矢量分析与场论 2018 年 2 月 13 日

原始定义

梯度：标量场某一点处增长率最大的方向（向量）

散度：向量场某一点处通量体密度（标量）

旋度：标量场某一点处最大的环量面密度（向量）

矢量场的 Jacobi 矩阵

设 $\mathbf{A} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$，其 Jacobi 矩阵为

$$D\mathbf{A} = \frac{\partial(P,Q,R)}{\partial(x,y,z)} = \begin{pmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} & \frac{\partial P}{\partial z} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} & \frac{\partial Q}{\partial z} \\ \frac{\partial R}{\partial x} & \frac{\partial R}{\partial y} & \frac{\partial R}{\partial z} \end{pmatrix}$$

类似单变量函数的导数。对角元之和为散度，非对角元组合成旋度。

Lamé 系数与空间元

从直角坐标系到任意正交曲线坐标系变换，已知 x, y, z 用新坐标变量的表达，可以求其 Lamé 系数：

$$H_i(q_1, q_2, q_3) = \sqrt{\left(\frac{\partial x}{\partial q_i}\right)^2 + \left(\frac{\partial y}{\partial q_i}\right)^2 + \left(\frac{\partial z}{\partial q_i}\right)^2},\ i = 1, 2, 3.$$

或计算式，先算 x, y, z 用 q_1, q_2, q_3 表示的全微分：

$$(dx)^2 + (dy)^x + (dz)^2 = H_1^2(dq_1)^2 + H_2^2(dq_2)^2 + H_3^2(dq_3)^2$$

曲线弧微分为

$$ds_i = H_i dq_i$$

面积元为

$$dS_{ij} = H_i H_j dq_i dq_j$$

体积元为

$$dV = H_1 H_2 H_3 dq_1 dq_2 dq_3$$

柱坐标

$$H_\rho = 1,\ H_\phi = \rho,\ H_z = 1,\ dV = \rho d\rho d\phi dz$$

球坐标

$$H_r = 1,\ H_\theta = r,\ H_\phi = r\sin\theta,\ dV = r^2 \sin\theta dr d\theta d\phi$$

基向量导数

基向量与其导数正交。

$$\frac{\partial \mathbf{e}_1}{\partial q_1} = -\frac{\mathbf{e}_2}{H_2}\frac{\partial H_1}{\partial q_2} - \frac{\mathbf{e}_3}{H_3}\frac{\partial H_1}{\partial q_3}$$

$$\frac{\partial \mathbf{e}_1}{\partial q_2} = \frac{\mathbf{e}_2}{H_1}\frac{\partial H_2}{\partial q_1} \qquad \frac{\partial \mathbf{e}_1}{\partial q_3} = \frac{\mathbf{e}_3}{H_1}\frac{\partial H_3}{\partial q_1}$$

柱坐标

$$\frac{\partial \mathbf{e}_\rho}{\partial \phi} = \mathbf{e}_\phi \quad \frac{\partial \mathbf{e}_\phi}{\partial \phi} = -\mathbf{e}_\rho$$

球坐标

$$\frac{\partial \mathbf{e}_r}{\partial \theta} = \mathbf{e}_r \quad \frac{\partial \mathbf{e}_r}{\partial \phi} = \mathbf{e}_\phi \sin\theta$$

$$\frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r \quad \frac{\partial \mathbf{e}_\theta}{\partial \phi} = \mathbf{e}_\phi \cos\theta \quad \frac{\mathbf{e}_\phi}{\partial \phi} = -\mathbf{e}_r \sin\theta - \mathbf{e}_\theta r\cos\theta$$

正交曲线坐标系 ∇

$$\nabla = \frac{\mathbf{e}_1}{H_1}\frac{\partial}{\partial q_1} + \frac{\mathbf{e}_2}{H_2}\frac{\partial}{\partial q_2} + \frac{\mathbf{e}_3}{H_3}\frac{\partial}{\partial q_3}$$

先求导再求内积。

柱坐标

$$\nabla u = \frac{\partial u}{\partial \rho}\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial u}{\partial \phi}\mathbf{e}_\phi + \frac{\partial u}{\partial z}\mathbf{e}_z$$

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho}\frac{\partial \rho A_\rho}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \left[\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right]\mathbf{e}_\rho + \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right]\mathbf{e}_\phi + \frac{1}{\rho}\left[\frac{\partial \rho A_\phi}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi}\right]\mathbf{e}_z$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2}$$

球坐标

$$\nabla u = \frac{\partial u}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial u}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial u}{\partial \phi}\mathbf{e}_\phi$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial r^2 A_r}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial \sin\theta A_\theta}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \frac{1}{r\sin\theta}\left[\frac{\partial \sin\theta A_\phi}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi}\right]\mathbf{e}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial r A_\phi}{\partial r}\right]\mathbf{e}_\theta + \frac{1}{r}\left[\frac{\partial r A_\theta}{\partial r} - \frac{\partial A_r}{\partial \theta}\right]\mathbf{e}_\phi$$

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial u}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}$$