10-1-1994

Generalized reverse theorems for multipass applications in matrix optics

Lee W. Casperson
Portland State University

Anthony A. Tovar

Citation Details

Anthony A. Tovar and Lee W. Casperson, "Generalized reverse theorems for multipass applications in matrix optics," J. Opt. Soc. Am. A 11, 2633-2642 (1994).

Generalized reverse theorems for multipass applications in matrix optics

Anthony A. Tovar and Lee W. Casperson

Department of Electrical Engineering, Portland State University, Portland, Oregon 97207-0751

Received January 3, 1994; revised manuscript received April 5, 1994; accepted April 14, 1994

The reverse theorem that governs backward propagation through optical systems represented by transfer matrices is examined for various matrix theories. We extend several reverse theorems to allow for optical systems represented by matrices that may or may not be unimodular and that may be 2×2 or take on an augmented 3×3 form. As an example, we use the 3×3 form of the reverse theorem to study a laser with intracavity misaligned optics. It is shown that, by tilting one of the laser's mirrors, we can align the laser output arbitrarily, and the mirror tilt angle is calculated.

1. INTRODUCTION

Simple 2×2 transfer-matrix methods are commonly used for studying a wide variety of problems in optics[1] and in other areas of engineering and physics. Such a matrix method exists, for example, to trace the position and the slope of paraxial light rays through optical systems that include lenses, mirrors, lenslike media, and other optical components. Similarly, a 2×2 matrix method[2,3] is used in Gaussian beam theory, where the beam's width and phase front curvature are propagated through more general optical systems that may include complex lenslike media[4,5] and Gaussian apertures.[6] The Jones calculus matrix method for polarization calculations may be used for propagating the two Cartesian electric-field components of TEM plane waves through optical systems that contain birefringent optical elements and polarizers.[7–9] The voltage and current transfer characteristics of an electric circuit may be obtained by the use of two-port network matrices. There are also 2×2 matrix theories that govern light propagation through thin films,[10,11] distributed-feedback waveguides and lasers,[12] and Gaussian light pulses through chirping elements.[13,14] Matrix methods may also be used in the study of quantum mechanics,[15] magnetic circuits, mechanical systems with springs, and computer graphics.[16] The use of a 2×2 matrix method has several advantages over the use of other analytical methods[17] and provides an orderly systems approach. Matrix methods encourage a standardization of notation and the use of diagrams. Highly sought-after analogies become transparent.

Many optical systems contain some type of reflecting element that causes the light signal to propagate through all or part of an optical system backward. For example, standing-wave and bidirectional ring laser oscillators contain optical signals that propagate through their intracavity optics in both directions. Reflective elements may also be used in optical system design in which some desired effect is to be enhanced. This is the case in multipass amplifier schemes for increased amplification[18] or for distortion correction with phase-conjugate mirrors. Similarly, multipass schemes may be used to decrease the transmission bandwidth of a filter. Another category of laser applications involves remote sensing and control,[19] which may require reverse propagation through the optical system. Examples include remote sensing of the atmosphere, nondestructive evaluation, adaptive optics, fiber-optic sensors, and microscopy. When the optical system is represented by a given matrix, then the corresponding matrix that represents backward propagation through the system is of interest. This reverse matrix is also important if there are established system symmetry requirements or if there is a need for experimental determination of a system matrix.

Based on the examination of several types of optical elements and systems, one is sometimes able to divine the form of the reverse matrix. However, such a methodology ought not to be necessary, and systematic procedures are demonstrated to yield the reverse matrix for most conventional matrix theories. The reverse matrix is often reported only for the special case of unimodular matrix theories. However, many matrix theories are unimodular only for some special case. For example, Jones calculus is unimodular only when the optical system is lossless and when absolute phase is ignored. A notable extension of the Jones calculus accounts, to first order, for polarization-dependent Fresnel reflection and refraction for nonnormal incidence. This extended Jones matrix method[20] retains the simple 2×2 form but is inherently nonunimodular. Of course, if the birefringent optical system contains polarizers, then the system is represented as a zero-determinant matrix, and there is no unique reverse matrix. The characteristic matrix method for light propagation in stratified media[11] is unimodular only when the media and the boundaries are lossless.[10] Similar restrictions are involved in transfer matrices used for distributed-feedback structures[12] and fiber ring resonators.[21] In the case of Gaussian beams and paraxial rays the unimodularity condition occurs only when the medium at the output has the same refractive properties as the medium at the input.[22] The generalization of the nonunimodu-

0740-3232/94/102633-10$06.00

lar reverse matrix concept to other matrix theories is addressed in this paper.

For every 2×2 matrix method there is an augmented matrix that corresponds to a 3×3 matrix method. The form of the 3×3 matrix of interest here is much simpler than the general 3×3 matrix. In both the paraxial ray matrix theory and the Gaussian beam theory the 3×3 matrix method permits the designer to trace paraxial light rays and Gaussian beams through misaligned optical systems.[1] This 3×3 formalism may be applied, for example, to the design of pulse compressors.[23–25] Similarly, 3×3 matrix methods are necessary for studying electrical circuits that contain intranetwork independent voltage and current sources. Another example exists in computer graphics, in which operations are performed on subfigures in a picture by means of 2×2 matrix multiplication. However, to perform translation one needs an augmented 3×3 matrix description.[16]

The purpose of this paper is to generalize the concept of a reverse matrix so that it applies to a variety of optical systems that are represented by matrices that may be nonunimodular, 3×3, or both. A unified overview of 2×2 transfer-matrix theory is given in Section 2. In Section 3 the reverse matrix is derived for several optical matrix theories. The reverse matrix is also generalized for unimodular 3×3 matrix theories. In Section 4 the importance of the results is highlighted by application of the theory to a practical example. In particular, it is demonstrated that a Fabry–Perot laser's output may be arbitrarily aligned, even though it contains tilted intracavity optics. The alignment procedure simply involves tilting one of the laser mirrors, and the mirror tilt angle is calculated.

2. GENERAL PROPERTIES OF 2×2 TRANSFER-MATRIX METHODS

There are several properties common to 2×2 transfer-matrix methods, and certain classes of matrices arise in several of the theories. In this section we give a unified overview of these general properties to exploit analogies between matrix theories. These analogies suggest several novel elements, including a cross-wiring circuit element, intranetwork independent voltage and current sources in electric circuits, and multiple-input–single-output birefringent and distributed-feedback optical systems. In addition to matrix properties, there are several universal matrix operations that may be used in system synthesis. These operations include raising a matrix to an arbitrary power, backward propagation through a matrix-represented component, and matrix factorization. For each 2×2 matrix theory there is also an associated bilinear transformation. Each of these properties is discussed in this section.

The propagation formulas for any given 2×2 transfer-matrix theory can be written in the form

$$\begin{pmatrix} X_2 \\ Y_2 \end{pmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}, \tag{1}$$

where X and Y are the two dependent parameters that change through the existence of the system. In our notation, whether the signal is injected in the forward or the reverse direction, X_2 and Y_2 represent the output and X_1 and Y_1 represent the input parameters. The A and D matrix elements are dimensionless. The units of B are the units of X divided by the units of Y. The units of C are the multiplicative inverse of the units of B.

The $ABCD$ matrix in Eq. (1) may represent forward propagation through a single system element, or it may refer to the overall system matrix. To obtain this system matrix given the individual element matrices, one need only multiply the system elements in reverse order. This can be easily seen from an example. The special case of a system consisting of two cascaded elements is considered. If the first element is given by Eq. (1), then the second is

$$\begin{pmatrix} X_3 \\ Y_3 \end{pmatrix} = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}. \tag{2}$$

The total system matrix may be obtained by the substitution of Eq. (1) into Eq. (2):

$$\begin{pmatrix} X_3 \\ Y_3 \end{pmatrix} = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}. \tag{3}$$

Then, as we stated above, a system matrix is defined as the product of individual element matrices in reverse order. It follows from induction that, if the system consists of n elements, then the total system matrix is

$$T_{\text{system}} = T_n T_{n-1} \cdots T_3 T_2 T_1 . \tag{4}$$

In many matrix theories the determinant of each of the system elements is unity, i.e., the matrix for the element is unimodular. Since the determinant of a product is the product of the determinants, it follows that, for such a matrix theory, any system matrix will be unimodular. For nonunimodular matrix theories the determinant usually carries important information.

It is sometimes of interest that certain properties of a given system be conserved. As an example, it may be desired to examine systems for which $X_2{}^*X_2 + Y_2{}^*Y_2 = X_1{}^*X_1 + Y_1{}^*Y_1$. The class of matrices for which this is true is called unitary. It can be shown that $T_T{}^* = T^{-1}$ for a unitary matrix, where the T subscript represents transposition (i.e., interchange of the B and C elements) and the asterisk represents complex conjugation. Unitary matrices have the properties that the complex magnitude of their determinants is unity and that the product of unitary matrices is a unitary matrix.

A. Specific Matrices

In the study of matrix optics one finds that there are individual matrices that commonly arise in several matrix theories. To reinforce analogies between systems, we find it useful to emphasize these matrices.

The first matrix to be considered is the identity matrix,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{5}$$

which changes neither the X component nor the Y component of the signal. The identity matrix can be used as a continuity condition. However, it is also sometimes highly desirable to design an overall system that does not change the input signal. In this case the system matrix has this identity matrix form. This design methodology is of interest in a Gaussian beam optical system when, for example, a flat untilted mirror is optimum but practical problems force one to position the mirror away from the end of the laser.[22] In the Jones calculus matrix method for optical polarization calculations one can imagine an optical system that, on account of its nature, distorts the input polarization. In this case it may be desired to synthesize a system so that the overall system matrix is the identity matrix.

The unimodular matrix

$$\begin{bmatrix} 1 & \chi \\ 0 & 1 \end{bmatrix} \tag{6}$$

changes the X component without changing the Y signal component. If χ is real and positive here, then expression (6) is the matrix representation of a uniform medium in Gaussian beam matrix theory.[2] In electrical circuit matrix theory, χ represents a series impedance.

A dual of expression (6) is the unimodular matrix

$$\begin{bmatrix} 1 & 0 \\ \chi & 1 \end{bmatrix}, \tag{7}$$

which changes the Y signal component without changing the X signal component. It is used to represent a thin lens and/or Gaussian aperture in the Gaussian beam theory and a shunt impedance in the electrical theory.

Because of the commonness and the importance of matrices (5)–(7), they are viable candidates as matrix primitives from which an arbitrary system may be synthesized.[22] In the paraxial ray and Gaussian beam theories many simple systems are made up of flat mirrors [expression (5)], uniform media [expression (6)], and lenses [expression (7)]. Similarly, in the electric circuit theory many two-port systems are composed only of series [expression (6)] and shunt [expression (7)] impedances.

Scaling can be obtained by the not necessarily unimodular matrix

$$\begin{bmatrix} \chi_X & 0 \\ 0 & \chi_Y \end{bmatrix}, \tag{8}$$

where χ_X and χ_Y are the scale factors for the X and Y signal components, respectively. In the Gaussian beam theory, expression (8) with $\chi_X = 1$ represents a dielectric boundary.[6] In the unimodular limit, $\chi_X = \chi_Y^{-1}$, and a matrix of this form is used to represent an ideal transformer in the electrical theory and an anisotropic medium[7] in the Jones calculus matrix method.

Symmetry is often considered a desirable property in a system. For our purposes a symmetric system is one that causes a signal injected backward to undergo the same transformation as one injected in the forward direction. As we will see below, for several types of unimodular systems the requirement of symmetry implies that $A = D$. A unimodular matrix in which the diagonal elements are equal ($A = D$) can be put in the form

$$\begin{bmatrix} \cos\theta & \chi \sin\theta \\ -\chi^{-1} \sin\theta & \cos\theta \end{bmatrix}, \tag{9}$$

where the potentially complex χ and θ are defined by the relationships $A = D \equiv \cos\theta$ and $\chi \equiv (-B/C)^{1/2}$. In many 2×2 transfer-matrix theories there are also individual elements that are represented by a matrix of this form. In particular, in the Gaussian beam theory, expression (9) is used to represent a complex lenslike medium.[4,5] Similarly, matrix (9) is used to represent a transmission line in the electrical circuit theory. Thus it follows that a symmetric Gaussian beam optical system can be synthesized with a single complex lenslike medium and a symmetric electrical system can be synthesized with a single transmission line. This matrix [expression (9)] is common in matrix theories derived from a second-order differential equation with constant coefficients. If the coefficients are nonconstant, then alternative solutions to the differential equations are of interest.[26,27]

A potentially important special case of expression (9) occurs when $\chi = -1$ and θ is real:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}. \tag{10}$$

This unitary matrix represents a pure rotation about the origin in the XY plane. Such rotation matrices are prevalent in Jones calculus and many other calculations.

Other operations in the XY plane include the nonunimodular matrices for mirror reflection across the X axis,

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{11}$$

and mirror reflection across the Y axis,

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{12}$$

The matrix for mirror reflection across both X and Y signal component axes is also a special case of expression (8). It is written as

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{13}$$

and it occurs in several unimodular matrix theories. In both the paraxial ray and Gaussian beam theories, expression (13) represents a retroreflecting mirror. If the analogy with paraxial ray theory is exploited, then it is suggested that this matrix can be used to represent cross wiring in the electrical circuit theory. Similarly, in the paraxial ray theory, matrix (11) is used to represent a phase conjugate mirror.

Given the possibility of rotations, mirror reflections, and scaling in the XY plane, the next natural operation that arises is translation. However, simple translation in the XY plane cannot be performed with 2×2 matrix theories. An elegant way to account for translation is to augment the 2×2 matrix as a 3×3 matrix of the form

$$\begin{pmatrix} X_2 \\ Y_2 \\ 1 \end{pmatrix} = \begin{bmatrix} A & B & E \\ C & D & F \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} X_1 \\ Y_1 \\ 1 \end{pmatrix}. \tag{14}$$

In this case the matrix for a simple translation is

$$\begin{bmatrix} 1 & 0 & \chi_X \\ 0 & 1 & \chi_Y \\ 0 & 0 & 1 \end{bmatrix}, \tag{15}$$

where χ_X and χ_Y are the amounts of translation in the X and Y axes, respectively. In the paraxial ray and Gaussian beam theories this translation matrix is interpreted physically as optical element or system misalignment.[1] Thus, with the 3×3 theory, a lens, for example, is allowed to be displaced from the optic axis. A 3×3 electric theory would permit ideal independent voltage and current sources distributed throughout the system. A 3×3 Jones calculus may include signal combining and could account for multiple system inputs.

An important property of matrix form (14) is that the E and F elements do not affect the A–D elements in matrix multiplication. Furthermore, the determinant of the matrix is simply $AD-BC$, the same as that of the corresponding 2×2 matrix.

B. System Synthesis

Design criteria for a given system can be realized as constraints on the system matrix. These constraints may often be written in terms of matrix operations. In this subsection several matrix operations are identified that permit the system matrix, based on these design criteria, to be found. Once the system matrix is known, then it is of interest to consider procedures to determine the optical components needed to fulfill these criteria, which is accomplished by factorization of the system matrix into matrix primitives. Each of these primitives represents an optical component available to the optical designer. In this subsection only, an emphasis is placed on unimodular matrix theories.

In addition to matrix multiplication of individual matrices, there are several other meaningful operations that may be performed on individual and system matrices. A first step in the synthesis process may include the interpretation of these operations. The first operation considered is the sth power of a unimodular matrix, which is given by the unimodular 2×2 special case of Sylvester's theorem[28]:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^s = \frac{1}{\sin\theta}\begin{bmatrix} A\sin(s\theta) - \sin[(s-1)\theta] & B\sin(s\theta) \\ C\sin(s\theta) & D\sin(s\theta) - \sin[(s-1)\theta] \end{bmatrix}, \tag{16}$$

where

$$\cos\theta \equiv \frac{A+D}{2}. \tag{17}$$

Sylvester's theorem is valid not only for positive integer powers of matrices but for negative integer powers and roots as well.[28] However, this corresponds to a potential design criterion. Suppose that it is desired to synthesize a known system matrix as the cascade of s identical subsystems. The design procedure amounts to taking the sth root ($1/s$ power) of the system matrix and writing it in terms of some set of defined matrix primitives.

Because of the somewhat complicated form of Sylvester's theorem, it is useful to consider the special case of Eq. (16) when $A = D \equiv \cos\theta$ and $\chi \equiv (-B/C)^{1/2}$. Here the system matrix is given by expression (9), and Sylvester's theorem takes on the simpler form

$$\begin{bmatrix} \cos\theta & \chi\sin\theta \\ -\chi^{-1}\sin\theta & \cos\theta \end{bmatrix}^s = \begin{bmatrix} \cos(s\theta) & \chi\sin(s\theta) \\ -\chi^{-1}\sin(s\theta) & \cos(s\theta) \end{bmatrix}. \tag{18}$$

Just as the matrix operation that corresponds to Sylvester's theorem has the physical interpretation of the cascade of s identical optical systems, other matrix operations can also be interpreted. It can be seen from Eq. (1) that a given system matrix yields the output signal given an input signal. If we multiply both sides of Eq. (1) by the inverse of the system matrix, it follows that the matrix inverse can be interpreted as the matrix that yields the input given the output. A reverse matrix may be defined as a matrix that yields the input going in the reverse direction given the output going in the reverse direction. Similarly, the inverse of a reverse matrix yields the output going in the reverse direction given the input going in the reverse direction. These matrix interpretations are summarized in Table 1.

In this way the reverse matrix is the appropriate matrix for propagating through a system backward. When a system matrix is equal to its own reverse, the system is (by our definition) symmetric. For our purposes this symmetry may be part of the given design criteria. However, different matrix theories may possess different reverse matrices. For paraxial light rays and Gaussian beams in first-order optical systems and for electrical signals in two-port networks the reverse matrix is[29]

$$T_R = \begin{bmatrix} D & B \\ C & A \end{bmatrix}. \tag{19}$$

As we will show below, this is also the reverse matrix for other unimodular matrix theories in which the Y parameter is the derivative of the X parameter. If a given matrix is equal to its reverse matrix, then the system is symmetric, and from Eq. (19) it follows that the condition for symmetry is $A = D$.

As opposed to these matrix theories, the reverse matrix for the unitary form of Jones calculus is[19]

$$T_R = \begin{bmatrix} A & C \\ B & D \end{bmatrix}. \tag{20}$$

Thus a lossless birefringent optical system is symmetric if its Jones matrix elements have the property that $B = C$.

As part of the design criteria for a given periodic system, the input signal may be required to repeat after propagating through s identical subsystems. Indeed, the sinusoidal nature of Sylvester's theorem [Eq. (16)] suggests such a repetitive signal condition. In particular, if

$$T^s = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{21}$$

then the input signal is reproduced after propagating through s systems or through a single system s times.

Table 1. Physical Interpretation of Some Simple Matrix Operations[a]

System	Throughput
T	Output given input
T^{-1}	Input given output
T_R	Input$\vert_{\text{going backward}}$ given output$\vert_{\text{going backward}}$
T_R^{-1}	Output$\vert_{\text{going backward}}$ given input$\vert_{\text{going backward}}$

[a]The R subscript represents the reverse operation. Input and output designations are independent of propagation direction.

It may be seen from Eq. (16) that this occurs when $s\theta = 2k\pi$, where k is an integer. In terms of matrix elements, the repetitive signal condition from Eq. (17) is

$$\frac{A+D}{2} = \cos(2k\pi/s), \tag{22}$$

where we make the restriction that

$$0 \leq k \leq s/2 \tag{23}$$

to avoid duplicating solutions. A graphic interpretation of the result is given in Ref. 30.

When the system matrix, based on design criteria, is known, one must factor the system matrix in terms of matrix primitives such as expressions (5)–(7). Each of these matrix primitives must represent a manufacturable optical component. If the system matrix is unimodular, then there exist two three-matrix factorizations[22]

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & (A-1)/C \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ C & 1 \end{bmatrix} \begin{bmatrix} 1 & (D-1)/C \\ 0 & 1 \end{bmatrix}, \tag{24}$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ (D-1)/B & 1 \end{bmatrix} \begin{bmatrix} 1 & B \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ (A-1)/B & 1 \end{bmatrix} \tag{25}$$

in terms of only matrix primitives of the form of expressions (6) and (7). As we mentioned above, if the system matrix has the property $A = D$, then a single matrix of the form of expression (9) may be used if it can be realized as a single component. Of course, the factorization in Eq. (24) is valid only when C is nonzero, and similarly the factorization in Eq. (25) is valid only when B is nonzero. If the system matrix elements B and C are both zero or if the system matrix is nonunimodular, then additional factorizations are necessary.[22] In the Jones calculus method other matrix primitives are of interest.[8]

As we suggested above, there exist design criteria that demand that the output signal be identical to the input signal when the system is periodic. Thus it is useful to consider other factorizations of the identity matrix:

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} &= \left(\begin{bmatrix} 1 & 0 \\ -3/l & 1 \end{bmatrix} \begin{bmatrix} 1 & l \\ 0 & 1 \end{bmatrix} \right)^3 \\ &= \left(\pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)^2 = \left(\pm \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right)^2 \\ &= \begin{bmatrix} 0 & \gamma \\ \gamma^{-1} & 0 \end{bmatrix}^2 = - \begin{bmatrix} 0 & \gamma \\ -\gamma^{-1} & 0 \end{bmatrix}^2 . \end{aligned} \tag{26}$$

Here γ is allowed to be complex. It is interesting to note from Eq. (26) that, as opposed to scalars, matrices may have an infinite number of roots. Not all the identity matrix factorizations above are given in terms of matrix primitives. However, in the last case factorization (24) and/or (25) may be inserted into Eq. (26) to accomplish this. These results lead to interesting optical systems such as the cat's-eye reflector,[1] which, by design, has the same system matrix as that for a retroreflector [expression (13)].

C. Bilinear Transformation

For every 2×2 matrix theory there exists an associated bilinear transformation. If a ratio parameter is defined as

$$Z \equiv \frac{X}{Y}, \tag{27}$$

then from Eq. (1) the corresponding transformation for the ratio parameter is

$$Z_2 = \frac{AZ_1 + B}{CZ_1 + D}. \tag{28}$$

This transformation is sometimes called the *ABCD* law. In electrical theory Z is physically interpreted as an impedance, and in Gaussian beam theory it is related to the width and phase-front radius of curvature of a Gaussian beam. In Jones calculus it is interpreted as the ellipse of polarization.[32] The ratio parameter is interpreted as a reflection coefficient in the distributed feedback and fiber ring resonator theories.[31]

Every system has a characteristic ratio parameter Z_∞ such that, if Z_∞ is input to the system, the same Z_∞ is output. Thus, if we constrain the output ratio parameter to be equal to the input ratio parameter, it follows from Eq. (28) and the unimodularity condition $AD - BC = 1$ that

$$A + BZ_\infty = \frac{A+D}{2} \pm i \left[1 - \left(\frac{A+D}{2} \right)^2 \right]^{1/2} \tag{29}$$

$$= \exp(\pm i\theta), \tag{30}$$

where

$$\cos\theta \equiv \frac{A+D}{2}. \tag{31}$$

It may be noted that Eq. (31) is identical to Eq. (17).

If a signal propagates through a system many times or through many systems, then Z may approach the value Z_∞. If Z approaches Z_∞, the system is said to be stable with respect to Z, and this occurs when the complex magnitude[32]

$$|A + BZ_\infty| > 1, \tag{32}$$

where $|A + BZ_\infty| = 1$ represents metastability and $|A + BZ_\infty| < 1$ represents instability.

3. DERIVATION OF GENERALIZED REVERSE THEOREMS

The purpose of this section is to demonstrate a systematic procedure to obtain the reverse matrix for a given matrix theory. The process does not require the usual inspection of individual system elements.[7,19] The secondary purpose of this section is to use these systematic procedures to obtain new reverse matrices for several matrix theories. In particular, the reverse matrix for 3 × 3 electric circuit matrices is found here. These results may be used in studies of reverse propagation through electric circuits with intranetwork independent current and voltage sources. In the Gaussian beam theory the 2 × 2 reverse matrices governing the beam's spot size and phase front curvature are known.[33] The reverse matrix for nonunimodular 3 × 3 paraxial ray matrices is also found. Previously, only the nonunimodular 2 × 2 form and the unimodular 3 × 3 form[34] had been discussed. The reverse matrix for the Jones calculus matrix method has been reported for unimodular 2 × 2 matrices.[7,19] Here the nonunimodular 3 × 3 reverse matrix is derived. This generalization may account for multiple-input optical systems with loss or gain. The nonunimodular 3 × 3 reverse matrix is also found for the matrix theories governing distributed-feedback lasers and waveguides and fiber ring resonators.

The reverse theorems do not apply to optical systems represented by a zero-determinant matrix. In these systems there may be several different inputs that yield the same output. Though there are different reverse matrices for different matrix methods, there are certain universal properties that these reverse matrices all share. For example, the reverse of a product of matrices is the product of the reverse of each of these matrices in reverse order. In equation form this may be written as

$$(T_s \cdots T_2 T_1)_R = T_{1R} T_{2R} \cdots T_{sR}\,. \tag{33}$$

The justification of Eq. (33) is suggested by Fig. 1. In this figure the matrix T, governing propagation from the left-hand side to the right-hand side, is defined as a product of submatrices: $T_4T_3T_2T_1$. Similarly, new matrices going from the right-hand side to the left-hand side may be defined. However, it is evident from the figure that these matrices correspond to the reverse matrices T_R, T_{4R}, T_{3R}, T_{2R}, and T_{1R}. It follows that, to obtain T_R, one must multiply the reverse submatrices in reverse order. This conclusion is independent of whether the matrix is 2 × 2 or of higher order and is independent of the matrix theory being studied. Similarly, there is no assumption about the determinant except that it is nonzero to ensure the existence of the inverse. For Gaussian beam theory this property of the unimodular 2 × 2 reverse matrix was discussed previously.[33]

As a special case of this theorem, suppose that each of the submatrices T_1–T_s is identical; then it follows that

$$(T^s)_R = (T_R)^s\,. \tag{34}$$

This is the intuitive result that propagation through s identical systems backward is the same as propagation backward through a single system s times. Though it was assumed that the exponent s in Eq. (34) is a positive integer, it is valid for any exponent s.

Another special case of Eq. (33) is

$$T_R T = (T T_R)_R\,. \tag{35}$$

The matrix T_RT represents forward propagation through a system followed by reverse propagation through that same system. In standing-wave laser theory this matrix often corresponds to a round trip. Now that some general properties of reverse matrices have been described, the specific reverse matrices for several matrix theories will be discussed.

A. Paraxial Ray Matrices

The purpose of this subsection is to derive the form of the reverse matrix that applies to paraxial light rays. At some position τ along the optic axis the X signal vector component represents the position of a light ray, and the Y component represents the slope of the light ray. For this derivation it is important to note that $Y = \mathrm{d}X/\mathrm{d}\tau$.

The reverse matrix governs the propagation of the signal going backward and starting from the output. Therefore the definition of T_R from Table 1 is

$$T_R \equiv \text{Input}\,|_{\text{going backward}} \text{ given Output}\,|_{\text{going backward}}\,. \tag{36}$$

As above, the propagation of X and $\mathrm{d}X/\mathrm{d}\tau$ is governed by the transformation matrix

$$\begin{bmatrix} X \\ \dfrac{\partial X}{\partial \tau} \end{bmatrix}_2 = T \begin{bmatrix} X \\ \dfrac{\partial X}{\partial \tau} \end{bmatrix}_1, \tag{37}$$

where T is the system matrix. Multiplying both sides of Eq. (37) by T^{-1} yields the formula for the input ray position and slope given the output ray position and slope:

$$\begin{bmatrix} X \\ \dfrac{\partial X}{\partial \tau} \end{bmatrix}_1 = T^{-1} \begin{bmatrix} X \\ \dfrac{\partial X}{\partial \tau} \end{bmatrix}_2. \tag{38}$$

The direction of the signal is $+\tau$. If the signal is propagating in the reverse direction, then τ is replaced with $-\tau$. Thus Eq. (38) may be rewritten as

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} X \\ \dfrac{\partial X}{\partial(-\tau)} \end{bmatrix}_1 = T^{-1} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} X \\ \dfrac{\partial X}{\partial(-\tau)} \end{bmatrix}_2. \tag{39}$$

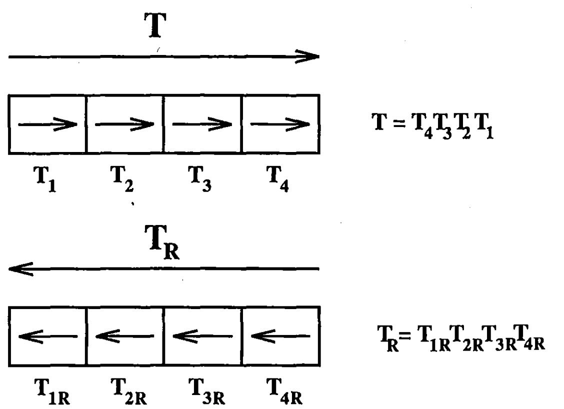

Fig. 1. Schematic demonstration that the reverse of a product of matrices is the product of reverse matrices in reverse order independent of matrix theory, i.e., $(T_4T_3T_2T_1)_R = T_{1R}T_{2R}T_{3R}T_{4R}$.

Multiplying both sides by the matrix in expression (11) and noting that

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{40}$$

reduce Eq. (39) to

$$\begin{bmatrix} X \\ \dfrac{\partial X}{\partial(-\tau)} \end{bmatrix}_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} T^{-1} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} X \\ \dfrac{\partial X}{\partial(-\tau)} \end{bmatrix}_2 . \tag{41}$$

However from the definition of the reverse matrix [Eq. (36)], it follows that the reverse matrix is

$$T_R = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} T^{-1} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{42}$$

Since $(T^{-1})^2 = (T^2)^{-1}$, it can be readily seen from Eq. (40) that $(T_R)^2 = (T^2)_R$. By induction, a general property of reverse matrices [Eq. (34)] follows. Similarly, Eq. (33) can be seen. Thus this specific reverse matrix has the same property that was shown to be a general property of reverse matrices.

The specific form of T^{-1} is well known, and Eq. (42) can be reduced to

$$\begin{aligned} T_R &= \frac{1}{AD - BC}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} D & -B \\ -C & A \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= \frac{1}{AD - BC}\begin{bmatrix} D & B \\ C & A \end{bmatrix}. \end{aligned} \tag{43}$$

In the special case in which the system matrix is unimodular, this is the known result. This result, along with the 2×2 matrix results from Section 2, is listed in Table 2. The symmetry condition is attained if a matrix is equal to its reverse matrix. If the matrix is unimodular, then $AD - BC = 1$, and the system is symmetric if $A = D$.

The use of 3×3 matrices to account for misaligned optical systems has become popular, and the 3×3 reverse matrix for unimodular ray optical systems is known.[34] However, the nonunimodular 3×3 reverse matrix was not previously reported. From the developments here, it is clear that retracing the steps in Eqs. (36)–(41) for systems of the form of expression (14) results in

$$\begin{aligned} T_R &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} T^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \frac{1}{AD - BC}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &\quad \times \begin{bmatrix} D & -B & BF - DE \\ -C & A & CE - AF \\ 0 & 0 & AD - BC \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \frac{1}{AD - BC}\begin{bmatrix} D & B & BF - DE \\ C & A & AF - CE \\ 0 & 0 & AD - BC \end{bmatrix}. \end{aligned} \tag{44}$$

This result is also listed in Table 2.

None of the properties unique to paraxial ray matrices was used in the derivation. Besides the postulating of the existence of an inverse, the only assumption was that the Y component of signal vector was the τ derivative of the X component of the signal vector. Thus this result is not unique to ray matrices. These reverse matrices also apply to the Gaussian beam matrix formalism, the electric circuit matrix theory, and other Wronskian-type matrix theories, where $Y = \mathrm{d}X/\mathrm{d}\tau$.

B. Jones Calculus

The reverse matrix has been found for the ray matrix formalism, the Gaussian beam matrix formalism, and the two-port electric circuit theory. The reverse matrix takes on the same form for each. However, for the Jones polarization calculus the reverse matrix is different and must be calculated separately.

Rather than derive the 2×2 and then the 3×3 reverse matrix, we derive the more general 3×3 case. The Jones matrix is allowed to have the form

$$T = \begin{bmatrix} A & B & E \\ C & D & F \\ 0 & 0 & 1 \end{bmatrix}. \tag{45}$$

The 2×2 reverse matrix becomes a simplified special case, where $E = F = 0$. The transformation of the augmented Jones vectors is

$$\begin{bmatrix} A_x \exp(i\phi_x) \\ A_y \exp(i\phi_y) \\ 1 \end{bmatrix}_2 = T \begin{bmatrix} A_x \exp(i\phi_x) \\ A_y \exp(i\phi_y) \\ 1 \end{bmatrix}_1 , \tag{46}$$

where A_x, A_y, ϕ_x, and ϕ_y are real. Proceeding as in the reverse matrix derivation in Subsection 3.A, we premultiply both sides of Eq. (46) by T^{-1}:

$$\begin{bmatrix} A_x \exp(i\phi_x) \\ A_y \exp(i\phi_y) \\ 1 \end{bmatrix}_1 = T^{-1} \begin{bmatrix} A_x \exp(i\phi_x) \\ A_y \exp(i\phi_y) \\ 1 \end{bmatrix}_2 . \tag{47}$$

When the optical system is lossless, the matrices are unitary, and reversal of the Jones vectors implies reversal of phase. Thus

$$\begin{bmatrix} A_x \exp[i(-\phi_x)] \\ A_y \exp[i(-\phi_y)] \\ 1 \end{bmatrix}_1^* = T^{-1} \begin{bmatrix} A_x \exp[i(-\phi_x)] \\ A_y \exp[i(-\phi_y)] \\ 1 \end{bmatrix}_2^* , \tag{48}$$

where the asterisk represents complex conjugation. Taking the complex conjugate of both sides yields

$$\begin{bmatrix} A_x \exp[i(-\phi_x)] \\ A_y \exp[i(-\phi_y)] \\ 1 \end{bmatrix}_1 = (T^{-1})^* \begin{bmatrix} A_x \exp[i(-\phi_x)] \\ A_y \exp[i(-\phi_y)] \\ 1 \end{bmatrix}_1 . \tag{49}$$

Thus from Eq. (36) it follows that the reverse matrix is

$$T_R = (T^{-1})^* . \tag{50}$$

When the matrix is 2×2 and unitary, this result becomes Eq. (20). However, Eq. (50) also applies to lossless systems represented by unitary 3×3 matrices. Care should be taken in the case of Faraday rotators and media with

Table 2. Reverse Matrices for Several Matrix Theories

Transfer Matrix Theory	2 × 2 Reverse Matrix	3 × 3 Reverse Matrix
Paraxial ray matrices Gaussian beam matrices Electric circuit matrices Miscellaneous Wronskian matrices	$\frac{1}{AD-BC}\begin{bmatrix} D & B \\ C & A \end{bmatrix}$	$\frac{1}{AD-BC}\begin{bmatrix} D & B & BF-DE \\ C & A & -(CE-AF) \\ 0 & 0 & AD-BC \end{bmatrix}$
Jones calculus	$\frac{1}{AD-BC}\begin{bmatrix} D & -B \\ -C & A \end{bmatrix}^*$	$\frac{1}{AD-BC}\begin{bmatrix} D & -B & BF-DE \\ -C & A & CE-AF \\ 0 & 0 & AD-BC \end{bmatrix}^*$
Distributed-feedback matrices Fiber ring resonator matrices	$\frac{1}{AD-BC}\begin{bmatrix} A & -B \\ -C & D \end{bmatrix}$	$\frac{1}{AD-BC}\begin{bmatrix} A & -B & CE-AF \\ -C & D & BF-DE \\ 0 & 0 & AD-BC \end{bmatrix}$

optical activity, however, because, though they have the same forward matrix, their reverse matrices may differ. Equation (50) is listed in Table 2. This result was previously reported only for unimodular 2×2 matrices.[19] If the matrix is unimodular, then the corresponding optical system is symmetric if B and C are pure imaginary and if $A = D^*$. For these Jones matrices the reverse matrix has been defined so that the transverse axes are unchanged. Thus the reverse coordinate system is left handed. When this is undesirable, one may, for example, change the sign of A_y in Eq. (48), and the resulting reverse matrix would be given by the complex conjugate of Eq. (44). As a final note, it is interesting that the transpose may be written for nonzero-determinant matrices as

$$T_T = (AD - BC)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{-1}\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{51}$$

C. Distributed-Feedback Matrices

As a further example of the methodology for finding the reverse matrix, the reverse matrix is found for the distributed-feedback matrix theory. Here the electric field is separated into rightward and leftward waves that form the signal vector. For generality, it is postulated that matrices take the more general form of Eq. (45). Thus the signal vector is augmented with unity as

$$\begin{bmatrix} A^+ \\ A^- \\ 1 \end{bmatrix}_2 = T\begin{bmatrix} A^+ \\ A^- \\ 1 \end{bmatrix}_1. \tag{52}$$

Proceeding as in Subsections 3.A and 3.B, we multiply both sides by T^{-1}:

$$\begin{bmatrix} A^+ \\ A^- \\ 1 \end{bmatrix}_1 = T^{-1}\begin{bmatrix} A^+ \\ A^- \\ 1 \end{bmatrix}_2. \tag{53}$$

The signal vector can again be written as a matrix multiplied by the corresponding signal vector traveling in the opposite direction:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} A^- \\ A^+ \\ 1 \end{bmatrix}_1 = T^{-1}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} A^- \\ A^+ \\ 1 \end{bmatrix}_2. \tag{54}$$

Multiplying both sides of Eq. (54) by the appropriate matrix yields

$$\begin{bmatrix} A^- \\ A^+ \\ 1 \end{bmatrix}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}T^{-1}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} A^- \\ A^+ \\ 1 \end{bmatrix}_2. \tag{55}$$

Thus it follows from Eq. (36) that

$$T_R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}T^{-1}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{56}$$

The result of this calculation is included in Table 2. In the 2×2 special case, Eq. (56) reduces to

$$T_R = \frac{1}{AD - BC}\begin{bmatrix} A & -C \\ -B & D \end{bmatrix}. \tag{57}$$

If $AD - BC = 1$, then the system represented by the matrix T is symmetric if $B = -C$.

4. EXAMPLE: MISALIGNED LASER

Standing-wave laser oscillators consist of optical elements that are inevitably out of perfect alignment. These misalignments, whether they are accidental or intentional, are crucial to the operation of the laser. Misalignment sensitivity may be examined with the 3×3 reverse matrix of paraxial ray optics.[34] The purpose of this section is to demonstrate that a laser with misaligned intracavity optics may be effectively aligned by a tilt of one of the laser's mirrors. This problem is well suited to paraxial ray optics and the 3×3 reverse matrix.

For simplicity only, it is assumed that the laser oscillator is operated in its fundamental Gaussian mode. Thus the design condition is that the Gaussian beam emerge from the laser perpendicular to the output coupler. For this example it is assumed that the flat untilted output coupler is the right-hand mirror and that the left-hand mirror is also flat but tilted at some angle θ. It is furthermore assumed that the round-trip Gaussian beam matrix for the laser consists of purely real elements. This is the case when there are no significant apertures and the gain per wavelength of the amplifying medium is small. In only these lossless optical systems the center of a Gaussian beam travels along paraxial light ray trajectories.[5] Thus one may use ray matrix techniques to trace the displacement and the slope of the center of the Gaussian beam.

The paraxial ray matrix for a mirror tilted at an angle θ is

$$T_{\text{mirror}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \tan(2\theta) \\ 0 & 0 & 1 \end{bmatrix}. \tag{58}$$

The intracavity optics, which may be misaligned, are represented by an *ABCDEF* matrix. If the reference plane is chosen at the output coupler, then the round-trip matrix is

$$T_{\text{round trip}} = T_{\text{system}} T_{\text{mirror}} (T_{\text{system}})_R \tag{59}$$

$$= \begin{bmatrix} A & B & E \\ C & D & F \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \tan(2\theta) \\ 0 & 0 & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} D & B & BF - DE \\ C & A & -(CE - AF) \\ 0 & 0 & 1 \end{bmatrix} \tag{60}$$

$$= \begin{bmatrix} AD + BC & 2AB & B[2(AF - CE) + \tan(2\theta)] \\ 2CD & AD + BC & D[2(AF - CE) + \tan(2\theta)] \\ 0 & 0 & 1 \end{bmatrix}, \tag{61}$$

where the unimodularity condition, valid for any laser oscillator,[22] has been used. The oscillation condition states that the displacement and the slope of the center of the Gaussian beam repeat after some number of round trips. Furthermore, the design condition requires that the final position and slope be zero, which is satisfied when the E and F elements of the round-trip matrix are identically zero. From Eq. (61) the E and F elements of the round-trip matrix are zero if $B = D = 0$. However, this would violate the unimodularity requirement of the system. The design condition would also be satisfied if

$$\tan(2\theta) = 2(CE - AF). \tag{62}$$

In this case the system is effectively aligned even though the intracavity optics are individually misaligned. It may be noted that a nonlaser reflective optical system can also be aligned in this fashion.

The fact that a complicated optical system with various misalignments can be aligned simply by the rotation of the feedback mirror is reasonable when the axial ray is considered. If the optical system is aligned, then an axial light ray (r, $r' = 0$) will remain an axial light ray at the output. However, in a misaligned optical system an axial light ray is displaced at the output, and the slope may be changed as well. If this output light ray strikes a mirror at normal incidence, then the light ray will retrace its path back through the optical system. Thus the axial light ray will return to its initial form, though the direction of the ray has changed. This means that the overall system is aligned. To ensure that the axial light ray strikes the mirror at normal incidence, we must tilt the mirror, and the tilt angle is given by Eq. (62).

5. CONCLUSION

Backward propagation through an optical system occurs in a large class of multipass applications, where the optical signal traverses the system at least twice. Such applications include remote sensing, nondestructive evaluation, and synthesis of optical delay lines and laser oscillators. In the ubiquitous matrix theories considered here, a 2×2 transfer matrix is used to represent forward propagation of light through the optical system, and a corresponding reverse matrix is used to represent backward propagation. A general procedure has been demonstrated to obtain reverse matrices. The reverse matrix has been found for several generalized theories that make use of matrices that may be nonunimodular, possess an important 3×3 form, or both. A possible application of the results has been demonstrated with an example. In particular, it was shown that, by tilting a mirror, one may align a laser even though it possesses misaligned intracavity optics.

ACKNOWLEDGMENT

This study was supported in part by the National Science Foundation under grant ECS-9014481.

REFERENCES

1. A. Gerrard and J. M. Burch, *Introduction to Matrix Methods in Optics* (Wiley, New York, 1975).
2. H. Kogelnik, "Imaging of optical modes—resonators with internal lenses," Bell Syst. Tech. J. **44**, 455–494 (1965).
3. H. Kogelnik, "On the propagation of Gaussian beams of light through lenslike media including those with a loss or gain variation," Appl. Opt. **4**, 1562–1569 (1965).
4. L. W. Casperson and A. Yariv, "The Gaussian mode in optical resonators with a radial gain profile," Appl. Phys. Lett. **12**, 355–357 (1968).
5. L. W. Casperson, "Gaussian light beams in inhomogeneous media," Appl. Opt. **12**, 2434–2441 (1973).
6. L. W. Casperson and S. Lunnam, "Gaussian modes in high loss laser resonators," Appl. Opt. **14**, 1193–1199 (1975).
7. R. C. Jones, "A new calculus for the treatment of optical systems. I. Description and discussion of the calculus," J. Opt. Soc. Am. **31**, 488–493 (1941).
8. R. C. Jones, "A new calculus for the treatment of optical systems. II. Proof of three general equivalence theorems," J. Opt. Soc. Am. **31**, 493–499 (1941).
9. R. C. Jones, "A new calculus for the treatment of optical systems. III. The Sohncke theory of optical activity," J. Opt. Soc. Am. **31**, 500–503 (1941).
10. M. Born and E. Wolf, *Principles of Optics*, 6th ed. (Pergamon, New York, 1980), pp. 66–70.
11. R. A. Craig, "Method for analysis of the characteristic matrix in optical systems," J. Opt. Soc. Am. A **4**, 1092–1096 (1987).
12. J. Hong, W. Huang, and T. Makino, "On the transfer matrix method for distributed-feedback waveguide devices," J. Lightwave Technol. **10**, 1860–1868 (1992).
13. S. P. Dijaili, A. Dienes, and J. S. Smith, "*ABCD* matrices for dispersive pulse propagation," IEEE J. Quantum Electron. **26**, 1158–1164 (1990).
14. J. L. A. Chilla and O. E. Martinez, "Spatial-temporal analysis of the self-mode-locked Ti:sapphire laser," J. Opt. Soc. Am. B **10**, 638–643 (1993).
15. J. L. Rosner, "The Smith chart and quantum mechanics," Am. J. Phys. **61**, 310–316 (1993).
16. R. A. Plastock and G. Kalley, *Schaum's Outline Series, Theory and Problems of Computer Graphics* (McGraw-Hill, New York, 1986).
17. See, for example, B.-G. Kim and E. Garmire, "Comparison between the matrix method and the coupled-wave method in

the analysis of Bragg reflector structures," J. Opt. Soc. Am. A **9,** 132–136 (1992).
18. J. Krasinski, D. F. Heller, and Y. B. Band, "Multipass amplifiers using optical circulators," IEEE J. Quantum Electron. **26,** 950–958 (1990).
19. N. Vansteenkiste, P. Vignolo, and A. Aspect, "Optical reversibility theorems for polarization: application to remote control of polarization," J. Opt. Soc. Am. A **10,** 2240–2245 (1993).
20. P. Yeh, "Extended Jones matrix method," J. Opt. Soc. Am. **72,** 507–513 (1982).
21. J. Capmany and M. A. Muriel, "A new transfer matrix formalism for the analysis of fiber ring resonators: compound coupled structures for FDMA demultiplexing," J. Lightwave Technol. **8,** 1904–1919 (1990).
22. L. W. Casperson, "Synthesis of Gaussian beam optical systems," Appl. Opt. **20,** 2243–2249 (1981).
23. O. E. Martinez, "Matrix formalism for pulse compressors," IEEE J. Quantum Electron. **24,** 2530–2536 (1988).
24. O. E. Martinez, P. Thiagarajan, M. C. Marconi, and J. J. Rocca, "Correction to magnified expansion and compression of subpicosecond pulses from a frequency-doubled Nd:YAG laser," IEEE J. Quantum Electron. **26,** 1676–1679 (1990).
25. O. E. Martinez, "Matrix formalism for dispersive laser cavities," IEEE J. Quantum Electron. **25,** 296–300 (1989).
26. L. W. Casperson, "Beam propagation in tapered quadratic-index waveguides: analytical solutions," J. Lightwave Technol. **LT-3,** 264–272 (1985).
27. L. W. Casperson, "Beam propagation in periodic quadratic-index waveguides," Appl. Opt. **24,** 4395–4403 (1985).
28. P. I. Richards, *Manual of Mathematical Physics* (Pergamon, New York, 1959).
29. A. E. Siegman, "Orthogonality properties of optical resonator eigenmodes," Opt. Commun. **31,** 369–373 (1979).
30. L. W. Casperson and P. M. Scheinert, "Multipass resonators for annular gain lasers," Opt. Quantum Electron. **13,** 193–199 (1981).
31. R. M. A. Azzam and N. M. Bashara, *Ellipsometry and Polarized Light* (North-Holland, New York, 1987), pp. 84–91.
32. L. W. Casperson, "Mode stability of lasers and periodic optical systems," IEEE J. Quantum Electron. **QE-10,** 629–634 (1974).
33. C. Fog, "Synthesis of optical systems," Appl. Opt. **21,** 1530–1531 (1982).
34. V. Magni, "Multielement stable resonators containing a variable lens," J. Opt. Soc. Am. A **4,** 1962–1969 (1987).