Stability of planar ring lasers with mirror misalignment

H. R. Bilger and G. E. Stedman

We consider an active cavity defined by a geometrically symmetric ring of N mirrors, each of which may have an arbitrary radius. The beam steering imposed by all possible mirror misalignment parameters is shown to diverge if the radii of curvature R_i of the mirrors satisfy two constraints of the form $\kappa_N(\{R_i\}) = 2$. By comparison, the standard conditions for the stability of rays departing from the design path in a perfectly aligned ring have the form $|\kappa_N(\{R_i\})| \leq 2$, so that only approximately half of the critical radii for the latter stability conditions is also critical radii for the former. The detailed solutions identify the most sensitive misalignment parameters regarding beam steering. Certain beam steering effects vanish for resonators with $\kappa_N = -2$; this can be of practical importance, as, for example, the $N = 2$ confocal resonator.

I. Introduction

As ring laser areas increase to, say, 1 m^2 or so (a current design of ours) and beyond, beam steering accuracy becomes increasingly important. For an active ring, it is desirable to maximize the plasma gain by using a narrow bore (a few millimeters) of the gain tube; at the same time diffraction losses for a 600-μm width laser beam need to be minimal, and the beam, therefore, needs placing in the bore accurately to 1 mm. Also, in ring interferometry, the optical interferometer coupling counterrotating beams at the output stage necessarily has its position fixed with respect to the output mirror so that beam divergence angles etc. are stable; however, this in turn requires similar close tolerances on the optical beam position. However, it is precisely for an active ring that the beam position is indeterminate to the extent that mirror positioning tolerances are finite. In addition, mirror misalignment introduces mode dispersion; we shall not discuss this.

The effect of mirror alignment on beam steering and mode dispersion has been considered for many years.[1–11] Nevertheless, many of our results, even the simpler ones, are apparently novel.

We find that while in certain choices of mirror radii and ring geometry the beam is inherently unstable with respect to mirror position and angle adjustments, these critical choices of mirror radii are also critical for instability with respect to beam deviations from the design path in a perfectly oriented ring, also that the latter instability, but not the former, is also achieved for other choices of mirror radii, which may, therefore, be relatively innocuous in practice. Indeed the linear confocal resonator is a familiar example of such partial but adequate stability; as Arnaud[11] discusses, alignment is noncritical for the confocal cavity. Since geometries with misalignment instabilities also have deviation instabilities, but not vice versa, we conclude with Siegman[10] that misalignment *per se* is not of critical importance in determining the stability of a ring. We discuss the two types of instability in Secs. III and II, respectively, and illustrate the general results in Sec. IV. In Sec. V we apply this to determining the most sensitive adjustments of certain resonators with two, three, or four mirrors.

II. Stability of Aligned Ring

Consider a planar ring of N mirrors, each having a spherical surface with radii of curvature R_i, $i = 1,2, \ldots, N$. In this section we place these mirrors precisely at the vertices of a symmetric N-polygon with side l and with their angular alignment again precisely adjusted for maximal symmetry. The N-polygon is itself the design ray path for the lasing beam. The question we address in this section is the extent to which a beam which suffers a first-order displacement from this in any way will wander uncontrollably or will remain within a reasonable (first-order) bound of the design path. While the answer is well known for a two-mirror cavity[12] and known for larger rings[13,14] also, a general development has the merit of establishing the funda-

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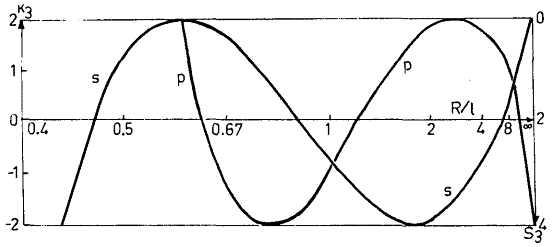

Fig. 1. Deviations $\kappa_N(\beta)$, $\kappa_N(\gamma)$ and stabilities $S_N(\beta)$ [$= \det D_N(\beta)$], $S_N(\gamma)$ in the in-plane (p) and sagittal (s) directions, for a resonator with $N = 3$ equivalent mirrors of radius R at the corners of an equilateral triangle of side l. Where the deviation is $+2, -2$, we have positive and negative criticality respectively. Both curves should be within these limits for optical path stability for a circulating beam deviating from the ideal ray in a perfectly aligned ring, while beam steering effects due to mirror misalignment are likely to be minimized if stability is maximized, and certainly are infinite for zero stability.

mental connections with the (physically distinct) stability criteria discussed in C. The development below, while not using ray matrix methods, is self-contained and concise.

Consider a single general reflection. The local axes at each mirror are chosen with x toward the radius of curvature, y the in-plane axis, and z the sagittal direction, or the out-of-plane axis tangent to the mirror surface. The ring axes have the center of the polygon as origin and are parallel to the axes on mirror I, say.

Let the beam hit mirror i at the point with local coordinates $\mathbf{r}_i = (0,\beta_i,\gamma_i)$ (to first order in the offsets β_i,γ_i, as always throughout this paper) and with ring axis coordinates $\mathbf{s}_i$, say. The normal to the mirror at this point is in local coordinates $\mathbf{n}_i = (1,-\beta_i/R_i,-\gamma_i/R_i)$. We now find the relation between three adjacent sets of $\{\beta_i,\gamma_i\}$ imposed by specular reflection.

Defining $\mathbf{s}_{i,j} \equiv \mathbf{s}_i - \mathbf{s}_j$, we have $\mathbf{s}_{I-1,I} = \mathrm{D}(\pi/N - \pi/2)\mathbf{s}_I'$, where $\mathrm{D}(\theta)$ is a matrix for anticlockwise rotation of a vector by θ or for the corresponding clockwise rotation of the axes about the z axis, and $\mathbf{s}_I' \equiv [l + (\beta_{I-1} - \beta_I)c,-(\beta_{I-1} - \beta_I)s/l,(\gamma_{I-1} - \gamma_I)/l]$ with $c = \cos(\pi/N)$, $s = \sin(\pi/N)$. Hence the unit vector parallel to $\mathbf{s}_{I-1,I}$ is $\hat{s}_{I-1,I} = \mathrm{D}(\pi/N - \pi/2)\ [1,-(\beta_{I-1} + \beta_I)s/l,(\gamma_{I-1} - \gamma_I)/l]$. Similarly, $\hat{s}_{I+1,I} = -\mathrm{D}(-\pi/2 - \pi/N)\hat{s}_{I+1}'$. The specular reflection condition may be written simply as

$$(\hat{s}_{I+1,I} + \hat{s}_{I-1,I}) \times \mathbf{n}_I = 0. \tag{1}$$

Hence

$$\gamma_{I+1} + \gamma_{I-1} = \Gamma_I\gamma_I, \quad \beta_{I+1} + \beta_{I-1} = B_I\beta_I, \tag{2}$$

where

$$\Gamma_i \equiv 2(1 - ls/R_i), \quad B_i \equiv 2[l/(sR_i) - 1]. \tag{3}$$

We note that in the first order for a planar ring the in-plane offsets β_i, and the out-of-plane offsets γ_i are governed by separated systems of equations and also that the different position of the angular factor s in B_I and Γ_I, respectively, reflects the astigmatism of an inclined mirror.

Iterating twice around the ring, we obtain a set of equations (for example, for the out-of-plane offsets)

$$\begin{aligned} &\gamma_{i-1} + \gamma_{i+1} = \Gamma_i\gamma_i, \qquad i = 2,3,\ldots,N-1,\\ &\gamma_{N-1} + \gamma_1' = \Gamma_N\gamma_N, \end{aligned} \tag{4}$$

$$\gamma_N + \gamma_2' = \Gamma_1\gamma_1', \tag{5}$$

$$\begin{aligned} &\gamma_{i-1}' + \gamma_{i+1}' = \Gamma_i\gamma_i', \qquad i = 2,3,\ldots,N-1,\\ &\gamma_{N-1}' + \gamma_1'' = \Gamma_N\gamma_N', \end{aligned} \tag{6}$$

where $\gamma_i' = \gamma_{N+i}$ etc. These $2N - 1$ equations may be used to find a linear relation between three offsets γ_1, γ_1', and γ_1'' in consecutive orbits at one mirror by eliminating the $2(N-1)$ intermediate offsets γ_i,γ_i' for $i = 2,3,\ldots,N$. For example, the linear combination $\Sigma_{i=1}^N c_i(\gamma_{i-1} + \gamma_{i+1}) = \Sigma_{i=1}^N \Gamma_i c_i\gamma_i$ from Eq. (4) and gives $\gamma_1 + c_N\gamma_1' = c_{N+1}\gamma_N$, where $c_{i-1} + c_{i+1} \equiv \Gamma_i c_i$, $i = 2,3,\ldots,N$ with $c_1 \equiv 0$, $c_2 \equiv 1$. Similarly from the last $N-1$ equations $\gamma_1'' + d_N\gamma_1 = d_{N+1}\gamma_2'$, where $d_{i-1} + d_{i+1} \equiv \Gamma_{N+2-i}d_i$. Hence by induction $c_{N+1} = d_M c_{N+3-M} - d_{M-1}c_{N+2-M}$ for all M, and in particular $d_{N+1} = c_{N+1}$. The various lower order c_i,d_i are given in Table I. On substitution of γ_2' and γ_N in Eq. (5), we get

$$\gamma_1 + \gamma_1'' = \kappa_N(\gamma)\gamma_1', \tag{7}$$

where the deviation $\kappa_N(\gamma)$ is defined by

$$\kappa_N(\gamma) \equiv c_{N+2} - d_N \tag{8}$$

and is listed in Table I along with the c_N and d_N. A similar relation is found for $\kappa_N(\beta)$ with B_i replacing Γ_i.

Table I. Coefficients in the Solutions of Difference Equations for Beam Stability in a Perfectly Aligned Ring. For Γ read B in the Case of In-plane Difference Equations.

N	c_N	d_N	κ_N
1	0	0	—
2	1	1	$\Gamma_1\Gamma_2 - 2$
3	Γ_2	Γ_N	$\Gamma_1\Gamma_2\Gamma_3 - \Gamma_1 - \Gamma_2 - \Gamma_3$
4	$\Gamma_3\Gamma_2 - 1$	$\Gamma_{N-1}\Gamma_N - 1$	$\Gamma_1\Gamma_2\Gamma_3\Gamma_4 - \Gamma_1\Gamma_2 - \Gamma_2\Gamma_3 - \Gamma_3\Gamma_4 - \Gamma_4\Gamma_1 + 2$
5	$\Gamma_4\Gamma_3\Gamma_2 - \Gamma_4 - \Gamma_2$	$\Gamma_{N-2}\Gamma_{N-1}\Gamma_N - \Gamma_{N-2} - \Gamma_N$	$\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5 - \Gamma_1\Gamma_2\Gamma_3 - \Gamma_2\Gamma_3\Gamma_4 - \Gamma_3\Gamma_4\Gamma_5 - \Gamma_4\Gamma_5\Gamma_1 - \Gamma_5\Gamma_1\Gamma_2 + \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 + \Gamma_5$
6	$\Gamma_5\Gamma_4\Gamma_3\Gamma_2\Gamma_1 - \Gamma_5\Gamma_4 - \Gamma_5\Gamma_2 - \Gamma_3\Gamma_2 + 1$	$\Gamma_{N-3}\Gamma_{N-2}\Gamma_{N-1}\Gamma_N - \Gamma_{N-1}\Gamma_N - \Gamma_{N-2}\Gamma_{N-1} - \Gamma_{N-3}\Gamma_{N-2} + 1$	$\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5\Gamma_6 - \Gamma_1\Gamma_2\Gamma_3\Gamma_4 - \Gamma_2\Gamma_3\Gamma_4\Gamma_5 - \Gamma_3\Gamma_4\Gamma_5\Gamma_6 - \Gamma_4\Gamma_5\Gamma_6\Gamma_1 - \Gamma_5\Gamma_6\Gamma_1\Gamma_2 - \Gamma_6\Gamma_1\Gamma_2\Gamma_3 + \Gamma_1\Gamma_2 + \Gamma_2\Gamma_3 + \Gamma_3\Gamma_4 + \Gamma_4\Gamma_5 + \Gamma_5\Gamma_6 + \Gamma_6\Gamma_1 + \Gamma_1\Gamma_4 + \Gamma_2\Gamma_5 + \Gamma_3\Gamma_6 - 2$

An alternative method uses matrices more explicitly. Let

$$M_{N-1}(\gamma) \equiv \begin{bmatrix} -\Gamma_2 & 1 & — & — & \cdots & — & — \\ 1 & \Gamma_3 & 1 & — & \cdots & — & — \\ — & 1 & -\Gamma_4 & — & \cdots & — & — \\ \cdot & \cdot & & & & & \\ \cdot & \cdot & & & 1 & -\Gamma_{N-1} & 1 \\ — & — & & & \cdots & 1 & -\Gamma_N \end{bmatrix}. \tag{9}$$

Then Eqs. (4) and (6) have the form $M_{N-1}(\gamma)\ (\gamma_2,\gamma_3, \ldots,\gamma_N)^T = (-\gamma_1,0,0,\ldots,0,-\gamma_1')^T$, and solving for γ_2, γ_N as in the above analysis shows that

$$\kappa_N(\gamma) = (-1)^{N+1}\{\Gamma_1 \det[M_{N-1}(\gamma)] + \mathcal{M}_{11} + \mathcal{M}_{N-1,N-1}\}, \tag{10}$$

where the cofactor matrix $\mathcal{M}_{ij} = \det[M_{N-1}(\gamma)]\text{-}[M_{N-1}(\gamma)^{-1}]_{ij}$; note that $\mathcal{M}_{1,N-1} = (-1)^N$. Hence

$$\kappa_N(\gamma) - 2 = (-1)^N \det D_N(\gamma), \tag{11}$$

where

$$D_N(\gamma) \equiv \begin{bmatrix} -\Gamma_1 & 1 & — & — & & & & \\ 1 & -\Gamma_2 & 1 & — & & & & \\ — & 1 & -\Gamma_3 & 1 & & & & \\ — & — & 1 & -\Gamma_4 & — & \cdots & — & 1 \\ \cdot & \cdot & & & — & \cdots & — & — \\ \cdot & \cdot & & & — & \cdots & — & — \\ 1 & — & — & & — & \cdots & — & — \\ & & & & & 1 & -\Gamma_{N-1} & 1 \\ & & & & & \cdots & 1 & -\Gamma_N \end{bmatrix}, \tag{12}$$

since by expansion of the determinant, $\det D_N(\gamma) = -\Gamma_1\mathcal{D}_{11} + \mathcal{D}_{12} + \mathcal{D}_{1N}$, where $\mathcal{D}_{ij}$ is the cofactor matrix for $D_N(\gamma)$, and $\mathcal{D}_{12} = \mathcal{M}_{N-1,1} - \mathcal{M}_{11}$ etc. The values of $\det D_N(\gamma)$ are, therefore, readily obtained from Table I; for example, $\det D_2(\gamma) = \Gamma_1\Gamma_2 - 4$; $\det D_3(\gamma) = -\Gamma_1\Gamma_2\Gamma_3 + \Gamma_1 + \Gamma_2 + \Gamma_3 + 2$. Borrowing the name and context from Ishchenko and Reshetin[7] and for reasons that become clear in Sec. III, we define the stability of the system as $\det D_N(\gamma)$. The stability and deviation are related by Eq. (11).

We also note for reference in Sec. V that

$$\begin{aligned} \det M_{N-1}(\gamma) &= -\Gamma_2\mathcal{M}_{11} + \mathcal{M}_{12} \\ &= -\Gamma_2 \det M_{N-2}(\gamma) - \det M_{N-3}(\gamma), \end{aligned} \tag{13}$$

where $M_{N-p}(\gamma)$ is a subdeterminant of $M_{N-1}(\gamma)$ and with the same form, containing $\Gamma_{p+1},\ldots,\Gamma_N$. Inserting this relation in Eqs. (10) and (11) and specializing in the case when all mirrors have the same radii ($\Gamma_i = \Gamma$) give the recurrence relations

$$\det D_{N+1}(\gamma) + \det D_{N-1}(\gamma) + \Gamma \det D_N(\gamma) = (-1)^N(4 - 2\Gamma), \tag{14}$$

$$\kappa_{N+1}(\gamma) + \kappa_{N-1}(\gamma) = \Gamma\kappa_N(\gamma) \tag{15}$$

For stability of the optical path, we seek a solution to Eq. (7) of the form $\gamma_1^{(p)} = X \sin(p\varphi + \alpha)$ and similarly for $\beta_1^{(p)}$; this requires the identification $\kappa_N(\gamma)$ [and $\kappa_N(\beta)$] $= 2\cos\varphi$. In particular, this requires that the modulus of the deviation in each plane be $\not> 2$:

$$|\kappa_N(\gamma)| \le 2 \ge |\kappa_N(\beta)|. \tag{16}$$

This is the general stability criterion for perfect alignment. Since from Eq. (8) the various κ_N are either even or odd in the quantities B_i,Γ_i the sign change between the definitions of each of these quantities is of no formal consequence in this section; the only difference between the stability criteria between in-plane β_i and out-of-plane γ_i offsets is that the critical radii differ by a factor of $\sin^2(\pi/N)$, the out-of-plane critical radii R_s being smaller than the in-plane critical radii R_p. However, for the purposes of Sec. IV.C we need to distinguish solutions of Eq. (16) for which either $\kappa_N(\beta)$ or $\kappa_N(\gamma)$ equals $+2$ from those when these quantities equal -2; among other things, this affects the kind of solution through the sign difference between B_i and Γ_i in Eq. (3). We name these types of solution as being of positive or negative criticality, respectively. We add a criticality superscript $\pm$ corresponding to the sign of $\kappa_N(\beta)$ or $\kappa_N(\gamma)$ to all roots in all following sections.

From Eq. (12), $\det D_N(\gamma)$ certainly vanishes if all $\Gamma_i = \Gamma = 2^+$. (Add all the rows or columns of the determinant to verify this.) Hence a ring of flat mirrors is always of critical stability for beam steering in a sagittal direction, although not necessarily in the in-plane direction (the generalization of this to nonplanar rings is considered by Ishchenko and Reshetin[7] and Smith[8] using Coxeter group theory), and a ring of mirrors with $B_i = B = 2^+$ i.e., $R = l/[4\sin(\pi/N)]$ is always unstable with respect to in-plane motion.

III. Stability of a Lasing Path under Mirror Misalignment

Suppose now that relative to the local axes defined in Sec. II, which are centered on the vertices of a regular polygon, the mirrors are displaced by offsets ξ_i,η_i,ζ_i in each of the three local spatial dimensions x,y,z and are tilted by angles θ_i,φ_i about the (local) y and z axes, respectively. We consider the first-order effects of these $5N$ offset parameters on the path of the lasing beam by searching for a path which closes on itself after one circuit of specular reflections. It transpires that very similar, but not identical, equations to those in Sec. II require solution and that the existence of a closed path different only to first order from the design path for optimum mirror alignment depends on the invertibility of a matrix which depends only on the mirror radii. For certain critical radii, this matrix is singular, and misalignment of the mirrors renders the lasing beam unstable. It is to be expected that these critical radii are related to those of Sec. II, but since the physics is somewhat different, the relationship is not trivial. In fact, the relevant matrices turn out to be $D_N(\beta),D_N(\gamma)$ for in-plane and out-of-plane misalignments, respectively.

Misalignment has the effect of introducing linear terms in all such geometric offsets in all the vectors discussed in Sec. II. For example, the new position of a laser spot, and the new normal at this point, would be

$$\begin{aligned} \mathbf{r}_i' &= (\xi_i,\beta_i + \eta_i,\gamma_i + \zeta_i), \\ \mathbf{n}_i' &= (1,-\beta_i/R_i + \varphi_i,-\gamma_i/R_i - \theta_i), \end{aligned} \tag{17}$$

respectively, and to first order in the local axes. These extra terms only induce additive effects in Eqs. (4)–(6), and again the equations for in-plane and out-of-plane beam motions separate.

Again we consider out-of-plane offsets explicitly, the in-plane results being given by substituting β_i for γ_i etc. Since we seek a closed path, we may set $\gamma_i^{(p)} = \gamma_i$ and conclude that $\gamma_{i-1} + \gamma_{i+1} - \Gamma_i\gamma_i = \mathcal{G}_i$, that is,

$$D_N(\gamma)\gamma = \mathcal{G}, \tag{18}$$

where $\mathcal{G} = (\mathcal{G}_i)$ is a vector of offsets controlling out-of-plane beam steering; the corresponding set for in-plane steering ($\gamma \rightarrow \beta$ etc.) is denoted $\mathcal{B}$. Their exact form is irrelevant for the moment (but is given in Sec. V), since the left-hand side alone of Eq. (18), which depends only on design parameters, suffices to determine the instability constraints. We note, however, that they have the typical size of linear offsets (misalignments) in position and of the arc length of an angular misalignment when projected over a typical distance l. In particular, this disproves the assertion of Ishchenko and Reshetin[7] that such tilts are of the order of l/a times the misalignment, a being the mirror size. In the first analysis, we may expect beam position changes of the order of magnitude of these quantities.

Hence if the stability, $\det D_N(\gamma)$, is nonzero, the matrix $D_N(\gamma)$ is invertible, and the beam shift parameters γ_i are simple linear combinations of the geometric offset parameters. If, on the other hand, the stability vanishes, a first-order misalignment of any mirror makes the beam unstable, and no closed path near the design ray is obtained. Ishchenko and Reshetin[7] generalize this to nonplanar rings using ray matrix methods.

This instability corresponds exactly to one choice of sign in Eq. (16), since the condition for the impossibility of inverting Eq. (18) reduces to

$$\kappa_N(\gamma) = 2 \text{ and/or } \kappa_N(\beta) = 2. \tag{19}$$

We may, therefore, expect that resonators with negative criticality, i.e., with either deviation $\kappa_N(\beta)$ or $\kappa_N(\gamma) = -2$, are less unstable than resonators with positive criticality, i.e., either deviation equal to 2. (Further support for this is given in the detailed assessment of beam steering from misalignment in Sec. V.) This is why we anticipated the distinguishing of these cases in Sec. II.

The value of the stability has some effect on the likely magnitudes of the deviations of the beam from its design position; the closer it is to the critically unstable value of zero, the more probable it is that at least some of the parameters describing the change in beam position is large, i.e., that certain γ_i is much larger than the corresponding offsets $\mathcal{G}_j$. A fuller analysis is given in Sec. V.

IV. Applications

In general, flat mirrors form a critical resonator with positive criticality [$R = \infty^+$; superscripts denote the criticality, i.e., whether $\kappa_N(\beta)$ or $\kappa_N(\gamma) = \pm 2$]. In the figures we plot the dependence of the stability $S_N = \det D_N(\beta,\gamma)$ and the deviation $\kappa_N(\beta,\gamma)$ on the mirror radii for the sagittal and in-plane directions for a few of the more interesting cases.

A. Two-Mirror Resonator

In-plane and out-of-plane analyses are, of course, indistinguishable. The stability condition reduces to the well-known form[4]

$$0^+ \leq (1 - l/R_1)(1 - l/R_2) \leq 1^-.$$

For completeness we list the various applications.

1. *Equal Radii*

We have $(1 - l/R)^2 = 0^-$ or 1^+ so that $\Gamma = 0^-$ (twice), $\Gamma = \pm 2^+$ or $R = l/2^+, l^-$ (twice), and ∞^+. The repeated root corresponds to a confocal cavity and thus to a second-order approach to a stability breakdown: in practice this does not preclude practical laser operation. Ewanisky,[12] for example, has remarked on the relative stability of a confocal resonator for other reasons. Our work indicates that the confocal resonator has the added advantage of negative criticality, i.e., of not contracting the instability discussed in Sec. III. The cases of flat mirrors ($R = \infty$) and the concentric cavity ($R = l/2$) give the more significant stability thresholds.

2. *One Flat, One Curved Mirror*

In this case $0^- \leq 1 - l/R \leq 1^+$, i.e., $l^- \leq R \leq \infty^+$.

B. Three-Mirror Resonator (Equilateral Triangle)

For this case (from Table I) $\kappa_3(\gamma) = \Gamma_1\Gamma_2\Gamma_3 - \Gamma_1 - \Gamma_2 - \Gamma_3$, where $\Gamma_i = 2(1 - \sqrt{3}l/2R_i)$. The in-plane critical radii (from B) are 4/3 of the corresponding out-of-plane critical radii: $R_p = 4R_s/3$. Since $\kappa_N(\{\Gamma_i\}) = (-1)^N\kappa_N(\{-\Gamma_i\})$, and because of the sign difference in Eq. (3), the superscript sign should be reversed in this subsection for in-plane stability considerations. For clarity, then, we list in-plane criteria separately.

1. *Three Equal Mirrors*

$\kappa_3(\gamma) = \Gamma^3 - 3\Gamma$ and has modulus 2 when $\Gamma = +2^+$, -2^- (once) and $+1^-$, -1^+ (twice), corresponding to radii $R_s = \infty^+$, $\sqrt{3}l/4^-$ and $\sqrt{3}l^-$, $l/\sqrt{3}^+$, respectively, the first two being first-order instability onsets and the second two second-order approaches to instability. These stability regions, with the scaled curve for in-plane offsets, are depicted in Fig. 1.

For in-plane stability the corresponding quantitites are $R_p = \infty^-$, $l/\sqrt{3}^+$ (first order), $4l/\sqrt{3}^-$, and $4l/3\sqrt{3}^+$.

2. *One Flat, Two Equal Curved Mirrors*

$\kappa_3(\gamma) = 2(\Gamma^2 - \Gamma - 1)$ and has modulus 2 when $\Gamma = 0,\pm1,2$; i.e., the two equal mirrors have radii $R_s = \infty^+,\sqrt{3}l^-,\sqrt{3}l/2^-,l/\sqrt{3}^+$. There are now two stability bands for each type of offset (Fig. 2); note the decidedly restricted passbands or regions of mirror radii choice where stability in both directions is possible.

For in-plane stability, the corresponding quantities are $R_p = \infty^-,4l/\sqrt{3}^+,2l/\sqrt{3}^+,4l/3\sqrt{3}^-$.

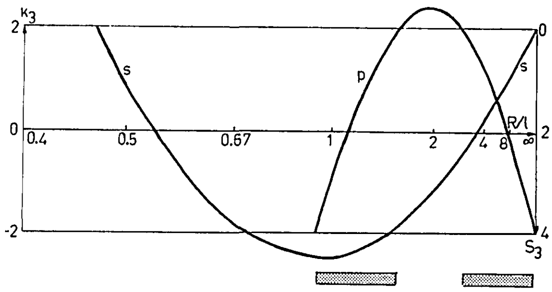

Fig. 2. Deviations and stabilities as for Fig. 1, except that one mirror is an optical flat ($R_3 = \infty$). Note the (hatched) pass bands for stable laser action in the parameter l/R.

3. *Two Flat Mirrors and One Curved Mirror*

$\kappa_3(\gamma) = 3\Gamma - 4$ and has modulus 2 when $R_s = 3\sqrt{3}l/4^{-},\infty^{+}$. For in-plane stability, $R_p = \infty^{-},\sqrt{3}l^{+}$.

C. Four-Mirror Square Resonator

For this case (Table I) $\kappa_4(\gamma) = \Gamma_1\Gamma_2\Gamma_3\Gamma_4 - \Gamma_1\Gamma_2 - \Gamma_2\Gamma_3 - \Gamma_3\Gamma_4 - \Gamma_4\Gamma_1 + 2$, where $\Gamma_i = 2(1 - \sqrt{2}l/R_i)$. The in-plane critical radii are increased by a factor of 2 compared to the out-of-plane critical radii derived below: $R_p = 2R_s$.

1. *Four Equal Mirrors*

$\kappa_4(\gamma) = \Gamma^4 - 4\Gamma^2 + 2$ and has modulus 2 if $\Gamma = \pm 2^{+}$ once and $\Gamma = \pm\sqrt{2}^{-},0^{+}$ twice. This corresponds to radii $R_s = \infty^{+},l/\sqrt{2}^{+}$ (first order) and $R_s = 2l/(\sqrt{2} \pm 1)^{-},\sqrt{2}l^{+}$ (second order), respectively (Fig. 3).

2. *Two Flat Mirrors and Two Equal Curved Mirrors Diagonally Arranged*

$\kappa_4(\gamma) = 4\Gamma^2 - 8\Gamma + 2$ and has modulus 2 if $\Gamma = 0^{+},2^{+}$ (once) and $\Gamma = 1^{-}$ (twice); i.e., $R_s = \sqrt{2}l^{+},\infty^{+}$ (first order), and $R_s = 2\sqrt{2}l^{-}$ (second order). This second-order instability onset also is of negative criticality and may be expected to correspond to a feasible lasing geometry much as for the confocal resonator with $N = 2$.

3. *Two Flat Mirrors and Two Equal Curved Mirrors Adjacent in Pairs*

$\kappa_4(\gamma) = 2(\Gamma^2 - 3)$, so that $\Gamma = \pm 2^{+},\pm\sqrt{2}^{-}$ and $R_s = l/(\sqrt{2} \pm 1)^{-},2\sqrt{2}l^{+},\infty^{+}$.

4. *One Curved and Three Flat Mirrors*

$\kappa_4(\gamma) = 4\Gamma - 6$ with modulus 2 if $\Gamma = 1,2$, i.e., $R_s = 2\sqrt{2}l^{-},\infty^{+}$.

V. Magnitude of Beam Steering Effects

A. Introduction

The discussion of Sec. IV was purely qualitative. It strongly suggests that regions of mirror radius parameter space ($\{R_i\}$) away from critical values of either sign of criticality, but particularly positive criticality, are likely to give usable systems. In practice this means using at least some curved mirrors with R_i values rather greater than l. The final choice may be made for different reasons entirely, such as securing such splitting of Hermite-Gaussian modes as will facilitate single-mode operation. However, since at least one practical design has negative criticality, it is of interest to study the beam deviations γ more quantitatively, for example, the degree of divergence of beam deviations at critical points Γ_0, parametrizing this by noting the maximal dependence $\gamma_j \sim (\Gamma - \Gamma_0)^{-p}$, where p is a critical index and Γ is the (assumed only) independent Γ factor.

A second reason for a fuller analysis is that in practice it may prove necessary to adjust mirrors to reduce beam deviations from the design positions. On an empirical approach, this would be a frustratingly complex flea-jumping exercise in a large ring requiring considerable experience, the adjustment at one mirror causing an apparently and grotesquely unrelated movement in the beam at another mirror. We are in a position to transform such an empirical black art into a science.

It will be shown by way of examples in small rings that when a resonator has negative criticality in second order, the diagonal elements of the beam steering matrix vanish. This singular and unexpected result, while probably not general, is certainly useful. A fuller analysis is at the end.

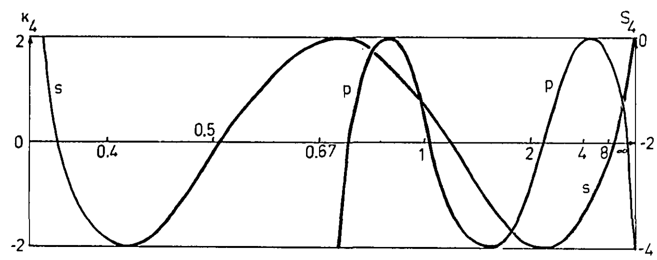

Fig. 3. Deviations and stabilities as for Fig. 1, now for a square resonator of side l with four equivalent mirrors.

B. General

When the revised geometry of Eq. (17) is fed into the analysis of Sec. I, we obtain Eq. (18) (and the β, $\mathcal{B}$ counterpart) with the identifications

$$\begin{aligned} \mathcal{G}_i &= -2sl\theta_i - (\zeta_{i+1} + \zeta_{i-1} - 2\zeta_i), \\ \mathcal{B}_i &= -2l\varphi_i/s + (c/s)(\xi_{i+1} - \xi_{i-1}) \\ &\quad - (\eta_{i+1} + \eta_{i-1} + 2\eta_i). \end{aligned} \tag{20}$$

This shows immediately several qualitative and quantitative results about beam steering. First, it is of the order of the linear displacements in $\mathbf{r}_i'$, i.e., of ξ_i, η_i, and ζ_i, and of the order of the angular displacements θ_i,φ_i when multiplied by a lever arm typically of length l. Hence stabilities of the order of 1 mm in beam position demand commensurate accuracy in the absolute positioning of the mirrors (say a factor of <10 or 100 μm) and angular precisions below typically 10^{-4} rad or 20 sec of arc.

It is interesting to note, however, the effect of the trigonometric factors in revising this qualitative guide. The system is more sensitive to φ_i than to θ_i by a factor $\csc^2(\pi/N)$. (This factor is 4/3 for a triangle and 2 for a square ring.) Hence on this consideration by itself it is somewhat more important to control accurately the angle made by the mirror with a line in the plane of the ring than to hold the mirrors accurately perpendicular to the plane of the ring. (φ is the misalignment angle with respect to rotations about the normal to the plane.)

All these estimates are scaled by the elements of the inverse matrix and so in particular by the stability $\det D_N(\gamma)$. These scaling factors may indeed give rise to order-of-magnitude magnifications of beam steering effects over the initial geometrical distortions discussed above, particularly as the stability decreases. Such magnification factors were discussed many years ago for unstable two-mirror confocal resonators by Krupke and Sooy.[1] Our analysis generalizes theirs to more complicated rings, and the connection with their formalism is indicated in Sec. V.C.

C. Two-Mirror Resonator

Particular solutions of the matrix inversion show more precisely which adjustments are most critical. For example, take $N = 2$. Inverting the matrix $D_2(\gamma)$, we have

$$\gamma = \begin{bmatrix} \Gamma_2 & 2 \\ 2 & \Gamma_1 \end{bmatrix} \mathcal{G}/(4 - \Gamma_1\Gamma_2). \tag{21}$$

We translate this into the language of Krupke and Sooy[1] as an illustration of the meaning of our equations in a familiar problem. The difference $\gamma_1 - \gamma_2$ in beam positions (more precisely, their projection in one dimension, the other being determined by the replacement $\gamma \rightarrow \beta$ etc.) at the two mirrors, when divided by their separation, measures the angular displacement of the final beam. The corresponding difference in the components of $\mathcal{G}$ similarly normalized gives the initial angular displacement. The ratio of these two angles is then a magnification factor coupling the initial to the final beam angular displacements. The (dimensionless) elements of the matrix $D_N(\gamma)^{-1}$ are a matrix generalization of such sets of magnification factors. In the case $N = 2$ and with the substitution $g_i \equiv \Gamma_i/2$, Eq. (21) gives

$$\gamma_1 - \gamma_2 = \left[\frac{1 - g_2}{g_1 g_2 - 1}\right]\mathcal{G}_1 - \left[\frac{1 - g_1}{g_1 g_2 - 1}\right]\mathcal{G}_2.$$

The factors here are clearly the magnification factors of Eq. (26) of Krupke and Sooy.[1] These authors pursued this analysis to determine the connections with output power, mirror diameters, and critical angles when, for example, the beams walked off the mirror faces entirely. While our analysis could obviously be extended to discuss such questions, we have in mind (see Sec. I) applications to stable high-Q systems, where the resonator becomes unusable well before the critical angles discussed in Ref. 1.

Since the various Γ_i in Eq. (21) all have modulus of the order of, and generally less than, 2 for stability the magnitudes of the beam steering effects are controlled principally by the offsets $\mathcal{G}$ and the denominator; the latter should, therefore, be maximal. Ishchenko and Reshetin[7] state that mirror curvature does not enhance focusing. However, one important qualification is when either Γ_1 or $\Gamma_2 = 0$; a diagonal term in the beam steering matrix then vanishes. Both vanish for the second-order critical point with negative criticality $R_1 = R_2 = l$, i.e., to a confocal resonator, which thus shows a tendency to stability in this regard! (For points of negative criticality, the denominator has the fixed magnitude of 4 in all systems, and the matrix elements in this and following expressions give a reliable estimate of beam steering effects.) In detail, for a confocal resonator, $\gamma_1 = -l\theta_2 + \zeta_2 - \zeta_1$ independently of θ_1; similarly γ_2 is independent of θ_2, β_1 is independent of φ_2, and β_2 of φ_1. Curved mirrors are likely to have smaller Γ_i and thus induce somewhat smaller beam steering effects in these diagonal terms than would flat mirrors (for which $\Gamma_i = 2$), assuming that the denominator maintains a similar size.

For equal mirrors, the critical index for both $\Gamma = \pm 2$ is $p = 1$. Beam steering diverges hyperbolically on approach to a point of positive criticality, i.e., the flat mirror case and the concentric cavity.

D. Three-Mirror Resonator

For $N = 3$ and with identical mirrors, the inverse matrix has the form

$$\gamma = \begin{bmatrix} \Gamma^2 - 1 & \Gamma + 1 & \Gamma + 1 \\ \Gamma + 1 & \Gamma^2 - 1 & \Gamma + 1 \\ \Gamma + 1 & \Gamma + 1 & \Gamma^2 - 1 \end{bmatrix} \mathcal{G}/(3\Gamma + 2 - \Gamma^3). \tag{22}$$

The case of $\Gamma = -1$, i.e., positive criticality in second order, corresponds to diverging beam steering effects in all matrix elements with an index $p = 1$, since the denominator diverges quadratically. However, diagonal elements genuinely vanish for the second-order points $\Gamma = 1$ with negative criticality. Once again, then, such second-order values are likely to be relatively stable for real systems.

E. Four-Mirror Resonator

First, consider four identical mirrors. We find

$$\gamma = \begin{bmatrix} 2 - \Gamma^2 & -\Gamma & -2 & -\Gamma \\ -\Gamma & 2 - \Gamma^2 & -\Gamma & -2 \\ -2 & -\Gamma & 2 - \Gamma^2 & -\Gamma \\ -\Gamma & -2 & -\Gamma & 2 - \Gamma^2 \end{bmatrix} \mathcal{G}/[\Gamma(\Gamma^2 - 4)]. \tag{23}$$

Again in this case the second-order instability $\Gamma = 0$ with positive criticality has a critical index $p = 1$, and the second-order negative criticality case $\Gamma = \pm\sqrt{2}$ corresponds to the absence of diagonal components in the beam steering matrix.

Consider finally the four-mirror resonator with two flat and two identical curved mirrors diagonally arranged. We obtain

$$\gamma = \begin{bmatrix} 4(1 - \Gamma) & -2\Gamma & -4 & -2\Gamma \\ -2\Gamma & 2\Gamma(1 - \Gamma) & -2\Gamma & -2\Gamma \\ -4 & -2\Gamma & 4(1 - \Gamma) & -2\Gamma \\ -2\Gamma & -2\Gamma & -2\Gamma & 2\Gamma(1 - \Gamma) \end{bmatrix} \mathcal{G}/[4\Gamma^2 - 8\Gamma], \tag{24}$$

which again shows the property that diagonal matrix elements vanish for second-order points of negative criticality ($\Gamma = 1$ in this case).

This suggests that a general result exists. However, proof has eluded us. Even if we consider systems in which all mirrors have the same radii and use the very symmetric recurrence relations of Eqs. (14) and (15), we soon find that not all deviations correspond to second-order negative criticality roots; indeed $N = 5$ gives a counterexample. One might still hope that should such a condition hold, it would be possible to show in general that diagonal beam steering matrix elements vanish. The vanishing of a typical diagonal component, say $D_N(\gamma)_{11}^{-1}$, means that $\mathcal{D}_{11} = 0$, i.e., $\det M_{N-1}(\gamma) = 0$. From Eqs. (10)–(13) this requires the conditions $\Gamma_2 = \mathcal{M}_{12}/\mathcal{M}_{11}$, and for equal mirror radii $\kappa_N(\gamma) = 2(-1)^{N+1}M_{N-2}(\gamma) = -2$, and so $\det M_{N-2}(\gamma) = (-1)^N$, $\mathcal{D}_{12} = 2(-1)^{N+1}$, etc. We have not been able to complete the argument by proving that $\det M_{N-1}(\gamma)$ must vanish. In any case it is unlikely to hold in more general systems; the last example considered above had a palindromic symmetry which is not general.

References

1. W. F. Krupke and W. R. Sooy, "Properties of an Unstable Confocal Resonator CO_2 Laser System," IEEE J. Quantum Electron. **QE-5**, 575 (1969).
2. R. J. Freiberg and A. S. Halsted, "Properties of Low Order Transverse Modes in Ar^+ Lasers," Appl. Opt. **8**, 355 (1969).
3. N. Yamauchi, "Resonant Modes in a Ring Laser Resonator and their Deformations by Mirror Rearrangements," Electron. Commun. Jpn. **57-C**, 92 (1974).
4. R. Hauck, H. P. Kortz, and H. Weber, "Misalignment Sensitivity of Optical Resonators," Appl. Opt. **19**, 598 (1980).
5. A. D. Schnurr, "Unstable Resonator Misalignment in Ring and Linear Toroidal Resonators," Appl. Opt. **22**, 298 (1983).
6. V. Magni, "Resonators for Solid State Lasers with Large-Volume Fundamental Mode and High Alignment Stability," Appl. Opt. **25**, 107 (1986).
7. E. F. Ishchenko and E. F. Reshetin, "Sensitivity to Misalignment of an Optical Ring Resonator," Opt. Spektrosk. **46**, 202 (1979) [Opt. Spectrosc. USSR **46**, 366 (1979)].
8. I. W. Smith, "Optical Resonator Axis Stability and Instability from First Principles," Proc. Soc. Photo-Opt. Instrum. Eng. **412**, 203 (1983).
9. T. Habib, M.Sc. Thesis, Oklahoma State U. (1986).
10. A. E. Siegman, *Lasers* (University Science Books, Mill Hill Valley, CA, 1986), p. 607.
11. J. A. Arnaud, "Degenerate Optical Cavities. 2: Effect of Misalignments," Appl. Opt. **8**, 1909 (1969).
12. J. T. Verdeyen, *Laser Electronics* (Prentice-Hall, Englewood Cliffs, N.J., 1981).
13. S. A. Collins, Jr., "Analysis of Optical Resonators Involving Focusing Elements," Appl. Opt. **3**, 1263 (1964).
14. H. R. Bilger, "Modelling a Large Ring Resonator Gyroscope," Report: AFOSR grant 84-0058 (1985).
15. T. F. Ewanizky, "Ray Transfer Matrix Approach to Unstable Resonator Analysis," Appl. Opt. **18**, 724 (1979).