta}$$
$$+(3b^2+4b+1)u_{\eta\eta},$$

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解得$a=-\dfrac{1}{3}, b=-1$ 或$a=-1, b=-\dfrac{1}{3}$.

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$$= \frac{\partial z}{\partial u}(\frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y) + \frac{\partial z}{\partial v}(\frac{\partial v}{\partial x}\mathrm{d}x + \frac{\partial v}{\partial y}\mathrm{d}y)$$

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