Laser stability and beam steering in a nonregular polygonal cavity

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Two laser stability criteria or lasing conditions for cavity geometry based on the standard *ABCD* matrix analysis and the Bilger and Stedman analysis [Appl. Opt. **26,** 3710 (1987)] are reconciled. Beam steering from mirror misalignment is discussed similarly, generalizing the Bilger and Stedman analysis to nonregular polygons by extending the standard *ABCD* matrix analysis to 3×3 matrices, which facilitates the thorough design of large rectangular ring lasers and is applied to a number of existing or planned ring lasers with perimeters of 77–120 m. © 2002 Optical Society of America

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1. Introduction

A stable lasing cavity must have a geometry that prevents a paraxial ray from eventually walking out of the cavity. In the term geometry we include the choice of radii of curvature for the mirrors as well as their placement. The judicious use of convex lenses or concave mirrors restores an errant light beam to the cavity, somewhat as a child needs periodic and firm guidance to be kept within bounds.

The standard stability condition (that trace S of the *ABCD* ring matrix has a modulus of less than 2)[1–3] is derived from a difference equation that describes the way in which such a ray deviates from the optical axis on repeated circuits of the ring. All the deviations of a beam from the optic axis of a stable cavity are periodic and so bounded, although in general the period does not correspond with an integral number of circuits of the ring. When the cavity is unstable, the deviations can be unbounded, and a given beam can bounce its way out of the cavity. This stability condition assumes that all the optical elements are perfectly aligned. An anthropomorphic mnemonic for the assumptions and analysis of this stability condition is: under which conditions will a child raised in a perfect environment kick off the traces and leave home?

We discuss three different but equivalent methods of deriving this result. The first method is the standard technique of deriving a difference equation from the *ABCD* matrix. The second is the vector-algebraic method of Bilger and Stedman.[4] Although these two methods have been reported and used for some time, a general proof of their compatibility is given here for the first time as far as we know. The third method is an adaptation of the *ABCD* matrix method and is based on a consideration of its eigenvalues and eigenvectors.

Bilger and Stedman[4] also introduced a second and unequivalent stability condition. Suppose not all optical elements are perfectly aligned; in general the lenses and mirrors will be tilted and shifted slightly. Which cavity geometry conditions are required for the beam to be certain of finding a lasing beam path close (for infinitesimal misalignments, infinitesimally close) to that for a perfect cavity? The corresponding anthropomorphic question is, under which conditions will a child raised in a less-than-perfect environment accommodate itself to that and make a fist of life? We give here a general formulation of beam steering from mirror misalignment, valid for any cavity and so extend the results of Bilger and Stedman[4] for regular polygons to any polygon. Rather than extend the method of Bilger and Stedman,[4] we use a matrix formalism that is as close as possible to the familiar *ABCD* matrices. This necessitates a novel extension to a 3×3 matrix formalism, which involves two new parameters, E and F. The extended theory might be called an *ABCDEF* matrix formalism; for brevity we call it *A2F*. We then aim

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for a similarly integrated analysis of beam steering. We reconcile the two different approaches by using novel theorems in the analysis of continuants (i.e., the determinants of certain quasi-tridiagonal matrices).

Our interest in this problem was rekindled by a goal of the Canterbury ring laser group in which large rectangular rings dubbed Ultra-G are now under construction at Cashmere, Christchurch, New Zealand (Section 4). The successful achievement of the first of these is reported in a companion paper.[5] The earlier paper[4] was written in anticipation of the smallest of these, which at that time held the record for area. A larger ring laser was built in 1997.[6] A further substantial area increase was achieved by Dunn.[7] With the currently considered perimeters in the 77–121-m range and 370–830-m^2 areas, such rings demand an increasingly good understanding of steering effects for optimization of alignment procedures. In particular the guidance must be more judicious as the cavity departs more from being a square. The Bilger and Stedman[4] steering analysis is inadequate for this task, since it applies only to a regular polygon, and the constraints of the underground location require each Ultra-G ring to be rectangular. The topicality of this kind of study is also indicated by other recent geometric analyses of ring cavities.[8–10] Additionally, the family of ring lasers at Christchurch, New Zealand, has been proved capable of measuring teleseismic rotation[11] and lunar Earth tides.[12]

2. Alternative Analyses of Stability

A. Existing Stability Criteria

The standard criterion for the stability of a lasing cavity is discussed in many books and papers[1–3] by use of the *ABCD* matrix formalism. In this, the nth optical element $X^{(n)}$, also including each open distance for propagation, is mapped onto a real 2×2 matrix which acts on a 2×1 vector $\mathbf{u}^{(n)}$ (which represents the incoming optical state) to give the vector $\mathbf{u}^{(n+1)}$ that represents the outgoing state:

$$X^{(n)} \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix}_n; \quad \mathbf{u}^{(n+1)} = X^{(n)}\mathbf{u}^{(n)}, \quad \mathbf{u} \equiv \begin{pmatrix} x \\ y \end{pmatrix}_n. \quad (1)$$

In the standard treatment of a one-dimensional cavity, lens L of focal length f, mirror M of radius of curvature (ROC) R, and open distance D of length d are mapped onto the matrices

$$L(f) = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}, \quad M(R) = \begin{pmatrix} 1 & 0 \\ -2/R & 1 \end{pmatrix},$$
$$D(d) = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}. \quad (2)$$

The total matrix is the product of these elemental matrices, which is justified within the principles of geometrical optics for the above optical elements by noting that these matrices so transform the state vector $\mathbf{u}$ with $x \equiv r, y \equiv r'$, where r, r' are the distance of the ray from the optical axis and the slope of this excursion. It is equally justified within Gaussian optics by noting that these matrices so transform the state vector $\mathbf{u}$ with x the complex ROC q and with $y \equiv 1$, up to an overall factor, so that $1/q = 1/\rho - 2i/kw^2$, $q^{(n+1)} = [Aq^{(n)} + B]/[Cq^{(n)} + D]$. Here ρ is the ROC of the appropriate TEM_{00} wave front and w is the width of its Gaussian profile [i.e., the intensity is $I = I_0 \exp(-2r^2/w^2)$]. When all the optical elements are reciprocal, the principle of reciprocity requires the ray or Gaussian beam to retrace its path under reversed propagation. For primitive (zero length or symmetric) optical elements, after allowing for the reversal of the sense of the optic axis, so that $r' \to -r'$ and $q \to -q^*$, the matrix Y for the reversed propagation must give the inverse of the matrix (X) for the forward propagation. Hence

$$X^{-1} = Y \equiv \begin{pmatrix} A & -B \\ -C & D \end{pmatrix}, \quad (3)$$

and so $A = D$, $\det X = AD - BC = 1$. In a combination multielement system such as a cavity, X and Y refer to systems with reversed ordering of the elements, and [as is readily seen from say $L(f)D(d)$, where $A = 1 \neq D$] it is not a requirement that $A = D$ in such a combination. However the determinant of a matrix product is the product of the determinants, so that even in combination all *ABCD* matrices are unimodular: $\det X = AD - BC = 1$. The comparable calculations for a polygonal (ring) cavity can be rendered in a similar form, with a complete to-and-fro excursion of a beam in the linear case corresponding to one circuit of a beam in the ring cavity. However several minor changes are necessary. First, stability criteria for both in-plane and out-of-plane deviations have to be satisfied (the two cases are equivalent only for axially symmetric components in the linear case). The in-plane and out-of-plane analyses are still cleanly decoupled because such analyses are first order in all deviations. Second, the axis systems have to be redefined for the in-plane analysis. In the standard development of a linear cavity, the same choice of axis for a sideways deviation is made for both the to-and-fro beams. This choice plus the above choice for M is adequate for the analysis of out-of-plane deviations. [However for the in-plane analysis this is perpendicular to the beams that change direction at each edge of the polygon.] It might seem that the most elegant choice is to make all these have a positive sense if they point in a similar sense, say, outward. However there is a still more elegant choice and that is made here: the axis sense should alternate from inward to outward for adjacent edges. Only this, for example, allows for the choice of a linear ring by use of the same axes for the to-and-fro beams. Other advantages are explained below. This choice carries the one complication that when closing an N-mirror ring we should add an overall sign $(-1)^N$ to the ring matrix to match input and output axes. Third, each mirror (with ROC R) at which the beam axis is turned through an angle 2θ is then mapped onto a matrix $M(Q)$, where

Table 1. Relevant Parameters for the Typical Ring Laser Gyroscope Situated in the Cashmere Cavern[a]

Parameter	Gain Tube Arm	Second Arm	Third Arm	Fourth Arm	Stability
Mirror radius of curvature (m)	20	20	20	20	
In-plane spot size (mm)	8.23	8.25	8.23	8.25	
Out-of-plane spot size (mm)	8.25	8.23	8.25	8.23	
Waist position relative to center of arm (m)	0	0	0	0	
In-plane waist (mm)	5.54	3.87	5.54	3.87	
Out-of-plane waist (mm)	5.97	7.26	5.97	7.26	
B&S stability in plane					0.37
B&S stability out of plane					0.58
Perimeter (m)					77.00
Area (m^2)					367.50

[a]These parameters can be optimized through the use of the MS Excel spreadsheet for stability analysis in a rectangular ring available at the Ring Laser Group website.[13] Mirror radii of curvature are given for the mirror that lies in the immediate path of a ray within the cavity, hence the symmetry of the cavity is preserved whether the ray travels clockwise or counterclockwise. The spot size at each mirror is given relative to the arm that it lies in, similarly for the waist size and position. Note the distinction between in-plane and out-of-plane cases. The Bilger and Stedman (B&S) stability parameter should lie between -2 and 2, with the case of positive criticality (i.e., 2) to be avoided.

$Q \equiv R/e$. Q is the effective ROC; e represents the effects of astigmatism, when the mirror affects in-plane (out-of-plane) deviations as if it had a smaller (larger, respectively) ROC; for the in-plane (p) and out-of-plane (s) cases, $e_p = \sin\theta$ and $e_s = \csc\theta$, respectively. As a result, in-plane and out-of-plane analyses have the same *ABCD* mirror matrices $M(Q)$:

$$M(Q) = \begin{pmatrix} 1 & 0 \\ -2/Q & 1 \end{pmatrix}. \tag{4}$$

The matrices remain unimodular. In a complete laser cavity $\mathscr{L}$ with total *ABCD* matrix $M(\mathscr{L})$, the standard criterion for stability is that the trace $A + D$ of matrix $M(\mathscr{L})$ has a modulus of less than 2:

$$|A + D| = |\chi(M(\mathscr{L}))| \leq 2. \tag{5}$$

This stability condition is often derived from a difference equation that describes the way in which such a ray deviates from the optical axis on repeated circuits of the ring. All the deviations of a beam from the optic axis of a stable cavity are periodic and so bounded, although the period might not correspond with an integral number of circuits of the ring. A spreadsheet implementation of this for a rectangular ring laser, which allows rapid analysis, is available from the Ring Laser Group's webpage.[13] A ring laser of geometry typical to the Cashmere Cavern is analyzed and the relevant parameters are displayed in Table 1. So far the standard theory for a linear cavity has been reviewed, and its application to a polygonal ring has been outlined.

An independent approach was taken by Bilger and Stedman[4] in 1987. They restricted their analysis to a regular polygon with N sides of length l (we label the mirrors in a counterclockwise sense). In this approach, parameters $\Gamma = 2\epsilon(1 - l/Q)$ are defined for out-of-plane p (where $\epsilon = 1$) and in-plane s cases (where $\epsilon = -1$, and now $e_p = \sin\pi/N$); so $\Gamma_s = 2[l/(R \sin\pi/N) - 1]$, $\Gamma_p = 2[1 - (l \sin\pi/N)/R]$ for each mirror i. The total ring behavior is described by an $N \times N$ matrix D_N, in the sense that

$$D_N\gamma = \mathbf{G}, \tag{6}$$

where $\mathbf{G}$ is a vector of misalignment parameters (position and angle) for all mirrors and γ is a vector of the resulting lasing beam walk parameters, each measured as a beam shift perpendicular to the beam and relative to the original poles of the mirrors, the corners of the regular polygon. (In the 1987 version as published[4] γ relates to the resulting lasing beam walk parameters relative to the new poles of the mirrors, also the shift as measured in the mirror plane, which is obliquely angled for in-plane deviations; hence the difference in some formulas below.) This matrix has the form

$$D_N = \begin{pmatrix} -\Gamma_1 & 1 & 0 & \ldots & 1 \\ 1 & -\Gamma_2 & 1 & \ldots & 0 \\ 0 & 1 & -\Gamma_3 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & 0 & 0 & \ldots & -\Gamma_N \end{pmatrix}. \tag{7}$$

For in-plane deviations

$$G_i = -2l(\eta_i/R_i + \phi_i) - \cos(\pi/N)(\mu_{i+1} - \mu_{i-1}), \tag{8}$$

and for the out-of-plane case

$$G_i = 2l \sin(\pi/N)(\nu_i/R_i - \theta_i), \tag{9}$$

where (μ, η, ν) is the physical displacement of a mirror pole (in axes where μ is radially inward, η is clockwise as seen from above, and ν is upward from the plane; η and ν are measured along the mirror) and ϕ, θ is its angular misalignment about the vertical and horizontal axes, with ϕ_i being positive in the counterclockwise direction as seen from above, and θ_i being positive if the top tilts toward the ring center. Note that η in particular is measured in the mirror plane and so is greater than the corresponding movement of the beam perpendicular to itself by a factor of

$\csc(\pi/N)$. If for the moment we ignore the radial deviations μ_i, the in-plane and out-of-plane formulas can be combined in the form

$$\mathbf{G} = -2l\alpha, \quad \alpha_i \equiv f(\theta_i - \epsilon\nu_i/R_i), \tag{10}$$

where $f = 1$ and $\epsilon = -1$ for in-plane deviations and $f = \sin \pi/N$ and $\epsilon = 1$ for out-of-plane deviations.

In 1987 Bilger and Stedman[4] showed that, for the beam walks to iterate within bounds, the so-called stability $\kappa_N(\gamma)$ must have a modulus of less than 2:

$$\kappa_N(\gamma) \equiv (-1)^N \det D_N + 2; \quad |\kappa_N(\gamma)| \leq 2. \tag{11}$$

As can be expected from the commonality of the physical model, Eqs. (5) and (11) were found to give equivalent results. We show here that the relationship between the stability criteria of the two methods is

$$S = A + D = (-\epsilon)^N \kappa_N. \tag{12}$$

A proof of this equivalence requires an analysis of recursion relations in the theory of continuants (the determinants of almost tridiagonal matrices) and is given in Appendix A.

The extrema or critical points of the stable region were recognized by Bilger and Stedman in 1987[4] as the condition $\kappa_N(\gamma) = 2$ (which was called positive criticality) and $\kappa_N(\gamma) = -2$ (negative criticality). At positive criticality the incipient cavity instability becomes serious, because [from Eq. (11)] the matrix D_N becomes singular, making Eq. (6) noninvertible, the beam steering parameters γ diverge, and a lasing beam is unsupportable in the ring.

The condition of positive criticality is illustrated by the case of certain cavities with flat mirrors. When $Q_i \to \infty$, $\Gamma_i = 2\epsilon(1 - l/Q_i-) \to 2\epsilon$ (that is, all Γ become equal to $+2$ for flat mirrors in the out-of-plane analysis, or all are equal to -2 for flat mirrors in the in-plane analysis). In these cases we find by inspection that $\det D_N = 0$ if $(-\epsilon)^N = 1$, i.e., if N is even and/or we consider out-of-plane stability.

The case of negative criticality is illustrated by the confocal linear cavity (in each direction) and by the in-plane stability analysis for an equilateral triangle with flat mirrors. There is no associated strong divergence in beam steering for these displacements in such cavities, which explains the practical value of a confocal cavity, despite its marginal stability, as opposed to a Fabry–Perot interferometer with flat mirrors, whose incipient criticality is positive. Along with the extra degrees of freedom in mirror adjustment, it also helps to explain (in accordance with our later application) the relative difficulty of alignment of a rectangular cavity over a triangular cavity.

The practical importance of this formal analysis for our application can now be stated. The obvious task in cavity design is to have a stable cavity in which the stability criteria of Eqs. (5) and (11) are satisfied. The less obvious but equally important task is to avoid regions in parameter space where steering effects become strong, making the mirror alignment task in a large multielement ring laser unduly difficult. Foremost among these regions is the neighborhood of the onset of positive criticality. We shall verify numerically for the case of large rectangular rings that the neighborhood of positive criticality is indeed the major area to be avoided. This helps to identify favorable regions of parameter space in ring design.

B. Eigenvalue and Waist Analysis

A third way of understanding this stability result is to consider the eigenvalues λ and eigenvectors $\mathbf{u}$ of the ring *ABCD* matrix $M(\mathscr{X})$, i.e., the quantities that satisfy $M(\mathscr{X})\,\mathbf{u} = \lambda\mathbf{u}$. Although this approach is dependent on and closely linked to the *ABCD* analysis, we have not found any discussion of this in the literature. From this eigenvalue equation, $\det[M(\mathscr{X}) - \lambda\mathbf{I}] = 0$ and so $\lambda^2 - (A + D)\lambda + 1 = 0$. A real eigenvalue with modulus greater than 1 indicates that the beam is liable to walk out of the cavity rapidly, so that the cavity is unstable. The product of the roots is unity; hence both roots must be pure phases for the cavity to be stable. This is guaranteed when the discriminant is negative, i.e., when Eq. (5) is obeyed, and the eigenvalues have the form $\exp(\pm i\phi)$.

Such roots guarantee stability since an appropriate if irrational power z (where $z = 2\pi/\phi$) of $M(\mathscr{X})$ then has unit eigenvalue; the beam will certainly restore itself.

The solution of this eigenvalue equation also shows that the conditions of positive and negative criticality correspond to $\lambda = +1, -1$, respectively; hence at positive criticality a beam will return on one circuit to its original deviation and slope, and at negative criticality it will return on one circuit to the negative of its original deviation and slope. (The latter condition, for example, is easily verified in the cases of a confocal linear cavity and of the in-plane deviations in an equilateral triangle with flat mirrors.) All this is consistent with the physical picture underlying all these stability analyses; beam deviations obey difference equations whose trigonometric solutions are periodic functions of the distance along the optical axis, if not at the period of the ring itself.

It is useful in this context to explore Kogelnik's method[1] (see also Yariv[14]) of determining the optical beam with w and ROC ρ at any point from the *ABCD* matrix $M(\mathscr{X})$ at that point. From the eigenvalue equation with $\mathbf{u} = (q, 1)^T$, which states that on performing one circuit of the ring an eigenmode must return to the same complex radius of curvature q, we have $Cq^2 + (D - A)q - B = 0$, and so

$$\frac{1}{q} = \frac{D - A}{2B} \pm \frac{i}{2B} \surd[4 - (A + D)^2].$$

Hence (choosing the appropriate sign) $kw^2 = 4|B|/\surd[4 - (A + D)^2]$ and $\rho = 2B/(D - A)$. At each waist $\rho_0 = \infty$ and so $A_0 = D_0$, $kw_0{}^2 = 2|B_0|/\surd[1 - D_0{}^2]$.

If a waist W exists at a distance d in free space beyond any reference point P used to determine the matrix $M(\mathscr{X})$, the *ABCD* matrix formed from

$D(d)M(\mathscr{X})D(-d)$ must have equal diagonal entries, which then become D. Hence the distances to the waist $d(PW)$ and the waist size w_0 are given by

$$d = \frac{D - A}{2C}; \quad w_0 = \sqrt{\left\{\frac{\lambda}{\pi} \frac{\sqrt{[4 - (A + D)^2]}}{2|C|}\right\}}. \quad (13)$$

This agrees with Rigrod's Eq. (35), and the modulus of Rigrod's Eq. (34) agrees with this.

C. Stability against Misalignment

Bilger and Stedman[4] considered the requirement of beam stability under cavity misalignment. This entailed taking into consideration whether any geometrical optical ray could be found in a misaligned ring, which is separate from the earlier problem as to whether all such rays displaced in first order within an aligned ring would have bounded orbits. This condition yielded

$$\kappa_N(\gamma) \leq 2. \quad (14)$$

From Eq. (12) the associated constraint on the trace of the $ABCD$ matrix is $(A + D)(-\epsilon)^N < 2$. This condition is contained within the first stability condition of Eq. (11). Although the stability condition of inequality (14) added no new restriction, it gave two new insights. First, it illuminated the physics of the standard stability condition. In the language of our earlier analogy, this proved that it is easier for a child to accommodate itself to a rough-and-ready environment than to avoid wandering from perfection. It confirms the seriousness of the limit of positive criticality, the fundamental reason for this link having been already noted: with the onset of positive criticality the matrix that describes beam steering becomes singular.

The second stability condition was also the natural prelude to the above formulation by Bilger and Stedman[4] of a full analysis of beam steering effects caused by misalignment. It was of practical use in dictating the effects of any particular mirror misalignment on the beam spots.

3. *A2F* Matrix Approach to Beam Steering under Misalignment

The $ABCD$ formalism has to be radically extended to cope with the effects of either displacive or angular misalignment, because the input ray deviation and slope are no longer homogeneously related to the output values of these quantities; they include an additive constant to ray deviation, under sideways displacive misalignment, and to ray slope, under angular misalignment. This will be illustrated below in the first of our simple examples.

The uniquely obvious extension is to enlarge the two-component formalism to a three-component formalism in which the state of a light beam is now defined by the vector $\mathbf{u} = (1, x, y)^T$ with (in the ray picture) $x = r$, $y = r'$ (the geometric deviation and slope) as before, and with each optical processing element correspondingly mapped to a 3×3 A2F matrix:

$$\bar{X} = \begin{pmatrix} 1 & 0 & 0 \\ E & A & B \\ F & C & D \end{pmatrix}, \quad (15)$$

where we distinguish the enlarged matrices by an overbar. In an aligned optical system where the elements share a common optic axis with the geometrical optical beam, $E = F = 0$, and the analysis reduces to the standard two-dimensional $ABCD$ matrix analysis. In the case of misaligned elements, the first (inhomogeneous) component of $\mathbf{u}$ allows the correct equation to be incorporated.

We illustrate this with the three key applications of this matrix. First, the free propagation by a distance l remains a homogeneous operation on r, r' and so is mapped by a trivial generalization $\bar{D}(l)$ of $D(l)$, with $E = F = C = 0$, $A = D = 1$, and $B = l$. Second, consider the ith mirror displaced sideways from the original optic axis by ν_i, by use of the sign conventions of Bilger and Stedman[4] as explained in Subsection 2.A. Now even an incident ray on the optic axis will emerge with a deviation in angle given by $\delta\bar{r}_1' = 2\nu_i/R_i$, with the negative sign occurring because of the alternation of the sense of the axis with an edge. Third, rotating a mirror (not a lens) by an angle θ_i gives a further angular deviation by $\delta\bar{r}_1' = -2\theta_i$. For the in-plane analysis, comparison with Eq. (10) and remembering the alternation of the sense of the axis with an edge, we find the same dependence on the two components of the in-plane α_i so that we can formulate these effects in the *A2F* matrix:

$$\bar{M}_i(\alpha_i, Q_i) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2\alpha_i & -2/Q_i & 1 \end{pmatrix},$$

$$\widehat{\bar{M}}_i = \bar{D}(l)\bar{M}_i = \begin{pmatrix} 1 & 0 & 0 \\ 2l\alpha_i & 1 - 2l/Q_i & l \\ 2\alpha_i & -2/Q_i & 1 \end{pmatrix}, \quad (16)$$

where α_i is defined in Eq. (10). [This determination of F requires a geometric analysis of the corresponding additive in-plane and out-of-plane contributions ρ, σ to the slope parameters of the outgoing beam. In outline, and using the mirror axes of Bilger and Stedman,[4] an on-axis incoming beam $(-\sin \pi/N, -\cos \pi/N, 0)$ reflects at a mirror point with normal $(1, \eta_i/R_i + \phi_i, \nu_i/R_i - \theta_i)$ and gives rise to a beam in the direction $\mathbf{o} = (\sin \pi/N + \rho \cos \pi/N, -\cos \pi/n + \rho \sin \pi/N, -\sigma)$; the laws of reflection then show that to first order $\rho = 2\alpha_y$, $\sigma = -2(\nu_i/R_i - \theta)\sin(\pi/N) = -2f\alpha_z$, and a rotation by $\sin \pi/N$ that $\mathbf{o} = (\rho, -1, -\sigma)$, which justifies the interpretations of ρ, σ as slope parameters r' for the in-plane and out-of-plane beam projections.] Note that the factor ϵ has canceled completely; this is the real value of the choice of alternating the in-plane axis system discussed above. Somewhat as for $ABCD$ matrices, we must reconcile the initial and final choices of axes for the in-plane analysis when there are an odd number of mirrors in a ring by including a matrix $\hat{N} = \mathrm{diag}[1, (-1)^N, (-1)^N]$. The total ring $A2F$ matrix $\bar{M}^{(1)}(\mathscr{L}) = \hat{N}\prod_{i=1}^{N} [\bar{D}(l)\bar{M}_i]$; the superscript denotes that we start from mirror 1; $\bar{M}^{(j)}(\mathscr{L})$ denotes a similar product starting from mirror j. These extended matrices are also unimodular.

The beam steering is defined by the new optic axis, which in turn is defined by the eigenvectors $\mathbf{u}^{(i)} = (1, r_i, r_i')'$ of $\bar{M}^{(i)}(\mathscr{L})$ that have a unit eigenvalue, since the optic axis has to retrace its path exactly on every circuit:

$$[\bar{M}^{(i)}(\mathscr{L}) - \mathbf{I}]\mathbf{u}^{(i)} = 0. \tag{17}$$

The requirement of a unit eigenvalue with a nontrivial eigenvector in turn requires, from Eq. (17), that $\det[\bar{M}^{(i)} - 1] = 0$. This is guaranteed for an $A2F$ matrix; from Eq. (15) $\bar{X} - 1$ has determinant zero. To define the optic axis (i.e., to find all r_i) we need to either diagonalize each $\bar{M}^{(i)}(\mathscr{L})$ or diagonalize, say, $i = 1$ and then transform $\mathbf{u}^{(1)}$ by all partial $A2F$ ring matrices $\bar{m}^{(i)} = \prod_{j=2}^{i} [\bar{D}(l)\bar{M}_i(Q)]$, $i = 2 \,..\, N$, whose nontrivial elements are denoted $a^{(i)}, \ldots, d^{(i)}$ and $e^{(i)}, f^{(i)}$. If $\bar{M}^{(i)}(\mathscr{L})$ has the form of Eq. (15) with subscripts i, Eq. (17) becomes

$$\begin{pmatrix} A_i - 1 & B_i \\ C_i & D_i - 1 \end{pmatrix}\begin{pmatrix} r_i \\ r_i' \end{pmatrix} = -\begin{pmatrix} E_i \\ F_i \end{pmatrix}. \tag{18}$$

Here E_i, F_i are the full ring misalignment parameters—the remaining nontrivial elements of the corresponding $A2F$ matrix $\bar{M}^{(i)}(\mathscr{L})$ and dependent on the misalignment parameters for all mirrors. We have the recurrence relations

$$\begin{pmatrix} e \\ f \end{pmatrix}^{(i)} = M_i\begin{pmatrix} e \\ f \end{pmatrix}^{(i-1)} + 2\alpha_i\begin{pmatrix} l \\ 1 \end{pmatrix}. \tag{19}$$

As a simple illustration, consider this approach in the linear system, with one flat mirror and one curved mirror separated by L and with the latter displaced and rotated as above. Astigmatic factors are trivial and the cavity $A2F$ matrix, $\bar{M} = \bar{M}_2(0, \infty)\bar{D}(L)\bar{M}_1(\alpha, R)\bar{D}(L)$, has an eigenvector with unit eigenvalue of $\mathbf{u}^{(1)} = (1 \quad \alpha R \quad 0)'$. As expected, the new axis $\hat{\mathbf{u}}$ is parallel to the old $\mathbf{u}$ and the optic axis, the system adapts to either distortion so as to find the new path that is still perpendicular to the flat mirror. The only effect of the misalignments is to displace this optic axis by αR so as to find the new point on the curved mirror that is parallel to the flat mirror.

As in the case of beam stability, the two formalisms are equivalent. Ignoring the radial or axial motions μ, the vectors $\boldsymbol{\gamma}(\boldsymbol{\nu})$ of Eq. (6) and the vector $\mathbf{r}(\boldsymbol{\nu}) = (r_1, r_2, \ldots, r_N)'$ defined from the $\{r_i\}$ of Eq. (17) are equal (see Appendix B):

$$\gamma(\{\alpha_i\}) = \mathbf{r}(\{\alpha_i\}). \tag{20}$$

4. Ultra-G Ring Lasers

We now outline the application of this formalism to the optical design of two proposed large rectangular ring lasers, Ultra-G1 and Ultra-G2 (for brevity UG1 and UG2). These are the latest devices planned in a set of large ring laser gyroscopes at Cashmere, Christchurch, New Zealand. Stedman[15] has given a full review of the general project aims, and an update is given in our companion paper.[5] A laser UG1, with a perimeter of 77 m and an area of 367.5 m^2, was made operational, a new record in ring laser gyroscopes.[5] The relatively stable environment of the underground Cashmere Cavern[15] is now being used to the full. UG1 occupies approximately half of the floor area of the cave, and the larger UG2, which would occupy the whole cave, is proposed. In each case the geometry of the cave forces these lasers to depart from the square (each laser encloses a substantial pillar of unexcavated basalt rock). In turn that necessity was the stimulus to develop the extension of the *ABCD* matrix, and in particular the method outlined in Section 2, which is evaluation of beam steering from mirror misalignments, since accurate containment of beam steering precision requirements is a vital need in such large devices.

Table 2. Design Parameters for the Ultra-G Family of Ring Lasers[a]

Design Parameter	UG1	UG2
a (m)	21.0	20.97
b (m)	17.5	39.57
P (m)	77.0	121.08
Free spectral range (MHz)	3.89	2.4760
A (m^2)	367.5	829.78
$\delta f_{\mathrm{Sagnac}}$ (kHz)	1.51	2.1766
χ	0.091	−0.31

[a]The a, b are side lengths, P, A are the perimeter and area. χ represents the departure from a square (as enforced by the underground location): $a, b = P(1 \pm \chi)/4$. The free spectral range and expected Sagnac frequency $\delta f_{\mathrm{Sagnac}}$ are listed.

The specifications of these ring laser gyroscopes are listed in Table 2. The departure from a square can be parameterized by χ: $a = P(1 + \chi)/4$ and $b = P(1 - \chi)/4$, where a, b are the lengths of the side (a is the one that contains the gain tube) and P is the

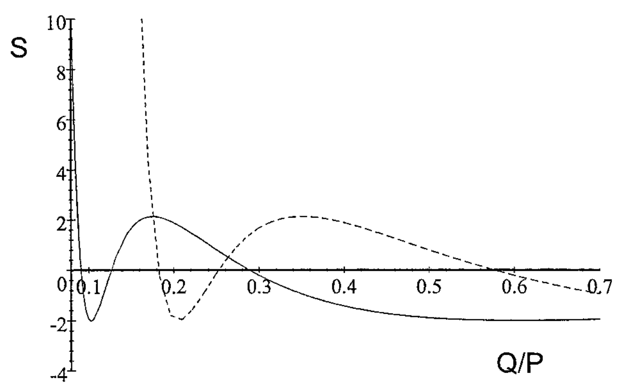

Fig. 1. In-plane (solid curve) and out-of-plane (dashed curve) stability, dimensionless parameter S, in a UG1 geometry. The abscissa is the astigmatically corrected ROC Q (normalized by perimeter P of 77 m) common to each mirror [*CCCC* configuration, Eq. (21)].

perimeter; $t\chi = 2(a - b)/P$. Also[15] $\delta f_{\text{Sagnac}} = 317.6A/P$, where A is the area.

We adopted a symmetrical cavity geometry $C_1C_2C_2C_1$, by which we mean that the system is reflection symmetric about the gain region; this undoubtedly contributed to the value of the December 1998 upgrade of Canterbury-II (C-II).[16,17] The side (length a) that contains the gain tube is therefore terminated by mirrors with the same ROC R_1, and the opposite side by mirrors with ROC R_2. For in-plane or out-of-plane stability, the astigmatic correction relates the real ROC R to an effective radius Q_{in}, Q_{out}, where $Q_{\text{in}} = R/\sqrt{2}$ and $Q_{\text{out}} = \sqrt{2}R$. With the definitions $\delta_1 = P/Q_1$, $\delta_2 = P/Q_2$, the overall ring $ABCD$ matrix has a trace $S = A + D$ of

$$S = 2 + \frac{1}{2}(7 - 2\chi - \chi^2)\delta_1\delta_2 + \frac{1}{4}(3 + 2\chi - \chi^2) \times (\delta_1{}^2 + \delta_2{}^2) + \frac{1}{2}(\chi^2\delta_1\delta_2 - 8 - \delta_1\delta_2)(\delta_1 + \delta_2) + \frac{1}{16}\delta_1{}^2\delta_2{}^2(1 - \chi^2)^2. \tag{21}$$

Note that the sign of χ affects the stability of the cavity, the physical distinction being whether the longer side is between similar or dissimilar mirrors.

As an illustration, in Fig. 1 we plot this stability function for the UG1 choice $\chi = 0.091$ and for four identical mirrors. The above theoretical development allows the susceptibility of these choices for beam steering to be thoroughly analyzed. The magnitude of the sum of the squares of the elements of the matrix $D_4{}^{-1}$ [Eq. (7)] for beam steering conveys the general effects on beam steering of a given choice of mirror radii of curvature. In particular the divergences, corresponding to singularities of D_4, heavily reinforce the need to avoid the regions of positive criticality $S = 2$. Stability regions are therefore in the regions of $R/P \sim 0.12, 0.25, 0.5+$. We require that $-2 < S < 2$, and the value $S = 2$ is to be particularly avoided because the steering diverges here. Thus, regions of in-plane stability are displayed alongside out-of-plane stability with the abscissa as the astigmatically corrected value of Q; for choice of a real mirror radius R (and for convenience with R normalized by the perimeter) to describe aspects of a particular cavity geometry. It is then necessary to choose a set of radii of curvature mirrors that can be used to avoid the divergences in both steering and stability.

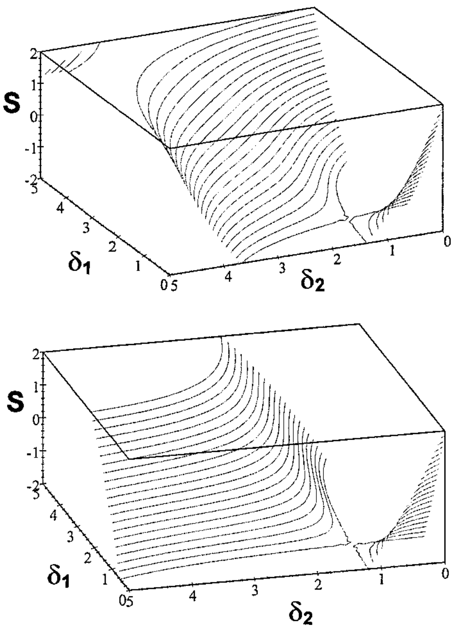

Fig. 2. Stability S for UG1 (upper) and UG2 (lower) in the $C_1C_2C_2C_1$ configuration plotted as a function of δ_1 and δ_2. The ratios of the perimeter versus the astigmatically corrected mirror radii are shown.

As another illustration, we take the mirrors to be of two kinds, those on the shorter side have the same radii of curvature R_1, R_2, and plot the stability in three dimensions as a function of $\delta_i \equiv P/Q_i$, $i = 1, 2$ for UG1 and for UG2 in Fig. 2. The resulting stability surfaces, although complex, show that the stability depends mostly on the choice of the smaller ROC. The origin corresponds to all flat mirrors (R_1, $R_2 \to \infty$) whereas the axes correspond to two flat mirrors ($CFFC$).

We compute the waist w_0 (as a local minimum in the width, related to a Rayleigh length L_R by $L_R = \pi w_0{}^2/\lambda$) from Eq. (13). In a rectangular symmetric cavity $C_1C_2C_2C_1$, with waist w_1 between mirrors C_1, of effective ROC Q_1 (where $q = ½\, kw_1{}^2 i \equiv iL_1$), to that ($w_2$, with Rayleigh length L_2) between mirrors C_2 of effective ROC Q_2 we find

$$\frac{L_2}{L_1} \equiv X = 1 - \frac{4(1 - \chi)(\delta_1 - \delta_2)}{8 + \delta_1\delta_2(1 - \chi^2) - 4(\delta_1 + \delta_2) + 2(1 - \chi)(\delta_1 - \delta_2)},$$

$$L_1 = \frac{P}{8}\sqrt{\left\{\frac{32 - (1 + \chi)[2(3 - \chi)(\delta_1 + \delta_2) - \delta_1\delta_2(1 - \chi^2)]}{X[2(\delta_1 + \delta_2) - (1 - \chi)\delta_1\delta_2]}\right\}}.$$

For example, in a square ring with all the mirrors equivalent, this reduces to $L = (P/8)\sqrt{[(8 - \delta)/\delta]}$, which agrees with the Gaussian optical requirement that $Q = P/8 + 8L^2/P$. As another example, in the case of a square ring with dissimilar mirrors ($\chi = 0$)

the stability and waist ratio are

$$S = -\left(4 + \frac{1}{2}\,\delta_1\delta_2\right)(\delta_1 + \delta_2) + \frac{3}{4}\,(\delta_1{}^2 + \delta_2{}^2) + \frac{1}{16}\,\delta_1{}^2\delta_2{}^2 + 2 + \frac{7}{2}\,\delta_1\delta_2,$$

$$X = 1 - \frac{4(\delta_1 - \delta_2)}{8 + \delta_1\delta_2 - 6\delta_2 - 2\delta_1}.$$

For moderately flat mirrors ($\delta < 0.5$, say) $X > 1$ if $\delta_1 < \delta_2$, i.e., if $R_1 > R_2$, making the smaller waist that between the curved mirrors. This relationship however reverses for regions of stability with smaller radii of curvature.

For the larger UG2 ring, smaller beam waists at the centers of the sides, an important consideration for radio-frequency excitation in a He–Ne plasma, are possible if the mirrors have somewhat different radii of curvature, for example 20 and 70 m. For identical mirrors, we could adopt $R_1 = R_2 \sim 70$ m; for two flat mirrors, we are restricted to an area near the origin ($P/R < 1, R > 121$ m) with significantly greater beam steering problems and fatter waists. The choices (20–70–70–20 and 70–70–70–70) minimize waists and beam steering problems. In the former case, the in-plane waists are 4.1 mm (between the 70-m ROC mirrors) and 0.7 mm (between the 20-m ROC mirrors), and the corresponding out-of-plane waists are 4.2 and 1.1 mm. When all the mirrors have 70-m ROC, the waists are 2.8–2.9 mm.

Appendix A: Stability Theorem

Here we prove Eq. (12), i.e., $(-\epsilon)^N\chi[M(\mathscr{L})] = (-1)^N \det D_N + 2$, where χ denotes the trace. For notational simplicity in this Appendix we write $A \equiv \Gamma_1$ (not to be confused with the A of the ring matrix), . . . $Z \equiv \Gamma_N$, $P_X \equiv M_\epsilon[2l/(2 + \epsilon X)]D(l))$ so that from Eqs. (2), (4), and (7) we obtain

$$D_N = \begin{pmatrix} -A & 1 & 0 & \dots & 1 \\ 1 & -B & 1 & \dots & 0 \\ 0 & 1 & -C & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & \dots & -Z \end{pmatrix},$$

$$\chi[M(\mathscr{L})] = \chi(P_A P_B P_C \dots P_Z),$$

$$P_X(\epsilon, l) \equiv \begin{pmatrix} 1 & l \\ (-\epsilon X - 2)/l & -\epsilon X - 1 \end{pmatrix}. \quad \text{(A1)}$$

The factors l cancel after formation of the trace in Eqs. (A1), and the factors $-\epsilon$ contribute an overall factor $(-\epsilon)^N$ since the trace involves odd or even powers of parameters Γ_i as N is odd or even. Hence $\chi[M(\mathscr{L})] = (-\epsilon)^N\chi(p_A \,..\, p_Z)$, where $p_X = P_X$ (1, 1). The required result is then

$$(-1)^N \det D_N + 2 = \chi\left[\begin{pmatrix} 1 & 1 \\ A-2 & A-1 \end{pmatrix} .. \begin{pmatrix} 1 & 1 \\ Z-2 & Z-1 \end{pmatrix}\right]. \quad \text{(A2)}$$

The proof of this theorem[18] is as follows. The trace is unchanged by cyclic permutation, and so unchanged by the transformation $p_I \to \bar{p}_I = s p_I s^{-1}$, where

$$s = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \bar{p}_I = \begin{pmatrix} I & -1 \\ 1 & 0 \end{pmatrix},$$

so that

$$\chi(\bar{P}) = \chi(\bar{p}_A\bar{p}_B \dots \bar{p}_Z) = \chi(p_A \dots p_Z).$$

Both M and $\bar{P}$ have a structure akin to the continuant (see Muir and Metzler[19]):

$$C(-A, -B, \dots, -Y, -Z) = \det\begin{pmatrix} -A & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & -B & 1 & \dots & 0 & 0 & 0 \\ 0 & 1 & -C & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -X & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & -Y & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & -Z \end{pmatrix}.$$

A Laplace expansion by the last row shows that this obeys the recurrence relation

$$C(-A, -B, \dots, -Y, -Z) = -ZC(-A, -B, \dots, -Y) - C(-A, -B, \dots, -X).$$

A similar expansion of $\det M$ first by the last row and then by the last column plus use of this recurrence relation shows that

$$\det M + 2(-1)^N = C(-A, -B, \dots, -Y, -Z) - C(-B, \dots, -Y).$$

Also from this recurrence relation it is straightforward to prove by induction that

$$\bar{P} = (-1)^N\begin{bmatrix} C(-A, -B, \dots, -Y, -Z) \\ -\,C(-B, \dots, -Y, -Z) \\ C(-A, -B, \dots, -Y) \\ -\,C\,(-B, \dots, -Y) \end{bmatrix}.$$

Together these equations yield Eq. (A2).

Appendix B: Steering Theorem

We present (without general proof, but confirming the result in lowest-order cases) the matrix relationship that links the Bilger and Stedman formulations[4] with the current $A2F$ formulations for beam steering (rather than for stability, as in Appendix A). We use Eqs. (16), (18), and (19) and define $\hat{M}_i$ as the right lower 2×2 block of $\widehat{\widehat{M}}_i$. For example, in the $N = 2$

case the first rows of the following two matrix equations agree:

$$\begin{pmatrix} r_1 \\ r_1' \end{pmatrix} = 2(\hat{M}_2\hat{M}_1 - \mathbf{1})^{-1}(\alpha_2 + \hat{M}_2\alpha_1)\begin{pmatrix} l \\ 1 \end{pmatrix}$$
$$= \begin{bmatrix} -Q_1Q_2(-\alpha_1 l/Q_2 + \alpha_1 + \alpha_2) \\ Q_1\alpha_1 - \alpha_2 Q_2 \end{bmatrix} \frac{1}{Q_1 + Q_2 - l},$$
$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{bmatrix} -2\left(1 - \dfrac{l}{Q_1}\right) & 2 \\ 2 & -2\left(1 - \dfrac{l}{Q_2}\right) \end{bmatrix}^{-1} (-2l\alpha)$$
$$= \begin{pmatrix} -Q_1Q_2(\alpha_1 + \alpha_2 - \alpha_1 l/Q_2) \\ -Q_1Q_2(\alpha_1 + \alpha_2 - \alpha_2 l/Q_1) \end{pmatrix} \frac{1}{Q_1 + Q_2 - l}.$$

More generically, the two rows of

$$-\begin{pmatrix} -A & 2 \\ 2 & -B \end{pmatrix}^{-1} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 B + 2\alpha_2 \\ A\alpha_2 + 2\alpha_1 \end{pmatrix} \frac{1}{BA - 4}$$

match the top rows on the right-hand sides of

$$\left[\begin{pmatrix} B-1 & 1 \\ B-2 & 1 \end{pmatrix}\begin{pmatrix} A-1 & 1 \\ A-2 & 1 \end{pmatrix} - 1\right]^{-1}\left[\alpha_2 + \begin{pmatrix} B-1 & 1 \\ B-2 & 1 \end{pmatrix}\alpha_1\right]$$
$$\times \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha_1 B + 2\alpha_2 \\ \alpha_1 B - A\alpha_2 - 2\alpha_1 + 2\alpha_2 \end{pmatrix} \frac{1}{BA - 4}$$

and of the similar equation with $A \to B \to A$, $\alpha_1 \to \alpha_2 \to \alpha_1$. For $N = 3$, the corresponding relations have been verified, where the rows of

$$-\begin{pmatrix} -A & 1 & 1 \\ 1 & -B & 1 \\ 1 & 1 & -C \end{pmatrix}^{-1} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} C\alpha_1 B - \alpha_1 + \alpha_2 + \alpha_3 + C\alpha_2 + B\alpha_3 \\ C\alpha_1 + \alpha_1 - \alpha_2 + CA\alpha_2 + \alpha_3 + \alpha_3 A \\ \alpha_1 + \alpha_2 - \alpha_3 + A\alpha_2 + \alpha_1 B + BA\alpha_3 \end{pmatrix} \frac{1}{CBA - C - A - B - 2}$$

match

$$\left[\begin{pmatrix} C-1 & 1 \\ C-2 & 1 \end{pmatrix}\begin{pmatrix} B-1 & 1 \\ B-2 & 1 \end{pmatrix}\begin{pmatrix} A-1 & 1 \\ A-2 & 1 \end{pmatrix} - 1\right]^{-1}\left\{\alpha_3 + \begin{pmatrix} C-1 & 1 \\ C-2 & 1 \end{pmatrix}\left[\alpha_2 + \begin{pmatrix} B-1 & 1 \\ B-2 & 1 \end{pmatrix}\alpha_1\right]\right\}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$= \frac{\begin{pmatrix} C\alpha_1 B - \alpha_1 + \alpha_2 + \alpha_3 + C\alpha_2 + B\alpha_3 \\ C\alpha_1 B - 2\alpha_1 + 2\alpha_3 - A\alpha_2 - \alpha_1 B + C\alpha_2 + B\alpha_3 - BA\alpha_3 \end{pmatrix}}{CBA - C - A - B - 2}$$

and its cyclically related counterparts ($A \to B \to C \to A$, $\alpha_1 \to \alpha_2 \to \alpha_3 \to \alpha_1$, etc.).

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